Towards a Data-Rate Theorem for Linear System State Estimation with Accuracy Constraints in Noisy Environments

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April 7, 2025

Abstract

In this work, we study the minimum data-rate to reconstruct the state trajectory of a perturbed discrete-time linear system with a prescribed accuracy. The data transmission happens periodically with a fixed transmission period. Among the results, we prove a data-rate lower bound that goes to infinity as a "disturbance-to-accuracy ratio" gets close to one. Further, we show that there exists a maximum transmission period above which the minimum data-rate becomes infinite. We also prove that there exists an estimator that reconstructs the state with a finite data-rate when both the transmission period and disturbance-to-accuracy ratio are small. This latter estimator's data-rate provides an upper bound for the minimum possible data-rate. We show that the gap between the upper and the lower bounds is tight when considered per dimension. Finally, we illustrate our results with an example.

1 Introduction

Control systems with distributed components, such as sensors and actuators, appear everywhere in practice. Those components often need to share measurement results among themselves, which requires communication channels. These channels, in turn, constrain the amount of data the transmitter can send over an interval of time to a finite number of symbols. This latter restriction, on the other hand, limits the accuracy with which the receiver can reconstruct the original message. Thus, understanding the relationship between this accuracy and the communication data-rate is relevant to solving distributed control and estimation problems efficiently.

It is now well-established that there is a minimum transmission data-rate below which some control [Bai02; HOV02; MS09; Nai13; CK09] and estimation ([Del89; Sav06; LM18; MP19; VL23]) problems have no solution. In particular, in the absence of perturbations, we know that the minimum data-rate to reconstruct the state with a prescribed uniform upper bound for the estimation error is the topological entropy of the system [Sav06]. However, data-rate theorems for state estimation of perturbed linear systems with accuracy guarantees are missing from the literature. The goal of the present document is to fill that gap.

We prove that, in general, there is a maximal time for the transmitter to send the data to keep the state estimation error within a prescribed bound. The reason is that if we wait too long to transmit, the cumulative effect of the disturbance eventually makes the set of possible states too large, which makes it impossible to build an estimator that satisfies the desired accuracy bound. Consequently, we cannot use asymptotic techniques, such as those presented in [Sav06; Yan+20; LM18; VL23], to analyze this problem. The present results tell us that the data-rate in noisy environments must be higher than the usually provided by the classical data-rate theorems. Remarkably, we prove that, as the cumulative effect of the disturbance becomes comparable with the required accuracy, the minimum data-rate to solve this problem grows unbounded. We arrive at this conclusion by proving a data-rate lower bound below which the state reconstruction problem has no solution. Also, we show that the problem always has a solution if some "disturbance-to-accuracy ratio" is smaller than one. We prove this latter fact by constructing such an estimator, which gives us a data-rate upper bound that is close, when considered per dimension, to the lower bound we provide.

This document is structured as follows: in Section 2, we informally pose and motivate the problem. Also, we present a motivating example that illustrates the types of results we obtain in this paper. Then, in Section 3, we introduce the uncertain model for the class of systems we study. This model gives us a way of understanding how the perturbed system dynamics generates uncertainty. We do that using the notion of uncertain variables introduced in [Nai13]. We also formally state the problem we want to study. Then, in Section 4, we present our main results. First, we show that when the transmission period and disturbance level are small enough, the state estimation problem has a solution, and we provide a data-rate upper bound. Next, we prove a data-rate lower bound for any transmission period and the existence of a maximal time to transmit. Afterward, we analyze our results and revisit our motivating example to illustrate the results obtained. Finally, we present our conclusions.

Notations: let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$ denote the sets of real, integer, nonnegative integer, and positive integer numbers, respectively. Given integers $a < b$, and $c > 0$, denote by $[a : b] \coloneqq \{a, \dots, b\}$ and by $[c] \coloneqq \{1, \dots, c\}$. Given a set $\mathcal{A}$, we define $a^{0:n} \coloneqq (a_0, \dots, a_n) \in \mathcal{A}^{n+1}$. For a vector $x \in \mathbb{R}^d$, we denote by $|x|$ its Euclidean norm. Given two sets $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{Y} \subseteq \mathbb{R}^d$, we denote by $\mathcal{X} \oplus \mathcal{Y}$ and $\mathcal{X} \ominus \mathcal{Y}$ their geometric sum and difference[1], respectively. A function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is Big-O of $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ if $\limsup_{n\to\infty} f(n)/g(n) \leq \infty$. Given $r \in \mathbb{R}$, we denote by $\lfloor r \rfloor$ and $\lceil r \rceil$ its floor and ceiling values, respectively. Given a positive definite matrix $Q \in \mathbb{R}^{d\times d}$, we define the ellipsoid $\mathcal{E}(0; Q) \coloneqq \{x \in \mathbb{R}^d : x'Q^{-1}x \leq 1\}$. Given $c \in \mathbb{R}^d$ and $r \in (0, \infty)$, we denote by $\mathbb{B}(c; r) \coloneqq \{x \in \mathbb{R}^d : |x - c| \leq r\}$. We denote by ω_k the k-dimensional Euclidean unit ball's volume. Finally, we denote by $\log(x)$ the logarithm in base 2 of $x \in (0, \infty)$.

2 The problem and motivation

In this work, we are interested in studying fundamental limitations on the data-rate associated with the problem of state reconstruction for discrete-time linear systems subject to the effect of unknown disturbances. Specifically, our goal is to understand what is the minimum data-rate associated with keeping the estimation error small when disturbances are present. We begin this session by introducing the model and by giving an informal description of the problem we want to address. Then, we present a motivating example that helps illustrate the theory.

Consider a perturbed discrete-time linear time-invariant (LTI) system. Explicitly, for each $n \in \mathbb{Z}_{\geq 0}$, we have that

$$x_{n+1} = Ax_n + Bu_n + v_n, \tag{1}$$

where $x_0 \in \mathbb{R}^d$ is the initial state, $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ are matrices, $(v_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of unknown disturbances, and $(u_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of known control actions. Also assume that A is invertible.

The informal description of our problem is as follows: for each time $n \in \mathbb{Z}_{\geq 0}$, we want to build an estimate $\hat{x}_n$ for the value of x_n. However, the estimator and the plant are physically far from each other. This forces us to transmit the state measurements over a communication channel, which we assume is a digital channel. Since digital channels can only transmit a finite number of symbols per unit of time, we must encode the measurement using a finite alphabet $\mathcal{A}$. Also, because the encoding step might be time consuming, we assume that we are only able to transmit data at certain transmission times[2] $j\ell$, where $\ell \in \mathbb{Z}_{>0}$ is the transmission period, for each $j \in \mathbb{Z}_{\geq 0}$. Note that, these two latter facts tell us that the channel transmits data at an average rate of $\log(\#\mathcal{A})/\ell$ bits per unit of time. Since the set of possible packets the estimator can receive from the channel at each time $j\ell$ is finite, we note that, even if we measure the state perfectly, the estimation error cannot be zero in general. Clearly, if the estimator knew x_0 exactly beforehand and there were no disturbances, it could use the recursion $\hat{x}_{n+1} = A\hat{x}_n + Bu_n$ with $\hat{x}_0 = x_0 \in \mathbb{R}^d$ to reconstruct the state since the estimator knows the control u_n. Unfortunately, the estimator does not know x_0 and there are disturbances affecting our system dynamics. We assume, nevertheless, that the estimator knows an estimate $p_0 \in \mathbb{R}^d$ and constants $\gamma \in [0, \infty)$ and $\varepsilon_0 \in (0, \infty)$ so that $|v_n| \leq \gamma$ and that $|x_0 - p_0| \leq \varepsilon_0$, i.e, it knows a bound for the noise and an approximation for the initial state. Note that, we can take $\hat{x}_0 = p_0$ and guarantee that $|x_0 - \hat{x}_0| \leq \varepsilon_0$. Thus, as an alternative goal to perfect state reconstruction, we require the estimator to keep the estimation error uniformly bounded by some prescribed accuracy level $\varepsilon \geq \varepsilon_0$ for all times. We know from the literature [Sav06] that this can be done if $v_n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$ when ε_0 is small enough.

This raises some questions: does there exist a minimum data-rate above which such an estimator exists for some value of ℓ? Assuming this problem has a solution, can we provide a simple lower bound for the minimum data-rate below which this problem has no solution and explain how it varies with ε_0, ε, γ, and ℓ? We answer the first question affirmatively for the case where $\ell = 1$[3] in Theorem 1. We address the second question in Theorem 2, where we provide a simple expression for the data-rate lower bound involving only the ratios γ/ε and $\varepsilon_0/\varepsilon$ and some constants that depend on the matrix A and ℓ. We also show that, unlike the case without noise, it is possible that a maximal transmission time above which this problem has no solution exists. Interestingly, when the transmission period is small, even when there are no disturbances, the minimum data-rate is generally higher than the intrinsic entropy, which is the minimum data-rate when we can take ℓ arbitrarily large and disturbances are not present. Consequently, the minimum data-rate to solve this problem for a small ℓ is typically strictly higher than the one reported in the literature for the case without disturbances and ℓ large.

[1] These are also known as Minkowski sum and Pontryagin difference [KV12]. And as dilation and erosion [Ser82].

[2] For simplicity, we assume that there is no transmission delay or packet losses.

[3] We will present the general case in a later submission.

To motivate our results, we consider the following example, where we provide an informal discussion. We revisit this same example once we have developed the theory in a more rigorous manner in Section 4.

Example 1 (Matrices with the same intrinsic entropy)
For simplicity, we assume that $\ell = 1$. We show in Theorem 1 that we can take $\varepsilon_0 = \varepsilon$ in this scenario. Our goal is to understand how the minimum data-rate for estimating the state of system (1) with a uniform estimation error bound varies as a function of the disturbance level γ, the accuracy ε, and the the system matrix A. To do that, consider matrices

$$A_1 = \begin{pmatrix} 0.9 & 2 \\ 0 & 0.5 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.5 \end{pmatrix}. \tag{2}$$

Note that both matrices are Schur stable, i.e., their eigenvalues lie inside the unit circle in the complex plane. Consequently, if we allowed ℓ to be arbitrarily large and there were no disturbances, the minimum data-rate for reconstructing the state of system (1) with a uniform estimation error bound when either $A = A_1$ or $A = A_2$ would equal zero bits per sample [Sav06]. When disturbances are present, however, we show in Corollary 1 that we must constrain ℓ. Moreover, we prove in Theorem 2 that the minimum data-rate for both matrices above is lower-bounded by functions that are $O(-\log(1 - \gamma/\varepsilon))$ when $\ell = 1$. Intuitively, this fact shows us that if the disturbance is similar in magnitude to the accuracy, the minimum data-rate becomes large. In the scenario where the ratio γ/ε is small, the minimum data-rate lower bound for the case where $A = A_2$ approaches the intrinsic entropy while the same data-rate is higher when $A = A_1$. Also, the upper bound in Theorem 1 is close to the lower bound for low dimensions. This allows us to conclude that the minimum data-rate differs for those two matrices, with the former being much smaller than the latter. This difference is remarkable since matrices A_1 and A_2 are similar, in the sense that there exists an invertible matrix $S \in \mathbb{R}^{2\times 2}$ so that $A_1 = SA_2S^{-1}$ since they have the same eigenvalues and are both diagonalizable. Thus, the minimum data-rate for estimating a trajectory with a given accuracy and a finite ℓ is a coordinate-dependent property. ▲

Model (1) describes the state evolution for a given initial state x_0, a given sequence of disturbances $(v_n)_{n\in\mathbb{Z}_{\geq 0}}$, and a given sequence of controls $(u_n)_{n\in\mathbb{Z}_{\geq 0}}$. To formally state and study our problem, we need a model that relates how the uncertainty in the sequence of disturbances and in the initial condition affects the uncertainty in the state. This is the goal of the next section.

3 Modeling uncertainty

Consider the following informal discussion where we let $\ell = 1$ for simplicity: at time n, the estimator knows that the state belongs to a known set, a fact it inferred from past received data. For this reason, we call this set the set of possible states at time n with prior knowledge. Without new information, the estimator's uncertainty about the state would grow since the system dynamics would make the set of possible states at time $n + 1$ larger. The encoder, in turn, transmits data that carries information about the state at time n. In this manner, the estimator can update its uncertainty about the state at time n and obtain a set of possible states with posterior knowledge that is "more accurate" than the set with prior knowledge. Finally, the estimator can infer where the state would be at time $n + 1$ using the set of possible states at time n and the system dynamics. Thus, in some sense, we can think that the system dynamics generate uncertainty while the symbols the encoder transmits reduce it. Therefore, to find the minimal data-rate for solving our problem, we must understand how fast system (1) generates uncertainty.

The goal of this section is to introduce tools that help us formalize the above discussion. First, we recall the notion of uncertain variables [Nai13] and show how we can use it to model how the uncertainty in system (1) evolves. Then, we formally state the informal assumptions and goals described in the previous section using the uncertain variable framework. Finally, we pose the estimation problem in this setting.

3.1 Uncertain variables

Our first step to formally describe how system (1) propagates uncertainty over time is to recall the notion of uncertain variables.

Definition 1 (Uncertain variables [Nai13])
Let $(\Omega, \mathscr{F})$ be a measurable space. We call Ω the *sample space* and any element $\omega \in \Omega$ is called a *sample*. A Borel measurable map $\mathbf{X} : \Omega \to \mathscr{X}$ is[4] called an *uncertain variable* (uv). We define an uncertain variable *(marginal)*

[4] We assume that the set $\mathscr{X}$ is endowed with a topology and with a σ-algebra provenient from that topology, i.e., its Borel σ-algebra.

range by

$$[\![\mathbf{X}]\!] \coloneqq \{\mathbf{X}(\omega) \in \mathcal{X} : \omega \in \Omega\}. \tag{3}$$

Also, given another uncertain variable $\mathbf{Y} : \Omega \to \mathcal{Y}$, we define the *conditional range of* $\mathbf{Y}$ *given* $\mathbf{X} = x$ as

$$[\![\mathbf{Y}|\mathbf{X} = x]\!] \coloneqq \{\mathbf{Y}(\omega) \in \mathcal{Y} : \omega \in \Omega \text{ and } \mathbf{X}(\omega) = x\} \tag{4}$$

Further, we say that $\mathbf{Y}$ and $\mathbf{X}$ are *unrelated*, which we denote by $\mathbf{Y} \perp\!\!\!\perp \mathbf{X}$, if

$$[\![\mathbf{Y}|\mathbf{X} = x]\!] = [\![\mathbf{Y}]\!] \tag{5}$$

for all $x \in [\![\mathbf{X}]\!]$. ▲

To better understand how uvs can help our discussion, consider the following example: the estimator can model the possible initial states as a uv $\mathbf{X}_0$. In this case, the true initial state is represented by $\mathbf{X}_0(\omega)$ for some unknown $\omega \in \Omega$, and the set $[\![\mathbf{X}_0]\!]$ corresponds to the set of possible initial states. Now, we can model the encoder as another uv $\mathbf{\Gamma}_0$. Similarly to the case of the initial state, we should have that $\mathbf{\Gamma}_0(\omega) = a_0$ corresponds to the encoded value of $\mathbf{X}_0(\omega)$ and that the set of possible symbols is contained in the coder alphabet, i.e., $[\![\mathbf{\Gamma}_0]\!] \subseteq \mathcal{A}$. Thus, even though the estimator does not know ω, it knows $\mathbf{\Gamma}_0(\omega) = a_0$, the encoded value of $\mathbf{X}_0(\omega)$. Hence, we need a way to model how receiving the information that $\mathbf{\Gamma}_0(\omega) = a_0$ affects the estimator uncertainty on $\mathbf{X}_0$. To do that, we use the conditional range $[\![\mathbf{X}_0|\mathbf{\Gamma}_0 = a_0]\!]$, which is the subset of elements of $[\![\mathbf{X}_0]\!]$ that are encoded as a_0, i.e., $[\![\mathbf{X}_0|\mathbf{\Gamma}_0 = a_0]\!]$ is our a posteriori knowledge about $\mathbf{X}_0(\omega)$ given a_0. Therefore, this framework gives us a concise and clear way of describing how new information affects the estimator uncertainty. Nonetheless, we still need to comprehend how the dynamics (1) generates uncertainty. To understand that is the goal of our next subsection.

We take this opportunity to remark that the idea of describing the set of possible states at a given time is common in the literature of uncertain dynamical systems (see, e.g., [Wit68; SK68; BR71; Sch73]). The advantage of using uvs instead of the usual treatment is twofold: we can borrow tools from measure theory [Nai13] and we can use simple and intuitive notations, e.g., the notation for conditional range.

3.2 The problem setting and the uncertain model

Denote by $\mathbf{X}_n$, $\hat{\mathbf{X}}_n$, $\mathbf{V}_n$, $\mathbf{U}_n$, and $\mathbf{\Gamma}_n$ the uvs representing the state, the state estimate, the disturbance, the control, and the encoder at time $n \in \mathbb{Z}_{\geq 0}$, respectively. With this notation, we can convert the informal setting described in Section 2 into properties of the uvs. We also impose some mild additional technical constraints. Before we do that, we first summarize the requirements in the informal list below.

i) the initial state x_0 is unknown but can be any point inside a known ball $\mathbb{B}(p_0; \varepsilon_0)$;

ii) the state $x_{\ell n}$ is encoded as a symbol a_n from a known alphabet $\mathcal{A}$ and transmitted instantaneously at times $\ell n \in \mathbb{Z}_{\geq 0}$ for each $n \in \mathbb{Z}_{\geq 0}$ and some fixed transmission period $\ell \in \mathbb{Z}_{>0}$;

iii) for each $n \in \mathbb{Z}_{\geq 0}$, the disturbance v_n is unknown and it can be any point in a known ball $\mathbb{B}(0; \gamma) \subset \mathbb{R}^d$. Additionally, v_n does not depend on x_k or a_k for any $k \in [0 : n]$;

iv) for each $n \in \mathbb{Z}_{\geq 0}$ and each $i \in [0 : \ell - 1]$, the value of $u_{n\ell+i}$ is computed using all of the past symbols received, i.e., the data coming from the encoding of x_k for $k \in [0 : n - 1]$;

v) the state estimate $\hat{x}_{n\ell+i}$ is a function of the past symbols $a^{0:n-1} \in \mathcal{A}^n$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $i \in [0 : \ell - 1]$, where $a^{0:-1}$ means no information;

vi) the region of the state space that is encoded as the symbol a_n after having encoded $a^{0:n-1}$ in the n previous transmission times is not too complicated.

Using the uvs we described, it is straight-forward to translate the requirements above into rigorous mathematical properties, which we state as Assumption 1.

Assumption 1

i) $[\![\mathbf{X}_0]\!] = \mathbb{B}(p_0; \varepsilon_0)$ and p_0 is known;

ii) $[\![\mathbf{\Gamma}_0|\mathbf{X}_0 = x_0]\!] \subseteq \mathcal{A}$ and $[\![\mathbf{\Gamma}_n|\mathbf{X}_{n\ell} = x_{n\ell}, \mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_{n-1} = a_{n-1}]\!] \subseteq \mathcal{A}$ are singletons for each $a^{0:n-1} \in \mathcal{A}^n$, each $x_{n\ell} \in [\![\mathbf{X}_{n\ell}|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_{n-1} = a_{n-1}]\!]$, each $n \in \mathbb{Z}_{>0}$, and some $\ell \in \mathbb{Z}_{>0}$;

iii) $[\![\mathbf{V}_n]\!] = \mathbb{B}(0;\gamma)$, $\mathbf{V}_n \perp\!\!\!\perp \mathbf{X}_k$, and $\mathbf{V}_{n\ell} \perp\!\!\!\perp \mathbf{\Gamma}_k$ for all $k \in [0:n]$ and all $n \in \mathbb{Z}_{\geq 0}$;

iv) $[\![\mathbf{U}_{n\ell+i}|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_{n-1} = a_{n-1}]\!]$ is a singleton for each $a^{0:n-1} \in \mathcal{A}^n$, each $i \in [0:\ell-1]$, and each $n \in \mathbb{Z}_{\geq 0}$. Also, $[\![\mathbf{U}_i|\mathbf{\Gamma}_{-1} = a_{-1}]\!] = [\![\mathbf{U}_i]\!]$ is a singleton for each $i \in [0:\ell-1]$;

v) $[\![\hat{\mathbf{X}}_{n\ell+i}|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_{n-1} = a_{n-1}]\!]$ is a singleton for each $a^{0:n-1} \in \mathcal{A}^n$, each $i \in [0:\ell-1]$, and each $n \in \mathbb{Z}_{\geq 0}$. Also, $[\![\hat{\mathbf{X}}_i|\mathbf{\Gamma}_{-1} = a_{-1}]\!] \coloneqq [\![\hat{\mathbf{X}}_i]\!]$ is a singleton for each $i \in [0:\ell-1]$;

vi) $[\![\mathbf{X}_{n\ell}|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_n = a_n]\!]$ is a Borel set with nonempty interior for each $n \in \mathbb{Z}_{\geq 0}$. ▲

Some observations are in order. First, ii) means that the symbol used to encode the state at time n is a function of the past symbols and the current state, i.e., it has memory. We remark that such coders are well-studied and used in practice. See, e.g., predictive quantizers in Chapter 7 from [GG92]. Second, conditions iv) and v) mean that the control and the estimator are functions of the past symbols. Third, condition vi) is less clear and requires some explaining: $[\![\mathbf{X}_{n\ell}|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_n = a_n]\!]$ is the set of possible states at time $n\ell$ after receiving the symbols $(a_0, \ldots, a_n)$. We interpret this set as a quantization region for a quantizer with memory, i.e., the encoder quantizes the state. To avoid pathological quantizers with quantization regions that are too convoluted, we impose vi). We note that half-open parallelepipeds, e.g., $\{\sum_{i=1}^{d} \beta_i e_i : \beta_i \in [0,1)\}$, are Borel sets. Next, we want to rewrite equation (1) using uvs. We do that as follows.

$$\mathbf{X}_{n+1} = A\mathbf{X}_n + \mathbf{U}_n + \mathbf{V}_n \tag{6}$$

and refer to is as the *uncertain model* for the dynamics (1).

3.3 The coder-estimator scheme and the problem statement

Before we mathematically formalize our problem statement, we first introduce the accuracy requirement state estimator as well as the estimator data-rate. From this point onward, we adopt the notation $\kappa(n) \coloneqq \lfloor n/\ell \rfloor - 1$ for clarity. In what follows, we denote $[\![\mathbf{X}_n - \hat{\mathbf{X}}_n|\mathbf{\Gamma}_{-1} = a_{-1}]\!] \coloneqq [\![\mathbf{X}_n - \hat{\mathbf{X}}_n]\!]$ for each $n \in [0:\ell-1]$.

Definition 2
A *state estimator* is a sequence of uvs $(\hat{\mathbf{X}}_n)_{n\in\mathbb{Z}_{\geq 0}}$ where the elements have range in $\mathbb{R}^d$. Given an estimator $(\hat{\mathbf{X}}_n)_{n\in\mathbb{Z}_{\geq 0}}$, a transmission period ℓ, and a coder $(\mathbf{\Gamma}_n)_{n\in\mathbb{Z}_{\geq 0}}$ with alphabet $\mathcal{A}$, we say that $\mathscr{S} = \big((\hat{\mathbf{X}}_n)_{n\in\mathbb{Z}_{\geq 0}}, (\mathbf{\Gamma}_n)_{n\in\mathbb{Z}_{\geq 0}}, \ell\big)$ is a *coder-estimator scheme*. We say that a coder-estimator scheme $\mathscr{S}$ is *ε-accurate* for $(\mathbf{X}_n)_{n\in\mathbb{Z}_{\geq 0}}$ if

$$[\![\mathbf{X}_n - \hat{\mathbf{X}}_n|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_{\kappa(n)} = a_{\kappa(n)}]\!] \subseteq \mathbb{B}(0;\varepsilon). \tag{7}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Finally, we define the coder-estimator scheme's *average data-rate* as

$$R(\mathscr{S}) \coloneqq \frac{1}{\ell}\log(\#\mathcal{A}). \tag{8}$$

▲

Note that the set $[\![\mathbf{X}_n - \hat{\mathbf{X}}_n|\mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_{\kappa(n)} = a_{\kappa(n)}]\!]$ corresponds to the set of possible estimation error at time n after receiving the symbols $a^{0:\kappa(n)} \in \mathcal{A}^{\kappa(n)+1}$. Thus, condition in (7) tells us that the estimation error must be contained in a ball of radius ε for all times. Now, we can finally formally state our problem.

Problem 1
Under Assumption 1. Let $\varepsilon_0 \in (0,\infty)$ be an initial accuracy, $\varepsilon \in [\varepsilon_0,\infty)$ be a prescribed estimation accuracy, $\gamma \in (0,\infty)$ be a disturbance level, and $\ell \in \mathbb{Z}_{>0}$ be a transmission period. What conditions we must impose on ε_0, ε, γ, and ℓ so that an ε-accurate coder-estimator scheme (Def. 2) that operates with a finite average data-rate exists for $(\mathbf{X}_n)_{n\in\mathbb{Z}_{\geq 0}}$? Under such conditions, what is the data-rate $R \in [0,\infty)$ so that i) there are no ε-accurate coder-estimator schemes with $R(\mathscr{S}) < R$ and ii) there are schemes with $R(\mathscr{S}) > R$?

We call the quantity R the *minimum data-rate* for solving Problem 1. The goal of next section is to provide a partial answer to the above problem and analyze what conclusions we can draw from our results.

4 Towards new data-rate theorems

In this section, we present our main results, which provide a partial answer to the questions posed in Section 2 and Problem 1. First, we state Theorem 1, which tells us that we always have an ε-accurate coder-estimator scheme with $\ell = 1$, $\varepsilon_0 \leq \varepsilon$, and $\varepsilon > \gamma$. We also provide a data-rate upper bound in this scenario. Then, we present a lower bound on the minimum data-rate as a function of the dynamics, the accuracy level, the size of the disturbance, and the transmission period. We show that the data-rate diverges when γ approaches ε. Also, when disturbances are not present, we show that the accuracy requirement together with the requirement that we must transmit data at times $n\ell$ forces the data-rate to be larger, in most cases, than the intrinsic entropy, which is what we need for stabilization and state reconstruction without noise [Sav06; Nai+07]. Finally, we analyze the results and illustrate them with an example.

The next quantities play a major role in what follows: given $r \in [0, 1)$, we define

$$k_r(\ell) \coloneqq \max\{i \in [d] : \sigma_i(A^\ell) > 1 - r\}, \tag{9}$$

with the convention that $\max\{\emptyset\} = 0$. Note that k_0^ℓ is the number of singular values of A^ℓ that are larger than one. For each $\ell \in \mathbb{Z}_{>0}$ and each $r \in [0, 1)$, we define the quantity[5]

$$H_\ell^r \coloneqq \frac{1}{\ell} \sum_{i=1}^{k_r} \log(\sigma_i(A^\ell)), \tag{10}$$

and the quantity $H_\ell \coloneqq H_\ell^0$, which we call that *$\ell$-th intrinsic entropy*. Informally, the ℓ-th intrinsic entropy is related to how much the uncertainty on the state of system (6) grows between times $n\ell$ and $(n+1)\ell$. With these notations, we state our main theorem.

4.1 Existence of a solution

In this subsection, we assume that $\ell = 1$, i.e., we transmit at every time instant. In addition to the previously defined quantities, we also define $J_r \coloneqq \max\{i \in [d] : \sigma > \frac{1-r}{\sqrt{d}}\}$. Further, to make the notation easier to read, we denote by $r \coloneqq \gamma/\varepsilon$ and $k_r \coloneqq k_r(1)$ in this subsection.

Theorem 1
Let $\varepsilon \in (0, \infty)$ be an accuracy, let $\varepsilon_0 \in (0; \varepsilon]$, and $\gamma \in [0, \varepsilon)$ be the radius of the disturbance set. Under Assumption 1, with $\ell = 1$, there exists an ε-accurate coder-estimator scheme $\mathscr{S}$ (Def. 2) that operates with a finite average data-rate that is upper-bounded by

$$\begin{aligned} R(\mathscr{S}) \leq H_1^r - k_r \log(1 - r) + \log(1.5 w_{k_r} k_r^{k_r/2}) \\ + O(\max\{k_r, J_r - k_r\} \log(d)) \end{aligned} \tag{11}$$

▲

Due to space limitations, we prove this theorem in the report [VN25]. The expression for the term $O(\max\{k_r, J_r - k_r\} \log(d)) = k_r \log(3/2) + \frac{k_r}{2} \log(d/k_r) + (J_r - k_r) \log(2\lceil \frac{\sqrt{d}+1}{2} \rceil)$. We remark that $\log(w_k k_r^{k/2}) \leq k \log(\sqrt{2e\pi})$ for all $k \in \mathbb{Z}_{>0}$, a fact we also prove in the technical report [VN25]. A corollary of the above result is that, if $k_r = J_r = d$, we have that

$$R(\mathscr{S}) \leq H_1^r - k_r \log(1 - r) + k_r \log(1.5\sqrt{2e\pi}), \tag{12}$$

This latter scenario happens, for example, when the system is expansive, i.e., $\sigma_d > 1$, or when the accuracy-to-disturbance ratio r is "large".

4.2 Necessary conditions: minimum data-rate lower bound and the maximum time to transmit

In the previous section, we showed that, when $\ell = 1$ and $\gamma < \varepsilon$, our problem has a solution. However, it is not clear how the average data-rate of that solution compares with the minimum one. In this subsection, we present a data-rate lower bound. To dot that, we first introduce some additional notation. For each $\ell \in \mathbb{Z}_{>0}$ and

[5] Note that $H_\ell^r \coloneqq 0$ if $k_r = 0$.

each unit vector $z \in \mathbb{R}^d$, define the positive-definite matrix $Q_z^-(\ell) := \left(\sum_{i=1}^{\ell} S_i Q_i^{1/2}\right)' \left(\sum_{i=1}^{\ell} S_i Q_i^{1/2}\right)$, where $Q_i := (A^{i-1}(A')^{i-1})$ and S_i is an orthogonal matrix so that $S_i Q_i^{1/2} z$ is parallel to z. This matrix gives an ellipsoid $\mathcal{E}(0; Q_z^-)$ that under-approximates the geometric sum[6] $\bigoplus_{i=0}^{\ell-1} A^i \mathbb{B}(0;\gamma)$ (see, e.g., Lemma 2.3.1 from [KV12]). Moreover, this approximation is tight along the direction z, i.e., the projections of $\bigoplus_{i=0}^{\ell-1} A^i \mathbb{B}(0;\gamma)$ and $\mathcal{E}(0; Q_z^-)$ along z are the same. Having this in mind, we define the auxiliary quantity $f(\ell) := \sup_{z \in \mathbb{R}^d, |z|=1} \{\sigma_d^{1/2}(Q_z^-)\}$, which coincides with the in-radius of $\mathcal{E}(0; Q_{z^*}^-)$ and of $\bigoplus_{i=0}^{\ell-1} A^i \mathbb{B}(0;\gamma)$ (see, e.g., Lemma 2.3.1 from [KV12]). Next, we define

$$r_\ell := \gamma f(\ell)/\varepsilon, \tag{13}$$

which we can interpret as a lower bound on the norm of the cumulative effect of the disturbance in (6).

Theorem 2
Let $\varepsilon \in (0,\infty)$ be an accuracy, let $\varepsilon_0 \in (0,\varepsilon]$, let $\gamma \in [0,\varepsilon)$ be the radius of the disturbance set, and let $\ell \in \mathbb{Z}_{>0}$ be a transmission period. Under Assumption 1, any coder-estimator scheme $\mathscr{S}$ (Def. 2) that is ε-accurate (Def. 2) for $(\mathbf{X}_n)_{n \in \mathbb{Z}_{\geq 0}}$ has its average data-rate lower-bounded by

$$H_\ell^{r_\ell} - \frac{k_r(\ell)}{\ell}\log(1-r_\ell) + \frac{k_r(\ell)}{\ell}\log(\varepsilon_0/\varepsilon) \leq R(\mathscr{S}). \tag{14}$$

Finally, if $r_\ell \geq 1$, there exists no state estimator that solves the problem with finite data-rate. ▲

Some remarks are in order. First, if $\ell = 1$, Theorem 1 shows that we can always take $\varepsilon = \varepsilon_0$. However, if $\ell > 1$, we might need to have $\varepsilon_0 < \varepsilon$. For example, let $\ell = 2$ and consider the scalar uncertain system $\mathbf{X}_{n+1} = 2\mathbf{X}_n$ with $[\![\mathbf{X}_0]\!] = \mathbb{B}(0;\varepsilon)$. Since $[\![\mathbf{X}_1]\!] = \mathbb{B}(0;2\varepsilon)$ and the estimator is such that $[\![\hat{\mathbf{X}}_1]\!]$ is a singleton by Assumption 1, it is impossible to solve Problem 1. Second, comparing (11) and (14) with $\ell = 1$, $\varepsilon_0 = \varepsilon$, and $k_r = d$, the gap is smaller than $d\log(1.5\sqrt{2e\pi}) \approx 2.65d$ bits. Thus, the number of extra bits per dimension is constant in this scenario.

When disturbances are not present, the results in [Sav06] show us that there always exists an ε-accurate coder-estimator scheme that operates with an average data-rate as close as desired to the minimum possible by taking ℓ large. Informally, waiting to transmit reduces the data-rate required to address our problem. Thus, a natural question is why we should consider small values of ℓ. The next example shows us that, unlike the case without perturbations, the disturbance might force us to transmit data "fast".

Example 2
Consider the system $\mathbf{X}_{n+1} = \mathbf{X}_n + \mathbf{V}_n$ where $[\![\mathbf{V}_n]\!] = \mathbb{B}(0;0.75\varepsilon)$ for some $\varepsilon > 0$ and assume that $[\![\mathbf{X}_0]\!] = \{p_0\}$, i.e., we know the initial state. If $\ell = 3$, we claim that no ε-accurate estimator exists. Indeed, using the facts that $[\![\mathbf{X}_2]\!] = \mathbb{B}(0;1.5\varepsilon)$ and that$[\![\hat{\mathbf{X}}_2]\!]$ is a singleton by Assumption 1, we conclude that $[\![\mathbf{X}_2 - \hat{\mathbf{X}}_2 | \mathbf{\Gamma}_0 = a_0]\!] = \mathbb{B}(\bar{x};1.5\varepsilon)$ for some $\bar{x} \in \mathbb{R}^d$. However, as we saw in Subsection 4.1, this problem has a solution for $\ell = 1$. ▲

This example motivates the next corollary of Theorem 2.

Corollary 1
Let $\varepsilon \in (0,\infty)$ be a precision, $\gamma \in [0,\varepsilon)$ be the radius of the disturbance set, and $\ell \in \mathbb{Z}_{>0}$ be a transmission period. Under Assumption 1, there are no ε-accurate coder-estimator schemes with $\ell \in \mathbb{Z}_{>0}$ such that $f(\ell) \geq \frac{\varepsilon}{\gamma}$. ▲

This result follows from the Theorem since, if $f(\ell) \geq \frac{\varepsilon}{\gamma}$, then $r(\ell) \geq 1$, which forces the data-rate lower bound to be infinite. We note that we can use this result to arrive at the conclusion from Example 2 as well. To do that, we compute $f(3)$ as follows: $Q_i = A^{i-1}(A^{i-1}) = 1$ and $S_i = 1$ for each $i \in [\ell]$. Thus, $Q_z^-(3) = 9$ and $f(\ell) = 3 > \varepsilon/0.75\varepsilon = 4/3$, which, by Corollary 1, gives the result.

4.3 Analysis of the result

Let $H := \sum_{\lambda \text{ eigenvalue of } A} \max\{\log(|\lambda|), 0\}$ be the *intrinsic entropy* of A [Nai+07], which is also the topological entropy of system (1) when $v_k = 0$ for all $k \in \mathbb{Z}_{\geq 0}$ and x_0 belongs to some compact set with nonempty interior [Sav06]. Then, we have that $\lim_{\ell \to \infty} H_\ell = H$. Thus, the result from Theorem 2 implies the classical lower bound for state estimation for linear systems without disturbances. Our result also shares some resemblance

[6]We have that $A^i\mathbb{B}(0;\gamma) = A^i\mathcal{E}(0;\gamma^2 I_d) = \mathcal{E}(0;\gamma^2 Q_i)$ for each $i \in [1:\ell]$. See Section 1 in Chapter 2 from [KV12].

with the works considering stabilization of linear systems with disturbances [Nai+07] and [NMD15; Nak+21]. To see why, we briefly recall that, using the uncertain variables notation, those results give a bound of the form $\lim_{n\to\infty} \sup_{m\in[0:n]}\{|x| : x \in [\![\mathbf{X}_m]\!]\} \geq \frac{\gamma}{1-2^{-(R-H)/d}}$, if all the eigenvalues of A are unstable, where $\ell = 1$ and R is the controller data-rate, i.e., the logarithm of the number symbols in the alphabet. Imposing the constraint $\lim_{n\to\infty} \sup_{m\in[0:n]}\{|x| : x \in [\![\mathbf{X}_m]\!]\} \leq \varepsilon$, we conclude that $R \geq H - d\log(1-r)$ after some algebraic steps. We note that the bound presented in that work is only asymptotic, i.e., we did not impose finite time constraints on the norm of $\mathbf{X}_n$. However, comparing $R \geq H - d\log(1-r)$ with (14), we see that, if $\varepsilon_0 = \varepsilon$ and all singular values are greater than one, we conclude that our bound is larger in general since $H \leq H_1^r$. Moreover, these two bounds coincide when A is a normal matrix. Exploring these connections further is a topic of future research.

We conclude this section by revisiting Example 1 and explaining the claims we made at that point.

Example 3 (Revisiting Example 1)
Consider the matrix A_1 from Example 1 and note that the eigenvalues of A_1 and A_2 are 0.9 and 0.5, which makes their intrinsic entropy equal to zero, i.e., $H = 0$. However, A_1's singular values are $\sigma_1 \approx 2.24$ and $\sigma_2 \approx 0.2$. Hence,[7] $H_1(A_1) \approx 1.164$ bits/sample. On the other hand, matrix A_2 is a normal matrix, which implies that its singular values equal the absolute value of its eigenvalues. Thus, $H_1(A_2) = 0$. This shows that the minimum data-rate is coordinate dependent. In Figure 1, we plot the upper and lower bounds for the the minimum data-rate when $A = A_1$, $\ell = 1$, and $\varepsilon_0 = \varepsilon$ as a function of $r = \gamma/\varepsilon$. We can see that the lower bound is an increasing function of r. Further, the lower bound is close to $H_1(A_1)$ when $r \approx 0$ and both bounds grow when $r \approx 1$ as expected. The latter fact happens because we need the encoder to use finer quantization regions to compensate the presence of noise and guarantee the estimation error bound. Finally, we note that the jumps we observe in the upper bound are due to the term multiplied by $J_r - k_r$, which stars as zero, becomes one, then equals zero again.

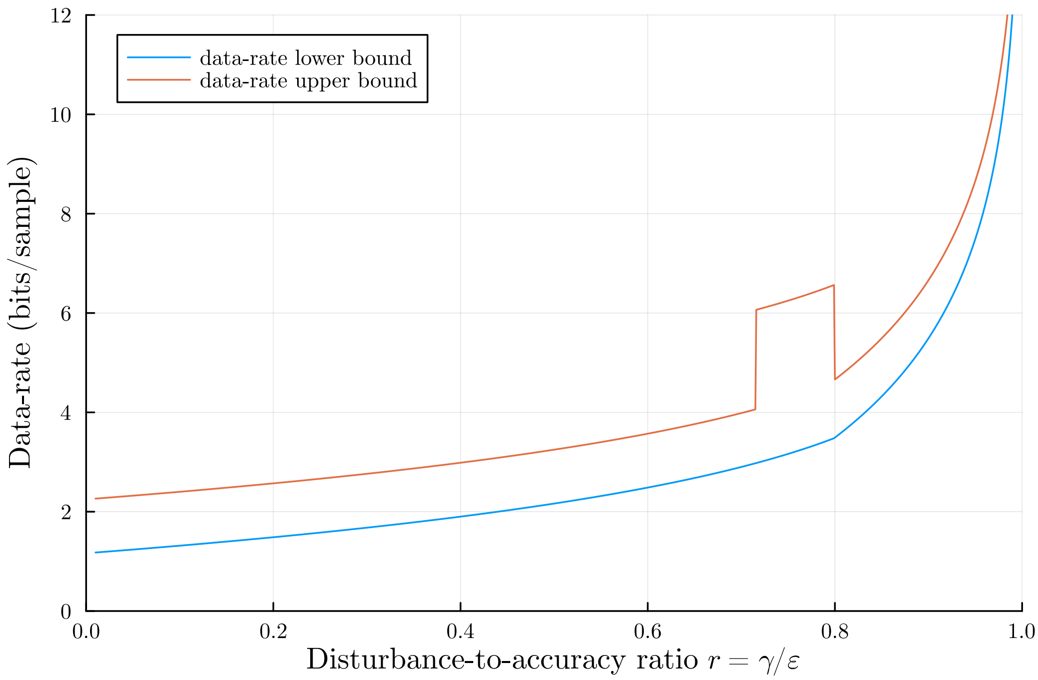

Figure 1: Upper and lower bounds for the minimum data-rate as a function of γ/ε when $A = A_1$

5 Conclusion

In this work, we described the problem of reconstructing the state of a perturbed linear system from encoded measurements with a prescribed accuracy when the information is transmitted with a finite data-rate. We explained how to model the quantities involved in this problem using uncertain variables and how we can formalize

[7] We abuse the notation and consider $H_1(A_i)$ as the 1-st intrinsic entropy of A_i for $i \in [1 : 2]$.

tasks of this kind. Next, we explained how the minimum data-rate varies with the the system dynamics, transmission period, initial state uncertainty, estimation accuracy, and the size of the disturbance. We did that by providing lower and upper bounds. We also proved that noise might force us to transmit data faster, unlike when no disturbances exist.

In future works, we will present the nonlinear case. Also, we want to extend this work to the problem of designing controllers that keep the state within a certain distance from a desired trajectory. Finally, we will provide an upper bound that works for all transmission periods for which the problem has a solution.

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A Proof of Theorem 2

First, we denote by $a^{0:m} \parallel a_{m+1}$ the concatenation of the (m)-tuple $a^{0:m-1} = (a_0, \dots, a_{m-1})$ and a_m, i.e., $a^{0:m-1} \parallel a_m = (a_0, \dots, a_{m-1}, a_m)$. Also, for any given uv $\mathbf{Y}$, define $[\![\mathbf{Y}|a^{0:m}]\!] := [\![\mathbf{Y}|\mathbf{\Gamma}_0 = a_0, \dots, \mathbf{\Gamma}_m = a_m]\!]$ for each $m \in \mathbb{Z}_{\geq 0}$. Further, define $[\![\mathbf{Y}|a^{0:-1}]\!] := [\![\mathbf{Y}]\!]$. Finally, we drop the subscript r in k_r and the superscript r in H_1^r as well as the dependency on ℓ since all those quantities are fixed. Finally, let[8] $A^\ell = V\Sigma W'$ be a singular value decomposition for A, i.e, $V \in \mathbb{R}^{d\times d}$ and $W \in \mathbb{R}^{d\times d}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{d\times d}$ is a diagonal matrix with its elements being the singular values of A. Recall that $[\![\mathbf{U}_{n\ell+k}|a^{0:n}]\!]$ is a singleton for each $k \in \mathbb{Z}_{\geq 0}$, which implies that $[\![\sum_{i=0}^{\ell-1} A^{\ell-i} B\mathbf{U}_{n\ell+i}|a^{0:n}]\!]$ is also a singleton. We denote by $b_{a^{0:n}} \in \mathbb{R}^d$ the element contained in the latter set. Also, denote by $\hat{x}_{a^{0:n}} \in \mathbb{R}^d$ the element contained in the singleton $[\![\hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!]$. Finally, define $c_{a^{0:n}} := b_{a^{0:n}} - \hat{x}_{a^{0:n}}$. Now, we state some auxiliary lemmas that we prove in the technical report [VN25].

Lemma 1
Under Assumption 1, we have that

$$\begin{aligned}&[\![\mathbf{X}_{(n+1)\ell} - \hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!] \\ &= A^\ell [\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \Big(\bigoplus_{i=0}^{\ell-1} \mathcal{E}(0;\gamma^2 Q_i)\Big) \oplus \{c_{a^{0:n}}\}.\end{aligned} \tag{15}$$

for every $n \in \mathbb{Z}_{\geq 0}$. ▲

Lemma 2
Let $m \in \mathbb{Z}_{\geq 0}$, let $A \in \mathbb{R}^{d\times d}$ be an invertible matrix with singular value decomposition $A^m = V\Sigma W'$, let $\mathcal{S} \subseteq \mathbb{R}^d$ be contained in the k-dimensional subspace $\mathcal{W} := \operatorname{span}\{We_1, \dots, We_k\}$, and denote by $\sigma_i(A^m)$ the i-th largest singular value of A^m. Then, we have that $\mathcal{H}^k(A^m\mathcal{S}) = \left(\prod_{i=1}^k \sigma_i(A^m)\right)\mathcal{H}^k(\mathcal{S})$. ▲

Proof of Theorem 2: we split the proof into four parts. In the first part, we prove a inclusion formula, which must hold for any estimator that satisfies the requirements in the theorem statement. Then, we use that inclusion together to obtain a k-dimensional volume lower bound for a section[9] of the set of possible states given the past measurements. Next, we use the previous k-dimensional volume lower bound to perform volume counting argument to obtain a lower bound for $\#\mathcal{A}$. Finally, we use the previous bound to prove the data-rate lower bound.

The inclusion: define $\tilde{\varepsilon}_\ell := \varepsilon - \gamma f(\ell)$. Our goal is to prove the inclusion

$$\mathbb{B}(-c_{a^{0:n}}; \tilde{\varepsilon}_\ell) \supseteq A^\ell [\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \tag{16}$$

for all $a^{0:n} \in \mathcal{A}^{n+1}$ and all $n \in \mathbb{Z}_{\geq 0}$. From this point onward, we denote $c := c_{a^{0:n}}$ to keep the notation light. First, equation (7) gives us that $[\![\mathbf{X}_{(n+1)\ell} - \hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!] \subseteq \mathbb{B}(0;\varepsilon)$. Note that $[\![\hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!]$ is a singleton by 2. Second, Lemma 1 gives us that $[\![\mathbf{X}_{(n+1)\ell} - \hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!] = A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \left(\bigoplus_{i=0}^{\ell-1} \mathcal{E}(0;\gamma^2 Q_i)\right) \oplus \{c\} \supseteq A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \mathcal{E}(0;\gamma^2 Q^-_{z^*}(\ell)) \oplus \{c\}$, where the inclusion follows from Lemma 2.3.1 from [KV12]. Combining the two previous inclusions we get that $A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \mathcal{E}(0;\gamma^2 Q^-_{z^*}(\ell)) \oplus \{c\} \subseteq \mathbb{B}(0;\varepsilon)$. Finally, taking the geometric sum with $\{-c\}$ and the geometric difference with $\mathcal{E}(0;\gamma^2 Q^-_{z^*}(\ell))$ on both sides, we arrive at $\mathbb{B}(-c;\varepsilon) \ominus \mathcal{E}(0;\gamma^2 Q^-_{z^*}(\ell)) \supseteq A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!]$, where we have used the fact that $\mathcal{X} \oplus \mathcal{Y} \ominus \mathcal{Y} \supseteq \mathcal{X}$ for any sets $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{Y} \subseteq \mathbb{R}^d$ (see, e.g., extensive property in equation II-23 [Ser82]). Note that $\mathcal{E}(0;\gamma^2 Q^-_{z^*}(\ell)) \supseteq \mathcal{E}(0;\gamma^2 \sigma_d(Q^-_{z^*})) = \mathbb{B}(0;\gamma f(\ell))$ by Lemma 2.3.1 from [KV12]). Thus, we conclude that $\mathbb{B}(-c;\varepsilon - \gamma f(\ell)) = \mathbb{B}(-c;\varepsilon) \ominus \mathcal{E}(0;\gamma^2 Q^-_{z^*}(\ell)) \supseteq A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!]$, where the equality follows from the fact that $\mathbb{B}(0;r_1) \ominus \mathbb{B}(0;r_2) = \mathbb{B}(0, r_1 - r_2)$.

Volume bounding: define $\mathcal{W} := \operatorname{span}\{We_1, \dots, We_k\}$, where $\{e_1, \dots, e_d\}$ is the canonical basis for $\mathbb{R}^d$. Let $x \in [\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]$ be an interior point, which exists by Assumption 1. The set $\mathcal{W} \oplus \{x\}$ is an affine space and, consequently,[10] $[\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \cap (A^{-\ell}\mathcal{W} \oplus \{A^{-\ell}x\})$ is a k-dimensional section of the set $[\![\mathbf{X}_{n\ell}|a^{0:n}]\!]$. In this part of the proof, we find a lower bound for the k-dimensional Hausdorff measure[11], of the section $A^\ell([\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \cap (A^{-\ell}\mathcal{W} \oplus \{A^{-\ell}x\})$. Explicitly, our goal is to prove that

$$\omega_k \tilde{\varepsilon}^k \geq \mathcal{H}^k(A^\ell([\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \cap (A^{-\ell}\mathcal{W} \oplus \{A^{-\ell}x\})). \tag{17}$$

[8] There was a typo in this definition in the submitted version of the paper.

[9] A k-dimensional section is the intersection of a set with a k-dimensional affine subspace.

[10] Recall that A is invertible, which implies that $A^{-\ell}\mathcal{W}$ is a k-dimensional subspace.

[11] This is a notion of k-dimensional volume of a subset of $\mathbb{R}^d$. See, e.g., Chapter 5 from [Gru07].

First, we can write that $\mathcal{H}^k(\mathbb{B}(-c;\tilde{\varepsilon}) \cap (\mathcal{W} \oplus \{x\})) \leq \sup_{z\in\mathbb{R}^d}\{\mathcal{H}^k(\mathbb{B}(-c;\tilde{\varepsilon}) \cap (\mathcal{W} \oplus \{z\}))\} = \omega_k\tilde{\varepsilon}^k$, which is true by the isodiametric inequality (see, e.g., Theorem 8.8 in [Gru07]), which is valid for the Hausdorff measure of convex sets (see, e.g., Chapter 5 from [Gru07]), $\mathcal{H}^k(\mathbb{B}(-c;\tilde{\varepsilon}) \cap (\mathcal{W} \oplus \{z\}))\} \leq (\mathrm{diam}(\mathbb{B}(-c;\tilde{\varepsilon}) \cap (\mathcal{W} \oplus \{z\}))/2)^k\omega_k \leq \tilde{\varepsilon}^k\omega_k$. Now, we can take the intersection on both sides of (16) with $\mathcal{W}\oplus\{x\}$ to get that $\mathbb{B}(-c;\tilde{\varepsilon}_\ell)\cap(\mathcal{W} \oplus \{x\}) \supseteq A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \cap (\mathcal{W} \oplus \{x\}) = A^\ell([\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \cap (A^{-\ell}\mathcal{W} \oplus \{A^{-\ell}x\}))$. Combining this inclusion with the inequality above, we get that $\omega_k\tilde{\varepsilon}^k \geq \mathcal{H}^k(A^\ell([\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \cap (A^{-\ell}\mathcal{W} \oplus \{A^{-\ell}x\})))$ for all $a^{0:n} \in \mathcal{A}^{n+1}$ since $\mathcal{H}^k(\mathcal{X}) \geq \mathcal{H}^k(\mathcal{Y})$ for any $\mathcal{X} \supseteq \mathcal{Y}$ since $\mathcal{H}^k(\cdot)$ is a measure.
The counting argument: write $[\![\mathbf{X}_n|a^{0:n}]\!] = [\![\mathbf{X}_n|a^{0:n-1} \parallel a_n]\!]$ and write $\mathcal{Z} := A^{-\ell}\mathcal{W} \oplus \{A^{-\ell}x\}$. Now, we prove that $\#\mathcal{A} \geq \left(\prod_{i=1}^k \sigma_i(A^\ell)\right) \frac{\mathcal{H}^k([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]\cap\mathcal{Z})}{\omega_k\tilde{\varepsilon}^k}$. First, we sum inequality (17) over all $a_n \in \mathcal{A}$ to conclude that $(\#\mathcal{A})\omega_k\tilde{\varepsilon}^k \geq \sum_{a_n\in\mathcal{A}} \mathcal{H}^k(A^\ell([\![\mathbf{X}_{n\ell}|a^{0:n-1} \parallel a_n]\!] \cap \mathcal{Z})) = \mathcal{H}^k(\cup_{a_n\in\mathcal{A}}(A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n-1} \parallel a_n]\!] \cap \mathcal{Z}) = \mathcal{H}^k(A^\ell(\cup_{a_n\in\mathcal{A}}[\![\mathbf{X}_{n\ell}|a^{0:n-1} \parallel a_n]\!]\cap\mathcal{Z}) = \mathcal{H}^k(A^\ell([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]\cap\mathcal{Z})) = \left(\prod_{i=1}^k \sigma_i(A^\ell)\right)\mathcal{H}^k([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]\cap\mathcal{Z})$ where the first equality follows from the fact that the sets $[\![\mathbf{X}_{n\ell}|a^{0:n} \parallel a]\!]$ are pairwise disjoint for distinct $a \in \mathcal{A}$, the second from the fact that the union distributes over intersection, the third from the fact that the sets $[\![\mathbf{X}_{n\ell}|a^{0:n-1}\|]\!]$ partition the set $[\![\mathbf{X}_{n\ell}|a^{n-1}]\!]$ (see, e.g., [Nai13]), and the last equality follows from Lemma 2. We remark that $\mathcal{H}^k([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!] \cap \mathcal{Z}) > 0$ since $A^{-\ell}x \in \mathcal{Z}$ is an interior point of $[\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]$, implying that there is a k-dimensional ball contained in the intersection $[\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!] \cap \mathcal{Z}$.
The data-rate lower bound: rearranging the terms and taking the logarithm on both sides, we can write that $\log(\#\mathcal{A}) \geq \ell H_\ell + \log\left(\frac{\mathcal{H}^k([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]\cap\mathcal{Z})}{\omega_k\tilde{\varepsilon}^k}\right)$. Finally, we can take the supremum over $n \in \mathbb{Z}_{\geq 0}$ and conclude that $\log(\#\mathcal{A}) \geq \sup_{n\in\mathbb{Z}_{\geq 0}}\left\{\ell H_\ell + \log\left(\frac{\mathcal{H}^k([\![\mathbf{X}_{n\ell}|a^{0:n-1}]\!]\cap\mathcal{Z})}{\omega_k\tilde{\varepsilon}^k}\right)\right\} \geq \ell H_\ell + k\log(\varepsilon/\tilde{\varepsilon}) + k\log(\varepsilon_0/\varepsilon)$, where we used the facts that for $n = 0$ we have that $[\![\mathbf{X}_0|a^{0:-1}]\!] = [\![\mathbf{X}_0]\!]$, that $x \in [\![\mathbf{X}_0|a^{0:-1}]\!]$ is an arbitrary interior point of $[\![\mathbf{X}_0]\!]$, and that we can pick the point $x = p_0$. ■

Now, we prove Lemmas 1 and 2.
Proof of Lemma 1: note that

$$\mathbf{X}_{(n+1)\ell} = A^\ell\mathbf{X}_{n\ell} + \sum_{i=0}^{\ell-1} A^{\ell-1}B\mathbf{U}_{n\ell+i} + \sum_{i=0}^{\ell-1} A^{\ell-i}\mathbf{V}_{n\ell+i}. \tag{18}$$

by recursively solving equation (6). Now, we have that

$$\begin{aligned}
&[\![\mathbf{X}_{(n+1)\ell} - \hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!]\\
&= [\![A^\ell\mathbf{X}_{n\ell} + \sum_{i=0}^{\ell-1} A^{\ell-i}B\mathbf{U}_{n\ell+i} + \sum_{i=0}^{\ell-1} A^{\ell-i}\mathbf{V}_{n\ell+i} - \hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!]\\
&= [\![A^\ell\mathbf{X}_{n\ell} + \sum_{i=0}^{\ell-1} A^{\ell-i}\mathbf{V}_{n\ell}|a^{0:n}]\!] \oplus \{b_{a^{0:n}}\} \oplus \{-\hat{x}_{a^{0:n}}\}\\
&= [\![A^\ell\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \left(\bigoplus_{i=0}^{\ell-1} A^{\ell-i}[\![\mathbf{V}_{n\ell}|a^{0:n}]\!]\right) \oplus \{c_{a^{0:n}}\}\\
&= [\![A^\ell\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \left(\bigoplus_{i=0}^{\ell-1} A^{\ell-i}\mathbb{B}(0;\gamma)\right) \oplus \{c_{a^{0:n}}\}\\
&= A^\ell[\![\mathbf{X}_{n\ell}|a^{0:n}]\!] \oplus \left(\bigoplus_{i=0}^{\ell-1} \mathcal{E}(0;\gamma^2 Q_i)\right) \oplus \{c_{a^{0:n}}\},
\end{aligned}$$

where the second equality follows from the facts that $[\![\sum_{i=0}^{\ell-1} A^{\ell-i}B\mathbf{U}_{n\ell+i}|a^{0:n}]\!]$ and $[\![\hat{\mathbf{X}}_{(n+1)\ell}|a^{0:n}]\!]$ are singletons and that a uv with constant range is always unrelated to any other uv, the third equality follows from the assumption that $\mathbf{X}_n \perp\!\!\!\perp \mathbf{V}_k$ for all $k \in [0:n]$, the fourth equality follows from the fact that $[\![V_n|a^{0:n}]\!] = [\![\mathbf{V}_n]\!] = \mathbb{B}(0;\gamma)$ for each $n \in \mathbb{Z}_{\geq 0}$, and the last equality comes from the fact that $A\mathbb{B}(0;\gamma) = \mathcal{E}(0, \gamma^2 AA')$ (see, e.g., Section 2.1 from [KV12]). ■

Proof of Lemma 2: let $A^m = V\Sigma W'$ be a singular value decomposition of A^m, let $\pi_k \in \mathbb{R}^{d\times k}$ be the matrix that $\pi_k e_i = e_i$ for each $i \in [k]$, and let $\mathcal{R} \coloneqq \text{span}\{e_1, \ldots, e_k\}$. Then, we have that

$$\begin{aligned}
\mathcal{H}^k(A^m \mathcal{S}) &= \mathcal{H}^k(V\Sigma\mathcal{R}) \\
&= \mathcal{H}^k(\Sigma\mathcal{R}) \\
&= \mathcal{H}^k(\Sigma\pi_k\mathcal{R}) \\
&= \sqrt{\det(\pi_k'\Sigma^2\pi_k)}\mathcal{H}^k(\mathcal{R}) \\
&= \prod_{i=1}^k \sigma_i(A^m)\mathcal{H}^k(\mathcal{R}),
\end{aligned}$$

where the first equality follows from the fact that $W'\mathcal{S} = W'W\mathcal{R} = \mathcal{R}$, the second equality follows from the fact that the Hausdorff measure is isometry-invariant (see, e.g., Proposition 11.18 in Section 2 of Chapter 11 from [Fol99]) and V is an orthogonal matrix, the third follows from the fact that $\pi_k\mathcal{R} = \mathcal{R}$, the fourth is the area-formula (see, e.g., Theorem 5.1.1 in Section 1.1 of Chapter 5 from [KP08] and Proposition 11.21 from [Fol99]), and the fifth follows from the fact that $\pi_k'\Sigma^2\pi_k = \text{diag}(\sigma_1^2(A^m), \ldots, \sigma_k^2(A^m))$. This concludes the proof.

■

B Proof of Upper Bound

Upper bound: we omit the subscript r in k_r to avoid cluttering the notation. Also, let $A = V\Sigma W'$ be a singular value decomposition (SVD) for A, where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$. To prove the upper bound, we construct a lattice vector quantizer (see, e.g., Section 10.5 from [GG92]). Consider the $d \times d$ matrix given by

$$T := 2\frac{(\varepsilon - \gamma)}{\sqrt{d}} W\Sigma^{-1}$$

and the lattice

$$\Lambda := \left\{ \sum_{i=1}^{d} q_i T e_i : (q_1, \ldots, q_d) \in \mathbb{Z}^d \right\}, \tag{19}$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis for $\mathbb{R}^d$. Let $\mathcal{P} \subset \mathbb{R}^d$ be

$$\mathcal{P} := \left\{ \sum_{i=1}^{d} \beta_i W e_i : -\frac{(\varepsilon - \gamma)}{\sqrt{d}} \sigma_i^{-1} \leq \beta_i < \frac{(\varepsilon - \gamma)}{\sqrt{d}} \sigma_i^{-1} \text{ for } i \in [d] \right\}. \tag{20}$$

We also define the partition

$$\mathscr{P} := \{\mathcal{P} \oplus \{\lambda\} : \lambda \in \Lambda\}. \tag{21}$$

Finally, we define the alphabet[12]

$$\mathcal{A} := \{\lambda \in \Lambda : (\mathcal{P} \oplus \{\lambda\}) \cap \mathbb{B}(0; \varepsilon) \neq \emptyset\}, \tag{22}$$

i.e., the lattice points that correspond to cells in $\mathscr{P}$ that intersect the ball of radius ε.

Accuracy: we claim that

$$A\mathcal{P} \subseteq \mathbb{B}(0; \varepsilon - \gamma).$$

Indeed, if $x \in \mathcal{P}$, we have that $x = \sum_{i=1}^{d} \beta_i(x) W e_i$ with $|\beta_i(x)| \leq \frac{\varepsilon - \gamma}{\sqrt{d}\sigma_i}$ for each $i \in [d]$ by the definition of the lattice. Now, we have that

$$\begin{aligned}
\left| A \sum_{i=1}^{d} \beta_i(x) W e_i \right|^2 &= \left| \sum_{i=1}^{d} \beta_i(x) V\Sigma W' W e_i \right|^2 \\
&= \left| \sum_{i=1}^{d} \beta_i(x) V\Sigma e_i \right|^2 \\
&= \left| \sum_{i=1}^{d} \beta_i(x) \sigma_i V e_i \right|^2 \\
&= \sum_{i=1}^{d} \beta_i^2(x) \sigma_i^2 \\
&\leq (\varepsilon - \gamma)^2.
\end{aligned}$$

where the first equality follows from the SVD of A, the second follows from the fact that W is an orthogonal matrix, the third follows from the fact that $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$, the fourth follows from the Pythagorean theorem and the fact that $\{Ve_1, \ldots, Ve_d\}$ is an orthonormal basis for $\mathbb{R}^d$, and the inequality follows from the restriction on $\beta_i(x)$ for each $i \in [d]$.

[12] This is known as codebook in the vector quantization literature.

The coder: denote by $u_{a^{0:n}} := [\![\mathbf{U}_n | \mathbf{\Gamma}_0 = a_0, \ldots, \mathbf{\Gamma}_n = a_n]\!]$. Define the coder and the estimator as follows: for each realization $\omega \in \Omega$, define

$$\begin{aligned}\hat{\mathbf{X}}_0(\omega) &:= p_0 \\ \mathbf{\Gamma}_0(\omega) &:= \lambda \text{ so that } \mathbf{X}_0(\omega) \in \mathcal{P} \oplus \{\lambda\} \oplus \{p_0\}.\end{aligned}$$

For each $\omega \in \Omega$, let $a_0 := \mathbf{\Gamma}_0(\omega)$ and define

$$\begin{aligned}\hat{\mathbf{X}}_1(\omega) &:= A(p_0 + a_0) + Bu_{a_0} \\ p_{a_0} &:= \hat{\mathbf{X}}_1(\omega) \\ \mathbf{\Gamma}_1(\omega) &:= \lambda \text{ so that } \mathbf{X}_0(\omega) \in \mathcal{P} \oplus \{\lambda\} \oplus \{p_{a_0}\}.\end{aligned}$$

For each realization $\omega \in \Omega$ and each time $n \in \mathbb{Z}_{>1}$, let $a^{0:n-1} := (\mathbf{\Gamma}_0(\omega), \ldots, \mathbf{\Gamma}_{n-1}(\omega))$. Now, recursively define the estimator and coder as

$$\begin{aligned}\hat{\mathbf{X}}_n(\omega) &:= A(p_{n-1} + a^{0:n-1}) + Bu_{a^{0:n-1}} \\ p_{a^{0:n-1}} &:= \hat{\mathbf{X}}_n(\omega) \\ \mathbf{\Gamma}_n(\omega) &:= \lambda \text{ so that } \mathbf{X}_n(\omega) \in \mathcal{P} \oplus \{\lambda\} \oplus \{p_{a^{0:n-1}}\}.\end{aligned}$$

First, we prove that $[\![\mathbf{X}_n - \hat{\mathbf{X}}_n | a^{0:n}]\!] \subseteq \mathbb{B}(0;\varepsilon)$ and that $[\![\mathbf{\Gamma}_n]\!] \subseteq \mathcal{A}$ for each $n \in \mathbb{Z}_{\geq 0}$.

Step $n = 0$: we have that $[\![\mathbf{X}_0 - \hat{\mathbf{X}}_0]\!] = \mathbb{B}(0;\varepsilon)$, which, together with (22), leads us to the conclusion that $[\![\mathbf{\Gamma}_0]\!] = \mathcal{A}$. For each $a_0 \in \mathcal{A}$, we have that

$$\begin{aligned}[\![\mathbf{X}_1 - \hat{\mathbf{X}}_1 | a_0]\!] &= A[\![\mathbf{X}_0 | a_0]\!] \oplus \{-A(p_0 + a_0) - Bu_{a_0}\} \oplus \{Bu_{a_0}\} \oplus \mathbb{B}(0;\gamma) \\ &= A(\mathcal{P} \oplus \{a_0\} \oplus \{p_0\}) \oplus \{-A(p_0 + a_0) - Bu_{a_0}\} \oplus \{Bu_{a_0}\} \oplus \mathbb{B}(0;\gamma) \\ &= A\mathcal{P} \oplus \mathbb{B}(0;\gamma) \\ &\subseteq \mathbb{B}(0;\varepsilon - \gamma) \oplus \mathbb{B}(0;\gamma) \\ &\subseteq \mathbb{B}(0;\varepsilon).\end{aligned}$$

Step $n = m+1$: our induction hypothesis is that, for each $a^{0:m-1} \in \mathcal{A}^{m-1}$, we have $[\![\mathbf{X}_m - \hat{\mathbf{X}}_m | a^{0:m-1}]\!] \subseteq \mathbb{B}(0;\varepsilon)$. For each $a_m \in \mathcal{A}$, we have that

$$\begin{aligned}[\![\mathbf{X}_{m+1} - \hat{\mathbf{X}}_{m+1} | a^{0:m}]\!] &= A[\![\mathbf{X}_m | a^{0:m}]\!] \oplus \{-A(p_{a^{0:m}} a_m) - Bu_{a^{0:m}}\} \oplus \{Bu_{a^{0:m}}\} \oplus \mathbb{B}(0;\gamma) \\ &= A(\mathcal{P} \oplus \{a_m\} \oplus \{p_{a^{0:m}}\}) \oplus \{-A(p_{a^{0:m}} + a_m) - Bu_{a^{0:m}}\} \oplus \{Bu_{a^{0:m}}\} \oplus \mathbb{B}(0;\gamma) \\ &= A\mathcal{P} \oplus \mathbb{B}(0;\gamma) \\ &\subseteq \mathbb{B}(0;\varepsilon - \gamma) \oplus \mathbb{B}(0;\gamma) \\ &= \mathbb{B}(0;\varepsilon).\end{aligned}$$

Upper bound for $\#\mathcal{A}$: for each $\bar{\lambda} \in \mathcal{A} \setminus \mathbb{B}(0;\varepsilon)$, there exists $y \in \mathcal{P}$, that depends on $\bar{\lambda}$, so that $|\bar{\lambda} + y| \leq \varepsilon$. Define $\bar{q}_i \in \mathbb{Z}$ via $\bar{\lambda} = \sum_{i=1}^{d} 2\frac{\bar{q}_i(\varepsilon-\gamma)}{\sqrt{d}\sigma_i} We_i$ for each $i \in [d]$ and recall that $y = \sum_{i=1}^{d} \beta_i(y) We_i$ with $|\beta_i(y)| \leq \frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_i}$. Next, for each $j \in [d]$ we can write that

$$\varepsilon \geq |\bar{\lambda} + y| \geq |(We_j)'(\bar{\lambda} + y)| = \left| 2\frac{q_j(\varepsilon - \gamma)}{\sqrt{d}\sigma_j} + \beta_j(y) \right|$$

where the first inequality follows from the discussion above and the second follows from the Cauchy-Schwarz inequality since W is an orthogonal matrix. If $q_j > 0$, we have that

$$\left| 2\frac{q_j(\varepsilon - \gamma)}{\sqrt{d}\sigma_j} + \beta_j(y) \right| = 2\frac{q_j(\varepsilon - \gamma)}{\sqrt{d}\sigma_j} + \beta_j(y) \geq (2q_j - 1)\frac{(\varepsilon - \gamma)}{\sqrt{d}\sigma_j}$$

and if $q_j < 0$, we have that

$$\left|2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j}+\beta_j(y)\right| = -2\frac{q_j(\varepsilon-\gamma)}{\sqrt{d}\sigma_j} - \beta_j(y) \geq -(2q_j+1)\frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_j}$$

since $|\beta_i(y)| \leq \frac{(\varepsilon-\gamma)}{\sqrt{d}\sigma_j}$. With this, we conclude that

$$q_j \leq \frac{\varepsilon}{2(\varepsilon-\gamma)}\sqrt{d}\sigma_j + \frac{1}{2} = (1-r)^{-1}\frac{\sqrt{d}\sigma_j}{2} + \frac{1}{2}$$

if $q_j > 0$ and

$$q_j \leq -\frac{\varepsilon}{2(\varepsilon-\gamma)}\sqrt{d}\sigma_j - \frac{1}{2} = -(1-r)^{-1}\frac{\sqrt{d}\sigma_j}{2} - \frac{1}{2}$$

if $q_j < 0$. From this we conclude that

$$|q_j| \leq (1-r)^{-1}\frac{\sqrt{d}\sigma_j}{2} + \frac{1}{2}.$$

For each $j \in [J+1:d]$, we have that $\sigma_j < \frac{(1-r)}{\sqrt{d}}$, which implies that

$$|q_j| < 1$$

which implies that $q_j = 0$. For each $j \in [k+1:J]$, we have that $\sigma_j \leq 1-r$, which implies that

$$|q_j| \leq \frac{\sqrt{d}}{2} + \frac{1}{2} = \frac{\sqrt{d}+1}{2}$$

which implies that $|q_j| \leq \lceil\frac{\sqrt{d}+1}{2}\rceil$ since $|q_j| \in \mathbb{Z}_{\geq 0}$. We claim that

$$\cup_{\lambda\in\mathcal{A}}(\{\lambda_k\} \oplus \mathcal{P}_k) \subseteq \mathbb{B}(0;3\varepsilon) \cap \mathcal{W}_k \tag{23}$$

where

$$\mathcal{P}_k := \left\{\sum_{i=1}^{k}\beta_i We_i : -\frac{(\varepsilon-\gamma)}{\sqrt{d}}\sigma_i^{-1} \leq \beta_i < \frac{(\varepsilon-\gamma)}{\sqrt{d}}\sigma_i^{-1} \text{ for } i \in [k]\right\}$$

and $\lambda_k = \sum_{i=1}^{k}(We_i)'\lambda$ for each $\lambda \in \mathcal{A}$.

For each $\lambda \in \mathcal{A}$, let $y \in \mathcal{P}$ be such that $\lambda + y \in \mathbb{B}(0;\varepsilon)$ and define $y_k := \sum_{i=1}^{k}((We_i)'y)We_i$. Also, let $z \in \mathcal{P}$ be arbitrary and define $z_k := \sum_{i=1}^{k}((We_i)'z)We_i$. First, note that

$$|y_k|^2 = \sum_{i=1}^{k}|(We_i)'y|^2 = \sum_{i=1}^{k}|\beta_i(y)|^2 \leq \sum_{i=1}^{k}\frac{(\varepsilon-\gamma)^2}{d\sigma_i^2} < \sum_{i=1}^{k}\frac{\varepsilon^2}{d} \leq \varepsilon^2,$$

where we have used the fact that $\sigma_i > 1-r$ for each $i \in [k]$. The same argument proves that $|z_k| < \varepsilon$ since we only used the fact that $y \in \mathcal{P}$ in the derivation above. Now, note that

$$|\lambda_k| = |\lambda_k + y_k - y_k| \leq |\lambda_k + y_k| + |y_k| \leq 2\varepsilon.$$

Finally, we conclude that

$$|\lambda_k + z_k| = |\lambda_k| + |z_k| \leq 3\varepsilon.$$

Noticing that $\lambda_k \in \mathcal{W}$ and $\mathcal{P}_k \subset \mathcal{W}_k$, we conclude that $\cup_{\lambda\in\mathcal{A}}(\{\lambda_k\} \oplus \mathcal{P}_k \subset \mathcal{W}_k$. Combining this fact with the previous result, we conclude that $\cup_{\lambda\in\mathcal{A}}(\{\lambda_k\} \oplus \mathcal{P}_k) \subseteq \mathbb{B}(0;3\varepsilon) \cap \mathcal{W}_k$.

Next, we can write that

$$\begin{aligned}\#\{\lambda \in \mathcal{A} : \lambda_k \neq 0\} + 1 &\leq \frac{\mathrm{vol}(\mathbb{B}(0; 3\varepsilon))}{\mathrm{vol}(\mathcal{P}_k)} \\ &= \left(\frac{3}{2}\right)^k \left(\frac{\varepsilon}{\varepsilon - \gamma}\right)^k w_k k^{k/2} (d/k)^{k/2} \prod_{i=1}^k \sigma_i\end{aligned}$$

where we have used the fact that $\mathcal{P}_k$ is a k-dimensional parallelepiped with k-dimensional volume

$$\mathrm{vol}(\mathcal{P}_k) = (\varepsilon - \gamma)^k d^{-k/2} \prod_{i=1}^k \sigma_i^{-1} = (\varepsilon - \gamma)^k k^{-k/2} (d/k)^{-k/2} \prod_{i=1}^k \sigma_i^{-1} \tag{24}$$

Note that $(\#\{\lambda \in \mathcal{A} : \lambda_k \neq 0\} + 1)(2\lceil\frac{\sqrt{d}+1}{2}\rceil)^{J-k} \geq \#\mathcal{A}$ by the discussion above. Thus, taking the logarithm, we get that

$$\log(\#\mathcal{A}) \leq H_1^k - k\log(1-r) + \log(w_k k^{k/2}) + k\log(3/2) + \frac{k}{2}\log(d/k) + (J-k)\log(2\lceil\frac{\sqrt{d}+1}{2}\rceil).$$

Finally, noticing that $\frac{k}{2}\log(d/k) + (J-k)\log(2\lceil\frac{\sqrt{d}+1}{2}\rceil) \leq \max\{k, J-k\}\max\{\log(d/k), \log(4\sqrt{d})\} = O(\max\{k/2, J-k\}\log(d))$.

■

Now, we finally prove that the gap per dimension is bounded.

Lemma 3
For each $k \in [d]$, we have that

$$\ln(k^{k/2}\omega_k) \leq k\ln(\sqrt{2e\pi}). \tag{25}$$

▲

Proof of Lemma 3: we denote by $\Gamma(\cdot)$ the Euler's gamma function and by $\psi(\cdot)$ the digamma function, which is defined as $\psi(x) \coloneqq \frac{\Gamma'(x)}{\Gamma(x)}$, i.e., it is the logarithmic derivative of Euler's Gamma function. We also define the function

$$g(x) \coloneqq \ln\left(\frac{(x\pi)^{x/2}}{\Gamma(x/2+1)}\right) - \frac{x}{2}\ln(2e\pi)$$

for $x \geq 1$. Our goal is to prove that $g(x) \leq 0$ for all $x \geq 1$. First, note that

$$g(1) = \ln\left(\frac{2\sqrt{\pi}}{\sqrt{\pi}}\right) - \frac{1}{2}\ln(2e\pi) = \ln(2) - \frac{1}{2}\ln(2e\pi) < 0.$$

Second, for $x \geq 1$, we have that

$$\begin{aligned} g'(x) &= \frac{1}{2}\left(\ln(e\pi x) + \psi(x/2+1)\right) - \frac{1}{2}\ln(2e\pi) \\ &\leq \frac{1}{2}\left(\ln(e\pi x) - \ln(x/2+1) + \frac{1}{x/2+1} - \ln(2e\pi)\right) \\ &= \frac{1}{2}\left(\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right) \\ &\leq 0, \end{aligned}$$

where the first inequality follows from the fact that $-\psi(x) < 1/x - \ln(x)$ for $x > 0$ (Eq. 2.2 from [Alz97]). The second inequality above follows from the next argument: define

$$h(x) \coloneqq \ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}$$

for $x \geq 1$ and note that

$$h'(x) = \frac{4}{x(x+2)^2} > 0$$

for all $x \geq 1$. Next, we note that

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \left(\ln \left(\frac{x}{x+2} \right) + \frac{2}{x+2} \right) = 0.$$

Thus, we have that $h(x) \leq 0$ for all $x \geq 1$. Therefore, we conclude that $g(x) \leq 0$ for all $x \geq 1$, which allows us to write that

$$\ln \left(\frac{(x\pi)^{x/2}}{\Gamma(x/2+1)} \right) \leq \frac{x}{2} \ln(2e\pi)$$

for all $x \geq 1$. Recall that the k-dimensional unit ball's volume is given by $\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$, which allows us to conclude that

$$\ln(k^{k/2}\omega_k) = \ln \left(\frac{(k\pi)^{k/2}}{\Gamma(k/2+1)} \right) \leq \frac{k}{2} \ln(2e\pi)$$

for each $k \in \mathbb{Z}_{>0}$. ■