Towards a Data-Rate Theorem for State Estimation with Guaranteed Accuracy in Bounded-Noise Environments

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Abstract— **In this work, we analyze the minimum data-rate to reconstruct the state trajectory of a perturbed discrete-time linear system with a prescribed accuracy. The data is transmitted periodically at every sample. Among the results, we prove data-rate lower and upper bounds that grow to infinity as a "disturbance-to-accuracy ratio" approaches one. In general, when considered per dimension, the gap between the upper and lower bounds grows at most logarithmically and, in many cases, remains constant. Furthermore, we discuss why such a result must be coordinate-dependent, a feature that differs from the noiseless case. Finally, we illustrate our results with an example.**

I. Introduction

Control systems with distributed components, such as sensors and actuators, are prevalent in practical applications. Those components often need to share measurement results, which requires communication channels. These channels, in turn, limit the amount of data the transmitter can send over a given time interval to a finite number of symbols. This latter restriction, on the other hand, limits the accuracy with which the receiver can reconstruct the original message. Thus, understanding the relationship between this accuracy and the communication data-rate is relevant to solving distributed control and estimation problems efficiently.

It is now well-established that there is a minimum transmission data-rate below which some control [1], [2], [3], [4], [5] and estimation ([6], [7], [8], [9], [10]) problems have no solution. In particular, in the absence of perturbations, we know that the minimum data-rate to reconstruct the state with a prescribed uniform upper bound for the estimation error is the topological entropy of the system [7]. When perturbations are considered, the work [11] provides a lower bound for such a minimal data-rate for nonlinear systems that satisfy a type of dissipation inequality. However, data-rate theorems providing upper and lower bounds for state estimation of perturbed linear systems with accuracy guarantees are missing from the literature. The goal of the present document is to fill that gap.

In general, there is a maximal time for the transmitter to send the data to keep the state estimation error within a prescribed bound. The reason is that if we wait too long to transmit, the cumulative effect of the disturbance eventually increases the set of possible states to a point where it is too large, making it impossible to build an estimator that is guaranteed to satisfy the desired accuracy bound. Consequently, we cannot use asymptotic techniques, such as those presented in [7], [8], [10], [12], to analyze this problem. The present results show that the data-rate in noisy environments must be higher than that usually provided by classical data-rate theorems. Remarkably, we prove that when the effect of the disturbance becomes comparable to the required accuracy, the minimum data-rate to solve this problem grows unbounded. We arrive at this conclusion by providing a data-rate lower bound below which the state reconstruction problem has no solution. Additionally, we show that the problem always has a solution if a"disturbance-to-accuracy ratio" is smaller than one.

It is worth mentioning that the work [11] also studies the problem of reconstructing the state of a nonlinear system under accuracy requirements and the effect of noise. Their result also utilizes a disturbance-to-accuracy ratio in its lower bound. Nevertheless, our work differs from that in a few aspects. First, although their result is stated for nonlinear systems, [11] does not provide a data-rate upper bound. In the present work, we restrict our attention to linear systems, a fact that allows us to provide both lower and upper bounds. Moreover, we show that the data-rate gap between the two bounds grows at most logarithmically and, when the accuracy is similar in size to the noise, is constant when considered per dimension, i.e., when dividing the gap by the state dimension. Second, as mentioned above, our lower bound grows unbounded when the disturbance-to-accuracy is close to one, a new feature not explicitly captured by the result in [11]. Third, our proof methods, presented in the appendix, are novel and might provide new tools, complementary to those in [11], for deriving new data-rate theorems.

This document is structured as follows: in Section II, we informally pose and motivate the problem. Also, we present a motivating example that illustrates the types of results obtained in this paper. Then, in Section III, we introduce the uncertain model for the class of systems we study. This model provides a way to understand how the perturbed system dynamics generates uncertainty. We do that using the notion of uncertain variables introduced in [4]. We also formally state the problem we want to study. Then, in Section IV, we present our main results: we claim that the state estimation problem has a solution, and we provide data-rate lower and upper bounds. Next, we analyze and illustrate the results by revisiting the motivating example. After that, we present our conclusions in section VI. Finally, we present the proofs of the main theorem and related results in the appendices.

Notations: let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$ denote the sets of real,

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integer, nonnegative integer, and positive integer numbers, respectively. Given integers $a < b$, and $c > 0$, denote by $[a : b] := \{a, \ldots, b\}$ and by $[c] := \{1, \ldots, c\}$. Given a set $\mathcal{A}$, we define $a^{0:n} := (a_0, \ldots, a_n) \in \mathcal{A}^{n+1}$. For a vector $x \in \mathbb{R}^d$, we denote by $|x|$ its Euclidean norm. A function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is Big-O of $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ if $\limsup_{n\to\infty} f(n)/g(n) \leq \infty$. Given $r \in \mathbb{R}$, we denote by $\lfloor r \rfloor$ and $\lceil r \rceil$ its floor and ceiling values, respectively. We denote by ω_k the k-dimensional Euclidean unit ball's volume. Finally, we denote by $\log(x)$ the logarithm in base 2 of $x \in (0, \infty)$.

II. The problem and motivation

In this work, we are interested in studying fundamental limitations on the data-rate associated with the problem of state reconstruction for discrete-time linear systems subject to the effect of unknown disturbances. Specifically, our goal is to understand what is the minimum data-rate associated with keeping the estimation error small when disturbances are present. We begin this session by introducing the model and by giving an informal description of the problem we want to address. Then, we present a motivating example that helps illustrate the theory.

Consider a perturbed discrete-time linear time-invariant (LTI) system. Explicitly, for each $n \in \mathbb{Z}_{\geq 0}$, we have that

$$x_{n+1} = Ax_n + Bu_n + v_n, \tag{1}$$

where $x_0 \in \mathbb{R}^d$ is the initial state, $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ are matrices, $(v_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of unknown disturbances, and $(u_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of known control actions. Also assume that A is invertible.

The informal description of our problem is as follows: for each time $n \in \mathbb{Z}_{\geq 0}$, we want to build an estimate $\hat{x}_n$ for the value of x_n. However, the estimator and the plant are physically far from each other. This forces us to transmit the state measurements over a communication channel, which we assume is a digital channel. Since digital channels can only transmit a finite number of symbols per unit of time, we must encode the measurement using a finite alphabet $\mathcal{A}$. This implies that the channel transmits data at an rate of $\log(\#\mathcal{A})$ bits per sample. Since the set of possible packets the estimator can receive from the channel at each time n is finite, we note that, even if we measure the state perfectly, the estimation error cannot be zero in general. Clearly, if the estimator knew x_0 exactly beforehand and there were no disturbances, it could use the recursion $\hat{x}_{n+1} = A\hat{x}_n + Bu_n$ with $\hat{x}_0 = x_0 \in \mathbb{R}^d$ to reconstruct the state since the estimator knows the control u_n. Unfortunately, the estimator does not know x_0 and there are disturbances affecting our system dynamics. We assume, nevertheless, that the estimator knows an estimate $p_0 \in \mathbb{R}^d$ and constants $\gamma \in [0, \infty)$ and $\varepsilon_0 \in (0, \infty)$ so that $|v_n| \leq \gamma$ and that $|x_0 - p_0| \leq \varepsilon_0$, i.e, it knows a bound for the noise and an approximation for the initial state. Note that, we can take $\hat{x}_0 = p_0$ and guarantee that $|x_0 - \hat{x}_0| \leq \varepsilon_0$. Thus, as an alternative goal to perfect state reconstruction, we require the estimator to keep the estimation error uniformly bounded by some prescribed accuracy level $\varepsilon \geq \varepsilon_0$ for all times. We know from the literature [7] that this can be done if $v_n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$ when ε_0 is small enough.

This raises some questions: does there exist a minimum data-rate above which such an estimator exists? Assuming this problem has a solution, can we provide a simple lower bound for the minimum data-rate below which this problem has no solution and explain how it varies with ε_0, ε, and γ? We answer both questions affirmatively in Theorem 1. In that theorem, we provide upper and lower bounds for the minimum data-rate. Also, we remark that the expressions obtained depend only on the noise-to-accuracy ratio γ/ε and on the dynamics, i.e., on A.

To motivate our results, consider the following example. We revisit this example in Section IV once we developed the theory.

Example 1 (Matrices with the same intrinsic entropy) Assume that the initial accuracy and the desired accuracy are the same $\varepsilon_0 = \varepsilon$. Our goal is to understand how the minimum data-rate for estimating the state of system (1) with a uniform estimation error bound varies as a function of the disturbance level γ, the accuracy ε, and the the system matrix A. To do that, consider matrices $A_1 = \begin{pmatrix} 0.9 & 2 \\ 0 & 0.5 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.5 \end{pmatrix}$. Both matrices are Schur stable, i.e., their eigenvalues lie inside the unit circle in the complex plane. We prove in Theorem 1 that the minimum data-rate for reconstructing the state of system (1) with an ε accuracy for each matrices above is lower-bounded by functions that are $O(\log(1/(1-r)))$ with $r = \gamma/\varepsilon$. Intuitively, this fact shows us that if the disturbance is similar in magnitude to the accuracy, the minimum data-rate becomes large. Also, Theorem 1 gives different minimum data-rates for A_1 and A_2. We analyze this observation in Subsection V-B. ▲

Model (1) describes the state evolution for a given initial state x_0, a given sequence of disturbances $(v_n)_{n\in\mathbb{Z}_{\geq 0}}$, and a given sequence of controls $(u_n)_{n\in\mathbb{Z}_{\geq 0}}$. To formally state and study our problem, we need a model that relates how the uncertainty in the sequence of disturbances and in the initial condition affects the uncertainty in the state. This is the goal of the next section.

III. Modeling uncertainty

Consider the following informal discussion: at time n, the estimator knows that the state belongs to a known set, a fact it inferred from past received data. For this reason, we call this set the set of possible states at time n with prior knowledge. Without new information, the estimator's uncertainty about the state would grow since the system dynamics would make the set of possible states at time $n+1$ larger. The encoder, in turn, transmits data that carries information about the state at time n. In this manner, the estimator can update its uncertainty about the state at time n and obtain a set of possible states with posterior knowledge that is "more accurate" than the set with prior knowledge. Finally, the estimator can infer where the state would be at time $n+1$

using the set of possible states at time n and the system dynamics. Thus, in some sense, we can think that the system dynamics generate uncertainty while the symbols the encoder transmits reduce it. Therefore, to find the minimal data-rate for solving our problem, we must understand how fast system (1) generates uncertainty.

The goal of this section is to introduce tools that help us formalize the above discussion. First, we recall the notion of uncertain variables [4] and show how we can use it to model how the uncertainty in system (1) evolves. Then, we formally state the informal assumptions and goals described in the previous section using the uncertain variable framework. Finally, we pose the estimation problem in this setting.

A. Uncertain variables

Our first step to formally describe how system (1) propagates uncertainty over time is to recall the notion of uncertain variables.

Definition 1 (Uncertain variables [4])
Let $(\Omega, \mathscr{F})$ be a measurable space. We call Ω the *sample space* and any element $\omega \in \Omega$ is called a *sample*. A Borel measurable map $\mathbf{X} : \Omega \to \mathcal{X}$ is called an *uncertain variable* (uv). We define an uncertain variable *(marginal) range* by $[\![\mathbf{X}]\!] := \{\mathbf{X}(\omega) \in \mathcal{X} : \omega \in \Omega\}$. Also, given another uncertain variable $\mathbf{Y} : \Omega \to \mathcal{Y}$, we define the *conditional range of* $\mathbf{Y}$ *given* $\mathbf{X} = x$ as $[\![\mathbf{Y}|\mathbf{X} = x]\!] := \{\mathbf{Y}(\omega) \in \mathcal{Y} : \omega \in \Omega \text{ and } \mathbf{X}(\omega) = x\}$. Further, we say that $\mathbf{Y}$ and $\mathbf{X}$ are *unrelated*, which we denote by $\mathbf{Y} \perp\!\!\!\perp \mathbf{X}$, if $[\![\mathbf{Y}|\mathbf{X} = x]\!] = [\![\mathbf{Y}]\!]$ for all $x \in [\![\mathbf{X}]\!]$. ▲

To better understand how uvs can help our discussion, consider the following example: the estimator can model the possible initial states as a uv $\mathbf{X}_0$. In this case, the true initial state is represented by $\mathbf{X}_0(\omega)$ for some unknown $\omega \in \Omega$, and the set $[\![\mathbf{X}_0]\!]$ corresponds to the set of possible initial states. Now, we can model the encoder as another uv $\mathbf{\Gamma}_0$. Similarly to the case of the initial state, we should have that $\mathbf{\Gamma}_0(\omega) = a_0$ corresponds to the encoded value of $\mathbf{X}_0(\omega)$ and that the set of possible symbols is contained in the coder alphabet, i.e., $[\![\mathbf{\Gamma}_0]\!] \subseteq \mathcal{A}$. Thus, even though the estimator does not know ω, it knows $\mathbf{\Gamma}_0(\omega) = a_0$, the encoded value of $\mathbf{X}_0(\omega)$. Hence, we need a way to model how receiving the information that $\mathbf{\Gamma}_0(\omega) = a_0$ affects the estimator uncertainty on $\mathbf{X}_0$. To do that, we use the conditional range $[\![\mathbf{X}_0|\mathbf{\Gamma}_0 = a_0]\!]$, which is the subset of elements of $[\![\mathbf{X}_0]\!]$ that are encoded as a_0, i.e., $[\![\mathbf{X}_0|\mathbf{\Gamma}_0 = a_0]\!]$ is our a posteriori knowledge about $\mathbf{X}_0(\omega)$ given a_0. Therefore, this framework gives us a concise and clear way of describing how new information affects the estimator uncertainty. Nonetheless, we still need to comprehend how the dynamics (1) generates uncertainty. To understand that is the goal of our next subsection.

B. The problem setting and the uncertain model

Denote by $\mathbf{X}_n$, $\hat{\mathbf{X}}_n$, $\mathbf{V}_n$, $\mathbf{U}_n$, and $\mathbf{\Gamma}_n$ the uvs representing the state, the state estimate, the disturbance, the control, and the encoder at time $n \in \mathbb{Z}_{\geq 0}$, respectively. With this notation, we can convert the informal setting described in Section II into properties of the uvs. First, we explain how our coding, decoding, and controller work: there exist sequences of functions $(\gamma_n)_{n \in \mathbb{Z}_{\geq 0}}$, $(c_n)_{n \in \mathbb{Z}_{\geq 0}}$, and $(\delta_n)_{n \in \mathbb{Z}_{\geq 0}}$ so that

$$\mathbf{\Gamma}_0 = \gamma_0(\mathbf{X}_0) \text{ and } \mathbf{\Gamma}_n = \gamma_n(\mathbf{X}_n, \mathbf{\Gamma}_0, \ldots, \mathbf{\Gamma}_{n-1}) \tag{2a}$$

$$\mathbf{U}_n = c_n(\mathbf{\Gamma}_0, \ldots, \mathbf{\Gamma}_{n-1}) \tag{2b}$$

$$\hat{\mathbf{X}}_n = \delta_n(\mathbf{\Gamma}_0, \ldots, \mathbf{\Gamma}_{n-1}) \tag{2c}$$

for each $n \in \mathbb{Z}_{>0}$. Here, we convention that $\mathbf{U}_0$ and $\hat{\mathbf{X}}_0$ are decided before operation. Equation (2a) means that the symbol used to encode the state at time n is a function of the past symbols and the current state, i.e., it has memory. These coders are well-studied and used in practice (see, e.g., predictive quantizers in Chapter 7 from [13]). Equations (2b) and (2c) imply that the control action and the state estimate are functions of the previously received symbols, respectively. We also impose some mild additional technical constraints.

Assumption 1

i) $[\![\mathbf{X}_0]\!] = \mathbb{B}(p_0; \varepsilon)$ where p_0 is known;
ii) $[\![\mathbf{V}_n]\!] = \mathbb{B}(0; \gamma)$, $\mathbf{V}_n \perp\!\!\!\perp \mathbf{X}_k$, and $\mathbf{V}_n \perp\!\!\!\perp \mathbf{\Gamma}_k$ for all $k \in [0 : n]$ and all $n \in \mathbb{Z}_{\geq 0}$;
iii) $[\![\mathbf{X}_n|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_n = a_n]\!]$ is a Borel set with nonempty interior for each $n \in \mathbb{Z}_{\geq 0}$. ▲

Some observations are in order. First, i) implies that we know an initial state estimate with accuracy ε. Second, condition iii) is less clear and requires some explaining: $[\![\mathbf{X}_n|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_n = a_n]\!]$ is the set of possible states at time n after receiving the symbols $(a_0, \ldots, a_n)$. We interpret this set as a quantization region for a quantizer with memory, i.e., the encoder quantizes the state. In the proof of the data-rate lower bound, we perform a k-dimensional volume counting, where $k \in [d]$. For that reason, we need to impose this measurability condition. Recall that all open sets, closed sets, and their countable unions and intersections are Borel sets (see, e.g., Section 1.2 from [14]). For example, half-open parallelepipeds, i.e., sets $\{\sum_{i=1}^{d} \beta_i e_i : \beta_i \in [\underline{a}_i, \overline{a}_i)\}$ with $\underline{a}_i < \overline{a}_i$ for each $i \in [d]$, which are common shapes for quantization regions, are Borel sets. We also remark that building sets that are not Borel requires some effort (see, e.g., Section 1.5 from [14]). Next, we want to rewrite equation (1) using uvs. We do that as follows.

$$\mathbf{X}_{n+1} = A\mathbf{X}_n + \mathbf{U}_n + \mathbf{V}_n \tag{3}$$

and refer to is as the *uncertain model* for the dynamics (1).

Remark 1 (Comparison with the literature)
The idea of representing the set of possible states at a given time is common in the literature of uncertain dynamical systems (see, e.g., [15], [16], [17], [18]) and set-membership, reachability analysis, and set-theoretic estimation (see, e.g., [19], [20]). In the following analysis, we utilize several tools from those fields. Nonetheless, our goal is to study a data-rate theorem for state estimation of systems of the form (1) and not to provide efficient algorithms that solve this problem[1].

[1]We do prove a data-rate upper bound constructively by providing an algorithm, but its purpose is merely theoretical and we do not claim it is efficient.

Having that in mind, we clarify that the purpose of introducing the uncertain variable framework to model uncertainty is to keep the notation concise and be consistent with previous results in nonstochastic information theory [4], [21], [22], [23]. To understand why the uncertain variables might make the notation clearer, recall that in the set-theoretic estimation framework [17], [20], the set of possible states compatible with the first k measurements at time k is often represented as $\hat{\mathcal{X}}_{k|k}$. This set contains the true state x_k and is compatible with a specific sequence past measurements. This notation, however, is insufficient for our analysis, as we need to be able to differentiate the distinct possible sets of possible states $\hat{\mathcal{X}}_{k|k}$ corresponding to the distinct possible encoded measurements. For example, the set of possible states corresponding to the k-tuples of symbols $(a_1, \ldots, a_k) \in \mathcal{A}^k$ might be different from the one corresponding to $(a'_1, \ldots, a'_k) \in \mathcal{A}^k$, where $(a_1, \ldots, a_k) \neq (a'_1, \ldots, a'_k)$. Clearly, we could write $\hat{\mathcal{X}}_{k|k}^{(a_1,\ldots,a_k)}$ and $\hat{\mathcal{X}}_{k|k}^{(a'_1,\ldots,a'_k)}$, respectively, but the uncertain variables framework naturally differentiates between those sets using the natural notion of conditional range. ▲

C. The coder-estimator scheme and the problem statement

Before we mathematically formalize our problem statement, we make a definition.

Definition 2
A *state estimator* is a sequence of uvs $(\hat{\mathbf{X}}_n)_{n\in\mathbb{Z}_{\geq 0}}$ where the elements have range in $\mathbb{R}^d$. Given a state estimator $(\hat{\mathbf{X}}_n)_{n\in\mathbb{Z}_{\geq 0}}$ and a coder $(\mathbf{\Gamma}_n)_{n\in\mathbb{Z}_{\geq 0}}$ with alphabet $\mathcal{A}$, we say that $\mathscr{S} = \big((\hat{\mathbf{X}}_n)_{n\in\mathbb{Z}_{\geq 0}}, (\mathbf{\Gamma}_n)_{n\in\mathbb{Z}_{\geq 0}}\big)$ is a *coder-estimator scheme*. We say that a coder-estimator scheme $\mathscr{S}$ is ε-*accurate* for $(\mathbf{X}_n)_{n\in\mathbb{Z}_{\geq 0}}$ if $[\![\mathbf{X}_n - \hat{\mathbf{X}}_n]\!] \subseteq \mathbb{B}(0;\varepsilon)$ for all $n \in \mathbb{Z}_{\geq 0}$. Finally, we define the coder-estimator scheme's *data-rate* as $R(\mathscr{S}) := \log(\#\mathcal{A})$. ▲

The ε-accuracy conditions corresponds to the set of possible estimation errors at time n after receiving the symbols $a^{0:n-1} \in \mathcal{A}^n$. Thus, ε-accuracy means that the estimation error must be contained in a ball of radius ε for all times. Now, we can finally formally state our problem.

Problem 1
Under Assumption 1. Let $\varepsilon \in (0,\infty)$ be a prescribed estimation accuracy and $\gamma \in (0,\infty)$ be a disturbance level. What conditions we must impose on ε and γ so that an ε-accurate coder-estimator scheme (Def. 2) that operates with a finite data-rate exists for $(\mathbf{X}_n)_{n\in\mathbb{Z}_{\geq 0}}$? Under such conditions, what is the data-rate $R \in [0,\infty)$ so that i) there are no ε-accurate coder-estimator schemes with $R(\mathscr{S}) < R$ and ii) there are schemes with $R(\mathscr{S}) > R$? ▲

We call the quantity R the *minimum data-rate* for solving Problem 1. The goal of next section is to provide a partial answer to the above problem and analyze what conclusions we can draw from our results.

IV. Towards a new data-rate theorem

In this section, we present our main results, which provide a partial answer to the questions posed in Section II and Problem 1. First, we state Theorem 1, which tells us that there exists an ε-accurate coder-estimator scheme when $\varepsilon > \gamma$. We also provide data-rate lower and upper bounds in this scenario. Also, we show that the data-rate diverges when γ approaches ε. Further, when disturbances are not present, we show that the accuracy requirement forces the data-rate to be larger, in most cases, than the intrinsic entropy, which is what we need for stabilization and state reconstruction without noise [7], [24]. Finally, we analyze the results and illustrate them with an example.

The next quantities play a major role in what follows: for $p \in [0,1)$, define

$$k_p := \max\{i \in [d] : \sigma_p(A) > 1-p\} \tag{4}$$

and define

$$H^p(A) := \sum_{i=1}^{k_r} \log(\sigma_i(A)) = \sum_{\sigma(A)>1-p} \log(\sigma(A)) \tag{5}$$

where $\sigma(A)$ represents a singular value of A. We call $H^0(A)$ the *first intrinsic entropy* of A. Informally, the first intrinsic entropy is related to how much the uncertainty on the state of system (3) grows between times n and $n+1$. We note that $H^p(A)$ is not the same as the usual intrinsic entropy, as we discuss in the next section. Also, define the *accuracy-to-noise ratio* as $r := \gamma/\varepsilon$. Finally, define the auxiliary quantity $J_p := \max\{i \in [d] : \sigma_p(A) > \frac{1-p}{\sqrt{d}}\}$ for $p \in [0,1)$. With these additional notations, we state our main theorem.

Theorem 1
Let $\varepsilon \in (0,\infty)$ be an accuracy and $\gamma \in [0,\varepsilon)$ be the radius of the disturbance set. Under Assumption 1, there exists an ε-accurate coder-estimator scheme $\mathscr{S}$ (Def. 2) that operates with a finite data-rate that is upper-bounded by

$$\begin{aligned} R(\mathscr{S}) \leq H^r(A) + k_r \log\Big(\frac{1}{1-r}\Big) + \log(1.5\omega_{k_r} k_r^{k_r/2}) \\ + O(\max\{k_r, J_r - k_r\}\log(d)) \end{aligned} \tag{6}$$

Conversely, any coder-estimator scheme $\mathscr{S}$ that is ε-accurate for $(\mathbf{X}_n)_{n\in\mathbb{Z}_{\geq 0}}$ has its data-rate lower-bounded by

$$H^r(A) + k_r \log\Big(\frac{1}{1-r}\Big) \leq R(\mathscr{S}). \tag{7}$$

Finally, if $r \geq 1$, there exists no state estimator that solves the problem with finite data-rate. ▲

The expression for the term $O(\max\{k_r, J_r - k_r\}\log(d)) = k_r \log(3/2) + \frac{k_r}{2}\log(d/k_r) + (J_r - k_r)\log(2\lceil\frac{\sqrt{d}+1}{2}\rceil)$. We remark that $\log(\omega_{k_r} k_r^{k/2}) \leq k\log(\sqrt{2e\pi})$ for all $k \in \mathbb{Z}_{>0}$. The proof of this theorem is in Appendix VII. The idea of the proof of the lower bound is to perform a k_r-dimensional volume counting inside the ambient space $\mathbb{R}^d$ using concepts from multi-linear algebra, such as compound matrices (see, e.g., [25]) and geometric analysis, via the area formula (see, e.g., Theorem 5.1.1 in

Section 1.1 of Chapter 5 from [26]). The proof of the upper bound involves designing an ε-accurate code-estimator scheme.

Remark 2 (Finiteness of the gap)
Note that, when $k_r = J_r = d$, inequality (6) becomes

$$R(\mathscr{S}) \leq H^r(A) + k_r \log\big(\frac{1}{1-r}\big) + k_r \log(1.5\sqrt{2e\pi}). \quad (8)$$

Comparing this to (7), when $k_r = d$, we see that the the gap between the bounds in Theorem 1 is smaller than or equal to $d\log(1.5\sqrt{2e\pi}) \approx 2.65d$ bits. Thus, the number of extra bits per dimension is constant in this case. Note that $k_r = J_r = d$ happens, for example, when the system is expansive, i.e., $\sigma_d > 1$, or when the accuracy-to-disturbance ratio r is "large". ▲

A. Comment on the transmission times

When disturbances are not present, the results in [7] show us that there always exists an ε-accurate coder-estimator scheme that operates with an average data-rate as close as desired to the minimum possible by taking the transmission periods to be multiples of a number ℓ chosen to be large. Informally, waiting to transmit reduces the data-rate required to address our problem. Thus, a natural question is why, as is the case in the present document, we should consider small values of ℓ. The next example shows us that, unlike the case without perturbations, the disturbance might force us to transmit data "fast".

Example 2
Consider the system $\mathbf{X}_{n+1} = \mathbf{X}_n + \mathbf{V}_n$ where $[\![\mathbf{V}_n]\!] = \mathbb{B}(0; 0.75\varepsilon)$ for some $\varepsilon > 0$ and assume that $[\![\mathbf{X}_0]\!] = \{p_0\}$, i.e., we know the initial state. If $\ell = 3$, we claim that no ε-accurate estimator exists. Indeed, using the facts that $[\![\mathbf{X}_2]\!] = \mathbb{B}(0; 1.5\varepsilon)$ and that$[\![\hat{\mathbf{X}}_2]\!]$ is a singleton by Assumption 1, we conclude that $[\![\mathbf{X}_2 - \hat{\mathbf{X}}_2 | \mathbf{\Gamma}_0 = a_0]\!] = \mathbb{B}(\bar{x}; 1.5\varepsilon)$ for some $\bar{x} \in \mathbb{R}^d$. However, as we saw in the statement of Theorem 1, this problem has a solution for $\ell = 1$. ▲

V. Analysis and Comparisons

In this section, we analyze and compare our results with the literature and explain how our result differs. Also, we explain the coordinate-dependence of our results an illustrate this fact by revisiting Example 1.

A. Comparison with the literature

Our result also shares some resemblance with the works considering stabilization of linear systems with disturbances [24], [27], [28]. To see why, we briefly recall that, using the uncertain variables notation, those results give a bound of the form $\lim_{n\to\infty} \sup_{m\in[0:n]}\{|x| : x \in [\![\mathbf{X}_m]\!]\} \geq \frac{\gamma}{1-2^{-(R-H(A))/d}}$, where $H(A) := \sum_{\lambda \text{ eigenvalue of } A} \max\{\log(|\lambda|), 0\}$ is the *intrinsic entropy* of A, R is the controller data-rate, i.e., the logarithm of the number symbols in the alphabet, and d is the dimension of the unstable subspace of A. Imposing the constraint $\lim_{n\to\infty} \sup_{m\in[0:n]}\{|x| : x \in [\![\mathbf{X}_m]\!]\} \leq \varepsilon$, we conclude that $R \geq H(A) + d\log(1/(1-r))$ after some algebraic steps. Note that the bound presented in that work is only asymptotic, i.e., it does not impose finite time constraints on the norm of $\mathbf{X}_n$. However, comparing $R \geq H(A) + d\log(1/(1-r))$ with (7), when $k_r = d$, we conclude that (7) is the same.[2] Remarkably, when A is a normal matrix and noise is not considered, we have that $H(A) = H^0(A)$ since normal matrices have the property that the absolute values of their eigenvalues equal their singular value (see, e.g. [29]). We dedicate the next subsection to discuss what is behind this latter observation.

B. Coordinate-dependence

One important feature of our result that differs considerably from the usual data-rate theorems is its clear coordinate-dependence. The usual data-rate theorem involves eigenvalues, which are invariant under linear coordinate changes, whereas our results involve singular values, which vary unless the linear coordinate change is also orthogonal. The reason for this discrepancy is due to three facts: the accuracy requirement, the presence of disturbances, and the fact that transients are coordinate-dependent. First, in most practical applications, we require accuracy in a specific coordinate system, for example, maintaining a system's position and velocity, quantities with physical meaning, within a certain range. Second, the presence of disturbances together with the accuracy requirement constrains the maximum time to transmit. This fact indicates that we must understand the finite-time behavior of the dynamics, i.e., we must study the transients. Finally, to see why transients are coordinate-dependent, consider the matrices $A_1 = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.5 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0.25 & 10 \\ 0 & 0.5 \end{pmatrix}$ and the dynamics

$$\mathbf{X}_{k+1,i} = A_i \mathbf{X}_{k,i} \quad (9)$$

for $i \in [2]$ and $k \in \mathbb{Z}_{\geq 0}$. The matrices A_1 and A_2 are similar, i.e., there exists a linear coordinate change that transforms one in the other, since they have the same eigenvalues 0.25 and 0.5. However, the finite-time behavior of (9) differs considerably for distinct $i \in [2]$. Indeed, for $i = 1$, we note that (9) is contractive since $\mathbf{X}_{1,1} = |A_1\mathbf{X}_{0,1}| \leq |\mathbf{X}_{0,1}|$ for all $\mathbf{X}_{0,1} \in \mathbb{R}^2$. For $i = 2$, however, note that $|\mathbf{X}_{1,2}| = |A_2\mathbf{X}_{0,2}| = |\sum_{i=1}^2 \langle \mathbf{X}_{0,i}, e_i\rangle A_2 e_i| = |(0.25\langle \mathbf{X}_{0,1}, e_1\rangle + 10\langle \mathbf{X}_{0,2}, e_2\rangle)e_1 + 0.5\langle \mathbf{X}_{0,2}, e_2\rangle e_2|$ since $\mathbf{X}_{0,2} = \sum_{i=1}^2 \langle \mathbf{X}_{0,i}, e_i\rangle e_i$, where $\{e_1, e_2\} \subset \mathbb{R}^2$ is the canonical basis. For example, if $\mathbf{X}_{0,2} = e_2$, we have that $|\mathbf{X}_{1,2}| = \sqrt{100.25}|e_2| \sim 10.1 > |\mathbf{X}_{0,2}|$. Informally, this result tell us that measurements in the first coordinate system require fewer symbols to be encoded whereas we need more data in the second to keep the same accuracy if we transmit at each sampling time. We remark, nonetheless, that if we are able to wait to transmit, as in the case where disturbances are not present, this feature disappears. Indeed, recall that

[2] In the conference draft there was a mistake. For an in-depth discussion of when the lower bound (7) is tighter than the previously reported bound, which includes an example, see Appendix VIII.

Gelfand's formula give us that (see, e.g., Corollary 5.6.14 from [30]) for each $\delta \in (0, \infty)$, there exists $\ell \in \mathbb{Z}_{\geq 0}$ so that $|x_{\ell,2}| = |A_2^\ell x_{0,2}| \leq (0.5 + \delta)^\ell |x_{0,2}|$, which is a contraction for $\delta < 0.5$. This explains why data-rate theorems for undisturbed systems are often related to topological entropy, an asymptotic coordinate-invariant property of a system.

This previous discussion also indicates that there might be preferential coordinate systems in terms of data-rate. For example, if a matrix A is diagonalizable over $\mathbb{C}$, there exists an invertible matrix $S \in \mathbb{R}^{d \times d}$ so that $S^{-1}AS$ is normal, e.g., the real Jordan normal form for such matrices is a normal matrix[3]. Normal matrices have the property that the absolute values of their eigenvalues equal their singular value (see, e.g. [29]). Therefore, in this case, we have that $H = H_1^0(A)$, which coincides with the usual data-rate theorem. Observe, nevertheless, that this coordinate system might not have any physical meaning.

We illustrate this analysis by revisiting Example 1 and explaining the claims we made at that point.

Example 3 (Revisiting Example 1)
Consider the matrix A_1 from Example 1 and note that the eigenvalues of A_1 and A_2 are 0.9 and 0.5, which makes their intrinsic entropy equal to zero, i.e., $H = 0$. However, A_1's singular values are $\sigma_1 \approx 2.24$ and $\sigma_2 \approx 0.2$. Hence, $H_1(A_1) \approx 1.164$ bits/sample. On the other hand, matrix A_2 is a normal matrix, which implies that its singular values equal the absolute value of its eigenvalues. Thus, $H_1(A_2) = 0$. This shows that the minimum data-rate is coordinate dependent. In Figure 1, we plot the upper and lower bounds for the the minimum data-rate when $A = A_1$, $\ell = 1$, and $\varepsilon_0 = \varepsilon$ as a function of $r = \gamma/\varepsilon$. We can see that the lower bound is an increasing function of r. Further, the lower bound is close to $H_1(A_1)$ when $r \approx 0$ and both bounds grow when $r \approx 1$ as expected. The latter fact happens because we need the encoder to use finer quantization regions to compensate the presence of noise and guarantee the estimation error bound. Finally, we note that the jumps we observe in the upper bound are due to the term multiplied by $J_r - k_r$, which stars as zero, becomes one, then equals zero again.

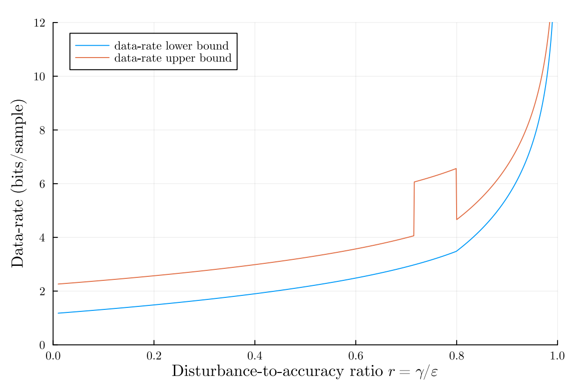

Fig. 1. Upper and lower bounds for the minimum data-rate as a function of γ/ε when $A = A_1$

[3] The word "normal" in the name of the Jordan normal form is unrelated to the property of the matrix being normal.

Finally, in future works, we hope to extend this work for input-output models and derive bounds for this scenario. These results can be compared with those obtained in [31], [32]. Also, we believe that analysis might extend other results from nonstochastic control theory from [4] and related to [33], [34].

VI. Conclusion

In this work, we stated the problem of reconstructing the state of a perturbed linear system from encoded measurements with a prescribed accuracy when the information is transmitted using a finite data-rate. We explained how to model the quantities involved in this problem using uncertain variables and how we can formalize tasks of this kind. Next, we explained how the minimum data-rate varies with the system dynamics, estimation accuracy, and the size of the state noise. We also showed that noise might force us to transmit data faster, unlike when no disturbances exist.

In the future, our goal will be to extend this analysis to consider transmission periods different from one. Additionally, we aim to extend this work to input-output models and derive bounds for this scenario. These results can be compared with those obtained in [31], [32]. Furthermore, we believe that this analysis might extend other results from nonstochastic control theory, as presented in [4] and related to [33], [34].

Appendix

VII. Proof of Theorem 1

First, we denote by $a^{0:m}; a_{m+1}$ the concatenation of the (m)-tuple $a^{0:m-1} = (a_0, \dots, a_{m-1})$ and a_m, i.e., $a^{0:m-1}; a_m := (a_0, \dots, a_{m-1}, a_m)$. Also, for any given uv $\mathbf{Y}$, define $[\![\mathbf{Y}|a^{0:m}]\!] := [\![\mathbf{Y}|\mathbf{\Gamma}_0 = a_0, \dots, \mathbf{\Gamma}_m = a_m]\!]$ for each $m \in \mathbb{Z}_{\geq 0}$. Further, define $[\![\mathbf{Y}|a^{0:-1}]\!] := [\![\mathbf{Y}]\!]$. Finally, we drop the subscript r in k_r and the superscript r in H^r since r is fixed. Finally, let $A = V \Sigma W'$ be a singular value decomposition for A, i.e, $V \in \mathbb{R}^{d \times d}$ and $W \in \mathbb{R}^{d \times d}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{d \times d}$ is a diagonal matrix with its elements being the singular values of A. Recall that since $\mathbf{U}_n$ and $\hat{\mathbf{X}}_{n+1}$ are functions of $\mathbf{\Gamma}_0, \dots, \mathbf{\Gamma}_n$, we have that $[\![\mathbf{U}_n|a^{0:n}]\!]$ and $[\![\hat{\mathbf{X}}_{n+1}|a^{0:n}]\!]$ are singletons. This fact implies that $[\![B\mathbf{U}_n|a^{0:n}]\!]$ is also a singleton; we denote by $b_{a^{0:n}} \in \mathbb{R}^d$ the element contained in the latter set. Also, denote by $\hat{x}_{a^{0:n}} \in \mathbb{R}^d$ the element contained in the singleton $[\![\hat{\mathbf{X}}_{n+1}|a^{0:n}]\!]$. Finally, define $c_{a^{0:n}} := b_{a^{0:n}} - \hat{x}_{a^{0:n}}$. Now, we state some auxiliary lemmas proven in Appendix IX.

The first lemma is similar to classical results in the theory of set-membership control and estimation (also in the uncertain systems literature) (see, e.g., [20]). We state and prove this result here for uncertain variables for completeness and convenience.

Lemma 1
Under Assumption 1, we have that

$$[\![\mathbf{X}_{n+1} - \hat{\mathbf{X}}_{n+1}|a^{0:n}]\!] = A[\![\mathbf{X}_n|a^{0:n}]\!] \oplus \mathbb{B}(0;\gamma) \oplus \{c_{a^{0:n}}\}. \tag{10}$$

for every $n \in \mathbb{Z}_{\geq 0}$. ▲

Informally, this lemma gives an expression for the estimation error and tells us that the system generates uncertainty via the propagation of the initial state uncertainty and the noise, which are the two non-singleton sets on the right-hand side of (10).

Lemma 2
Let $A \in \mathbb{R}^{d\times d}$ be an invertible matrix with singular value decomposition $A = V\Sigma W'$. Also, let $\mathcal{S} \subseteq \mathbb{R}^d$ be contained in the k-dimensional subspace $\mathcal{W} := \text{span}\{We_1, \ldots, We_k\}$ and denote by $\sigma_i(A)$ the i-th largest singular value of A. Then, we have that $\mathcal{H}^k(A\mathcal{S}) = \left(\prod_{i=1}^k \sigma_i(A)\right) \mathcal{H}^k(\mathcal{S})$. ▲

We remark that the subspace $\mathcal{W}$ is the subspace corresponding to the k first singular vectors, i.e., the singular vectors that correspond to the k largest singular values.

Lemma 3
Let $k \in [d]$, $\mathcal{S} \subseteq \mathbb{R}^d$ be a k-dimensional subspace, and $x \in \mathbb{R}^d$ be a point. Also, let $b \in \mathbb{R}^d$ be a center of a ball and $r \in (0, \infty)$ its radius. Then,

$$\mathcal{H}^k(\mathbb{B}(b; r) \cap (\mathcal{S} \oplus \{x\}) \leq r^k\omega_k.$$

▲

We split the proof in two parts: the poof of the lower bound (7) and of the upper bound (6).

A. The lower bound

Proof of (7) **in Theorem 1**: the main idea of the proof is to perform a lower-dimensional volume counting argument, similar to the proofs of lower bounds for the metric entropy in [35]. To keep the ideas clear, we split the proof into four parts. In the first part, prove a inclusion formula, which must hold for any estimator that satisfies the requirements in the theorem statement. Informally, we build an overapproximation for the reachable set for the estimation error, which we use to upper bound $A[\![\mathbf{X}_n|a^{0:n}]\!]$. In the second part, we use that inclusion to obtain a k-dimensional volume lower bound for a section[4] of the set of possible states given the past measurements. Next, we use the previous k-dimensional volume lower bound to perform volume counting argument to obtain a lower bound for $\#\mathcal{A}$. Finally, we use the previous bound to prove the data-rate lower bound.

The inclusion: define $\tilde{\varepsilon} := \varepsilon - \gamma$. Our goal is to prove the inclusion

$$\mathbb{B}(-c_{a^{0:n}}; \tilde{\varepsilon}) \supseteq A[\![\mathbf{X}_n|a^{0:n}]\!] \tag{11}$$

for all $a^{0:n} \in \mathcal{A}^{n+1}$ and all $n \in \mathbb{Z}_{\geq 0}$. From this point onward, we denote $c := c_{a^{0:n}}$ to keep the notation light. First, Definition 2 gives us that $[\![\mathbf{X}_{n+1} - \hat{\mathbf{X}}_{n+1}|a^{0:n}]\!] \subseteq \mathbb{B}(0; \varepsilon)$. Note that $[\![\hat{\mathbf{X}}_{n+1}|a^{0:n}]\!]$ is a singleton by Definition 2. Second, Lemma 1 gives us that

$$[\![\mathbf{X}_{n+1} - \hat{\mathbf{X}}_{n+1}|a^{0:n}]\!] = A[\![\mathbf{X}_n|a^{0:n}]\!] \oplus (\mathbb{B}(0; \gamma)) \oplus \{c\}$$

Combining the two previous inclusions we get that

$$A[\![\mathbf{X}_n|a^{0:n}]\!] \oplus \mathbb{B}(0; \gamma) \oplus \{c\} \subseteq \mathbb{B}(0; \varepsilon).$$

Finally, taking the geometric sum with $\{-c\}$ and the geometric difference[5] with $\mathbb{B}(0; \gamma)$ on both sides, we arrive at $\mathbb{B}(-c; \varepsilon) \ominus \mathbb{B}(0; \gamma) \supseteq A[\![\mathbf{X}_n|a^{0:n}]\!]$, where we have used the fact that $\mathcal{X} \oplus \mathcal{Y} \ominus \mathcal{Y} \supseteq \mathcal{X}$ for any sets $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{Y} \subseteq \mathbb{R}^d$ (see, e.g., extensive property in equation II-23 [36]).
Thus, we conclude that $\mathbb{B}(-c; \tilde{\varepsilon}) = \mathbb{B}(-c; \varepsilon) \ominus \mathbb{B}(0; \gamma) \supseteq A[\![\mathbf{X}_n|a^{0:n}]\!]$, where the equality follows from the fact that $\mathbb{B}(0; r_1) \ominus \mathbb{B}(0; r_2) = \mathbb{B}(0, r_1 - r_2)$.

The intersection: define $\mathcal{W} := \text{span}\{We_1, \ldots, We_k\}$, where $\{e_1, \ldots, e_d\}$ is the canonical basis for $\mathbb{R}^d$. Let $x \in [\![\mathbf{X}_n|a^{0:n-1}]\!]$ be an interior point, which exists by Assumption 1. Define $\mathcal{Z} := A^{-1}\mathcal{W} \oplus \{A^{-1}x\}$, where we leave the x-dependency implicit. Note that, by taking the intersection with $A\mathcal{Z}$ on both sides of (11), we arrive at the inclusion

$$\mathbb{B}(-c; \tilde{\varepsilon}) \cap A\mathcal{Z} \supseteq A([\![\mathbf{X}_n|a^{0:n}]\!] \cap \mathcal{Z}), \tag{12}$$

where we used the fact that $A\mathcal{X} \cap A\mathcal{Y} = A(\mathcal{X} \cap \mathcal{Y})$ for any two subsets $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbb{R}^d$. We, note that $A\mathcal{Z} = \mathcal{W} \oplus \{x\}$ is an affine space and, consequently,[6] the sets $[\![\mathbf{X}_n|a^{0:n}]\!] \cap \mathcal{Z}$ and $\mathbb{B}(-c; \tilde{\varepsilon}) \cap A\mathcal{Z}$ are k-dimensional sections of the sets $[\![\mathbf{X}_n|a^{0:n}]\!]$ and $\mathbb{B}(-c; \tilde{\varepsilon})$, respectively.

k-dimensional volume bounding: in this part of the proof, we find a lower bound for the k-dimensional Hausdorff measure[7], of the section $A([\![\mathbf{X}_n|a^{0:n}]\!] \cap \mathcal{Z})$. Explicitly, our goal is to prove that

$$\omega_k\tilde{\varepsilon}^k \geq \mathcal{H}^k(A([\![\mathbf{X}_n|a^{0:n}]\!] \cap \mathcal{Z})). \tag{13}$$

First, Lemma 3 tells us that $\mathcal{H}^k(\mathbb{B}(-c; \tilde{\varepsilon}) \cap A\mathcal{Z}) \leq \tilde{\varepsilon}^k\omega_k$. Combining this inequality with (12), we arrive at (13) for each $a^{0:n} \in \mathcal{A}^{n+1}$ since $\mathcal{H}^k(\mathcal{X}) \geq \mathcal{H}^k(\mathcal{Y})$ for any $\mathcal{X} \supseteq \mathcal{Y}$ (see, e.g., Section 11.2 from [14]).

The counting argument: write $[\![\mathbf{X}_n|a^{0:n}]\!] = [\![\mathbf{X}_n|a^{0:n-1}; a_n]\!]$. Now, we prove that

$$\#\mathcal{A} \geq \left(\prod_{i=1}^k \sigma_i(A)\right) \frac{\mathcal{H}^k([\![\mathbf{X}_n|a^{0:n-1}]\!] \cap \mathcal{Z})}{\omega_k\tilde{\varepsilon}^k}. \tag{14}$$

[4] A k-dimensional section is the intersection of a set with a k-dimensional affine subspace.

[5] Also known as the Pontryagin difference [19] or morphological erosion [36].

[6] Recall that A is invertible, which implies that $A^{-1}\mathcal{W}$ is a k-dimensional subspace.

[7] This is a notion of k-dimensional volume of a subset of $\mathbb{R}^d$. See, e.g., Chapter 5 from [37].

First, we sum inequality (13) over all $a_n \in \mathcal{A}$ to conclude that

$$\begin{aligned}(\#\mathcal{A})\omega_k\tilde{\varepsilon}^k &\geq \sum_{a_n\in\mathcal{A}} \mathcal{H}^k(A([\![\mathbf{X}_n|a^{0:n-1};a_n]\!]\cap\mathcal{Z}))\\ &= \mathcal{H}^k(\cup_{a_n\in\mathcal{A}}(A[\![\mathbf{X}_n|a^{0:n-1};a_n]\!]\cap\mathcal{Z})\\ &= \mathcal{H}^k(A(\cup_{a_n\in\mathcal{A}}[\![\mathbf{X}_n|a^{0:n-1};a_n]\!]\cap\mathcal{Z})\\ &= \mathcal{H}^k(A([\![\mathbf{X}_n|a^{0:n-1}]\!]\cap\mathcal{Z}))\\ &= \left(\prod_{i=1}^k \sigma_i(A)\right)\mathcal{H}^k([\![\mathbf{X}_n|a^{0:n-1}]\!]\cap\mathcal{Z})\end{aligned}$$

where the first equality follows from the fact that the sets $[\![\mathbf{X}_n|a^{0:n};a]\!]$ are pairwise disjoint for distinct $a\in\mathcal{A}$, the second from the fact that the union distributes over intersection, the third from the fact that the sets $[\![\mathbf{X}_n|a^{0:n-1};a_n]\!]$ partition the set $[\![\mathbf{X}_n|a^{n-1}]\!]$ (see, e.g., [4]), and the last equality follows from Lemma 2. We remark that $\mathcal{H}^k([\![\mathbf{X}_n|a^{0:n-1}]\!]\cap\mathcal{Z})>0$ since $A^{-1}x\in\mathcal{Z}$ is an interior point of $[\![\mathbf{X}_n|a^{0:n-1}]\!]$, implying that there is a k-dimensional ball contained in the intersection $[\![\mathbf{X}_n|a^{0:n-1}]\!]\cap\mathcal{Z}$.

The data-rate lower bound: rearranging the terms and taking the logarithm on both sides, we can write that

$$\log(\#\mathcal{A}) \geq H^r + \log\left(\frac{\mathcal{H}^k([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]\cap\mathcal{Z})}{\omega_k\tilde{\varepsilon}^k}\right)$$

since

$$H^r = \sum_{\sigma(A)>1-r}\log(\sigma(A)) = \sum_{i=1}^k \log(\sigma_i(A))$$

and the fact that[8] $k=1-r$.

Finally, we can take the supremum over $n\in\mathbb{Z}_{\geq 0}$ and conclude that

$$\begin{aligned}\log(\#\mathcal{A}) &\geq \sup_{n\in\mathbb{Z}_{\geq 0}}\left\{H^r + \log\left(\frac{\mathcal{H}^k([\![\mathbf{X}_n|a^{0:n-1}]\!]\cap\mathcal{Z})}{\omega_k\tilde{\varepsilon}^k}\right)\right\}\\ &\geq H^r + k\log(\varepsilon/\tilde{\varepsilon})\\ &= H^r + k\log(1/(1-r)),\end{aligned}$$

where the second inequality follows from the facts that for $n=0$ we have that $[\![\mathbf{X}_0|a^{0:-1}]\!]=[\![\mathbf{X}_0]\!]=\mathbb{B}(p_0;\varepsilon)$, that $x\in[\![\mathbf{X}_0|a^{0:-1}]\!]$ is an arbitrary interior point of $[\![\mathbf{X}_0]\!]$, and that we can pick the point $x=p_0$. ■

B. The upper bound

Proof of (6) **in Theorem 1**: we omit the subscript r in k_r to avoid cluttering the notation. Also, let $A=V\Sigma W'$ be a singular value decomposition (SVD) for A, where $\Sigma=\text{diag}(\sigma_1,\ldots,\sigma_d)$. To prove the upper bound, we construct a lattice vector quantizer (see, e.g., Section 10.5 from [13]). Consider the $d\times d$ matrix given by

$$T := 2\frac{(\varepsilon-\gamma)}{\sqrt{d}}W\Sigma^{-1}$$

and the lattice

$$\Lambda := \left\{\sum_{i=1}^d q_i T e_i : (q_1,\ldots,q_d)\in\mathbb{Z}^d\right\}, \tag{15}$$

where $\{e_1,\ldots,e_d\}$ is the canonical basis for $\mathbb{R}^d$. Let $\mathcal{P}\subset\mathbb{R}^d$ be

$$\mathcal{P} := \left\{\sum_{i=1}^d \beta_i W e_i : -\frac{(\varepsilon-\gamma)}{\sigma_i\sqrt{d}} \leq \beta_i < \frac{(\varepsilon-\gamma)}{\sigma_i\sqrt{d}} \text{ for } i\in[d]\right\}. \tag{16}$$

We also define the partition

$$\mathscr{P} := \{\mathcal{P}\oplus\{\lambda\} : \lambda\in\Lambda\}. \tag{17}$$

Finally, we define the alphabet[9]

$$\mathcal{A} := \{\lambda\in\Lambda : (\mathcal{P}\oplus\{\lambda\})\cap\mathbb{B}(0;\varepsilon)\neq\emptyset\}, \tag{18}$$

i.e., the lattice points that correspond to cells in $\mathscr{P}$ that intersect the ball of radius ε.

Accuracy: we claim that

$$A\mathcal{P}\subseteq\mathbb{B}(0;\varepsilon-\gamma).$$

Indeed, if $x\in\mathcal{P}$, we have that $x=\sum_{i=1}^d\beta_i(x)We_i$ with $|\beta_i(x)|\leq\frac{\varepsilon-\gamma}{\sqrt{d}\sigma_i}$ for each $i\in[d]$ by the definition of the lattice. Now, we have that

$$\begin{aligned}\left|A\sum_{i=1}^d\beta_i(x)We_i\right|^2 &= \left|\sum_{i=1}^d\beta_i(x)V\Sigma W'We_i\right|^2\\ &= \left|\sum_{i=1}^d\beta_i(x)V\Sigma e_i\right|^2\\ &= \left|\sum_{i=1}^d\beta_i(x)\sigma_iVe_i\right|^2\\ &= \sum_{i=1}^d\beta_i^2(x)\sigma_i^2\\ &\leq(\varepsilon-\gamma)^2.\end{aligned}$$

where the first equality follows from the SVD of A, the second follows from the fact that W is an orthogonal matrix, the third follows from the fact that $\Sigma=\text{diag}(\sigma_1,\ldots,\sigma_d)$, the fourth follows from the Pythagorean theorem and the fact that $\{Ve_1,\ldots,Ve_d\}$ is an orthonormal basis for $\mathbb{R}^d$, and the inequality follows from the restriction on $\beta_i(x)$ for each $i\in[d]$.

The coder: denote by $u_{a^{0:n}} := [\![\mathbf{U}_n|\mathbf{\Gamma}_0=a_0,\ldots,\mathbf{\Gamma}_n=a_n]\!]$. Define the coder and the estimator as follows: for each realization $\omega\in\Omega$, define

$$\begin{aligned}&\hat{\mathbf{X}}_0(\omega) := p_0\\ &\mathbf{\Gamma}_0(\omega) := \lambda \text{ so that } \mathbf{X}_0(\omega)\in\mathcal{P}\oplus\{\lambda\}\oplus\{p_0\}.\end{aligned}$$

[8]Remember that we omitted the dependency in r on k to avoid cluttering the notation.

[9]This is known as codebook in the vector quantization literature.

For each $\omega \in \Omega$, let $a_0 \coloneqq \mathbf{\Gamma}_0(\omega)$ and define

$$\begin{aligned}\hat{\mathbf{X}}_1(\omega) &\coloneqq A(p_0 + a_0) + Bu_{a_0}\\ p_{a_0} &\coloneqq \hat{\mathbf{X}}_1(\omega)\\ \mathbf{\Gamma}_1(\omega) &\coloneqq \lambda \text{ so that } \mathbf{X}_0(\omega) \in \mathcal{P} \oplus \{\lambda\} \oplus \{p_{a_0}\}.\end{aligned}$$

For each realization $\omega \in \Omega$ and each time $n \in \mathbb{Z}_{>1}$, let $a^{0:n-1} \coloneqq (\mathbf{\Gamma}_0(\omega), \ldots, \mathbf{\Gamma}_{n-1}(\omega))$. Now, recursively define the estimator and coder as

$$\begin{aligned}\hat{\mathbf{X}}_n(\omega) &\coloneqq A(p_{n-1} + a^{0:n-1}) + Bu_{a^{0:n-1}}\\ p_{a^{0:n-1}} &\coloneqq \hat{\mathbf{X}}_n(\omega)\\ \mathbf{\Gamma}_n(\omega) &\coloneqq \lambda \text{ so that } \mathbf{X}_n(\omega) \in \mathcal{P} \oplus \{\lambda\} \oplus \{p_{a^{0:n-1}}\}.\end{aligned}$$

First, we prove that $[\![\mathbf{X}_n - \hat{\mathbf{X}}_n | a^{0:n}]\!] \subseteq \mathbb{B}(0;\varepsilon)$ and that $[\![\mathbf{\Gamma}_n]\!] \subseteq \mathcal{A}$ for each $n \in \mathbb{Z}_{\geq 0}$.

Step $n = 0$: we have that $[\![\mathbf{X}_0 - \hat{\mathbf{X}}_0]\!] = \mathbb{B}(0;\varepsilon)$, which, together with (18), leads us to the conclusion that $[\![\mathbf{\Gamma}_0]\!] = \mathcal{A}$. For each $a_0 \in \mathcal{A}$, we have that

$$\begin{aligned}&[\![\mathbf{X}_1 - \hat{\mathbf{X}}_1 | a_0]\!] =\\ &= A[\![\mathbf{X}_0|a_0]\!] \oplus \{-A(p_0 + a_0) - Bu_{a_0}\} \oplus\\ &\quad \oplus \{Bu_{a_0}\} \oplus \mathbb{B}(0;\gamma)\\ &= A(\mathcal{P} \oplus \{a_0\} \oplus \{p_0\}) \oplus \{-A(p_0 + a_0) - Bu_{a_0}\} \oplus\\ &\quad \oplus \{Bu_{a_0}\} \oplus \mathbb{B}(0;\gamma)\\ &= A\mathcal{P} \oplus \mathbb{B}(0;\gamma)\\ &\subseteq \mathbb{B}(0;\varepsilon-\gamma) \oplus \mathbb{B}(0;\gamma)\\ &\subseteq \mathbb{B}(0;\varepsilon).\end{aligned}$$

Step $n = m + 1$: our induction hypothesis is that, for each $a^{0:m-1} \in \mathcal{A}^{m-1}$, we have $[\![\mathbf{X}_m - \hat{\mathbf{X}}_m | a^{0:m-1}]\!] \subseteq \mathbb{B}(0;\varepsilon)$. For each $a_m \in \mathcal{A}$, we have that

$$\begin{aligned}&[\![\mathbf{X}_{m+1} - \hat{\mathbf{X}}_{m+1} | a^{0:m}]\!]\\ &= A[\![\mathbf{X}_m | a^{0:m}]\!] \oplus \{-A(p_{a^{0:m}} a_m) - Bu_{a^{0:m}}\} \oplus\\ &\quad \oplus \{Bu_{a^{0:m}}\} \oplus \mathbb{B}(0;\gamma)\\ &= A(\mathcal{P} \oplus \{a_m\} \oplus \{p_{a^{0:m}}\}) \oplus\\ &\quad \oplus \{-A(p_{a^{0:m}} + a_m) - Bu_{a^{0:m}}\} \oplus \{Bu_{a^{0:m}}\} \oplus \mathbb{B}(0;\gamma)\\ &= A\mathcal{P} \oplus \mathbb{B}(0;\gamma)\\ &\subseteq \mathbb{B}(0;\varepsilon-\gamma) \oplus \mathbb{B}(0;\gamma)\\ &= \mathbb{B}(0;\varepsilon).\end{aligned}$$

Upper bound for $\#\mathcal{A}$: for each $\bar{\lambda} \in \mathcal{A} \setminus \mathbb{B}(0;\varepsilon)$, there exists $y \in \mathcal{P}$, that depends on $\bar{\lambda}$, so that $|\bar{\lambda} + y| \leq \varepsilon$. Define $\bar{q}_i \in \mathbb{Z}$ via $\bar{\lambda} = \sum_{i=1}^d 2\frac{\bar{q}_i(\varepsilon-\gamma)}{\sqrt{d}\sigma_i} We_i$ for each $i \in [d]$ and recall that $y = \sum_{i=1}^d \beta_i(y) We_i$ with $|\beta_i(y)| \leq \frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_i}$. Next, for each $j \in [d]$ we can write that

$$\varepsilon \geq |\bar{\lambda} + y| \geq |(We_j)'(\bar{\lambda} + y)| = \left|2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j} + \beta_j(y)\right|$$

where the first inequality follows from the discussion above and the second follows from the Cauchy-Schwarz inequality since W is an orthogonal matrix. If $q_j > 0$, we have that

$$\begin{aligned}\left|2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j} + \beta_j(y)\right| &= 2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j} + \beta_j(y)\\ &\geq (2q_j - 1)\frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_j}\end{aligned}$$

and if $q_j < 0$, we have that

$$\begin{aligned}\left|2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j} + \beta_j(y)\right| &= -2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j} - \beta_j(y)\\ &\geq -(2q_j + 1)\frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_j}\end{aligned}$$

since $|\beta_i(y)| \leq \frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_j}$. With this, we conclude that

$$q_j \leq \frac{\varepsilon}{2(\varepsilon-\gamma)}\sqrt{d}\sigma_j + \frac{1}{2} = (1-r)^{-1}\frac{\sqrt{d}\sigma_j}{2} + \frac{1}{2}$$

if $q_j > 0$ and

$$q_j \leq -\frac{\varepsilon}{2(\varepsilon-\gamma)}\sqrt{d}\sigma_j - \frac{1}{2} = -(1-r)^{-1}\frac{\sqrt{d}\sigma_j}{2} - \frac{1}{2}$$

if $q_j < 0$. From this we conclude that

$$|q_j| \leq (1-r)^{-1}\frac{\sqrt{d}\sigma_j}{2} + \frac{1}{2}.$$

For each $j \in [J+1:d]$, we have that $\sigma_j < \frac{(1-r)}{\sqrt{d}}$, which implies that

$$|q_j| < 1$$

which implies that $q_j = 0$. For each $j \in [k+1:J]$, we have that $\sigma_j \leq 1 - r$, which implies that

$$|q_j| \leq \frac{\sqrt{d}}{2} + \frac{1}{2} = \frac{\sqrt{d}+1}{2}$$

which implies that $|q_j| \leq \lceil\frac{\sqrt{d}+1}{2}\rceil$ since $|q_j| \in \mathbb{Z}_{\geq 0}$. We claim that

$$\cup_{\lambda \in \mathcal{A}}(\{\lambda_k\} \oplus \mathcal{P}_k) \subseteq \mathbb{B}(0;3\varepsilon) \cap \mathcal{W}_k \tag{19}$$

where

$$\mathcal{P}_k \coloneqq \left\{\sum_{i=1}^k \beta_i We_i : -\frac{(\varepsilon-\gamma)}{\sigma_i\sqrt{d}} \leq \beta_i < \frac{(\varepsilon-\gamma)}{\sigma_i\sqrt{d}} \text{ for } i \in [k]\right\}$$

and $\lambda_k = \sum_{i=1}^k (We_i)'\lambda$ for each $\lambda \in \mathcal{A}$.

For each $\lambda \in \mathcal{A}$, let $y \in \mathcal{P}$ be such that $\lambda + y \in \mathbb{B}(0;\varepsilon)$ and define $y_k \coloneqq \sum_{i=1}^k ((We_i)'y)We_i$. Also, let $z \in \mathcal{P}$ be arbitrary and define $z_k \coloneqq \sum_{i=1}^k ((We_i)'z)We_i$. First, note that

$$\begin{aligned}|y_k|^2 &= \sum_{i=1}^k |(We_i)'y|^2 = \sum_{i=1}^k |\beta_i(y)|^2\\ &\leq \sum_{i=1}^k \frac{(\varepsilon-\gamma)^2}{d\sigma_i^2} < \sum_{i=1}^k \frac{\varepsilon^2}{d} \leq \varepsilon^2,\end{aligned}$$

where we have used the fact that $\sigma_i > 1-r$ for each $i \in [k]$. The same argument proves that $|z_k| < \varepsilon$ since we only used the fact that $y \in \mathcal{P}$ in the derivation above. Now, note that

$$|\lambda_k| = |\lambda_k + y_k - y_k| \leq |\lambda_k + y_k| + |y_k| \leq 2\varepsilon.$$

Finally, we conclude that

$$|\lambda_k + z_k| = |\lambda_k| + |z_k| \leq 3\varepsilon.$$

Noticing that $\lambda_k \in \mathcal{W}$ and $\mathcal{P}_k \subset \mathcal{W}_k$, we conclude that $\cup_{\lambda\in\mathcal{A}}(\{\lambda_k\} \oplus \mathcal{P}_k \subset \mathcal{W}_k$. Combining this fact with the previous result, we conclude that $\cup_{\lambda\in\mathcal{A}}(\{\lambda_k\} \oplus \mathcal{P}_k) \subseteq \mathbb{B}(0;3\varepsilon) \cap \mathcal{W}_k$.
Next, we can write that

$$\begin{aligned}
\#\{\lambda \in \mathcal{A} : \lambda_k \neq 0\} + 1 &\leq \frac{\mathrm{vol}(\mathbb{B}(0;3\varepsilon))}{\mathrm{vol}(\mathcal{P}_k)} \\
&= \left(\frac{3}{2}\right)^k \left(\frac{\varepsilon}{\varepsilon-\gamma}\right)^k w_k k^{k/2} (d/k)^{k/2} \prod_{i=1}^k \sigma_i
\end{aligned}$$

where we have used the fact that $\mathcal{P}_k$ is a k-dimensional parallelepiped with k-dimensional volume

$$\begin{aligned}
\mathrm{vol}(\mathcal{P}_k) &= (\varepsilon-\gamma)^k d^{-k/2} \prod_{i=1}^k \sigma_i^{-1} \\
&= (\varepsilon-\gamma)^k k^{-k/2} (d/k)^{-k/2} \prod_{i=1}^k \sigma_i^{-1} \qquad (20)
\end{aligned}$$

Note that $(\#\{\lambda \in \mathcal{A} : \lambda_k \neq 0\} + 1)(2\lceil \frac{\sqrt{d}+1}{2} \rceil)^{J-k} \geq \#\mathcal{A}$ by the discussion above. Thus, taking the logarithm, we get that $\log(\#\mathcal{A}) \leq H_1^r(A) - k\log(1-r) + \log(w_k k^{k/2}) + k\log(3/2) + \frac{k}{2}\log(d/k) + (J-k)\log(2\lceil \frac{\sqrt{d}+1}{2} \rceil)$. Finally, noticing that $\frac{k}{2}\log(d/k) + (J-k)\log(2\lceil \frac{\sqrt{d}+1}{2} \rceil) \leq \max\{k, J-k\}\max\{\log(d/k), \log(4\sqrt{d})\} = O(\max\{k/2, J-k\}\log(d))$.

■

C. The gap per dimension

Now, we finally prove that the gap per dimension is bounded.

Lemma 4
For each $k \in [d]$, we have that

$$\ln(k^{k/2}\omega_k) \leq k\ln(\sqrt{2e\pi}). \qquad (21)$$

▲

Proof of Lemma 4: we denote by $\Gamma(\cdot)$ the Euler's gamma function and by $\psi(\cdot)$ the digamma function, which is defined as $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$, i.e., it is the logarithmic derivative of Euler's Gamma function. We also define the function

$$g(x) := \ln\left(\frac{(x\pi)^{x/2}}{\Gamma(x/2+1)}\right) - \frac{x}{2}\ln(2e\pi)$$

for $x \geq 1$. Our goal is to prove that $g(x) \leq 0$ for all $x \geq 1$. First, note that

$$g(1) = \ln\left(\frac{2\sqrt{\pi}}{\sqrt{\pi}}\right) - \frac{1}{2}\ln(2e\pi) = \ln(2) - \frac{1}{2}\ln(2e\pi) < 0.$$

Second, for $x \geq 1$, we have that

$$\begin{aligned}
g'(x) &= \frac{1}{2}\left(\ln(e\pi x) + \psi(x/2+1)\right) - \frac{1}{2}\ln(2e\pi) \\
&\leq \frac{1}{2}\left(\ln(e\pi x) - \ln(x/2+1) + \frac{1}{x/2+1} - \ln(2e\pi)\right) \\
&= \frac{1}{2}\left(\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right) \\
&\leq 0,
\end{aligned}$$

where the first inequality follows from the fact that $-\psi(x) < 1/x - \ln(x)$ for $x > 0$ (Eq. 2.2 from [38]). The second inequality above follows from the next argument: define

$$h(x) := \ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}$$

for $x \geq 1$ and note that

$$h'(x) = \frac{4}{x(x+2)^2} > 0$$

for all $x \geq 1$. Next, we note that

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left(\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right) = 0.$$

Thus, we have that $h(x) \leq 0$ for all $x \geq 1$. Therefore, we conclude that $g(x) \leq 0$ for all $x \geq 1$, which allows us to write that

$$\ln\left(\frac{(x\pi)^{x/2}}{\Gamma(x/2+1)}\right) \leq \frac{x}{2}\ln(2e\pi)$$

for all $x \geq 1$. Recall that the k-dimensional unit ball's volume is given by $\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$, which allows us to conclude that

$$\ln(k^{k/2}\omega_k) = \ln\left(\frac{(k\pi)^{k/2}}{\Gamma(k/2+1)}\right) \leq \frac{k}{2}\ln(2e\pi)$$

for each $k \in \mathbb{Z}_{>0}$. ■

VIII. Tightness

First, recall that $\prod_{i=1}^d \sigma_i(A) = \prod_{i=1}^d |\lambda_i(A)| = |\det(A)|$. This implies that, when $k_r = d$, we have that $H^r = H$ since for our comparison with the result in [24], we require that A has all of its eigenvalues with their absolute values greater than one.

Second, Horn inequality gives us that $\sum_{i=1}^{k_r} \log(|\lambda_i(A)|) \leq \sum_{i=1}^{k_r} \log(\sigma_i(A))$ for A invertible (see, e.g., equation (3.6.1) in Chapter 3 of [29]). When $r \approx 0$, we have that both $k_r \log(1/(1-r)) \approx 0 \approx d\log(1/(1-r))$ and the lower bound in (7) is approximately H^r and in [24] is H . Consequently, when Horn inequality holds strictly, there exists r small enough so that (7) is larger than the one presented in the literature.

For example, the matrix $A = \begin{pmatrix} 1.1 & 100 \\ 0 & 1.1 \end{pmatrix}$ is such that $\lambda_1(A) = \lambda_2(A) = 1.1$ and $\sigma_1(A) \approx 100.012$ and $\sigma_2(A) \approx$

0.012. Thus, for $r < 1 - \sigma_2(A)$, we have that $k_r = 1$, which implies that $H^r \approx 6.644 - \log(1/(1-r))$ and $H \approx 0.275$. As long as $r < 0.988$ ($\log(1/(1-r)) < 6.369$), we have that (7) is larger than $H + d\log(1/(1-r))$ for $d = 2$ (dimension of the state space in this example).

The full characterization of when the result is tighter will be addressed in a future submission.

IX. Auxiliary results

Now, we prove Lemmas 1, 2, 3, and 4.

Proof of Lemma 1: this lemma shows results similar to those presented in [20]. However, for convenience and completeness, we show a short proof of this result. First, note that

$$\mathbf{X}_{n+1} = A\mathbf{X}_n + B\mathbf{U}_n + \mathbf{V}_n. \tag{22}$$

by recursively solving equation (3). Now, we have that

$$\begin{aligned}
[\![\mathbf{X}_{n+1} - \hat{\mathbf{X}}_{n+1} | a^{0:n}]\!] &= [\![\mathbf{X}_n + B\mathbf{U}_n + \mathbf{V}_n - \hat{\mathbf{X}}_{n+1} | a^{0:n}]\!] \\
&= [\![A\mathbf{X}_n + \mathbf{V}_n | a^{0:n}]\!] \oplus \{b_{a^{0:n}}\} \oplus \{-\hat{x}_{a^{0:n}}\} \\
&= [\![A\mathbf{X}_n | a^{0:n}]\!] \oplus [\![\mathbf{V}_n | a^{0:n}]\!] \oplus \{c_{a^{0:n}}\} \\
&= [\![A\mathbf{X}_n | a^{0:n}]\!] \oplus \mathbb{B}(0;\gamma) \oplus \{c_{a^{0:n}}\}
\end{aligned}$$

where the second equality follows from the facts that $[\![B\mathbf{U}_n | a^{0:n}]\!]$ and $[\![\hat{\mathbf{X}}_{n+1} | a^{0:n}]\!]$ are singletons by (2b) and (2c) and that a uv with constant range is always unrelated to any other uv, the third equality follows from the assumption that $\mathbf{X}_n \perp\!\!\!\perp \mathbf{V}_k$ for all $k \in [0:n]$, and the fourth equality follows from the fact that $[\![V_n | a^{0:n}]\!] = [\![\mathbf{V}_n]\!] = \mathbb{B}(0;\gamma)$ for each $n \in \mathbb{Z}_{\geq 0}$. ■

Proof of Lemma 2: let $A = V\Sigma W'$ be a singular value decomposition of A, let $\pi_k \in \mathbb{R}^{d\times k}$ be the matrix that $\pi_k e_i = e_i$ for each $i \in [k]$, and let $\mathcal{R} := \operatorname{span}\{e_1, \ldots, e_k\}$. Then, we have that $\mathcal{H}^k(A\mathcal{S}) = \mathcal{H}^k(V\Sigma\mathcal{R}) = \mathcal{H}^k(\Sigma\mathcal{R}) = \mathcal{H}^k(\Sigma\pi_k\mathcal{R}) = \sqrt{\det(\pi_k'\Sigma^2\pi_k)}\mathcal{H}^k(\mathcal{R}) = \prod_{i=1}^k \sigma_i(A)\mathcal{H}^k(\mathcal{R})$, where the first equality follows from the fact that $W'\mathcal{S} = W'W\mathcal{R} = \mathcal{R}$, the second equality follows from the fact that the Hausdorff measure is isometry-invariant (see, e.g., Proposition 11.18 in Section 2 of Chapter 11 from [14]) and V is an orthogonal matrix, the third follows from the fact that $\pi_k\mathcal{R} = \mathcal{R}$, the fourth is the area-formula (see, e.g., Theorem 5.1.1 in Section 1.1 of Chapter 5 from [39] and Proposition 11.21 from [14]), and the fifth follows from the fact that $\pi_k'\Sigma^2\pi_k = \operatorname{diag}(\sigma_1^2(A), \ldots, \sigma_k^2(A))$. This concludes the proof.

■

Proof of Lemma 3: let $c := b - x$ and note that $\mathbb{B}(b;r) \cap (\mathcal{S} \oplus \{x\} = (\mathbb{B}(c;r) \cap \mathcal{S}) \oplus \{x\}$. Since the Hausdorff measure is invariant under translations, we have that $\mathcal{H}^k(\mathbb{B}(c;r) \cap \mathcal{S}) = \mathcal{H}^k(\mathbb{B}(b;r) \cap (\mathcal{S} \oplus \{x\})$. Thus, we focus on studying the Hausdorff measure of $\mathbb{B}(c;r) \cap \mathcal{S}$.

Choose coordinates so that $\mathcal{S} = \{(x,y) \in \mathbb{R}^d : x \in \mathbb{R}^k \text{ and } y = (0,\ldots,0) \in \mathbb{R}^{d-k}\}$. We can also represent the center of the ball $c = (c_1, \ldots, c_k, \bar{c}_1, \ldots, \bar{c}_{d-k})$ in that same coordinate system. Thus, $z = (z_1, \ldots, z_k, \bar{z}_1, \ldots, \bar{z}_k) \in \mathbb{B}(c;r) \cap \mathcal{S}$ if, and only if,

$$\begin{aligned}
\sum_{i=1}^k |z_i - c_i|^2 + \sum_{j=1}^{d-k} |\bar{z}_j - \bar{c}_j|^2 \leq r^2 \\
\bar{z}_j = 0 \text{ for all } j \in [d-k].
\end{aligned}$$

This implies that $z \in \mathbb{B}(c;r) \cap \mathcal{S}$ if, and only if,

$$\sum_{i=1}^k |z_i - c_i|^2 \leq r^2 - \sum_{j=1}^{d-k} |\bar{c}_j|^2.$$

If $r^2 - \sum_{j=1}^{d-k} |\bar{c}_j|^2 > 0$, this set of points describes a ball with center in $(c_1, \ldots, c_k)$ and radius $\sqrt{r^2 - \sum_{j=1}^{d-k} |\bar{c}_j|^2}$ in $\mathbb{R}^k$. In this case, the k-dimensional Hausdorff measure of this ball is $\mathcal{H}^k(\mathbb{B}(c;r) \cap \mathcal{S}) = (\sqrt{r^2 - \sum_{j=1}^{d-k} |\bar{c}_j|^2})^k\omega_k \leq r^k\omega_k$. If $r^2 - \sum_{j=1}^{d-k} |\bar{c}_j|^2 \leq 0$ the set is either a singleton or empty. In this case, $\mathcal{H}^k(\mathbb{B}(c;r) \cap \mathcal{S}) = 0 < r^k\omega_k$. ■

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