Proofs

A proof is a valid argument that establishes the truth of a mathematical statement.

L P U

- # Direct Proofs ✓
- # Proof By Contraposition ✓
- # Vacuous And Trivial Proofs ✓
- # Proofs By Contradiction
- # Proofs Of Equivalence
- # Counterexamples
- # Mistakes In Proofs

L P U

Theorems, Facts, Axioms, Lemma, Corollary, Proposition, Postulate

L P U

https://www.geogebra.org/m/czUQweag

Program is correct, O.S. Security, System Consistency

$1+2+3+\cdots n = \frac{n(n+1)}{2}$

L P U

A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

p → fact, theorem, postulate → q

L P U

DEFINITION: The integer n is even if there exists an integer k such that $n = 2k$, ✓
and n is odd if there exists an integer k such that $n = 2k + 1$.

L P U

Q: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

p: n is odd
q: n^2 is odd

If n is odd then n^2 odd

L P U

Let p is true, ⇒ p: $n = 2k+1 \quad \forall k \in \mathbb{Z}$

q: $n^2 = (2k+1)^2 = 4k^2+1+4k$
$= 2(\underbrace{2k^2+2k}) + 1$
$= 2t+1 \qquad t = 2k^2+2k$

Q. Show that the square of an even number is an even number using a direct proof.

If n is even then n^2 is even

$n = 2k,\ k \in \mathbb{Z} \Rightarrow n^2 = (2k)^2 = 4k^2 = 2\cdot(2k^2)$
$= 2\cdot t$

L P U

The conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true.

$$p \to q \equiv \neg q \to \neg p$$

Q: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

$n \in \mathbb{Z}$

If $3n+2$ is odd

$p \to q \equiv \neg q \to \neg p$

If $3n+2$ is odd then n is odd

$p: 3n+2$ is odd

Let p is true

$3n+2 = 2k+1 \quad k \in \mathbb{Z}$

$3n = 2k-1$

$\boxed{n = \frac{2k-1}{3}}$

$\forall k \in \mathbb{Z}$

odd

$k=0 \Rightarrow -1/3$ ✓

$k=1 \Rightarrow 1/3$ ✓

$\sim p: 3n+2$ is even

$\sim q: n$ is even

$\sim q \to \sim p$: If n is even then $3n+2$ is even

$n = 2k \quad \forall k \in \mathbb{Z}$

$3n+2 = 3\cdot(2k)+2 = 6k+2$

$= 2\cdot(3k+1)$

$= 2\cdot t \qquad t \in \mathbb{Z}$

Hence $3n+2$ is even

Vacuous proof

&

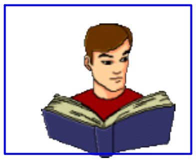
Trivial proof

p	q	p→q
T	T	T
T	F	
F	T	T
F	F	T

"If $\boxed{\text{Monkey can fly}}$ then $\boxed{\text{donkey visits the Bar on Moon}}$"

$\boxed{\text{If I study hard}}$ then I will get a Job

→ T



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"If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

$(-\infty, 0) \cup (0, \infty)$

Proof by Contradiction

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\boxed{\neg p \rightarrow q}$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

$p \rightarrow q$ is true by assuming $\boxed{(p \quad \neg q)}$ is true .

Q Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction

Let p: $\sqrt{2}$ is irrational

$\neg p$: $\sqrt{2}$ is rational

we will prove that assuming $\neg p$ is true leads to Contradiction.

If n^2 is even then n is also even

$\neg p$ is true $\sqrt{2}$ is rational

$\sqrt{2} = \frac{a}{b}$

$\exists\, a, b$ s.t. $\sqrt{2} = \frac{a}{b}$ [a, b have no common factors]

Irrational Number is a real No. that can't be expressed as a ratio of integers

$\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even $\Rightarrow$

Every rational No. can be written in lowest term

a is even $\Rightarrow a = 2K \Rightarrow a^2 = 4K^2$

2,4,

$2b^2 = 4K^2 \Rightarrow b^2 = 2K^2$ is even

b is even

Give a proof by contradiction of the theorem "If 3n + 2 is odd, then n is odd."

p: 3n+2 is odd

q: n is odd

$p \rightarrow q$

contradiction.

CONTRA

$p \wedge \neg q$ is true

p: $3n+2$ is odd

$\neg q$: n is even

$n = 2k$, $k \in \mathbb{Z}$

$p = 3n+2 = 3 \cdot 2k + 2 = 6k+2 = 2(3k+1) = 2 \cdot t$ even

CONTRA

OR

$\neg p \to \neg q \equiv \neg q \to p$

$\neg p$: $3n+2$ is even	$\neg q$: n is even
$3n+2 = 2k$	$n = 2k$
$n = \dfrac{2(k-1)}{3}$	$3n+2 = 6k+2$
Can't conclude	$= 2(t)$
	$=$ even

PROOFS OF EQUIVALENCE

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology

$(p \leftrightarrow q) \leftrightarrow (p \to q) \quad (q \to p)$.

propositions p1, p2, p3,..., pn are equivalent. This can be written as

$p_1 \leftrightarrow p_2 \leftrightarrow \quad \leftrightarrow p_n \leftrightarrow [(p_1 \to p_2) \quad (p_2 \to p_3) \quad\quad (p_n \to p_1).]$

If $\boxed{P_1 \to P_2}$, $\boxed{P_2 \to P_3}$, ..., $\boxed{P_n \to P_1}$ are true then $P_1, P_2, \dots, P_n$ are Equivalent

Q: Show that these statements about the integer n are equivalent:

p1: n is even.

p2: $n - 1$ is odd.

p3: n^2 is even.

$P_1 \to P_2$, $P_2 \to P_3$, $P_3 \to P_1$ are true

Solution:

$P_1 \to P_2$

P_1: $n = 2k$, $k \in \mathbb{Z}$

P_2: $n-1$ is odd

$n-1 = 2k-1$

$= 2k-1+2-2$

$= 2k-2+1$

$= 2(k-1)+1$

$= 2t+1$

$P_2 \to P_3$

P_2: $n-1$ is odd

$n-1 = 2k+1$

$n = 2k+2$

$n = \underbrace{2(k+1)}_{2t}$

$n^2 = 4t^2$

$= 2 \cdot (2t^2)$

even

$P_3 \to P_1$

P_3: n^2 is even

P_1: n is even

$P_3 \to P_1$

To show that a statement of the form $xP(x)$ is false, we need only find a counterexample that is, an example x for which $P(x)$ is false.

$\forall x, y, z \in \mathbb{N}$

$x^2 + y^2 \neq z^2$

$1^2 + 2^2 \neq 3^2$

$x = 1$

$y = 2$

$z = 3$

$\forall n\, P(n)$

$\exists n \neg P(n)$

Mistakes in Proofs:

https://www.youtube.com/watch?v=WMPw4PZRbOk

$1 = 2$

$1 = -1$

$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$

$x = y \quad x - y = 0$

$x = y$ $x - y = 0$	A
$x^2 = xy$	B
$x^2 - y^2 = xy - y^2$	C
$(x+y)(x-y) = y(x-y)$	D
$x + y = y$	E
$2y = y$	
$1 = 2$	

$1 = 1$	A
$1 = (-1) * (-1)$	B
$\sqrt{1} = \sqrt{(-1)\cdot(-1)}$	C
$1 = \sqrt{-1} \cdot \sqrt{-1}$	D
$1 = i \cdot i$	E
$1 = i^2$	F
$1 = -1$	G

$$
\begin{aligned}
x &= (\pi + 3)/2 \\
2x &= \pi + 3 \\
2x(\pi - 3) &= (\pi + 3)(\pi - 3) \\
2\pi x - 6x &= \pi^2 - 9 \\
9 - 6x &= \pi^2 - 2\pi x \\
9 - 6x + x^2 &= \pi^2 - 2\pi x + x^2 && \text{D} \\
(3 - x)^2 &= (\pi - x)^2 && \text{C} \\
3 - x &= \pi - x && \text{B} \\
\pi &= 3 && \text{A}
\end{aligned}
$$

$(3 - x) = \pm(\pi - x)$

$+$

$3 - x = \pi - x$ or $3 - x = -(\pi - x)$

$\pi = 3$ $3 - x = -\pi + x$

$3 + \pi = 2x$

$x = \dfrac{\pi + 3}{2}$