Minimum Data-Rate for Linear Feedback Emulation

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Abstract

In this work, we ask what the minimum data-rate is for a possibly nonlinear control law acting on a linear system to emulate the closed-loop behaviour of a desired linear control law. This result gives an implicit estimate for the information flow in the feedback path of the closed-loop linear system. We formally introduce the notion of emulation and control law, and we present a data-rate theorem. Remarkably, we note that the minimum data-rate varies discontinuously with the parameters of the open and closed-loop systems. This feature is new since the usual data-rate theorem, which only requires stabilization instead of emulation, gives a continuously varying minimum data-rate.

1 Introduction

Quantization and sampling are ubiquitous in control systems since the advent of digital circuits. These features, however, constraints the class of problems we can solve since the number of possible distinct controls functions we can implement on any given time interval is finite. Moreover, there exists a minimum data-rate, informally understood as the number of distinct possible controls on a given time interval, below which we cannot solve several control problems (see, e.g., [Del90; WB99; NE00; Bai02; João02; MS04; TM04; MS09; Kaw13]). Generally, results that present data-rate restrictions are called Data-Rate Theorems (see, e.g., [Nai+07] for an overview).

In this document, we address the following question: given a linear feedback system, what is the minimum asymptotic average data-rate to emulate that system's behaviour? This question has an immediate practical consequence since, often, the designer ignores the data-rate constraints and simply designs the controller for a given plant assuming perfect information. Our paper addresses what is the data-rate we need to use for our control system to behave similarly to the desired designed behaviour.

In most of the previous literature, authors addressed the problem of stabilizing the origin in some sense (see, e.g., [Del90; WB99; BL00; NE00]). In our problem, we require the system's state trajectory to closely mimic a desired behaviour. This additional requirement leads us to a data-rate theorem that takes into consideration both the eigenstructure of the open-loop system and the that of the desired closed-loop one, which we want to emulate. Surprisingly, unlike the usual data-rate theorem, the minimum data-rate to solve the emulation problem varies discontinuously with the system parameters. Part of the proof of this result uses tools from geometric measure theory, namely the Hausdorff measure, instead of the usual volume (Lebesgue measure) counting argument, and is one the contributions of this paper.

The structure of our work is as follows: in Section 2, we introduce the emulation problem and related notions. We also recall the notion of Lyapunov indices and flag, which appear in our data-rate theorem. Still in Section 2, we present Theorem 1, characterizing the minimum data-rate to solve the emulation problem with the Lyapunov indices of an associated time-varying matrix. Then, in Section 3, we show how to compute those Lyapunov indices in terms of the open and closed-loop systems' eigenstructures. Next, in Section 4, we present part of the proof of the data-rate theorem, the one that proves that our expression for the minimum data-rate is a lower bound. Finally, we conclude the paper and prove some technical results in the appendix.

Notations: We denote by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ the sets of complex, real, and integer numbers, respectively. We also denote by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) the set of positive (nonnegative) integers. Given $n \in \mathbb{Z}_{>0}$, we denote $[n] := \{1, \ldots, n\}$. For a number $\lambda \in \mathbb{C}$, we denote by $\Re(\lambda)$ its real part. For a set S, we denote by $\#S$ its cardinality. Given a vector subspace $V \subset \mathbb{R}^d$, a basis $\{v_1, \ldots, v_d\}$ is a linearly independent set of vectors of maximal cardinality together with a total order relation on that set. We denote by $|\cdot|$ the usual Euclidean norm in $\mathbb{R}^d$ and by $\|\cdot\|$ the respective induced norm on $d \times d$ matrices. We denote by I_d the $d \times d$ identity matrix. Given a matrix $A \in \mathbb{R}^{d \times m}$ we denote by $A' \in \mathbb{R}^{m \times d}$ its transpose. Finally, given a $d \times d$ matrix A, we denote by $\Sigma(A) := \{\Re(z) : z \text{ is an eigenvalue of } A\}$ the set of real parts of the eigenvalues of A.

2 The Emulation Problem

In this section, we introduce the model and the problem we want to solve. Next, we introduce several notions necessary to make our problem rigorous. After that, we state our main result, which we prove in subsequent sections.

2.1 The problem's informal description

Let A and B be real matrices of dimensions $d \times d$ and $d \times m$, respectively. We assume that we want to design a controller for system

$$\dot{z}(t) = Az(t) + Bv(t) \tag{1}$$

with the informal goal of mimicking the behavior of a target closed-loop system

$$\dot{x}(t) = (A + BF)x(t), \tag{2}$$

where F is a $m \times d$ matrix. The idea is the following: suppose we designed a full-state feedback controller $u(x(t), t) = Fx(t)$ for system (1) and obtained the closed-loop system (2). Now, we want to imitate the target closed-loop dynamics (2) over a communication channel, which will force the controller to operate with a finite data-transmission rate. What is the minimum data-rate for us to implement such controller with an error that vanishes asymptotically? We address this problem as the emulation problem since we are interested in emulating the behavior of a system that operates with perfect information by using a controller that is subject to data-rate constraints. This problem statement is only informal; the goal of the remainder of this section is to make it precise and to present a data-rate theorem for it.

We take this opportunity to introduce some notation[1]: $\xi(x, t)$ is the solution of system (2) at time $t \in [0, \infty)$ when the initial condition is $x \in \mathbb{R}^d$. Also, $\phi(x, t, v(\cdot))$ is the solution of (1) at time $t \in [0, \infty)$, when the initial condition is $x \in \mathbb{R}^d$, and the control is $v(\cdot)$. Finally, $\eta(x, t) := \xi(x, t) - \phi(x, t, v(\cdot))$ is the *emulation error* at time t when we start at the state x.

2.2 The notion of control law

Now, to talk about a data-rate associated to a controller, we introduce the notion of control law. To get an informal understanding of such idea, let $T \in (0, \infty)$ be a time horizon, $\mathcal{K} \subset \mathbb{R}^d$ be a set of possible initial conditions, and let $u(\cdot, \cdot)$ be a function that associates an initial condition to a function $u(x, \cdot)$ on $[0, \infty)$ with image on $\mathbb{R}^m$, which we interpret as a control. We notice that several initial conditions in $\mathcal{K}$ can have the same function $u(\cdot, \cdot)$ when restricted to the time interval $[0, T]$. If the number of distinct function restrictions on $[0, T]$ is finite, we can encode them on an alphabet and use its growth-rate as a data-rate measure. That is the idea behind the next definition.

Definition 1 (Control law)
A *control law* is a function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^m$ so that $u(x, \cdot)$ is locally bounded and càdlàg for each $x \in \mathbb{R}^d$. Given a set of possible initial conditions $\mathcal{K} \subset \mathbb{R}^d$, which we assume compact, we associate an *asymptotic average data-rate* as

$$b(u) := \limsup_{t \to \infty} \frac{1}{t} \log(\#u(\mathcal{K}, [0, t])). \tag{3}$$

If the asymptotic average data-rate is finite, the control law induces a natural finite partition of $\mathcal{K} \subset \mathbb{R}^d$ for each $T \in (0, \infty)$. We describe the partition using an equivalence relation: two possible initial conditions x and y are equivalent on the time interval $[0, T]$ if, and only if, $u(x, t) = u(y, t)$ for all $t \in [0, T]$. We denote this partition of $\mathcal{K}$ by $\mathscr{P}_u(T)$ and we denote its cardinality by $N_u(T) \in \mathbb{Z}_{>0}$. We further assume that the cells $\mathcal{P} \in \mathscr{P}_u(T)$ are Borel and have nonempty interior. ▲

Remark 1
Our definition of control law is inspired by the definition of control set from [CK00] (see Chapter 3). We recall that a control set is a set of admissible functions indexed by elements from $\mathcal{K}$. Equivalently, we can say that a control set is a function indexed by its first entry. Thus, both definitions of control law and control set give the same object. Nonetheless, we believe that introducing the control law makes the discussion clearer and more intuitive.

[1] The initial time is always equal to zero.

2.3 Formal problem statement

We now can state formally what we mean by emulation.

Definition 2 (Emulation)
Let $\varepsilon \in (0, \infty)$ and $\alpha \in [0, \infty)$ be constants. Also, let $\mathcal{K} \subset \mathbb{R}^d$ be a set of possible initial condition, which is compact and has a nonempty interior. We say that a control law $u(\cdot, \cdot)$ makes system (1) $(\varepsilon, \mathcal{K}, \alpha)$-*emulate* system (2) if

$$|\eta(x, t)| < \varepsilon e^{-\alpha t}$$

for all $x \in \mathcal{K}$ and all $t \in [0, \infty)$. The *minimum data-rate* to solve the emulation problem for any $\varepsilon \in (0, \infty)$ is

$$h(\alpha, \mathcal{K}) := \sup_{\varepsilon > 0} \{\inf\{b(u) : u(\cdot, \cdot) \text{ makes system (1)} \\ (\varepsilon, \alpha, \mathcal{K})\text{-emulate system (2)}\}\} \tag{4}$$

▲

We note that, although we call it the minimum data-rate, the quantity (4) is actually an infimum in general. The supremum over ε reflects that we are not interested in bounding the initial transients of the emulation error $\eta(\cdot, \cdot)$, so long as it converges to zero faster than $e^{-\alpha t}$.

The goal of the rest of this section is to state Theorem 1, which gives the minimum data-rate to solve the emulation problem. Nonetheless, to do that, we must introduce a geometric notion, which is related to the eigenstructure of systems (1) and (2).

We take this opportunity to introduce some additional notation. We define

$$\varphi(t) := e^{(A+BF)t} - e^{At} \tag{5}$$

for all $t \in [0, \infty)$. Note that we can explicitly solve equations (1) and (2) to compute $\eta(x, t)$ as

$$\eta(x, t) = \varphi(t)x + \int_0^t e^{A(t-s)} B v(s) ds \tag{6}$$

for all $t \in [0, \infty)$ and any integrable control $v(\cdot)$. Thus, we can interpret $\varphi(t)$ as a "free response" of the emulation error. In this sense, $\varphi(\cdot)$ is similar to semi-flows that arise from differential equations (see, e.g., Chapter 1 from [SY02]). We note, however, that $\varphi(\cdot)$ does not satisfy the semi-group property, since it comprises the difference of two exponentials.

2.4 Lyapunov indices and flag

In [Lia07], the concept of *characteristic number* of a function was introduced (see Chapter 1 Section 6), which is known today as the Lyapunov index of a function[2] (see, e.g., [Arn98]) and, in a slightly less general setting, as *Lyapunov exponent*. Lyapunov used that concept to study the asymptotic behavior of solutions of linear time-varying differential equations that appeared in his first method of stability. In our work, we analyse the Lyapunov indices of $\varphi(\cdot)$ since they are related to the solution of the minimum average asymptotic data-rate emulation problem. Explicitly, we use these indices to provide lower and upper bounds to the minimum data-rate for the emulation problem.

The next definition is adapted from Chapter 3 of [Arn98]. The difference is in the fact that we parametrize the functions by a vector $v \in \mathbb{R}^d$ to make our discussion clearer.

Definition 3 (Lyapunov index)
The *Lyapunov index* of a function $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ parametrized by its second argument is the functional

$$\lambda(f, v) := \limsup_{t \to \infty} \frac{1}{t} \log (|f(t, v)|) \in \mathbb{R} \cup \{-\infty, \infty\}, \tag{7}$$

where $\log(0) = -\infty$. ▲

[2]Lyapunov considered the quantity with the sign flipped. We follow the current sign convention in our work.

We are interested in studying the Lyapunov indices when $f(t, v) = \varphi(t)v$. Recall that $\varphi(t)$ is a $d \times d$ real matrix for each $t \in [0, \infty)$ (see equation (5)). We denote this Lyapunov index, when evaluated at $v \in \mathbb{R}^d$, by $\lambda(\varphi, v)$, to keep the notation simple. Now, we need the following two properties in our analysis. First, we denote by

$$\boldsymbol{\chi} := \lambda(\varphi, \mathbb{R}^d) \cap \mathbb{R} \tag{8}$$

the set of real values $\lambda(\varphi, \cdot)$ can take. Further, we denote the cardinality of $\boldsymbol{\chi}$ by

$$q := \#\boldsymbol{\chi}. \tag{9}$$

We remark that $q \in [d]$ (see Section 2.1 in Chapter 3 from [Arn98]). Further, we order the set $\boldsymbol{\chi}$ so that $\boldsymbol{\chi} = \{\chi_1, \dots, \chi_q\}$ where $\chi_i > \chi_{i+1}$ for $i \in [q-1]$ and define $\chi_{q+1} := -\infty$.

The other property of $\lambda(\varphi, \cdot)$ is geometric in nature. More clearly, the set

$$V_\beta := \{v \in \mathbb{R}^d : \lambda(\varphi, v) \leq \beta\} \tag{10}$$

is a vector subspace of $\mathbb{R}^d$ for each $\beta \in \mathbb{R} \cup \{-\infty, \infty\}$. Notice that the set of vector spaces $\mathcal{F}' := \{V_\beta : \beta \in \{\chi_1, \dots, \chi_q, \chi_{q+1}\}\}$ forms a chain with respect to strict set inclusion, i.e.,

$$V_{\chi_{q+1}} \subsetneq V_{\chi_q} \subsetneq \cdots \subsetneq V_{\chi_1} = \mathbb{R}^d.$$

Recall that a set of subspaces $\tilde{\mathcal{F}} := \{F_0, \dots, F_p\}$ of $\mathbb{R}^d$ is called a *flag* (see, e.g., Chapter 7 from [SR13]) if $\{0\} \in \tilde{\mathcal{F}}$, $\mathbb{R}^d \in \tilde{\mathcal{F}}$, and if

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_p = \mathbb{R}^d.$$

We see that, if $V_{\chi_{q+1}} = \{0\}$, then $\mathcal{F}'$ is a flag. When $\varphi(t)$ is not invertible, however, we might have $V_{-\infty} \neq \{0\}$. To remedy that, we define $\mathcal{F} := \mathcal{F}' \cup \{0\}$ and note that this is always a flag. This is called the *Lyapunov flag* of $\varphi(\cdot)$ in $\mathbb{R}^d$. In some works, this is called the *Oseledets flag*[3]. In our next definition, we introduce a class of bases of $\mathbb{R}^d$, which are compatible to a flag.

Definition 4 (Adapted basis)
Given a flag $\mathcal{F}$ for $\mathbb{R}^d$ with $k+1$ elements, we say that a basis $\mathcal{B} = \{v_1, \dots, v_d\}$ is *adapted* to it if $\{v_i, \dots, v_d\}$ is a basis for F_i for each $i \in [k]$. If $\mathcal{F}$ is a Lyapunov flag and $\mathcal{B}$ is an orthonormal basis, we call $\mathcal{B}$ a *Lyapunov basis* and we denote it by $\mathcal{V}$. Furthermore, we define $\mathcal{V}_{\geq \alpha} := \{v \in \mathcal{V} : \lambda(\varphi, v) \geq \alpha\}$ for any $\alpha \in \mathbb{R}$. ▲

Another concept that plays a role in our discussion is that of Lyapunov spectrum. We briefly recall that a multiset is a "set" where an element can occur more than once[4]. The next definition is an adaptation from the one given in Chapter 3 from [Arn98].

Definition 5 (Lyapunov spectrum)
The ordered multiset $\bar{\boldsymbol{\chi}} := \{\bar{\chi}_1, \dots, \bar{\chi}_d\}$ is defined by the properties:

1. $\bar{\chi}_i \geq \bar{\chi}_{i+1}$ for each $i \in [d]$,
2. $\bar{\chi}_i \in \boldsymbol{\chi}$ for each $i \in [d]$,
3. there are exactly $d_j := \dim(V_{\chi_j}) - \dim(V_{\chi_{j-1}})$ copies of χ_j in $\bar{\boldsymbol{\chi}}$ for each $j \in [q]$.

We call $\bar{\boldsymbol{\chi}}$ the *Lyapunov spectrum* of φ and the quantity d_j is the *multiplicity* of the Lyapunov index χ_j for each $j \in [q]$. ▲

[3] After V.I. Oseledets who studied such flags in [Ose68]. The nomenclature *Oseledets filtration* is also standard in the dynamical systems' literature for such flag.

[4] We believe that this informal description of a multiset is enough for our purposes. We refer to page 694 from [Knu97] for a discussion on this terminology and for a rigorous definition.

2.5 A data-rate theorem

Finally, we can state our main result.

Theorem 1
Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set of possible initial conditions with nonempty interior. Also, let $\alpha \in [0,\infty)$ and $\varepsilon \in (0,\infty)$ be constants. Further, assume that[5] $\Sigma(A) \cap \Sigma(A+BF) = \emptyset$. Then, any control law $u(\cdot,\cdot)$ (Definition 1) that makes system (1)$(\varepsilon,\mathcal{K},\alpha)$-emulate (Definition 2) the behavior of system (2) must satisfy

$$b(u) \geq \sum_{j=1}^{d} \max\{\bar{\chi}_j + \alpha, 0\} \tag{11}$$

Moreover, for each $\delta \in (0,\infty)$ there exists a control law $u_\delta(\cdot,\cdot)$ so that

$$b(u_\delta) < \sum_{j=1}^{d} \max\{\bar{\chi}_j + \alpha, 0\} + \delta. \tag{12}$$

Consequently, the minimum asymptotic average data-rate is given by

$$h(\alpha,\mathcal{K}) = \sum_{j=1}^{d} \max\{\bar{\chi}_j + \alpha, 0\}. \tag{13}$$

▲

Note that inequalities (11) and (12) are tight, since (12) holds for each $\delta \in (0,\infty)$. We note, however, that this is not a satisfactory result in itself since we need to know the Lyapunov flag and the Lyapunov indices for $\varphi(\cdot)$. In the next section, we characterize the values that $\bar{\chi}_j$ can have for each $j \in [d]$ in terms of the eigenvalues and eigenspaces of the matrices A and $A+BF$ that appear in equations (1) and (2), respectively.

Remark 2
The reader might wonder if we are able to make system (1) mimic more general linear dynamics than those described by (2), i.e., is it possible to emulate $\dot{y}(t) = Gy(t)$ for some $d \times d$ real matrix $G \neq A+BF$ for any $d \times m$ matrix F? The answer to that question is negative. Therefore, the feedback form we chose is the most general type of finite-dimensional linear time-invariant system that can be emulated by (1).

We end this subsection by presenting a motivating example.

Example 1
For concreteness, let

$$\bar{A} := \begin{pmatrix} -1.5 & 0 \\ 0 & -0.5 \end{pmatrix} \text{ and } \bar{B} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

and consider the system

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) \tag{15}$$

for all $t \in [0,\infty)$. We assume that the initial state $x(0)$ belongs to the compact set $\mathcal{K} = [-1,1]^2$. Let $\alpha = 2$. Further, define the matrices

$$\Lambda := \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } R_\varpi := \begin{pmatrix} \cos(\varpi) & -\sin(\varpi) \\ \sin(\varpi) & \cos(\varpi) \end{pmatrix} \tag{16a}$$

$$A_\varpi := R_\varpi \Lambda R'_\varpi. \tag{16b}$$

In this example, our goal is to compute the minimum data-rate $h(\alpha,\mathcal{K})$ for system (15) to $(\alpha,\mathcal{K})$-emulate the behavior of the closed-loop system

$$\dot{z}(t) = A_\varpi z(t), \tag{17}$$

[5]See the notations part of the paper.

for each fixed choice of $\varpi \in [0, 2\pi)$. Note that $A_\varpi = \bar{A} + \bar{B}F$ with $F = A_\varpi - \bar{A}$.
Case $\varpi = 0$: in this case, we have that

$$A_0 = \Lambda. \tag{18}$$

In this scenario, we have

$$h(\alpha, \mathcal{K}) = 2.0 \text{ nats/unit of time.} \tag{19}$$

Case $\varpi \neq 0$: in this case, it can be shown that

$$h(\alpha, \mathcal{K}) = 2.5 \text{ nats/unit of time.} \tag{20}$$

See Example 2 for the calculations leading to (19) and (20). We note that in both the previous cases, the matrices of the closed-loop system have the same eigenvalues. However, their eigenspaces differ. In the next section, we show that $h(\alpha, \mathcal{K})$ depends not only on the eigenvalues of the open and closed-loop systems, but on the relative position between their eigenspaces. Moreover, notice that this example shows us that the minimum data-rate does not vary continuously with the target mode.

3 How to compute the Lyapunov indices

In this section, we relate the Lyapunov indices of $\varphi(\cdot)$ with the eigenvalues of the open and closed-loop modes for a class of systems. Explicitly, we express the Lyapunov indices of $\varphi(\cdot)$ in terms of the eigenvalues of A and $A + BF$ when the root spaces of common eigenvalues of A and $A + BF$ have a trivial intersection.

First, we recall the notion of root space[6] (see, e.g., Chapter 12 from [GLR86]).

Definition 6 (Root space)
Let $\lambda \in \mathbb{C}$ be an eigenvalue of a $d \times d$ real matrix A. Further, let $q_\lambda(x) = x - \lambda$, if $\lambda \in \mathbb{R}$, and $q_\lambda(x) = x^2 - 2\Re(\lambda)x + |\lambda|^2$, otherwise. We define the *root space* $\mathcal{R}(\lambda, A) \subseteq \mathbb{R}^d$ associated with the eigenvalue λ as

$$\mathcal{R}(\lambda, A) := \ker\{(q_\lambda(A))^p\}, \tag{21}$$

where $p \in \mathbb{Z}_{>0}$ is the smallest integer so that $\ker\{(q_\lambda(A))^k\}$
$\subseteq \ker\{(q_\lambda(A))^p\}$, for all $k \in \mathbb{Z}_{>0}$. ▲

With this notation at hand, we can state the main result of this section.

Proposition 1
Assume that if A and $A + BF$ have a common eigenvalue $\lambda \in \mathbb{C}$, then $\mathcal{R}(\lambda, A) \cap \mathcal{R}(\lambda, A + BF) = \{0\}$. Then, we have that

$$\lambda(\varphi, x) = \max\{\lambda(e^{At}, x), \lambda(e^{(A+BF)t}, x)\}. \tag{22}$$

▲

We take this opportunity to briefly remark that the inequality $\lambda(\varphi, x) \leq \max\{\lambda(e^{At}, x), \lambda(e^{(A+BF)t}, x)\}$ always holds and was proven by Lyapunov in Chapter 7 of [Lia07] in a more general setting. We also note that the converse inequality does not always hold, however. Clearly, if $BF = 0$, we see that $\lambda(\varphi, x) = -\infty$ for any $x \in \mathbb{R}^d$. The condition we assumed on the root space of common eigenvalues is sufficient for the equality in (22). Also, we note that $\lambda(e^{At}, x)$ equals the real part of some eigenvalue of A (see, e.g., Example 3.2.3 from [Arn98]).

Due to space considerations, we prove this proposition in the appendix. Now, we revisit Example 1 and present the calculations that yield the quantities (19) and (20).

Example 2
Proposition 1 allows us to calculate the Lyapunov indices of φ easily for any given fixed $\varpi \in [0, 2\pi)$. Indeed, we have that the eigenpairs for the open-loop system are $(-1.5, e_1)$ and $(-0.5, e_2)$ and the ones for the closed-loop system are $(-2, R'_\varpi e_1)$ and $(-1, R'_\varpi e_2)$. Clearly, we satisfy the conditions of our previous proposition since the eigenvalues of the open and closed-loop system are distinct.

[6]When the field of scalars is $\mathbb{C}$, some authors call the root spaces the generalized eigenspaces. See, e.g., Chapter 8 Section B in [Axl24].

Case $\varpi = 0$: in this case, we have that

$$\begin{aligned}\lambda(\varphi, e_1) &= \max\{\lambda(e^{A_0 t}, e_1), \lambda(e^{\bar{A}t}, e_1)\} \\ &= \max\{-2, -1.5\} = -1.5 \text{ and} \\ \lambda(\varphi, e_2) &= \max\{\lambda(e^{A_0 t}, e_2), \lambda(e^{\bar{A}t}, e_2)\} \\ &= \max\{-1, -0.5\} = -0.5.\end{aligned}$$

Since $\lambda(\varphi, \cdot)$ can only have at most 2 distinct values (see the remark after (9)), we have found χ_1 and χ_2. Therefore, the bound in Theorem 1 gives us that the minimum data-rate for $\alpha = 2$ is given by $\max\{2 - 1.5, 0\} + \max\{2 - 0.5, 0\} = 2.0$, which is the bound we presented in equation (19).
Case $\varpi \neq 0$: in this case, we have that

$$\begin{aligned}\lambda(\varphi, e_1) &= \max\{\lambda(e^{A_\varpi t}, e_1), \lambda(e^{\bar{A}t}, e_1)\} \\ &= \max\{-1, -1.5\} = -1 \text{ and} \\ \lambda(\varphi, e_2) &= \max\{\lambda(e^{A_\varpi t}, e_2), \lambda(e^{\bar{A}t}, e_2)\} \\ &= \max\{-1, -0.5\} = -0.5.\end{aligned}$$

where we concluded that $\lambda(e^{A_\varpi t}, e_1) = -1$ as a consequence of the fact that e_1 belongs only to the largest element of the Lyapunov flag for A_ϖ for $\varpi \neq 0$. Therefore, the bound in Theorem 1 gives us that the minimum data-rate for $\alpha = 2$ is given by $\max\{2 - 1, 0\} + \max\{2 - 0.5, 0\} = 2.5$, which is the bound we presented in equation (20).

4 Proof of the lower bound

In this section, we present the proof of the lower bound (11) when $V_{\chi_{q+1}} = \{0\}$. We remark that the proof of the general case is analogous, but we focus on this case due to space constraints.

First, let $\mathcal{V}$ be a Lyapunov basis for φ (see Definition 4) and let $\mathcal{V}_\alpha = \{v \in \mathcal{V} : \lambda(\varphi, v) \geq -\alpha\}$. Notice that the existence of $\mathcal{V}$ is a consequence of our assumption that $V_{\chi_0} = \{0\}$ for such a basis for $\mathbb{R}^d$ to exist[7]. Unless stated otherwise, we assume that $\mathcal{V}_\alpha \neq \emptyset$. Under this assumption, let $k := \#\mathcal{V}_\alpha$ and note that $\mathcal{V}_\alpha = \{v_1, \ldots, v_k\}$. Now, we can define an "inclusion matrix" from $\mathbb{R}^k$ into span$\{\mathcal{V}_\alpha\}$. Let

$$\iota_\alpha := (v_1 \cdots v_k) \tag{23}$$

be a $d \times k$ block-column matrix that maps vectors in $\mathbb{R}^k$ into vectors in $\mathbb{R}^d$. More specifically, the range of ι_α is the vectors subspace span$\{\mathcal{V}_\alpha\}$. This matrix is semi-orthogonal, i.e., $\iota_\alpha' \iota_\alpha = I_k$. This has two immediate implications: first, ι_α' is a left-inverse for ι_α. Second, we have that $|\iota_\alpha x| = (x' \iota_\alpha' \iota_\alpha x)^{1/2} = |x|$, i.e., ι_α is an isometry with respect to the Euclidean norm from $\mathbb{R}^k$ into span$\{\mathcal{V}_\alpha\}$. Next, we define

$$Q_\alpha(t) := (\iota_\alpha' \varphi'(t) \varphi(t) \iota_\alpha)^{1/2} \tag{24}$$

for all $t \in [0, \infty)$, which is an auxiliary matrix that we use in our proof. In our analysis, we need a technical lemma related to the concept of Lyapunov regularity (See Chapter 1 Section 9 from [Lia07]).

Lemma 1
Let α be so that $\mathcal{V}_\alpha \neq \emptyset$ and let $k := \#\mathcal{V}_\alpha$. Then, if $\Sigma(A) \cap \Sigma(A + BF) = \emptyset$, we have that

$$\limsup_{t \to \infty} \frac{1}{t} \log(\sqrt{\det(Q_\alpha^2(t))}) = \sum_{i=1}^{k} \bar{\chi}_i. \tag{25}$$

where $\bar{\chi}_i$ is the i-th Lyapunov index with multiplicity. ▲

Proof 1
Choose an arbitrary $T \in [0, \infty)$ and recall that $\mathscr{P}_u(T)$ is the partition of $\mathcal{K}$ that corresponds to the control law $u(\cdot, \cdot)$ on the time interval $[0, T]$ (see Definition 1). Recall that any arbitrary cell in the partition

[7] Otherwise, we only get a basis for a subspace of $\mathbb{R}^d$.

$\mathcal{P} \in \mathscr{P}_u(T)$ is a Borel set and has nonempty interior. We split our proof into five parts: the restriction to $\mathcal{V}_\alpha$, the diameter upper bound, the area formula, the volume upper bound, and the counting argument.
The restriction to $\mathcal{V}_\alpha$: let $\bar{x} \in \mathcal{K}$ be an interior point. We define the restriction of the set of possible initial conditions along the subspace $\mathcal{V}_\alpha$ through $\bar{x}$ as the set

$$\mathcal{K}_\alpha(\bar{x}) \coloneqq \mathcal{K} \cap (\bar{x} + \text{span}\{\mathcal{V}_\alpha\}) \,, \tag{26}$$

where the plus sign corresponds to the Minkowisky sum. Since $\bar{x}$ is an interior point, there exists $r \in (0, \text{diam}(\mathcal{K}))$ so that $\mathbb{B}_d(\bar{x}, r) \subseteq \mathcal{K}$, which implies that

$$\mathbb{B}_d(\bar{x}, r) \cap (\bar{x} + \text{span}\{\mathcal{V}_\alpha\}) \subseteq \mathcal{K}_\alpha(\bar{x}) \tag{27}$$

by taking the intersection with $(\bar{x} + \text{span}\{\mathcal{V}_\alpha\})$ on both sides. Thus, we can compute the k-dimensional Huasdorff measure of this set to get that

$$\begin{aligned}
\mathcal{H}^k(\mathcal{K}_\alpha(\bar{x})) &\geq \mathcal{H}^k(\mathbb{B}_d(\bar{x}, r) \cap (\bar{x} + \text{span}\{\mathcal{V}_\alpha\})) \\
&= \mathcal{H}^k(\mathbb{B}_d(0, r) \cap \text{span}\{\mathcal{V}_\alpha\} + \bar{x}) \\
&= \mathcal{H}^k(\mathbb{B}_d(0, r) \cap \text{span}\{\mathcal{V}_\alpha\}) \\
&= \mathcal{H}^k(\mathbb{B}_k(0, r)) > 0
\end{aligned} \tag{28}$$

where the first inequality follows from the set inclusion (27), the first equality follows from properties of thew Minkowisky addition, and the second equality follows from the fact that the Hausdorff measure is translation-invariant. To see why the third equality holds, we recall that the range of the matrix ι_α equals $\text{span}\{\mathcal{V}_\alpha\}$. Hence, we can write $\text{span}\{\mathcal{V}_\alpha\} = \iota_\alpha \mathbb{R}^k$. Therefore, we have that

$$\begin{aligned}
\mathbb{B}_d(0, r) \cap \text{span}\{\mathcal{V}_\alpha\} &= \{x \in \text{span}\{\mathcal{V}_\alpha\} : |x| \leq r\} \\
&= \{\iota_\alpha y : y \in \mathbb{R}^k \text{ and } |\iota_\alpha y| \leq r\} \\
&= \{\iota_\alpha y : y \in \mathbb{R}^k \text{ and } |y| \leq r\} \\
&= \iota_\alpha \mathbb{B}_k(0, r),
\end{aligned}$$

where the second equality follows from the fact that each element $x \in \text{span}\{\mathcal{V}_\alpha\}$ can be written as $x = \iota_\alpha y$ for some $y \in \mathbb{R}^k$, the third equality follows from the fact that ι_α is an isometry.
Next, we define the partition induced by $\mathscr{P}_u(T)$ on $\mathcal{K}_\alpha(\bar{x})$ as

$$\begin{aligned}
\mathscr{P}_u^\alpha(T) \coloneqq \{\mathcal{P} \subseteq \mathcal{K}_\alpha(\bar{x}) : \mathcal{P} = \mathcal{P}' \cap \mathcal{K}_\alpha(\bar{x}) \\
\text{where } \mathcal{P}' \in \mathscr{P}_u(T)\}.
\end{aligned} \tag{29}$$

We note that $\mathscr{P}_u^\alpha(T)$ partitions $\mathcal{K}_\alpha(\bar{x})$ and that its cardinality, which we denote by $M_u(T)$, is less than or equal to $N_u(T)$ (see Definition 1).
The diameter upper bound: choose a cell $\mathcal{P} \in \mathscr{P}_u^\alpha(T)$. Let $w(t) \coloneqq u(x, t)$, where $x \in \mathcal{P}$ is arbitrary[8]. Our goal is to find an upper bound for the diameter of the set $\{\varphi(t)x : x \in \mathcal{P}\}$ for each $t \in [0, T]$. To do that, chose points $x \in \mathcal{P}$ and $y \in \mathcal{P}$ and note that equation (6) allows us to write that $|\varphi(t)x - \varphi(t)y| = |\eta(x, t) - \eta(y, t)| \leq |\eta(x, t)| + |\eta(y, t)| < 2\varepsilon e^{-\alpha t}$ for all $t \in [0, T]$, where we used the triangle inequality to obtain the first inequality and we used the fact that the control law is (ε, K, α)-emulating to get the second. This allows us to conclude that the inequality

$$\begin{aligned}
\text{diam}(\{\varphi(t)x : x \in \mathcal{P}\}) &= \text{diam}(\{\eta(x, t) : x \in \mathcal{P}\}) \\
&= \sup\{|\eta(x, t) - \eta(y, t)| : (x, y) \in \mathcal{P}^2\} \leq 2\varepsilon e^{-\alpha t}
\end{aligned} \tag{30}$$

must hold for all $t \in [0, T]$.
The area formula: the area formula for the Hausdorff measure (see, e.g., Section 8 of Chapter 2 from [Sim83]), gives that

$$\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) = \sqrt{\det(Q_\alpha^2(t))}\mathcal{H}^k(\iota'_\alpha \mathcal{P}),$$

where $\iota'_\alpha \mathcal{P} \coloneqq \{x \in \mathbb{R}^k : x = \iota'_\alpha y \text{ for some } y \in \mathcal{P}\}$ when $Q_\alpha(t)$ is invertible. Since ι'_α is an isometry, we can write

$$\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) = \sqrt{\det(Q_\alpha^2(t))}\mathcal{H}^k(\mathcal{P}).$$

[8]Note that $u(\cdot, t)$ is constant in $\mathcal{P}$.

When $Q_\alpha(t)$ is not invertible, we have that $\sqrt{\det(Q_\alpha^2(t))} = 0$. Noticing that $\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) \geq 0$ for all $t \in [0, \infty)$, we have the following inequality valid for all times

$$\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) \geq \sqrt{\det(Q_\alpha^2(t))}\mathcal{H}^k(\mathcal{P}). \tag{31}$$

The volume upper bound: now, we want to find an upper bound for $\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P}))$. We start by noticing that, for any $x \in \iota'_\alpha \mathcal{P}$ and any $y \in \iota'_\alpha \mathcal{P}$, we have that $|Q_\alpha(t)(x-y)| = |(x-y)'\iota'_\alpha \varphi'(t)\varphi(t)\iota_\alpha(x-y)|^{1/2} = |\varphi(t)\iota_\alpha(x-y)|$, for all $t \in [0, T]$. This equality tells us that

$$\begin{aligned}\text{diam}(\{Q_\alpha(t)x : x \in \iota'_\alpha \mathcal{P}\}) &= \text{diam}(\{\varphi(t)\iota_\alpha x : x \in \iota'_\alpha \mathcal{P}\}) \\ &= \text{diam}(\{\varphi(t)y : y \in \mathcal{P}\}) \leq 2\varepsilon e^{-\alpha t}\end{aligned}$$

for all $t \in [0, T]$, where the second equality follows from the fact that, for each $x \in \iota'_\alpha \mathcal{P}$,there exists $y \in \mathcal{P}$ so that $x = \iota'_\alpha y$. Also, the inequality follows from (30). Now, the k-dimensional Hausdorff measure coincides with the Lebesgue measure of Borel sets in $\mathbb{R}^k$ (see, e.g., Theorem 2.6 from [Sim83]). Thus, $\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) = \mathcal{L}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P}))$. Therefore, we can use the isodiametric inequality (see, e.g., Theorem[9] 8.8 from [Gru07]) to conclude that $\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) \leq (\text{diam}(Q_\alpha(t)(\iota'_\alpha \mathcal{P}))/2)^k \mathcal{H}^k(\mathbb{B}_k(0;1))$, which implies that

$$\mathcal{H}^k(Q_\alpha(t)(\iota'_\alpha \mathcal{P})) \leq \mathcal{H}^k(\mathbb{B}_k(0;1))\varepsilon^k e^{-k\alpha t}. \tag{32}$$

We adopt the notation $\omega_k \coloneqq \mathcal{H}^k(\mathbb{B}_k(0;1))$.

The counting argument: combining the inequality (32) with the equality (31), we get that

$$\omega_k \varepsilon^k e^{-k\alpha t} \geq \sqrt{\det(Q_\alpha^2(t))}\mathcal{H}^k(\mathcal{P})$$

for all $t \in [0, T]$. We can rearrange the terms to get that

$$1 > \sqrt{\det(Q_\alpha^2(t))} e^{k\alpha t} \frac{\mathcal{H}^k(\mathcal{P})}{\varepsilon^k \omega_k}.$$

Now, we can sum over all elements of the partition $\mathscr{P}_u^\alpha(T)$ to get that

$$M_u(T) > \sqrt{\det(Q_\alpha^2(t))} e^{k\alpha t} \frac{\mathcal{H}^k(\mathcal{K}_\alpha(\bar{x}))}{\varepsilon^k \omega_k}$$

since $\sum_{\mathcal{P} \in \mathscr{P}_u^\alpha(T)} \mathcal{H}^k(\mathcal{P}) = \mathcal{H}(\mathcal{K}_\alpha(\bar{x}))$ and $M_u(T)$ is the cardinality of $\mathscr{P}_u^\alpha(T)$. Here, we note that $N_u(T)$, the cardinality of $\mathscr{P}_u(T)$ is larger than or equal to $M_u(T)$. Thus, we can write that

$$N_u(T) > \sqrt{\det(Q_\alpha^2(t))} e^{k\alpha t} \frac{\mathcal{H}^k(\mathcal{K}_\alpha(\bar{x}))}{\varepsilon^k \omega_k}$$

Recall that inequality (28) gives us that the left-hand side of the previous inequality is greater than zero. Therefore, we can evaluate the right-hand side of the last inequality at $t = T$, take the logarithm on both sides, and divide by T to get that

$$\begin{aligned}\frac{1}{T}\log(N_u(T)) >& \frac{1}{T}\log(\sqrt{\det(Q_\alpha^2(T))}) + k\alpha + \\ &+ \frac{1}{T}\log\left(\frac{\mathcal{H}^k(\mathcal{K}_\alpha(\bar{x}))}{\varepsilon^k \omega_k}\right).\end{aligned}$$

We take the limit superior of T going to infinity to get the lower bound for the asymptotic average data-rate

$$b(u) \geq k\alpha + \limsup_{T\to\infty} \frac{1}{T}\log(\sqrt{\det(Q_\alpha^2(T))}),$$

where we have used the fact that $\limsup_{T\to\infty}(f(T) + g(T)) = \lim_{T\to\infty} g(T) + \limsup_{T\to\infty} f(T)$ for any function $g(\cdot)$ that has a limit when T goes to infinity. We now use Lemma 1 to evaluate the limit superior and get that

$$b(u) \geq k\alpha + \sum_{i=1}^k \bar{\chi}_i = \sum_{i=1}^k (\bar{\chi}_i + \alpha) = \sum_{i=1}^d \max\{\bar{\chi}_i + \alpha, 0\}.$$

This concludes the proof. ■

[9] That result is stated only for convex bodies. However, we see trivially that it holds for any Lebesgue measurable set since $S \subseteq \text{co}(S)$ implies that $\text{vol}(S) \leq \text{vol}(\text{co}(S))$ and $\text{diam}(S) = \text{diam}(\text{co}(S))$.

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A Appendix of Section 3

To prove Proposition 1, we need some lemmas. First, let $\tilde{A} := \begin{pmatrix} A & 0 \\ 0 & A+BF \end{pmatrix}$, $\tilde{C} := \begin{pmatrix} I_d & -I_d \end{pmatrix}$, and $M = \begin{pmatrix} I_d & I_d \end{pmatrix}'$. Our goal is to characterize the unobservable subspace of the pair $(\tilde{A}, \tilde{C})$ in terms of the root spaces of A and $A + BF$. To do that, we state the next lemma.

Lemma 2
The unobservable subspace of the pair $(\tilde{A}, \tilde{C})$ is given by $\mathcal{N} = \{(x, y)' \in \mathbb{R}^{2d} : x = y \text{ and } x \in V\}$, where V is the largest A-invariant subspace contained in ker$\{BF\}$. Moreover, if $V \neq \{0\}$, we have that A and $A + BF$ have a common eigenvalue $\bar{\lambda}$ so that $\mathcal{R}(\bar{\lambda}, A) \cap \mathcal{R}(\bar{\lambda}, A + BF) \neq \{0\}$. ▲

This lemma tells us that if A and $A+BF$ have no common eigenvalues with intersecting root spaces, the pair $(\tilde{A}, \tilde{C})$ is observable. In particular, if all the eigenvalues of A are distinct from those of $A+BF$, we get the observability of the pair. Our next goal is to associate the Lyapunov indices of the output function $\tilde{y}(\cdot)$ of the auxiliary system

$$\tilde{x}(t) = \tilde{A}\tilde{x}(t) \tag{33a}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) \tag{33b}$$

with the Lyapunov indices of the state $\tilde{x}(\cdot)$. The following lemma states that, when the pair $(\tilde{A}, \tilde{C})$ is observable, we have that $\lambda(\tilde{C}e^{\tilde{A}t}, x) = \lambda(e^{\tilde{A}t}, x)$. We prove this result in the technical report due to space considerations.

Lemma 3
Let A and C be a $k \times k$ and $p \times k$ real matrices. Also, assume that (A, C) is an observable pair. Then, we have that

$$\limsup_{t\to\infty} \frac{1}{t} \log(|Ce^{At}x|) = \limsup_{t\to\infty} \frac{1}{t} \log(|e^{At}x|) \tag{34}$$

for all $x \in \mathbb{R}^k$. ▲

We finally have all tools necessary to prove Proposition 1
Proof of Proposition 1: first, let $x \in \mathbb{R}^{2d}$ be written as $x = (x_1, x_2)'$ for vectors $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^d$ in some basis. Then, consider the norm

$$|x|_{\max} := \max\{|x_1|, |x_2|\}$$

in $\mathbb{R}^{2d}$. Note that the Lyapunov index value $\lambda(\varphi, x)$ does not change for equivalent norms. That follows from the fact that if $k_1|x|_1 \leq |x|_2 \leq k_2|x|_1$ for norms $|\cdot|_1$ and $|\cdot|_2$ and positive constants k_1 and k_2, then $\frac{1}{t}\big(\log(k_1) + \log(|\varphi(t)x|_1)\big) \leq \frac{1}{t}\log(|\varphi(t)x|_2) \leq \frac{1}{t}\big(\log(k_2) + \log(|\varphi(t)x|_1)\big)$ for all $t \in (0, \infty)$ since the logarithm is increasing and t is positive. Hence, after taking the limit superior, we conclude our claim. Now, since all norms in a finite-dimensional vector space are equivalent, we can use the norm $|\cdot|_{\max}$ in our analysis. Thus, for any $x \in \mathbb{R}^d$, we have that

$$\begin{aligned}
\lambda(\varphi, x) &= \limsup_{t\to\infty} \frac{1}{t} \log(|\tilde{C}e^{\tilde{A}t}Mx|_{\max}) \\
&= \limsup_{t\to\infty} \frac{1}{t} \log(|e^{\tilde{A}t}Mx|_{\max}) \\
&= \limsup_{t\to\infty} \frac{1}{t} \log(\max\{|e^{At}x|, |e^{(A+BF)t}x|\}) \\
&= \max\Big\{ \limsup_{t\to\infty} \frac{1}{t} \log(|e^{At}x|), \\
&\qquad \limsup_{t\to\infty} \frac{1}{t} \log(|e^{(A+BF)t}x|)\Big\} \\
&= \max\{\lambda(e^{At}x), \lambda(e^{(A+BF)t}, x)\}
\end{aligned}$$

where the second equality follows from Lemmas 2 and 3. Indeed, Lemma 2 implies that the pair $(\tilde{A}, \tilde{C})$ is observable, which, by its turn, allows us to apply Lemma 3 and conclude that the second equality holds. The fourth equality follows from the facts that the logarithm is increasing, t is positive, and the fact that the limit superior of a maximum of two functions equals the maximum of the limit superiors. This concludes the proof of the Proposition

■

Before we prove Lemma 2, we introduce some notation and recall some algebraic properties of polynomials; we refer to [HK71], in particular Chapter 4, for an introduction on the subject. We denote by $\mathbb{R}[x]$ the ring of polynomials with real coefficients. Given a vector $v \in \mathbb{R}^d \setminus \{0\}$ and a $d \times d$ matrix A, we define the A-annihilator of v as the set $S(A, v) \subseteq \mathbb{R}[x]$ of all polynomials $p(x) \in \mathbb{R}[x]$ so that $p(A)v = 0$. The set $S(A, v)$ is a polynomial ideal (see, e.g., the second lemma in Section 4 of Chapter 6 in [HK71]), which implies that there exists a unique monic polynomial $q_{(A,v)}(x) \in \mathbb{R}[x]$ that divides every polynomial in $S(A, v)$. A polynomial is irreducible in $\mathbb{R}[x]$ if it cannot be divided by a nonconstant polynomial with coefficients in $\mathbb{R}$ (see Section 5 of Chapter 4 from [HK71]). Clearly, polynomials of degree one and polynomials of degree two with a pair of complex conjugate roots are irreducible in $\mathbb{R}[x]$. Also, note that if $q(x) \in \mathbb{R}[x]$ is irreducible, $p \in \mathbb{Z}_{>0}$, and $r(x) \in \mathbb{R}[x]$ divides $(q(x))^p$, then $r(x) = (q(x))^s$ for some $s \in [p]$ (this follows from Theorem 4 from [HK71]).

Proof of Lemma 2: we split this proof into two parts. First, we prove that $\mathcal{N}$ is the unobservable subspace of the pair $(\tilde{A}, \tilde{C})$. Second, we prove that if $V \neq \{0\}$, there exists an eigenvalue $\bar{\lambda}$ so that $\mathcal{R}(\bar{\lambda}, A) \cap \mathcal{R}(\bar{\lambda}, A + BF)$.

First part: recall that $\mathcal{N}$ is the largest $\tilde{A}$-invariant subspace contained in $\ker\{\tilde{C}\}$ (see, e.g., Section 2 of Chapter 3 from [Won79]). Firstly, we have that $\mathcal{N} \subseteq \ker\{\tilde{C}\}$ since $(x, y)' \in \mathcal{N}$ only if $x = y$, which implies that $\tilde{C}(x, y)' = x - y = 0$. Secondly, we have that $\mathcal{N}$ is $\tilde{A}$-invariant because for each vector $(x, y)' \in \mathcal{N}$, we have that $\tilde{A}(x, y)' = (Ax, (A + BF)y)' = (Ax, Ay) = (Ax, Ax) \in \mathcal{N}$, where the second equality follows from the fact that $y \in V$, which is contained in $\ker\{BF\}$, and the last follows from the fact that $x = y$.

Next, assume that $\mathcal{M} \subseteq \mathbb{R}^{2d}$ is an $\tilde{A}$-invariant subspace contained in the kernel of $\tilde{C}$. We want to show that $\mathcal{M} \subseteq \mathcal{N}$. Recall that the projection $\pi_i : \mathbb{R}^{2d} \to \mathbb{R}^d$ onto the i-th coordinate is a linear map for each $i \in [2]$. Therefore $V_i := \pi_i \mathcal{M}$ is a vector subspace of $\mathbb{R}^d$ for each $i \in [2]$. Since we assumed that $\mathcal{M} \subseteq \ker\{\tilde{C}\} = \{(x, y)' \in \mathbb{R}^{2d} : x = y\}$, we conclude that $\mathcal{M} = \{(x, y)' \in \mathbb{R}^{2d} : x \in V_1, y \in V_2\} \cap \{(x, y)' \in \mathbb{R}^{2d} : x = y\} = \{(x, x)' \in \mathbb{R}^{2d} : x \in V_1\}$. Now, for any $(x, x)' \in \mathcal{M}$, we have that $\tilde{A}(x, x)' = (Ax, (A + BF)x)' = (Ax, Ax)' \in \mathcal{M}$, where the last equality follows from the fact that $Ax = (A + BF)x$. Thus, if $x \in V_1$, we have that $Ax \in V_1$, proving that V_1 is A-invariant. Also, the equality $Ax = (A + BF)x$ implies that $BFx = 0$, which proves that $V_1 \subseteq \ker\{BF\}$. Thus, $\mathcal{M} = \{(x, y) \in \mathbb{R}^{2d} : x = y \text{ and } x \in V_1\}$ with V_1 and A-invariant subspace contained in $\ker\{BF\}$. Finally, the fact that V is the largest among all such subspaces of $\mathbb{R}^d$ tells us that $\mathcal{N} \supseteq \mathcal{M}$, which concludes the proof of the first part.

Second part: since V is A-invariant, we have that $V = \bigoplus_{\lambda \in \Sigma(A)} \mathcal{R}(\lambda, A) \cap V$ (see, e.g., Theorem 12.2.1 from [GLR86]). Thus, if $V \neq \{0\}$, there exists $\bar{\lambda} \in \Sigma(A)$ so that $\mathcal{R}(\bar{\lambda}, A) \cap V \neq \{0\}$. Pick a nonzero $v \in \mathcal{R}(\bar{\lambda}, A) \cap V$. Recall that (see Definition 6) $\mathcal{R}(\bar{\lambda}, A) = \ker\{(q(A))^p\}$, where $q(x) = x - \bar{\lambda}$, if $\bar{\lambda} \in \mathbb{R}$, or $q(x) = x^2 - 2\Re(\bar{\lambda})x + |\bar{\lambda}|^2$, if $\bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$, and $p \in \mathbb{Z}_{>0}$. Note that $q(x)$ is irreducible in $\mathbb{R}[x]$. Since $v \in \mathcal{R}(\bar{\lambda}, A)$, we have that $(q(A))^p v = 0$, which implies that $(q(x))^p \in S(A, v)$ and $q_{(A,v)}(x)$ must divide $(q(x))^p$. Hence, $q_{(A,v)}(x) = (q(x))^s$ for some $s \in [p]$.

Next, we note that $A^i v = (A + BF)^i v$ for all $i \in \mathbb{Z}_{\geq 0}$. We prove this fact by induction. We proved the base case $Av = (A + BF)v$ in the first part. Assume that $A^i v = (A + BF)^i v$ for all $i \in [\ell]$ for some $\ell \in \mathbb{Z}_{>0}$. Thus, we have that $A^{i+1} v = A(A^i v) = A((A + BF)^i v) = (A + BF)^{i+1} v$, where the second equality follows from the induction hypothesis and the last one comes from the fact that $(A + BF)^i v = A^i v \in V$ since V is A-invariant. This implies that $S(A, v) = S(A + BF, v)$. To see why this latter fact holds, pick an arbitrary polynomial $r(x) = \sum_{j=1}^n a_j x^j$ for some $n \in \mathbb{Z}_{>0}$ and $a_j \in \mathbb{R}$ for each $j \in [n]$. Then, if $r(x) \in S(A + BF, v)$, we have that $0 = r(A + BF)v = \sum_{j=1}^n a_j (A + BF)^j v = \sum_{j=1}^n a_j A^j v = r(A)v$, which implies that $S(A + BF, v) \subseteq S(A, v)$. The proof of the reverse inclusion is analogous. This implies that $q_{(A+BF,v)}(x) = q_{(A,v)}(x)$ and that $q_{(A,v)}(x)$ divides the minimal polynomial of $A + BF$ since the minimal polynomial is contained in $S(A + BF, v)$. Hence, $\bar{\lambda}$ is a common eigenvalue of $A + BF$ and A. Therefore, $\mathcal{R}(\bar{\lambda}, A + BF) \neq \{0\}$ and there exists an integer $\bar{p} \in \mathbb{Z}_{>0}$ so that $\ker\{(q(A + BF))^{\bar{p}}\} = \mathcal{R}(\bar{\lambda}, A + BF)$. Moreover, $v \in \mathcal{R}(\bar{\lambda}, A + BF)$ since $q_{(A+BF,v)}(A + BF)v = q_{(A,v)}(A)v = 0$, $q_{(A,v)}(x) = (q(x))^s$ for $s \in \mathbb{Z}_{>0}$, and $\bar{p} \in \mathbb{Z}_{>0}$ is the smallest integer so that $\ker\{(q(A + BF))^{\bar{p}}\} = \ker\{(q(A + BF))^n\}$ for all $n \geq \bar{p}$. The first two facts we mentioned imply that $v \in \ker\{(q(A+BF))^s\}$ and the latter implies that $\ker\{(q(A+BF))^s\} \subseteq \ker\{(q(A+BF))^{\bar{p}}\}$. Hence, $v \in \mathcal{R}(\bar{\lambda}, A) \cap \mathcal{R}(\bar{\lambda}, A+BF) \neq \{0\}$, which concludes the proof. ■

To prove Lemma 3, we need to state and prove the following auxiliary result.

Lemma 4
Let $f : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ be a function so that $f(\cdot, x)$ is upper semi-continuous function for each $x \in \mathbb{R}^d$. Also let $g(t,x) \coloneqq \sup\{f(s,x) : s \in [t, t+\tau]\}$ for some $\tau \in [0,\infty)$. Then, $\lambda(g,x) = \lambda(f,x)$. ▲

Proof of Lemma 4: first, we note that

$$g(t,x) \geq f(t,x) \tag{35}$$

for each $t \in [0,\infty)$. Thus,

$$\limsup_{t\to\infty} \frac{1}{t} \log(f(x,t)) \leq \limsup_{t\to\infty} \frac{1}{t} \log(g(x,t)). \tag{36}$$

To prove the other inequality, we pick an increasing sequence of times $(t_n)_{n\in\mathbb{Z}_{\geq 0}}$ so that $\lim_{n\to\infty} t_n = \infty$ and

$$\lambda(f,x) = \lim_{n\to\infty} \frac{1}{t_n} \log(g(x,t_n)) = \limsup_{t\to\infty} \frac{1}{t} \log(g(x,t)) = \lambda(g,x), \tag{37}$$

which always exists (see, e.g., Theorem 2 from [Sib08]). Since $f(\cdot,x)$ is upper semi-continuous, for each $n \in \mathbb{Z}_{\geq 0}$, there exists some $\tau_n \in [t_n, t_n+\tau]$ so that $f(\tau_n, x) = g(t_n, x)$. Hence, we have that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{\tau_n} \log(f(\tau_n,x)) &= \limsup_{n\to\infty} \frac{1}{\tau_n} \log(g(t_n,x)) \\
&= \lim_{n\to\infty} \frac{1}{\tau_n} \log(g(t_n,x)) \\
&= \lim_{n\to\infty} \frac{t_n}{\tau_n} \frac{1}{t_n} \log(g(t_n,x)) \\
&= \lim_{n\to\infty} \frac{1}{t_n} \log(g(t_n,x)) \\
&= \limsup_{t\to\infty} \frac{1}{t} \log(g(t,x)),
\end{aligned}$$

where the second equality follows from the fact that the limit on the right-hand side exists by our choice of sequence $(t_n)_{n\in\mathbb{Z}_{\geq 0}}$, the fourth equality follows from the fact that $\lim_{n\to\infty} \frac{t_n}{\tau_n} = 1$ since $\tau_n \in [t_n, t_n+\tau]$ for each $n \in \mathbb{Z}_{\geq 0}$, and the last equality follows from equation (37). Recalling that

$$\limsup_{t\to\infty} \frac{1}{t} \log(f(x,t)) \geq \limsup_{n\to\infty} \frac{1}{\tau_n} \log(f(\tau_n,x))$$

for any increasing sequence $(\tau_n)_{n\in\mathbb{Z}_{\geq 0}}$ so that $\lim_{n\to\infty} \tau_n = \infty$, we get hat $\lambda(f,x) \geq \lambda(g,x)$, which concludes the proof of the lemma. ■

Proof of Lemma 3: first, we prove that $\lambda(e^{At},x) \geq \lambda(Ce^{At},x)$. This follows by observing that

$$\begin{aligned}
\lambda(Ce^{At},x) &= \limsup_{t\to\infty} \frac{1}{t} \log(|Ce^{At}x|) \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \left(\log(\|C\|) + \log(|e^{At}x|)\right) \\
&= \limsup_{t\to\infty} \frac{1}{t} \log(|e^{At}x|) \\
&= \lambda(e^{At},x).
\end{aligned}$$

Now, we use Lemma 4 to prove that $\lambda(e^{At},x) \leq \lambda(Ce^{At},x)$. Pick $\tau \in (0,\infty)$ and $x \in \mathbb{R}^k \setminus \{0\}$. Next, note that

$$\sup\{|Ce^{As}x| : s \in [t,t+\tau]\} \geq \omega(\tau)|e^{At}x| \tag{38}$$

for some $\omega(\tau) \in (0, \infty)$ for all $t \in [0, \infty)$. This follows from the fact that the observability Gramian satisfies

$$\int_0^\tau e^{A's} C' C e^{As} ds > \nu(\tau) I_d \tag{39}$$

for each $\tau \in [0, \infty)$ and some $\nu(\tau) > 0$ when (A, C) is observable. Then, we can write

$$\begin{aligned} \int_0^\tau |Ce^{A(t+s)} x|^2 ds &= (e^{At} x)' \int_0^\tau e^{A's} C' C e^{As} ds (e^{At} x) \\ &> \nu(\tau) |e^{At} x|^2, \end{aligned}$$

which gives us that

$$\begin{aligned} \nu(\tau) |e^{At} x|^2 &< \int_0^\tau |Ce^{A(t+s)} x|^2 ds \\ &\leq \sup\{|Ce^{As} x| : s \in [t, t+\tau]\}^2 \tau. \end{aligned}$$

Defining $\omega(\tau) := \sqrt{\frac{\nu(\tau)}{\tau}}$, we prove inequality (38). To simplify the notation, we write $g(t, x) := \sup\{|Ce^{As} x| : s \in [t, t+\tau]\}$. Taking the logarithm, divding by $t \in (0, \infty)$, and taking the limit superior as t goes to infinity, we get

$$\begin{aligned} \lambda(g(t), x) &= \limsup_{t \to \infty} \frac{1}{t} \log(g(t, x)) \\ &\geq \limsup_{t \to \infty} \frac{1}{t} \log(\omega(\tau) |e^{At} x|) \\ &= \limsup_{t \to \infty} \frac{1}{t} \big(\log(\omega(\tau)) + \log(|e^{At} x|)\big) \\ &= \limsup_{t \to \infty} \frac{1}{t} \log(|e^{At} x|) \\ &= \lambda(e^{At}, x), \end{aligned}$$

where the inequality holds because the logarithm is an increasing function and $t > 0$. By Lemma 4, we have that $\lambda(g(t), x) = \lambda(Ce^{At}, x)$. This concludes the proof of the lemma.

■

B Appendix of Section 4

In this section of the appendix, we use the Courant-Fischer Theorem some times. For convenience, we state it here (see, e.g., Theorem 4.2.11 from [HJ12] for a proof[10]).

Theorem 2 (Courant-Fischer)
Let A be a $d \times d$ symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_d$. Then, for each $i \in [d]$, we have that

$$\begin{aligned}\lambda_i = \min\{&\max\{v'Av : v \in V, |v| = 1\} : \\ &V \subset \mathbb{R}^d, \dim(V) = d - i + 1\} \qquad (40a)\\ = \max\{&\min\{v'Av : v \in V, |v| = 1\} : \\ &V \subset \mathbb{R}^d, \dim(V) = i\}. \qquad (40b)\end{aligned}$$

▲

To prove Lemma 1, we recall that $V \subseteq$ is the maximal A-invariant subspace contained in $\ker(BF)$ and note that $\lambda(\varphi, v) = -\infty$ for each $v \in V$ since $e^{At}v = e^{(A+BF)t}v$ for all $t \in [0, \infty)$ (see the proof of Lemma 2). Before we prove Lemma 1, we state the next two lemmas. The first one proves that $\Sigma(A) \cap \Sigma(A+BF) = \emptyset$ is sufficient to ensure that the limit superior in the definition of the Lyapunov index $\lambda(\cdot, \cdot)$ is actually a limit. Lemma 6 shows us that the exponential growth-rate of the singular values of $Q_\alpha(t)$ converge.

Lemma 5
Assume that $\Sigma(A) \cap \Sigma(A + BF) = \emptyset$. Then, we have that

$$\lambda(\varphi, x) = \lim_{t\to\infty} \frac{1}{t} \log(|\varphi(t)x|), \qquad (41)$$

for all $x \in \mathbb{R}^d \setminus V$. ▲

Lemma 6
Assume that $\Sigma(A) \cap \Sigma(A + BF) = \emptyset$. Then, we have that

$$\lambda(\varphi, v_i) = \lim_{t\to\infty} \frac{1}{t} \log(\sigma_i(Q_\alpha(t))), \qquad (42)$$

for all $x \in \mathbb{R}^d \setminus V$. ▲

Now, we can finally prove Lemma 1.

Proof of Lemma 1: since $Q_\alpha^2(t)$ is positive definite for all $t \in [0, \infty)$, we have that $\det(Q_\alpha^2(t)) = \prod_{i=1}^k \sigma_i(Q_\alpha^2(t))$ for all $t \in [0, \infty)$. Finally, we note that

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \log(\sqrt{\det(Q_\alpha^2(t))}) &= \limsup_{t\to\infty} \frac{1}{2t} \log\Big(\prod_{i=1}^k \sigma_i(Q_\alpha^2(t))\Big)\\ &= \limsup_{t\to\infty} \sum_{i=1}^k \frac{1}{2t} \log\Big(\sigma_i(Q_\alpha^2(t))\Big)\\ &= \limsup_{t\to\infty} \sum_{i=1}^k \frac{1}{t} \log\Big(\sigma_i(Q_\alpha(t))\Big)\\ &= \sum_{i=1}^k \lim_{t\to\infty} \frac{1}{t} \log\Big(\sigma_i(Q_\alpha^2(t))\Big)\\ &= \sum_{i=1}^k \bar{\chi}_i,\end{aligned}$$

[10] We note that our definition is slightly different since we order the singular values in decreasing order and [HJ12] does it in increasing order.

where the third equality follows from the fact that $\sigma_i(A^2) = \sigma(A)^2$ (see, e.g., Corollary 7.2.3 from [HJ12]), the fourth equality folows from Lemma 6. This concludes the proof of the lemma. ■

Here, we prove the auxiliary lemmas 5 and 6.

Proof of Lemma 5: to keep the notation simple, let $\gamma_1 := \lambda(e^{At}, x)$, $\gamma_2 := \lambda(e^{(A+BF)t}, x)$, and $\gamma := \max\{\gamma_1, \gamma_2\}$. The reverse triangle inequality applied to $|\varphi(t)x|$ gives us that

$$|\varphi(t)x| \geq ||e^{At}x| - |e^{(A+BF)t}x|| \tag{43}$$

for all $t \in [0, \infty)$. Next, recall that for each $\delta \in (0, \infty)$, there exists $T(\delta) \in [0, \infty)$ so that

$$e^{(\gamma_1-\delta)t} \leq |e^{At}x| \leq e^{(\gamma_1+\delta)t} \tag{44a}$$

$$e^{(\gamma_2-\delta)t} \leq |e^{(A+BF)t}x| \leq e^{(\gamma_2+\delta)t} \tag{44b}$$

for all $t \in (T(\delta), \infty)$ (see, e.g., Example 3.2.3 from [Arn98]). Combining the inequalities (43), (44a), and (44b), we get that

$$\begin{aligned}|\varphi(t)x| &\geq \max\{e^{(\gamma_1-\delta)t} - e^{(\gamma_2+\delta)t}, e^{(\gamma_2-\delta)t} - e^{(\gamma_1+\delta)t}\} \\ &= e^{(\gamma-\delta)t} \max\{e^{(\gamma_1-\gamma)t} - e^{(\gamma_2-\gamma+2\delta)t}, e^{(\gamma_2-\gamma)t} - e^{(\gamma_1-\gamma+2\delta)t}\}.\end{aligned}$$

for all $t \in (T(\delta), \infty)$. Recall that our choice of $\delta \in (0, \infty)$ was arbitrary. In particular, the previous inequality is valid for any $\delta \in (0, |\gamma_1 - \gamma_2|/2)$. Now, assume that $\gamma_1 - \gamma = 0$. Under this assumption, we have that $\gamma_2 - \gamma + 2\delta < 0$, which implies that

$$|\varphi(t)x| \geq e^{(\gamma-\delta)t}(1 - e^{(\gamma_2-\gamma+2\delta)t})$$

for all $t \in (T(\delta), \infty)$. Therefore, we conclude

$$\frac{1}{t}\log(|\varphi(t)x|) \geq \gamma - \delta + \frac{1}{t}\log(1 - e^{(\gamma_2-\gamma+2\delta)t}). \tag{45}$$

On the other hand, we always have the upper bound

$$|\varphi(t)x| \leq 2\max\{|e^{At}|, |e^{(A+BF)t}|\}$$

for all $t \in [0, \infty)$ as a consequence of the triangle inequality. This, combined with the upper bounds from inequalities (44a) and (44b) give us that

$$|\varphi(t)x| \leq 2\max\{e^{(\gamma_1+\delta)t}, e^{(\gamma_2+\delta)t}\} = 2e^{(\gamma+\delta)t}$$

for all $t \in [T(\delta), \infty)$. Clearly, we have that

$$\gamma - \delta + \frac{1}{t}\log(1 - e^{(\gamma_2-\gamma+2\delta)t}) \leq \frac{1}{t}\log(|\varphi(t)x|) \leq \gamma + \delta + \frac{1}{t}\log(2). \tag{46}$$

We note that

$$\gamma - \delta \leq \liminf_{t\to\infty} \frac{1}{t}\log(|\varphi(t)x|) \leq \gamma + \delta \tag{47a}$$

$$\gamma - \delta \leq \limsup_{t\to\infty} \frac{1}{t}\log(|\varphi(t)x|) \leq \gamma + \delta. \tag{47b}$$

for each $\delta \in (0, |\gamma_1 - \gamma_2|/2)$. This implies that

$$\gamma = \liminf_{t\to\infty} \frac{1}{t}\log(|\varphi(t)x|) = \limsup_{t\to\infty} \frac{1}{t}\log(|\varphi(t)x|) = \lim_{t\to\infty} \frac{1}{t}\log(|\varphi(t)x|), \tag{48}$$

proving that the limit exists. The case where $\gamma_2 = \gamma$ is analogous, thus omitted.

■

Proof of Lemma 6: we split this proof into several parts.

Lower bound: let $Z_i \coloneqq \text{span}\{\iota'_\alpha v_1, \ldots, \iota'_\alpha v_i\}$ and observe that we can use the Courant-Fischer Theorem to write that

$$\begin{aligned}\sigma_i(Q_\alpha(t)) &= \max\{\min\{x'Q_\alpha(t)x : x \in Z, |x| = 1\} : Z \subseteq \mathbb{R}^k, \dim(Z) = i\} \\ &\geq \min\{x'Q_\alpha(t)x : x \in Z_i, |x| = 1\} \\ &= \min\{y'\varphi'(t)\varphi(t)y : y \in \text{span}\{v_1, \ldots, v_i\}, |y| = 1\} \\ &= \min\left\{|\varphi(t)y|^2 : y \in \text{span}\{v_1, \ldots, v_i\}, |y| = 1\right\}\end{aligned}$$

where the inequality comes from the fact that $\dim(Z_i) = i$ and the second equality comes from the fact that for each $x \in Z_i$ with $|x| = 1$, we can write $y = \iota_\alpha x \in \text{span}\{v_1, \ldots, v_i\}$ where $|y| = 1$ since ι_α is an isometry from Z_i to $\mathbb{R}^i$, and vice-versa.
Upper bound: analogously, we can again use the Courant-Fischer Theorem to conclude that

$$\sigma_i(Q_\alpha(t)) \leq \max\left\{|\varphi(t)y|^2 : y \in \text{span}\{v_i, \ldots, v_k\}, |y| = 1\right\}$$

for all $t \in [0, \infty)$.
Squeezing lemma: Lemma 5 tells us that the limit $\lim_{t\to\infty} \frac{1}{t}\log(|\varphi(t)y|)$ exists for each $y \in \text{span}\{v_1, \ldots, v_i\}$ since $\{v_1, \ldots, v_k\} \cap V = \emptyset$. Otherwise, we would have $\lambda(\varphi, v_i) = -\infty$ for some $i \in [k]$, which cannot happen given that $\lambda(\varphi, v_i) \geq -\alpha$. Thus, we have that

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t}\log(Q_\alpha(t)) &\geq \liminf_{t\to\infty} \min\left\{\frac{2}{t}\log(|\varphi(t)y|) : y \in \text{span}\{v_1, \ldots, v_i\}, |y| = 1\right\} \\ &= \min\left\{\lim_{t\to\infty} \frac{2}{t}\log(|\varphi(t)y|) : y \in \text{span}\{v_1, \ldots, v_i\}, |y| = 1\right\} \\ &= 2\min\{\lambda(\varphi, y) : y \in \text{span}\{v_1, \ldots, v_i\}, |y| = 1\} \\ &= 2\lambda(\varphi, v_i),\end{aligned}$$

where the first equality holds because the limit of the functions inside the minimum on the right-hand side exist and the fact that the minimum is a continuous function. Analogously, we have that

$$\limsup_{t\to\infty} \max\left\{\frac{2}{t}\log(|\varphi(t)y|) : y \in \text{span}\{v_i, \ldots, v_k\}, |y| = 1\right\} \leq \lambda(\varphi, v_i),$$

following a similar reasoning as before. Thus, by the squeezing lemma, we conclude that

$$\lim_{t\to\infty} \frac{1}{t}\log(\sigma_i(Q_\alpha(t))) = \lambda(\varphi, v_i) \tag{49}$$

for each $i \in [k]$. ■