correspondence between themselves ultimately solved this problem and this correspondence laid the first foundation of the science of probability. Next stalwart in this field was J. Bernoulli (1654-1705) whose *'Treatise on Probability'* was published posthumously by his nephew N. Bernoulli in 1713. De-Moivre (1667-1754) also did considerable work in this field and published his famous *'Doctrine of Chances'* in 1718. Other main contributors are : T. Bayes (Inverse probability), P.S. Laplace (1749-1827) who after extensive research over a number of years finally published *'Theoric analytique des probabilities'* in 1812. In addition to these, other outstanding contributors are Levy, Mises and R.A. Fisher.

Russian mathematician also have made very valuable contributions to the modern theory of probability. Chief contributors, to mention only a few of them are : Chebychev (1821-94) who founded the Russian School of Statisticians; A. Markoff (1856-1922); Liapounoff (Central Limit Theorem) ; A. Khintchine (Law of Large Numbers) and A. Kolmogorov, who axiomised the calculus of probability.

3·3. BASIC TERMINOLOGY

In this section we shall explain the various terms which are used in the definition of probability under different approaches.

1. *Random Experiment.* *If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a random experiment.*

Examples of random experiments are : tossing a coin, throwing a die, selecting a card from a pack of playing cards, selecting a family out of a given group of families, etc. In all these cases, there are a number of possible results which can occur but there is an uncertainty as to which one of them will actually occur.

Notes : (*i*) A *die* is a small cube used in gambling. On its six faces, dots are marked as

Plural of die is *dice*. The outcome of throwing a die is the number of dots on its uppermost face.

(*ii*) A *pack of cards* consists of four suits called *Spades, Hearts, Diamonds* and *Clubs*. Each suit consists of 13 cards, of which nine cards are numbered from 2 to 10, an ace, king, a queen and a jack (or knave). Spades and clubs are black-faced cards, while hearts and diamonds are red-faced cards.

2. *Outcome.* The result of a random experiment will be called an *outcome*.

3. *Trial and Event.* Any particular performance of a random experiment is called a *trial* and outcome or combination of outcomes are termed as *events*. For example,

(*i*) If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces, head or tail. Thus tossing of a coin is a random experiment or trial and getting of a head or tail is an event.

(*ii*) In an experiment which consists of the throw of a six-faced die and observing the number of points that appear, the possible outcomes are 1, 2, 3, 4, 5, 6

In the same experiment, the possible events could also be stated as

'Odd number of points' ; *'Even number of points'*; *'Getting a point greater than 4'*; and so on.

Event is called *simple* if it corresponds to a single possible outcome of the experiment otherwise it is known as a *compound* or *composite* event. Thus in tossing of a single die the event of getting '6' is a simple event but the event of getting an even number is a composite event.

4. ***Exhaustive Events or Cases.*** The total number of possible outcomes of a random experiment is known as the *exhaustive events or cases*. For example,

(*i*) In tossing of a coin, there are two exhaustive cases, *viz*., head and tail (the possibility of the coin standing on an edge being ignored).

(*ii*) In throwing of a die, there are 6 exhaustive cases since any one of the 6 faces 1, 2, ..., 6 may come uppermost.

(*iii*) In drawing two cards from a pack of cards, the exhaustive number of cases is ${}^{52}C_2$, since 2 cards can be drawn out of 52 cards in ${}^{52}C_2$ ways.

(*iv*) In throwing of two dice, the exhaustive number of cases is $6^2 = 36$, since any of the numbers 1 to 6 on the first die can be associated with any of the 6 numbers on the other die. In general, in throwing of n dice, the exhaustive number of cases is 6^n.

5. ***Favourable Events or Cases.*** The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event. For example,

(*i*) In drawing a card from a pack of cards the number of cases favourable to drawing of an ace is 4, for drawing a spade is 13 and for drawing a red card is 26.

(*ii*) In throwing of two dice, the number of cases favourable to getting the sum 5 is :

$$(1, 4), (4, 1), (2, 3), (3, 2), \textit{i.e.}, 4.$$

6. ***Mutually Exclusive Events.*** Events are said to be *mutually exclusive* or *incompatible* if the happening of any one of them precludes the happening of all the others, *i.e.*, if no two or more of them can happen simultaneously in the same trial. For example,

(*i*) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of others, in the same trial, is ruled out.

(*ii*) Similarly in tossing a coin the events head and tail are mutually exclusive.

7. ***Equally Likely Events.*** Outcomes of trial are said to be *equally likely* if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others. For example,

(*i*) In a random toss of an unbiased or uniform coin, head and tail are equally likely events.

(*ii*) In throwing an unbiased die, all the six faces are equally likely to come.

8. ***Independent Events.*** Several events are said to be independent if the happening (or non-happening) of an event is not affected by the supplementary knowledge concerning the occurrence of any number of the remaining events. For example,

(*i*) In tossing an unbiased coin, the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(*ii*) When a die is thrown twice, the result of the first throw does not affect the result of the second throw.

(*iii*) If we draw a card from a pack of well-shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second draw is dependent on the first draw.

3·4. MATHEMATICAL (OR CLASSICAL OR 'A PRIORI') PROBABILITY

Definition. *If a random experiment or a trial results in 'n' exhaustive, mutually exclusive and equally likely outcomes (or cases), out of which m are favourable to the occurrence of an event E, then the probability 'p' of occurrence (or happening) of E, usually denoted by P(E), is given by :*

$$p = P(E) = \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}} = \frac{m}{n} \qquad \dots (3\cdot1)$$

This definition was given by James Bernoulli who was the first person to obtain a quantitative measure of uncertainty.

Remarks 1. Since $m \geq 0, n > 0$ and $m \leq n$, we get from (3·1) :

$$P(E) \geq 0 \text{ and } P(E) \leq 1 \quad \Rightarrow \quad 0 \leq P(E) \leq 1$$

2. Sometimes we express (3·1) by saying that 'the odds in favour of E are $m : (n - m)$ or the odds against E are $(n - m) : m$.

3. The non-happening of the event E is called the *complementary event of* E and is denoted by $\overline{E}$ or E^c. The number of cases favourable to $\overline{E}$, *i.e.*, non-happening of E is $(n - m)$. Then the probability q that E will not happen is given by :

$$q = P(\overline{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p \quad \Rightarrow \quad p + q = 1 \qquad \dots (3\cdot1a)$$

$$\therefore \quad q = P(\overline{E}) = 1 - P(E) \quad \Rightarrow \quad P(E) = 1 - P(\overline{E}) \quad \text{or} \quad P(E) + P(\overline{E}) = 1 \qquad \dots (3\cdot1b)$$

If the event E represents the happening of at least one of the events $E_1, E_2, \dots, E_n$, then its complementary event $\overline{E}$ represents the happening of none of the events $E_1, E_2, \dots, E_n$. Substituting in (3·1 *b*), we get

$$P\left(\text{Happening of at least one of the events } E_1, E_2, \dots, E_n\right)$$
$$= 1 - P\left(\text{None of the events } E_1, E_2, \dots, E_n \text{ happens}\right) \qquad \dots (3\cdot1c)$$

4. Probability 'p' of the happening of an event is also known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure, *i.e.*, $(p + q = 1)$.

5. If $P(E) = 1$, E is called a *certain event* and if $P(E) = 0$, E is called an *impossible* event.

6. We can compute the probability in (3·1) by logical reasoning, without conducting any experiment. Since, the probability in (3·1) can be computed prior to obtaining any experimental data, it is also termed as *'a priori'* or *mathematical probability*.

3·4·1. Limitations of Classical Definition. This definition of classical probability breaks down in the following cases :

(*i*) If the various outcomes of the random experiment are not equally likely or equally probable. For example,

(*a*) The probability that a candidate will pass in a certain test is not 50%, since the two possible outcomes, *viz.*, success and failure (excluding the possibility of a compartment) are not equally likely.

(*b*) The probability that a ceiling fan in a room will fall is not 1/2, since the events of the fan 'falling' and 'not falling' though mutually exclusive and exhaustive, are not equally likely. In fact, the probability of the fan falling will be almost zero.

(*c*) If a person jumps from a running train, then the probability of his survival will not be 50%, since in this case the events survival and death, though exhaustive and mutually exclusive, are not equally likely.

(*ii*) If the exhaustive number of outcomes of the random experiment is infinite or unknown.

3.5. STATISTICAL (OR EMPIRICAL) PROBABILITY

Definition. (VON MISES). *If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.*

Symbolically, if in N trials an event E happens M times, then the probability of the happening of E, denoted by $P(E)$, is given by :

$$P(E) = \lim_{N \to \infty} \frac{M}{N} \qquad \ldots (3{\cdot}2)$$

Remarks 1. Since in the relative frequency approach, the probability is obtained objectively by repetitive empirical observations, it is also known as '*Empirical Probability*'.

2. An experiment is unique and non-repeating only in the case of *subjective probability*. In other cases, there are a large number of experiments or trials to establish the chance of occurrence of an event. This is particularly so in case of empirical probability. In classical probability also, repeated experiments may be made to verify whether a deduction on the basis of certain axioms or undisputed laws is justified. Only after repeated trials it can be established that the chance of head in a toss of a coin is 1/2. J. E. Kerrich conducted coin tossing experiment with 10 sets of 1,000 tosses each during his confinement in World War II. The number of heads found by him were : 502, 511, 497, 529, 504, 476, 507, 520, 504, 529.

This gives the probability of getting a head in a toss of a coin as : $\frac{5,079}{10,000} = 0{\cdot}5079 \simeq \frac{1}{2}$.
Thus, the empirical probability approaches the classical probability as the number of trials becomes indefinitely large.

3·5·1. Limitations of Empirical Probability.

(*i*) If an experiment is repeated a large number of times, the experimental conditions may not remain identical and homogeneous.

(*ii*) The limit in (3·2) may not attain a unique value, however large N may be.

Example 3·1. *What is the chance that a leap year selected at random will contain 53 Sundays ?*

Solution. In a leap year (which consists of 366 days), there are 52 complete weeks and 2 days over. The following are the possible combinations for these two '*over*' days : (*i*) Sunday and Monday, (*ii*) Monday and Tuesday, (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, and (*vii*) Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two 'over' days must be Sunday. Since out of the above 7 possibilities, 2, *viz.*, (*i*) and (*vii*), are favourable to this event.

$\therefore$ Required probability $= \frac{2}{7}$.

Example 3·2. *Two unbiased dice are thrown. Find the probability that :*

(*i*) *both the dice show the same number,*

(*ii*) *the first die shows 6,*

(iii) *the total of the numbers on the dice is* 8.

(iv) *the total of the numbers on the dice is greater than* 8,

(v) *the total of the numbers on the dice is* 13, *and*

(vi) *the total of the numbers on the dice is any number from* 2 *to* 12, *both inclusive.*

Solution. In a random throw of two dice, since each of the six faces of one die can be associated with each of six faces of the other die, the total number of cases is $6 \times 6 = 36$, as given below :

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Here, the expression, say, (i, j) means that the first die shows the number i and the second die shows the number j. Obviously, $(i, j) \neq (j, i)$ if $i \neq j$.

$\therefore$ Exhaustive number of cases $(n) = 36$.

(*a*) The favourable cases that both the dice show the same number are :

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6), *i.e.*, $m = 6$.

$\therefore$ Probability that the two dice show the same number $= \dfrac{6}{36} = \dfrac{1}{6}$.

(*b*) The favourable cases that the first die shows 6 are :

(6, 1), (6, 2), (6, 3), (6, 4), (6, 5) and (6, 6), *i.e.*, 6 in all.

$\therefore$ Probability that the first die shows '6' $= \dfrac{6}{36} = \dfrac{1}{6}$.

(*c*) The cases favourable to getting a total of 8 on the two dice are :

(2, 6), (3, 5), (4, 4), (5, 3), (6, 2), *i.e.*, $m = 5$.

$\therefore$ Probability that total of numbers on two dice is $8 = \dfrac{5}{36}$.

(*d*) The cases favourable to getting a total of more than 8 are :

(3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6), *i.e.*, $m = 10$.

$\therefore$ Probability that the total of numbers on two dice is greater than $8 = \dfrac{10}{36} = \dfrac{5}{18}$.

(*e*) This is an example of an *impossible event*, since the maximum total can be $6 + 6 = 12$. Therefore, the required probability is 0.

(*f*) The probability is 1, as the total of the numbers on the two dice certainly ranges from 2 to 12. The given event is called a *certain event*.

Example 3·3. (*a*) *Among the digits 1, 2, 3, 4, 5 at first one is chosen and then a second selection is made among the remaining four digits. Assuming that all twenty possible outcomes have equal probabilities, find the probability that an odd digit will be selected*

(*i*) *the first time,* (*ii*) *the second time, and* (*iii*) *both times.*

(*b*) *From 25 tickets, marked with first 25 numerals, one is drawn at random. Find the chance that* (*i*) *it is multiple of 5 or 7, and* (*ii*) *it is a multiple of 3 or 7.*

Solution. (*a*) Total number of cases $= 5 \times 4 = 20$.

(i) Now there are 12 cases in which the first digit drawn is odd, *viz.*, (1, 2), (1, 3), (1, 4), (1, 5), (3, 1), (3, 2), (3, 4), (3, 5), (5, 1), (5, 2), (5, 3) and (5, 4).

$\therefore$ The probability that the first digit drawn is odd $= \frac{12}{20} = \frac{3}{5}$.

(ii) Also there are 12 cases in which the second digit drawn is odd, *viz.*, (2, 1), (3, 1), (4, 1), (5, 1), (1, 3), (2, 3), (4, 3), (5, 3), (1, 5), (2, 5), (3, 5) and (4, 5).

$\therefore$ The probability that the second digit drawn is odd $= \frac{12}{20} = \frac{3}{5}$.

(iii) There are six cases in which both the digits drawn are odd, *viz.*, (1, 3), (1, 5), (3, 1), (3, 5), (5, 1) and (5, 3).

$\therefore$ The probability that both the digits drawn are odd $= \frac{6}{20} = \frac{3}{10}$.

(b) (i) Numbers (out of the first 25 numerals) which are multiples of 5 are 5, 10, 15, 20 and 25, *i.e.*, 5 in all and the numbers which are multiples of 7 are 7, 14 and 21, *i.e.*, 3 in all. Hence required number of favourable cases are $5 + 3 = 8$.

$\therefore$ Required probability $= \frac{8}{25}$.

(ii) Numbers (among the first 25 numerals) which are multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, *i.e.*, 8 in all ; and the numbers, which are multiples of 7 are 7, 14, 21, *i.e.*, 3 in all. Since the number 21 is common in both the cases, the required number of distinct favourable cases is $8 + 3 - 1 = 10$.

$\therefore$ Required probability $= \frac{10}{25} = \frac{2}{5}$.

Example 3·4. (a) *Four cards are drawn at random from a pack of 52 cards. Find the probability that*

(i) They are a king, a queen, a jack and an ace.

(ii) Two are kings and two are queens.

(iii) Two are black and two are red.

(iv) There are two cards of hearts and two cards of diamonds.

(b) *In shuffling a pack of cards, four are accidentally dropped, find the chance that the missing cards should be one from each suit.*

Solution. Four cards can be drawn from a well-shuffled pack of 52 cards in ${}^{52}C_4$ ways, which gives the exhaustive number of cases.

(i) 1 king can be drawn out of the 4 kings in ${}^4C_1$ ways. Similarly, 1 queen, 1 jack and an ace can each be drawn in ${}^4C_1 = 4$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable number of cases are ${}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1$.

Hence the required probability $= \dfrac{{}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1}{{}^{52}C_4} = \dfrac{256}{{}^{52}C_4}$.

(ii) Required probability $= \dfrac{{}^4C_2 \times {}^4C_2}{{}^{52}C_4}$.

(iii) Since there are 26 black cards (of spades and clubs) and 26 red cards (of diamonds and hearts) in a pack of cards, the required probability $= \dfrac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$.

(iv) Required probability $= \dfrac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}$.

(*b*) There are ${}^{52}C_4$ possible ways in which four cards can slip while shuffling a pack of cards. The favourable number of cases in which the four cards can be one from each suit is : ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$.

$$\therefore \quad \text{The required probability} = \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4} = \frac{2197}{20825}.$$

Example 3·5. *What is the probability of getting 9 cards of the same suit in one hand at a game of bridge ?*

Solution. Since one hand in a bridge game consists of 13 cards, the exhaustive number of cases is ${}^{52}C_{13}$.

The number of ways in which 9 cards of a suit can come out of 13 cards of the suit $= {}^{13}C_9$. The number of ways in which balance $13 - 9 = 4$ cards can come in one hand out of a balance of 39 cards of other suits is ${}^{39}C_4$.

Since there are four different suits and 9 cards of any suit can come, by the principle of counting, the total number of favourable cases of getting 9 cards of suit $= {}^{13}C_4 \times {}^{39}C_4 \times 4$.

$$\therefore \quad \text{Required probability} = \frac{{}^{13}C_9 \times {}^{39}C_4 \times 4}{{}^{52}C_{13}}.$$

Example 3·6. *A man is dealt 4 spade cards from an ordinary pack of 52 cards. If he is given three more cards, find the probability p that at least one of the additional cards is also a spade.*

Solution. After a man has dealt 4 spade cards from an ordinary pack of 52 cards, there are $52 - 4 = 48$ cards left in the pack, out of which 9 are spade cards and 39 are non-spade cards.

Since 3 more cards can be dealt to the same man out of the 48 cards in ${}^{48}C_3$ ways, the exhaustive number of outcomes $= {}^{48}C_3$.

If none of these 3 additional cards is a spade card, then the 3 additional cards must be drawn out of the 39 non-spade cards, which can be done in ${}^{39}C_3$ ways. The probability that none of the three additional cards dealt to the man is a spade card is given by ${}^{39}C_3 / {}^{48}C_3$.

Hence, the probability p that at least one of the three additional cards is also a spade is given by :

$$p = 1 - P \text{ [None of the three additional cards is a spade.]}$$

$$= 1 - \frac{{}^{39}C_3}{{}^{48}C_3} = 1 - \frac{39 \times 38 \times 37}{3\,!} \times \frac{3\,!}{48 \times 47 \times 46} = 1 - \frac{13 \times 19 \times 37}{16 \times 47 \times 23} = 0{\cdot}4718.$$

Example 3·7. *A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manner :*

(*i*) *There must be one from each category.*

(*ii*) *It should have at least one from the purchase department.*

(*iii*) *The chartered accountant must be in the committee.*

Solution. There are $3 + 4 + 2 + 1 = 10$ persons in all and a committee of 4 people can formed out of them in ${}^{10}C_4$ ways. Hence, exhaustive number of cases is :

$${}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4\,!} = 210$$

(*i*) Favourable number of cases for the committee to consist of 4 members, one from each category, is : ${}^4C_1 \times {}^3C_1 \times {}^2C_1 \times 1 = 4 \times 3 \times 2 = 24$

$\therefore$ Required probability $= \dfrac{24}{210} = \dfrac{4}{35}$.

(*ii*) P [Committee has at least one purchase officer]

$= 1 - P$ (Committee has no purchase officer)

In order that the committee has no purchase officer, all the 4 members are to be selected from amongst officers of production department, sales department and chartered accountant, *i.e.*, out of 3 + 2 + 1 = 6 members and this can be done in ${}^6C_4 = \dfrac{6 \times 5}{1 \times 2} = 15$ ways. Hence

P (Committee has no purchase officer) $= \dfrac{15}{210} = \dfrac{1}{14}$

$\therefore$ P (Committee has at least one purchase officer) $= 1 - \dfrac{1}{14} = \dfrac{13}{14}$.

(*iii*) Favourable number of cases that the committee consists of a chartered accountant as a member and three others are : $1 \times {}^9C_3 = \dfrac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84$ ways, since a chartered accountant can be selected out of one chartered accountant in only 1 way and the remaining 3 members can be selected out of the remaining 10 – 1 = 9 persons in ${}^9C_3$ ways. Hence the required probability $= \dfrac{84}{210} = \dfrac{2}{5}$.

Example 3·8. *An urn contains 6 white, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that : (i) two of the balls drawn are white, (ii) one is of each colour, (iii) none is red, (iv) at least one is white.*

Solution. Total number of balls in the urn is 6 + 4 + 9 = 19. Since 3 balls can be drawn out of 19 in ${}^{19}C_3$ ways, the exhaustive number of cases are ${}^{19}C_3$.

(*i*) If 2 balls of the 3 drawn balls are to be white, these two balls should be drawn out of 6 white balls which can be done in ${}^6C_2$ ways, and the third ball can be drawn out of the remaining 19 – 6 = 13 balls, which can be done in ${}^{13}C_1$ ways. Since any of the former ways can be associated with any one of the later ways, the number of favourable cases = ${}^6C_2 \times {}^{13}C_1$.

Hence, required probability $= \dfrac{{}^6C_2 \times {}^{13}C_1}{{}^{19}C_3}$.

(*ii*) Since the number of favourable cases of getting one ball of each colour is ${}^6C_1 \times {}^4C_1 \times {}^9C_1$, the required probability $= \dfrac{{}^6C_1 \times {}^4C_1 \times {}^9C_1}{{}^{19}C_3}$.

(*iii*) If none of the drawn balls is red, then all the 3 balls must be out of the white and black balls, *viz.*, out of 6 + 9 = 15 balls. Hence the number of favourable cases for this event is ${}^{15}C_3$.

$\therefore$ Required probability $= \dfrac{{}^{15}C_3}{{}^{19}C_3}$.

(*iv*) P (at least one ball is white) $= 1 - P$ (none of the three balls is white) ... (*)

In order that none of the three balls is white, all the three balls must be drawn out of the red and black balls, *i.e.*, out of 4 + 9 = 13 balls and this can be done in ${}^{13}C_3$ ways.

Hence P (none of the three balls is white) $= \dfrac{{}^{13}C_3}{{}^{19}C_3}$.

Substituting in (*), we obtain $\quad P$ (at least one ball is white) $= 1 - \dfrac{^{13}C_3}{^{19}C_3}$.

Example 3·9. *In a random arrangement of the letters of the word 'COMMERCE', find the probability that all the vowels come together.*

Solution. The total number of permutations of the letters of the word 'COMMERCE' are $(8\,!)/(2\,!\,2\,!\,2\,!)$, because it contains 8 letters of which 2 are C's, 2 M's, and 2 E's, and remaining are all different.

The word COMMERCE contains 3 vowels, *viz.*, *OEE* (2 *E*'s being identical). To obtain the total number of arrangements in which these 3 vowels come together, we regard them as tied together, forming only one letter so that total number of letters in COMMERCE may be taken as $8 - 2 = 6$, out of which 2 are *C*'s, 2 are *M*'s and rest distinct and, therefore, their number of arrangement is given by $(6\,!)/(2\,!\,2\,!)$.

Further, the three vowels *OEE*, two of which are identical, can be arranged among themselves in $3\,!/2\,!$ ways. Hence, the total number of arrangements favourable to getting all vowels together $= \dfrac{6\,!}{2\,!\,2\,!} \times \dfrac{3\,!}{2\,!}$.

Hence, the required probability $= \dfrac{6\,!\,3\,!}{2\,!\,2\,!\,2} \div \dfrac{8\,!}{2\,!\,2\,!\,2\,!} = \dfrac{3}{28}$.

Example 3·10. (a) *If the letters of the word 'REGULATIONS' be arranged at random, what is the chance that there will be exactly 4 letters between R and E ?*

(b) *What is the probability that four S's come consecutively in the word 'MISSISSIPPI' ?*

Solution. (a) The word 'REGULATIONS' consists of 11 letters. The two letters *R* and *E* can occupy $^{11}P_2$, *i.e.*, $11 \times 10 = 110$ positions.

The number of ways in which there will be exactly 4 letters between *R* and *E* are enumerated below :

(i) *R* is in the 1st place and *E* is in the 6th place.

(ii) *R* is in the 2nd place and *E* is in the 7th place.

... ...

... ...

... ...

(vi) *R* is in the 6th place and *E* is in the 11th place.

Since *R* and *E* can interchange their positions, the required number of favourable cases is $2 \times 6 = 12$.

$\therefore$ The required probability $= \dfrac{12}{110} = \dfrac{6}{55}$.

(b) Total number of permutations of the 11 letters of the word '*MISSISSIPPI*' in which 4 are of one kind (*viz.*, *S*), 4 of other kind (*viz.*, *I*), 2 of third kind (*viz.*, *P*) and 1 of fourth kind (*viz.*, *M*) are $11\,!/4\,!\,4\,!\,2\,!\,1\,!$.

Following are the 8 possible combinations of 4 *S*'s coming consecutively :

(i) S S S S

(ii) — S S S S

(iii) — — S S S S

⋮

(viii) — — — — — — — S S S S

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, *viz.*, *MIIIPPI* of which 4 are of one kind, 2 of other kind and one of third kind are $\frac{7\,!}{4\,!\,2\,!\,1\,!}$, the required number of favourable cases $= \frac{8 \times 7\,!}{4\,!\,2\,!\,1\,!}$.

$$\therefore \quad \text{Required probability} = \frac{8 \times 7\,!}{4\,!\,2\,!\,1\,!} \div \frac{11\,!}{4\,!\,4\,!\,2\,!\,1\,!} = \frac{8 \times 7\,! \times 4\,!}{11\,!} = \frac{4}{165}.$$

Example 3·11. *Twenty-five books are placed at random in a shelf. Find the probability that a particular pair of books shall be : (i) Always together, and (ii) Never together.*

Solution. Since 25 books can be arranged among themselves in 25 ! ways, the exhaustive number of cases is 25 !

(*i*) Let us now regard that the two particular books are tagged together so that we shall regard them as a single book. Thus, now we have $(25 - 1) = 24$ books which can be arranged among themselves in 24 ! ways. But the two books which are fastened together can be arranged among themselves in 2 ! ways. Hence, associating these two operations, the number of favourable cases for getting a particular pair of books always together is $24\,! \times 2\,!$.

$$\therefore \quad \text{Required probability} = \frac{24\,! \times 2\,!}{25\,!} = \frac{2}{25}.$$

(*ii*) Total number of arrangements of 25 books among themselves is 25 ! and the total number of arrangements that a particular pair of books will always be together is $24\,! \times 2$. Hence, the number of arrangements in which a particular pair of books is never together is : $25\,! - 2 \times 24\,! = (25 - 2) \times 24\,! = 23 \times 24\,!$

$$\therefore \quad \text{Required probability} = \frac{23 \times 24\,!}{25\,!} = \frac{23}{25}.$$

Aliter :

P [A particular pair of books shall never be together.]

$= 1 - P$ [A particular pair of books is always together.]

$$= 1 - \frac{2}{25} = \frac{23}{25}.$$

Example 3·12. *n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.*

Solution. Since n persons can be seated in n chairs at a round table in $(n - 1)\,!$ ways, the exhaustive number of cases $= (n - 1)\,!$.

Assuming the two specified persons A and B who sit together as one, we get $(n - 1)$ persons in all, who can be seated at a round table in $(n - 2)\,!$ ways. Further, since A and B can interchange their positions in 2 ! ways, total number of favourable cases of getting A and B together is $(n - 2)\,! \times 2\,!$.

$$\therefore \quad \text{Required probability} = \frac{(n-2)\,! \times 2\,!}{(n-1)\,!} = \frac{2}{n-1}.$$

Example 3·13. *A five-figure number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.*

Solution. The total number of ways in which the five digits 0, 1, 2, 3, 4 can be arranged among themselves is 5 !. Out of these, the number of arrangements which begin with 0 (and, therefore, will give only 4-digited numbers) is 4 !. Hence the total number of five-digited numbers that can be formed from the digits 0, 1, 2, 3, 4 is :

$$5\,! - 4\,! = 120 - 24 = 96.$$

The number formed will be divisible by 4 if number formed by the two digits on extreme right (*i.e.*, the digits in the unit and tens places) is divisible by 4. Such numbers are :

$$04, \quad 12, \quad 20, \quad 24, \quad 32, \quad \text{and} \quad 40.$$

If the numbers end in 04, the remaining three digits, *viz.*, 1, 2 and 3 can be arranged among themselves in $3!$ ways. Similarly, the number of arrangements of the numbers ending with 20 and 40 is $3!$ in each case.

If the numbers end with 12, the remaining three digits 0, 3, 4 can be arranged in $3!$ ways. Out of these we shall reject those numbers which start with 0 (*i.e.*, have 0 as the first digit). There are $(3-1)! = 2!$ such cases. Hence, the number of five-digited numbers ending with 12 is $3! - 2! = 6 - 2 = 4$.

Similarly, the number of five digited numbers ending with 24 and 32 each is 4. Hence the total number of favourable cases is $3 \times 3! + 3 \times 4 = 18 + 12 = 30$.

$$\text{Hence, the required probability} = \frac{30}{96} = \frac{5}{16}.$$

Example 3·14. *(a) Twelve balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls ?*

(b) If n biscuits be distributed among N persons, find the chance that a particular person receives $r\ (< n)$ biscuits.

Solution. (*a*) Since each ball can go to any one of the there boxes, there are 3 ways in which a ball can go to any one of the three boxes. Hence there are 3^{12} ways in which 12 balls can be placed in the three boxes.

Number of ways in which 3 balls out of 12 can go to the first box is ${}^{12}C_3$. Now the remaining 9 balls are to be placed in remaining 2 boxes and this can be done in 2^9 ways. Hence, the total number of favourable cases $= {}^{12}C_3 \times 2^9$.

$$\therefore \qquad \text{Required probability} = \frac{{}^{12}C_3 \times 2^9}{3^{12}}.$$

(*b*) Take any one biscuit. This can be given to any one of the N beggars so that there are N ways of distributing any one biscuit. Hence the total number of ways in which n biscuits can be distributed at random among N beggars $= N. N \ldots (n \text{ times}) = N^n$.

'r' biscuits can be given to any particular beggar in ${}^nC_r$ ways. Now we are left with $(n - r)$ biscuits which are to be distributed among the remaining $(N - 1)$ beggars and this can be done in $(N-1)^{n-r}$ ways.

$$\therefore \qquad \text{Number of favourable cases} = {}^nC_r \,.\, (N-1)^{n-r}$$

$$\text{Hence, required probability} = \frac{{}^nC_r\,(N-1)^{n-r}}{N^n}.$$

Example 3·15. *A car is parked among N cars in a row, not at either end. On his return the owner finds that exactly r of the N places are still occupied. What is the probability that both neighbouring places are empty ?*

Solution. Since the owner finds on return that exactly r of the N places (including owner's car) are occupied, the exhaustive number of cases for such an arrangement is ${}^{N-1}C_{r-1}$ [since the remaining $r-1$ cars are to be parked in the remaining $N-1$ places and this can be done in ${}^{N-1}C_{r-1}$ ways].

Let A denote the event that both the neighbouring places to owner's car are empty. This requires the remaining $(r-1)$ cars to be parked in the remaining $N-3$ places and hence the number of cases favourable to A is ${}^{N-3}C_{r-1}$. Hence,

$$P(A) = \frac{{}^{N-3}C_{r-1}}{{}^{N-1}C_{r-1}} = \frac{(N-r)(N-r-1)}{(N-1)(N-2)}.$$

Example 3·16. *What is the probability that at least two out of n people have the same birthday ? Assume 365 days in a year and that all days are equally likely.*

Solution. Since the birthday of any person can fall on any one of the 365 days, the exhaustive number of cases for the birthdays of n persons is 365^n.

If the birthdays of all n persons fall on different days, then the number of favourable cases is : $365\,(365-1)\,(365-2)\ldots[365-(n-1)]$,

because in this case the birthday of the first person can fall on any one of 365 days, the birthday of the second person can fall on any one of the remaining 364 days, and so on. Hence, the probability (p) that birthdays of all the n persons are different is given by :

$$p = \frac{365\,(365-1)\,(365-2)\ldots\{365-(n-1)\}}{365^n}$$

$$= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right)$$

Hence, the required probability that at least two persons have same birthday is :

$$1-p = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right).$$

Example 3·17. *Compare the chances of throwing 4 with one die, 8 with two dice and 12 with three dice.*

Solution. *(i) Probability of throwing 4 with one die* : There are 6 possible ways in which the die can fall, and of these one is favourable to the required event .

$\therefore$ Required probability $(p_1) = \frac{1}{6}$.

(ii) Probability of throwing 8 with two dice : Exhaustive number of cases in single throw with two dice is $6^2 = 36$. Now the sum of '8' can be obtained on the two dice in the following ways : (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), *i.e.*, 5 cases in all, where the first and second number in the brackets ( ) refer to the numbers on the 1st and 2nd die respectively.

$\therefore$ Required probability $(p_2) = \frac{5}{36}$.

(iii) Probability of throwing 12 with three dice : The exhaustive number of ways in a single throw of three dice $= 6 \times 6 \times 6 = 216$.

To make a throw of 12, the three dice must show the faces either (6, 1, 5) or (6, 2, 4) or (6, 3, 3) or (5, 2, 5) or (5, 3, 4) or (4, 4, 4). The first two of these arrangements can occur in $3! = 6$ ways each, the second two (*i.e*, third and fourth arrangement) in $\frac{3!}{2!\,1!} = 3$ ways each, the fifth in $3! = 6$ ways and the last in one way only. Thus, the total number of favourable cases $= 6 + 6 + 3 + 3 + 6 + 1 = 25$.

$\therefore$ Required probability $(p_3) = \frac{25}{216}$.

Hence the chances of throwing 4 with one die, 8 with two dice, and 12 with three dice are :

$$p_1 : p_2 : p_3 :: \frac{1}{6} : \frac{5}{36} : \frac{25}{216} \quad \text{or} \quad 36 : 30 : 25.$$

3·7·3. Algebra of Sets. Now we state certain important properties concerning operations on sets. If A, B and C are the subsets of a universal set S, then the following laws hold :

Commutative Laws : $A \cup B = B \cup A, \quad A \cap B = B \cap A$

Associative Law : $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Complementary Law : $A \cup \overline{A} = S, \quad A \cap \overline{A} = \phi$

$A \cup S = S,$ and $A \cap S = A \quad (\because A \subset S),$

$A \cup \phi = A, \quad A \cap \phi = \phi.$

Difference Law : $A - B = A \cap \overline{B}$.

$A - B = A - (A \cap B) = (A \cup B) - B$

$A - (B - C) = (A - B) \cup (A - C)$

$(A \cup B) - C = (A - C) \cup (B - C)$

$A - (B \cup C) = (A - B) \cap (A - C)$

$(A \cap B) \cup (A - B) = A, \quad (A \cap B) \cap (A - B) = \phi$

De-Morgan's Law

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

More generally

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i} \quad \text{and} \quad \overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \overline{A_i}$$

Involution Law : $\overline{(\overline{A})} = A$

Idempotency Law : $A \cup A = A, \quad A \cap A = A$

3·7·4. Limit of Sequence of Sets. Let $\{A_n\}$ be a sequence of sets in S. The *limit supremum or limit superior* of the sequence, usually written as lim sup A_n, is the set of all those elements which belong to A_n for infinitely many n. Thus

$$\lim_{n \to \infty} \sup A_n = \{x : x \in A_n \text{ for infinitely many } n\} \quad \dots (3{\cdot}3)$$

The set of all those elements which belong to A_n for all but a finite number of n is called *limit infinimum or limit inferior* of the sequence and is denoted by lim inf A_n.

TABLE – GLOSSARY OF PROBABILITY TERMS

	Statement	*Meaning in terms of set theory*
1.	At least one of the events A or B occurs	$\omega \in A \cup B$
2.	Both the events A and B occur.	$\omega \in A \cap B$
3.	Neither A nor B occurs	$\omega \in \overline{A} \cap \overline{B}$
4.	Event A occurs and B does not occur.	$\omega \in A \cap \overline{B}$
5.	Exactly one of the events A or B occurs.	$\omega \in A \Delta B$
6.	Not more than one of the events A or B occurs.	$\omega \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$
7.	If event A occurs, so does B.	$A \subset B$
8.	Events A and B are mutually exclusive.	$A \cap B = \phi$
9.	Complementary event of A.	$\overline{A}$
10.	Sample space	Universal set S.

Example 3·23. *A, B and C are three arbitrary events. Find expression for the events noted below, in the context of A, B and C :*

(i) *Only A occurs,*
(ii) *Both A and B, but not C, occur,*
(iii) *All three events occur,*
(iv) *At least one occurs,*
(v) *At least two occur,*
(vi) *One and no more occurs,*
(vii) *Two and no more occur,*
(viii) *None occurs.*

Solution.

(i) $A \cap \overline{B} \cap \overline{C}$, (ii) $A \cap B \cap \overline{C}$, (iii) $A \cap B \cap C$ (iv) $A \cup B \cup C$,

(v) $A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$

(viii) $(\overline{A} \cap \overline{B} \cap \overline{C})$ or $\overline{A \cup B \cup C}$

3·9. SOME THEOREMS ON PROBABILITY

In this section, we shall prove a few simple theorems which help us evaluate the probabilities of some complicated events in a rather simple way. In proving these theorems, we shall follow the axiomatic approach, based on the three axioms, discussed in § 3·8·5.

Theorem 3·2. *Probability of the impossible event is zero, i.e.,* $P(\phi) = 0$. ... (3·8)

Proof. Impossible event contains no sample point and hence the certain event S and the impossible event ϕ are mutually exclusive.

$$\therefore \qquad S \cup \phi = S \quad \Rightarrow \quad P(S \cup \phi) = P(S)$$

Hence, using Axiom 2 of probability, *i.e.,* Axiom of Additivity, we get

$$P(S) + P(\phi) = P(S) \quad \Rightarrow \quad P(\phi) = 0$$

Remark. It may be noted $P(A) = 0$, does not imply that A is necessarily an empty set. In practice, probability '0' is assigned to the events which are so rare that they happen only once in a lifetime. For example, if a person who does not know typing is asked to type one page of the manuscript of a book, the probability of the event that he will type it correctly without any mistake is 0.

As another illustration, let us consider the random tossing of a coin. The event that the coin will stand erect on its edge, is assigned the probability 0.

The study of continuous random variable provides another illustration to the fact that $P(A) = 0$, does not imply $A = \phi$, because in case of continuous random variable X, the probability at a point is always zero, *i.e.*, $P(X = c) = 0$ [See Chapter 5].

Theorem 3·3. *Probability of the complementary event* $\bar{A}$ *of* A *is given by*

$$P(\bar{A}) = 1 - P(A). \qquad \ldots (3{\cdot}9)$$

Proof. A and $\bar{A}$ are mutually disjoint events, so that

$$A \cup \bar{A} = S \quad \Rightarrow \quad P(A \cup \bar{A}) = P(S)$$

Hence, from Axioms 2 and 3 of probability, we have

$$P(A) + P(\bar{A}) = P(S) = 1 \quad \Rightarrow \quad P(\bar{A}) = 1 - P(A)$$

Cor. 1. We have $P(A) = 1 - P(\bar{A}) \le 1$ $\quad [\because P(\bar{A}) \ge 0$, by Axiom 1]

Further, since $P(A) \ge 0$ (Axiom 1)

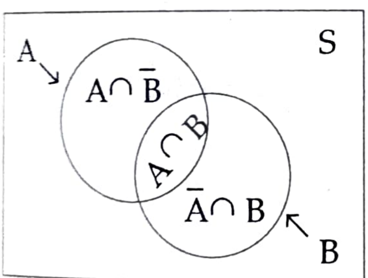
$$\therefore \quad 0 \le P(A) \le 1. \qquad \ldots (3{\cdot}9a)$$

Cor. 2. $P(\phi) = 0$, since $\phi = \bar{S}$ and $P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0$.

Theorem 3·4. *For any two events A and B, we have*

(*i*) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$ $\qquad$ (*ii*) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Proof. From the Venn diagram, we get $\quad B = (A \cap B) \cup (\bar{A} \cap B)$,



where $\bar{A} \cap B$ and $A \cap B$ are disjoint events.

Hence by Axiom (3), we get

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow \quad P(\bar{A} \cap B) = P(B) - P(A \cap B) \qquad \ldots (3{\cdot}10)$$

(*ii*) Similarly, we have $\quad A = (A \cap B) \cup (A \cap \bar{B})$,

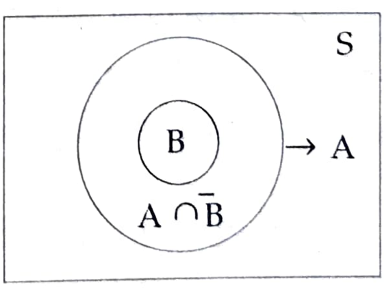
where $(A \cap B)$ and $A \cap \bar{B}$ are disjoint events. Hence, by Axiom 3, we get

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) \quad \Rightarrow \quad P(A \cap \bar{B}) = P(A) - P(A \cap B) \qquad \ldots (3{\cdot}11)$$

Theorem 3·5. *If* $B \subset A$, *then*

(*i*) $P(A \cap \bar{B}) = P(A) - P(B)$,

(*ii*) $P(B) \le P(A)$



Proof. (*i*) When $B \subset A$, B and $A \cap \bar{B}$ are mutually exclusive events so that $A = B \cup (A \cap \bar{B})$

$$\Rightarrow \quad P(A) = P[B \cup (A \cap \bar{B})]$$

$$= P(B) + P(A \cap \bar{B}) \quad \text{(By Axiom 3)}$$

$$\Rightarrow \quad P(A \cap \bar{B}) = P(A) - P(B)$$

(ii) $P(A \cap \overline{B}) \geq 0 \quad \Rightarrow \quad P(A) - P(B) \geq 0 \quad \Rightarrow \quad P(B) \leq P(A)$

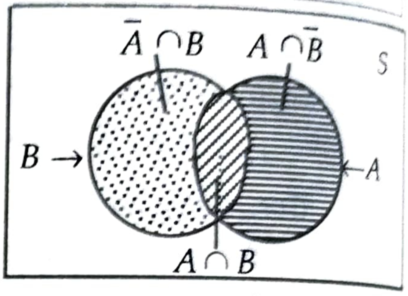
Hence $\quad B \subset A \quad \Rightarrow \quad P(B) \leq P(A)$ ... (3·12)

3·9·1. Addition Theorem of Probability

Theorem 3·6. *If A and B are any two events (subsets of sample space S) and are not disjoint, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{... (3·13)}$$

Proof. From the Venn diagram, we have



$$A \cup B = A \cup (\overline{A} \cap B),$$

where A and $\overline{A} \cap B$ are mutually disjoint.

$$\begin{aligned}
\therefore \quad P(A \cup B) &= P[A \cup (\overline{A} \cap B)] \\
&= P(A) + P(\overline{A} \cap B) && \text{[By Axiom 3]} \quad \text{... (*)} \\
&= P(A) + P(B) - P(A \cap B) && \text{[From Theorem 3·3 (i)]}
\end{aligned}$$

OR From (*) onwards.

$$\begin{aligned}
P(A \cup B) &= P(A) + [P(\overline{A} \cap B) + P(A \cap B)] - P(A \cap B) \\
&= P(A) + P[(\overline{A} \cap B) \cup (A \cap B)] - P(A \cap B)
\end{aligned}$$

[$\because$ $(\overline{A} \cap B)$ and $(A \cap B)$ are disjoint]

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Aliter. $P(A \cup B) = \dfrac{n(A \cup B)}{n(S)} = \dfrac{n(A) + n(B) - n(A \cap B)}{n(S)}$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} = P(A) + P(B) - P(A \cap B)$$

Cor. 1. If the events A and B are mutually disjoint, then

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0$$

$\therefore \quad P(A \cup B) = P(A) + P(B)$, which is Axiom 3 of probability.

Cor. 2. For three non-mutually exclusive events A, B and C, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C). \quad \text{(3·13a)}$$

Proof.
$$\begin{aligned}
P(A \cup B \cup C) &= P[A \cup (B \cup C)] \\
&= P(A) + P(B \cup C) - P[A \cap (B \cup C)] && \text{[From (3·13)]} \\
&= P(A) + [P(B) + P(C) - P(B \cap C)] - P[(A \cap B) \cup (A \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)] \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).
\end{aligned}$$

3·9·2. Extension of Addition Theorem of Probability to n Events.

Theorem 3·7. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k)$$

$$+ \ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n) \qquad \ldots (3{\cdot}14)$$

Proof. For two events A_1 and A_2, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \qquad (*)$$

Hence (3·14) is true for $n = 2$.

Let us now suppose that (3·14) is true for $n = r$, (say) so that

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P(A_i) - \sum\sum_{i \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r) \ldots (**)$$

Now

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left\{\left(\bigcup_{i=1}^{r} A_i\right) \cup A_{r+1}\right\}$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right\} \qquad \ldots [\text{Using } (*)]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} \left(A_i \cap A_{r+1}\right)\right\} \qquad (\text{By Distributive Law})$$

$$= \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$

$$+ P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right\} \qquad [\text{From } (**)]$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$

$$- \left\{\sum_{i=1}^{r} P(A_i \cap A_{r+1}) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j \cap A_{r+1})\right.$$

$$\left. + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r \cap A_{r+1})\right\} \qquad \ldots [\text{From } (**)]$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r+1} A_i\right) = \sum_{i=1}^{r+1} P(A_i) - \left\{\sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \sum_{i=1}^{r} P(A_i \cap A_{r+1})\right\}$$

$$+ \ldots + (-1)^r P\{(A_1 \cap A_2 \cap \ldots \cap A_{r+1}\}$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le (r+1)} P(A_i \cap A_j) + \ldots + (-1)^r P(A_1 \cap A_2 \cap \ldots \cap A_{r+1})$$

Hence if (3·14) is true for $n = r$, it is also true for $n = (r+1)$. But we have proved in (*) that (3·14) is true for $n = 2$. Hence by the principle of mathematical induction, it follows that (3·14) is true for all positive integral values of n.

Example 3·24. *A letter of the English alphabet is chosen at random. Calculate the probability that the letter so chosen*

(i) is a vowel, (ii) precedes m and is a vowel, (iii) follows m and is a vowel.

Solution. The sample space of the experiment is :

$$S = (a, b, c, d, \ldots, x, y, z), \qquad n(S) = 26.$$

(*i*) Let E_1 be the event that the letter chosen is a vowel, Then

$$E_1 = \{a, e, i, o, u\}\ ; \quad n(E_1) = 5 \qquad \therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{5}{26}$$

(ii) Let E_2 be the event that the letter precedes m and is a vowel, Then

$$E_2 = \{a, e, i\}\ ; \quad n(E_2) = 3 \qquad \therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{26}$$

(iii) Let E_3 be the event that the letter follows m and is a vowel. Then,

$$E_3 = \{o, u\}\ ; \quad n(E_3) = 2 \qquad \therefore \quad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{26} = \frac{1}{13}$$

Example 3·25. *Five salesmen of B, C, D and E of a company are considered for a three-member trade delegation to represent the company in an international trade conference. Construct the sample space and find the probability that :*

(i) A is selected. (ii) A is not selected, and (iii) Either A or B (not both) is selected. (Assume the natural assignment of probability.)

Solution. The sample space for selecting three salesmen out of 5 salesmen A, B, C, D and E for the trade delegation is given by :

$$S = \{ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE\} \Rightarrow n(S) = 10$$

Under the assumption of natural assignment of probabilities, each of these outcomes (elementary events) has an equal chance of being selected.

Let us define the following events :

$E_1 : A$ is selected and $E_2 : A$ or D (not both) is selected.

(i) $E_1 = (ABC, ABD, ABE, ACD, ACE, ADE\} \quad \Rightarrow \quad n(E_1) = 6.$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{10} = \frac{3}{5}$$

(ii) $\overline{E_1} = A$ is not selected $= \{BCD, BCE, BDE, CDE\} \quad \Rightarrow \quad n(E_2) = 4$

$$\therefore \quad P(\overline{E_1}) = \frac{n(\overline{E_1})}{n(S)} = \frac{4}{10} = \frac{2}{5} \quad \text{or} \quad P(\overline{E_1}) = 1 - P(E_1) = \frac{2}{5}.$$

(iii) $E_2 = \{ABC, ABE, ACE, BCD, BDE, CDE\} \quad \Rightarrow \quad n(E_2) = 6$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{10} = \frac{3}{5}.$$

Important Remark. In all the problems that follow, we shall always assume natural assignment of probabilities to the elementary events, unless specified otherwise.

Example 3·26. *A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A) given that :*

$$P(B) = \frac{3}{2}P(A) \quad \text{and} \quad P(C) = \frac{1}{2}P(B)$$

Solution. Let $P(A) = p$, then $P(B) = \frac{3}{2}p$ and $P(C) = \frac{1}{2} \times \frac{3}{2}p = \frac{3}{4}p$.

Since A, B, C are mutually exclusive and exhaustive events,

$$P(A) + P(B) + P(C) = 1 \Rightarrow p + \frac{3}{2}p + \frac{3}{4}p = 1 \Rightarrow \frac{13}{4}p = 1 \Rightarrow p = \frac{4}{13}.$$

Example 3·27. *If* $p_1 = P(A)$, $p_2 = P(B)$, $p_3 = P(A \cap B)$, $(p_1, p_2, p_3 > 0)$, *express the following in terms of* p_1, p_2, p_3

(a) $P(\overline{A \cup B})$, (b) $P(\overline{A} \cup \overline{B})$, (c) $P(\overline{A} \cap B)$, (d) $P(\overline{A} \cup B)$, (e) $P(\overline{A} \cap \overline{B})$

(f) $P(A \cap \overline{B})$, (g) $P(A \mid B)$, (h) $P(B \mid \overline{A})$, (i) $P[\overline{A} \cap (A \cup B)]$.

Solution.

(a) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - p_1 - p_2 + p_3.$

(b) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - p_3$

(c) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = p_2 - p_3$

(d) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B) = 1 - p_1 + p_2 - (p_2 - p_3) = 1 - p_1 + p_3$

(e) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - p_1 - p_2 + p_3.$ [Part (a)]

(f) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = p_1 - p_3$

(g) $P(A \mid B) = P(A \cap B) / P(B) = p_3/p_2$

(h) $P(B \mid \bar{A}) = P(\bar{A} \cap B)/P(\bar{A}) = (p_2 - p_3)/(1 - p_1)$ [Part (c)]

(i) $P(\bar{A} \cap (A \cup B)] = P[(\bar{A} \cap A) \cup (\bar{A} \cap B)] = P(\bar{A} \cap B) = p_2 - p_3$ $[\because A \cap \bar{A} = \phi]$

Example 3·28. *Let $P(A) = p$, $P(A \mid B) = q$, $P(B \mid A) = r$. Find relations between the number p, q, r for the following cases :*

(a) *Events A and B are mutually exclusive.*

(b) *A and B are mutually exclusive and collectively exhaustive.*

(c) *A is sub-event of B ; B is a sub-event of A.*

(d) *$\bar{A}$ and $\bar{B}$ are mutually exclusive.*

Solution. From the given data : $P(A) = p,\ P(A \cap B) = P(A)\,P(B \mid A) = rp$

$$\therefore \quad P(B) = \frac{P(A \cap B)}{P(A \mid B)} = \frac{rp}{q} \quad \text{and} \quad P(A) + P(B) = p + \frac{rp}{q} = \frac{p(q+r)}{q}$$

(a) Since A and B are mutually exclusive

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0 \quad \Rightarrow \quad rp = 0.$$

(b) Since A and B are mutually exclusive and collectively exhaustive,

$$P(A \cap B) = 0 \quad \text{and} \quad P(A) + P(B) = 1$$

$$\Rightarrow \quad p(q + r) = q\,;\ rp = 0 \quad \text{or} \quad pq = q \quad \Rightarrow \quad p = 1 \quad \text{or} \quad q = 0$$

(c) $A \subseteq B \Rightarrow A \cap B = A$ or $P(A \cap B) = P(A) \Rightarrow rp = p$, *i.e.*, $r = 1$ or $p = 0$.

$B \subseteq A \Rightarrow A \cap B = B$ or $P(A \cap B) = P(B)$

$$\Rightarrow \quad rp = (rp/q) \Rightarrow rp\,(q - 1) = 0 \quad \Rightarrow \quad r = 0 \quad \text{or} \quad p = 0.$$

(d) Since $\bar{A}$ and $\bar{B}$ are mutually exclusive, $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 0$

$$\Rightarrow \quad 1 - [P(A) + P(B) - P(A \cap B)] = 0 \quad \Rightarrow \quad P(A) + P(B) = 1 + P(A \cap B)]$$

$$\Rightarrow \quad p\,[1 + (r/q)] = 1 + rp \quad \Rightarrow \quad p\,(q + r) = q\,(1 + pr).$$

Example 3·29. *A die is loaded in such a manner that for $n = 1, 2, 3, 4, 5, 6$, the probability of the face marked n, landing on top when the die is rolled is proportional to n. Find the probability that an odd number will appear on tossing the die.*

Solution. Here we are given :

$P(n) \propto n \quad \Rightarrow \quad P(n) = kn \quad \ldots (*)$, where k is the constant of proportionality.

Also $P(1) + P(2) + \ldots + P(6) = 1 \quad \Rightarrow \quad k(1+2+3+4+5+6) = 1 \quad \Rightarrow \quad k = \dfrac{1}{21}$

$$\text{Required Probability} = P(1) + P(3) + P(5) = \frac{1+3+5}{21} = \frac{3}{7} \qquad \text{[Using (*)]}$$

Example 3·30. *If two dice are thrown, what is the probability that the sum is (a) greater than 8, and (b) neither 7 nor 11 ?*

Solution. (*a*) If S denotes the sum on the two dice, then we want $P(S > 8)$.

The required event can happen in the following mutually exclusive ways :

(*i*) $S = 9$ (*ii*) $S = 10$ (*iii*) $S = 11$ (*iv*) $S = 12$.

Hence by addition theorem of probability

$$P(S > 8) = P(S = 9) + P(S = 10) + P(S = 11) + P(S = 12) \qquad \ldots (*)$$

In a throw of two dice, the sample space contains $6^2 = 36$ points. The number of favourable cases can be enumerated as follows :

$S = 9$: (3, 6), (6, 3), (4, 5), (5, 4), *i.e.*, 4 sample points $\quad \therefore P(S = 9) = \dfrac{4}{36}$

$S = 10$: (4, 6), (6, 4), (5, 5), *i.e.*, 3 sample points. $\quad \therefore P(S = 10) = \dfrac{3}{36}$

$S = 11$: (5, 6), (6, 5), *i.e.*, 2 sample points $\quad \therefore P(S = 11) = \dfrac{2}{36}$

$S = 12$: (6, 6), *i.e.*, 1 sample point. $\quad \therefore P(S = 12) = \dfrac{1}{36}$

$$\therefore \quad P(S > 8) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18} \qquad \text{[From (*)]}$$

(*b*) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11 with a pair of dice.

$S = 7$: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), *i.e.*, 6 distinct sample points.

$$\therefore \quad P(A) = P(S = 7) = \frac{6}{36} = \frac{1}{6}$$

$S = 11$: (5, 6), (6, 5), *i.e.*, 2 distinct sample points.

$$\therefore \quad P(B) = P(S = 11) = \frac{2}{36} = \frac{1}{18}$$

$$\begin{aligned}\therefore \text{ Required probability} &= P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B)] \qquad (\because A \text{ and } B \text{ are disjoint events.}) \\ &= 1 - \frac{1}{6} - \frac{1}{18} = \frac{7}{9}.\end{aligned}$$

Example 3·31. *Two dice are tossed. Find the probability of getting 'an even number on the first die or a total of 8'.*

Solution. In a random toss of two dice, sample space S is given by :

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(S) = 6 \times 6 = 36$$

Let us define the events :

A : Getting an even number on the first dice

B : The sum of the points obtained on the two dice 8.

These events are represented by the following subsets of S.

$$A = \{2, 4, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(A) = 3 \times 6 = 18$$

$$B = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\} \quad \Rightarrow \quad n(B) = 5$$

Also $A \cap B = \{(2, 6), (6, 2), (4, 4)\} \quad \Rightarrow \quad n(A \cap B) = 3.$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}, \quad P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}, \text{ and } P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

Hence, the required probability is given by :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}.$$

Example 3·32. *An integer is chosen at random from two hundred digits. What is the probability that the integer is divisible by 6 or 8 ?*

Solution. The sample space of the random experiment is :

$$S = \{1, 2, 3, \ldots, 199, 200\} \quad \Rightarrow \quad n(S) = 200$$

The event A : 'integer chosen is divisible by 6' has the sample points given by :

$$A = \{6, 12, 18, \ldots, 198\} \quad \Rightarrow \quad n(A) = \frac{198}{6} = 33. \quad \therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{33}{200}$$

Similarly the event B : 'integer chosen is divisible by 8' has the sample points given by :

$$B = \{8, 16, 24, \ldots, 200\} \quad \Rightarrow \quad n(B) = \frac{200}{8} = 25 \quad \therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{25}{200}$$

The LCM of 6 and 8 is 24. Hence, a number is divisible by both 6 and 8, if it is divisible by 24.

$$\therefore \quad A \cap B = \{24, 48, 72, \ldots, 192\} \quad \Rightarrow \quad n(A \cap B) = \frac{192}{24} = 8 \quad \Rightarrow \quad P(A \cap B) = \frac{8}{200}$$

Hence, the required probability is :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{1}{4}.$$

Example 3·33. *The probability that a student passes a Physics test is $\frac{2}{3}$ and the probability that he passes both a Physics test and an English test is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that he passes the English test ?*

Solution. Let us define the following events :

A : The student passes a Physics test ; B : The student passes an English test

In the usual notations, we are given :

$$P(A) = \frac{2}{3}, \; P(A \cap B) = \frac{14}{45}, \quad P(A \cup B) = \frac{4}{5} \text{ and we want, } P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \Rightarrow \quad \frac{4}{5} = \frac{2}{3} + P(B) - \frac{14}{45}$$

$$\therefore \quad P(B) = \frac{4}{5} + \frac{14}{45} - \frac{2}{3} = \frac{36 + 14 - 30}{45} = \frac{4}{9}.$$

Example 3·34. *An investment consultant predicts that the odds against the price of a certain stock will go up during the next week are 2 : 1 and the odds in favour of the price* *remaining the same are 1 : 3. What is the probability that the price of the stock will go down during the next week ?*

Solution. Let A denote the event that 'stock price will go up', and B be the event 'stock price will remain same'. Then $P(A) = \frac{1}{2+1} = \frac{1}{3}$ and $P(B) = \frac{1}{1+3} = \frac{1}{4}$.

$\therefore$ P(stock price will either go up or remain same) is given by :

$$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Hence, the probability that stock price will go down is given by :

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}.$$

Example 3·35. *An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0·7 and being rejected at Y is 0·5. The probability of at least one of his applications being rejected is 0·6. What is probability that he will be selected in one of the firms ?*

Solution. Let A and B denote the events that the person is selected in firms X and Y respectively. Then in the usual notations, we are given :

$$\left.\begin{aligned} P(A) &= 0{\cdot}7 \quad \Rightarrow \quad P(\bar{A}) = 1 - 0{\cdot}7 = 0{\cdot}3 \\ P(\bar{B}) &= 0{\cdot}5 \quad \Rightarrow \quad P(B) = 1 - 0{\cdot}5 = 0{\cdot}5 \end{aligned}\right\} \quad \ldots (*)$$

and $$P(\bar{A} \cup \bar{B}) = 0{\cdot}6 = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B}) \quad \ldots (**)$$

The probability that the persons will be selected in one of the two firms X or Y is given by :

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B}) = 1 - \{P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cup \bar{B})\} \quad \text{[From (**)]}$$
$$= 1 - (0{\cdot}3 + 0{\cdot}5 - 0{\cdot}6) = 0{\cdot}8. \quad \text{[From (*)]}$$

Example 3·36. *Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult population : 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers ?*

Solution. Let E, F and G denote the events that the adult reads newspapers A, B and C respectively. Then we are given :

$$P(E) = \frac{20}{100}, \quad P(F) = \frac{16}{100}, \quad P(G) = \frac{14}{100}, \quad P(E \cap F) = \frac{8}{100}$$
$$P(E \cap G) = \frac{5}{100}, \quad P(F \cap G) = \frac{4}{100}, \quad \text{and} \quad P(E \cap F \cap G) = \frac{2}{100}$$

The required probability that an adult reads at least one of the newspapers (by addition theorem) is given by :

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(E \cap G) + P(E \cap F \cap G)$$
$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{4}{100} - \frac{5}{100} + \frac{2}{100} = \frac{35}{100} = 0{\cdot}35$$

Hence 35% of the adult population reads at least one of the newspapers.

Example 3·37. *A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.*

Solution. Let us define the following events :

A : the card drawn is a king ; B : the card drawn is a heart ;

C : the card drawn is a red card.

Then A, B and C are not mutually exclusive.

$A \cap B$: the card drawn is the king of hearts $\Rightarrow$ $n(A \cap B) = 1$

$B \cap C = B$: the card drawn is a heart $(\because B \subset C)$ $\Rightarrow$ $n(B \cap C) = 13$

$C \cap A$: the card drawn is a red king $\Rightarrow$ $n(C \cap A) = 2$

$A \cap B \cap C = A \cap B$: the card drawn is the king of hearts $\Rightarrow n(A \cap B \cap C) = 1$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}; \quad P(B) = \frac{13}{52}; \quad P(C) = \frac{26}{52}$$

$$P(A \cap B) = \frac{1}{52}; \quad P(B \cap C) = \frac{13}{52}; \quad P(C \cap A) = \frac{2}{52}; \quad P(A \cap B \cap C) = \frac{1}{52}$$

The required probability of getting a king or heart or a red card is given by :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}.$$

3·10. CONDITIONAL PROBABILITY

As discussed earlier, the probability $P(A)$ of an event A represents the likelihood that a random experiment will result in an outcome in the set A relative to the sample space S of the random experiment. However, quite often, while evaluating some event probability, we already have some information stemming from the experiment. For example, if we have prior information that the outcome of the random experiment must be in a set B of S, then this information must be used to re-appraise the likelihood that the outcome will also be in B. This re-appraised probability is denoted by $P(A \mid B)$ and is read as the conditional probability of the event A, given that the event B has already happened.

We give below some illustrations to explain this concept.

Illustrations 1. Let us consider a random experiment of drawing a card from a pack of cards. Then the probability of happening of the event A : "The card drawn is a king", is given by :
$$P(A) = \frac{4}{52} = \frac{1}{13}.$$

Now suppose that a card is drawn and we are informed that the drawn card is red. How does this information affect the likelihood of the event A ?

Obviously, if the event B : 'The card drawn is red', has happened, the event 'Black card' is not possible. Hence the probability of the event A must be computed relative to the new sample space 'B' which consists of 26 sample points (red cards only), *i.e.*, $n(B) = 26$. Among these 26 red cards, there are two (red) kings so that $n(A \cap B) = 2$. Hence, the required probability is given by :

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{26} = \frac{1}{13}.$$

2. Consider a random experiment of tossing three fair coins. Then, as explained earlier, the sample space S is :

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$
$$= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \text{ (say)},$$

so that : $$P(\{\omega_i\}) = \frac{1}{8}; i = 1, 2, \ldots 8.$$

Now suppose that the same experiment is performed by another person and nothing is known about its outcome. However, we have the information that he obtained 'at least two heads'. We are interested to find how this additional information affects the probabilities of the elementary outcomes.

This means that if the event A : 'At least two heads are obtained', has happened then the elementary outcomes ω_4, ω_6, ω_7 and ω_8 could not have happened. However, the remaining four outcomes ω_1, ω_2, ω_3, and ω_5 are still possible and we assign the probability $\frac{1}{4}$ to each one of them.

$$\therefore \quad P(\omega_1 \mid A) = P(\omega_2 \mid A) = P(\omega_3 \mid A) = P(\omega_5 \mid A) = \frac{1}{4}.$$

From the above illustrations we observe that some additional information may change the probability of the happening of some event. We now proceed to develop procedure to calculate the probabilities of events when we know some additional information.

Remark. When we know that a particular event B has occurred, instead of S, we concentrate our attention on B only and the conditional probability of A given B will be analogously the ratio of the probability of that part of A which is included in B (*i.e.*, $A \cap B$) to the probability of B. It, therefore, reflects the change of viewpoint only, namely, instead of S we have to concentrate on B only.

3·11. MULTIPLICATION THEOREM OF PROBABILITY

Theorem 3·9. *For two events A and B,*

$$\left.\begin{aligned} P(A \cap B) &= P(A) . P(B \mid A), \quad P(A) > 0 \\ &= P(B) . P(A \mid B), \quad P(B) > 0 \end{aligned}\right\} \qquad \ldots (3\cdot 17)$$

where $P(B \mid A)$ represents conditional probability of occurrence of B when the event A has already happened and $P(A \mid B)$ is the conditional probability of happening of A, given that B has already happened.

Proof. In the usual notations, we have

$$P(A) = \frac{n(A)}{n(S)}; \qquad P(B) = \frac{n(B)}{n(S)}. \quad \text{and} \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad \ldots (*)$$

For the conditional event $A \mid B$, the favourable outcomes must be one of the sample points of B, *i.e.*, for the event $A \mid B$, the sample space is B and out of the $n(B)$ sample points, $n(A \cap B)$ pertain to the occurrence of the event A. Hence

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (*), we get

$$P(A \cap B) = \frac{n(B)}{n(S)} \times \frac{n(A \cap B)}{n(B)} = P(B) . P(A \mid B) \qquad \ldots (**)$$

Similarly, we get from (*) :

$$P(A \cap B) = \frac{n(A)}{n(S)} \times \frac{n(A \cap B)}{n(A)} = P(A) . P(B \mid A) \qquad \ldots (***)$$

From (**) and (***), we get the result (3·17).

Thus, we have proved that "*the probability of the simultaneous occurrence of two events*

A and B is equal to the product of the probability of one of these events and the conditional probability of the other, given that the first one has occurred". Any of the events may be called the first event.

Remarks 1. $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ and $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ ... (3·18)

Thus the conditional probabilities $P(B \mid A)$ and $P(A \mid B)$ are defined if and only if $P(A) \neq 0$ and $P(B) \neq 0$, respectively.

2. (*i*) For $P(B) > 0$, $P(A \mid B) \leq P(A)$

Proof. $n(A \cap B) \leq n(A)$ and $n(B) \leq n(S)$. Dividing, we get

$$\frac{n(A \cap B)}{n(B)} \leq \frac{n(A)}{n(S)} \quad \Rightarrow \quad P(A \mid B) \leq P(A).$$

(*ii*) The conditional probability $P(A \mid B)$ is not defined if $P(B) = 0$.

(*iii*) $P(B \mid B) = 1$.

3·12. INDEPENDENT EVENTS

Two or more events are said to be *independent* if the happening or non-happening of any one of them, does not, in any way, affect the happening of others.

Consider the experiment of throwing two dice, say die 1 and die 2. It is obvious that the occurrence of a certain number of dots on the die 1 has nothing to do with a similar event for the die 2. The two are quite independent of each other, so to say. But suppose, the two dice were connected with a piece of thread before being thrown. The situation changes. This time the two events are not independent in as much as that the uppermost face of one die will have something to do in causing a particular face of the other die to be uppermost ; and the shorter the thread the more is this influence or dependence.

Similarly, if we draw two cards from a pack of cards in succession, then the results of the two draws are independent if the cards are drawn with replacement (*i.e.*, if the first card drawn is placed back in the pack before drawing the second card) and the results of the two draws are not independent if the cards are drawn without replacement.

Definition. *An event A is said to be independent (or statistically independent) of another event B, if the conditional probability of A given B, i.e., $P(A \mid B)$ is equal to the unconditional probability of B, i.e., if* $P(A \mid B) = P(A)$. ... (3·19)

It may be noted that the above definition is meaningful only when $P(A \mid B)$ is defined, *i.e.*, if $P(B) \neq 0$.

Similarly, an event B is said to be independent (or statistically independent) of event A, if

$$P(B \mid A) = P(B); \quad P(A) \neq 0. \qquad \text{... (3·20)}$$

Theorem 3·10. *If the events A and B are such that $P(A) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is independent of A.*

Proof. Since the event A is independent of B, we have

$$P(A \mid B) = P(A) \quad \Rightarrow \quad \frac{P(A \cap B)}{P(B)} = P(A) \quad \Rightarrow \quad P(A \cap B) = P(A)\,P(B)$$

$\therefore \quad \dfrac{P(B \cap A)}{P(A)} = P(B) \qquad [\because \; P(A) \neq 0 \text{ and } A \cap B = B \cap A]$

$\Rightarrow \quad P(B \mid A) = P(B) \quad \Rightarrow \quad B$ is independent of A.

Remarks 1. Thus, we see that if A is independent of B, then B is independent of A. Hence, instead of saying that 'A is independent of B' or 'B is independent of A', we may say that A and B are independent events.

2. For any event A in S,

(a) A and the null event ϕ are independent

(b) A and S are independent.

Proof. (a) $P(A \cap \phi) = P(\phi) = 0 = P(A) \cdot P(\phi) \quad \Rightarrow \quad A$ and ϕ are independent.

(b) $P(A \cap S) = P(A) = P(A) \,.\, 1 = P(A)\, P(S) \qquad [\because A \subset S \text{ and } P(S) = 1]$

$\Rightarrow \quad A$ and S are independent.

3·13. MULTIPLICATION THEOREM OF PROBABILITY FOR INDEPENDENT EVENTS

Theorem 3·11. *If A and B are two events with positive probabilities $\{P(A) \neq 0,\ P(B) \neq 0\}$, then A and B are independent if and only if* $P(A \cap B) = P(A).\, P(B)$ ... (3·21)

Proof. We have :

$$P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)\,;\, P(A) \neq 0, \quad P(B) \neq 0 \qquad \text{... (*)}$$

If A and B are independent, *i.e.*, A is independent of B and B is independent of A, then, we have

$$P(A \mid B) = P(A) \qquad \text{and} \qquad P(B \mid A) = P(B) \qquad \text{... (**)}$$

From (*) and (**), we get $\qquad P(A \cap B) = P(A)\, P(B)$, as required.

Conversely, if (3·21) holds, then we get

$$\left.\begin{aligned} \frac{P(A \cap B)}{P(B)} &= P(A) \quad \Rightarrow \quad P(A \mid B) = P(A) \\ \text{and} \quad \frac{P(A \cap B)}{P(A)} &= P(B) \quad \Rightarrow \quad P(B \mid A) = P(B) \end{aligned}\right\} \qquad \text{...(***)}$$

(***) implies that A and B are independent events.

Hence, for independent events A and B, the probability that both of these occur simultaneously is the product of their respective probabilities.

This rule is known as the ***Multiplication Rule of Probability***.

3·14. EXTENSION OF MULTIPLICATION THEOREM OF PROBABILITY TO n EVENTS

Theorem 3·12. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \ldots \times P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \qquad \text{... (3·22)}$$

where $P(A_i \mid A_j \cap A_k \cap \ldots \cap A_l)$ represents the conditional probability of the event A_i given that the events $A_j, A_k, \ldots, A_l$ have already happened.

Proof. For two events A_1 and A_2, $P(A_1 \cap A_2) = P(A_1)\, P(A_2 \mid A_1)$

We have for three events A_1, A_2, and A_3

$$P(A_1 \cap A_2 \cap A_3) = P\{A_1 \cap (A_2 \cap A_3)\}$$

Theorem 3·14. *For a fixed B with P(B) > 0, P (A | B) is probability function.*

Proof. (i) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

(ii) $$P(S \mid B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(iii) If $\{A_n\}$ is any finite or infinite sequences of disjoint events, then

$$P[\bigcup_n A_n \mid B] = \frac{P[(\bigcup_n A_n) \cap B]}{P(B)} = \frac{P[\bigcup_n (A_n \cap B)]}{P(B)}$$

$$= \frac{\sum_n P(A_n \cap B)}{P(B)} = \sum_n \left[\frac{P(A_n \cap B)}{P(B)}\right] = \sum_n P(A_n \mid B)$$

Hence the theorem.

Remark. For given B satisfying $P(B) > 0$, the conditional probability $P[\cdot \mid B]$ also enjoys the same properties as the unconditional probability.

For example, in the usual notations, we have

(i) $P(\phi \mid B) = 0$

(ii) $P(\overline{A} \mid B) = 1 - P(A \mid B)$,

(iii) $P\left[\bigcup_{i=1}^{n} A_i \mid B\right] = \sum_{i=1}^{n} P(A_i \mid B)$,

where $A_1, A_2, \ldots, A_n$ are mutually disjoint events.

(iv) $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

(v) $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$

(vi) If $E \subset F$, then $P(E \mid B) \leq P(F \mid B)$

and so on.

The proofs of results (*iv*), (*v*) and (*vi*) are given in Theorems 3·15, 3·16 and 3·17 respectively. Others are left as exercises to the reader.

Theorem 3·15. *For any three events A, B and C,*

$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Proof. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Dividing both sides by P (C), we get

$$\frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}, \; P(C) > 0$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad \frac{P[(A \cup B) \cap C]}{P(C)} = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

$$\Rightarrow \quad P[(A \cup B) \mid C] = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Theorem 3·16. *For any three events A, B and C,*

$$P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$$

Proof. $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C)$

$$= \frac{P(A \cap \overline{B} \cap C)}{P(C)} + \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap \overline{B} \cap C) + P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} = P(A \mid C).$$

Theorem 3·17. *For any three events A, B and C defined on the sample space S such that $B \subset C$ and $P(A) > 0$, $P(B \mid A) \le P(C \mid A)$.*

Proof. $$P(C \mid A) = \frac{P(C \cap A)}{P(A)} = \frac{P[(B \cap C \cap A) \cup (\overline{B} \cap C \cap A)}{P(A)}$$

$$= \frac{P[(B \cap C \cap A)}{P(A)} + \frac{P(\overline{B} \cap C \cap A)}{P(A)} \qquad \text{(Using Axiom 3)}$$

$$= P(B \cap C \mid A) + P(\overline{B} \cap C \mid A)$$

$$= P(B \mid A) + P(\overline{B} \cap C \mid A) \qquad [\because B \subset C \Rightarrow B \cap C = B]$$

$$\Rightarrow \quad P(C \mid A) \ge P(B \mid A) \qquad [\because P(\overline{B} \cap C \mid A) \ge 0]$$

Theorem 3·18. *If A and B are independent events, then*

(i) A and $\overline{B}$ (ii) $\overline{A}$ and B (iii) $\overline{A}$ and $\overline{B}$, are also independent

Proof. Since A and B are independent, $P(A \cap B) = P(A)\,P(B)$ ... (*)

(a) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)\,P(B)$ [From (*)

$= P(A)\,[1 - P(B)] = P(A)\,P(\overline{B})$

$\Rightarrow$ A and $\overline{B}$ are independent events.

Aliter. $P(A \cap B) = P(A)\,P(B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B)$

i.e., $P(B \mid A) = P(B) \Rightarrow B$ is independent of A.

and $P(A \mid B) = P(A) \Rightarrow A$ is independent of B.

Also $P(B \mid A) + P(\overline{B} \mid A) = 1 \Rightarrow P(B) + P(\overline{B} \mid A) = 1$

$\therefore$ $P(\overline{B} \mid A) = 1 - P(B) = P(\overline{B})$

$\therefore$ $\overline{B}$ is independent of A and by symmetry we say that A is independent of $\overline{B}$. Hence, A and $\overline{B}$ are independent events.

(ii) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\,P(B)$ [From (*)]

$= P(B)\,[1 - P(A)] = P(\overline{A})\,P(B)$

$\Rightarrow$ $\overline{A}$ and B are independent events.

(iii) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A)\,P(B)$ [From (*)]

$= [1 - P(B)] - P(A)\,[1 - P(B)]$

$= [1 - P(A)]\,[1 - P(B)] = P(\overline{A})\,P(\overline{B})$

$\therefore$ $\overline{A}$ and $\overline{B}$ are independent events.

Aliter. We know that : $P(\overline{A} \mid \overline{B}) + P(A \mid \overline{B}) = 1$ ... (**)

Since A and B are independent, by Part (i) of the above theorem A and $\overline{B}$ are also independent.

$\therefore$ $P(A \mid \overline{B}) = P(A)$. Hence from (**), we get

$$P(\overline{A} \mid \overline{B}) + P(A) = 1 \quad \Rightarrow \quad P(\overline{A} \mid \overline{B}) = 1 - P(A) = P(\overline{A})$$

Hence $\overline{A}$ and $\overline{B}$ are independent events.

3·15. PAIRWISE INDEPENDENT EVENTS

Consider n events $A_1, A_2, \ldots, A_n$ defined on the same sample space so that $P(A_i) > 0;\ i = 1, 2, \ldots, n$. These *events are said to be pairwise independent if every pair of two events is independent in the sense of the definition given in* § 3·13.

Definition. *The events $A_1, A_2, \ldots, A_n$ are said to be pairwise independent if and only if :*

$$P(A_i \cap A_j) = P(A_i)\, P(A_j),\ i \neq j = 1, 2, \ldots, n \qquad \ldots (3\cdot25)$$

In particular, three events A_1, A_2, A_3 are pairwise independent if and only if :

$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\, P(A_2) \\ P(A_1 \cap A_3) &= P(A_1)\, P(A_3) \\ P(A_2 \cap A_3) &= P(A_2)\, P(A_3) \end{aligned}\right\} \qquad \ldots (3\cdot26)$$

3·15·1. Mutually Independent Events. Let S denote the sample space for a number of events. *The events in S are said to be mutually independent if the probability of the simultaneous occurrence of (any) finite number of them is equal to the product of their separate probabilities.*

Definition. *The n events $A_1, A_2, \ldots, A_n$ in a sample space S are said to be mutually independent if*

$$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})\, P(A_{i2}) \ldots P(A_{ik})\ ;\ k = 2, 3, \ldots, n \qquad \ldots (3\cdot27)$$

Hence, the events are mutually independent if they are independent by pairs, and by triplets, and by quadruples, and so on.

Conditions for mutual independence of n events. Mathematically, n events $A_1, A_2, \ldots A_n$ are mutually independent if and only if the following conditions hold.

$$\left.\begin{aligned} &(i)\quad P(A_i \cap A_j) = P(A_i)\, P(A_j),\ (i \neq j\ ;\ i, j = 1, 2, \ldots, n) \\ &(ii)\quad P(A_i \cap A_j \cap A_k) = P(A_i)\, P(A_j)\, P(A_k),\ (i \neq j \neq k\ ;\ i\ ;\ j\ ;\ k = 1, 2, \ldots, n) \\ &\qquad \vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}\right\} \ldots (3\cdot28)$$

$$(n-1) : P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$$

It is interesting to note that the above equations in (3·28) give respectively ${}^nC_2, {}^nC_3, {}^nC_n$ conditions to be satisfied by $A_1, A_2, \ldots A_n$.

Hence the total number of conditions for mutual independence of $A_1, A_2, \ldots, A_n$ is :

$${}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n - ({}^nC_0 + {}^nC_1) = 2^n - 1 - n$$

In particular for three events A_1, A_2 and A_3, ($n = 3$), we have the following $2^3 - 1 - 3 = 4$, conditions for their mutual independence.

$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\,P(A_2) \\ P(A_2 \cap A_3) &= P(A_2)\,P(A_3) \\ P(A_1 \cap A_3) &= P(A_1)\,P(A_3) \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)\,P(A_2)\,P(A_3) \end{aligned}\right\} \quad \ldots (3\cdot29)$$

Remarks 1. It may be observed that pairwise or mutual independence of events : $A_1, A_2, \ldots, A_n$, is defined only when $P(A_i) \neq 0$, for $i = 1, 2, \ldots, n$.

2. From (3·26) and (3·29), it is obvious that mutual independence of events implies that they are pairwise independent. However, the converse is not true, *i.e.*, the events may be pairwise independent but not mutually independent. For illustrations, see Examples 3·54 and 3·55.

Theorem 3·9. *If A, B, C are mutually independent events then $A \cup B$ and C are also independent.*

Proof. We are required to prove :

$$P[(A \cup B) \cap C] = P(A \cup B)\,P(C)$$

$$\begin{aligned} \text{L.H.S.} &= P[(A \cap C) \cup P(B \cap C)] && \text{[By Distributive Law]} \\ &= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A)\,P(C) + P(B)\,P(C) - P(A)\,P(B)\,P(C) \end{aligned}$$

[$\because$ A, B and C are mutually independent]

$$= P(C)\,[P(A) + P(B) - P(A \cap B)] = P(C)\,P(A \cup B) = \text{R.H.S.}$$

Hence $(A \cup B)$ and C are independent.

Theorem 3·20. *If A, B and C are random events in a sample space and if A, B and C are pairwise independent and A is independent of $(B \cup C)$, then A, B and C are mutually independent.*

Proof. We are given

$$\left.\begin{aligned} P(A \cap B) &= P(A)\,P(B) \\ P(B \cap C) &= P(B)\,P(C) \\ P(A \cap C) &= P(A)\,P(C) \\ P[A \cap (B \cup C)] &= P(A)\,P(B \cup C) \end{aligned}\right\} \quad \ldots (*)$$

Now
$$\begin{aligned} P[A \cap (B \cup C)] &= P[(A \cap B) \cup (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)] \\ &= P(A)\,.\,P(B) + P(A)\,.\,P(C) - P(A \cap B \cap C) \end{aligned}$$

[From (*)] ... (**)

and
$$\begin{aligned} P(A)\,P(B \cup C) &= P(A)\,[P(B) + P(C) - P(B \cap C)] \\ &= P(A)\,.\,P(B) + P(A)\,P(C) - P(A)\,P(B \cap C) \quad \ldots (***) \end{aligned}$$

From (**) and (***), on using (*), we get

$$P(A \cap B \cap C) = P(A)\,P(B \cap C) = P(A)\,P(B)\,P(C) \quad \text{[From (*)]}$$

Hence A, B, C are mutually independent.

3·15·2. Given n independent events A_i, $(i = 1, 2, \ldots, n)$ with respective probabilities of occurrence p_i, to find the probability of occurrence of at least one of them.

We have $P(A_i) = p_i \quad \Rightarrow \quad P(\bar{A}_i) = 1 - p_i\,;\, i = 1, 2, \ldots, n \quad \ldots (*)$

Hence the probability 'p' of happening of at least one of the events is given by :

$$\left.\begin{aligned} p = P(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n}) \\ &= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) = 1 - P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n) \end{aligned}\right\} \quad \ldots (^{**})$$

lie

($\because$ $A_1, A_2, \ldots, A_n$ are independent $\Rightarrow \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ are also independent)

$$= 1 - [(1 - p_1)(1 - p_2) \ldots (1 - p_n)]$$

$$= \left[\sum_{i=1}^{n} p_i - \sum\sum_{\substack{i, j = 1 \\ i < j}}^{n} (p_i p_j) + \sum\sum\sum_{\substack{i, j, k = 1 \\ i < j < k}}^{n} (p_i p_j p_k) - \ldots + (-1)^{n-1} (p_1 p_2 \ldots p_n) \right]$$

Remark. The results in (**) are very important and are used quite often in numerical problems. Result (**) stated in words gives :

P [happening of at least one of the events $A_1, A_2, \ldots, A_n$]

$= 1 - P$ (none of the events $A_1, A_2, \ldots, A_n$ happens) ... (3·30)

or equivalently,

P {none of the given events happens} $= 1 - P$ {at least one of them happens}. ... (3·30a)

Example 3.38. *If $A \cap B = \phi$, then show that $P(A) \leq P(\bar{B})$.*

Solution. We have

$$A = (A \cap B) \cup (A \cap \bar{B}) = \phi \cup (A \cap \bar{B}) = A \cap \bar{B} \qquad [\because A \cap B = \phi \text{ (Given)}]$$

$$\therefore \qquad A \subseteq \bar{B} \quad \Rightarrow \quad P(A) \leq P(\bar{B}), \text{ as desired.}$$

Aliter. Since $A \cap B = \phi$, we have $A \subset \bar{B}$, which implies that $P(A) \leq P(\bar{B})$.

Example 3·39. *Let A and B be two events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$, show that*

(a) $P(A \cup B) \geq \frac{3}{4}$, and (b) $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$.

Solution. (a) we have

$$A \subset (A \cup B) \;\Rightarrow\; P(A) \leq P(A \cup B) \;\Rightarrow\; \frac{3}{4} \leq P(A \cup B) \Rightarrow P(A \cup B) \geq \frac{3}{4}$$

(b) $A \cap B \subseteq B \;\Rightarrow\; P(A \cap B) \leq P(B) = \frac{5}{8}$ ... (i)

Also $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1 \;\Rightarrow\; \frac{3}{4} + \frac{5}{8} - 1 \leq P(A \cap B)$

$\therefore \quad \frac{6 + 5 - 8}{8} \leq P(A \cap B) \;\Rightarrow\; \frac{3}{8} \leq P(A \cap B)$ ... (ii)

From (i) and (ii), we get $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$.

Example 3·40. *For any two events A and B,*

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Proof. We have

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

Using axiom 3, we have

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)] = P(A \cap \bar{B}) + P(A \cap B)$$

Now $\quad P[(A \cap \bar{B}) \geq 0$ (From axiom 1)

$\therefore$

$\quad P(A) \geq P(A \cap B)$ ... (*)

Similarly $\quad P(B) \geq P(A \cap B)$

$\Rightarrow \quad P(B) - P(A \cap B) \geq 0$

Now $\qquad P(A \cup B) = P(A) + [P(B) - P(A \cap B)]$ ...(**)

$\therefore \qquad P(A \cup B) \geq P(A) \quad \Rightarrow \quad P(A) \leq P(A \cup B)$ ...(***)

Also $\qquad P(A \cup B) \leq P(A) + P(B) \quad$ [From (**)] ...(****)

Hence from (*), (***) and (****), we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Aliter. Since $A \cap B \subset A$, by *Theorem 4·6 (ii)*, we get

$$P(A \cap B) \leq P(A).$$

Also $A \subset (A \cup B) \quad \Rightarrow \quad P(A) \leq P(A \cup B)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &\leq P(A) + P(B) \qquad [\because P(A \cap B) \geq 0] \end{aligned}$$

Combining the above results, we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

EXAMPLES ON ADDITION AND MULTIPLICATION THEOREMS OF PROBABILITY

Example 3·41. *The odds against Manager X settling the wage dispute with the workers are 8 : 6 and odds in favour of manager Y settling the same dispute are 14 : 16.*

(i) What is the chance that neither settles the dispute, if they both try, independently of each other ?

(ii) What is the probability that the dispute will be settled ?

Solution. Let A be the event that the manager X will settle the dispute and B be the event that the Manager Y will settle the dispute. Then clearly

$$P(\bar{A}) = \frac{8}{8+6} = \frac{4}{7} \quad \Rightarrow \quad P(A) = 1 - P(\bar{A}) = \frac{6}{14} = \frac{3}{7}$$

$$P(B) = \frac{14}{14+16} = \frac{7}{15} \quad \Rightarrow \quad P(\bar{B}) = 1 - P(B) = \frac{16}{14+16} = \frac{8}{15}$$

The required probability that neither settles the dispute is given by :

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B}) = \frac{4}{7} \times \frac{8}{15} = \frac{32}{105}$$

[Since A and B independent $\Rightarrow$ $\bar{A}$ and $\bar{B}$ are also independent]

(ii) The dispute will be settled if at least one of the managers X and Y settles the dispute. Hence the required probability is given by :

$$\begin{aligned} P(A \cup B) &= \text{Prob. [At least one of X and Y settles the dispute.]} \\ &= 1 - \text{Prob. [None settles the dispute.]} \\ &= 1 - P(\bar{A} \cap \bar{B}) = 1 - \frac{32}{105} = \frac{73}{105}. \qquad \text{[From Part (i)]} \end{aligned}$$

Example 3·42. *The odds that person X speaks the truth are 3 : 2 and the odds that person Y speaks the truth are 5 : 3. In what percentage of cases are they likely to contradict each other on an identical point.*

Solution. Let us define the events :

A : X speaks the truth, $\qquad B$: Y speaks the truth

Then $\bar{A}$ and $\bar{B}$ represent the complementary events the X and Y tell a lie respectively. We are given :

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\bar{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and
$$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\bar{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways :

(*i*) X speaks the truth and Y tells a lie, *i.e.*, the event $A \cap \bar{B}$ happens,

(*ii*) X tells a lie and Y speaks the truth, *i.e.*, the event $\bar{A} \cap B$ happens.

Hence by addition theorem of probability, the required probability is given by :

$$P(E) = P(i) + P(ii) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B) \qquad \text{[Since } A \text{ and } B \text{ are independent]}$$

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0{\cdot}475$$

Hence A and B are likely to contradict each other on an identical point in 47·5% of the cases.

Example 3·43. *One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0{\cdot}5$ $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$ and E_1, E_2, E_3, are independent events, find the probability that (a) exactly one hit is registered, and (b) at least two hits are registered.*

Solution. We are given : $P(E_1) = 0{\cdot}5$, $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$

$\Rightarrow$ $P(\bar{E}_1) = 0{\cdot}5$, $P(\bar{E}_2) = 0{\cdot}4$ and $P(\bar{E}_3) = 0{\cdot}2$

(*a*) Exactly one hit can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap \bar{E}_2 \cap \bar{E}_3$ happens, (*ii*) $\bar{E}_1 \cap E_2 \cap \bar{E}_3$ happens, (*iii*) $\bar{E}_1 \cap \bar{E}_2 \cap E_3$ happens.

Hence by addition probability theorem, the required probability 'p' is given by :

$$p = P(E_1 \cap \bar{E}_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap E_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap \bar{E}_2 \cap E_3)$$

$$= P(E_1)\, P(\bar{E}_2)\, P(\bar{E}_3) + P(\bar{E}_1)\, P(E_2)\, P(\bar{E}_3) + P(\bar{E}_1)\, P(\bar{E}_2)\, P(E_3)$$

[Since E_1, E_2 and E_3 are independent.]

$$= 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}26.$$

(*b*) At least two hits can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap E_2 \cap \bar{E}_3$ happens (*ii*) $E_1 \cap \bar{E}_2 \cap E_3$ happens

(*iii*) $\bar{E}_1 \cap E_2 \cap E_3$ happens, (*iv*) $E_1 \cap E_2 \cap E_3$ happens.

$\therefore$ Required probability

$$= P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8$$

$$= 0{\cdot}06 + 0{\cdot}16 + 0{\cdot}24 + 0{\cdot}24 = 0{\cdot}70.$$

Example 3·44. *An urn contains 4 tickets numbered 1, 2, 3, 4 and another contains 6 tickets numbered 2, 4, 6, 7, 8, 9. If one of the two urns is chosen at random and a ticket is* *drawn at random from the chosen urn, find the probabilities that the ticket drawn bears the number (i) 2 or 4, (ii) 3, (iii) 1 or 9.*

Solution. (*i*) Required event can happen in the following mutually exclusive ways :

(*I*) First urn is chosen and then a ticket is drawn.

(*II*) Second urn is chosen and then a ticket is drawn.

Since the probability of choosing any urn is $\frac{1}{2}$, required probability '*p*' is given by :

$$p = P(I) + P(II) = \frac{1}{2} \times \frac{2}{4} + \frac{1}{2} \times \frac{2}{6} = \frac{5}{12}$$

(*ii*) Required probability $= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times 0 = \frac{1}{8}$

($\because$ In the 2nd urn there is no ticket with number 3.)

(*iii*) Required probability $= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{24}$.

Example 3·45. *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.*

Solution. The required event E that 'in a draw of 4 balls from the box at random there is at least one ball of each colour', can materialise in the following mutually disjoint ways :

(*i*) 1 Red, 1 White, 2 Black balls ; (*ii*) 2 Red, 1 White, 1 Black balls; (*iii*) 1 Red, 2 White, 1 Black balls.

Hence by addition theorem of probability, the required probability is given by :

$$\begin{aligned} P(E) &= P(i) + P(ii) + P(iii) \\ &= \frac{{}^6C_1 \times {}^4C_1 \times {}^5C_2}{{}^{15}C_4} + \frac{{}^6C_2 \times {}^4C_1 \times {}^5C_1}{{}^{15}C_4} + \frac{{}^6C_1 \times {}^4C_2 \times {}^5C_1}{{}^{15}C_4} \\ &= \frac{1}{{}^{15}C_4}[6 \times 4 \times 10 + 15 \times 4 \times 5 + 6 \times 6 \times 5] = \frac{4!}{15 \times 14 \times 13 \times 12}(240 + 300 + 180) \\ &= \frac{24 \times 720}{15 \times 14 \times 13 \times 12} = 0{\cdot}5275. \end{aligned}$$

Example 3·46. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is 13/32.*

Solution. The required event of getting 1 girl and two boys among the three selected children can materialise in the following three mutually disjoint cases :

Group No. →	I	II	III
(*i*)	Girl	Boy	Boy
(*ii*)	Boy	Girl	Boy
(*iii*)	Boy	Boy	Girl

Hence by addition theorem of probability,

Required probability $= P(i) + P(ii) + P(iii)$ ...(*)

Since the probability of selecting a girl from the first group is 3/4, of selecting a boy from the second group is 2/4, and of selecting a boy from the third group is 3/4, and since these three events of selecting children from three groups are independent of each other, by compound probability theorem, we have

$$P(i) = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}; \quad P(ii) = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}; \quad \text{and} \quad P(iii) = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$$

Substituting in (*), we get

$$\text{Required probability} = \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}.$$

Example 3·47. *It is 8 : 5 against the wife who is 40 years old living till she is 70 and 4 : 3 against her husband now 50 living till he is 80. Find the probability that*

(i) Both will be alive, *(ii) None will be alive,*
(iii) Only wife will be alive, *(iv) Only husband will be alive,*
(v) Only one will be alive, *(vi) At least one will be alive,*

30 years hence.

Solution. Let us define the events :

A : Wife will be alive, and B : Husband will be alive; 30 years hence.

Then, we are given :

$$P(A) = \frac{5}{8+5} = \frac{5}{13} \quad \Rightarrow \quad P(\bar{A}) = 1 - P(A) = \frac{8}{13}$$

$$P(B) = \frac{3}{4+3} = \frac{3}{7} \quad \Rightarrow \quad P(\bar{B}) = 1 - P(B) = \frac{4}{7}$$

If we assume that A and B are independent so that A and $\bar{B}$, $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$ are also independent, then the required probabilities are given by :

(i) $P(A \cap B) = P(A)\, P(B) = \frac{5}{13} \times \frac{3}{7} = \frac{15}{91}$ ($\because$ A and B are independent.)

(ii) $P(\bar{A} \cap \bar{B}) = P(\bar{A})\, P(\bar{B}) = \frac{8}{13} \times \frac{4}{7} = \frac{32}{91}$ ($\because$ $\bar{A}$, $\bar{B}$ are independent.)

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{5}{13} - \frac{15}{91} = \frac{20}{91}$ [From Part (i)]

(iv) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{3}{7} - \frac{15}{91} = \frac{24}{91}$ [From Part (i)]

(v) $P(A \cap \bar{B}) + P(\bar{A} \cap B) = \frac{20}{91} + \frac{24}{91} = \frac{44}{91}$ [From Parts (iii) and (iv)]

(vi) $P(A \cup B) = 1 - (\bar{A} \cap \bar{B}) = 1 - \frac{32}{91} = \frac{59}{91}$. [From Part (ii)]

Example 3·48. *A problem in Statistics is given to three students A, B and C whose chances of solving it are 1/2, 3/4 and 1/4 respectively.*

What is the probability that the problem will be solved if all of them try independently ?

Solution. Let A, B, C denote the events that the problem is solved by the students A, B, C respectively. Then

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{3}{4}, \quad \text{and} \quad P(C) = \frac{1}{4}$$

The problem will be solved if at least one of them solves the problem. Thus we have to calculate the probability of occurrence of at least one of the three events A, B, C, *i.e.*, $P(A \cup B \cup C)$.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$= P(A) + P(B) + P(C) - P(A)\,P(B) - P(A)\,P(C) - P(B)\,P(C) + P(A)\,P(B)\,P(C)$$
($\because$ A, B, C are mutually independent events.)
$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2}\cdot\frac{3}{4} - \frac{3}{4}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{29}{32}.$$

Aliter. $P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$
$$= 1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C})$$
[$\because$ A, B, C are mutually independent $\Rightarrow \bar{A}, \bar{B}$ and $\bar{C}$ are mutually independent.]
$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right) = 1 - \frac{3}{32} = \frac{29}{32}.$$

Example 3·49. *A manager has two assistants and he bases his decision on information supplied independently by each one of them. The probability that he makes a mistake in his thinking is 0·005. The probability that an assistant gives wrong information is 0·3. Assuming that the mistakes made by the manager are independent of the information given by the assistants, find the probability that he reaches a wrong decision.*

Solution. Let us define the following events :

A : The manager makes a mistake in his thinking.
B : The 1st assistant gives him wrong information.
C : The 2nd assistant gives him wrong information.

In usual notations, we are given :
$$P(A) = 0{\cdot}005,\ P(B) = 0{\cdot}3 = P(C) \quad \Rightarrow \quad P(\bar{A}) = 0{\cdot}995,\ P(\bar{B}) = P(\bar{C}) = 0{\cdot}7$$

Assuming that the mistakes made by the manager are independent of the information supplied independently by each of the two assistants, we conclude that A, B and C, and consequently $\bar{A}$, $\bar{B}$ and $\bar{C}$ are mutually independent.

$\therefore \quad p = P$ [Manager reaches a wrong decision]

$= 1 - P$ [Manager reaches a correct decision] $= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$; because manager will reach a correct decision if he does not make a mistake in his thinking (*i.e.*, $\bar{A}$ happens) *and* both the assistants supply him correct information (*i.e.*, $\bar{B} \cap \bar{C}$ happens).

$\therefore \quad p = 1 - P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$

[Since the events $\bar{A}$, $\bar{B}$ and $\bar{C}$ are independent.]

$= 1 - 0{\cdot}995 \times 0{\cdot}7 \times 0{\cdot}7 = 1 - 0{\cdot}48755 = 0{\cdot}51245.$

Example 3·50. *The odds that a book on Statistics will be favourably reviewed by 3 independent critics are 3 to 2, 4 to 3 and 2 to 3 respectively. What is the probability that of the three reviews :*

(i) *All will be favourable,*
(ii) *Majority of the reviews will be favourable,*
(iii) *Exactly one review will be favourable,*
(iv) *Exactly two reviews will be favourable, and*
(v) *At least one of the reviews will be favourable.*

Solution. Let A, B and C denote respectively the events that the book is favourably reviewed by first, second and third critic respectively. Then we are given :

$$P(A) = \frac{3}{5},\ P(B) = \frac{4}{7} \text{ and } P(C) = \frac{2}{5} \quad \Rightarrow \quad P(\bar{A}) = \frac{2}{5},\ P(\bar{B}) = \frac{3}{7} \text{ and } P(\bar{C}) = \frac{3}{5}$$

(i) The probability that all the three reviews will be favourable is :

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C) = \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{24}{175}$$

($\because$ A, B and C are mutually independent events.)

(ii) The event that majority, *i.e.*, at least 2 reviews are favourable can materialise in the following mutually exclusive ways :

(a) $A \cap B \cap \bar{C}$ happens, (b) $A \cap \bar{B} \cap C$ happens, (c) $\bar{A} \cap B \cap C$ happens, and (d) $A \cap B \cap C$ happens.

Hence, the required probability is :

$$P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(\bar{B})\,P(C) + P(\bar{A})\,P(B)\,P(C) + P(A)\,P(B)\,P(C)$$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} + \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{94}{175}$$

(iii) Arguing as in case (ii), the probability that exactly one review will be favourable is

$$P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$$

$$= \frac{3}{5} \times \frac{3}{7} \times \frac{3}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{7} \times \frac{2}{5} = \frac{63}{175}$$

(iv) Similarly, the probability that exactly two reviews will be favourable is :

$$P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$$

$$= P(A) \times P(B) \times P(\bar{C}) + P(A) \times P(\bar{B}) \times P(C) + P(\bar{A}) \times P(B) \times P(C)$$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{70}{175}$$

(v) The probability that at least one of the reviews will be favourable is :

$$P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap (\bar{C}) = 1 - P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$$

$$= 1 - \frac{2}{5} \times \frac{3}{7} \times \frac{3}{5} = \frac{157}{175}$$

In (ii) to (v) we have used that A, B and C are mutually independent.

Example 3·51. *A and B alternately cut a pack of cards and the pack is shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what are the respective chances of A and B first cutting a diamond ?*

Solution. Let A_i and B_i denote the events of A and B cutting a diamond respectively in the ith trial. Then, we are given :

$$P(A_i) = P(B_i) = \frac{13}{52} = \frac{1}{4} \quad \Rightarrow \quad P(\bar{A}_i) = P(\bar{B}_i) = \frac{3}{4};\ i = 1, 2, 3, \ldots$$

If A starts the game, he can first cut the diamond in the following mutually exclusive ways :

(i) A_1 happens, (ii) $\bar{A}_1 \cap \bar{B}_2 \cap A_3$ happens, (iii) $\bar{A}_1 \cap \bar{B}_2 \cap \bar{A}_3 \cap \bar{B}_4 \cap A_5$ happens,

Example 3·54. *An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i, $(i = 1, 2, 3)$ be the event that ith digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.*

Solution. A_1 is the event that the first digit of the number of the ticket drawn is 1 and the favourable cases for this are 112 and 121, *i.e.*, two cases.

$$\therefore \quad P(A_1) = \frac{2}{4} = \frac{1}{2}. \quad \text{Similarly, we get} \quad P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2}$$

$A_1 \cap A_2$ is the event that the first two digits in the number which the selected ticket bears are each equal to unity and the only favourable case is ticket with number 112.

$$\therefore \quad P(A_1 \cap A_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A_1)\, P(A_2)$$

Similarly, $P(A_2 \cap A_3) = \frac{1}{4} = P(A_2)\, P(A_3)$ and $P(A_3 \cap A_1) = \frac{1}{4} = P(A_3)\, P(A_1)$

Thus we conclude that the events A_1, A_2 and A_3 are pairwise independent. Now

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P\{\text{all the three digits in the number selected are 1's}\} \\ &= P(\phi) = 0 \\ &\neq P(A_1)\, P(A_2)\, P(A_3) \end{aligned}$$

Hence A_1, A_2 and A_3, though pairwise independent are not mutually independent.

Example 3·55. *Two fair dice are thrown independently. Three events A, B and C are defined as follows :*

A : Odd face with first dice

B : Odd face with second dice

C : Sum of points on two dice is odd.

Are the events A, B and C (i) Pairwise independent, (ii) Mutually independent ?

Solution. In a random toss of two dice

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(S) = 6 \times 6 = 36$$

Event	*Favourable cases*	*No. of favourable cases*
A	$\{1, 3, 5\} \times \{1, 2, 3, 4, 5, 6\}$	$3 \times 6 = 18$
B	$\{1, 2, 3, 4, 5, 6\} \times \{1, 3, 5\}$	$6 \times 3 = 18$
*C	$\{1, 3, 5\} \times \{2, 4, 6\} \cup \{2, 4, 6\} \times \{1, 3, 5\}$	$3 \times 3 + 3 \times 3 = 18$
$A \cap B$	$\{1, 3, 5\} \times \{1, 3, 5\}$	$3 \times 3 = 9$
** $A \cap C$	$\{1, 3, 5\} \times \{2, 4, 6\}$	$3 \times 3 = 9$
** $B \cap C$	$\{2, 4, 6\} \times \{1, 3, 5\}$	$3 \times 3 = 9$
$A \cap B \cap C$	ϕ	0

* The sum of points on two dice be odd if one shows odd number and the other shows even number.

** If one die shows odd number and the sum is also odd, then the other die must show even number.

$$\left.\begin{aligned} P(A) &= \frac{18}{36} = \frac{1}{2} = P(B) = P(C) \\ \therefore \quad P(A \cap B) &= \frac{9}{36} = \frac{1}{4} = P(A)\,P(B) \\ P(A \cap C) &= \frac{9}{36} = \frac{1}{4} = P(A)\,P(C) \\ P(B \cap C) &= \frac{9}{36} = \frac{1}{4} = P(B)\,P(C) \end{aligned}\right\} \quad \ldots (*)$$

and $\quad P(A \cap B \cap C) = P(\phi) = 0 \neq P(A)\,P(B)\,P(C) \quad \ldots (**)$

Hence (*) implies that the events A, B and C are pairwise independent but (**) implies that they not mutually independent.

Example 3·56. *Why does it pay to bet consistently on seeing 6 at least once in 4 throws of a die, but not on seeing a double six at least once in 24 throws with two dice ? (de Mere's Problem).*

Solution. The probability of getting a '6' in throw of die $= \frac{1}{6}$

$\therefore$ The probability of not getting a '6' in a throw of die $= 1 - \frac{1}{6} = \frac{5}{6}$

By compound probability theorem, the probability that in 4 throws of die no '6' is obtained $= \left(\frac{5}{6}\right)^4$

Hence, the probability of obtaining '6' at least once in 4 throws of a die

$$= 1 - \left(\frac{5}{6}\right)^4 = 0{\cdot}516$$

Now, if a trial consists of throwing two dice at a time, then the probability of getting a 'double' of '6' in a trial $= \frac{1}{36}$

Thus the probability of not getting a 'double of 6' in a trial $= \frac{35}{36}$

The probability that in 24 throws, with two dice each, no 'double of 6' is obtained

$$= \left(\frac{35}{36}\right)^{24}$$

Hence the probability of getting a 'double of 6' at least once in 24 throws

$$= 1 - \left(\frac{35}{36}\right)^{24} = 0{\cdot}491$$

Since the probability in the first case is greater than the probability in the second case, the result follows.

Example 3·57. *Let $A_1, A_2, \ldots A_n$ be independent events and $P(A_k) = p_k$. Further, let p be the probability that none of the events occurs ; then show that $p \leq \exp\left(-\sum_k p_k\right)$.*

Solution. We have $\quad p_i = P(A_i)\ ;\ i = 1, 2, \ldots, n$

CHAPTER CONCEPTS QUIZ

1. Find out the correct answer from group Y for each item of group X.

Group X

(*a*) At least one of the events A or B occurs.

(*b*) Neither A nor B occurs.

(*c*) Exactly one of the events A or B occurs.

(*d*) If event A occurs, so does B.

(*e*) Not more than one of the events A or B occurs.

Group Y

(*i*) $(\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B})$ e

(*ii*) $(A \cup B) - (A \cap B)$

(*iii*) $A \subset B$ d

(*iv*) $B \subset A$

(*v*) $[A - (A \cap B)] \cup [B - A \cap B)]$ c

(*vi*) $\bar{A} \cap \bar{B}$ b

(*vii*) $A \cup B$ a

(*viii*) $S - (A \cup B)$

2. Match the correct expression of probabilities on the left :

(*a*) $P(\phi)$, where ϕ is null set

(*b*) $P(A \mid B)\, P(B)$

(*c*) $P(\bar{A})$

(*d*) $P(\bar{A} \cap \bar{B})$

(*e*) $P(A - B)$

(*i*) $1 - P(A)$ c

(*ii*) $P(A \cap B)$ b

(*iii*) $P(A) - P(A \cap B)$ e

(*iv*) 0 a

(*v*) $1 - P(A) - P(B) + P(A \cap B)$ d

(*f*) $P(A \cup B)$ (*vi*) $P(A) + P(B) - P(A \cap B)$ f

3. Given that A, B and C are mutually exclusive events, explain why the following are not permissible assignments of probabilities :

(*i*) $P(A) = 0{\cdot}24$, $P(B) = 0{\cdot}4$ and $P(A \cup C) = 0{\cdot}2$

(*ii*) $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}61$

4. In each of the following, indicate whether events A and B are :
(*i*) independent, (*ii*) mutually exclusive, (*iii*) dependent but not mutually exclusive.

(*a*) $P(A \cap B) = 0$ (*b*) $P(A \cap B) = 0{\cdot}3$, $P(A) = 0{\cdot}45$

(*c*) $P(A \cup B) = 0{\cdot}85$, $P(A) = 0{\cdot}3$, $P(B) = 0{\cdot}6$

(*d*) $P(A \cup B) = 0{\cdot}70$, $P(A) = 0{\cdot}5$, $P(B) = 0{\cdot}4$

(*e*) $P(A \cup B) = 0{\cdot}90$, $P(A \mid B) = 0{\cdot}8$, $P(B) = 0{\cdot}5$.

5. Give the correct label as answer like *a* or *b* etc., for the following questions :

(*i*) The probability of drawing any one spade card from a pack of cards is

(*a*) $\frac{1}{52}$ (*b*) $\frac{1}{13}$ (*c*) $\frac{4}{13}$ (*d*) $\frac{1}{4}$

(*ii*) The probability of drawing one white ball from a bag containing 6 red, 8 black, 10 yellow and 1 green balls is

(*a*) $\frac{1}{25}$ (*b*) 0 (*c*) 1 (*d*) $\frac{24}{25}$ (*e*) $\frac{15}{20}$

(*iii*) A coin is tossed three times in succession, the number of sample points in sample space is

(*a*) 6 (*b*) 8 (*c*) 3 (*d*) 9

(*iv*) In the simultaneous tossing of two perfect coins, the probability of having at least one head is

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{3}{4}$ (*d*) 1

(*v*) In the simultaneous tossing of two perfect dice, the probability of obtaining 4 as the sum of the resultant faces is

(*a*) $\frac{4}{12}$ (*b*) $\frac{1}{12}$ (*c*) $\frac{3}{12}$ (*d*) $\frac{2}{12}$

(*vi*) A single letter is selected at random from the word 'probability'. The probability that it is a vowel is

(*a*) $\frac{3}{11}$ (*b*) $\frac{2}{11}$ (*c*) $\frac{4}{11}$ (*d*) 0

(*vii*) An urn contains 9 balls, two of which are red, three blue and four black. Three balls are drawn at random. The chance that they are of the same colour is

(*a*) $\frac{5}{84}$ (*b*) $\frac{3}{9}$ (*c*) $\frac{3}{7}$ (*d*) $\frac{7}{17}$

(*viii*) A number is chosen at random among the first 120 natural numbers. The probability of the number chosen being a multiple of 5 or 15 is

(*a*) $\frac{1}{5}$ (*b*) $\frac{1}{8}$ (*c*) $\frac{1}{16}$ (*d*) $\frac{1}{9}$.

(*ix*) If A and B are mutually exclusive events, then

(*a*) $P(A \cup B) = P(A) \cdot P(B)$, (*b*) $P(A \cup B) = P(A) + P(B)$, (*c*) $P(A \cup B) = 0$.

(*x*) If A and B are two independent events, the probability that both A and B occrs is $\frac{1}{8}$ and the probability that neither of them occurs is $\frac{3}{8}$. If $P(A) < P(B)$, then the probability of the occurrence of A is :

(*a*) $\frac{1}{2}$, (*b*) $\frac{1}{3}$, (*c*) $\frac{1}{4}$, (*d*) $\frac{1}{5}$.

(*xi*) If A and B are two events, the probability that exactly one of them occurs is given by :

(*a*) $P(A) + P(B) - 2P(A \cap B)$, (*b*) $P(A) + P(B) - P(A \cap B)$

(*c*) $P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$, (*d*) $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

(*xii*) If $P(A \cap B) = \frac{1}{2}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$, and $P(A) = P(B) = p$, then the value of p is given by :

(*a*) $\frac{1}{2}$, (*b*) $\frac{7}{8}$, (*c*) $\frac{1}{3}$, (*d*) $\frac{7}{12}$.

(*xiii*) If $P(A \cap B) = \frac{1}{2}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{2}$ and $2P(A) = P(B) = p$, then the value of p is given by :

(*a*) $\frac{1}{4}$, (*b*) $\frac{1}{2}$, (*c*) $\frac{1}{3}$, (*d*) $\frac{2}{3}$.

(*xiv*) A and B are two independent events such that $P(A \cap \bar{B}) = \frac{3}{25}$ and $P(\bar{A} \cap B) = \frac{8}{25}$. If $P(A) < P(B)$, then $P(A)$ is :

(*a*) $\frac{1}{5}$, (*b*) $\frac{2}{5}$, (*c*) $\frac{3}{5}$, (*d*) $\frac{4}{5}$.

(*xv*) A and B are two independent events such that $P(\bar{A}) = 0{\cdot}7$, $P(\bar{B}) = k$ and $P(A \cup B) = 0{\cdot}8$, then k is

(*a*) $\frac{5}{7}$, (*b*) 1, (*c*) $\frac{2}{7}$, (*d*) none of these

(*xvi*) The probability that a 3-card hand drawn at random and without replacement from an ordinary deck consists entirely of black cards is :

(*a*) $\frac{1}{17}$, (*b*) $\frac{2}{17}$, (*c*) $\frac{1}{8}$, (*d*) $\frac{3}{17}$, (*e*) $\frac{4}{17}$.

(*xvii*) What is the probability that a bridge hand contains one card of each denomination (*i.e.*, one ace, one king, one queen, ..., one three, one two) ?

(*a*) $\frac{13!}{13^{13}}$, (*b*) $\frac{4^{13}}{^{52}C_{13}}$, (*c*) $\frac{^{52}C_4}{^{52}C_{13}}$ (*d*) $\left(\frac{1}{13}\right)^{13}$ (*e*) $\frac{13^4}{^{52}C_{13}}$

(*xviii*) If the events S and T have equal probability and are independent with $P(S \cap T) = p > 0$, then $P(S)$

(*a*) $\sqrt{p}$, (*b*) p^2, (*c*) $\frac{p}{2}$, (*d*) p, (*e*) $2p$

(*xix*) The probability that both S and T occur, the probability that S occurs and T does not, and the probability that T occurs and S does not are all equal to p. The probability that either S or T occurs is :

(*a*) p, (*b*) $2p$, (*c*) $3p$, (*d*) $3p^2$, (*e*) p^3

(*x*) Events S and T are independent with $P(S) < P(T)$, $P(S \cap T) = 6/25$, and $P(S \mid T) + P(T \mid S) = 1$. Then $P(S)$ is

(*a*) $\frac{1}{25}$, (*b*) $\frac{1}{5}$, (*c*) $\frac{6}{25}$, (*d*) $\frac{2}{5}$, (*e*) $\frac{3}{5}$

(*xxi*) An unbiased die is thrown two independent times. Given that the first throw resulted in an even number, the probability that the sum obtained is 8 is :

(*a*) $\frac{5}{36}$, (*b*) $\frac{1}{6}$, (*c*) $\frac{4}{21}$, (*d*) $\frac{7}{36}$, (*e*) $\frac{1}{3}$

6. Fill in the blanks :

(i) Two events are said to be equally likely if ......

(ii) A set of events is said to be independent if .....

(iii) If $P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C)$, then the events A, B, C are .....

(iv) Two events A and B are mutually exclusive if $P(A \cap B) = \ldots$ and are independent if $P(A \cap B) = \ldots$

(v) The probability of getting a multiple of 2 in a throw of a dice is 1/2 and of getting a multiple of 3 is 1/3. Hence probability of getting a multiple of 2 or 3 is ......

(vi) Let A and B be independent events and suppose the event C has probability 0 or 1. Then A, B and C are ...... events.

(vii) If A, B, C are pairwise independent and A is independent of $B \cup C$, then, A, B, C are ...... independent.

(viii) A man has tossed 2 fair dice. The conditional probability that he has tossed two sixes, given that he has tossed at least one six is ......

(ix) Let A and B be two events such that $P(A) = 0{\cdot}3$ and $P(A \cup B) = 0{\cdot}8$. If A and B are independent events than $P(B) = \ldots$

(x) If $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are probabilities of three mutually exclusive events, then the set of all values of p is ...

(xi) If A and B are two events, then $P(A \cup B) = P(A \cap B)$ if and only if relation between $P(A)$ and $P(B)$ is ...

(xii) A bag contains tickets numbered 1 to 100. Ten tickets are drawn at random and arranged in ascending order. The probability that fourth and sixth ticket bear numbers 50 and 60 respectively is ......

7. Each of the following statements is either true or false. If it is true prove it, otherwise, give a counter example to show that it is false :

(i) The probability of occurrence of at least one of two events is the sum of the probability of each of the two events.

(ii) Mutually exclusive events are independent.

(iii) For any two events A and B, $P(A \cap B)$ cannot be less than either $P(A)$ or $P(B)$.

(iv) The conditional probability of A given B is always greater than $P(A)$.

(v) If the occurrence of an event A implies the occurrence of another event B then $P(A)$ cannot exceed $P(B)$.

(vi) For any two events A and B, $P(A \cup B)$ cannot be greater than either $P(A)$ or $P(B)$.

(vii) Mutually exclusive events are not independent.

(viii) Pairwise independence does not necessarily imply mutual independence.

(ix) Let A and B be events neither of which has probability zero. Then if A and B are disjoint, A and B are independent.

(x) The probability of any event is always a proper fraction.

(xi) If $0 < P(B) < 1$ so that $P(A \mid B)$ and $P(A \mid \bar{B})$ are both defined, then

$$P(A) = P(B)\, P(A \mid B) + P(\bar{B})\, P(A \mid \bar{B}).$$

(xii) For two events A and B if $P(A) = P(A \mid B) = 1/4$ and $P(A \mid \bar{B}) = 1/2$, then

(a) A and B are mutually exclusive. (b) A and B are independent.

(c) A is a sub-event of B. (d) $P(\bar{A} \mid B) = 3/4$.

(*xiii*) Two events can be independent and mutually exclusive simultaneously.

(*xiv*) Let A and B be events, neither of which has probability zero. Prove or disprove the following :

(*a*) If A and B are disjoint, A and B are independent.

(*b*) If A and B are independent, A and B are disjoint.

(*xv*) If $P(A) = 0$, then $A = \phi$.

(*xvi*) For two events A and B we have the following probabilities :

$$P(A) = P(A \mid B) = \frac{1}{4} \text{ and } P(B \mid A) = \frac{1}{2}.$$

Check whether the following statements are true or false :

(*a*) A and B are mutually exclusive, (*b*) A and B are independent,

(*c*) A is a sub-event of B, (*d*) $P(\bar{A} \mid B) = \frac{3}{4}$.

(*xvii*) Consider two events A and B such that $P(A) = \frac{1}{4}$, $P(B \mid A) = \frac{1}{2}$, $P(A \mid B) = \frac{1}{4}$. For each of the following statements, ascertain whether it is true or false :

(*i*) A is a sub-event of B, (*ii*) $P(\bar{A} \mid \bar{B}) = \frac{3}{4}$ (*iii*) $P(A \mid B) + P(A \mid \bar{B}) = 1$

(*xviii*) If A and B are two events none of which has probability zero, then A and B are disjoint implies that A and B are independent.

DISCUSSION AND REVIEW QUESTIONS

1. (*a*) Give the classical and statistical definitions of probability. What are the objections raised in these definitions ?

(*b*) When are a number of cases said to be equally likely ? Give an example of the following :

(*i*) the equally likely cases,

(*ii*) four cases which are not equally likely, and

(*iii*) five cases in which one case is more likely than the other four.

(*c*) What is meant by mutually exclusive events ? Give an example of

(*i*) three mutually exclusive events,

(*ii*) three events which are not mutually exclusive.

(*d*) Can

(*i*) events be mutually exclusive and exhaustive ?

(*ii*) events be exhaustive and independent ?

(*iii*) events be mutually exclusive and independent ?

(*iv*) events be exhaustive, mutually exclusive and independent ?

2. (*i*) If A, B and C are any three events, write down the theoretical expressions for the following events :

(*a*) Only A occurs, (*b*) A and B occur but C does not,

(*c*) A, B, and C all the three occur, (*d*) At least one occurs

(*e*) At least two occur, (*f*) One does not occur,

(*g*) Two do not occur, (*h*) None occurs.

4·2. BAYES' THEOREM

In the discussion of conditional probability we indicated that revising probability when new information is obtained is an important phase of probability analysis. Often, we begin our analysis with initial or *prior* probability estimates for specific events of interest. Then, from sources such as a sample, a special report, a product test, and so on we obtain some additional information about the events. Given this new information, we update the prior probability values by calculating revised probabilities, referred to as *posterior probabilities*. Bayes' theorem *which was given by Thomas Bayes, a British Mathematician,* in 1763, provides a means for making these probability calculations. The steps in this probability revision process are shown in the following diagram :

Prior Probabilities	→	New Information	→	Application of Bayes' Theorem	→	Posterior Probabilities

Theorem 4·2. *Bayes' Theorem. If* $E_1, E_2, E_2 \ldots, E_n$ *are mutually disjoint events with* $P(E_i) \neq 0$, $(i = 1, 2, \ldots, n)$, *then for any arbitrary event A which is a subset of* $\bigcup_{i=1}^{n} E_i$ *such that* $P(A) > 0$, *we have*

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} = \frac{P(E_i)\, P(A \mid E_i)}{P(A)}\ ; i = 1, 2, \ldots, n \qquad \ldots (4·9)$$

Proof. Since $A \subset \bigcup_{i=1}^{n} E_i$, we have, $A = A \cap \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (A \cap E_i)$

[By distributive law]

Since $(A \cap E_i) \subset E_i$, $(i = 1, 2, \ldots, n)$ are mutually disjoint events, we have by addition theorem of probability :

$$P(A) = P\left\{ \bigcup_{i=1}^{n} (A \cap E_i) \right\} = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i), \qquad \ldots(*)$$

by multiplication theorem of probability.

Also we have $\quad P(A \cap E_i) = P(A)\, P(E_i \mid A)$

$$\Rightarrow \qquad P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \text{[From (*)]}$$

Example 4·4. *A factory produces a certain type of outputs by three types of machine. The respective daily production figures are :*

Machine I : 3,000 Units; Machine II : 2,500 Units; Machine III : 4,500 Units

Past experience shows that 1 per cent of the output produced by Machine I is defective. The corresponding fraction of defectives for the other two machines are 1·2 per-cent and 2 per-cent respectively. An item is drawn at random from the day's production run and is found to be defective. What is probability that it comes from the output of

(i) Machine I, *(ii) Machine II,* *(iii) Machine (iii) ?*

Solution. Let E_1, E_2 and E_3 denote the events that the output is produced by machines I, II and III respectively and let A denote the event that the output is defective. Then we have :

$$P(E_1) = \frac{3000}{10{,}000} = 0{\cdot}30, \quad P(E_2) = \frac{2500}{10{,}000} = 0{\cdot}25, \quad P(E_3) = \frac{4500}{10{,}000} = 0{\cdot}45$$

$$P(A \mid E_1) = 1\% = 0{\cdot}01, \quad P(A \mid E_2) = 1{\cdot}2\% = 0{\cdot}012, \quad P(A \mid E_3) = 2\% = 0{\cdot}02$$

The probability that an item selected at random from day's production is defective is given by :

$$P(A) = \sum_{i=1}^{3} P(E_i \cap A) = \sum_{i=1}^{3} P(E_i) \cdot P(A \mid E_i)$$

$$= 0{\cdot}30 \times 0{\cdot}01 + 0{\cdot}25 \times 0{\cdot}012 + 0{\cdot}45 \times 0{\cdot}02 = 0{\cdot}015$$

By Baye's rule, the required probabilities are given by :

(i) $$P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{P(A)} = \frac{0{\cdot}003}{0{\cdot}015} = \frac{1}{5}$$

(ii) $$P(E_2 \mid A) = \frac{P(E_2) \cdot P(A \mid E_3)}{P(A)} = \frac{0{\cdot}003}{0{\cdot}015} = \frac{1}{5}$$

(iii) $$P(E_3 \mid E) = \frac{P(E_3) \cdot P(A \mid E_3)}{P(A)} = \frac{0{\cdot}009}{0{\cdot}015} = \frac{3}{5}$$

The probabilities in (i), (ii) and (iii) are known as posterior probabilities of events E_1, E_2 and E_3 respectively.

Aliter. The posterior probabilities can be obtained elegantly in a tabular form as given below.

Events E_i	*Prior Probabilities $P(E_i)$*	*Conditional Probabilities $P(A \mid E_i)$*	*Joint Probabilities $P(E_i \cap A)$*	*Posterior Probabilities $P(E_i \mid A)$*
(1)	(2)	(3)	(4) = (2) × (3)	(5) = (4) ÷ P (A)
E_1	0·30	0·010	0·003	$\frac{0{\cdot}003}{0{\cdot}015} = \frac{1}{5}$
E_2	0·25	0·012	0·003	$\frac{0{\cdot}003}{0{\cdot}015} = \frac{1}{5}$
E_3	0·45	0·020	0·009	$\frac{0{\cdot}009}{0{\cdot}015} = \frac{3}{5}$
Total	1·00		$P(A) = 0{\cdot}015$	1

Example 4·5. *There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is $\frac{6}{7}$, find the value of n.*

Solution. Let E_1 denote the event that bag A is selected and E_2 denote the event that bag B is selected. Let E be the event that two balls drawn are white. We have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E \mid E_1) = \frac{{}^nC_2}{{}^{n+2}C_2} = \frac{n(n-1)}{(n+2)(n+1)}$$

and $$P(E \mid E_2) = \frac{{}^2C_2}{{}^{n+2}C_2} = \frac{2}{(n+2)(n+1)}$$

Using Baye's Theorem, the probability that the two white balls drawn are from the bag A, is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\, P(E \mid E_1)}{P(E_1)\, P(E \mid E_1) + P(E_2)\, P(E \mid E_2)} = \frac{6}{7} \quad \text{(Given)}$$

$$\Rightarrow \quad \frac{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)}}{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)} + \frac{1}{2} \cdot \frac{2}{(n+2)(n+1)}} = \frac{6}{7} \quad \Rightarrow \quad \frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$$

$$\therefore \quad 7n(n-1) = 6n(n-1) + 12 \quad \Rightarrow \quad n^2 - n - 12 = 0 \quad \Rightarrow \quad n = 4, -3.$$

Since n cannot be negative, we get $n = 4$.

Example 4·6. *A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA ?*

Solution. Let E_1 and E_2 denote the events that the letter came from TATANAGAR and CALCUTTA respectively. Let A denote the event that two consecutive visible letters on the envelop are TA. We have

$$P(E_1) = P(E_2) = \frac{1}{2},\ P(A \mid E_1) = \frac{2}{8} \quad \text{and} \quad P(A \mid E_2) = \frac{1}{7}.$$

Using the Bayes' theorem, we get

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)} = \frac{\frac{1}{2}\cdot\frac{1}{7}}{\frac{1}{2}\cdot\frac{2}{8} + \frac{1}{2}\cdot\frac{1}{7}} = \frac{4}{11}.$$

Example 4·7. *The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly ?*

Solution. Let us define the following events :

E_1 : Disease X is diagnosed correctly by doctor A.
E_2 : Disease X is not diagnosed correctly by doctor A.
E : A patient (of Dr A) who had disease X dies.

Then, we are given :

$$P(E_1) = 0{\cdot}6 \quad ; \quad P(E \mid E_1) = 0{\cdot}4$$

$$P(E_2) = P(\bar{E}_1) = 1 - P(E_1) = 0{\cdot}4 \quad ; \quad P(E \mid E_2) = 0{\cdot}7$$

$$\therefore \quad P(E) = \sum_{i=1}^{2} P(E_i)\,P(E \mid E_i) = 0{\cdot}6 \times 0{\cdot}4 + 0{\cdot}4 \times 0{\cdot}7 = 0{\cdot}52$$

Using the Bayes' thereon, the required probability is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E)} = \frac{0{\cdot}6 \times 0{\cdot}4}{0{\cdot}52} = \frac{0{\cdot}24}{0{\cdot}52} = \frac{6}{13}.$$

Example 4·8. *The contents of urns I, II and III are as follows :*

1 white, 2 black and 3 red balls,
2 white, 1 black and 1 red balls, and
4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn from it. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solution. Let E_1, E_2, and E_3 denote the events that the urn, I, II and III is chosen, respectively, and let A be the event that the two balls taken from the selected urn are white and red.

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A \mid E_1) = \frac{1 \times 3}{{}^6C_2} = \frac{1}{5},\ P(A \mid E_2) = \frac{2 \times 1}{{}^4C_2} = \frac{1}{3},\ \text{and}\ P(A \mid E_3) = \frac{4 \times 3}{{}^{12}C_2} = \frac{2}{11}$$

$$\therefore \quad P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{\sum_{i=1}^{3} P(E_i)\,P(A \mid E_i)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{11}} = \frac{55}{118}$$

Similarly,
$$P(E_3 \mid A) = \frac{\frac{1}{3} \times \frac{2}{11}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} \times + \frac{1}{3} \times \frac{2}{11}} = \frac{30}{118}$$

$\therefore$
$$P(E_1 \mid A) = 1 - \frac{55}{118} - \frac{30}{118} = \frac{33}{118}.$$

Example 4·9. *From a vessel containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. From this vessel a ball is drawn and is found to be white. What is the probability that out of four balls transferred 3 are white and 1 is black ?*

Solution. Let us define the following events :

E_1 : Transfer of 0 white and 4 black balls

E_2 : Transfer of 1 white and 3 black balls

E_3 : Transfer of 2 white and 2 black balls

E_4 : Transfer of 3 white and 1 black balls

(Since the urn contains 3 white balls, more than 3 white balls cannot be transferred from the vessel)

E : Drawing of a white ball from the second vessel.

Then $\quad P(E_1) = \frac{{}^5C_4}{{}^8C_4} = \frac{1}{14}, \qquad P(E_2) = \frac{{}^3C_1 \times {}^5C_3}{{}^8C_4} = \frac{3}{7}$

$P(E_3) = \frac{{}^3C_2 \times {}^5C_2}{{}^8C_4} = \frac{3}{7}, \qquad P(E_4) = \frac{{}^3C_3 \times {}^5C_1}{{}^8C_4} = \frac{1}{14}$

Also $\quad P(E \mid E_1) = 0, \; P(E \mid E_2) = \frac{1}{4}, \quad P(E \mid E_3) = \frac{2}{4} \quad$ and $\quad P(E \mid E_4) = \frac{3}{4}$

Hence, by Bayes Theorem, the probability that out of four balls transferred, 3 are white and is black is :

$$P(E_4 \mid E) = \frac{\frac{1}{14} \times \frac{3}{4}}{\frac{1}{14} \times 0 + \frac{3}{7} \times \frac{1}{4} + \frac{3}{7} \times \frac{1}{2} + \frac{1}{14} \times \frac{3}{4}} = \frac{3}{6 + 12 + 3} = \frac{1}{7} = 0·14.$$

Example 4·10. *A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer, find the probability that their answer is correct.*

Solution. Let us define the following events :

E_1 : Both A and B solve the problem correctly.

E_2 : Exactly one of them solves the problem correctly.

E_3 : Neither of them solves the problem correctly.

E : They get the same answer.

Then, according to the data given in the problem, assuming that A and B try the problem independently, we have.

$P(E_1) = \frac{1}{6} \times \frac{1}{8} = \frac{1}{48} \qquad ; \quad P(E \mid E_1) = 1$

$P(E_2) = \frac{1}{6} \times \frac{7}{8} + \frac{5}{6} \times \frac{1}{8} = \frac{12}{48} \qquad ; \quad P(E \mid E_2) = 0$