Recurrence Relation

Many counting problems cannot be solved easily using the simple counting techniques

How many bit strings of length n do not contain two consecutive zeros?

0 1

1 | 1

1 0
0 1
1 1

0 1 0
0 1 1
1 1 0
1 0 1
1 1 1

$a_1 = 2$
$a_2 = 3$
$a_3 = 5$
$a_4 = 8$
$\boxed{a_5 = 13}$
$a_6 = 21$

$a_{100} = a_{99} + a_{98}$

$a_n = a_{n-1} + a_{n-2}$

or

$\boxed{a_{n+2} = a_{n+1} + a_n}$

Recurrence relations play important roles in the study of algorithms.

For example, suppose that the number of CORONA VIRUS in a colony doubles every hour. If a colony begins with 2 VIRUS, how many will be present in n hours?

let an be the number of CORONA VIRUS at the end of n hours.

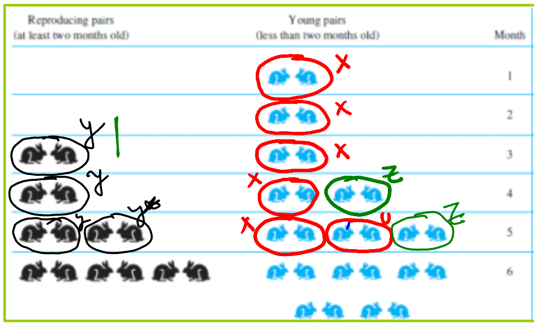
whenever n is a positive integer

$a_n = 2 \cdot a_{n-1}$
$a_1 = 2$

$a_1 = 2$
$a_2 = 2 \cdot a_1 = 2 \cdot 2 = 2^2$
$a_3 = 2 \cdot a_2 = 2 \cdot 2^2 = 2^3$
$a_4 = 2 \cdot a_3 = 2^4$
$\vdots$
$a_n = 2^n$

Modeling With Recurrence Relations

A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another Links pair each month



$f_1 = 1$
$f_2 = 1$
$f_3 = 2$
$f_4 = 3$
$\boxed{f_5 = 5}$
$\boxed{f_6 = 8}$
$f_7 = 13$

F_{50}

⋮

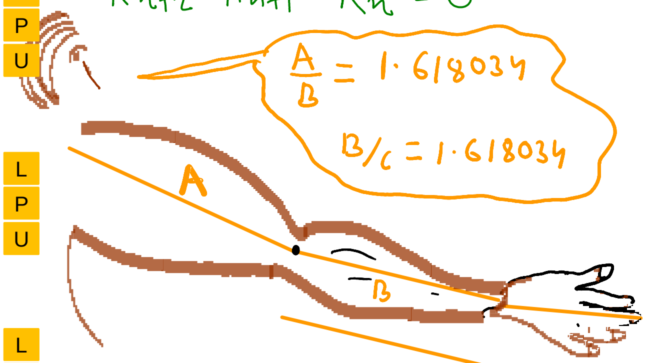
$F_{100} = ?$

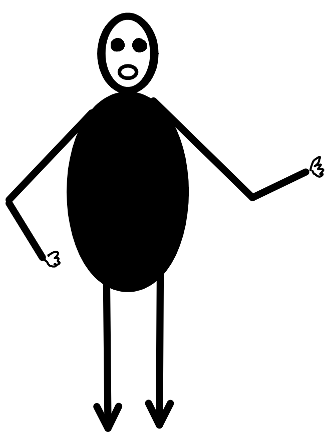
$R_n = R_{n-1} + R_{n-2}$

$R_{n+2} = R_{n+1} + R_n$

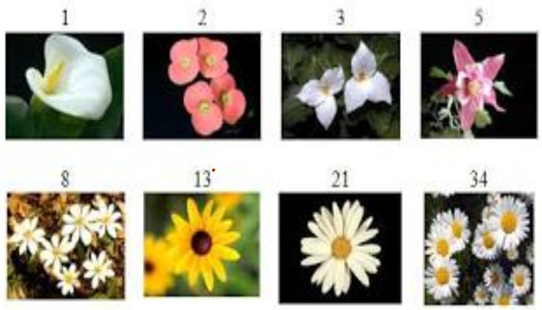
$n+2 = n$

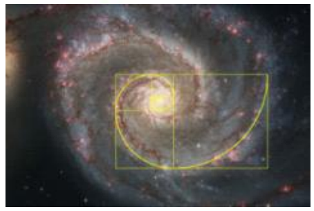
$R_{n+2} - R_{n+1} - R_n = 0$





Month	No.of Pairs	Golden ratio
F1	1	1
F2	1	1
F3	2	2
F4	3	1.5
F5	5	1.666667
F6	8	1.6
F7	13	1.625
F8	21	1.615385
F9	34	1.619048
F10	55	1.617647
F11	89	1.618182
F13	233	1.618056
F14	377	1.618026
F15	610	1.618037
F16	987	1.618033
F17	1597	1.618034
F18	2584	1.618034
F19	4181	1.618034
F20	6765	1.618034
F21	10946	1.618034
F22	17711	1.618034
F23	28657	1.618034
F24	46368	1.618034
F25	75025	1.618034





The Tower of Hanoi Puzzle

$M_0 = 00$
$M_1 = 01$
$M_2 = 03$
$M_3 = 07$
$M_4 = 15$
$M_5 = 31$
$M_6 = 63$
$\vdots$

$2^n - 1$

$M_n = 2\cdot M_{n-1} + 1$ ✓

or

$M_{n+1} - 2M_n = 1$ ✓

Solving Linear Recurrence Relations

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = R(n) \quad \text{—} (*)$$

$c_0, c_1, \ldots, c_k$ are real No.

$c_k \neq 0$

$\boxed{a_n = 2\cdot a_{n-1}}$

$a_n = 2\cdot n\cdot a_{n-1}$

$a_{n+1} = 2^n \cdot a_n$

Any one of is variable →

Every term has single (≤ 1) seq. with ≤ 1 occurrence

$\boxed{\text{Linear}}$: $a_{n+2} = a_{n+1} + a_n$, $H_n = 2H_{n-1} + 1$

1 1 1 1 1 0

$2x + 2y = 9$
$4x - 5y = 10$
$xy + 2y^2 = 0$ ✓
$x^2 + y^2 = 9$ ✓

$\boxed{\text{Non linear}}$: $a_n = 2(a_{n-1})^2$, $a_{n+2} + a_n\cdot a_{n-1} = 0$

$a_{n+2} = 5a_{n+1} - 6a_n$ — $\boxed{\text{linear}}$

$a_{n+2} = 5a_{n+1} \cdot a_n + 6a_{n-1}$ — $\boxed{\text{Nonlinear}}$

$a_n - a_{n-1} - a_{n-2} = 10^n$ — $\boxed{\text{linear}}$

OR $R(n) = 0$ in — (*)

$\boxed{\text{Homogeneous}}$: Every term is multiplied with a_j's

$a_n = 2a_{n-1} \Rightarrow a_n - 2a_{n-1} = 0$ ✓

$\boxed{\text{Nonhomogeneous}}$: Some terms are not multiplied with a_j's

- $H_n = 2H_{n-1} + 1 \Rightarrow H_n - 2H_{n-1} = 1$ ✓
- $H_{n+1} - 5H_n = 3^n$

$\boxed{\text{Degree}}$: Highest Subscript − least Subscript

$H_n = 2H_{n-1} + 1$ Degree $= n - (n-1) = 1$

$F_{n+3} - 3F_{n+2} - 3F_{n+1} = 6^n$, Degree $= \underline{\underline{2}}$

$n+3 - (n+1) = \cancel{n}+3-\cancel{n}-1 = 2$

- Linear homogeneous recurrence relation with Constant Co-efficient

$a_{n+2} - 2a_n = 0$, $a_{n+2} - 5a_{n+1} + 6a_n = 0$, $a_n = 5a_{n-1}$

$C_0 = 1, C_1 = -2$, $C_0 = 1, C_1 = -5, C_2 = 6$, $C_0 = 1, C_2 = -5$

- Linear Nonhomogeneous recurrence relation with Constant Co-efficient

$a_{n+2} - 2a_n = 5^n$, $a_{n+2} - 5a_{n+1} + 6a_n - 10 = 0$, $a_n = 5a_{n-1} + 10^n$

- Linear homogeneous recurrence relation with Variable Co-efficient

$a_{n+2} - 2^n a_n = 0$, $a_{n+2} - 5n \cdot a_{n+1} + 6a_n = 0$, $\frac{1}{n} a_n = 5a_{n-1}$

$C_1 = 1, C_2 = -2^n$, $C_1 = 1, C_2 = -5n \cdot C_3 = 6$, $C_1 = 1/n \cdot C_2 = -5$

- Linear Nonhomogeneous recurrence relation with Variable Co-efficient

$a_{n+2} - 2^n a_n = 5^n$, $a_{n+2} - 5n\,a_{n+1} + 6 = 0$, $\frac{1}{n} a_n = 5a_{n-1} + 3^n$

- Non Linear homogeneous recurrence relation with Constant Co-efficient

$a_{n+2} - 2(a_n)^2 = 0$, $a_{n+2} - 5(a_{n+1})^2 + 6a_n = 0$, $a_n = 5a_{n-1} \cdot a_{n-2}$

$C_0 = 1, C_1 = -2$, $C_0 = 1, C_1 = -5, C_2 = 6$, $C_0 = 1, C_2 = -5$

- Non

- Non Linear Nonhomogeneous recurrence relation with Constant Co-efficient

$(a_{n+2})^2 - 2a_n = 5^n, \quad a_{n+2} \cdot a_{n+1} + 5a_n + 10 = 0, \quad a_n = 5(a_{n-1})^2 + 10^n$

- Non Linear homogeneous recurrence relation with Variable Co-efficient

$a_{n+2} - 2^n (a_n)^2 = 0, \quad a_{n+2} - 5n \cdot a_{n+1} \cdot a_n = 0, \quad \frac{1}{n} a_n = 5(a_{n-1})^2$

$C_1 = 1,\ C_2 = -2^n, \quad C_1 = 1,\ C_2 = -5n \cdot C_3 = 6, \quad C_1 = 1/n \cdot C_2 = -5$

- Non Linear Nonhomogeneous recurrence relation with Variable Co-efficient

$a_{n+2} - 2^n (a_n)^2 = 5^n, \quad a_{n+2} - 5n\, a_{n+1} \cdot a_n + 6 = 0, \quad \frac{1}{n} a_n = 5(a_{n-1})^2 + 3^n$

$E f(x) = f(x+1)$

$E^2 f(x) = f(x+2)$

$\Delta f(x) = f(x+1) - f(x)$

$= E f(x) - f(x)$

$\Delta f(x) = (E-1) f(x)$

SOLUTION

$\boxed{\Delta \text{ operator}}$ $\quad \Delta = E - 1$

$\boxed{\text{Shift operator}}$ $\quad E(a_n) = a_{n+1}, \quad E^2(a_n) = a_{n+2}$

$\boxed{\Delta a_n = a_{n+1} - a_n}$

$E(5^n) = 5^{n+1} = 5 \cdot 5^n$

$E^2(5^n) = 25 \cdot 5^n$

$\Delta(5^n) = 5^{n+1} - 5^n$

Solution of Linear Homogeneous Recurrence Relations with CC

Solution of Linear Non-homogeneous Recurrence Relations with CC

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots, c_k a_{n-k} + R(n) \quad \text{--- (1)}$$

or $\qquad n \geq 0$

$$\boxed{C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots, C_k a_{n-k} = R(n)} \quad \text{--- (2)}$$

Apply E operator k times

the a_n be the general sol. of --- (2)

degree k

$$C_0 a_{n+k} + C_1 a_{n+k-1} + \cdots, C_k a_n = R(n+k)$$

$$C_0 E^k a_n + C_1 E^{k-1} a_n + \cdots, C_k E^0 a_n = R(n+k)$$

USING

$$C_0 E^k a_n + C_1 E^{k-1} a_n + \cdots, C_k E^0 a_n = 0 \quad [\text{Homogeneous}]$$

Let the general solution of $-①$ is a_n

HELP OF

$$\boxed{a_n = C.F. + P.I.}$$

Charactestic equation/ Auxilliary equation

$$C_0 E^k + C_1 E^{k-1} + \cdots, C_k E^0 = 0 \quad — ⊛$$

Roots

Are real & different → $\gamma_1, \gamma_2, \ldots, \gamma_k$

$$C.F. = A_1 \gamma_1^n + A_2 \gamma_2^n + \cdots, A_n \gamma_k^n$$

Are real & same → $\gamma_1 = \gamma_2$, $\gamma_3 \ldots, \gamma_k$

$$C.F. = (A_1 + A_2 n)\gamma_1^n + A_3 \gamma_3^n + \cdots, A_n \gamma_k^n$$

$\alpha \pm i\beta$

$$\rho = \sqrt{\alpha^2 + \beta^2}$$

$$\theta = \tan^{-1} \beta/\alpha$$

$$C.F. = \rho^n (A_1 \cos n\theta + A_2 \sin n\theta)$$

① Do check All Subscript's are Positive with $(n=0)$

Yes → ③

No → ② Appy E operator K(degree) time to both Side of relation

③ Find the charatrstic equation

④ Find the roots of characterstic eqn

⑤ Find the C.F. According to the roots

⑥

$\boxed{\text{Find the P.I. According to R.H.S}}$

Q: Find the general Sol. of Fibbonacci seq.

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n - a_{n-1} - a_{n-2} = 0 \qquad n \geq 0 \quad \text{———} \; (*)$$

$$\text{degree} = n - (n-2) = 2$$

Apply E^2 both side to — $(*)$

$$E^2 a_n - E^2 a_{n-1} - E^2 a_{n-2} = 0$$

$$\boxed{a_{n+2} - a_{n+1} - a_n = 0}$$

$$E^2 a_n - E^1 a_n - E^0 a_n = 0$$

$$E^2 - E - 1 = 0$$

$$E = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{C.F. = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

P.I. $= 0$

$$\boxed{a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

$$F_n - 5F_{n-1} + 6F_{n-2} = 0 \quad \text{—} \; (*)$$

$$n \geq 0$$

$$\text{degree} = n - (n-2) = 2$$

Apply E^2 both side to — $(*)$

$$E^2 F_n - 5E^2 F_{n-1} + 6E^2 F_{n-2} = 0$$

$$F_{n+2} - 5F_{n+2-1} + 6F_{n+2-2} = 0$$

$$\boxed{F_{n+2} - 5F_{n+1} + 6F_n = 0}$$

$$E^2 F_n - 5E^1 F_n + 6E^0 F_n = 0$$

$$\boxed{E^2 - 5E + 6 = 0}$$

$$E = 3, 2$$

$$\boxed{C.F. = c_1 \cdot 3^n + c_2 \cdot 2^n}$$

$$P.I = 0$$

$$\boxed{F_n = c_1 3^n + c_2 \cdot 2^n}$$

$$F_0 = c_1 + c_2 = 1$$

$$F_1 = 3c_1 + 2c_2 = 4$$

$$2c_1 + 2c_2 = 2$$

$$3c_1 + 2c_2 = 4$$

$$c_1 = 2$$

$$c_2 = -1$$

Solve the recurrence relation – $a_{n+2} = 10a_{n+1} - 25a_n$

$$a_{n+2} - 10a_{n+1} + 25a_n = 0$$

$$E^2 a_n - 10E^1 a_n + 25E^0 a_n = 0 \qquad n \geq 0$$

$$E^2 - 10E + 25 = 0$$

$$E = 5, 5$$

$$C.F. = (c_1 + c_2 \cdot n)\, 5^n$$

$$P.I. = 0$$

$$\boxed{a_n = (c_1 + c_2 n)\, 5^n}$$

Solve $a_n = 2a_{n-1} - 2a_{n-2}$

$$a_n - 2a_{n-1} + 2a_{n-2} = 0 \qquad n \geq 0 \quad — (*)$$

degree $= n - (n-2) = 2$

Apply E^2 both side — (*)

$$E^2 a_n - 2E^2 a_{n-1} + 2E^2 a_{n-2} = 0$$

$$a_{n+2} - 2a_{n+2-1} + 2a_{n+2-2} = 0$$

$$\boxed{a_{n+2} - 2a_{n+1} + 2a_n = 0} \checkmark$$

$$E^2 a_n - 2E^1 a_n + 2E^0 a_n = 0$$

$$E^2 - 2E + 2 = 0$$

$$E = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = \boxed{1 \pm i}$$

$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$$

$$C.F. = (\sqrt{2})^n \cdot \left[c_1 \cos \frac{n\pi}{4} + c_2 \sin \frac{n\pi}{4}\right]$$

$$P.I. = 0$$

$$a_n$$

Exercises:

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b) $a_n = 2na_{n-1} + a_{n-2}$

c) $a_n = a_{n-1} + a_{n-4}$

d) $a_n = a_{n-1} + 2$

e) $a_n = a_{n-1}^2 + a_{n-2}$

f) $a_n = a_{n-2}$

g) $a_n = a_{n-1} + n$

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$

b) $a_n = 3$

c) $a_n = a_{n-1}^2$

d) $a_n = a_{n-1} + 2a_{n-3}$

e) $a_n = a_{n-1}/n$

f) $a_n = a_{n-1} + a_{n-2} + n + 3$

g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$

b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$

c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

Solution of Linear Non-homogeneous Recurrence Relations with Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots, c_K a_{n-K} + R(n) \quad — (1)$$

or

$$C_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots, C_K a_{n-K} = R(n) \quad — (2)$$

Apply E operator K time $\boxed{\frac{1}{\phi(E)} \cdot R(n)}$

$$a_n = C.F. + P.I.$$

the a_n be the general sol. of — (2) order K

$$C_0 a_{n+k} + C_1 a_{n+k-1} + \cdots, C_k a_n = R(n+k)$$

R.H.S.

- $\beta^n \longrightarrow$ P.I. $= \boxed{\frac{1}{\phi(E)}\cdot\beta^n = \beta^n\cdot\frac{1}{\phi(\beta)}}$, $\phi(\beta)\neq 0$
- $\sin\alpha n / \cos\alpha n$
- Polynomial $= P(k) \longrightarrow \boxed{\text{P.I.} = \frac{1}{\phi(E)}\cdot P(k) = \frac{1}{\phi(1+\Delta)}P(k)} = (1 \pm f(\Delta))^{-1}\cdot P(k)$
- $\beta^n\cdot P(k) \longrightarrow$ P.I. $= \frac{1}{\phi(E)}\cdot\beta^n\cdot \text{Poly}(k) = \beta^n\cdot\frac{1}{\phi(\beta\cdot E)}\,\text{Poly}(k)$

Case of failure: If $\phi(\beta)=0$ then Apply the last Case

Q: $a_n = 7a_{n-1} - 12a_{n-2} = 5^n$, $n\geq 2$ — *

$n \geq 0$

degree $= n-(n-2) = 2$

Apply E^2 both side of — *

$$E^2(a_n) - 7E^2 a_{n-1} + 12E^2 a_{n-2} = E^2(5^n)$$

$$a_{n+2} - 7a_{n+2-1} + 12a_{n+2-2} = 5^{n+2}$$

$$\boxed{a_{n+2} - 7a_{n+1} + 12a_n = 5^n\cdot 5^2}$$

$$E^2 a_n - 7E^1 a_n + 12E^0 a_n = 5^n\cdot 5^2$$

$\boxed{E^2 - 7E + 12 = 0}$ $\quad E = 4, 3$

$\boxed{\text{C.F.} = c_1 4^n + c_2 3^n}$

$$\text{P.I.} = \frac{1}{E^2-7E+12}\cdot 25\cdot 5^n = 25\cdot\boxed{\frac{1}{E^2-7E+12}\cdot 5^n} = \frac{25\cdot 5^n}{2}$$

$25 - 7\cdot 5 + 12$

$\boxed{\text{P.I.} = \frac{25\cdot 5^n}{2}}$

$$a_n = c_1\cdot 4^n + c_2\cdot 3^n + \frac{25}{2}\cdot 5^n$$

Q: $\boxed{a_{n+1} - 5a_n = \sin 3n}$ $\quad n \geq 0$

$E - 5 = 0 \Rightarrow \boxed{E = 5}$

$\boxed{\text{C.F.} = c_1 5^n}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}}$

$\sin\theta = e^{i\theta} - e^{-i\theta}$

L P U

Q: $\boxed{a_{n+1} - 5a_n = \sin 3n}$

$n \geq 0$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$

$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$

$E - 5 = 0 \Rightarrow \boxed{E = 5}$

$\boxed{C.F. = c_1 5^n}$

L P U

$\sin 3n = \frac{e^{i3n} - e^{-i3n}}{2i} = \frac{1}{2i}\left[(e^{3i})^n - (e^{-3i})^n\right]$

L P U

$P.I. = \frac{1}{E-5}\left[\frac{1}{2i}\left((e^{3i})^n - (e^{-3i})^n\right)\right] = \frac{1}{2i}\left[\frac{1}{E-5}(e^{3i})^n - \frac{1}{E-5}(e^{-3i})^n\right]$

$= \frac{1}{2i}\left[\frac{(e^{3i})^n}{e^{3i}-5} - \frac{(e^{-3i})^n}{e^{-3i}-5}\right]$

L P U

$a_{n+1} = 2a_n + 1, \; a_0 = 0$

$2^n - 1$

$a_{n+1} - 2a_n = 1$

$E - 2 = 0$

$E = 2$

$\boxed{C.F. = c_1 2^n}$

$a_n = c_1 2^n - 1$

$a_0 = c_1 2^0 - 1 = 0 \Rightarrow \boxed{c_1 = 1}$

$a_n = 2^n - 1$ ✓

$P.I. = \frac{1}{E-2} \cdot 1 = \frac{1}{E-2} \cdot 1^n = 1^n \cdot \frac{1}{1-2} = -1$

$\boxed{P.I = -1}$

L P U

Factorial Polynomial: $n^{(k)} = n(n-1)(n-2)\cdots, n-(k-1)$

$n^0 = n^{(0)}$

$n^{(1)} = n$

$n = n^{(1)}$

$n^{(2)} = n(n-1) = n^2 - n$

$n^2 = n^{(2)} + n^{(1)}$

$n^{(3)} = n(n-1)(n-2)$

L P U

Δ and $\frac{1}{\Delta}$ can be treated as differentiation and integration operator for $n^{(k)}$

L P U

$\Delta n = \Delta n^{(1)} = 1$

$\Delta n = n+1 - n = 1$

$\Delta n^2 = \Delta\left[n^{(2)} + n^{(1)}\right] = 2n^{(1)} + 1$

$\Delta^2 n^2 = \Delta^2\left[n^{(2)} + n^{(1)}\right] = \Delta\left[2n^{(1)} + 1\right] = 2$

L P U

$\Delta^3 n^2 = 0$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

Q: $a_{n+2} - 5a_{n+1} + 6a_n = n^2$

+ve subscript, $n \geq 0$

$E^2 - 5E + 6 = 0$

$E = 3, 2$

$\boxed{C.F. = c_1 3^n + c_2 2^n}$

$P.I. = \frac{1}{E^2 - 5E + 6} \cdot n^2$

$= \frac{1}{(1+\Delta)^2 - 5(1+\Delta) + 6} \cdot n^2$

$= \frac{1}{1 + \Delta^2 + 2\Delta - 5 - 5\Delta + 6} \cdot n^2$

$= \frac{1}{2 + \Delta^2 - 3\Delta} \cdot n^2$

$P.I. = \frac{1}{2\left[1 - \left(\frac{3\Delta - \Delta^2}{2}\right)\right]} \cdot n^2$

$= \frac{1}{2}\left[1 - \boxed{\left(\frac{3\Delta - \Delta^2}{2}\right)}\right]^{-1} \cdot n^2$

$= \frac{1}{2}\left[1 + \left(\frac{3\Delta - \Delta^2}{2}\right) + \boxed{\left(\frac{3\Delta - \Delta^2}{2}\right)^2} + \cdots\right] n^2$

$= \frac{1}{2}\left[1 + \frac{3}{2}\Delta - \frac{\Delta^2}{2} + \frac{9\Delta^2}{4}\right] n^2$

$= \frac{1}{2}\left[1 + \frac{3}{2}\Delta - \frac{\Delta^2}{2} - \frac{9}{4}\Delta^2\right] n^{(2)} + n^{(1)}$

$= \frac{1}{2}\left[n^{(2)} + n^{(1)} + \frac{3}{2}\left(2n^{(1)} + 1\right) - \frac{1}{2} \cdot 2 - \frac{9}{4} \cdot 2\right]$

$= \frac{1}{2}\left[n^2 + \frac{3}{2}(2n+1) - 1 - \frac{9}{2}\right]$

$\frac{1}{E-2} \cdot n^2 = \frac{1}{1+\Delta-2} \cdot n^2 = \frac{1}{\Delta - 1} \cdot n^2$

$\Rightarrow -(1-\Delta)^{-1} \cdot n^2 = -\left[1 + \Delta + \Delta^2 + \cdots\right] n^2$

$\Rightarrow -\left[1 + \Delta + \Delta^2\right] n^{(2)} + n^{(1)}$

$\Rightarrow -\left[n^{(2)} + n^{(1)} + 2n^{(1)} + 1 + 2\right]$

Q: $a_{n+1} - 2a_n = 2^n$, $n \geq 0$

Q: $a_n = 2a_{n-1} + 3^n \cdot n$

$Ea_n - 2a_n = 0$

$E - 2 = 0$

$E = 2$

$\boxed{C.F. = c_2 2^n}$

$P.I = \frac{1}{E-2} \cdot 2^n$

$= 2^n \cdot \frac{1}{2-2}$, $\phi(B) = 0$ [Case of failure]

$P.I. = \frac{1}{E-2} \cdot 2^n \cdot n^0$

$= 2^n \cdot \boxed{\frac{1}{2 \cdot E - 2} \cdot n^0} = \frac{2^n}{2} \cdot \left[\frac{1}{E-1} \cdot n^0\right] = \frac{2^n}{2} \cdot \left[\frac{1}{1+\Delta-1}\right]^{n} 2$

$= \frac{2^n}{2} \cdot \frac{1}{\Delta} n^0 = \frac{2^n}{2} \cdot \frac{1}{\Delta} n^{(0)} = \boxed{\frac{n}{2} 2^n}$

GENERATING FUNCTION

Let $a_0, a_1, \ldots, a_n$ are the sequence of real No. the generating function is defined as

$$\boxed{G(x) = \sum_{k=0}^{\infty} a_k x^k}$$

$$\boxed{G(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots, + a_nx^n + \cdots,}$$

$$\boxed{G(x) - a_0x^0 = a_1x^1 + \cdots a_nx^n + \cdots = \sum_{k=1}^{\infty} a_k x^k}$$

$$\boxed{G(x) - a_0x^0 - a_1x^1 = \sum_{k=2}^{\infty} a_k x^k}$$

The G.F. of the sequence $\left\{ \frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \cdots, \frac{1}{k!}, \cdots, \right\}$

$a_0 = \frac{1}{0!}, \quad a_1 = \frac{1}{1!}, \quad a_2 = \frac{1}{2!} \quad \cdots,$

e^x

$$G(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots,$$

$$= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 = \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

G.F. of the $\{1, 1, 1, \cdots, 1 \cdots\}$

$\quad a_0 \;\; a_1 \;\; a_2 \cdots, a_k$

$$G(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots,$$

$$= 1 + x + x^2 + x^3 + \cdots,$$

$$= (1-x)^{-1}$$

G.F. of $a_k = a^k$ $\qquad a_0 = 1, \; a_1 = a^1, \; a_2 = a^2, \; a_3 = a^3$

$$G(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots,$$

$$= 1 + ax + (ax)^2 + (ax)^3 + \cdots = \underline{\underline{(1-ax)^{-1}}}$$

G.F. of $\{1, 1, 1\}$ where $a_0 = 1, \; a_1 = 1, \; a_2 = 1$

$$G(x) = a_0x^1 + a_1x^1 + a_2x^2 = 1 + x + x^2$$

$$(1-x)^{-1} = \underbrace{1 + x + x^2} + x^3 + x^4 + \cdots,$$

$$1+x+x^2 = (1-x)^{-1} - x^3 - x^4 - x^5 - \cdots,$$
$$= (1-x)^{-1} - x^3[1+x+x^2+\cdots,]$$
$$= \frac{1}{1-x} - \frac{x^3}{1-x}$$
$$= \frac{1-x^3}{1-x}$$

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$\boxed{\text{Solution of Recurrence Relation using G.F.}}$

$E=3 \quad = \; C_1 3^n \qquad 3^n$

Solve using G.F. $a_{n+1} - 3a_n = 0$, $a_0 = 1$ ✓

$a_n = 4a_{n-1} + 5^n$, $a_0 = 1$

$a_{n+1} - 2a_n = 1$, $a_0 = 5$

$$a_{n+1} - 3a_n = 0$$
$$x^n \cdot a_{n+1} - 3x^n a_n = x^n \cdot 0 \qquad [\text{Multiply } x^n \text{ both side}]$$
$$\sum_{n=0}^{\infty} a_{n+1} x^{n} - 3 \boxed{\sum_{n=0}^{\infty} a_n x^n} = 0$$
$$x[a_1 x^0 + a_2 x^1 + a_3 x^2 + \cdots] - 3[a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots] = 0$$
$$\frac{[a_1 x + a_2 x^2 + a_3 x^3 + \cdots]}{x} - 3G(x) = 0$$
$$\frac{G(x) - a_0 x^0}{x} - 3G(x) = 0$$
$$G(x) - a_0 x^0 - 3x G(x) = 0$$
$$G(x) - 1 - 3x\,G(x) = 0$$
$$G(x)[1-3x] = 1$$
$$\boxed{G(x) = \frac{1}{}} = (1-3x)^{-1} = 1 + 3x + (3x)^2 + (3x)^3 + \cdots,$$

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$\boxed{\dots \quad 1-3x}$ $\quad a_0=3^0,\ a_1=3^1,\ a_2=3^2,\dots,\ a_n=3^n$

$$a_n-4a_{n-1}=5^n \qquad a_0=1$$

$\boxed{n\geq 0,\ -\text{ve subscript},\ \text{degree}=1}$

Apply E both side

$$E\,a_n-4E\,a_{n-1}=E\,5^n$$

$$a_{n+1}-4a_n=5^{n+1}$$

$$x^n\cdot a_{n+1}-4x^na_n=x^n\cdot 5^n\cdot 5=5\cdot(5x)^n$$

$$\sum_{n=0}^{\infty}a_{n+1}x^n-4\sum_{n=0}^{\infty}a_nx^n=5\sum_{n=0}^{\infty}(5x)^n$$

$$\frac{x}{x}[a_1x^0+a_2x^1+\cdots]-4[a_0x^0+a_1x^1+a_2x^2+\cdots]=5[1+5x+(5x)^2+\cdots]$$

$$\frac{G(x)-a_0x^0}{x}-4G(x)=5\cdot(1-5x)^{-1}$$

$$G(x)-1-4x\,G(x)=5x(1-5x)^{-1}$$

$$G(x)(1-4x)=\frac{5x}{1-5x}+1$$

$$\boxed{G(x)=\frac{5x}{(1-5x)(1-4x)}+\frac{1}{1-4x}}$$

$5x=A(1-4x)+B(1-5x)$

$\frac{5}{4}=B(1-5/4)$

$\boxed{B=-5}$

$1=A(1-\frac{4}{5})$

$\boxed{A=5}$

$$=\frac{A}{1-5x}+\frac{B}{1-4x}-\frac{1}{1-4x}$$

$$=5(1-5x)^{-1}-5(1-4x)^{-1}-(1-4x)^{-1}$$

$$=5(1-5x)^{-1}-6(1-4x)^{-1}$$

$$=5[1+5x+(5x)^2+\cdots]-6[1+4x+(4x)^2+\cdots]$$

$= a_0=5\cdot 5^0 \qquad a_0=-6\cdot 4^0$

$a_1=5\cdot 5^1 \qquad a_1=-6\cdot 4^1$

$a_2=5\cdot 5^2 \qquad a_2=-6\cdot 4^2$

$$\boxed{a_n=5\cdot 5^n-6\cdot 4^n}$$

<u>verification</u>

$$a_{n+1}-4a_n=5^{n+1}$$

$E-4=0 \Rightarrow E=4$, $\boxed{C.F.=c_1 4^n}$

$$P.I.=\frac{1}{\quad}\cdot 5\cdot 5^n=\frac{5\cdot 5^n}{\quad}=5\cdot 5^n$$

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E-4 $\qquad\qquad$ 5-4

$a_n = c_1 4^n + 5 \cdot 5^n$

$a_0 = 1 \Rightarrow c_1 \cdot 4 + 25 = 1 \Rightarrow 4c_1 = -24 \Rightarrow \boxed{c_1 = -6}$

$a_n = -6\,4^n + 5 \cdot 5^n$