correspondence between themselves ultimately solved this problem and this correspondence laid the first foundation of the science of probability. Next stalwart in this field was J. Bernoulli (1654-1705) whose *'Treatise on Probability'* was published posthumously by his nephew N. Bernoulli in 1713. De-Moivre (1667-1754) also did considerable work in this field and published his famous *'Doctrine of Chances'* in 1718. Other main contributors are : T. Bayes (Inverse probability), P.S. Laplace (1749-1827) who after extensive research over a number of years finally published *'Theoric analytique des probabilities'* in 1812. In addition to these, other outstanding contributors are Levy, Mises and R.A. Fisher.

Russian mathematician also have made very valuable contributions to the modern theory of probability. Chief contributors, to mention only a few of them are : Chebychev (1821-94) who founded the Russian School of Statisticians; A. Markoff (1856-1922); Liapounoff (Central Limit Theorem) ; A. Khintchine (Law of Large Numbers) and A. Kolmogorov, who axiomised the calculus of probability.

3·3. BASIC TERMINOLOGY

In this section we shall explain the various terms which are used in the definition of probability under different approaches.

1. ***Random Experiment.*** *If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a random experiment.*

Examples of random experiments are : tossing a coin, throwing a die, selecting a card from a pack of playing cards, selecting a family out of a given group of families, etc. In all these cases, there are a number of possible results which can occur but there is an uncertainty as to which one of them will actually occur.

Notes : (*i*) A *die* is a small cube used in gambling. On its six faces, dots are marked as

Plural of die is *dice*. The outcome of throwing a die is the number of dots on its uppermost face.

(*ii*) A *pack of cards* consists of four suits called *Spades, Hearts, Diamonds* and *Clubs*. Each suit consists of 13 cards, of which nine cards are numbered from 2 to 10, an ace, king, a queen and a jack (or knave). Spades and clubs are black-faced cards, while hearts and diamonds are red-faced cards.

2. ***Outcome.*** The result of a random experiment will be called an *outcome*.

3. ***Trial and Event.*** Any particular performance of a random experiment is called a *trial* and outcome or combination of outcomes are termed as *events*. For example,

(*i*) If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces, head or tail. Thus tossing of a coin is a random experiment or trial and getting of a head or tail is an event.

(*ii*) In an experiment which consists of the throw of a six-faced die and observing the number of points that appear, the possible outcomes are 1, 2, 3, 4, 5, 6

In the same experiment, the possible events could also be stated as

'Odd number of points' ; *'Even number of points'*; *'Getting a point greater than 4'* ; and so on.

Event is called *simple* if it corresponds to a single possible outcome of the experiment otherwise it is known as a *compound* or *composite* event. Thus in tossing of a single die the event of getting '6' is a simple event but the event of getting an even number is a composite event.

4. ***Exhaustive Events or Cases.*** The total number of possible outcomes of a random experiment is known as the *exhaustive events or cases*. For example,

(*i*) In tossing of a coin, there are two exhaustive cases, *viz.*, head and tail (the possibility of the coin standing on an edge being ignored).

(*ii*) In throwing of a die, there are 6 exhaustive cases since any one of the 6 faces 1, 2, ..., 6 may come uppermost.

(*iii*) In drawing two cards from a pack of cards, the exhaustive number of cases is $^{52}C_2$, since 2 cards can be drawn out of 52 cards in $^{52}C_2$ ways.

(*iv*) In throwing of two dice, the exhaustive number of cases is $6^2 = 36$, since any of the numbers 1 to 6 on the first die can be associated with any of the 6 numbers on the other die. In general, in throwing of n dice, the exhaustive number of cases is 6^n.

5. ***Favourable Events or Cases.*** The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event. For example,

(*i*) In drawing a card from a pack of cards the number of cases favourable to drawing of an ace is 4, for drawing a spade is 13 and for drawing a red card is 26.

(*ii*) In throwing of two dice, the number of cases favourable to getting the sum 5 is :

$$(1, 4), (4, 1), (2, 3), (3, 2), \textit{i.e.}, 4.$$

6. ***Mutually Exclusive Events.*** Events are said to be *mutually exclusive* or *incompatible* if the happening of any one of them precludes the happening of all the others, *i.e.*, if no two or more of them can happen simultaneously in the same trial. For example,

(*i*) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of others, in the same trial, is ruled out.

(*ii*) Similarly in tossing a coin the events head and tail are mutually exclusive.

7. ***Equally Likely Events.*** Outcomes of trial are said to be *equally likely* if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others. For example,

(*i*) In a random toss of an unbiased or uniform coin, head and tail are equally likely events.

(*ii*) In throwing an unbiased die, all the six faces are equally likely to come.

8. ***Independent Events.*** Several events are said to be independent if the happening (or non-happening) of an event is not affected by the supplementary knowledge concerning the occurrence of any number of the remaining events. For example,

(*i*) In tossing an unbiased coin, the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(*ii*) When a die is thrown twice, the result of the first throw does not affect the result of the second throw.

(*iii*) If we draw a card from a pack of well-shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second draw is dependent on the first draw.

3·4. MATHEMATICAL (OR CLASSICAL OR 'A PRIORI') PROBABILITY

Definition. *If a random experiment or a trial results in 'n' exhaustive, mutually exclusive and equally likely outcomes (or cases), out of which m are favourable to the occurrence of an event E, then the probability 'p' of occurrence (or happening) of E, usually denoted by P(E), is given by :*

$$p = P(E) = \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}} = \frac{m}{n} \qquad \dots (3·1)$$

This definition was given by James Bernoulli who was the first person to obtain a quantitative measure of uncertainty.

Remarks 1. Since $m \geq 0$, $n > 0$ and $m \leq n$, we get from (3·1) :

$$P(E) \geq 0 \text{ and } P(E) \leq 1 \quad \Rightarrow \quad 0 \leq P(E) \leq 1$$

2. Sometimes we express (3·1) by saying that 'the odds in favour of E are $m : (n - m)$ or the odds against E are $(n - m) : m$.

3. The non-happening of the event E is called the *complementary event of* E and is denoted by $\overline{E}$ or E^c. The number of cases favourable to $\overline{E}$, *i.e.*, non-happening of E is $(n - m)$. Then the probability q that E will not happen is given by :

$$q = P(\overline{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p \quad \Rightarrow \quad p + q = 1 \qquad \dots (3·1a)$$

$$\therefore \quad q = P(\overline{E}) = 1 - P(E) \quad \Rightarrow \quad P(E) = 1 - P(\overline{E}) \quad \text{or} \quad P(E) + P(\overline{E}) = 1 \qquad \dots (3·1b)$$

If the event E represents the happening of at least one of the events $E_1, E_2, \dots, E_n$, then its complementary event $\overline{E}$ represents the happening of none of the events $E_1, E_2, \dots, E_n$. Substituting in (3·1 *b*), we get

$$P\left(\text{Happening of at least one of the events } E_1, E_2, \dots, E_n\right)$$
$$= 1 - P\left(\text{None of the events } E_1, E_2, \dots, E_n \text{ happens}\right) \qquad \dots (3·1c)$$

4. Probability 'p' of the happening of an event is also known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure, *i.e.*, $(p + q = 1)$.

5. If $P(E) = 1$, E is called a *certain event* and if $P(E) = 0$, E is called an *impossible* event.

6. We can compute the probability in (3·1) by logical reasoning, without conducting any experiment. Since, the probability in (3·1) can be computed prior to obtaining any experimental data, it is also termed as '*a priori*' or *mathematical probability*.

3·4·1. Limitations of Classical Definition. This definition of classical probability breaks down in the following cases :

(*i*) If the various outcomes of the random experiment are not equally likely or equally probable. For example,

(*a*) The probability that a candidate will pass in a certain test is not 50%, since the two possible outcomes, *viz.*, success and failure (excluding the possibility of a compartment) are not equally likely.

(*b*) The probability that a ceiling fan in a room will fall is not 1/2, since the events of the fan 'falling' and 'not falling' though mutually exclusive and exhaustive, are not equally likely. In fact, the probability of the fan falling will be almost zero.

(*c*) If a person jumps from a running train, then the probability of his survival will not be 50%, since in this case the events survival and death, though exhaustive and mutually exclusive, are not equally likely.

(*ii*) If the exhaustive number of outcomes of the random experiment is infinite or unknown.

3.5. STATISTICAL (OR EMPIRICAL) PROBABILITY

Definition. (VON MISES). *If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.*

Symbolically, if in N trials an event E happens M times, then the probability of the happening of E, denoted by $P(E)$, is given by :

$$P(E) = \lim_{N \to \infty} \frac{M}{N} \qquad \ldots (3{\cdot}2)$$

Remarks 1. Since in the relative frequency approach, the probability is obtained objectively by repetitive empirical observations, it is also known as '*Empirical Probability*'.

2. An experiment is unique and non-repeating only in the case of *subjective probability*. In other cases, there are a large number of experiments or trials to establish the chance of occurrence of an event. This is particularly so in case of empirical probability. In classical probability also, repeated experiments may be made to verify whether a deduction on the basis of certain axioms or undisputed laws is justified. Only after repeated trials it can be established that the chance of head in a toss of a coin is 1/2. J. E. Kerrich conducted coin tossing experiment with 10 sets of 1,000 tosses each during his confinement in World War II. The number of heads found by him were : 502, 511, 497, 529, 504, 476, 507, 520, 504, 529.

This gives the probability of getting a head in a toss of a coin as : $\frac{5{,}079}{10{,}000} = 0{\cdot}5079 \simeq \frac{1}{2}$. Thus, the empirical probability approaches the classical probability as the number of trials becomes indefinitely large.

3·5·1. Limitations of Empirical Probability. (*i*) If an experiment is repeated a large number of times, the experimental conditions may not remain identical and homogeneous.

(*ii*) The limit in (3·2) may not attain a unique value, however large N may be.

Example 3·1. *What is the chance that a leap year selected at random will contain 53 Sundays ?*

Solution. In a leap year (which consists of 366 days), there are 52 complete weeks and 2 days over. The following are the possible combinations for these two '*over*' days : (*i*) Sunday and Monday, (*ii*) Monday and Tuesday, (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, and (*vii*) Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two 'over' days must be Sunday. Since out of the above 7 possibilities, 2, *viz.*, (*i*) and (*vii*), are favourable to this event.

$$\therefore \qquad \text{Required probability} = \frac{2}{7}\cdot$$

Example 3·2. *Two unbiased dice are thrown. Find the probability that :*

(*i*) *both the dice show the same number,*

(*ii*) *the first die shows 6,*

(iii) the total of the numbers on the dice is 8.

(iv) the total of the numbers on the dice is greater than 8,

(v) the total of the numbers on the dice is 13, and

(vi) the total of the numbers on the dice is any number from 2 to 12, both inclusive.

Solution. In a random throw of two dice, since each of the six faces of one die can be associated with each of six faces of the other die, the total number of cases is $6 \times 6 = 36$, as given below :

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Here, the expression, say, (i, j) means that the first die shows the number i and the second die shows the number j. Obviously, $(i, j) \neq (j, i)$ if $i \neq j$.

$\therefore$ Exhaustive number of cases $(n) = 36$.

(*a*) The favourable cases that both the dice show the same number are :

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6), *i.e.*, $m = 6$.

$\therefore$ Probability that the two dice show the same number $= \dfrac{6}{36} = \dfrac{1}{6}\cdot$

(*b*) The favourable cases that the first die shows 6 are :

(6, 1), (6, 2), (6, 3), (6, 4), (6, 5) and (6, 6), *i.e.*, 6 in all.

$\therefore$ Probability that the first die shows '6' $= \dfrac{6}{36} = \dfrac{1}{6}\cdot$

(*c*) The cases favourable to getting a total of 8 on the two dice are :

(2, 6), (3, 5), (4, 4), (5, 3), (6, 2), *i.e.*, $m = 5$.

$\therefore$ Probability that total of numbers on two dice is $8 = \dfrac{5}{36}\cdot$

(*d*) The cases favourable to getting a total of more than 8 are :

(3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6), *i.e.*, $m = 10$.

$\therefore$ Probability that the total of numbers on two dice is greater than $8 = \dfrac{10}{36} = \dfrac{5}{18}\cdot$

(*e*) This is an example of an *impossible event*, since the maximum total can be $6 + 6 = 12$. Therefore, the required probability is 0.

(*f*) The probability is 1, as the total of the numbers on the two dice certainly ranges from 2 to 12. The given event is called a *certain event*.

Example 3·3. (*a*) *Among the digits 1, 2, 3, 4, 5 at first one is chosen and then a second selection is made among the remaining four digits. Assuming that all twenty possible outcomes have equal probabilities, find the probability that an odd digit will be selected*

(i) the first time, (ii) the second time, and (iii) both times.

(*b*) *From 25 tickets, marked with first 25 numerals, one is drawn at random. Find the chance that (i) it is multiple of 5 or 7, and (ii) it is a multiple of 3 or 7.*

Solution. (*a*) Total number of cases $= 5 \times 4 = 20$.

(i) Now there are 12 cases in which the first digit drawn is odd, *viz.*, (1, 2), (1, 3), (1, 4), (1, 5), (3, 1), (3, 2), (3, 4), (3, 5), (5, 1), (5, 2), (5, 3) and (5, 4).

$\therefore$ The probability that the first digit drawn is odd $= \dfrac{12}{20} = \dfrac{3}{5}$.

(ii) Also there are 12 cases in which the second digit drawn is odd, *viz.*, (2, 1), (3, 1), (4, 1), (5, 1), (1, 3), (2, 3), (4, 3), (5, 3), (1, 5), (2, 5), (3, 5) and (4, 5).

$\therefore$ The probability that the second digit drawn is odd $= \dfrac{12}{20} = \dfrac{3}{5}$.

(iii) There are six cases in which both the digits drawn are odd, *viz.*, (1, 3), (1, 5), (3, 1), (3, 5), (5, 1) and (5, 3).

$\therefore$ The probability that both the digits drawn are odd $= \dfrac{6}{20} = \dfrac{3}{10}$.

(b) (i) Numbers (out of the first 25 numerals) which are multiples of 5 are 5, 10, 15, 20 and 25, *i.e.*, 5 in all and the numbers which are multiples of 7 are 7, 14 and 21, *i.e.*, 3 in all. Hence required number of favourable cases are $5 + 3 = 8$.

$\therefore$ Required probability $= \dfrac{8}{25}$.

(ii) Numbers (among the first 25 numerals) which are multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, *i.e.*, 8 in all ; and the numbers, which are multiples of 7 are 7, 14, 21, *i.e.*, 3 in all. Since the number 21 is common in both the cases, the required number of distinct favourable cases is $8 + 3 - 1 = 10$.

$\therefore$ Required probability $= \dfrac{10}{25} = \dfrac{2}{5}$.

Example 3·4. (a) *Four cards are drawn at random from a pack of 52 cards. Find the probability that*

(i) They are a king, a queen, a jack and an ace.

(ii) Two are kings and two are queens.

(iii) Two are black and two are red.

(iv) There are two cards of hearts and two cards of diamonds.

(b) In shuffling a pack of cards, four are accidentally dropped, find the chance that the missing cards should be one from each suit.

Solution. Four cards can be drawn from a well-shuffled pack of 52 cards in ${}^{52}C_4$ ways, which gives the exhaustive number of cases.

(i) 1 king can be drawn out of the 4 kings in ${}^4C_1$ ways. Similarly, 1 queen, 1 jack and an ace can each be drawn in ${}^4C_1 = 4$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable number of cases are ${}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1$.

Hence the required probability $= \dfrac{{}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1}{{}^{52}C_4} = \dfrac{256}{{}^{52}C_4}$.

(ii) Required probability $= \dfrac{{}^4C_2 \times {}^4C_2}{{}^{52}C_4}$.

(iii) Since there are 26 black cards (of spades and clubs) and 26 red cards (of diamonds and hearts) in a pack of cards, the required probability $= \dfrac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$.

(iv) Required probability $= \dfrac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}$.

(*b*) There are ${}^{52}C_4$ possible ways in which four cards can slip while shuffling a pack of cards. The favourable number of cases in which the four cards can be one from each suit is : ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$.

$$\therefore \quad \text{The required probability} = \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4} = \frac{2197}{20825}.$$

Example 3·5. *What is the probability of getting 9 cards of the same suit in one hand at a game of bridge ?*

Solution. Since one hand in a bridge game consists of 13 cards, the exhaustive number of cases is ${}^{52}C_{13}$.

The number of ways in which 9 cards of a suit can come out of 13 cards of the suit $= {}^{13}C_9$. The number of ways in which balance $13 - 9 = 4$ cards can come in one hand out of a balance of 39 cards of other suits is ${}^{39}C_4$.

Since there are four different suits and 9 cards of any suit can come, by the principle of counting, the total number of favourable cases of getting 9 cards of suit $= {}^{13}C_4 \times {}^{39}C_4 \times 4$.

$$\therefore \quad \text{Required probability} = \frac{{}^{13}C_9 \times {}^{39}C_4 \times 4}{{}^{52}C_{13}}.$$

Example 3·6. *A man is dealt 4 spade cards from an ordinary pack of 52 cards. If he is given three more cards, find the probability p that at least one of the additional cards is also a spade.*

Solution. After a man has dealt 4 spade cards from an ordinary pack of 52 cards, there are $52 - 4 = 48$ cards left in the pack, out of which 9 are spade cards and 39 are non-spade cards.

Since 3 more cards can be dealt to the same man out of the 48 cards in ${}^{48}C_3$ ways, the exhaustive number of outcomes $= {}^{48}C_3$.

If none of these 3 additional cards is a spade card, then the 3 additional cards must be drawn out of the 39 non-spade cards, which can be done in ${}^{39}C_3$ ways. The probability that none of the three additional cards dealt to the man is a spade card is given by ${}^{39}C_3/{}^{48}C_3$.

Hence, the probability p that at least one of the three additional cards is also a spade is given by :

$$p = 1 - P\ [\text{None of the three additional cards is a spade.}]$$

$$= 1 - \frac{{}^{39}C_3}{{}^{48}C_3} = 1 - \frac{39 \times 38 \times 37}{3\,!} \times \frac{3\,!}{48 \times 47 \times 46} = 1 - \frac{13 \times 19 \times 37}{16 \times 47 \times 23} = 0{\cdot}4718.$$

Example 3·7. *A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manner :*

(*i*) *There must be one from each category.*

(*ii*) *It should have at least one from the purchase department.*

(*iii*) *The chartered accountant must be in the committee.*

Solution. There are $3 + 4 + 2 + 1 = 10$ persons in all and a committee of 4 people can formed out of them in ${}^{10}C_4$ ways. Hence, exhaustive number of cases is :

$${}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4\,!} = 210$$

(*i*) Favourable number of cases for the committee to consist of 4 members, one from each category, is : ${}^4C_1 \times {}^3C_1 \times {}^2C_1 \times 1 = 4 \times 3 \times 2 = 24$

$\therefore$ Required probability $= \dfrac{24}{210} = \dfrac{4}{35}$.

(*ii*) P [Committee has at least one purchase officer]

$= 1 - P$ (Committee has no purchase officer)

In order that the committee has no purchase officer, all the 4 members are to be selected from amongst officers of production department, sales department and chartered accountant, *i.e.*, out of 3 + 2 + 1 = 6 members and this can be done in ${}^6C_4 = \dfrac{6 \times 5}{1 \times 2} = 15$ ways. Hence

P (Committee has no purchase officer) $= \dfrac{15}{210} = \dfrac{1}{14}$

$\therefore$ P (Committee has at least one purchase officer) $= 1 - \dfrac{1}{14} = \dfrac{13}{14}$.

(*iii*) Favourable number of cases that the committee consists of a chartered accountant as a member and three others are : $1 \times {}^9C_3 = \dfrac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84$ ways, since a chartered accountant can be selected out of one chartered accountant in only 1 way and the remaining 3 members can be selected out of the remaining $10 - 1 = 9$ persons in ${}^9C_3$ ways. Hence the required probability $= \dfrac{84}{210} = \dfrac{2}{5}$.

Example 3·8. *An urn contains 6 white, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that : (i) two of the balls drawn are white, (ii) one is of each colour, (iii) none is red, (iv) at least one is white.*

Solution. Total number of balls in the urn is 6 + 4 + 9 = 19. Since 3 balls can be drawn out of 19 in ${}^{19}C_3$ ways, the exhaustive number of cases are ${}^{19}C_3$.

(*i*) If 2 balls of the 3 drawn balls are to be white, these two balls should be drawn out of 6 white balls which can be done in ${}^6C_2$ ways, and the third ball can be drawn out of the remaining $19 - 6 = 13$ balls, which can be done in ${}^{13}C_1$ ways. Since any of the former ways can be associated with any one of the later ways, the number of favourable cases $= {}^6C_2 \times {}^{13}C_1$.

Hence, required probability $= \dfrac{{}^6C_2 \times {}^{13}C_1}{{}^{19}C_3}$.

(*ii*) Since the number of favourable cases of getting one ball of each colour is ${}^6C_1 \times {}^4C_1 \times {}^9C_1$, the required probability $= \dfrac{{}^6C_1 \times {}^4C_1 \times {}^9C_1}{{}^{19}C_3}$.

(*iii*) If none of the drawn balls is red, then all the 3 balls must be out of the white and black balls, *viz.*, out of 6 + 9 = 15 balls. Hence the number of favourable cases for this event is ${}^{15}C_3$.

$\therefore$ Required probability $= \dfrac{{}^{15}C_3}{{}^{19}C_3}$.

(*iv*) P (at least one ball is white) $= 1 - P$ (none of the three balls is white) ... (*)

In order that none of the three balls is white, all the three balls must be drawn out of the red and black balls, *i.e.*, out of 4 + 9 = 13 balls and this can be done in ${}^{13}C_3$ ways.

Hence P (none of the three balls is white) $= \dfrac{{}^{13}C_3}{{}^{19}C_3}$.

Substituting in (*), we obtain $\quad P$ (at least one ball is white) $= 1 - \dfrac{{}^{13}C_3}{{}^{19}C_3}$.

Example 3·9. *In a random arrangement of the letters of the word 'COMMERCE', find the probability that all the vowels come together.*

Solution. The total number of permutations of the letters of the word 'COMMERCE' are $(8\,!)/(2\,!\,2\,!\,2\,!)$, because it contains 8 letters of which 2 are C's, 2 M's, and 2 E's, and remaining are all different.

The word COMMERCE contains 3 vowels, *viz.*, *OEE* (2 *E*'s being identical). To obtain the total number of arrangements in which these 3 vowels come together, we regard them as tied together, forming only one letter so that total number of letters in COMMERCE may be taken as $8 - 2 = 6$, out of which 2 are *C*'s, 2 are *M*'s and rest distinct and, therefore, their number of arrangement is given by $(6\,!)/(2\,!\,2\,!)$.

Further, the three vowels *OEE*, two of which are identical, can be arranged among themselves in $3\,!/2\,!$ ways. Hence, the total number of arrangements favourable to getting all vowels together $= \dfrac{6\,!}{2\,!\,2\,!} \times \dfrac{3\,!}{2\,!}$.

Hence, the required probability $= \dfrac{6\,!\,3\,!}{2\,!\,2\,!\,2} \div \dfrac{8\,!}{2\,!\,2\,!\,2\,!} = \dfrac{3}{28}$.

Example 3·10. (a) *If the letters of the word 'REGULATIONS' be arranged at random, what is the chance that there will be exactly 4 letters between R and E ?*

(b) *What is the probability that four S's come consecutively in the word 'MISSISSIPPI' ?*

Solution. (*a*) The word 'REGULATIONS' consists of 11 letters. The two letters *R* and *E* can occupy ${}^{11}P_2$, *i.e.*, $11 \times 10 = 110$ positions.

The number of ways in which there will be exactly 4 letters between *R* and *E* are enumerated below :

(*i*) *R* is in the 1st place and *E* is in the 6th place.

(*ii*) *R* is in the 2nd place and *E* is in the 7th place.

... ...

... ...

... ...

(*vi*) *R* is in the 6th place and *E* is in the 11th place.

Since *R* and *E* can interchange their positions, the required number of favourable cases is $2 \times 6 = 12$.

$\therefore$ The required probability $= \dfrac{12}{110} = \dfrac{6}{55}$.

(*b*) Total number of permutations of the 11 letters of the word '*MISSISSIPPI*' in which 4 are of one kind (*viz.*, *S*), 4 of other kind (*viz.*, *I*), 2 of third kind (*viz.*, *P*) and 1 of fourth kind (*viz.*, *M*) are $11\,!/4\,!\,4\,!\,2\,!\,1\,!$.

Following are the 8 possible combinations of 4 *S*'s coming consecutively :

(*i*)	S	S	S	S							
(*ii*)	—	S	S	S	S						
(*iii*)	—	—	S	S	S	S					
⋮											
(*viii*)	—	—	—	—	—	—	—	S	S	S	S

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, *viz.*, *MIIIPPI* of which 4 are of one kind, 2 of other kind and one of third kind are $\frac{7!}{4!2!1!}$, the required number of favourable cases $= \frac{8 \times 7!}{4!2!1!}$.

$$\therefore \quad \text{Required probability} = \frac{\frac{8 \times 7!}{4!2!1!}}{\frac{11!}{4!4!2!1!}} = \frac{8 \times 7! \times 4!}{11!} = \frac{4}{165}.$$

Example 3·11. *Twenty-five books are placed at random in a shelf. Find the probability that a particular pair of books shall be : (i) Always together, and (ii) Never together.*

Solution. Since 25 books can be arranged among themselves in 25 ! ways, the exhaustive number of cases is 25 !

(*i*) Let us now regard that the two particular books are tagged together so that we shall regard them as a single book. Thus, now we have $(25 - 1) = 24$ books which can be arranged among themselves in 24 ! ways. But the two books which are fastened together can be arranged among themselves in 2 ! ways. Hence, associating these two operations, the number of favourable cases for getting a particular pair of books always together is $24! \times 2!$.

$$\therefore \quad \text{Required probability} = \frac{24! \times 2!}{25!} = \frac{2}{25}.$$

(*ii*) Total number of arrangements of 25 books among themselves is 25 ! and the total number of arrangements that a particular pair of books will always be together is $24! \times 2$. Hence, the number of arrangements in which a particular pair of books is never together is : $25! - 2 \times 24! = (25 - 2) \times 24! = 23 \times 24!$

$$\therefore \quad \text{Required probability} = \frac{23 \times 24!}{25!} = \frac{23}{25}.$$

Aliter :

P [A particular pair of books shall never be together.]

$= 1 - P$ [A particular pair of books is always together.]

$$= 1 - \frac{2}{25} = \frac{23}{25}.$$

Example 3·12. *n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.*

Solution. Since n persons can be seated in n chairs at a round table in $(n - 1)!$ ways, the exhaustive number of cases $= (n - 1)!$.

Assuming the two specified persons A and B who sit together as one, we get $(n - 1)$ persons in all, who can be seated at a round table in $(n - 2)!$ ways. Further, since A and B can interchange their positions in 2 ! ways, total number of favourable cases of getting A and B together is $(n - 2)! \times 2!$.

$$\therefore \quad \text{Required probability} = \frac{(n-2)! \times 2!}{(n-1)!} = \frac{2}{n-1}.$$

Example 3·13. *A five-figure number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.*

Solution. The total number of ways in which the five digits 0, 1, 2, 3, 4 can be arranged among themselves is 5 !. Out of these, the number of arrangements which begin with 0 (and, therefore, will give only 4-digited numbers) is 4 !. Hence the total number of five-digited numbers that can be formed from the digits 0, 1, 2, 3, 4 is :

$$5! - 4! = 120 - 24 = 96.$$

The number formed will be divisible by 4 if number formed by the two digits on extreme right (*i.e.*, the digits in the unit and tens places) is divisible by 4. Such numbers are :

$$04, \quad 12, \quad 20, \quad 24, \quad 32, \quad \text{and} \quad 40.$$

If the numbers end in 04, the remaining three digits, *viz.*, 1, 2 and 3 can be arranged among themselves in 3 ! ways. Similarly, the number of arrangements of the numbers ending with 20 and 40 is 3 ! in each case.

If the numbers end with 12, the remaining three digits 0, 3, 4 can be arranged in 3 ! ways. Out of these we shall reject those numbers which start with 0 (*i.e.*, have 0 as the first digit). There are $(3 - 1)! = 2 !$ such cases. Hence, the number of five-digited numbers ending with 12 is $3 ! - 2 ! = 6 - 2 = 4$.

Similarly, the number of five digited numbers ending with 24 and 32 each is 4. Hence the total number of favourable cases is $3 \times 3 ! + 3 \times 4 = 18 + 12 = 30$.

Hence, the required probability $= \dfrac{30}{96} = \dfrac{5}{16}$.

Example 3·14. (*a*) *Twelve balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls ?*

(*b*) *If n biscuits be distributed among N persons, find the chance that a particular person receives r (< n) biscuits.*

Solution. (*a*) Since each ball can go to any one of the there boxes, there are 3 ways in which a ball can go to any one of the three boxes. Hence there are 3^{12} ways in which 12 balls can be placed in the three boxes.

Number of ways in which 3 balls out of 12 can go to the first box is ${}^{12}C_3$. Now the remaining 9 balls are to be placed in remaining 2 boxes and this can be done in 2^9 ways. Hence, the total number of favourable cases $= {}^{12}C_3 \times 2^9$.

$$\therefore \quad \text{Required probability} = \frac{{}^{12}C_3 \times 2^9}{3^{12}}.$$

(*b*) Take any one biscuit. This can be given to any one of the N beggars so that there are N ways of distributing any one biscuit. Hence the total number of ways in which n biscuits can be distributed at random among N beggars $= N. N\ldots(n \text{ times}) = N^n$.

'r' biscuits can be given to any particular beggar in ${}^nC_r$ ways. Now we are left with $(n - r)$ biscuits which are to be distributed among the remaining $(N - 1)$ beggars and this can be done in $(N - 1)^{n-r}$ ways.

$$\therefore \quad \text{Number of favourable cases} = {}^nC_r \,.\, (N-1)^{n-r}$$

$$\text{Hence, required probability} = \frac{{}^nC_r\,(N-1)^{n-r}}{N^n}.$$

Example 3·15. *A car is parked among N cars in a row, not at either end. On his return the owner finds that exactly r of the N places are still occupied. What is the probability that both neighbouring places are empty ?*

Solution. Since the owner finds on return that exactly r of the N places (including owner's car) are occupied, the exhaustive number of cases for such an arrangement is ${}^{N-1}C_{r-1}$ [since the remaining $r - 1$ cars are to be parked in the remaining $N - 1$ places and this can be done in ${}^{N-1}C_{r-1}$ ways].

Let A denote the event that both the neighbouring places to owner's car are empty. This requires the remaining $(r - 1)$ cars to be parked in the remaining $N - 3$ places and hence the number of cases favourable to A is ${}^{N-3}C_{r-1}$. Hence,

$$P(A) = \frac{^{N-3}C_{r-1}}{^{N-1}C_{r-1}} = \frac{(N-r)(N-r-1)}{(N-1)(N-2)}.$$

Example 3·16. *What is the probability that at least two out of n people have the same birthday ? Assume 365 days in a year and that all days are equally likely.*

Solution. Since the birthday of any person can fall on any one of the 365 days, the exhaustive number of cases for the birthdays of n persons is 365^n.

If the birthdays of all n persons fall on different days, then the number of favourable cases is : $365\,(365-1)\,(365-2)\,\ldots\,[365-(n-1)]$,

because in this case the birthday of the first person can fall on any one of 365 days, the birthday of the second person can fall on any one of the remaining 364 days, and so on. Hence, the probability (p) that birthdays of all the n persons are different is given by :

$$p = \frac{365\,(365-1)\,(365-2)\,\ldots\,\{365-(n-1)\}}{365^n}$$

$$= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right)$$

Hence, the required probability that at least two persons have same birthday is :

$$1-p = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right).$$

Example 3·17. *Compare the chances of throwing 4 with one die, 8 with two dice and 12 with three dice.*

Solution. *(i) Probability of throwing 4 with one die* : There are 6 possible ways in which the die can fall, and of these one is favourable to the required event .

$\therefore$ Required probability $(p_1) = \frac{1}{6}$.

(ii) Probability of throwing 8 with two dice : Exhaustive number of cases in single throw with two dice is $6^2 = 36$. Now the sum of '8' can be obtained on the two dice in the following ways : (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), *i.e.*, 5 cases in all, where the first and second number in the brackets ( ) refer to the numbers on the 1st and 2nd die respectively.

$\therefore$ Required probability $(p_2) = \frac{5}{36}$.

(iii) Probability of throwing 12 with three dice : The exhaustive number of ways in a single throw of three dice $= 6\times6\times6 = 216$.

To make a throw of 12, the three dice must show the faces either (6, 1, 5) or (6, 2, 4) or (6, 3, 3) or (5, 2, 5) or (5, 3, 4) or (4, 4, 4). The first two of these arrangements can occur in $3! = 6$ ways each, the second two (*i.e*, third and fourth arrangement) in $\frac{3!}{2!\,1!} = 3$ ways each, the fifth in $3! = 6$ ways and the last in one way only. Thus, the total number of favourable cases $= 6+6+3+3+6+1 = 25$.

$\therefore$ Required probability $(p_3) = \frac{25}{216}$.

Hence the chances of throwing 4 with one die, 8 with two dice, and 12 with three dice are :

$$p_1 : p_2 : p_3 :: \frac{1}{6} : \frac{5}{36} : \frac{25}{216} \quad \text{or} \quad 36 : 30 : 25.$$

3·7·3. Algebra of Sets. Now we state certain important properties concerning operations on sets. If A, B and C are the subsets of a universal set S, then the following laws hold :

Commutative Laws : $A \cup B = B \cup A, \quad A \cap B = B \cap A$

Associative Law : $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Complementary Law : $A \cup \bar{A} = S, \quad A \cap \bar{A} = \phi$

$A \cup S = S$, and $A \cap S = A \quad (\because A \subset S)$,

$A \cup \phi = A, \quad A \cap \phi = \phi.$

Difference Law : $A - B = A \cap \bar{B}$.

$A - B = A - (A \cap B) = (A \cup B) - B$

$A - (B - C) = (A - B) \cup (A - C)$

$(A \cup B) - C = (A - C) \cup (B - C)$

$A - (B \cup C) = (A - B) \cap (A - C)$

$(A \cap B) \cup (A - B) = A, \quad (A \cap B) \cap (A - B) = \phi$

De-Morgan's Law

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B} \quad \text{and} \quad \overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

More generally

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \bar{A_i} \quad \text{and} \quad \overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \bar{A_i}$$

Involution Law : $\overline{(\bar{A})} = A$

Idempotency Law : $A \cup A = A, \quad A \cap A = A$

3·7·4. Limit of Sequence of Sets. Let $\{A_n\}$ be a sequence of sets in S. The *limit supremum* or *limit superior* of the sequence, usually written as lim sup A_n, is the set of all those elements which belong to A_n for infinitely many n. Thus

$$\lim_{n \to \infty} \sup A_n = \{x : x \in A_n \text{ for infinitely many } n\} \qquad \ldots (3\cdot3)$$

The set of all those elements which belong to A_n for all but a finite number of n is called *limit infinimum or limit inferior* of the sequence and is denoted by lim inf A_n.

TABLE – GLOSSARY OF PROBABILITY TERMS

Statement	*Meaning in terms of set theory*
1. At least one of the events A or B occurs	$\omega \in A \cup B$
2. Both the events A and B occur.	$\omega \in A \cap B$
3. Neither A nor B occurs	$\omega \in \overline{A} \cap \overline{B}$
4. Event A occurs and B does not occur.	$\omega \in A \cap \overline{B}$
5. Exactly one of the events A or B occurs.	$\omega \in A \Delta B$
6. Not more than one of the events A or B occurs.	$\omega \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$
7. If event A occurs, so does B.	$A \subset B$
8. Events A and B are mutually exclusive.	$A \cap B = \phi$
9. Complementary event of A.	$\overline{A}$
10. Sample space	Universal set S.

Example 3·23. *A, B and C are three arbitrary events. Find expression for the events noted below, in the context of A, B and C :*

(i) Only A occurs,
(ii) Both A and B, but not C, occur,
(iii) All three events occur,
(iv) At least one occurs,
(v) At least two occur,
(vi) One and no more occurs,
(vii) Two and no more occur,
(viii) None occurs.

Solution.

(i) $A \cap \overline{B} \cap \overline{C}$, (ii) $A \cap B \cap \overline{C}$, (iii) $A \cap B \cap C$ (iv) $A \cup B \cup C$,

(v) $A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$

(viii) $(\overline{A} \cap \overline{B} \cap \overline{C})$ or $\overline{A \cup B \cup C}$

3·9. SOME THEOREMS ON PROBABILITY

In this section, we shall prove a few simple theorems which help us evaluate the probabilities of some complicated events in a rather simple way. In proving these theorems, we shall follow the axiomatic approach, based on the three axioms, discussed in § 3·8·5.

Theorem 3·2. *Probability of the impossible event is zero, i.e.,* $P(\phi) = 0$. ... (3·8)

Proof. Impossible event contains no sample point and hence the certain event S and the impossible event ϕ are mutually exclusive.

$$\therefore \quad S \cup \phi = S \quad \Rightarrow \quad P(S \cup \phi) = P(S)$$

Hence, using Axiom 2 of probability, *i.e.,* Axiom of Additivity, we get

$$P(S) + P(\phi) = P(S) \quad \Rightarrow \quad P(\phi) = 0$$

Remark. It may be noted $P(A) = 0$, does not imply that A is necessarily an empty set. In practice, probability '0' is assigned to the events which are so rare that they happen only once in a lifetime. For example, if a person who does not know typing is asked to type one page of the manuscript of a book, the probability of the event that he will type it correctly without any mistake is 0.

As another illustration, let us consider the random tossing of a coin. The event that the coin will stand erect on its edge, is assigned the probability 0.

The study of continuous random variable provides another illustration to the fact that $P(A) = 0$, does not imply $A = \phi$, because in case of continuous random variable X, the probability at a point is always zero, *i.e.*, $P(X = c) = 0$ {See Chapter 5].

Theorem 3·3. *Probability of the complementary event $\overline{A}$ of A is given by*

$$P(\overline{A}) = 1 - P(A). \qquad \ldots (3·9)$$

Proof. A and $\overline{A}$ are mutually disjoint events, so that

$$A \cup \bar{A} = S \quad \Rightarrow \quad P(A \cup \bar{A}) = P(S)$$

Hence, from Axioms 2 and 3 of probability, we have

$$P(A) + P(\overline{A}) = P(S) = 1 \quad \Rightarrow \quad P(\overline{A}) = 1 - P(A)$$

Cor. 1. We have $P(A) = 1 - P(\overline{A}) \le 1$ $\quad [\because P(\overline{A}) \ge 0$, by Axiom 1]

Further, since $P(A) \ge 0$ (Axiom 1)

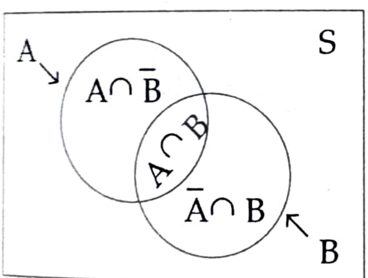
$$\therefore \qquad 0 \le P(A) \le 1. \qquad \ldots (3·9a)$$

Cor. 2. $P(\phi) = 0$, since $\phi = \overline{S}$ and $P(\phi) = P(\overline{S}) = 1 - P(S) = 1 - 1 = 0$.

Theorem 3·4. *For any two events A and B, we have*

(*i*) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$ $\qquad$ (*ii*) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Proof. From the Venn diagram, we get $\quad B = (A \cap B) \cup (\overline{A} \cap B)$,



where $\overline{A} \cap B$ and $A \cap B$ are disjoint events.

Hence by Axiom (3), we get

$$P(B) = P(A \cap B) + P(\overline{A} \cap B)$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B) \qquad \ldots (3·10)$$

(*ii*) Similarly, we have $\qquad A = (A \cap B) \cup (A \cap \overline{B})$,

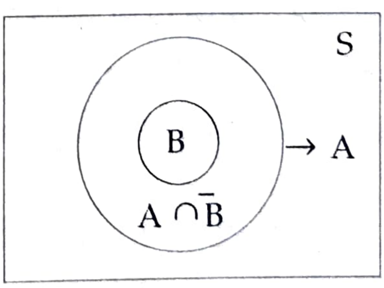
where $(A \cap B)$ and $A \cap \overline{B}$ are disjoint events. Hence, by Axiom 3, we get

$$P(A) = P(A \cap B) + P(A \cap \overline{B}) \quad \Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(A \cap B) \qquad \ldots (3·11)$$

Theorem 3·5. *If* $B \subset A$, *then*

(*i*) $P(A \cap \overline{B}) = P(A) - P(B)$,

(*ii*) $P(B) \le P(A)$



Proof. (*i*) When $B \subset A$, B and $A \cap \overline{B}$ are mutually exclusive events so that $A = B \cup (A \cap \overline{B})$

$$\Rightarrow \quad P(A) = P[B \cup (A \cap \overline{B})]$$

$$= P(B) + P(A \cap \overline{B}) \qquad \text{(By Axiom 3)}$$

$$\Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(B)$$

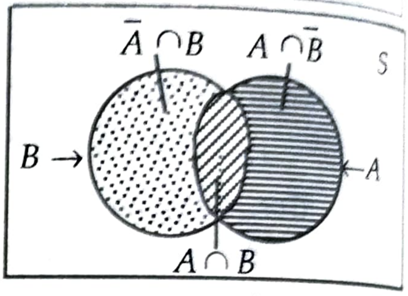
(ii) $P(A \cap \overline{B}) \geq 0 \Rightarrow P(A) - P(B) \geq 0 \Rightarrow P(B) \leq P(A)$

Hence $B \subset A \Rightarrow P(B) \leq P(A)$ ...(3·12)

3·9·1. Addition Theorem of Probability

Theorem 3·6. *If A and B are any two events (subsets of sample space S) and are not disjoint, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{...(3·13)}$$



Proof. From the Venn diagram, we have

$$A \cup B = A \cup (\overline{A} \cap B),$$

where A and $\overline{A} \cap B$ are mutually disjoint.

$\therefore \quad P(A \cup B) = P[A \cup (\overline{A} \cap B)]$

$= P(A) + P(\overline{A} \cap B)$ [By Axiom 3] ...(*)

$= P(A) + P(B) - P(A \cap B)$ [From Theorem 3·3 (i)]

OR From (*) onwards.

$P(A \cup B) = P(A) + [P(\overline{A} \cap B) + P(A \cap B)] - P(A \cap B)$

$= P(A) + P[(\overline{A} \cap B) \cup (A \cap B)] - P(A \cap B)$

[$\because (\overline{A} \cap B)$ and $(A \cap B)$ are disjoint]

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Aliter. $P(A \cup B) = \dfrac{n(A \cup B)}{n(S)} = \dfrac{n(A) + n(B) - n(A \cap B)}{n(S)}$

$= \dfrac{n(A)}{n(S)} + \dfrac{n(B)}{n(S)} - \dfrac{n(A \cap B)}{n(S)} = P(A) + P(B) - P(A \cap B)$

Cor. 1. If the events A and B are mutually disjoint, then

$$A \cap B = \phi \Rightarrow P(A \cap B) = P(\phi) = 0$$

$\therefore \quad P(A \cup B) = P(A) + P(B)$, which is Axiom 3 of probability.

Cor. 2. For three non-mutually exclusive events A, B and C, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C). \quad \text{(3·13a)}$$

Proof. $P(A \cup B \cup C) = P[A \cup (B \cup C)]$

$= P(A) + P(B \cup C) - P[A \cap (B \cup C)]$ [From (3·13)]

$= P(A) + [P(B) + P(C) - P(B \cap C)] - P[(A \cap B) \cup (A \cap C)]$

$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$

3·9·2. Extension of Addition Theorem of Probability to n Events.

Theorem 3·7. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k)$$
$$+ \ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n) \qquad \ldots (3{\cdot}14)$$

Proof. For two events A_1 and A_2, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \qquad (*)$$

Hence (3·14) is true for $n = 2$.

Let us now suppose that (3·14) is true for $n = r$, (say) so that

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P(A_i) - \sum\sum_{i \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r) \ldots (**)$$

Now

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left\{\left(\bigcup_{i=1}^{r} A_i\right) \cup A_{r+1}\right\}$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right)\right\} \qquad \ldots [\text{Using } (*)]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right\} \qquad \text{(By Distributive Law)}$$

$$= \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$
$$+ P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right\} \qquad [\text{From } (**)]$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$
$$- \left\{\sum_{i=1}^{r} P(A_i \cap A_{r+1}) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j \cap A_{r+1})\right.$$
$$\left. + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r \cap A_{r+1})\right\} \qquad \ldots [\text{From } (**)]$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r+1} A_i\right) = \sum_{i=1}^{r+1} P(A_i) - \left\{\sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \sum_{i=1}^{r} P(A_i \cap A_{r+1})\right\}$$
$$+ \ldots + (-1)^r P\{(A_1 \cap A_2 \cap \ldots \cap A_{r+1}\}$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le (r+1)} P(A_i \cap A_j) + \ldots + (-1)^r P(A_1 \cap A_2 \cap \ldots \cap A_{r+1})$$

Hence if (3·14) is true for $n = r$, it is also true for $n = (r+1)$. But we have proved in (*) that (3·14) is true for $n = 2$. Hence by the principle of mathematical induction, it follows that (3·14) is true for all positive integral values of n.

Example 3·24. *A letter of the English alphabet is chosen at random. Calculate the probability that the letter so chosen*

(i) is a vowel, (ii) precedes m and is a vowel, (iii) follows m and is a vowel.

Solution. The sample space of the experiment is :

$$S = (a, b, c, d, \ldots, x, y, z), \qquad n(S) = 26.$$

(*i*) Let E_1 be the event that the letter chosen is a vowel, Then

$$E_1 = \{a, e, i, o, u\}; \quad n(E_1) = 5 \qquad \therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{5}{26}$$

(ii) Let E_2 be the event that the letter precedes m and is a vowel, Then

$$E_2 = \{a, e, i\}\ ; \qquad n(E_2) = 3 \qquad \therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{26}$$

(iii) Let E_3 be the event that the letter follows m and is a vowel. Then,

$$E_3 = \{o, u\}\ ; \qquad n(E_3) = 2 \qquad \therefore \quad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{26} = \frac{1}{13}$$

Example 3·25. *Five salesmen of B, C, D and E of a company are considered for a three-member trade delegation to represent the company in an international trade conference. Construct the sample space and find the probability that :*

(i) A is selected. (ii) A is not selected, and (iii) Either A or B (not both) is selected. (Assume the natural assignment of probability.)

Solution. The sample space for selecting three salesmen out of 5 salesmen A, B, C, D and E for the trade delegation is given by :

$$S = \{ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE\} \Rightarrow n(S) = 10$$

Under the assumption of natural assignment of probabilities, each of these outcomes (elementary events) has an equal chance of being selected.

Let us define the following events :

E_1 : A is selected and E_2 : A or D (not both) is selected.

(i) $E_1 = (ABC, ABD, ABE, ACD, ACE, ADE\} \Rightarrow n(E_1) = 6.$

$$\therefore \qquad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{10} = \frac{3}{5}$$

(ii) $\overline{E_1}$ = A is not selected = $\{BCD, BCE, BDE, CDE\} \Rightarrow n(E_2) = 4$

$$\therefore \qquad P(\overline{E_1}) = \frac{n(\overline{E_1})}{n(S)} = \frac{4}{10} = \frac{2}{5} \quad \text{or} \quad P(\overline{E_1}) = 1 - P(E_1) = \frac{2}{5}.$$

(iii) $E_2 = \{ABC, ABE, ACE, BCD, BDE, CDE\} \Rightarrow n(E_2) = 6$

$$\therefore \qquad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{10} = \frac{3}{5}.$$

Important Remark. In all the problems that follow, we shall always assume natural assignment of probabilities to the elementary events, unless specified otherwise.

Example 3·26. *A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A) given that :*

$$P(B) = \frac{3}{2} P(A) \quad \text{and} \quad P(C) = \frac{1}{2} P(B)$$

Solution. Let $P(A) = p$, then $P(B) = \frac{3}{2}p$ and $P(C) = \frac{1}{2} \times \frac{3}{2} p = \frac{3}{4} p$.

Since A, B, C are mutually exclusive and exhaustive events,

$$P(A) + P(B) + P(C) = 1 \Rightarrow p + \tfrac{3}{2}p + \tfrac{3}{4}p = 1 \Rightarrow \tfrac{13}{4}p = 1 \Rightarrow p = \tfrac{4}{13}.$$

Example 3·27. *If* $p_1 = P(A)$, $p_2 = P(B)$, $p_3 = P(A \cap B)$, $(p_1, p_2, p_3 > 0)$, *express the following in terms of* p_1, p_2, p_3

(a) $P(\overline{A \cup B})$, (b) $P(\overline{A} \cup \overline{B})$, (c) $P(\overline{A} \cap B)$, (d) $P(\overline{A} \cup B)$, (e) $P(\overline{A} \cap \overline{B})$

(f) $P(A \cap \overline{B})$, (g) $P(A \mid B)$, (h) $P(B \mid \overline{A})$, (i) $P[\overline{A} \cap (A \cup B)]$.

Solution.

(a) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - p_1 - p_2 + p_3.$

(b) $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - p_3$

(c) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = p_2 - p_3$

(d) $P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B) = 1 - p_1 + p_2 - (p_2 - p_3) = 1 - p_1 + p_3$

(e) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - p_1 - p_2 + p_3.$ [Part (a)]

(f) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = p_1 - p_3$

(g) $P(A \mid B) = P(A \cap B)/P(B) = p_3/p_2$

(h) $P(B \mid \overline{A}) = P(\overline{A} \cap B)/P(\overline{A}) = (p_2 - p_3)/(1 - p_1)$ [Part (c)]

(i) $P(\overline{A} \cap (A \cup B)] = P[(\overline{A} \cap A) \cup (\overline{A} \cap B)] = P(\overline{A} \cap B) = p_2 - p_3$ $[\because A \cap \overline{A} = \phi]$

Example 3·28. *Let $P(A) = p$, $P(A \mid B) = q$, $P(B \mid A) = r$. Find relations between the number p, q, r for the following cases :*

(a) Events A and B are mutually exclusive.

(b) A and B are mutually exclusive and collectively exhaustive.

(c) A is sub-event of B ; B is a sub-event of A.

(d) $\overline{A}$ and $\overline{B}$ are mutually exclusive.

Solution. From the given data : $P(A) = p,\ P(A \cap B) = P(A)\,P(B \mid A) = rp$

$$\therefore \quad P(B) = \frac{P(A \cap B)}{P(A \mid B)} = \frac{rp}{q} \quad \text{and} \quad P(A) + P(B) = p + \frac{rp}{q} = \frac{p(q+r)}{q}$$

(a) Since A and B are mutually exclusive

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0 \quad \Rightarrow \quad rp = 0.$$

(b) Since A and B are mutually exclusive and collectively exhaustive,

$$P(A \cap B) = 0 \quad \text{and} \quad P(A) + P(B) = 1$$

$$\Rightarrow \quad p(q + r) = q\,;\, rp = 0 \quad \text{or} \quad pq = q \quad \Rightarrow \quad p = 1 \quad \text{or} \quad q = 0$$

(c) $A \subseteq B \Rightarrow A \cap B = A$ or $P(A \cap B) = P(A) \Rightarrow rp = p$, *i.e.*, $r = 1$ or $p = 0$.

$B \subseteq A \Rightarrow A \cap B = B$ or $P(A \cap B) = P(B)$

$$\Rightarrow \quad rp = (rp/q) \Rightarrow rp(q-1) = 0 \quad \Rightarrow \quad r = 0 \quad \text{or} \quad p = 0.$$

(d) Since $\overline{A}$ and $\overline{B}$ are mutually exclusive, $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 0$

$$\Rightarrow \quad 1 - [P(A) + P(B) - P(A \cap B)] = 0 \quad \Rightarrow \quad P(A) + P(B) = 1 + P(A \cap B)]$$

$$\Rightarrow \quad p[1 + (r/q)] = 1 + rp \quad \Rightarrow \quad p(q + r) = q(1 + pr).$$

Example 3·29. *A die is loaded in such a manner that for $n = 1, 2, 3, 4, 5, 6$, the probability of the face marked n, landing on top when the die is rolled is proportional to n. Find the probability that an odd number will appear on tossing the die.*

Solution. Here we are given :

$P(n) \propto n \Rightarrow P(n) = kn$ ... (*), where k is the constant of proportionality.

Also $P(1) + P(2) + \ldots + P(6) = 1 \Rightarrow k(1+2+3+4+5+6) = 1 \Rightarrow k = \frac{1}{21}$

Required Probability $= P(1) + P(3) + P(5) = \frac{1+3+5}{21} = \frac{3}{7}$ [Using (*)]

Example 3·30. *If two dice are thrown, what is the probability that the sum is (a) greater than 8, and (b) neither 7 nor 11 ?*

Solution. (a) If S denotes the sum on the two dice, then we want $P(S > 8)$.

The required event can happen in the following mutually exclusive ways :

(i) $S = 9$ (ii) $S = 10$ (iii) $S = 11$ (iv) $S = 12$.

Hence by addition theorem of probability

$$P(S > 8) = P(S = 9) + P(S = 10) + P(S = 11) + P(S = 12) \quad \ldots (*)$$

In a throw of two dice, the sample space contains $6^2 = 36$ points. The number of favourable cases can be enumerated as follows :

$S = \mathbf{9}$: (3, 6), (6, 3), (4, 5), (5, 4), *i.e.*, 4 sample points $\therefore P(S = 9) = \frac{4}{36}$

$S = \mathbf{10}$: (4, 6), (6, 4), (5, 5), *i.e.*, 3 sample points. $\therefore P(S = 10) = \frac{3}{36}$

$S = \mathbf{11}$: (5, 6), (6, 5), *i.e.*, 2 sample points $\therefore P(S = 11) = \frac{2}{36}$

$S = \mathbf{12}$: (6, 6), *i.e.*, 1 sample point. $\therefore P(S = 12) = \frac{1}{36}$

$$\therefore \quad P(S > 8) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18} \quad \text{[From (*)]}$$

(b) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11 with a pair of dice.

$S = \mathbf{7}$: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), *i.e.*, 6 distinct sample points.

$$\therefore \quad P(A) = P(S = 7) = \frac{6}{36} = \frac{1}{6}$$

$S = \mathbf{11}$: (5, 6), (6, 5), *i.e.*, 2 distinct sample points.

$$\therefore \quad P(B) = P(S = 11) = \frac{2}{36} = \frac{1}{18}$$

$$\therefore \text{ Required probability} = P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B)] \quad (\because A \text{ and } B \text{ are disjoint events.})$$
$$= 1 - \frac{1}{6} - \frac{1}{18} = \frac{7}{9}.$$

Example 3·31. *Two dice are tossed. Find the probability of getting 'an even number on the first die or a total of 8'.*

Solution. In a random toss of two dice, sample space S is given by :

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6 \times 6 = 36$$

Let us define the events :

A : Getting an even number on the first dice

B : The sum of the points obtained on the two dice 8.

These events are represented by the following subsets of S.

$$A = \{2, 4, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(A) = 3 \times 6 = 18$$

$$B = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\} \quad \Rightarrow \quad n(B) = 5$$

Also $A \cap B = \{(2, 6), (6, 2), (4, 4)\} \quad \Rightarrow \quad n(A \cap B) = 3.$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}, \; P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}, \text{ and } P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

Hence, the required probability is given by :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}.$$

Example 3·32. *An integer is chosen at random from two hundred digits. What is the probability that the integer is divisible by 6 or 8 ?*

Solution. The sample space of the random experiment is :

$$S = \{1, 2, 3, \ldots, 199, 200\} \quad \Rightarrow \quad n(S) = 200$$

The event A : 'integer chosen is divisible by 6' has the sample points given by :

$$A = \{6, 12, 18, \ldots, 198\} \quad \Rightarrow \quad n(A) = \frac{198}{6} = 33. \quad \therefore \; P(A) = \frac{n(A)}{n(S)} = \frac{33}{200}$$

Similarly the event B : 'integer chosen is divisible by 8' has the sample points given by :

$$B = \{8, 16, 24, \ldots, 200\} \quad \Rightarrow \quad n(B) = \frac{200}{8} = 25 \; \therefore \; P(B) = \frac{n(B)}{n(S)} = \frac{25}{200}$$

The LCM of 6 and 8 is 24. Hence, a number is divisible by both 6 and 8, if it is divisible by 24.

$$\therefore \quad A \cap B = \{24, 48, 72, \ldots, 192\} \Rightarrow n(A \cap B) = \frac{192}{24} = 8 \quad \Rightarrow \quad P(A \cap B) = \frac{8}{200}$$

Hence, the required probability is :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{1}{4}.$$

Example 3·33. *The probability that a student passes a Physics test is $\frac{2}{3}$ and the probability that he passes both a Physics test and an English test is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that he passes the English test ?*

Solution. Let us define the following events :

A : The student passes a Physics test ; B : The student passes an English test

In the usual notations, we are given :

$$P(A) = \frac{2}{3}, \; P(A \cap B) = \frac{14}{45}, \quad P(A \cup B) = \frac{4}{5} \text{ and we want, } P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \Rightarrow \quad \frac{4}{5} = \frac{2}{3} + P(B) - \frac{14}{45}$$

$$\therefore \quad P(B) = \frac{4}{5} + \frac{14}{45} - \frac{2}{3} = \frac{36 + 14 - 30}{45} = \frac{4}{9}.$$

Example 3·34. *An investment consultant predicts that the odds against the price of a certain stock will go up during the next week are 2 : 1 and the odds in favour of the price* *remaining the same are 1 : 3. What is the probability that the price of the stock will go down during the next week ?*

Solution. Let A denote the event that 'stock price will go up', and B be the event 'stock price will remain same'. Then $P(A) = \frac{1}{2+1} = \frac{1}{3}$ and $P(B) = \frac{1}{1+3} = \frac{1}{4}$.

$\therefore$ P(stock price will either go up or remain same) is given by :

$$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Hence, the probability that stock price will go down is given by :

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}.$$

Example 3·35. *An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0·7 and being rejected at Y is 0·5. The probability of at least one of his applications being rejected is 0·6. What is probability that he will be selected in one of the firms ?*

Solution. Let A and B denote the events that the person is selected in firms X and Y respectively. Then in the usual notations, we are given :

$$\left.\begin{aligned} P(A) &= 0{\cdot}7 \quad \Rightarrow \quad P(\bar{A}) = 1 - 0{\cdot}7 = 0{\cdot}3 \\ P(\bar{B}) &= 0{\cdot}5 \quad \Rightarrow \quad P(B) = 1 - 0{\cdot}5 = 0{\cdot}5 \end{aligned}\right\} \quad \ldots (*)$$

and $$P(\bar{A} \cup \bar{B}) = 0{\cdot}6 = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B}) \quad \ldots (**)$$

The probability that the persons will be selected in one of the two firms X or Y is given by :

$$\begin{aligned} P(A \cup B) &= 1 - P(\bar{A} \cap \bar{B}) = 1 - \{P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cup \bar{B})\} && \text{[From (**)]} \\ &= 1 - (0{\cdot}3 + 0{\cdot}5 - 0{\cdot}6) = 0{\cdot}8. && \text{[From (*)]} \end{aligned}$$

Example 3·36. *Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult population : 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers ?*

Solution. Let E, F and G denote the events that the adult reads newspapers A, B and C respectively. Then we are given :

$$P(E) = \frac{20}{100}, \quad P(F) = \frac{16}{100}, \quad P(G) = \frac{14}{100}, \quad P(E \cap F) = \frac{8}{100}$$

$$P(E \cap G) = \frac{5}{100}, \quad P(F \cap G) = \frac{4}{100}, \quad \text{and} \quad P(E \cap F \cap G) = \frac{2}{100}$$

The required probability that an adult reads at least one of the newspapers (by addition theorem) is given by :

$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(E \cap G) + P(E \cap F \cap G) \\ &= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{4}{100} - \frac{5}{100} + \frac{2}{100} = \frac{35}{100} = 0{\cdot}35 \end{aligned}$$

Hence 35% of the adult population reads at least one of the newspapers.

Example 3·37. *A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.*

Solution. Let us define the following events :

A : the card drawn is a king ; B : the card drawn is a heart ;

C : the card drawn is a red card.

Then A, B and C are not mutually exclusive.

$A \cap B$: the card drawn is the king of hearts $\Rightarrow$ $n(A \cap B) = 1$

$B \cap C = B$: the card drawn is a heart $(\because B \subset C)$ $\Rightarrow$ $n(B \cap C) = 13$

$C \cap A$: the card drawn is a red king $\Rightarrow$ $n(C \cap A) = 2$

$A \cap B \cap C = A \cap B$: the card drawn is the king of hearts $\Rightarrow n(A \cap B \cap C) = 1$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}; \quad P(B) = \frac{13}{52}; \quad P(C) = \frac{26}{52}$$

$$P(A \cap B) = \frac{1}{52}; \quad P(B \cap C) = \frac{13}{52}; \quad P(C \cap A) = \frac{2}{52}; \quad P(A \cap B \cap C) = \frac{1}{52}$$

The required probability of getting a king or heart or a red card is given by :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}.$$

3·10. CONDITIONAL PROBABILITY

As discussed earlier, the probability $P(A)$ of an event A represents the likelihood that a random experiment will result in an outcome in the set A relative to the sample space S of the random experiment. However, quite often, while evaluating some event probability, we already have some information stemming from the experiment. For example, if we have prior information that the outcome of the random experiment must be in a set B of S, then this information must be used to re-appraise the likelihood that the outcome will also be in B. This re-appraised probability is denoted by $P(A \mid B)$ and is read as the conditional probability of the event A, given that the event B has already happened.

We give below some illustrations to explain this concept.

Illustrations 1. Let us consider a random experiment of drawing a card from a pack of cards. Then the probability of happening of the event A : "The card drawn is a king", is given by :
$$P(A) = \frac{4}{52} = \frac{1}{13}.$$

Now suppose that a card is drawn and we are informed that the drawn card is red. How does this information affect the likelihood of the event A ?

Obviously, if the event B : 'The card drawn is red', has happened, the event 'Black card' is not possible. Hence the probability of the event A must be computed relative to the new sample space 'B' which consists of 26 sample points (red cards only), *i.e.*, $n(B) = 26$. Among these 26 red cards, there are two (red) kings so that $n(A \cap B) = 2$. Hence, the required probability is given by :

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{26} = \frac{1}{13}.$$

2. Consider a random experiment of tossing three fair coins. Then, as explained earlier, the sample space S is :

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$
$$= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \text{ (say)},$$

so that : $$P(\{\omega_i\}) = \frac{1}{8}; i = 1, 2, \ldots 8.$$

Now suppose that the same experiment is performed by another person and nothing is known about its outcome. However, we have the information that he obtained 'at least two heads'. We are interested to find how this additional information affects the probabilities of the elementary outcomes.

This means that if the event A : 'At least two heads are obtained', has happened then the elementary outcomes ω_4, ω_6, ω_7 and ω_8 could not have happened. However, the remaining four outcomes ω_1, ω_2, ω_3, and ω_5 are still possible and we assign the probability $\frac{1}{4}$ to each one of them.

$$\therefore \quad P(\omega_1 \mid A) = P(\omega_2 \mid A) = P(\omega_3 \mid A) = P(\omega_5 \mid A) = \frac{1}{4}.$$

From the above illustrations we observe that some additional information may change the probability of the happening of some event. We now proceed to develop procedure to calculate the probabilities of events when we know some additional information.

Remark. When we know that a particular event B has occurred, instead of S, we concentrate our attention on B only and the conditional probability of A given B will be analogously the ratio of the probability of that part of A which is included in B (*i.e.*, $A \cap B$) to the probability of B. It, therefore, reflects the change of viewpoint only, namely, instead of S we have to concentrate on B only.

3·11. MULTIPLICATION THEOREM OF PROBABILITY

Theorem 3·9. *For two events A and B,*

$$\left.\begin{aligned} P(A \cap B) &= P(A) \,.\, P(B \mid A), \quad P(A) > 0 \\ &= P(B) \,.\, P(A \mid B), \quad P(B) > 0 \end{aligned}\right\} \qquad \ldots (3\cdot17)$$

where $P(B \mid A)$ represents conditional probability of occurrence of B when the event A has already happened and $P(A \mid B)$ is the conditional probability of happening of A, given that B has already happened.

Proof. In the usual notations, we have

$$P(A) = \frac{n(A)}{n(S)}; \qquad P(B) = \frac{n(B)}{n(S)}, \quad \text{and} \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad \ldots (*)$$

For the conditional event $A \mid B$, the favourable outcomes must be one of the sample points of B, *i.e.*, for the event $A \mid B$, the sample space is B and out of the $n(B)$ sample points, $n(A \cap B)$ pertain to the occurrence of the event A. Hence

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (*), we get

$$P(A \cap B) = \frac{n(B)}{n(S)} \times \frac{n(A \cap B)}{n(B)} = P(B) \,.\, P(A \mid B) \qquad \ldots (**)$$

Similarly, we get from (*) :

$$P(A \cap B) = \frac{n(A)}{n(S)} \times \frac{n(A \cap B)}{n(A)} = P(A) \,.\, P(B \mid A) \qquad \ldots (***)$$

From (**) and (***), we get the result (3·17).

Thus, we have proved that "*the probability of the simultaneous occurrence of two events*