A and B is equal to the product of the probability of one of these events and the conditional probability of the other, given that the first one has occurred". Any of the events may be called the first event.

Remarks 1. $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ and $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ ... (3·18)

Thus the conditional probabilities $P(B \mid A)$ and $P(A \mid B)$ are defined if and only if $P(A) \neq 0$ and $P(B) \neq 0$, respectively.

2. (*i*) For $P(B) > 0$, $P(A \mid B) \leq P(A)$

Proof. $n(A \cap B) \leq n(A)$ and $n(B) \leq n(S)$. Dividing, we get

$$\frac{n(A \cap B)}{n(B)} \leq \frac{n(A)}{n(S)} \quad \Rightarrow \quad P(A \mid B) \leq P(A).$$

(*ii*) The conditional probability $P(A \mid B)$ is not defined if $P(B) = 0$.

(*iii*) $P(B \mid B) = 1$.

3·12. INDEPENDENT EVENTS

Two or more events are said to be *independent* if the happening or non-happening of any one of them, does not, in any way, affect the happening of others.

Consider the experiment of throwing two dice, say die 1 and die 2. It is obvious that the occurrence of a certain number of dots on the die 1 has nothing to do with a similar event for the die 2. The two are quite independent of each other, so to say. But suppose, the two dice were connected with a piece of thread before being thrown. The situation changes. This time the two events are not independent in as much as that the uppermost face of one die will have something to do in causing a particular face of the other die to be uppermost ; and the shorter the thread the more is this influence or dependence.

Similarly, if we draw two cards from a pack of cards in succession, then the results of the two draws are independent if the cards are drawn with replacement (*i.e.*, if the first card drawn is placed back in the pack before drawing the second card) and the results of the two draws are not independent if the cards are drawn without replacement.

Definition. *An event A is said to be independent (or statistically independent) of another event B, if the conditional probability of A given B, i.e., $P(A \mid B)$ is equal to the unconditional probability of B, i.e., if* $P(A \mid B) = P(A)$. ... (3·19)

It may be noted that the above definition is meaningful only when $P(A \mid B)$ is defined, *i.e.*, if $P(B) \neq 0$.

Similarly, an event B is said to be independent (or statistically independent) of event A, if

$$P(B \mid A) = P(B) \,; \quad P(A) \neq 0. \qquad \text{... (3·20)}$$

Theorem 3·10. *If the events A and B are such that $P(A) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is independent of A.*

Proof. Since the event A is independent of B, we have

$$P(A \mid B) = P(A) \quad \Rightarrow \quad \frac{P(A \cap B)}{P(B)} = P(A) \quad \Rightarrow \quad P(A \cap B) = P(A)\,P(B)$$

$$\therefore \quad \frac{P(B \cap A)}{P(A)} = P(B) \qquad [\because \; P(A) \neq 0 \text{ and } A \cap B = B \cap A]$$

$$\Rightarrow \quad P(B \mid A) = P(B) \quad \Rightarrow \quad B \text{ is independent of } A.$$

Remarks 1. Thus, we see that if A is independent of B, then B is independent of A. Hence, instead of saying that 'A is independent of B' or 'B is independent of A', we may say that A and B are independent events.

2. For any event A in S,

(*a*) A and the null event ϕ are independent

(*b*) A and S are independent.

Proof. (*a*) $P(A \cap \phi) = P(\phi) = 0 = P(A) \cdot P(\phi) \quad \Rightarrow \quad A$ and ϕ are independent.

(*b*) $P(A \cap S) = P(A) = P(A) \,.\, 1 = P(A)\, P(S) \qquad [\because A \subset S \text{ and } P(S) = 1]$

$\Rightarrow \quad A$ and S are independent.

3·13. MULTIPLICATION THEOREM OF PROBABILITY FOR INDEPENDENT EVENTS

Theorem 3·11. *If A and B are two events with positive probabilities $\{P(A) \neq 0$, $P(B) \neq 0\}$, then A and B are independent if and only if* $P(A \cap B) = P(A) \,.\, P(B)$ ... (3·21)

Proof. We have :

$$P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B) \,;\, P(A) \neq 0, \;\; P(B) \neq 0 \qquad \text{... (*)}$$

If A and B are independent, *i.e.*, A is independent of B and B is independent of A, then, we have

$$P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B) \qquad \text{... (**)}$$

From (*) and (**), we get $\quad P(A \cap B) = P(A)\, P(B)$, as required.

Conversely, if (3·21) holds, then we get

$$\left.\begin{aligned} \frac{P(A \cap B)}{P(B)} &= P(A) \quad \Rightarrow \quad P(A \mid B) = P(A) \\ \text{and} \quad \frac{P(A \cap B)}{P(A)} &= P(B) \quad \Rightarrow \quad P(B \mid A) = P(B) \end{aligned}\right\} \qquad \text{...(***)}$$

(***) implies that A and B are independent events.

Hence, for independent events A and B, the probability that both of these occur simultaneously is the product of their respective probabilities.

This rule is known as the ***Multiplication Rule of Probability***.

3·14. EXTENSION OF MULTIPLICATION THEOREM OF PROBABILITY TO *n* EVENTS

Theorem 3·12. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \ldots \times P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \qquad \text{... (3·22)}$$

where $P(A_i \mid A_j \cap A_k \cap \ldots \cap A_l)$ represents the conditional probability of the event A_i given that the events $A_j, A_k, \ldots, A_l$ have already happened.

Proof. For two events A_1 and A_2, $P(A_1 \cap A_2) = P(A_1)\, P(A_2 \mid A_1)$

We have for three events A_1, A_2, and A_3

$$P(A_1 \cap A_2 \cap A_3) = P\{A_1 \cap (A_2 \cap A_3)\}$$

Theorem 3·14. *For a fixed B with P(B) > 0, P (A | B) is probability function.*

Proof. (i) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

(ii) $$P(S \mid B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(iii) If $\{A_n\}$ is any finite or infinite sequences of disjoint events, then

$$P[\bigcup_n A_n \mid B] = \frac{P[(\bigcup_n A_n) \cap B]}{P(B)} = \frac{P[\bigcup_n (A_n \cap B)]}{P(B)}$$

$$= \frac{\sum_n P(A_n \cap B)}{P(B)} = \sum_n \left[\frac{P(A_n \cap B)}{P(B)}\right] = \sum_n P(A_n \mid B)$$

Hence the theorem.

Remark. For given B satisfying $P(B) > 0$, the conditional probability $P[\cdot \mid B]$ also enjoys the same properties as the unconditional probability.

For example, in the usual notations, we have

(i) $P(\phi \mid B) = 0$

(ii) $P(\overline{A} \mid B) = 1 - P(A \mid B)$,

(iii) $P\left[\bigcup_{i=1}^{n} A_i \mid B\right] = \sum_{i=1}^{n} P(A_i \mid B)$,

where $A_1, A_2, \ldots, A_n$ are mutually disjoint events.

(iv) $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

(v) $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$

(vi) If $E \subset F$, then $P(E \mid B) \leq P(F \mid B)$

and so on.

The proofs of results (*iv*), (*v*) and (*vi*) are given in Theorems 3·15, 3·16 and 3·17 respectively. Others are left as exercises to the reader.

Theorem 3·15. *For any three events A, B and C,*

$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Proof. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Dividing both sides by $P(C)$, we get

$$\frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}, \quad P(C) > 0$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad \frac{P[(A \cup B) \cap C]}{P(C)} = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

$$\Rightarrow \quad P[(A \cup B) \mid C] = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Theorem 3·16. *For any three events A, B and C,*

$$P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$$

Proof. $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C)$

$$= \frac{P(A \cap \overline{B} \cap C)}{P(C)} + \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap \overline{B} \cap C) + P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} = P(A \mid C).$$

Theorem 3·17. *For any three events A, B and C defined on the sample space S such that $B \subset C$ and $P(A) > 0$, $P(B \mid A) \le P(C \mid A)$.*

Proof. $$P(C \mid A) = \frac{P(C \cap A)}{P(A)} = \frac{P[(B \cap C \cap A) \cup (\overline{B} \cap C \cap A)}{P(A)}$$

$$= \frac{P[(B \cap C \cap A)}{P(A)} + \frac{P(\overline{B} \cap C \cap A)}{P(A)} \quad \text{(Using Axiom 3)}$$

$$= P(B \cap C \mid A) + P(\overline{B} \cap C \mid A)$$

$$= P(B \mid A) + P(\overline{B} \cap C \mid A) \quad [\because B \subset C \Rightarrow B \cap C = B]$$

$$\Rightarrow \quad P(C \mid A) \ge P(B \mid A) \quad [\because P(\overline{B} \cap C \mid A) \ge 0]$$

Theorem 3·18. *If A and B are independent events, then*

(i) A and $\overline{B}$ (ii) $\overline{A}$ and B (iii) $\overline{A}$ and $\overline{B}$, are also independent

Proof. Since A and B are independent, $P(A \cap B) = P(A)\, P(B)$ ... (*)

(*a*) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)\, P(B)$ [From (*)

$= P(A)\,[1 - P(B)] = P(A)\, P(\overline{B})$

$\Rightarrow$ A and $\overline{B}$ are independent events.

Aliter. $P(A \cap B) = P(A)\, P(B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)$

i.e., $P(B \mid A) = P(B) \Rightarrow B$ is independent of A.

and $P(A \mid B) = P(A) \Rightarrow A$ is independent of B.

Also $P(B \mid A) + P(\overline{B} \mid A) = 1 \Rightarrow P(B) + P(\overline{B} \mid A) = 1$

$\therefore$ $P(\overline{B} \mid A) = 1 - P(B) = P(\overline{B})$

$\therefore$ $\overline{B}$ is independent of A and by symmetry we say that A is independent of $\overline{B}$. Hence, A and $\overline{B}$ are independent events.

(*ii*) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\, P(B)$ [From (*)]

$= P(B)\,[1 - P(A)] = P(\overline{A})\, P(B)$

$\Rightarrow$ $\overline{A}$ and B are independent events.

(*iii*) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A)\, P(B)$ [From (*)]

$= [1 - P(B)] - P(A)\,[1 - P(B)]$

$= [1 - P(A)]\,[1 - P(B)] = P(\overline{A})\, P(\overline{B})$

$\therefore$ $\overline{A}$ and $\overline{B}$ are independent events.

Aliter. We know that : $P(\overline{A} \mid \overline{B}) + P(A \mid \overline{B}) = 1$ ... (**)

Since A and B are independent, by Part (*i*) of the above theorem A and $\bar{B}$ are also independent.

$\therefore \quad P(A \mid \bar{B}) = P(A)$. Hence from (**), we get

$$P(\bar{A} \mid \bar{B}) + P(A) = 1 \quad \Rightarrow \quad P(\bar{A} \mid \bar{B}) = 1 - P(A) = P(\bar{A})$$

Hence $\bar{A}$ and $\bar{B}$ are independent events.

3·15. PAIRWISE INDEPENDENT EVENTS

Consider n events $A_1, A_2, \ldots, A_n$ defined on the same sample space so that $P(A_i) > 0; i = 1, 2, \ldots, n$. These *events are said to be pairwise independent if every pair of two events is independent in the sense of the definition given in* § 3·13.

Definition. *The events $A_1, A_2, \ldots, A_n$ are said to be pairwise independent if and only if:*

$$P(A_i \cap A_j) = P(A_i)\, P(A_j),\ i \neq j = 1, 2, \ldots, n \qquad \ldots (3·25)$$

In particular, three events A_1, A_2, A_3 are pairwise independent if and only if :

$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\, P(A_2) \\ P(A_1 \cap A_3) &= P(A_1)\, P(A_3) \\ P(A_2 \cap A_3) &= P(A_2)\, P(A_3) \end{aligned}\right\} \qquad \ldots (3·26)$$

3·15·1. Mutually Independent Events.

Let S denote the sample space for a number of events. *The events in S are said to be mutually independent if the probability of the simultaneous occurrence of (any) finite number of them is equal to the product of their separate probabilities.*

Definition. *The n events $A_1, A_2, \ldots, A_n$ in a sample space S are said to be mutually independent if*

$$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})\, P(A_{i2}) \ldots P(A_{ik})\ ;\ k = 2, 3, \ldots, n \qquad \ldots (3·27)$$

Hence, the events are mutually independent if they are independent by pairs, and by triplets, and by quadruples, and so on.

Conditions for mutual independence of n events. Mathematically, n events $A_1, A_2, \ldots A_n$ are mutually independent if and only if the following conditions hold.

$$\left.\begin{aligned} &(i) \quad P(A_i \cap A_j) = P(A_i)\, P(A_j),\ (i \neq j\ ;\ i, j = 1, 2, \ldots, n) \\ &(ii) \quad P(A_i \cap A_j \cap A_k) = P(A_i)\, P(A_j)\, P(A_k),\ (i \neq j \neq k\ ;\ i\ ;\ j\ ;\ k = 1, 2, \ldots, n) \\ &\qquad \vdots \qquad\qquad\qquad \vdots \end{aligned}\right\} \ldots (3·28)$$

$$(n-1) : P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$$

It is interesting to note that the above equations in (3·28) give respectively ${}^nC_2, {}^nC_3, {}^nC_n$ conditions to be satisfied by $A_1, A_2, \ldots A_n$.

Hence the total number of conditions for mutual independence of $A_1, A_2, \ldots, A_n$ is :

$${}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n - ({}^nC_0 + {}^nC_1) = 2^n - 1 - n$$

In particular for three events A_1, A_2 and A_3, $(n = 3)$, we have the following $2^3 - 1 - 3 = 4$, conditions for their mutual independence.

$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\,P(A_2) \\ P(A_2 \cap A_3) &= P(A_2)\,P(A_3) \\ P(A_1 \cap A_3) &= P(A_1)\,P(A_3) \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)\,P(A_2)\,P(A_3) \end{aligned}\right\} \quad \ldots (3\cdot29)$$

Remarks 1. It may be observed that pairwise or mutual independence of events : $A_1, A_2, \ldots, A_n$, is defined only when $P(A_i) \neq 0$, for $i = 1, 2, \ldots, n$.

2. From (3·26) and (3·29), it is obvious that mutual independence of events implies that they are pairwise independent. However, the converse is not true, *i.e.*, the events may be pairwise independent but not mutually independent. For illustrations, see Examples 3·54 and 3·55.

Theorem 3·9. *If A, B, C are mutually independent events then $A \cup B$ and C are also independent.*

Proof. We are required to prove :

$$P[(A \cup B) \cap C] = P(A \cup B)\,P(C)$$

$$\begin{aligned} \text{L.H.S.} &= P[(A \cap C) \cup P(B \cap C)] && \text{[By Distributive Law]} \\ &= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A)\,P(C) + P(B)\,P(C) - P(A)\,P(B)\,P(C) \\ & && [\because A, B \text{ and } C \text{ are mutually independent}] \\ &= P(C)\,[P(A) + P(B) - P(A \cap B)] = P(C)\,P(A \cup B) = \text{R.H.S.} \end{aligned}$$

Hence $(A \cup B)$ and C are independent.

Theorem 3·20. *If A, B and C are random events in a sample space and if A, B and C are pairwise independent and A is independent of $(B \cup C)$, then A, B and C are mutually independent.*

Proof. We are given

$$\left.\begin{aligned} P(A \cap B) &= P(A)\,P(B) \\ P(B \cap C) &= P(B)\,P(C) \\ P(A \cap C) &= P(A)\,P(C) \\ P[A \cap (B \cup C)] &= P(A)\,P(B \cup C) \end{aligned}\right\} \quad \ldots (*)$$

Now

$$\begin{aligned} P[A \cap (B \cup C)] &= P[(A \cap B) \cup (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)] \\ &= P(A)\,.\,P(B) + P(A)\,.\,P(C) - P(A \cap B \cap C) \end{aligned}$$

[From (*)] ... (**)

and

$$\begin{aligned} P(A)\,P(B \cup C) &= P(A)\,[P(B) + P(C) - P(B \cap C)] \\ &= P(A)\,.\,P(B) + P(A)\,P(C) - P(A)\,P(B \cap C) \quad \ldots (***) \end{aligned}$$

From (**) and (***), on using (*), we get

$$P(A \cap B \cap C) = P(A)\,P(B \cap C) = P(A)\,P(B)\,P(C) \quad \text{[From (*)]}$$

Hence A, B, C are mutually independent.

3·15·2. Given n independent events A_i, $(i = 1, 2, \ldots, n)$ with respective probabilities of occurrence p_i, to find the probability of occurrence of at least one of them.

We have $P(A_i) = p_i \quad \Rightarrow \quad P(\overline{A}_i) = 1 - p_i\,;\, i = 1, 2, \ldots, n \quad \ldots (*)$

Hence the probability 'p' of happening of at least one of the events is given by :

$$\left.\begin{aligned} p = P(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n}) \\ &= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) = 1 - P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n) \end{aligned}\right\} \quad \ldots (**)\text{lie}$$

($\because A_1, A_2, \ldots, A_n$ are independent $\Rightarrow \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ are also independent)

$$= 1 - [(1 - p_1)(1 - p_2) \ldots (1 - p_n)]$$

$$= \left[\sum_{i=1}^{n} p_i - \sum\sum_{\substack{i,j=1 \\ i<j}}^{n} (p_i p_j) + \sum\sum\sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} (p_i p_j p_k) - \ldots + (-1)^{n-1} (p_1 p_2 \ldots p_n)\right]$$

Remark. The results in (**) are very important and are used quite often in numerical problems. Result (**) stated in words gives :

P [happening of at least one of the events $A_1, A_2, \ldots, A_n$]

$= 1 - P$ (none of the events $A_1, A_2, \ldots, A_n$ happens) ... (3·30)

or equivalently,

P {none of the given events happens} $= 1 - P$ {at least one of them happens}. ... (3·30a)

Example 3.38. *If $A \cap B = \phi$, then show that $P(A) \leq P(\bar{B})$.*

Solution. We have

$$A = (A \cap B) \cup (A \cap \bar{B}) = \phi \cup (A \cap \bar{B}) = A \cap \bar{B} \qquad [\because A \cap B = \phi \text{ (Given)}]$$

$$\therefore \quad A \subseteq \bar{B} \quad \Rightarrow \quad P(A) \leq P(\bar{B}), \text{ as desired.}$$

Aliter. Since $A \cap B = \phi$, we have $A \subset \bar{B}$, which implies that $P(A) \leq P(\bar{B})$.

Example 3·39. *Let A and B be two events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$, show that*

(a) $P(A \cup B) \geq \frac{3}{4}$, *and* *(b)* $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$.

Solution. (*a*) we have

$$A \subset (A \cup B) \Rightarrow P(A) \leq P(A \cup B) \Rightarrow \frac{3}{4} \leq P(A \cup B) \Rightarrow P(A \cup B) \geq \frac{3}{4}$$

(*b*) $A \cap B \subseteq B \Rightarrow P(A \cap B) \leq P(B) = \frac{5}{8}$ ... (*i*)

Also $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1 \Rightarrow \frac{3}{4} + \frac{5}{8} - 1 \leq P(A \cap B)$

$\therefore \quad \frac{6 + 5 - 8}{8} \leq P(A \cap B) \Rightarrow \frac{3}{8} \leq P(A \cap B)$ ... (*ii*)

From (*i*) and (*ii*), we get $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$.

Example 3·40. *For any two events A and B,*

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Proof. We have

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

Using axiom 3, we have

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)] = P(A \cap \bar{B}) + P(A \cap B)$$

Now $P[(A \cap \bar{B}) \geq 0$ (From axiom 1)

$\therefore$ $P(A) \geq P(A \cap B)$ ... (*)

Similarly $P(B) \geq P(A \cap B)$

$\Rightarrow$ $P(B) - P(A \cap B) \geq 0$

Now $P(A \cup B) = P(A) + [P(B) - P(A \cap B)]$ ...(**)

$\therefore$ $P(A \cup B) \geq P(A) \Rightarrow P(A) \leq P(A \cup B)$ ...(***)

Also $P(A \cup B) \leq P(A) + P(B)$ [From (**)] ...(****)

Hence from (*), (***) and (****), we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Aliter. Since $A \cap B \subset A$, by *Theorem 4·6 (ii)*, we get

$$P(A \cap B) \leq P(A).$$

Also $A \subset (A \cup B) \Rightarrow P(A) \leq P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\leq P(A) + P(B) \qquad [\because P(A \cap B) \geq 0]$$

Combining the above results, we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

EXAMPLES ON ADDITION AND MULTIPLICATION THEOREMS OF PROBABILITY

Example 3·41. *The odds against Manager X settling the wage dispute with the workers are 8 : 6 and odds in favour of manager Y settling the same dispute are 14 : 16.*

(i) What is the chance that neither settles the dispute, if they both try, independently of each other ?

(ii) What is the probability that the dispute will be settled ?

Solution. Let A be the event that the manager X will settle the dispute and B be the event that the Manager Y will settle the dispute. Then clearly

$$P(\bar{A}) = \frac{8}{8+6} = \frac{4}{7} \Rightarrow P(A) = 1 - P(\bar{A}) = \frac{6}{14} = \frac{3}{7}$$

$$P(B) = \frac{14}{14+16} = \frac{7}{15} \Rightarrow P(\bar{B}) = 1 - P(B) = \frac{16}{14+16} = \frac{8}{15}$$

The required probability that neither settles the dispute is given by :

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B}) = \frac{4}{7} \times \frac{8}{15} = \frac{32}{105}$$

[Since A and B independent $\Rightarrow$ $\bar{A}$ and $\bar{B}$ are also independent]

(ii) The dispute will be settled if at least one of the managers X and Y settles the dispute. Hence the required probability is given by :

$$P(A \cup B) = \text{Prob. [At least one of X and Y settles the dispute.]}$$

$$= 1 - \text{Prob. [None settles the dispute.]}$$

$$= 1 - P(\bar{A} \cap \bar{B}) = 1 - \frac{32}{105} = \frac{73}{105}. \qquad \text{[From Part (i)]}$$

Example 3·42. *The odds that person X speaks the truth are 3 : 2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point.*

Solution. Let us define the events :

A : X speaks the truth, B : Y speaks the truth

Then $\bar{A}$ and $\bar{B}$ represent the complementary events the X and Y tell a lie respectively. We are given :

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\bar{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and
$$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\bar{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways :

(*i*) X speaks the truth and Y tells a lie, *i.e.*, the event $A \cap \bar{B}$ happens,

(*ii*) X tells a lie and Y speaks the truth, *i.e.*, the event $\bar{A} \cap B$ happens.

Hence by addition theorem of probability, the required probability is given by :

$$P(E) = P(i) + P(ii) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$
$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B) \qquad \text{[Since } A \text{ and } B \text{ are independent]}$$
$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0{\cdot}475$$

Hence A and B are likely to contradict each other on an identical point in 47·5% of the cases.

Example 3·43. *One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0{\cdot}5$ $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$ and E_1, E_2, E_3, are independent events, find the probability that (a) exactly one hit is registered, and (b) at least two hits are registered.*

Solution. We are given : $P(E_1) = 0{\cdot}5$, $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$

$\Rightarrow$ $P(\bar{E}_1) = 0{\cdot}5$, $P(\bar{E}_2) = 0{\cdot}4$ and $P(\bar{E}_3) = 0{\cdot}2$

(*a*) Exactly one hit can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap \bar{E}_2 \cap \bar{E}_3$ happens, (*ii*) $\bar{E}_1 \cap E_2 \cap \bar{E}_3$ happens, (*iii*) $\bar{E}_1 \cap \bar{E}_2 \cap E_3$ happens.

Hence by addition probability theorem, the required probability '*p*' is given by :

$$p = P(E_1 \cap \bar{E}_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap E_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap \bar{E}_2 \cap E_3)$$
$$= P(E_1)\, P(\bar{E}_2)\, P(\bar{E}_3) + P(\bar{E}_1)\, P(E_2)\, P(\bar{E}_3) + P(\bar{E}_1)\, P(\bar{E}_2)\, P(E_3)$$

[Since E_1, E_2 and E_3 are independent.]

$$= 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}26.$$

(*b*) At least two hits can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap E_2 \cap \bar{E}_3$ happens (*ii*) $E_1 \cap \bar{E}_2 \cap E_3$ happens

(*iii*) $\bar{E}_1 \cap E_2 \cap E_3$ happens, (*iv*) $E_1 \cap E_2 \cap E_3$ happens.

$\therefore$ Required probability

$$= P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$
$$= 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8$$
$$= 0{\cdot}06 + 0{\cdot}16 + 0{\cdot}24 + 0{\cdot}24 = 0{\cdot}70.$$

Example 3·44. *An urn contains 4 tickets numbered 1, 2, 3, 4 and another contains 6 tickets numbered 2, 4, 6, 7, 8, 9. If one of the two urns is chosen at random and a ticket is* *drawn at random from the chosen urn, find the probabilities that the ticket drawn bears the number* (i) 2 or 4, (ii) 3, (iii) 1 or 9.

Solution. (*i*) Required event can happen in the following mutually exclusive ways :

(*I*) First urn is chosen and then a ticket is drawn.

(*II*) Second urn is chosen and then a ticket is drawn.

Since the probability of choosing any urn is $\frac{1}{2}$, required probability 'p' is given by :

$$p = P(I) + P(II) = \frac{1}{2}\times\frac{2}{4} + \frac{1}{2}\times\frac{2}{6} = \frac{5}{12}$$

(*ii*) Required probability $= \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times 0 = \frac{1}{8}$

($\because$ In the 2nd urn there is no ticket with number 3.)

(*iii*) Required probability $= \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{6} = \frac{5}{24}$.

Example 3·45. *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.*

Solution. The required event E that 'in a draw of 4 balls from the box at random there is at least one ball of each colour', can materialise in the following mutually disjoint ways :

(*i*) 1 Red, 1 White, 2 Black balls ; (*ii*) 2 Red, 1 White, 1 Black balls; (*iii*) 1 Red, 2 White, 1 Black balls.

Hence by addition theorem of probability, the required probability is given by :

$$P(E) = P(i) + P(ii) + P(iii)$$

$$= \frac{{}^6C_1 \times {}^4C_1 \times {}^5C_2}{{}^{15}C_4} + \frac{{}^6C_2 \times {}^4C_1 \times {}^5C_1}{{}^{15}C_4} + \frac{{}^6C_1 \times {}^4C_2 \times {}^5C_1}{{}^{15}C_4}$$

$$= \frac{1}{{}^{15}C_4}[6\times4\times10 + 15\times4\times5 + 6\times6\times5] = \frac{4!}{15\times14\times13\times12}(240+300+180)$$

$$= \frac{24\times720}{15\times14\times13\times12} = 0{\cdot}5275.$$

Example 3·46. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is 13/32.*

Solution. The required event of getting 1 girl and two boys among the three selected children can materialise in the following three mutually disjoint cases :

Group No. →	I	II	III
(*i*)	Girl	Boy	Boy
(*ii*)	Boy	Girl	Boy
(*iii*)	Boy	Boy	Girl

Hence by addition theorem of probability,

Required probability $= P(i) + P(ii) + P(iii)$ ...(*)

Since the probability of selecting a girl from the first group is 3/4, of selecting a boy from the second group is 2/4, and of selecting a boy from the third group is 3/4, and since these three events of selecting children from three groups are independent of each other, by compound probability theorem, we have

$$P\,(i) = \frac{3}{4}\times\frac{2}{4}\times\frac{3}{4} = \frac{9}{32}; \quad P\,(ii) = \frac{1}{4}\times\frac{2}{4}\times\frac{3}{4} = \frac{3}{32}; \quad \text{and} \quad P\,(iii) = \frac{1}{4}\times\frac{2}{4}\times\frac{1}{4} = \frac{1}{32}$$

Substituting in (*), we get

$$\text{Required probability} = \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}.$$

Example 3·47. *It is 8 : 5 against the wife who is 40 years old living till she is 70 and 4 : 3 against her husband now 50 living till he is 80. Find the probability that*

(i) Both will be alive, *(ii) None will be alive,*
(iii) Only wife will be alive, *(iv) Only husband will be alive,*
(v) Only one will be alive, *(vi) At least one will be alive,*

30 years hence.

Solution. Let us define the events :

A : Wife will be alive, and B : Husband will be alive; 30 years hence.

Then, we are given :

$$P\,(A) = \frac{5}{8+5} = \frac{5}{13} \quad \Rightarrow \quad P\,(\bar{A}) = 1 - P\,(A) = \frac{8}{13}$$

$$P\,(B) = \frac{3}{4+3} = \frac{3}{7} \quad \Rightarrow \quad P\,(\bar{B}) = 1 - P\,(B) = \frac{4}{7}$$

If we assume that A and B are independent so that A and $\bar{B}$, $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$ are also independent, then the required probabilities are given by :

(i) $P\,(A \cap B) = P\,(A)\,P\,(B) = \frac{5}{13}\times\frac{3}{7} = \frac{15}{91}$ $(\because A$ and B are independent.)

(ii) $P\,(\bar{A} \cap \bar{B}) = P\,(\bar{A})\,P\,(\bar{B}) = \frac{8}{13}\times\frac{4}{7} = \frac{32}{91}$ $(\because \bar{A}, \bar{B}$ are independent.)

(iii) $P\,(A \cap \bar{B}) = P\,(A) - P\,(A \cap B) = \frac{5}{13} - \frac{15}{91} = \frac{20}{91}$ [From Part (i)]

(iv) $P\,(\bar{A} \cap B) = P\,(B) - P\,(A \cap B) = \frac{3}{7} - \frac{15}{91} = \frac{24}{91}$ [From Part (i)]

(v) $P\,(A \cap \bar{B}) + P\,(\bar{A} \cap B) = \frac{20}{91} + \frac{24}{91} = \frac{44}{91}$ [From Parts (iii) and (iv)]

(vi) $P\,(A \cup B) = 1 - (\bar{A} \cap \bar{B}) = 1 - \frac{32}{91} = \frac{59}{91}.$ [From Part (ii)]

Example 3·48. *A problem in Statistics is given to three students A, B and C whose chances of solving it are 1/2, 3/4 and 1/4 respectively.*

What is the probability that the problem will be solved if all of them try independently ?

Solution. Let A, B, C denote the events that the problem is solved by the students A, B, C respectively. Then

$$P\,(A) = \frac{1}{2}, \quad P\,(B) = \frac{3}{4}, \quad \text{and} \quad P\,(C) = \frac{1}{4}$$

The problem will be solved if at least one of them solves the problem. Thus we have to calculate the probability of occurrence of at least one of the three events A, B, C, *i.e.*, $P\,(A \cup B \cup C)$.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$= P(A) + P(B) + P(C) - P(A)\,P(B) - P(A)\,P(C) - P(B)\,P(C) + P(A)\,P(B)\,P(C)$$
($\because$ A, B, C are mutually independent events.)
$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2}\cdot\frac{3}{4} - \frac{3}{4}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{29}{32}.$$

Aliter. $P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$
$$= 1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C})$$
[$\because$ A, B, C are mutually independent $\Rightarrow \bar{A}, \bar{B}$ and $\bar{C}$ are mutually independent.]
$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right) = 1 - \frac{3}{32} = \frac{29}{32}.$$

Example 3·49. *A manager has two assistants and he bases his decision on information supplied independently by each one of them. The probability that he makes a mistake in his thinking is 0·005. The probability that an assistant gives wrong information is 0·3. Assuming that the mistakes made by the manager are independent of the information given by the assistants, find the probability that he reaches a wrong decision.*

Solution. Let us define the following events :

A : The manager makes a mistake in his thinking.

B : The 1st assistant gives him wrong information.

C : The 2nd assistant gives him wrong information.

In usual notations, we are given :
$$P(A) = 0{\cdot}005,\ P(B) = 0{\cdot}3 = P(C) \quad \Rightarrow \quad P(\bar{A}) = 0{\cdot}995,\ P(\bar{B}) = P(\bar{C}) = 0{\cdot}7$$

Assuming that the mistakes made by the manager are independent of the information supplied independently by each of the two assistants, we conclude that A, B and C, and consequently $\bar{A}, \bar{B}$ and $\bar{C}$ are mutually independent.

$\therefore$ $p = P$ [Manager reaches a wrong decision]

$= 1 - P$ [Manager reaches a correct decision] $= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$; because manager will reach a correct decision if he does not make a mistake in his thinking (*i.e.*, $\bar{A}$ happens) *and* both the assistants supply him correct information (*i.e.*, $\bar{B} \cap \bar{C}$ happens).

$\therefore$ $p = 1 - P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$

[Since the events $\bar{A}, \bar{B}$ and $\bar{C}$ are independent.]

$= 1 - 0{\cdot}995 \times 0{\cdot}7 \times 0{\cdot}7 = 1 - 0{\cdot}48755 = 0{\cdot}51245.$

Example 3·50. *The odds that a book on Statistics will be favourably reviewed by 3 independent critics are 3 to 2, 4 to 3 and 2 to 3 respectively. What is the probability that of the three reviews :*

(*i*) *All will be favourable,*

(*ii*) *Majority of the reviews will be favourable,*

(*iii*) *Exactly one review will be favourable,*

(*iv*) *Exactly two reviews will be favourable, and*

(*v*) *At least one of the reviews will be favourable.*

Solution. Let A, B and C denote respectively the events that the book is favourably reviewed by first, second and third critic respectively. Then we are given :

$$P(A)=\frac{3}{5},\ P(B)=\frac{4}{7} \text{ and } P(C)=\frac{2}{5} \quad \Rightarrow \quad P(\bar{A})=\frac{2}{5},\ P(\bar{B})=\frac{3}{7} \text{ and } P(\bar{C})=\frac{3}{5}$$

(*i*) The probability that all the three reviews will be favourable is :

$$P(A\cap B\cap C)=P(A)\times P(B)\times P(C)=\frac{3}{5}\times\frac{4}{7}\times\frac{2}{5}=\frac{24}{175}$$

($\because$ A, B and C are mutually independent events.)

(*ii*) The event that majority, *i.e.*, at least 2 reviews are favourable can materialise in the following mutually exclusive ways :

(*a*) $A\cap B\cap \bar{C}$ happens, (*b*) $A\cap \bar{B}\cap C$ happens, (*c*) $\bar{A}\cap B\cap C$ happens, and (*d*) $A\cap B\cap C$ happens.

Hence, the required probability is :

$$P(A\cap B\cap \bar{C})+P(A\cap \bar{B}\cap C)+P(\bar{A}\cap B\cap C)+P(A\cap B\cap C)$$

$$=P(A)\,P(B)\,P(\bar{C})+P(A)\,P(\bar{B})\,P(C)+P(\bar{A})\,P(B)\,P(C)+P(A)\,P(B)\,P(C)$$

$$=\frac{3}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{3}{5}\times\frac{3}{7}\times\frac{2}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{2}{5}+\frac{3}{5}\times\frac{4}{7}\times\frac{2}{5}=\frac{94}{175}$$

(*iii*) Arguing as in case (*ii*), the probability that exactly one review will be favourable is

$$P(A\cap \bar{B}\cap \bar{C})+P(\bar{A}\cap B\cap \bar{C})+P(\bar{A}\cap \bar{B}\cap C)$$

$$=\frac{3}{5}\times\frac{3}{7}\times\frac{3}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{2}{5}\times\frac{3}{7}\times\frac{2}{5}=\frac{63}{175}$$

(*iv*) Similarly, the probability that exactly two reviews will be favourable is :

$$P(A\cap B\cap \bar{C})+P(A\cap \bar{B}\cap C)+P(\bar{A}\cap B\cap C)$$

$$=P(A)\times P(B)\times P(\bar{C})+P(A)\times P(\bar{B})\times P(C)+P(\bar{A})\times P(B)\times P(C)$$

$$=\frac{3}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{3}{5}\times\frac{3}{7}\times\frac{2}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{2}{5}=\frac{70}{175}$$

(*v*) The probability that at least one of the reviews will be favourable is :

$$P(A\cup B\cup C)=1-P(\bar{A}\cap \bar{B}\cap(\bar{C})=1-P(\bar{A})\times P(\bar{B})\times P(\bar{C})$$

$$=1-\frac{2}{5}\times\frac{3}{7}\times\frac{3}{5}=\frac{157}{175}$$

In (*ii*) to (*v*) we have used that A, B and C are mutually independent.

Example 3·51. *A and B alternately cut a pack of cards and the pack is shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what are the respective chances of A and B first cutting a diamond ?*

Solution. Let A_i and B_i denote the events of A and B cutting a diamond respectively in the ith trial. Then, we are given :

$$P(A_i)=P(B_i)=\frac{13}{52}=\frac{1}{4} \quad \Rightarrow \quad P(\bar{A}_i)=P(\bar{B}_i)=\frac{3}{4};\ i=1,2,3,\ldots$$

If A starts the game, he can first cut the diamond in the following mutually exclusive ways :

(*i*) A_1 happens, (*ii*) $\bar{A}_1\cap \bar{B}_2\cap A_3$ happens, (*iii*) $\bar{A}_1\cap \bar{B}_2\cap \bar{A}_3\cap \bar{B}_4\cap A_5$ happens,

Example 3·54. *An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i, ($i = 1, 2, 3$) be the event that ith digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.*

Solution. A_1 is the event that the first digit of the number of the ticket drawn is 1 and the favourable cases for this are 112 and 121, *i.e.*, two cases.

$\therefore \quad P(A_1) = \frac{2}{4} = \frac{1}{2}$. Similarly, we get $P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2}$

$A_1 \cap A_2$ is the event that the first two digits in the number which the selected ticket bears are each equal to unity and the only favourable case is ticket with number 112.

$$\therefore \quad P(A_1 \cap A_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A_1)\, P(A_2)$$

Similarly, $P(A_2 \cap A_3) = \frac{1}{4} = P(A_2)\, P(A_3)$ and $P(A_3 \cap A_1) = \frac{1}{4} = P(A_3)\, P(A_1)$

Thus we conclude that the events A_1, A_2 and A_3 are pairwise independent. Now

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P\{\text{all the three digits in the number selected are 1's}\} \\ &= P(\phi) = 0 \\ &\neq P(A_1)\, P(A_2)\, P(A_3) \end{aligned}$$

Hence A_1, A_2 and A_3, though pairwise independent are not mutually independent.

Example 3·55. *Two fair dice are thrown independently. Three events A, B and C are defined as follows :*

A : Odd face with first dice

B : Odd face with second dice

C : Sum of points on two dice is odd.

Are the events A, B and C (i) Pairwise independent, (ii) Mutually independent ?

Solution. In a random toss of two dice

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(S) = 6 \times 6 = 36$$

Event	*Favourable cases*	*No. of favourable cases*
A	$\{1, 3, 5\} \times \{1, 2, 3, 4, 5, 6\}$	$3 \times 6 = 18$
B	$\{1, 2, 3, 4, 5, 6\} \times \{1, 3, 5\}$	$6 \times 3 = 18$
*C	$\{1, 3, 5\} \times \{2, 4, 6\} \cup \{2, 4, 6\} \times \{1, 3, 5\}$	$3 \times 3 + 3 \times 3 = 18$
$A \cap B$	$\{1, 3, 5\} \times \{1, 3, 5\}$	$3 \times 3 = 9$
** $A \cap C$	$\{1, 3, 5\} \times \{2, 4, 6\}$	$3 \times 3 = 9$
** $B \cap C$	$\{2, 4, 6\} \times \{1, 3, 5\}$	$3 \times 3 = 9$
$A \cap B \cap C$	ϕ	0

* The sum of points on two dice be odd if one shows odd number and the other shows even number.

** If one die shows odd number and the sum is also odd, then the other die must show even number.

$$P(A) = \frac{18}{36} = \frac{1}{2} = P(B) = P(C)$$

$\therefore$
$$\left.\begin{aligned} P(A \cap B) &= \frac{9}{36} = \frac{1}{4} = P(A)\,P(B) \\ P(A \cap C) &= \frac{9}{36} = \frac{1}{4} = P(A)\,P(C) \\ P(B \cap C) &= \frac{9}{36} = \frac{1}{4} = P(B)\,P(C) \end{aligned}\right\} \quad \ldots (*)$$

and $$P(A \cap B \cap C) = P(\phi) = 0 \neq P(A)\,P(B)\,P(C) \quad \ldots (**)$$

Hence (*) implies that the events A, B and C are pairwise independent but (**) implies that they not mutually independent.

Example 3·56. *Why does it pay to bet consistently on seeing 6 at least once in 4 throws of a die, but not on seeing a double six at least once in 24 throws with two dice ? (de Mere's Problem).*

Solution. The probability of getting a '6' in throw of die $= \frac{1}{6}$

$\therefore$ The probability of not getting a '6' in a throw of die $= 1 - \frac{1}{6} = \frac{5}{6}$

By compound probability theorem, the probability that in 4 throws of die no '6' is obtained $= \left(\frac{5}{6}\right)^4$

Hence, the probability of obtaining '6' at least once in 4 throws of a die

$$= 1 - \left(\frac{5}{6}\right)^4 = 0{\cdot}516$$

Now, if a trial consists of throwing two dice at a time, then the probability of getting a 'double' of '6' in a trial $= \frac{1}{36}$

Thus the probability of not getting a 'double of 6' in a trial $= \frac{35}{36}$

The probability that in 24 throws, with two dice each, no 'double of 6' is obtained

$$= \left(\frac{35}{36}\right)^{24}$$

Hence the probability of getting a 'double of 6' at least once in 24 throws

$$= 1 - \left(\frac{35}{36}\right)^{24} = 0{\cdot}491$$

Since the probability in the first case is greater than the probability in the second case, the result follows.

Example 3·57. *Let $A_1, A_2, \ldots A_n$ be independent events and $P(A_k) = p_k$. Further, let p be the probability that none of the events occurs ; then show that $p \le \exp\left(-\sum_k p_k\right)$.*

Solution. We have $$p_i = P(A_i)\ ;\ i = 1, 2, \ldots, n$$

CHAPTER CONCEPTS QUIZ

1. Find out the correct answer from group Y for each item of group X.

Group X	*Group Y*
(a) At least one of the events A or B occurs.	(i) $(\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B})$ e
(b) Neither A nor B occurs.	(ii) $(A \cup B) - (A \cap B)$
(c) Exactly one of the events A or B occurs.	(iii) $A \subset B$ d
(d) If event A occurs, so does B.	(iv) $B \subset A$
(e) Not more than one of the events A or B occurs.	(v) $[A - (A \cap B)] \cup [B - A \cap B)]$ c
	(vi) $\bar{A} \cap \bar{B}$ b
	(vii) $A \cup B$ a
	(viii) $S - (A \cup B)$

2. Match the correct expression of probabilities on the left :

(a) $P(\phi)$, where ϕ is null set	(i) $1 - P(A)$ c
(b) $P(A \mid B)\, P(B)$	(ii) $P(A \cap B)$ b
(c) $P(\bar{A})$	(iii) $P(A) - P(A \cap B)$ e
(d) $P(\bar{A} \cap \bar{B})$	(iv) 0 a
(e) $P(A - B)$	(v) $1 - P(A) - P(B) + P(A \cap B)$ d

(*f*) $P(A \cup B)$ (*vi*) $P(A) + P(B) - P(A \cap B)$

3. Given that A, B and C are mutually exclusive events, explain why the following are not permissible assignments of probabilities :

(*i*) $P(A) = 0·24$, $P(B) = 0·4$ and $P(A \cup C) = 0·2$

(*ii*) $P(A) = 0·4$, $P(B) = 0·61$

4. In each of the following, indicate whether events A and B are :
(*i*) independent, (*ii*) mutually exclusive, (*iii*) dependent but not mutually exclusive.

(*a*) $P(A \cap B) = 0$ (*b*) $P(A \cap B) = 0·3$, $P(A) = 0·45$

(*c*) $P(A \cup B) = 0·85$, $P(A) = 0·3$, $P(B) = 0·6$

(*d*) $P(A \cup B) = 0·70$, $P(A) = 0·5$, $P(B) = 0·4$

(*e*) $P(A \cup B) = 0·90$, $P(A \mid B) = 0·8$, $P(B) = 0·5$.

5. Give the correct label as answer like *a* or *b* etc., for the following questions :

(*i*) The probability of drawing any one spade card from a pack of cards is

(*a*) $\frac{1}{52}$ (*b*) $\frac{1}{13}$ (*c*) $\frac{4}{13}$ (*d*) $\frac{1}{4}$

(*ii*) The probability of drawing one white ball from a bag containing 6 red, 8 black, 10 yellow and 1 green balls is

(*a*) $\frac{1}{25}$ (*b*) 0 (*c*) 1 (*d*) $\frac{24}{25}$ (*e*) $\frac{15}{20}$

(*iii*) A coin is tossed three times in succession, the number of sample points in sample space is

(*a*) 6 (*b*) 8 (*c*) 3 (*d*) 9

(*iv*) In the simultaneous tossing of two perfect coins, the probability of having at least one head is

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{3}{4}$ (*d*) 1

(*v*) In the simultaneous tossing of two perfect dice, the probability of obtaining 4 as the sum of the resultant faces is

(*a*) $\frac{4}{12}$ (*b*) $\frac{1}{12}$ (*c*) $\frac{3}{12}$ (*d*) $\frac{2}{12}$

(*vi*) A single letter is selected at random from the word 'probability'. The probability that it is a vowel is

(*a*) $\frac{3}{11}$ (*b*) $\frac{2}{11}$ (*c*) $\frac{4}{11}$ (*d*) 0

(*vii*) An urn contains 9 balls, two of which are red, three blue and four black. Three balls are drawn at random. The chance that they are of the same colour is

(*a*) $\frac{5}{84}$ (*b*) $\frac{3}{9}$ (*c*) $\frac{3}{7}$ (*d*) $\frac{7}{17}$

(*viii*) A number is chosen at random among the first 120 natural numbers. The probability of the number chosen being a multiple of 5 or 15 is

(*a*) $\frac{1}{5}$ (*b*) $\frac{1}{8}$ (*c*) $\frac{1}{16}$ (*d*) $\frac{1}{9}$.

(*ix*) If A and B are mutually exclusive events, then

(*a*) $P(A \cup B) = P(A) \cdot P(B)$, (*b*) $P(A \cup B) = P(A) + P(B)$, (*c*) $P(A \cup B) = 0$.

(*x*) If A and B are two independent events, the probability that both A and B occrs is $\frac{1}{8}$ and the probability that neither of them occurs is $\frac{3}{8}$. If $P(A) < P(B)$, then the probability of the occurrence of A is :

(*a*) $\frac{1}{2}$, (*b*) $\frac{1}{3}$, (*c*) $\frac{1}{4}$, (*d*) $\frac{1}{5}$.

(*xi*) If A and B are two events, the probability that exactly one of them occurs is given by :

(*a*) $P(A) + P(B) - 2P(A \cap B)$, (*b*) $P(A) + P(B) - P(A \cap B)$
(*c*) $P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$, (*d*) $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

(*xii*) If $P(A \cap B) = \frac{1}{2}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$, and $P(A) = P(B) = p$, then the value of p is given by :

(*a*) $\frac{1}{2}$, (*b*) $\frac{7}{8}$, (*c*) $\frac{1}{3}$, (*d*) $\frac{7}{12}$.

(*xiii*) If $P(A \cap B) = \frac{1}{2}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{2}$ and $2P(A) = P(B) = p$, then the value of p is given by :

(*a*) $\frac{1}{4}$, (*b*) $\frac{1}{2}$, (*c*) $\frac{1}{3}$, (*d*) $\frac{2}{3}$.

(*xiv*) A and B are two independent events such that $P(A \cap \bar{B}) = \frac{3}{25}$ and $P(\bar{A} \cap B) = \frac{8}{25}$. If $P(A) < P(B)$, then $P(A)$ is :

(*a*) $\frac{1}{5}$, (*b*) $\frac{2}{5}$, (*c*) $\frac{3}{5}$, (*d*) $\frac{4}{5}$.

(*xv*) A and B are two independent events such that $P(\bar{A}) = 0{\cdot}7$, $P(\bar{B}) = k$ and $P(A \cup B) = 0{\cdot}8$, then k is

(*a*) $\frac{5}{7}$, (*b*) 1, (*c*) $\frac{2}{7}$, (*d*) none of these

(*xvi*) The probability that a 3-card hand drawn at random and without replacement from an ordinary deck consists entirely of black cards is :

(*a*) $\frac{1}{17}$, (*b*) $\frac{2}{17}$, (*c*) $\frac{1}{8}$, (*d*) $\frac{3}{17}$, (*e*) $\frac{4}{17}$.

(*xvii*) What is the probability that a bridge hand contains one card of each denomination (*i.e.*, one ace, one king, one queen, ..., one three, one two) ?

(*a*) $\frac{13!}{13^{13}}$, (*b*) $\frac{4^{13}}{{}^{52}C_{13}}$, (*c*) $\frac{{}^{52}C_4}{{}^{52}C_{13}}$ (*d*) $\left(\frac{1}{13}\right)^{13}$ (*e*) $\frac{13^4}{{}^{52}C_{13}}$

(*xviii*) If the events S and T have equal probability and are independent with $P(S \cap T) = p > 0$, then $P(S)$

(*a*) $\sqrt{p}$, (*b*) p^2, (*c*) $\frac{p}{2}$, (*d*) p, (*e*) $2p$

(*xix*) The probability that both S and T occur, the probability that S occurs and T does not, and the probability that T occurs and S does not are all equal to p. The probability that either S or T occurs is :

(*a*) p, (*b*) $2p$, (*c*) $3p$, (*d*) $3p^2$, (*e*) p^3

(*x*) Events S and T are independent with $P(S) < P(T)$, $P(S \cap T) = 6/25$, and $P(S \mid T) + P(T \mid S) = 1$. Then $P(S)$ is

(*a*) $\frac{1}{25}$, (*b*) $\frac{1}{5}$, (*c*) $\frac{6}{25}$, (*d*) $\frac{2}{5}$, (*e*) $\frac{3}{5}$

(*xxi*) An unbiased die is thrown two independent times. Given that the first throw resulted in an even number, the probability that the sum obtained is 8 is :

(*a*) $\frac{5}{36}$, (*b*) $\frac{1}{6}$, (*c*) $\frac{4}{21}$, (*d*) $\frac{7}{36}$, (*e*) $\frac{1}{3}$

6. Fill in the blanks :

(i) Two events are said to be equally likely if ......

(ii) A set of events is said to be independent if .....

(iii) If $P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C)$, then the events A, B, C are .....

(iv) Two events A and B are mutually exclusive if $P(A \cap B) = \ldots$ and are independent if $P(A \cap B) = \ldots$

(v) The probability of getting a multiple of 2 in a throw of a dice is $1/2$ and of getting a multiple of 3 is $1/3$. Hence probability of getting a multiple of 2 or 3 is ......

(vi) Let A and B be independent events and suppose the event C has probability 0 or 1. Then A, B and C are ...... events.

(vii) If A, B, C are pairwise independent and A is independent of $B \cup C$, then, A, B, C are ...... independent.

(viii) A man has tossed 2 fair dice. The conditional probability that he has tossed two sixes, given that he has tossed at least one six is ......

(ix) Let A and B be two events such that $P(A) = 0{\cdot}3$ and $P(A \cup B) = 0{\cdot}8$. If A and B are independent events than $P(B) = \ldots$

(x) If $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are probabilities of three mutually exclusive events, then the set of all values of p is ...

(xi) If A and B are two events, then $P(A \cup B) = P(A \cap B)$ if and only if relation between $P(A)$ and $P(B)$ is ...

(xii) A bag contains tickets numbered 1 to 100. Ten tickets are drawn at random and arranged in ascending order. The probability that fourth and sixth ticket bear numbers 50 and 60 respectively is ......

7. Each of the following statements is either true or false. If it is true prove it, otherwise, give a counter example to show that it is false :

(i) The probability of occurrence of at least one of two events is the sum of the probability of each of the two events.

(ii) Mutually exclusive events are independent.

(iii) For any two events A and B, $P(A \cap B)$ cannot be less than either $P(A)$ or $P(B)$.

(iv) The conditional probability of A given B is always greater than $P(A)$.

(v) If the occurrence of an event A implies the occurrence of another event B then $P(A)$ cannot exceed $P(B)$.

(vi) For any two events A and B, $P(A \cup B)$ cannot be greater than either $P(A)$ or $P(B)$.

(vii) Mutually exclusive events are not independent.

(viii) Pairwise independence does not necessarily imply mutual independence.

(ix) Let A and B be events neither of which has probability zero. Then if A and B are disjoint, A and B are independent.

(x) The probability of any event is always a proper fraction.

(xi) If $0 < P(B) < 1$ so that $P(A \mid B)$ and $P(A \mid \bar{B})$ are both defined, then

$$P(A) = P(B)\, P(A \mid B) + P(\bar{B})\, P(A \mid \bar{B}).$$

(xii) For two events A and B if $P(A) = P(A \mid B) = 1/4$ and $P(A \mid \bar{B}) = 1/2$, then

(a) A and B are mutually exclusive.
(b) A and B are independent.
(c) A is a sub-event of B.
(d) $P(\bar{A} \mid B) = 3/4$.

(xiii) Two events can be independent and mutually exclusive simultaneously.

(xiv) Let A and B be events, neither of which has probability zero. Prove or disprove the following :

(*a*) If A and B are disjoint, A and B are independent.

(*b*) If A and B are independent, A and B are disjoint.

(xv) If $P(A) = 0$, then $A = \phi$.

(xvi) For two events A and B we have the following probabilities :

$$P(A) = P(A \mid B) = \frac{1}{4} \text{ and } P(B \mid A) = \frac{1}{2}.$$

Check whether the following statements are true or false :

(*a*) A and B are mutually exclusive, (*b*) A and B are independent,

(*c*) A is a sub-event of B, (*d*) $P(\bar{A} \mid B) = \frac{3}{4}$.

(xvii) Consider two events A and B such that $P(A) = \frac{1}{4}$, $P(B \mid A) = \frac{1}{2}$, $P(A \mid B) = \frac{1}{4}$.
For each of the following statements, ascertain whether it is true or false :

(*i*) A is a sub-event of B, (*ii*) $P(\bar{A} \mid \bar{B}) = \frac{3}{4}$ (*iii*) $P(A \mid B) + P(A \mid \bar{B}) = 1$

(xviii) If A and B are two events none of which has probability zero, then A and B are disjoint implies that A and B are independent.

DISCUSSION AND REVIEW QUESTIONS

1. (*a*) Give the classical and statistical definitions of probability. What are the objections raised in these definitions ?

(*b*) When are a number of cases said to be equally likely ? Give an example of the following :

(*i*) the equally likely cases,

(*ii*) four cases which are not equally likely, and

(*iii*) five cases in which one case is more likely than the other four.

(*c*) What is meant by mutually exclusive events ? Give an example of

(*i*) three mutually exclusive events,

(*ii*) three events which are not mutually exclusive.

(*d*) Can

(*i*) events be mutually exclusive and exhaustive ?

(*ii*) events be exhaustive and independent ?

(*iii*) events be mutually exclusive and independent ?

(*iv*) events be exhaustive, mutually exclusive and independent ?

2. (*i*) If A, B and C are any three events, write down the theoretical expressions for the following events :

(*a*) Only A occurs, (*b*) A and B occur but C does not,

(*c*) A, B, and C all the three occur, (*d*) At least one occurs

(*e*) At least two occur, (*f*) One does not occur,

(*g*) Two do not occur, (*h*) None occurs.

4·2. BAYES' THEOREM

In the discussion of conditional probability we indicated that revising probability when new information is obtained is an important phase of probability analysis. Often, we begin our analysis with initial or *prior* probability estimates for specific events of interest. Then, from sources such as a sample, a special report, a product test, and so on we obtain some additional information about the events. Given this new information, we update the prior probability values by calculating revised probabilities, referred to as *posterior probabilities*. Bayes' theorem *which was given by Thomas Bayes, a British Mathematician,* in 1763, provides a means for making these probability calculations. The steps in this probability revision process are shown in the following diagram :

Prior Probabilities	→	New Information	→	Application of Bayes' Theorem	→	Posterior Probabilities

Theorem 4·2. *Bayes' Theorem. If $E_1, E_2, E_2 \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0$, $(i = 1, 2, \ldots, n)$, then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have*

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} = \frac{P(E_i)\, P(A \mid E_i)}{P(A)}\ ; i = 1, 2, \ldots, n \qquad \ldots (4{\cdot}9)$$

Proof. Since $A \subset \bigcup_{i=1}^{n} E_i$, we have, $A = A \cap \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (A \cap E_i)$

[By distributive law]

Since $(A \cap E_i) \subset E_i$, $(i = 1, 2, \ldots, n)$ are mutually disjoint events, we have by addition theorem of probability :

$$P(A) = P\left\{ \bigcup_{i=1}^{n} (A \cap E_i) \right\} = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i), \qquad \ldots(*)$$

by multiplication theorem of probability.

Also we have $\quad P(A \cap E_i) = P(A)\, P(E_i \mid A)$

$$\Rightarrow \qquad P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \text{[From (*)]}$$

Example 4·4. *A factory produces a certain type of outputs by three types of machine. The respective daily production figures are :*

Machine I : 3,000 Units; Machine II : 2,500 Units; Machine III : 4,500 Units

Past experience shows that 1 per cent of the output produced by Machine I is defective. The corresponding fraction of defectives for the other two machines are 1·2 per-cent and 2 per-cent respectively. An item is drawn at random from the day's production run and is found to be defective. What is probability that it comes from the output of

(i) Machine I, *(ii) Machine II,* *(iii) Machine (iii) ?*

Solution. Let E_1, E_2 and E_3 denote the events that the output is produced by machines I, II and III respectively and let A denote the event that the output is defective. Then we have :

$$P(E_1) = \frac{3000}{10,000} = 0 \cdot 30, \quad P(E_2) = \frac{2500}{10,000} = 0 \cdot 25, \quad P(E_3) = \frac{4500}{10,000} = 0 \cdot 45$$

$$P(A \mid E_1) = 1\% = 0 \cdot 01, \quad P(A \mid E_2) = 1 \cdot 2\% = 0 \cdot 012, \quad P(A \mid E_3) = 2\% = 0 \cdot 02$$

The probability that an item selected at random from day's production is defective is given by :

$$P(A) = \sum_{i=1}^{3} P(E_i \cap A) = \sum_{i=1}^{3} P(E_i) \cdot P(A \mid E_i)$$

$$= 0 \cdot 30 \times 0 \cdot 01 + 0 \cdot 25 \times 0 \cdot 012 + 0 \cdot 45 \times 0 \cdot 02 = 0 \cdot 015$$

By Baye's rule, the required probabilities are given by :

(i) $$P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{P(A)} = \frac{0.003}{0.015} = \frac{1}{5}$$

(ii) $$P(E_2 \mid A) = \frac{P(E_2) \cdot P(A \mid E_3)}{P(A)} = \frac{0.003}{0.015} = \frac{1}{5}$$

(iii) $$P(E_3 \mid E) = \frac{P(E_3) \cdot P(A \mid E_3)}{P(A)} = \frac{0.009}{0.015} = \frac{3}{5}$$

The probabilities in (i), (ii) and (iii) are known as posterior probabilities of events E_1, E_2 and E_3 respectively.

Aliter. The posterior probabilities can be obtained elegantly in a tabular form as given below.

Events E_i	Prior Probabilities $P(E_i)$	Conditional Probabilities $P(A \mid E_i)$	Joint Probabilities $P(E_i \cap A)$	Posterior Probabilities $P(E_i \mid A)$
(1)	(2)	(3)	(4) = (2) × (3)	(5) = (4) ÷ P (A)
E_1	0·30	0·010	0·003	$\frac{0.003}{0.015} = \frac{1}{5}$
E_2	0·25	0·012	0·003	$\frac{0.003}{0.015} = \frac{1}{5}$
E_3	0·45	0·020	0·009	$\frac{0.009}{0.015} = \frac{3}{5}$
Total	1·00		P (A) = 0·015	1

Example 4·5. *There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is $\frac{6}{7}$, find the value of n.*

Solution. Let E_1 denote the event that bag A is selected and E_2 denote the event that bag B is selected. Let E be the event that two balls drawn are white. We have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E \mid E_1) = \frac{{}^nC_2}{{}^{n+2}C_2} = \frac{n(n-1)}{(n+2)(n+1)}$$

and $$P(E \mid E_2) = \frac{{}^2C_2}{{}^{n+2}C_2} = \frac{2}{(n+2)(n+1)}$$

Using Baye's Theorem, the probability that the two white balls drawn are from the bag A, is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E_1)\,P(E \mid E_1) + P(E_2)\,P(E \mid E_2)} = \frac{6}{7} \qquad \text{(Given)}$$

$$\Rightarrow \quad \frac{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)}}{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)} + \frac{1}{2} \cdot \frac{2}{(n+2)(n+1)}} = \frac{6}{7} \quad \Rightarrow \quad \frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$$

$$\therefore \quad 7n(n-1) = 6n(n-1) + 12 \quad \Rightarrow \quad n^2 - n - 12 = 0 \quad \Rightarrow \quad n = 4, -3.$$

Since n cannot be negative, we get $n = 4$.

Example 4·6. *A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA ?*

Solution. Let E_1 and E_2 denote the events that the letter came from TATANAGAR and CALCUTTA respectively. Let A denote the event that two consecutive visible letters on the envelop are TA. We have

$$P(E_1) = P(E_2) = \frac{1}{2},\ P(A \mid E_1) = \frac{2}{8} \quad \text{and} \quad P(A \mid E_2) = \frac{1}{7}.$$

Using the Bayes' theorem, we get

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)} = \frac{\frac{1}{2}\cdot\frac{1}{7}}{\frac{1}{2}\cdot\frac{2}{8}+\frac{1}{2}\cdot\frac{1}{7}} = \frac{4}{11}.$$

Example 4·7. *The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly ?*

Solution. Let us define the following events :

E_1 : Disease X is diagnosed correctly by doctor A.
E_2 : Disease X is not diagnosed correctly by doctor A.
E : A patient (of Dr A) who had disease X dies.

Then, we are given :

$$P(E_1) = 0{\cdot}6 \quad ; \quad P(E \mid E_1) = 0{\cdot}4$$

$$P(E_2) = P(\bar{E}_1) = 1 - P(E_1) = 0{\cdot}4 \quad ; \quad P(E \mid E_2) = 0{\cdot}7$$

$$\therefore \quad P(E) = \sum_{i=1}^{2} P(E_i)\,P(E \mid E_i) = 0{\cdot}6 \times 0{\cdot}4 + 0{\cdot}4 \times 0{\cdot}7 = 0{\cdot}52$$

Using the Bayes' thereon, the required probability is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E)} = \frac{0{\cdot}6 \times 0{\cdot}4}{0{\cdot}52} = \frac{0{\cdot}24}{0{\cdot}52} = \frac{6}{13}.$$

Example 4·8. *The contents of urns I, II and III are as follows :*

1 white, 2 black and 3 red balls,
2 white, 1 black and 1 red balls, and
4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn from it. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solution. Let E_1, E_2, and E_3 denote the events that the urn, I, II and III is chosen, respectively, and let A be the event that the two balls taken from the selected urn are white and red.

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A \mid E_1) = \frac{1 \times 3}{{}^6C_2} = \frac{1}{5},\ P(A \mid E_2) = \frac{2 \times 1}{{}^4C_2} = \frac{1}{3},\ \text{and}\ P(A \mid E_3) = \frac{4 \times 3}{{}^{12}C_2} = \frac{2}{11}$$

$$\therefore \quad P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{\sum_{i=1}^{3} P(E_i)\,P(A \mid E_i)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{11}} = \frac{55}{118}$$

Similarly,
$$P(E_3 \mid A) = \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}\times+\frac{1}{3}\times\frac{2}{11}} = \frac{30}{118}$$

$\therefore$
$$P(E_1 \mid A) = 1-\frac{55}{118}-\frac{30}{118} = \frac{33}{118}.$$

Example 4·9. *From a vessel containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. From this vessel a ball is drawn and is found to be white. What is the probability that out of four balls transferred 3 are white and 1 is black ?*

Solution. Let us define the following events :

E_1 : Transfer of 0 white and 4 black balls

E_2 : Transfer of 1 white and 3 black balls

E_3 : Transfer of 2 white and 2 black balls

E_4 : Transfer of 3 white and 1 black balls

(Since the urn contains 3 white balls, more than 3 white balls cannot be transferred from the vessel)

E : Drawing of a white ball from the second vessel.

Then $P(E_1) = \frac{^5C_4}{^8C_4} = \frac{1}{14}$, $\quad P(E_2) = \frac{^3C_1 \times {^5C_3}}{^8C_4} = \frac{3}{7}$

$P(E_3) = \frac{^3C_2 \times {^5C_2}}{^8C_4} = \frac{3}{7}$, $\quad P(E_4) = \frac{^3C_3 \times {^5C_1}}{^8C_4} = \frac{1}{14}$

Also $P(E \mid E_1) = 0$, $P(E \mid E_2) = \frac{1}{4}$, $P(E \mid E_3) = \frac{2}{4}$ and $P(E \mid E_4) = \frac{3}{4}$

Hence, by Bayes Theorem, the probability that out of four balls transferred, 3 are white and is black is :

$$P(E_4 \mid E) = \frac{\frac{1}{14}\times\frac{3}{4}}{\frac{1}{14}\times 0+\frac{3}{7}\times\frac{1}{4}+\frac{3}{7}\times\frac{1}{2}+\frac{1}{14}\times\frac{3}{4}} = \frac{3}{6+12+3} = \frac{1}{7} = 0{\cdot}14.$$

Example 4·10. *A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer, find the probability that their answer is correct.*

Solution. Let us define the following events :

E_1 : Both A and B solve the problem correctly.

E_2 : Exactly one of them solves the problem correctly.

E_3 : Neither of them solves the problem correctly.

E : They get the same answer.

Then, according to the data given in the problem, assuming that A and B try the problem independently, we have.

$P(E_1) = \frac{1}{6}\times\frac{1}{8} = \frac{1}{48}$; $\quad P(E \mid E_1) = 1$

$P(E_2) = \frac{1}{6}\times\frac{7}{8} + \frac{5}{6}\times\frac{1}{8} = \frac{12}{48}$; $\quad P(E \mid E_2) = 0$