Report: Controllability with a finite data-rate of switched linear systems

Guilherme S. Vicinansa and Daniel Liberzon‡

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Abstract

In this work, we argue that the usual notion of controllability is unfit for systems that operate with finite data-rate constraints. We deal with this issue by defining a new concept of controllability with a finite data-rate. Then, we specialize our discussion to the case of switched linear systems. We state a necessary condition and a sufficient condition for our new controllability notion to hold. Next, we take advantage of the switched linear system's structure to present a simple sufficient condition for controllability with a finite data-rate that only involves the controllable subspace of the individual modes and some mild assumptions about the switching signal that guarantee that our sufficient condition holds. We also present another sufficient condition for systems that activate some controllable mode often enough. In particular, we illustrate the power of this result by deriving relations between the sampling time and the Average Dwell-Time (ADT) of the switching signal that guarantee that the switched system is controllable with a finite data-rate. Finally, we discuss the gap between the necessary and the sufficient conditions and show that the sufficient condition is not necessary.

1 INTRODUCTION

Computers and electronic circuits appear everywhere in modern control systems practice. The digital nature of modern controllers and sensors forces our control system not only to work with discretized time but also with quantized measurements. Moreover, since these digital controllers only have a finite number of possible outputs for any given clock cycle, they must operate with a finite data-rate. This latter fact restricts what control problems we can solve with such controllers. Historically, the discovery of new fundamental limitations in control systems guided us to new ideas that helped develop new controller design techniques [21]. For instance, in [13], Kalman introduced the concept of controllability to explain what plant properties hinder our ability to design controllers that solve specific problems. In that same work, he showed how to construct a controller for a controllable plant that sends the system's state to zero as fast as possible, extending the work [4].

In light of this discussion, we ask a natural question: what new constraints arise from the fact that our controller must operate with a finite data-rate? The so-called data-rate theorems [19], which provide the minimum data-rates for stabilizing plants, give part of the answer. Indeed, the control over communication networks community devoted much of its attention to studying such theorems [24, 18, 14] since communication channels restrict the data-rate of the control laws used. Nonetheless, these theorems are not the only restrictions to finite data-rate control. In this article, we argue that, in general, a finite data-rate controller can only make the system's state norm decay exponentially at the fastest in a well-defined sense. This fact shows us that the usual concept of controllability, as defined in [13], is unfit for studying the problem of making the state go to the origin as quickly as possible when data-rate constraints are present. Thus, this motivates us to introduce a new controllability notion suited to this case. We do so with the help of concepts from [6]. In that article, the author introduced a concept of stabilization with a finite data-rate, which, loosely speaking, is the ability to drive the state of a system to zero with a prescribed exponential rate of decay. In our work, we strengthen that notion to allow for arbitrary exponential rates of decay. This latter concept is compatible with the idea of being able to drive the state to zero as fast as possible, as we discuss later.

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†Guilherme S. Vicinansa and Daniel Liberzon are are with the Coordinated Science Laboratory, University of Illinois Urbana-Champaign, U.S.A. {gs16,liberzon}@illinois.edu

To continue our analysis, we specialize to switched linear systems [16]. For this case, we provide a necessary condition and a sufficient condition for our system to be controllable with a finite data-rate. The sufficient condition we present, however, might be hard to check. To address this issue, we exploit the switched system's structure to help us derive simple conditions that we can verify by imposing mild assumptions on the modes and on the switching signal. One of these conditions has a geometric nature, involving the controllable subspace of the modes and an easily checked assumption on the switching signal. Another condition requires our system to switch to a controllable mode frequently enough, in a sense specified later. When we assume that all the modes are controllable and our system satisfies an average dwell-time condition, this latter result allows us to derive a simple inequality involving the average dwell-time, the sampling frequency, and the chatter bound that guarantees controllability with a finite data-rate. Finally, we discuss the gap between our necessary condition and the sufficient one. Closing this gap is the topic of future research.

We take this opportunity to connect this work with [25], where we discussed controllability with a finite data-rate for linear time-varying (LTV) systems. In that document, we also presented a necessary condition and a sufficient condition for a general LTV system to be controllable with a finite data-rate. The necessary condition we state here is the same as the one from [25]. On the other hand, the sufficient condition we present here is different and more directly checked than the one we stated in [25]. In fact, our new condition is a consequence of the one we discussed in [25], as we prove in the present paper. Nonetheless, by asking more of our system, i.e., by imposing the switched linear structure, we get several elementary conditions for controllability with a finite data-rate that are, in some sense, more realistically verified. Thus, we can see the current paper as a continuation of that one where we impose more structure to our system to get readily checked sufficient conditions for the class of switched linear systems.

The structure of this document is as follows: First, in Section 2, we present the model we want to study and describe why we need a new controllability notion. Still in Section 2, we define controllability with a finite data-rate. Next, in Section 3, we state a necessary condition and a sufficient condition for controllability with a finite data-rate. Then, by constraining the switching signal and the sampling times, we derive a sufficient condition for controllability using the controllable subspaces of the modes. After that, in Section 4, we present another sufficient condition that requires our system to activate some controllable mode often enough. In this section, we present our condition that involves average dwell-time and the sampling frequency. We finish this section with a discussion on the gap between the necessary condition and the sufficient one. In Section 5, we conclude and present future research directions. Finally, in the appendix, we prove the results we state in the paper.

Notation: We denote by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) the set of the positive (nonnegative) integers. We denote by $\mathbb{R}$ the set of real numbers. We denote by $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) the set of positive (nonnegative) real numbers. Given $m \in \mathbb{Z}_{>0}$, we define the set $[m] := \{1, \ldots, m\}$. Given a set S, we denote by $\#S$ its cardinality. Let $d_x \in \mathbb{Z}_{>0}$ and $d_u \in \mathbb{Z}_{>0}$, we denote by $\mathcal{M}^{d_x \times d_u}$ the set of $d_x \times d_u$ real matrices. Let $d_x \in \mathbb{Z}_{>0}$, then we denote by I_{d_x} the $d_x \times d_x$ identity matrix. We denote the transpose of a matrix $A \in \mathcal{M}^{d_x \times d_u}$ by $A' \in \mathcal{M}^{d_u \times d_x}$. Given a pair of matrices (A, B) with $A \in \mathcal{M}^{d_x \times d_x}$ and $B \in \mathcal{M}^{d_x \times d_u}$, we denote by $\langle A|B \rangle$ their controllable subspace. Given $A \in \mathcal{M}^{d_x \times d_x}$ and $B \in \mathcal{M}^{d_x \times d_x}$ two symmetric positive semi-definite matrices, we write that $A \geq B$ ($A > B$) if $A - B$ is positive semidefinite (definite). If A is a $d_x \times d_x$ real matrix and $|\cdot|$ is a vector norm[1] in $\mathbb{R}^{d_x}$, we denote by $\|A\| := \max\{|Ax| : |x| = 1, x \in \mathbb{R}^{d_x}\}$ the norm induced by that vector norm. For a set $S \subset \mathbb{R}^{d_x}$, we define its maximum distance from the origin as $\mathrm{dist}(S) := \sup\{|x| : x \in S\}$. We denote by $\log(a)$ the natural logarithm of $a \in \mathbb{R}_{>0}$. We denote by $L^{\infty}_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^{d_u})$ the set of all Lebesgue integrable (see, e.g., Chapter 2 of [7]) locally essentially bounded functions from $[t_0, \infty)$ to $\mathbb{R}^{d_u}$ where $t_0 \in \mathbb{R}_{\geq 0}$ and $d_u \in \mathbb{Z}_{>0}$. Finally, given a function $u : I \subset \mathbb{R} \to \mathbb{R}^{d_u}$ and a set $J \subset I$, we denote by $u_{|J}(\cdot)$ the restriction of the function $u(\cdot)$ to the subset J.

2 MODEL AND MOTIVATION

2.1 The Model

In this work, we study the controllability with a finite data-rate of switched linear systems, i.e., systems described by equation:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{1}$$

where the current time is $t \in [t_0, \infty)$, the initial time is $t_0 \in \mathbb{R}_{\geq 0}$, the initial state is $x(t_0) = x_0$ and it belongs to a compact set with nonempty interior $K \subset \mathbb{R}^{d_x}$, $m \in \mathbb{Z}_{>0}$ is the number of modes, $\sigma : [t_0, \infty) \to [m]$ is the switching

[1]If not stated otherwise, we assume that $|\cdot|$ is the Euclidean norm.

signal, $u : [t_0, \infty) \to \mathbb{R}^{d_u}$ is the control function, and $A_p \in \mathcal{M}^{d_x \times d_x}$ and $B_p \in \mathcal{M}^{d_x \times d_u}$ are the matrices of each mode $p \in [m]$. We also assume that $u(\cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^{d_u})$ and that $\sigma(\cdot)$ is a càdlàg function[2]. We denote by[3] t_n the n-th discontinuity point of $\sigma(\cdot)$ and we call such points the *switching times*. Finally, we define by $\Phi_\sigma(t, \tau)$ for $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$ the *state-transition matrix* associated with the autonomous part of system (1), i.e., $\Phi_\sigma(t, \tau)$ is the unique solution to the differential equation $\dot{\Phi}_\sigma(t, \tau) = A_{\sigma(t)} \Phi_\sigma(t, \tau)$ with $\Phi_\sigma(\tau, \tau) = I_{d_x}$.

A *control law* is a set $\mathscr{U}(K)$ of functions $u(x, \cdot)$ indexed by initial conditions $x \in K \subset \mathbb{R}^{d_x}$, i.e., each initial state $x \in K$ corresponds to a unique control $u(x, \cdot) \in \mathscr{U}(K)$. Denote by $\mathscr{U}_T(K) \coloneqq \{v_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0, T], \mathbb{R}^{d_u}) : v(\cdot) \in \mathscr{U}(K)\}$ the set of restrictions of functions from our control law. We define the *data-rate of the control law* $\mathscr{U}(K)$ as $b(\mathscr{U}(K)) \coloneqq \limsup_{T\to\infty} \frac{1}{T} \log(\#\mathscr{U}_T(K))$ and we say that the control law $\mathscr{U}(K)$ *operates with a finite data-rate* if it satisfies $b(\mathscr{U}(K)) < \infty$.

2.2 The Need for a New Controllability Notion

In this subsection, we explain why the usual notion of controllability of LTV systems is not suitable when we consider control systems that operate with a finite data-rate. To do that, we start by recalling the usual controllability notion (see, e.g., Chapter 9 of [20]) for LTV systems.

Definition 2.1. We say that system (1) is *controllable in the usual sense* on $[t_0, T]$, where $T \geq t_0$, if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^{d_x}$ there exists a function $u : [t_0, T] \to \mathbb{R}^{d_u}$ such that $x(T) = 0$.

To see why this notion is unfit when we work with a finite data-rate, we consider the following simple Example 2.1.

Example 2.1. Let $\dot{x}(t) = u(t)$ where $t \in \mathbb{R}$, $x_0 \in K \subset \mathbb{R}$ with K compact with a nonempty interior, and $u(t) \in \mathbb{R}^{d_u}$. We can easily solve this equation to get that $x(T) = x_0 + \int_{t_0}^T u(\tau) d\tau$. Note that, if $u(t) \in \mathbb{R}^{d_u}$, this system is controllable in the usual sense on the interval $[t_0, T]$. If we impose that this control function comes from a control law that operates with a finite data-rate, we have that the set of possible controls $u_{[t_0,T]}(\cdot)$ on any interval of time $t \in [t_0, T]$ has a finite cardinality. Therefore, the integral $\int_{t_0}^T u(\tau) d\tau$ attains at most finitely many values, but x_0 belongs to the set K, which has infinitely many points. Hence, it is not possible to make $x(T) = 0$ for an arbitrary initial condition in K.

The goal of the previous example is to make the straightforward observation that we cannot have $x(T) = 0$ for an arbitrary initial condition in K, which supports the claim that the usual controllability notion is unfit for the case where we have a finite data-rate. Thus, we must define a new notion of controllability in this setting. One way of doing so, is to think of controllability as the property of being able to drive the state as fast as possible to the origin. This was Kalman's original idea when he introduced the concept of controllability [13]. The following Proposition 2.2 shows that, in general, the fastest mode of decay for the norm of the state of system (1) using finite data-rate is exponentially fast. Indeed, a stronger claim is true for a much larger class of systems.

Proposition 2.2. Let the set of possible initial states $K \subset \mathbb{R}^{d_x}$ have a nonempty interior, let $m \in \mathbb{Z}_{>0}$ be the number of modes, and let $t_0 \in \mathbb{R}_{\geq 0}$ be the initial time. Consider the switched nonlinear time-varying dynamics given by

$$\dot{x}(t) = f(t, \sigma(t), u(x_0, t), x(t)), \tag{2}$$

where $x(t_0) = x_0 \in K$ is a initial state, $u(x_0, \cdot) \in \mathscr{U}(K)$ is the control function that corresponds to the initial state x_0, $\mathscr{U}(K)$ is a control law that operates with a finite data-rate, $\sigma : [t_0, \infty) \to [m]$ is a càdlàg switching signal, and $f : \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$. Also, define $\mathcal{R}_u \coloneqq \overline{\{u(x, t) \in \mathbb{R}^{d_u} : u(\cdot, \cdot) \in \mathcal{U}(K), (x, t) \in P\}}$, where[4] $P \coloneqq \{(x, t) \in K \times [t_0, \infty) : |u(x, t)| < \infty\}$. We assume that:

- Equation (2) has a unique forward-complete[5] Caratheodory solution for each initial state $x_0 \in K$ and the initial time t_0. We denote by[6] $\xi(t, t_0, x_0)$ the Caratheodory solution of (2) at time t when the initial time is t_0 and the initial state is x_0.
- There exists a compact set $B_x \subset \mathbb{R}^{d_x}$ such that[7] $\{|\xi(t, t_0, x_0)| : x_0 \in K, t \in [t_0, \infty)\} \subset B_x$.

[2] A function that is right-continuous and has a left limit everywhere.
[3] We consider t_0 the 0-th discontinuity point of $\sigma(\cdot)$ to keep the notation simple.
[4] Note that $(K \times [0, \infty)) \setminus P$ has measure zero since $u(\cdot, \cdot)$ is locally essentially bounded.
[5] This means that the solution is defined for all $t \in [t_0, \infty)$. See, e.g., Section 1.5 from [9] for sufficient conditions on $f(\cdot, \cdot, \cdot, \cdot)$ for this assumption to hold.
[6] Note that the control is defined by the initial state.
[7] Informally, we are asking the control law to keep the state bounded uniformly over all possible initial states.

- $f(\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable in its fourth argument. Define the Jacobian of $f(\cdot,\cdot,\cdot,\cdot)$ in its fourth argument as $f_x : \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_x}$ where $(f_x(\cdot,\cdot,\cdot,\cdot))_{(i,j)} := \frac{\partial f_i}{\partial x_j}(\cdot,\cdot,\cdot,\cdot)$ for each pair $(i,j) \in [d_x]^2$. We assume that $f_x(\cdot,\cdot,\cdot,\cdot)$ is a continuous function. Further, the quantity $\underline{a} := \operatorname{ess\,sup}\{\|f_x(p_1,p_2,p_3,p_4)\| : p_1 \in [t_0,\infty), p_2 \in [m], p_3 \in \mathcal{R}_u, p_4 \in B_x\}$ is finite.

Denote by $\operatorname{dist}(t,t_0,K) := \sup\{|\xi(t,t_0,x_0)| : x_0 \in K\}$ the maximum distance from a point in the reachable set of (2) at time $t \in [t_0,\infty)$ and the origin of $\mathbb{R}^{d_x}$ when the initial condition belongs to K. Then, we have that

$$\liminf_{t\to\infty} \frac{\log(\operatorname{dist}(t,t_0,K))}{t} > -\infty.$$

In particular, if $f(t,\sigma(t),u(x_0,t),x(t)) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(x_0,t)$, $\mathcal{R}_u$ is a bounded subset of $\mathbb{R}^{d_u}$, and the second bullet above is true, then this result holds.

Thus, it seems natural to relax the usual controllability notion by asking the norm of the state to converge to zero with an arbitrary exponential rate of decay instead of asking the state to equal zero in finite time. To formally state our controllability notion, we use the following Definition 2.2, which is an adaptation from the definitions given in [6] about stabilization with finite data-rate. To improve readability, we name some sets and properties that were not named in [6].

Definition 2.2. We say that system (1) satisfies the *exponential decay condition* with rate $\alpha \in \mathbb{R}_{\geq 0}$, with $M \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$ if for each $x_0 \in K \subset \mathbb{R}^{d_x}$ there exists $u(\cdot) \in L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^{d_u})$ such that the corresponding solution satisfies

$$|x(t)| \leq \big(M|x_0| + \epsilon\big)e^{-\alpha(t-t_0)} \tag{3}$$

for all $t \in \mathbb{R}_{\geq t_0}$. For given $\alpha \in \mathbb{R}_{\geq 0}$, $M \in \mathbb{R}_{>0}$, $\epsilon \in \mathbb{R}_{>0}$, and $K \subset \mathbb{R}^{d_x}$ as above, we call a set[8] $\mathscr{R}(\epsilon,M,K,\alpha) \subset L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^{d_u})$ a *stabilizing control set* of system (1) if for every $x_0 \in K$, there exists a control function $u(\cdot) \in \mathscr{R}(\epsilon,M,K,\alpha)$ such that (3) holds for the corresponding solution. Furthermore, we denote by

$$\begin{aligned}\mathscr{R}_T(\epsilon,M,K,\alpha) := \{u_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0,T],\mathbb{R}^{d_u}) :\\ u(\cdot) \in \mathscr{R}(\epsilon,M,K,\alpha)\}\end{aligned} \tag{4}$$

a *set of restrictions of stabilizing controls*, where $T > t_0$ is arbitrary. We define the *data-rate* associated with system (1) in the following manner. First, given a stabilizing control set $\mathscr{R}(\epsilon,M,K,\alpha)$, we define the *data-rate of the stabilizing control set* $\mathscr{R}(\epsilon,M,K,\alpha)$ as[9]

$$b(\mathscr{R}(\epsilon,M,K,\alpha)) := \limsup_{T\to\infty} \frac{1}{T}\log(\#\mathscr{R}_T(\epsilon,M,K,\alpha)).$$

Next, we define the *data-rate of system* (1) as[10]

$$\begin{aligned}b(M,\alpha) := \lim_{\epsilon\to 0}\big(\inf\{b(\mathscr{R}(\epsilon,M,K,\alpha)) : \mathscr{R}(\epsilon,M,K,\alpha)\\ \text{is a stabilizing control set of (1)}\}\big).\end{aligned} \tag{5}$$

Finally, we say that system (1) can be *stabilized with finite data-rate* with $M \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$ if $b(M,\alpha) < \infty$.

The reader might wonder if we can remove the ϵ term from inside inequality (3) and still get a reasonable notion of stabilizability with finite data-rate. The answer is negative, and is proved in Proposition 2.2 of [6] where the author showed that, for any pair $(\alpha,M) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, LTI systems with poles with a nonnegative real part cannot satisfy (3) with $\epsilon = 0$ and have $b(M,\alpha) < \infty$. Also, we take this opportunity to note that the limit on the right-hand side of equation (5) exists. That happens because the infimum on the right-hand side of that equality is a monotonically decreasing function of ϵ. Consequently, that limit can be replaced by the supremum over $\epsilon \in \mathbb{R}_{>0}$. We also note that $\mathscr{R}(\epsilon,M,K,\alpha)$ is a control law[11] that operates with the data-rate $b(\mathscr{R}(\epsilon,M,K,\alpha))$. Now, we are ready to define controllability with a finite data-rate, which is one of the contributions of this paper.

[8]We note that this set can be infinite in general.

[9]The corresponding quantity in [6] uses the limit inferior instead of limit superior. Because of that, if the quantity given in [6] is infinite, ours is also infinite.

[10]Note that $b(M,\alpha)$ also depends on the set of initial conditions K. We drop that dependence to make the notation simpler.

[11]See Subsection 2.1

Definition 2.3. We say that system (1) is *controllable with a finite data-rate* if for every $\alpha \in \mathbb{R}_{>0}$, there exists $M \in \mathbb{R}_{\geq 0}$ such that system (1) can be stabilized with finite data-rate $b(M, \alpha) < \infty$.

In light of our discussion, Definition 2.3 captures the property of the norm of the state converging to zero as fast as possible in our setting. We believe that it is a natural candidate for extending the concept of controllability to switched linear systems with finite data-rate. It is important to remark that the previous definition is new and it differs from the definition of stabilization with a finite data-rate, originally given in [6], in the sense that it captures the possibility of stabilization with an arbitrary convergence rate $\alpha \in \mathbb{R}_{\geq 0}$, while in [6] α was taken to be a fixed parameter.

3 CHARACTERIZING CONTROLLABILITY WITH A FINITE DATA-RATE

In this section, we characterize controllability with a finite data-rate of switched linear systems using classical controllability notions. We recall the concept of complete controllability to state our necessary condition. Next, we use the well-known uniform complete controllability notion to present our sufficient condition. After that, we briefly discuss the gap between our two conditions. Finally, we provide a geometric characterization for uniform complete controllability of switched linear systems using the controllable subspaces of each mode and the switching signal.

3.1 The Necessary Condition

We start this subsection by recalling the notion of complete controllability, first stated in [13] and [12].

Definition 3.1. We say that system (1) is *completely controllable* if, for each $\bar{t} \in [t_0, \infty)$, there exists $t_1 \in (\bar{t}, \infty)$ such that (1) is controllable in the usual sense[12] on the time interval $[\bar{t}, t_1]$.

We have two remarks about this definition. First, some authors, such as [22] in Chapter 4, use the term "complete controllability" to refer to usual controllability on a given time interval. The difference is that Definition 3.1 requires system (1) to be controllable over infinitely many intervals, while the definition given in [22] requires the system to be controllable on a single time interval. Second, recall that the *controllability Gramian* of system (1) is given by $W(t, s) := \int_s^t \Phi_\sigma(t, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(t, \tau) d\tau$ for any $t \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}_{>0}$. Then, it is a well-known fact (see, e.g., [13]) that complete controlabillity is equivalent to the statement: for every $\bar{t} \in \mathbb{R}_{\geq 0}$ there exists some $t_1 > \bar{t}$ such that $W(t_1, \bar{t})$ is invertible. This result gives us an operational way to check if a system is completely controllable. Now, we are ready to state our necessary condition in Theorem 3.1.

Theorem 3.1. System (1) is controllable with finite data-rate only if it is completely controllable.

This statement is interesting because it gives a simple condition that guarantees that, if not satisfied, we can rule out the possibility of our system being controllable with a finite data-rate. This theorem appears in a slightly different form, stated for a more general class of LTV systems, in [25]. We also refer to [25] for an example of a system that does not satisfy the assumption of Theorem 3.1.

3.2 The Sufficient Condition

To state the sufficient condition, we must first recall a classical controllability notion for LTV systems.

Definition 3.2. We say that system (1) is *uniformly completely controllable* (UCC) if there exist $T \in \mathbb{R}_{>0}$ and some $\underline{w} \in \mathbb{R}_{>0}$ such that the controllability Gramian satisfies $\underline{w} I_{d_x} \leq W(t+T, t)$ for all $t \in \mathbb{R}_{>0}$, where the inequality here denotes the partial order relation on symmetric positive definite matrices.

We remark that this concept was introduced by Kalman in works [13] and [12] using different conditions from the one we stated. It was [2] who proved that, if $A_{\sigma(\cdot)}$ and $B_{\sigma(\cdot)}$ are uniformly bounded for all times, then the condition we present in Definition 3.2 is equivalent to UCC. Now, we are ready to state our sufficient condition:

Theorem 3.2. System (1) is controllable with a finite data-rate if it is UCC.

[12] See Definiton 2.1.

This result is a consequence of Theorem 3.1 from [25]. In that work, we came up with a different sufficient condition for general LTV systems to be controllable with a finite data-rate. It happens that being UCC is a stronger condition than the one state in that paper. Thus, an LTV UCC system is controllable with a finite data-rate. We prove this latter fact in the appendix.

The previous result applies to any LTV system[13], and it requires us to prove that our system is UCC, which might be difficult in general. However, assuming that our system is given by the switched linear dynamics (1), we can prove results that involve the controllable subspaces of the modes and some properties of the switching signal.

3.3 Geometric Characterization of UCC for switched systems

Up to this point, we did not mention sampling. The reason is because all the previous results do not need an explicit sampling strategy to hold. In this subsection, however, the relationship between the sampling times and the switching times will be instrumental. Thus, we start it by defining the sampling times and some related concepts. Then, we present a controllability notion that allows us to give a geometric criteria for a switched linear system to be controllable with a finite data-rate.

We define the sequence of *sampling times* $(\tau_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [t_0, \infty)$ by

$$\tau_n \coloneqq t_0 + nT_p, \tag{6}$$

where $T_p \in \mathbb{R}_{>0}$ is the *sampling period*. Next, recall that $(t_n)_{n\in\mathbb{Z}\geq 0}$ is the sequence of switching times. Note that the sequences $(\tau_n)_{n\in\mathbb{Z}_{>0}}$ and $(t_n)_{n\in\mathbb{Z}_{>0}}$ are not, in principle, related. When $(\tau_n)_{n\in\mathbb{Z}_{\geq 0}} \subset (t_n)_{n\in\mathbb{Z}_{\geq 0}}$, we say that the switching happens *synchronously* with the sampling. At this point, it is convenient to introduce some notation. Let $\mathcal{S} \coloneqq \{n \in \mathbb{Z}_{\geq 0} : \sigma(\tau_n) \neq \sigma(t) \text{ for some } t \in [\tau_n, \tau_{n+1})\}$, i.e., $n \in \mathcal{S}$ if a switching occurs in the interior of the time interval $[\tau_n, \tau_{n+1})$. Note that $\mathcal{S} = \emptyset$ only if the switchings happen synchronously with the samplings, or if there are no switchings.

Now, to state our condition for uniform complete controllability, we must introduce a new controllability definition. We briefly recall that $\langle A|B\rangle$ denotes the controllable subspace of the pair (A, B).

Definition 3.3. Let $\ell \in \mathbb{Z}_{>0}$ be a discrete time-horizon and let $\mathcal{S} = \emptyset$. For each $k \in \mathbb{Z}_{\geq 0}$, let $n = n(k) \coloneqq \lfloor \frac{k}{\ell} \rfloor$. Define $\mathcal{V}_k \coloneqq \Phi_\sigma^{-1}(\tau_k, \tau_{\ell n})\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$. We say that system (1) is $\ell-$*uniformly completely controllable* if

$$\sum_{j=\ell n(k)}^{\ell(n(k)+1)-1} \mathcal{V}_j = \mathbb{R}^{d_x} \tag{7}$$

for each[14] $k \in \mathbb{Z}_{\geq 0}$.

To help the reader better understand the idea behind Definition 3.3, we first discuss its relationship with classical controllability notions. Notice that equation (7) is the same as the condition for complete controllability on the interval $[\tau_{\ell n}, \tau_{\ell(n+1)}]$ given in Chapter 4 of [22] for some fixed $n \in \mathbb{Z}_{\geq 0}$[15]. In fact, more is true. Since equation (7) holds for each $n \in \mathbb{Z}_{\geq 0}$, a stronger controllability property must hold. The following lemma shows that Definition 3.3 and UCC are equivalent when the switchings are synchronous. Therefore, the existence of $\ell \in \mathbb{Z}_{\geq 0}$ such that our system is ℓ-uniformly completely controllable is sufficient for our system to be controllable with a finite data-rate.

Lemma 3.3. Let $\mathcal{S} = \emptyset$. Then, there exists some $\ell \in \mathbb{Z}_{>0}$ such that system (1) is $\ell-$uniformly completely controllable if, and only if, system (1) is UCC.

The following example should help us illustrate how we can apply Lemma 3.3 to show a nontrivial result.

Example 3.4. Let $\ell \in \mathbb{Z}_{>0}$, let $m = 2$, and let $t_0 = 0$. Let $\{e_1, e_2\} \subset \mathbb{R}^2$ be the canonical basis. Assume that, for each $n \in \mathbb{Z}_{\geq 0}$, there exists at least one integer $k_i(n)$ such that $\ell n \leq k_i(n) < \ell(n+1)$ and that $\sigma(k_i(n)) = i$ for $i \in [2]$. Also, let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $B_1 = e_1$, and $B_2 = e_2$. Note that each individual mode is

[13] Any system with $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, where the function $u(\cdot)$ is integrable and locally essentially bounded an the matrix functions $A(\cdot)$ and $B(\cdot)$ are locally integrable and bounded.

[14] Note that for each $l \in \mathbb{Z}_{\geq 0}$ there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $l = \lfloor \frac{k}{\ell} \rfloor$. Thus, $n(k)$ is a surjective function.

[15] We also notice that there exists an analogous characterization for the concept of complete observability, given in [23].

unstabilizable. A simple calculation shows that $\langle A_i|B_i\rangle = \text{span}\{e_i\}$ for $i \in [2]$. Also, since the matrix A_i is diagonal for each $i \in [2]$, we have that $\Phi_\sigma(t,s)$ is diagonal for each $t \in \mathbb{R}_{\geq 0}$ and $s \in [t,\infty)$. This latter fact implies that $\Phi_\sigma^{-1}(t,s)\langle A_i|B_i\rangle = \langle A_i|B_i\rangle = \text{span}\{e_i\}$ for each $i \in [2]$, all $t \in \mathbb{R}_{\geq 0}$, and all $s \in [t,\infty)$. In particular, for each $n \in \mathbb{Z}_{\geq 0}$ and each $i \in [2]$, we have that $\mathcal{V}_{k_i(n)} = \text{span}\{e_i\}$. Thus, we conclude that $\sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j \supset \mathcal{V}_{k_1(n)} + \mathcal{V}_{k_2(n)} = \mathbb{R}^2$, which implies that our system is ℓ-uniformly completely controllable. Thus, by Lemma 3.3, our system is controllable with a finite data-rate.

The previous example used the fact that the switchings are synchronous to conclude that the switched system is controllable with a finite data-rate, even though the modes are unstabilizable. In the next section, we deal with switching signals that might not be synchronous.

4 APPLICATIONS AND THE GAP BETWEEN CONDITIONS

4.1 Average Dwell-Time and Sampling

At this point, the reader might wonder if there are simple conditions that ensure that the conditions from Theorem 3.2 hold when we do not require the switchings to be synchronous. The next proposition answers this questions affirmatively.

Proposition 4.1. Let $\ell \in \mathbb{Z}_{\geq 0}$. If, for each index $n \in \mathbb{Z}_{\geq 0}$, there exists some index $k(n) \in \mathbb{Z}_{\geq 0}$ such that $\ell n \leq k(n) < \ell(n+1)$, that $k(n) \notin \mathcal{S}$, and that $\langle A_{\sigma(\tau_{k(n)})}|B_{\sigma(\tau_{k(n)})}\rangle = \mathbb{R}^d$, then system (1) is UCC.

Informally, the last proposition is saying the following: if each interval of the form $[\tau_{\ell n}, \tau_{\ell(n+1)})$, where $n \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 0}$ is given, has a sampling subinterval without a switching in its interior and a controllable mode is active on that subinterval, then the conditions of Theorem 3.2 hold. This latter condition is verified, for instance, when we have a “safe" mode, which we visit at least once in each time interval $[\tau_{\ell n}, \tau_{\ell(n+1)})$, i.e., we vist the controllable mode “frequently enough".

Interestingly, Proposition 4.1 has an immediate corollary of practical interest. First, we recall the definition of average dwell-time.

Definition 4.1 (Average Dwell-Time [10])**.** We say that system (1) satisfies an *average dwell-time condition* [10] if there exist a *chatter bound* $N_0 \in \mathbb{Z}_{\geq 0}$ and an *average dwell-time* $\tau_D \in \mathbb{R}_{>0}$ such that the number of switches $N_\sigma(t,\tau)$ on any time interval of the form $[\tau,t) \subset [t_0,\infty)$ satisfies $N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}$.

The next result gives us a simple relation between the sampling period, the chatter bound, and the dwell-time of our switching signal that ensures that system (1) is controllable with a finite data-rate. We prove this corollary in the appendix.

Corollary 4.1. Assume that system (1) satisfies the ADT condition with average dwell-time $\tau_D \in \mathbb{R}_{>0}$ and chatter bound $N_0 \in \mathbb{Z}_{\geq 0}$. Further, assume that system (1) modes' are controllable. If $\frac{\tau_D}{N_0+2} \geq T_p$, then the system is controllable with a finite data-rate.

4.2 The Gap Between Conditions

We see that the difference between the assumptions of Theorem 3.1 and 3.2 is just uniformity. It seems natural to ask if the sufficient condition is actually necessary. The answer is negative as Example 4.2 shows. Before we formally state that example, we take this opportunity to recall some concepts and results presented in [25]. We recall that system (1) is called *persistently completely controllable* if there exists an increasing sequence of times $(s_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [t_0,\infty)$ such that $s_0 = t_0$, that $\lim_{n\to\infty} s_n = \infty$, that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$, and that $W(s_{n+1},s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. We also recall that system (1) satisfies the *exponential energy-growth condition* if there exist constants $N \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}_{\geq 0}$ such that $\|W^{-1}(s_{n+1},s_n)\| \leq Ne^{\theta s_{n+1}}$ for each $n \in \mathbb{Z}_{\geq 0}$. The latter condition is related to the minimum control energy needed to drive the state $x(s_n)$ at time s_n to zero at time s_{n+1} for each $n \in \mathbb{Z}_{\geq 0}$. We refer to [25] for a discussion on this latter point. Now, Theorem 3.1 from [25] says that if an LTV system is persistently completely controllable and satisfies the exponential energy-growth condition, then it is controllable with a finite data-rate. We use this result in our next example to show that UCC is not a necessary condition.

Example 4.2. Let $t_0 = 2$ and consider the equation $\dot{x}(t) = b_{\sigma(t)}u(t)$ with $b_{\sigma(t)} = 1$, when $t \in \cup_{n\geq 1}[2^n, 2^n + 1)$, and $b_{\sigma(t)} = 0$, otherwise. We claim that this system is controllable with a finite data-rate but it is not UCC. We start by choosing a sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [2, \infty)$ such that $s_n = 2^{n+1}$ for $n \in \mathbb{Z}_{\geq 0}$. Naturally, $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ is an increasing sequence that grows to infinity. Also, we have that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = 2$. Further, for each $n \in \mathbb{Z}_{\geq 0}$, on the interval $[2^{n+1}, 2^{n+2})$, we have that $b_{\sigma(t)} = 1$ only on the time subinterval $[2^{n+1}, 2^{n+1} + 1)$ and $b_{\sigma(t)} = 0$ for the remainder of the total interval. Therefore, we get that $W(s_{n+1}, s_n) = \int_{s_n}^{s_{n+1}} b^2_{\sigma(\tau)} d\tau = \int_{2^{n+1}}^{2^{n+1}+1} 1 d\tau = 1$ for each $n \in \mathbb{Z}_{\geq 0}$, i.e., $W(s_{n+1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. Finally, we can easily see that $|W^{-1}(s_{n+1}, s_n)| = 1$ for every $n \in \mathbb{Z}_{\geq 0}$, which implies that our system satisfies the exponential energy-growth condition with $N = 1$ and $\theta = 0$. Thus, our system satisfies all the conditions for Theorem 3.1 from [25] to hold. We therefore conclude that this system is controllable with a finite data-rate. Nonetheless, this system is not UCC. To see that, note that for every $T \in \mathbb{R}_{>0}$ there exists some $n \in \mathbb{Z}_{\geq 0}$ so that $W(s_n + 1 + T, s_n + 1) = 0$. Indeed, this follows from the fact that $b_{\sigma(t)} = 0$ for all $t \in [s_n + 1, s_n + 1 + T)$ if $T < 2^{n+1} - 1$ since $s_n + 1 + T < 2^{n+2}$. This proves the claim.

5 CONCLUSIONS

In this work, we discussed why we need a new controllability notion for systems that operate with a finite data-rate. Then, we presented a necessary condition and a sufficient condition for switched linear systems to be controllable with a finite data-rate. Next, we took advantage of the switched linear system's structure to get simpler sufficient conditions. The first condition, stated in Lemma 3.3, uses the controllable subspaces of the modes and a mild assumption on the switching signal to establish controllability with a finite data-rate. The second one, stated in Proposition 4.1, required us to activate some controllable mode frequently enough. In particular, when all the modes are controllable, this latter condition boils down to a simple inequality for the sampling frequency that guarantees that a system that satisfies an ADT condition is controllable with a finite data-rate.

In future works, we want to study similar conditions for nonlinear systems. Also, we want understand and close the gap between the necessary and the sufficient conditions presented.

Appendix

A Auxiliary Results and Notions

To prove Proposition 2.2, we define the notations $\mathbb{B}(x, r) := \{y \in \mathbb{R}^{d_x} : |x - y| < r\}$ and $\mathbb{B}[x, r] := \{y \in \mathbb{R}^{d_x} : |x - y| \leq r\}$. We use the following two auxiliary lemmas in the proofs of Proposition 2.2 and Lemma 3.3.

Lemma A.1. Let $|\cdot| : \mathbb{R}^{d_x} \to \mathbb{R}_{\geq 0}$ be the Euclidean norm. Consider equation

$$\dot{X}(t, t_0) = A(t)X(t, t_0), \tag{8}$$

where $X(t_0, t_0) = I_{d_x}$, $A(\cdot)$ is locally integrable, and the quantity $\bar{a} := \sup\{\|A(t)\| : t \in [t_0, \infty)\}$ is such that $\bar{a} < \infty$. Then, $X(t, t_0)$ satisfies $e^{-\bar{a}(t-t_0)} \leq |X(t, t_0)v| \leq e^{\bar{a}(t-t_0)}$ for all $t \geq t_0$ and all $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. In particular, it is also true that $\|X(t, t_0)\| \leq e^{\bar{a}(t-t_0)}$.

Proof of Lemma A.1. Theorem 1 from [3] tells us that $X(t, t_0)$ exists and it is given by the uniform limit of the Peano-Baker series on any arbitrary time interval of the form $[t_0, t_1]$ for which $t_1 > t$. More explicitly,

$$X(t, t_0) = \lim_{k\to\infty} M_k(t, t_0),$$

where the limit on the right-hand side is a uniform limit on the time interval $[t_0, t_1]$, and the sequence[16]

$$(M_k(\cdot, t_0))_{k\in\mathbb{Z}_{\geq 0}} \subset L^\infty([t_0, t_1], \mathbb{R}^{d_x\times d_x})$$

is such that $M_0(t, t_0) := I_{d_x}$ and $M_k(t, t_0) := I_{d_x} + \int_{t_0}^{t} A(\tau)M_{k-1}(\tau, t_0)d\tau$ for each $k \in \mathbb{Z}_{>0}$ and each $t \in [t_0, t_1]$.

[16]Recall that the set $L^\infty([t_0, t_1], \mathbb{R}^{d_x\times d_x})$ is the set of integrable essentially bounded functions from $[t_0, t_1]$ with image on $\mathbb{R}^{d_x\times d_x}$.

Our goal now is to prove that $\|X(t,t_0)\| \leq e^{\bar{a}(t-t_0)}$ for all $t \in [t_0,t_1]$. We do that by proving that $\|M_k(t,t_0)\| \leq \sum_{i=0}^{k} \bar{a}^i \frac{(t-t_0)^i}{i!}$ holds for every $k \in \mathbb{Z}_{\geq 0}$ and all $t \in [t_0,t_1]$ using induction. The base case $\|M_0(t,t_0)\| \leq 1$ is trivially true[17]. Now, assume that $\|M_{k-1}(t,t_0)\| \leq \sum_{i=0}^{k-1} \bar{a}^i \frac{(t-t_0)^i}{i!}$ is true. Then,

$$\|M_k(t,t_0)\| \leq 1 + \int_{t_0}^{t} \bar{a}\|M_{k-1}(\tau,t_0)d\tau\| \leq 1 + \sum_{i=0}^{k-1} \bar{a}^{i+1} \frac{(t-t_0)^{i+1}}{(i+1)!} = \sum_{j=0}^{k} \bar{a}^j \frac{(t-t_0)^j}{j!}$$

where $j = i+1$ and the inequality holds for all $t \in [t_0,t_1]$. Thus,

$$\|X(t,t_0)\| = \|\lim_{N\to\infty} M_N(t,t_0)\| = \lim_{N\to\infty} \|M_N(t,t_0)\| \leq e^{\bar{a}(t-t_0)}.$$

for all $t \in [t_0,t_1]$, where the second equality follow from the fact that the norm is a continuous function and $\lim_{N\to\infty} M_N(t,t_0)$ exists. Since $t_1 > t_0$ was arbitrary, $\|X(t,t_0)\| \leq e^{\bar{a}(t-t_0)}$ holds for every $t \geq t_0$. Moreover, by definition of induced norm, we have that $\|X(t,t_0)\| \geq |X(t,t_0)v|$ for any $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. Thus, we get $|X(t,t_0)v| \leq e^{\bar{a}(t-t_0)}$ for all $t \geq t_0$ and all $|v| = 1$, which proves the upper bound.
For the lower bound, pick any $v \in \mathbb{R}^{d_x}$ with $|v| = 1$ and note that

$$1 = |v'v| = |v'X(t_0,t)X(t,t_0)v| \leq |v'X(t_0,t)||X(t,t_0)v|,$$

where the first equality comes from the fact that $|\cdot|$ is the Euclidean norm, the second comes from the fact that $X(t,t_0)X(t_0,t) = I_{d_x}$ (this is due to the semigroup property of transfer matrices, see, e.g., Chapter 4 from [20]), and the last inequality follows from the Cauchy–Schwarz inequality. Now, divide the leftmost and the rightmost terms in the inequality above by[18] $|v'X(t_0,t)|$ to get

$$|X(t,t_0)v| \geq |v'X(t_0,t)|^{-1}.$$

Next, note that

$$\begin{aligned}|X(t,t_0)v| &\geq \min\{|X(t,t_0)v| : |v|=1\} \geq \min\{|v'X(t_0,t)|^{-1} : |v|=1\} = \Big(\max\{|v'X(t_0,t)| : |v|=1\}\Big)^{-1} \\ &= \Big(\max\{|X'(t_0,t)v| : |v|=1\}\Big)^{-1} = \|X(t_0,t)\|^{-1},\end{aligned}$$

where the last equality follows from the definition of induced norm of matrices and fact that $\|A\| = \|A'\|$ for each the matrix $A \in \mathbb{R}^{d_x \times d_x}$ when the norm is the matrix 2-norm, i.e., the matrix norm induced by the Euclidean norm. Now, let $Z(t,t_0) = X'(t_0,t)$. It is a well-known fact that [19]

$$\frac{dZ(t,t_0)}{dt} = -A'(t)Z(t,t_0)$$

with $Z(t_0,t_0) = I_{d_x}$. Thus, we can apply an analogous reasoning to what we did before to get that $\|X'(t_0,t)\| \leq e^{\bar{a}(t-t_0)}$ since $\bar{a} = \sup\{\|-A'(t)\| : t \in [t_0,\infty)\}$ as well. This latter fact implies that $\|X(t_0,t)\|^{-1} = \|X'(t_0,t)\|^{-1} \geq e^{-\bar{a}(t-t_0)}$ for all $t \geq t_0$. So, we have that

$$|X(t,t_0)v| \geq \|X(t_0,t)\|^{-1} \geq e^{-\bar{a}(t-t_0)}$$

for any $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. Therefore, we concluded the proof.

□

Lemma A.2. Let $t_0 \in \mathbb{R}_{\geq 0}$, let $F(t) = \int_0^1 (1-\tau) f_x(t,\sigma(t),v(t),\tau\xi(t,t_0,x_{k_1}) + (1-\tau)\xi(t,t_0,x_{k_2}))d\tau$, where $f_x : [t_0,\infty) \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_x}$ is continuous, $\sigma : [t_0,\infty) \to [m]$ is a càdlàg function, $u(x_{k_1},\cdot) : [t_0,\infty) \to \mathbb{R}^{d_u}$ is an integrable locally essentially bounded function that belongs to a control law $\mathcal{U}(K)$, which operates with a finite data-rate. Also, let $x_{k_1} \in K$, let $x_{k_2} \in K$, and let $\xi(t,t_0,x) \in \mathbb{R}^{d_x}$ be the same as in the statement of Proposition 2.2 for each $x \in K$ and $t \geq t_0$. Then, $F(\cdot)$ is a locally integrable function.

[17] We are using the convention that, for $t = t_0$, $(t-t_0)^0 = \lim_{t\to t_0}(t-t_0)^0 = 1$.
[18] $X(\cdot,\cdot)$ is always invertible, so $|v'X(t_0,t)|$ cannot be zero for any v with $|v| = 1$.
[19] See, e.g., Chapter 4 of [20] for the proof when $A(\cdot)$ is continuous. Once can adapt that argument for the case where $A(\cdot)$ is locally integrable using the arguments from [3].

Proof. Let $D \subset \mathbb{R}$ be a compact set and let $d \in \mathbb{Z}_{>0}$ be arbitrary. We endow $\mathbb{R}$ with the usual Lebesgue σ-algebra $\mathcal{L}(\mathbb{R})$ and $\mathbb{R}^d$ with the product σ-algebra $\mathcal{L}(\mathbb{R}^d) := \otimes_{i=1}^d \mathbb{R}$. Further, we endow any Lebesgue-measurable subset $S \subset \mathbb{R}^d$ with the subset σ-algebra, i.e., the σ-algebra $\mathcal{L}(S, \mathbb{R}^d) := \{X \cap S : X \in \mathcal{L}(\mathbb{R}^d)\}$. Furthermore, we treat any subset of $\mathbb{Z}_{\geq 0}$ as a subset of $\mathbb{R}$. Additionally, we denote by $\lambda_{\mathbb{R}}(\cdot)$ the Lebesgue measure and by $\lambda_{\mathbb{R}^d}(\cdot)$ the d-dimensional Lebesgue measure. We recall that that $\lambda_{\mathbb{R}^d}(\prod_{i=1}^d X_i) = \prod_{i=1}^d \lambda_{\mathbb{R}}(X_i)$ when $X_i \in \mathcal{L}(\mathbb{R})$ for each $i \in [d]$. Finally, we define the Lebesgue measure $\lambda_S(\cdot)$ on a subset $S \subset \mathbb{R}^d$ as the restriction of the function $\lambda_{\mathbb{R}^d}(\cdot)$ to the subspace σ-algebra $\mathcal{L}(S, \mathbb{R}^d)$. We also note that we endow the Cartesian product of two topological spaces with the product topology.

Let $h_1(\cdot) := \sigma_{|D}(\cdot)$, $h_2(\cdot) := u_{|D}(x_{k_1}, \cdot)$, and $h_3(t,\tau) := \tau\xi(t, t_0, x_{k_1}) + (1-\tau)\xi(t, t_0, x_{k_2})$ for $(t,\tau) \in D \times [0,1]$. Note that $h_1(\cdot)$ is bounded and measurable[20], $h_2(\cdot)$ is essentially bounded and measurable, and $h_3(\cdot,\cdot)$ is continuous. The latter fact follows from the facts that $\xi(\cdot, t_0, x_{k_1})$ is an absolutely continuous function (see, e.g., Section 1.5 from [9]) and usual properties of sums and products of continuous functions. We claim that the function $G : D \times [0,1] \to \mathbb{R}^{d_x \times d_x}$ defined as $G(t,\tau) := f_x(t, h_1(t), h_2(t), h_3(t,\tau))$ is a measurable function. Indeed, if we define the function $g : D \times [0,1] \to \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x}$ as $g(t,\tau) := (t, h_1(t), h_2(t), h_3(t,\tau))$, Proposition 2.4 from [7] tells us that $g(\cdot,\cdot)$ is measurable. Next, note that $G(t,\tau) = (f_x \circ g)(t,\tau)$ for each $(t,\tau) \in D \times [0,1]$. Since $g(\cdot,\cdot)$ is measurable and $f_x(\cdot,\cdot,\cdot,\cdot)$ is continuous, it follows that $G(\cdot,\cdot)$ is measurable (see, e.g., Section 2.1 from [7]).

We claim that $G(\cdot,\cdot)$ is essentially bounded. To see that, we make some definitions: let $B_i := \overline{\{h_i(t) : (D \setminus N)\}}$ for each $i \in [2]$ and note that these sets are compact. This latter claim follows from the fact that $h_i(\cdot)$ is bounded on $D \setminus N$ and the fact that B_i is a closed of an Euclidean space for each $i \in [2]$. Also, let $B_3 := \{h_3(t,\tau) : (t,\tau) \in D \times [0,1]\}$, which is also compact because $D \times [0,1]$ is compact and $h_3(\cdot,\cdot)$ is continuous. Since $f_x(\cdot,\cdot,\cdot,\cdot)$ is continuous and $D \times B_1 \times B_2 \times B_3$ is compact (the finite product of compact sets is compact, see, e.g., Theorem 26.7 from [17]), we get that $M_G := \sup\{\|f_x(t, p_1, p_2, p_3)\| : (t, p_1, p_2, p_3) \in D \times B_1 \times B_2 \times B_3\} < \infty$. Now, note that

$$\begin{aligned}\sup\{\|G(t,\tau)\| : (t,\tau) \in (D \setminus N) \times [0,1]\} =& \sup\{\|f_x(t, h_1(t), h_2(t), h_3(t,\tau))\| : (t,\tau) \in (D \setminus N) \times [0,1]\} \leq \\ & \sup\{\|f_x(t, p_1, p_2, p_3)\| : (t, p_1, p_2, p_3) \in D \times B_1 \times B_2 \times B_3\} = M_G.\end{aligned}$$

Thus, $\|G(\cdot,\cdot)\| < \infty$ on $(D \setminus N) \times [0,1]$. All that we need to do now is prove that $N \times [0,1]$ is a null set in $D \times [0,1]$. This follows from the fact that $\lambda_{D \times [0,1]}(N \times [0,1]) = \lambda_D(N)\lambda_{[0,1]}([0,1]) = 0$ since $\lambda_D(N) = 0$, proving the claim. We also note that, since $G(\cdot,\cdot)$ is measurable and essentially bounded, it is integrable.

Finally, note that $F(t) := \int_0^1 (1-\tau)G(t,\tau)d\tau$. Since $G(\cdot,\cdot)$ and $1-\tau$ are integrable, we get that $(1-\tau)G(t,\tau)$ is integrable. Then, the Fubini-Tonelli Theorem (see, e.g., Theorem 2.37 from [7]) guarantees that $F(\cdot)$ is integrable as well. Further, note that $F(\cdot)$ is bounded. Indeed, $\|F(t)\| = \|\int_0^1 (1-\tau)G(t,\tau)d\tau\| \leq \int_0^1 \|G(t,\tau)\|d\tau \leq M_G$, proving the lemma.

□

B Proposition 2.2

Proof of Proposition 2.2. Since $b(\mathscr{U}(K)) = \limsup_{t\to\infty} \frac{1}{t-t_0} \log(\#\mathscr{U}_t(K))$, we know that, for each $\epsilon \in \mathbb{R}_{>0}$, there exists $s^1 = s^1(\epsilon) \in [t_0, \infty)$ such that

$$\frac{1}{t-t_0} \log(\mathscr{U}_t(K)) < \epsilon + b(\mathscr{U}(K)) \tag{9}$$

for all $t \geq s^1$.

Since $K \subset \mathbb{R}^{d_x}$ has nonempty interior, there exists some point $\bar{x} \in K$ and some $\delta = \delta(\bar{x}) \in \mathbb{R}_{>0}$ such that the open ball $\mathbb{B}(\bar{x}, \delta)$ is contained in K. Further, since K is closed, the closed ball $\mathbb{B}[\bar{x}, \delta]$ is also contained in K.

Next, choose an arbitrary $s \in \mathbb{R}_{>0}$ so that $s > s^1 = s^1(\epsilon)$ and define $N_s := \#\mathscr{U}_s(K)$. Also, choose $N_s + 1$ points $S := \{x_1, \ldots, x_{N_s+1}\} \subset \mathbb{B}[\bar{x}, \delta]$ such that $|x_i - x_j| \geq \frac{\delta}{N_s+1}$ for each pair $(i,j) \in [N_s+1]^2$ with $i \neq j$. Note that we can always choose a set of $N_s + 1$ points with this property. One example is the set $S = \{x \in \mathbb{R}^{d_x} : x = \bar{x} + \frac{\delta}{N_s+1} p e_1 \text{ for } p \in [N_s+1]\}$, where $e_1 = (1, 0, \ldots, 0)' \in \mathbb{R}^{d_x}$. It is easy to verify that given $x \in S$ and $y \in S$ distinct, we have $|x-y| = \frac{\delta}{N_s+1} q \geq \frac{\delta}{N_s+1}$ for some $q \in [N_s]$. Also, note that any $x \in S$ is such that $|\bar{x} - x| = \frac{\delta}{N_s+1} p \leq \delta$, proving that $S \subset \mathbb{B}[\bar{x}, \delta]$. Now, by the pigeonhole principle, there are at least two indices $k_1 \in [N_s+1]$ and $k_2 \in [N_s+1]$ of points in S such that $u(x_{k_1}, t) = u(x_{k_2}, t)$ for all $t \in [t_0, s]$. For simplicity, we define $v(t) := u(x_{k_1}, t)$ for each $t \in [t_0, s]$.

[20] Every bounded càdlàg function is Lebesgue measurable since its discontinuity set is countable. See, e.g., Chapter 3 from [5].

Now, for each fixed $t \in [t_0, s]$, we can apply the Taylor Theorem with remainder in its integral form (see, e.g., Theorem 2.68 from [8]) to get

$$f(t,\sigma(t),v(t),\xi(t,t_0,x_{k_1})) = f(t,\sigma(t),v(t),\xi(t,t_0,x_{k_2})) + F(t)(\xi(t,t_0,x_{k_1}) - \xi(t,t_0,x_{k_2})) \tag{10}$$

where $F(t) = \int_0^1 (1-\tau) f_x(t,\sigma(t),v(t),\tau\xi(t,t_0,x_{k_1}) + (1-\tau)\xi(t,t_0,x_{k_2}))d\tau$ is the remainder. To continue the proof, let $\mathscr{X}(t) \coloneqq \xi(t,t_0,x_{k_1}) - \xi(t,t_0,x_{k_2})$ for each $t \in [t_0, s]$. Then, we can write

$$\dot{\mathscr{X}}(t) = \dot{\xi}(t,t_0,x_{k_1}) - \dot{\xi}(t,t_0,x_{k_2}) = f(t,v(t),\xi(t,t_0,x_{k_1})) - f(t,v(t),\xi(t,t_0,x_{k_2})) = F(t)\mathscr{X}(t). \tag{11}$$

We can interpret equation (11) as a variational equation (see, e.g., [15] Section 4.2.4) for nonlinear time-varying controlled switched systems. We notice three things about equation (11). First, we prove in Lemma A.2 that $F(\cdot)$ is locally integrable. Second, note that $\|F(\cdot)\|$ is bounded on $[t_0, \infty)$. This follows from

$$\|F(t)\| \le \operatorname{ess\,sup}\{\|f_x(p_1,p_2,p_3,p_4)\| : p_1 \in [t_0,\infty), p_2 \in [m], p_3 \in \mathcal{R}_u, p_4 \in B_x\} \int_0^1 (1-\tau)d\tau \le \underline{a}.$$

Third, the right-hand side of (11) is a linear time-varying system. Thus, we know that this system has a unique Caratheodory solution (see, e.g., Section 1.5 from [9]). In fact, more is true: we can write $\mathscr{X}(t) = \Phi(t,t_0)\mathscr{X}(t_0)$ for each $t \in [t_0, s]$, where $\Phi(t,t_0)$ is the uniform limit over compact sets of the Peano-Baker series (see, e.g., Theorem 1 from [3]).

Since $F(\cdot)$ is locally integrable and its norm is uniformly bounded by $\underline{a}$, Lemma A.1 tells us that $|\Phi(t,t_0)\mathscr{X}(t_0)| \ge e^{-\underline{a}(t-t_0)}|\mathscr{X}(t_0)|$ for each $t \in [t_0,\infty)$. Further, recall that, by our previous definitions of $\mathscr{X}(\cdot)$ and the set S, we have $|\mathscr{X}(t_0)| = |x_{k_1} - x_{k_2}| \ge \frac{\delta}{N_s} \ge \frac{\delta}{2N_s}$. By (9) and the fact that $s > s^1$, we know that $\frac{1}{s-t_0}\log(N_s) < \epsilon + b(\mathscr{U}(K))$, which implies that $N_s < e^{(s-t_0)(b(\mathscr{U}(K))+\epsilon)}$, which, by its turn, implies that $\frac{1}{N_s} > e^{-(s-t_0)(b(\mathscr{U}(K))+\epsilon)}$. Therefore,

$$|\mathscr{X}(s)| = |\Phi(s,t_0)\mathscr{X}(t_0)| \ge \frac{\delta}{2} e^{-a(s-t_0)} e^{-(s-t_0)(b(\mathscr{U}(K))+\epsilon)}. \tag{12}$$

Note that $|\xi(s,t_0,x_{k_1}) - \xi(s,t_0,x_{k_2})| \le 2\max\{|\xi(s,t_0,x_{k_1})|, |\xi(s,t_0,x_{k_2})|\} \le 2\sup\{|\xi(s,t_0,x)| : x \in K\} = 2\mathrm{diam}(s,t_0,K)$, where the first inequality follows from the triangle inequality and the second follows from the fact that $x_{k_1} \in K$ and $x_{k_2} \in K$. Hence, combining inequality (12) and the above, we get that

$$2\mathrm{diam}(s,t_0,K) \ge \frac{\delta}{2} e^{-\underline{a}(s-t_0)} e^{-(s-t_0)(b(\mathscr{U}(K))+\epsilon)}. \tag{13}$$

Finally, we write

$$\frac{1}{s-t_0}\log(\mathrm{diam}(s,t_0,K)) \ge \frac{1}{s-t_0}\log(\delta/4) - \underline{a} - b(\mathscr{U}(K)) + \epsilon, \tag{14}$$

for $s > s^1(\epsilon)$ arbitrary. This implies that

$$\liminf_{s\to\infty} \frac{1}{s-t_0}\log(\mathrm{diam}(s,t_0,K)) \ge -\underline{a} - b(\mathscr{U}(K)) + \epsilon > -\infty \tag{15}$$

proving the proposition.

To see why the switched linear case, i.e., $f(t,\sigma(t),u(x_0,t),x(t)) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(x_0,t)$, is a particular case, we have to show that the three bullets that appear in the proposition statement hold. To prove that the first bullet holds, note that the functions $A_{\sigma(\cdot)}$ and $B_{\sigma(\cdot)}$ are integrable since $\sigma(\cdot)$ is measurable and there are only finitely many modes. Further, since $\mathcal{R}_u \subset \mathbb{R}^{d_u}$ is bounded, we have that $f(\cdot,\cdot,\cdot,\cdot)$ satisfies conditions for existence and uniqueness for Caratheodory solutions (see, e.g., Section 1.5 from [9]). All that is left for us is to prove the third bullet. Since there are finitely many modes, we know that the quantity $f_x(t,\sigma(t),u(x_0,t),x(t)) = A_{\sigma(t)}$ remains bounded for all times, proving that all assumptions hold for the switched linear case. This concludes the proof of this proposition. □

C Theorem 3.2

Proof of Theorem 3.2. We prove that if system (1) is UCC, then it satisfies the assumptions from Theorem 3.1 from [25]. The assumptions of that theorem are: system (1) is persistently completely controllable and it satisfies the exponential energy-growth condition[21]. First, we prove that UCC implies persistent complete controllability. Since

[21] See subsection 4.2 for the formal statement of these conditions.

systems (1) is UCC, there exists $T \in \mathbb{R}_{>0}$ and $\underline{w} \in \mathbb{R}_{>0}$ such that $W(t+T,t) \geq \underline{w} I_{d_x}$ for all $t \in [t_0, \infty)$. Choose $s_n = t_0 + Tn$ for each $n \in \mathbb{Z}_{\geq 0}$. Note that $W(s_{n+1}, s_n)$ is invertible since $s_{n+1} - s_n = T$ for each $n \in \mathbb{Z}_{\geq 0}$. Further, $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$ since $\lim_{n\to\infty} \frac{t_0+(n+1)T_p}{t_0+nTp} = 1$. Thus, we proved the first claim. Next, note that $W(t+T,t) \geq \underline{w} I_{d_x}$ implies that $\|W^{-1}(s_{n+1}, s_n)\| = \max\{|W^{-1}(s_{n+1}, s_n)v| : |v| = 1\} \leq \underline{w}^{-1}$ for each $n \in \mathbb{Z}_{\geq 0}$, proving the second claim. Thus, we proved our theorem. □

D Lemma 3.3

To prove Lemma 3.3, we need the following auxiliary result:

Lemma D.1. Let $\ell \in \mathbb{Z}_{>0}$. System (1) is $\ell-$uniformly completely controllable if, and only if, for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in \mathbb{R}^{d_x}$, there exist a control $u_n(\cdot) \in L^\infty_{\text{loc}}([\tau_{\ell n}, \tau_{\ell(n+1)}), \mathbb{R}^{d_u})$ with the following property: if $x(\tau_{\ell n}) = x$, then we have $x(\tau_{\ell(n+1)}) = 0$.

Proof. We prove the sufficiency and necessity parts separately. We take this opportunity to make a few definitions and prove some auxiliary results. For each $k \in \mathbb{Z}_{\geq 0}$, define the linear operator

$$L_k : L^\infty_{\text{loc}}([\tau_k, \tau_{k+1}), \mathbb{R}^{d_u}) \to \mathbb{R}^{d_x}$$

given by $L_k(u(\cdot)) := \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1}, \tau) B_{\sigma(\tau)} u(\tau) d\tau$. Now, given two real vector spaces V and W and a linear operator $L : V \to W$, we define the *range* of L to be the subspace $\mathscr{R}(L) := \{w \in W : L(v) = w \text{ for some } v \in V\}$. The next result will be instrumental in what follows: the range of $L_k(\cdot)$ equals $\Phi_\sigma(\tau_{k+1}, \tau_{\ell n})\mathcal{V}_k$, i.e.,

$$\mathscr{R}(L_k(\cdot)) = \Phi_\sigma(\tau_{k+1}, \tau_{\ell n})\mathcal{V}_k. \tag{16}$$

Indeed, note that there are no switchings inside the interval $[\tau_k, \tau_{k+1})$, i.e., system (1) behaves as an LTI system on this interval. It is a well-known fact that $\mathscr{R}(L_k(\cdot)) = \mathscr{R}(W(\tau_{k+1}, \tau_k))$ (see[22], e.g., Lemma 2.1 from [1]). Furthermore, $\mathscr{R}(W(\tau_{k+1}, \tau_k)) = \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$ (see, e.g., Lemma 2.10 from [1]) since $A_{\sigma(t)}$ and $B_{\sigma(t)}$ are constant on the time interval $[\tau_k, \tau_{k+1})$. Thus, $\mathscr{R}(L_k(\cdot)) = \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle = \Phi_\sigma(\tau_{k+1}, \tau_{\ell n})\mathcal{V}_k$ and we conclude the proof of our remark.
To aid the reader, we now prove some well-known facts about the range of linear operators. Let V and W be real vector spaces, let $L : V \to W$ be a linear operator, and let $S \subset V$ be a subset. We define $LS := \{w \in W : L(v) = w \text{ for some } v \in S\}$. Let $L_1 : V \to W$ and $L_2 : W \to Z$ be linear operators, where Z is a real vector space. Then, we see that

$$\mathscr{R}(L_2 L_2) = \{z \in Z : L_2 L_1(v) = z \text{ for some } v \in V\} = \{z \in Z : L_2(y) = z \text{ for some } y \in \mathscr{R}(L_1)\} = L_2 \mathscr{R}(L_1). \tag{17}$$

Finally, let $L_1 : V \to W$ and $L_2 : V \to W$ be linear operators. Then, we see that

$$\begin{aligned} \mathscr{R}(L_1 + L_2) &= \{w \in W : L_1(v) + L_2(v) \text{ for some } v \in V\} \\ &\subset \{w \in W : y_1 + y_2 = w \text{ for some } y_1 \in \mathscr{R}L_1 \text{ and some } y_2 \in \mathscr{R}L_2\} = \mathscr{R}L_1 + \mathscr{R}L_2 \end{aligned} \tag{18}$$

Sufficiency: for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in \mathbb{R}^{d_x}$, there exist a control $u_n(\cdot) \in L^\infty_{\text{loc}}([\tau_{\ell n}, \tau_{\ell(n+1)}), \mathbb{R}^{d_u})$ with the following property: if $x(\tau_{\ell n}) = x$, then we have $x(\tau_{\ell(n+1)}) = 0$. We start this proof by noticing that the variation of constants formula lets us write that

$$0 = x(\tau_{\ell(n+1)}) = \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n})x + \int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u_n(\tau) d\tau.$$

[22]Although it is not explicitly mentioned, the proof of Lemma 2.1 provided in [1] shows that the control is integrable locally essentially bounded.

We can rewrite the above integral as

$$
\begin{aligned}
\int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)},\tau)B_{\sigma(\tau)}u_n(\tau)d\tau &= \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{\ell(n+1)},\tau)B_{\sigma(\tau)}u_n(\tau)d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1}) \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1},\tau)B_{\sigma(\tau)}u_n(\tau)d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot)). \qquad (19)
\end{aligned}
$$

Since $x \in \mathbb{R}^{d_x}$ is arbitrary and $\Phi_\sigma(\tau_{\ell(n+1)},\tau_{\ell n})$ is invertible, our initial assumption is equivalent to the fact that

$$
\mathbb{R}^{d_x} = \mathscr{R}\left(\int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)},\tau)B_{\sigma(\tau)}u_n(\tau)d\tau\right),
$$

which, by equation (19), is equivalent to the following condition

$$
\mathbb{R}^{d_x} = \mathscr{R}\left(\sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))\right). \qquad (20)
$$

Using the set inclusion (18), we can write

$$
\mathscr{R}\left(\sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))\right) \subset \sum_{k=\ell n}^{\ell(n+1)-1} \mathscr{R}(\Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))).
$$

Next, using (17), we conclude that

$$
\mathscr{R}\left(\sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))\right) \subset \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})\mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))).
$$

Thus, the previous inclusion together with the equality (20), lets us write that

$$
\mathbb{R}^{d_x} \subset \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n},\tau_{k+1})\mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))).
$$

We multiply both terms in the previous relation by $\Phi_\sigma(\tau_{\ell n},\tau_{\ell(n+1)})$ on the left to get that

$$
\begin{aligned}
\mathbb{R}^{d_x} &\subset \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n},\tau_{\ell(n+1)})\Phi_\sigma(\tau_{\ell(n+1)},\tau_{k+1})\mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n},\tau_{k+1})\mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n},\tau_{k+1})\Phi_\sigma(\tau_{k+1},\tau_{\ell n})\mathcal{V}_k = \sum_{k=\ell n}^{\ell(n+1)-1} \mathcal{V}_k,
\end{aligned}
$$

where the first equality follows from the semi-group property of the state-transition matrix, and the second equality follows from equation (16). Finally, we note that the inclusion is actually an equality because the range of the operator on the right is $\mathbb{R}^{d_x}$. Hence,

$$
\mathbb{R}^{d_x} = \sum_{k=\ell n}^{\ell(n+1)-1} \mathcal{V}_k.
$$

Since this holds for any $n \in \mathbb{Z}_{\geq 0}$, we conclude the proof of the sufficiency part.
Necessity: there exists $\ell \in \mathbb{Z}_{\geq 0}$ such that system (1) is ℓ-uniformly completely controllable. So, for each $x \in \mathbb{R}^{d_x}$ and each $n \in \mathbb{Z}_{\geq 0}$, we can write $x = \sum_{k=\ell n}^{\ell(n+1)-1} x_j$, where $x_j \in \mathcal{V}_j$ for each $j \in \{\ell n, \ldots, \ell(n+1)-1\}$. To organize our ideas, we split the proof of necessity into four parts.
First, we have that $\Phi_\sigma(\tau_{k+1}, \tau_{\ell n})x_k \in \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$. Indeed, we know that $x_k \in \Phi_\sigma^{-1}(\tau_k, \tau_{\ell n})\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$. Thus, $\Phi_\sigma(\tau_k, \tau_{\ell n})x_k \in \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$. Furthermore, we also have that $\Phi_\sigma(\tau_{k+1}, \tau_k) = e^{A_{\sigma(\tau_k)} T_p}$, which implies that $\Phi_\sigma(\tau_{k+1}, \tau_k)\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle = \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$ since $\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$ is $A_{\sigma(\tau_k)}$-invariant (see, e.g., Chapter 1 from [26]).
Second, equation (16) gives us that $\mathscr{R}(L_k(\cdot)) = \Phi_\sigma(\tau_{k+1}, \tau_{\ell n})\mathcal{V}_k$ for each $k \in \mathbb{Z}_{\geq 0}$. This implies that there exists a $L^\infty_{\text{loc}}([\tau_k, \tau_{k+1}), \mathbb{R}^{d_u})$ function $u_k : [\tau_k, \tau_{k+1}) \to \mathbb{R}^{d_u}$ such that $-\Phi_\sigma(\tau_{k+1}, \tau_{\ell n})x_k = L_k(u_k(\cdot))$.
Third, define the function $u : [\tau_{\ell n}, \tau_{\ell(n+1)}) \to \mathbb{R}^{d_u}$ as $u(t) = u_k(t)$ for $t \in [\tau_k, \tau_{k+1})$ and each $k \in \{\ell n, \ldots, \ell(n+1)-1\}$. We note that $u(\cdot)$ is $L^\infty_{\text{loc}}([\tau_{\ell n}, \tau_{\ell(n+1)}), \mathbb{R}^{d_u})$. This follows from the facts that each $u_k(\cdot)$ is integrable on its own domain and that the set where $u(\cdot)$ is not bounded is a null Lebesgue set. This latter fact follows from the simple observation that $\{t \in [\tau_{\ell n}, \tau_{\ell(n+1)}) : |u(t)| = \infty\} = \cup_{k=\ell n}^{\ell(n+1)-1} \{t \in [\tau_k, \tau_{k+1}) : |u_k(t)| = \infty\}$ and that the finite union of null sets is a null set. Therefore, $u(\cdot)$ is locally essentially bounded. Fourth, let $x(\tau_{\ell n}) = x$. Then, we can use the variation of constants formula to get that

$$
\begin{aligned}
x(\tau_{\ell(n+1)}) &= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \sum_{k=\ell n}^{\ell(n+1)-1} x_k + \int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u(\tau) d\tau \\
&= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \sum_{k=\ell n}^{\ell(n+1)-1} x_k + \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u_k(\tau) d\tau \\
&= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \left(\sum_{k=\ell n}^{\ell(n+1)-1} x_k + \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1}, \tau) B_{\sigma(\tau)} u_k(\tau) d\tau \right) \\
&= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \left(\sum_{k=\ell n}^{\ell(n+1)-1} x_k + L_k(u_k(\cdot)) \right) \\
&= 0.
\end{aligned}
$$

Since $n \in \mathbb{Z}_{\geq 0}$ is arbitrary, we conclude the proof.

□

Now, we prove Lemma 3.3.

Proof. We need the following classical identity for the controllability Gramian: let $t_2 > t_1 > t_0$ for $t_0 \in \mathbb{R}_{\geq 0}$, then

$$W(t_2, t_0) = \Phi_\sigma(t_2, t_1) W(t_1, t_0) \Phi'_\sigma(t_2, t_1) + W(t_2, t_1). \tag{21}$$

Sufficiency: there exist $T \in \mathbb{R}_{>0}$ and $\underline{w} \in \mathbb{R}_{>0}$ such that $W(t+T, t) \geq I_{d_x}\underline{w}$ for each $t \in \mathbb{R}_{\geq 0}$. Now, let $\ell = \lceil \frac{T}{T_p} \rceil$ and fix some arbitrary $t \in \mathbb{R}_{\geq 0}$. Note that $\ell T_p \geq T$ since $\lceil \frac{T}{T_p} \rceil \geq \frac{T}{T_p}$. Let $t_2 = \ell T_p + t$, $t_1 = t + T$, and $t_0 = t$ in equation (21) to get that $W(t + \ell T_p, t) = \Phi_\sigma(t + \ell T_p, t + T) W(t + T, t) \Phi'_\sigma(t + \ell T_p, t + T) + W(t + \ell T_p, t + T)$. Since the controllability Gramian is always positive semi-definite, i.e., $W(t + \ell T_p, t + T) \geq 0$, and $W(t+T, t) \geq I_{d_x}\underline{w}$ we conclude that $W(t + \ell T_p, t) \geq \Phi_\sigma(t + \ell T_p, t + T)\Phi'_\sigma(t + \ell T_p, t)\underline{w}$.
Now, recall that the Rayleigh-Ritz Theorem (see, e.g., Theorem 4.2.2 from [11]) gives us that $AA' \geq \underline{\lambda}(AA')I_{d_x}$ for an arbitrary matrix $A \in \mathbb{R}^{d_x \times d_x}$, where $\underline{\lambda}(AA') \in \mathbb{R}_{\geq 0}$ is the minimum eigenvalue of AA'. Further, remember that $\underline{\lambda}(AA')$ equals the square of the smallest singular value of A (see, e.g. Theorem 2.6.3 from [11]), which we denote by $\underline{\varsigma}(A)$. Thus, $AA' \geq \underline{\varsigma}^2(A)I_{d_x}$. Hence, we can write $W(t + \ell T_p, t) \geq \underline{\varsigma}^2(\Phi_\sigma(t + \ell T_p, t + T))\underline{w}I_{d_x}$. Noticing that $\Phi_\sigma(t + \ell T_p, t + T)$ is always invertible, we conclude that $W(t + \ell T_p, t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$.
For each $t \in \mathbb{R}_{\geq 0}$ and each $x_t \in \mathbb{R}^{d_x}$, the condition $W(t + \ell T_p, t) > 0$ implies that there exists a control $u : [t, t + \ell T_p] \to \mathbb{R}^{d_u}$ such that $\phi(t + \ell T_p, t, x_t, u(\cdot)) = 0$ by Proposition 5.2 from [12]. Thus, Lemma D.1 tells us that system (1) is ℓ-uniformly completely controllable.
Necessity: there exists $\ell \in \mathbb{Z}_{>0}$ such that $\sum_{j=\ell n(k)}^{\ell(n(k)+1)-1} \mathcal{V}_j = \mathbb{R}^d$ holds for each $k \in \mathbb{Z}_{\geq 0}$, where $n(k) = \lfloor \frac{k}{\ell} \rfloor$. We note that ℓ-uniform complete controllability with losses in $\mathcal{R}^c$ implies, by Lemma D.1, that system (1) is controllable

in the usual sense on each interval $[\tau_{\ell n}, \tau_{\ell(n+1)}]$ for $n \in \mathbb{Z}_{\geq 0}$. Again by Proposition 5.2 from [12], we know that $W(\tau_{\ell(n+1)}, \tau_{\ell n}) > 0$ for each $n \in \mathbb{Z}_{\geq 0}$. Now, we prove that there exists $\underline{\tilde{w}} \in \mathbb{R}_{>0}$ such that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) > \underline{\tilde{w}} I_{d_x} \tag{22}$$

for every $n \in \mathbb{Z}_{\geq 0}$.

First, recall that $\Phi_\sigma(\tau_{k+1}, \tau_k) = e^{A_{\sigma(\tau_k)} T_p}$. Also, define $W_p(0, T_p) \coloneqq \int_0^{T_p} e^{A_p(T_p - \tau)} B_p B_p e^{A_p'(T_p-\tau)} d\tau$ for each $p \in [m]$. Now, we can rewrite $W(\tau_{\ell(n+1)}, \tau_{\ell n})$ in the following manner:

$$\begin{aligned}
W(\tau_{\ell(n+1)}, \tau_{\ell n}) &= \int_{\tau_{\ell n}}^{\ell(n+1)} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(\tau_{\ell(n+1)}, \tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{k+1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(\tau_{\ell(n+1)}, \tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \int_{\tau_k}^{k+1} \Phi_\sigma(\tau_{k+1}, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(\tau_{k+1}, \tau) d\tau \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) W_{\sigma(k)}(0, T_p) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} e^{A_{\sigma(\ell(n+1)-1)} T_p} \cdots e^{A_{\sigma(k+1)} T_p} W_{\sigma(k)}(0, T_p) e^{A'_{\sigma(k+1)} T_p} \cdots e^{A'_{\sigma(\ell(n+1)-1)} T_p}.
\end{aligned} \tag{23}$$

Note that the last term in equation (23) can only attain at most m^ℓ possible values, each one corresponding to a tuple $(\sigma(\tau_{\ell(n+1)-1}), \ldots, \sigma(\tau_{\ell n})) \in [m]^\ell$. This motivates us to define the simpligying notation

$$W_{(p_1, \ldots, p_\ell)} \coloneqq \sum_{k=1}^{\ell} e^{A_{p_\ell} T_p} \cdots e^{A_{p_{k+1}} T_p} W_{\sigma(k)}(0, T_p) e^{A'_{p_{k+1}} T_p} \cdots e^{A'_{p_\ell} T_p}$$

for each tuple $(p_1, \ldots, p_\ell) \in [m]^\ell$. Thus, we can write $W(\tau_{\ell(n+1)-1}, \tau_{\ell n}) = W_{\sigma(\tau_{\ell n}), \ldots, \sigma(\tau_{\ell(n+1)-1})}$ for each $n \in \mathbb{Z}_{\geq 0}$. This latter equality implies that there exist constants $\underline{w}_{(p_1, \ldots, p_\ell)} \in \mathbb{R}_{>0}$ such that $W_{(p_1, \ldots, p_\ell)} > \underline{w}_{(p_1, \ldots, p_\ell)} I_{d_x}$ for each $(p_1, \ldots, p_\ell) \in [m]^\ell$, since $W(\tau_{\ell(n+1)-1}, \tau_{\ell n}) > 0$ for every $n \in \mathbb{Z}_{\geq 0}$. Hence, if we choose $\underline{\tilde{w}} \coloneqq \min\{\underline{w}_{(p_1, \ldots, p_\ell)} : (p_1, \ldots, p_\ell) \in [m]^\ell\}$, we prove our claim that $W(\tau_{\ell(n+1)}, \tau_{\ell n}) > \underline{\tilde{w}} I_{d_x}$ for every $n \in \mathbb{Z}_{\geq 0}$.

Choose $T = 2\ell T_p$ and fix an arbitrary $t \in [t_0, \infty)$. Next, define $q \coloneqq \lceil \frac{t - \tau_0}{T_p \ell} \rceil$. We claim that $t \leq \tau_{\ell q}$ and $\tau_{\ell(q+1)} < T + t$. The first claim follows from $\tau_{\ell q} = \tau_0 + q\ell T_p = \tau_0 + \lceil \frac{t-\tau_0}{T_p \ell} \rceil \ell T_p \geq \tau_0 + (\frac{t-\tau_0}{T_p \ell}) \ell T_p = t$ and the second claim follows from $\tau_{\ell(q+1)} = \tau_0 + (q+1)\ell T_p = \tau_0 + \lceil \frac{t-\tau_0}{T_p \ell} \rceil \ell T_p + \ell T_p < \tau_0 + (\frac{t-\tau_0}{T_p \ell} + 1)\ell T_p + \ell T_p = t + 2\ell T_p = t + T$.

We use (21) with the choices $t_2 = t + T$, $t_1 = \tau_{\ell(q+1)}$, $t_0 = t$ to get that

$$\begin{aligned}
W(t+T, t) &= \Phi_\sigma(t+T, \tau_{\ell(q+1)}) W(\tau_{\ell(q+1)}, t) \Phi'_\sigma(t+T, \tau_{\ell(q+1)}) + W(t+T, \tau_{\ell(q+1)}) && (24) \\
&\geq \Phi_\sigma(t+T, \tau_{\ell(q+1)}) W(\tau_{\ell(q+1)}, t) \Phi'_\sigma(t+T, \tau_{\ell(q+1)}), && (25)
\end{aligned}$$

where the inequality comes from the fact that any Gramian matrix is always positive semi-definite. Similarly, we use (21) with the choices $t_2 = \tau_{\ell(q+1)}$, $t_1 = \tau_{\ell q}$, $t_0 = t$ to get that

$$W(\tau_{\ell(q+1)}, t) = \Phi_\sigma(\tau_{\ell(q+1)}, \tau_{\ell q}) W(\tau_{\ell q}, t) \Phi'_\sigma(\tau_{\ell(q+1)}, t_1) + W(\tau_{\ell(q+1)}, \tau_{\ell q}) \geq W(\tau_{\ell(q+1)}, \tau_{\ell q}) \tag{26}$$

where the inequality follows from the fact that any Gramian matrix is positive semi-definite and by Sylvester law of Inertia (see, e.g., Theorem 4.5.8 from [11]), which states that two congruent[23] symmetric matrices have the same number of positive, negative, and zero eigenvalues. Hence, combining both inequalities (25) and (26), we get that

$$W(t+T, t) \geq \Phi_\sigma(t+T, \tau_{\ell(q+1)}) W(\tau_{\ell(q+1)}, \tau_{\ell q}) \Phi'_\sigma(t+T, \tau_{\ell(q+1)}) \geq \Phi_\sigma(t+T, \tau_{\ell(q+1)}) \Phi'_\sigma(t+T, \tau_{\ell(q+1)}) \underline{\tilde{w}}, \tag{27}$$

[23]The matrices $A \in \mathbb{R}^{d_x \times d_x}$ and $B \in \mathbb{R}^{d_x \times d_x}$ are congruent to each other if there exists an invertible matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that $A = PBP'$. See, e.g., Definition 4.5.4 from [11].

where the last inequality follows from inequality (22).
Recall that $a := \sup\{\|A_{\sigma(t)}\| : t \in [t_0, \infty)\}$ is finite. Thus, we can apply Lemma A.1 to get that $|\Phi_\sigma(t+T, \tau_{\ell(q+1)})v| \geq e^{-a(t+T-\tau_{\ell(q+1)})}$ for each $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. Now, we note that $T + t - \tau_{\ell(q+1)} \leq \ell T_p$. This follows from the fact that $T + t - \tau_{\ell(q+1)} \leq 2\ell T_p + \tau_{\ell q} - \tau_{\ell(q+1)} = \ell T_p$. Thus, $|\Phi_\sigma(t+T, \tau_{\ell(q+1)})v| \geq e^{-a\ell T_p}$ for each $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. Next, note that $e^{-a2\ell T_p} \leq |\Phi_\sigma(t+T, \tau_{\ell(q+1)})v|^2 = v'\Phi'_\sigma(t+T, \tau_{\ell(q+1)})\Phi_\sigma(t+T, \tau_{\ell(q+1)})v$ for each $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. This last remark implies that $e^{-a2\ell T_p} I_{d_x} \leq \Phi'_\sigma(t+T, \tau_{\ell(q+1)})\Phi_\sigma(t+T, \tau_{\ell(q+1)})$ (see, e.g. Section 7.1 from [11]).
Let $A \in \mathbb{R}^{d_x \times d_x}$. Once again Rayleigh-Ritz Theorem tells us that a number $r \in \mathbb{R}$ satisfies $AA' \geq rI_{d_x}$ if, and only if, r lower-bounds the minimum eigenvalue of AA'. Consequently, the previous inequality is true if, and only if, r lower-bound the square of the minimum singular value of A. Recalling that the singular values of A and A' are the same (see, e.g., Theorem 2.6.3 from [11]), we conclude that $AA' \geq rI_{d_x}$ if, and only if $A'A \geq rI_{d_x}$. Thus, since $e^{-a2\ell T_p} I_{d_x} \leq \Phi'_\sigma(t+T, \tau_{\ell(q+1)})\Phi_\sigma(t+T, \tau_{\ell(q+1)})$, we have that $e^{-a2\ell T_p} I_{d_x} \leq \Phi_\sigma(t+T, \tau_{\ell(q+1)})\Phi'_\sigma(t+T, \tau_{\ell(q+1)})$. This latter inequality, together with (27), lets us write

$$W(t+T, t) \geq \underline{\tilde{w}} e^{-a2\ell T_p} I_{d_x}.$$

Defining $\underline{w} := \underline{\tilde{w}} e^{-a2\ell T_p}$, we get that $W(t+T, t) \geq \underline{w} I_{d_x}$. Since $t \in [t_0, \infty)$ is arbitrary, we conlcude the proof of the lemma.

□

E Proposition 4.1

Proof of Proposition 4.1. Once again, we need the following classical identity for the controllability Gramian: let $t_2 > t_1 > t_0$ for $t_0 \in \mathbb{R}_{\geq 0}$, then

$$W(t_2, t_0) = \Phi_\sigma(t_2, t_1)W(t_1, t_0)\Phi'_\sigma(t_2, t_1) + W(t_2, t_1). \tag{28}$$

First, fix an arbitrary $n \in \mathbb{Z}_{\geq 0}$. Let $t_2 = \tau_{\ell(n+1)}$, $t_1 = \tau_{k(n)}$, and $t_0 = \tau_{\ell n}$. Then, equation (28) gives us that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) = \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)})W(\tau_{k(n)}, \tau_{\ell n})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)}) + W(\tau_{\ell(n+1)}, \tau_{k(n)}) \geq W(\tau_{\ell(n+1)}, \tau_{k(n)}), \tag{29}$$

where the last inequality follows from the fact that $\Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)})W(\tau_{k(n)}, \tau_{\ell n})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)}) \geq 0$ (see proof of Lemma 3.3).
Next, let $t_2 = \tau_{\ell(n+1)}$, $t_1 = \tau_{k(n)+1}$, and $t_0 = \tau_{k(n)}$. Then, equation (28) gives us that

$$\begin{aligned} W(\tau_{\ell(n+1)}, \tau_{k(n)}) &= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})W(\tau_{k(n)+1}, \tau_{k(n)})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) + W(\tau_{\ell(n+1)}, \tau_{k(n)+1}) \\ &\geq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})W(\tau_{k(n)+1}, \tau_{k(n)})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}). \end{aligned} \tag{30}$$

Let $W_p(0, T) := \int_0^{T_p} e^{A_p(T_p-\tau')}B_pB'_pe^{A'_p(T_p-\tau')}d\tau$ for each $p \in [m]$. Also, for each $p \in [m]$ choose $\underline{w}_p \in \mathbb{R}_{>0}$ so that $W_p(0, T) \geq \underline{w}_p I_{d_x}$. Define $\bar{\underline{w}} := \min\{\underline{w}_p : p \in [m]\}$ and note that $\bar{\underline{w}} \in \mathbb{R}_{>0}$. It is easy to see that

$$\begin{aligned} W(\tau_{k(n)+1}, \tau_{k(n)}) &= \int_{\tau_{k(n)}}^{\tau_{k(n)+1}} e^{A_{\sigma(\tau_{k(n)})}(\tau_{k(n)+1}-\tau)}B_{\sigma(\tau_{k(n)})}B'_{\sigma(\tau_{k(n)})}e^{A'_{\sigma(\tau_{k(n)})}(\tau_{k(n)+1}-\tau)}d\tau \\ &= \int_0^{T_p} e^{A_{\sigma(\tau_{k(n)})}(T_p-\tau')}B_{\sigma(\tau_{k(n)})}B'_{\sigma(\tau_{k(n)})}e^{A'_{\sigma(\tau_{k(n)})}(T_p-\tau')}d\tau = W_{\sigma(\tau_{k(n)})}(0, T_p) \geq \bar{\underline{w}} I_{d_x}, \end{aligned}$$

where the second equality follows from the change of variables $\tau' = \tau - \tau_{k(n)}$ and the inequality follows from the definition of $\bar{\underline{w}}$. Combining the previous inequality with inequalities (29) and (30), we get that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) \geq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})\bar{\underline{w}}.$$

Let $a := \sup\{\|A_{\sigma(t)}\| : t \in [t_0, \infty)\}$. Then, following similar steps as in the necessity part of the proof of Lemma 3.3, we get that $e^{-2a(\tau_{\ell(n+1)}-\tau_{k(n)+1})} \leq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})$. Since $\tau_{\ell(n+1)} - \tau_{k(n)+1} \leq \ell T_p$, we conclude that $e^{-2a\ell T_p} I_{d_x} \leq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})\Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})$. Therefore, we know that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) \geq \bar{\underline{w}} e^{-2a\ell T_p} I_{d_x}.$$

Since $n \in \mathbb{Z}_{\geq 0}$ is arbitrary, we proved that $W(\tau_{\ell(n+1)}, \tau_{\ell n}) \geq \bar{\underline{w}} e^{-2a\ell T_p} I_{d_x}$ for all $n \in \mathbb{Z}_{\geq 0}$.
To finish this proof, we take $T = 2\ell T_p$ and let $t \in [t_0, \infty)$ be arbitrary. Further, define $\tilde{\underline{w}} := \bar{\underline{w}} e^{-2a\ell T_p}$. Then, we follow the exact same steps as in the proof of Lemma 3.3 from the paragraph that starts with "Chose $T = 2\ell T_p$". We get that $W(t+T, t) \geq \tilde{\underline{w}} e^{-2a\ell T_p} I_{d_x} = \bar{\underline{w}} e^{-4a\ell T_p} I_{d_x}$. Choosing $\underline{w} := \tilde{\underline{w}} e^{-2a\ell T_p}$, we conclude that system (1) is UCC. This concludes the proof of the Proposition.

□

F Corollary 4.1

Proof of Corollary 4.1. Assume that $\frac{\tau_D}{N_0+2} \geq T_p$. Choose some $\ell \in \mathbb{Z}_{>0}$ such that $\ell < \frac{\tau_D}{T_p}$. Then, for each $n \in \mathbb{Z}_{\geq 0}$, the time interval $[\tau_{\ell n}, \tau_{\ell(n+1)})$ has at most N_0+1 switchings and contains at least N_0+2 sampling intervals. Therefore, by the pigeonhole principle, for each $n \in \mathbb{Z}_{>0}$, we know that there exists some $k(n) \in \mathbb{Z}_{>0}$ such that $k(n) \notin \mathcal{S}$, i.e., there is no switching on the time interval $[\tau_{k(n)}, \tau_{(k(n)+1)})$. Consequently, since all modes are controllable, we have that $\langle A_{\sigma(k(n))} | B_{\sigma(k(n))} \rangle = \mathbb{R}^d$. Thus, all assumption from Theorem 4.1 hold, concluding the proof of the corollary. □

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