# Butcher tables to specify Runge–Kutta methods

Butcher tables are used to simplify the presentation of Runge-Kutta (RK) methods. RK are numerical methods for solving first-order ordinary differential equations of the form

$$[1] \quad \frac{dy}{dx} = f(x, y)$$

A Butcher table has the general form

[2]

$$\begin{array}{c|cccc}
c_1 & a_{1,1} & a_{1,2} & \dots & a_{1,s} \\
c_2 & a_{2,1} & a_{2,2} & \dots & a_{2,s} \\
\dots & \dots & \dots & \dots & \dots \\
c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s} \\
\hline
. & b_1 & b_2 & \dots & b_s
\end{array}$$

and is a simple memory aid device for specifying the coefficients in a Runge–Kutta method defined by

$$[3] \quad y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i$$

where, n is the node's running index (n=0,1,2,…), $i$ is the index used to label stages ( $1 \leq i \leq s$), and $s$ is the number of stages for a given method. A stage $s$ is a function evaluation per step of size $h$. To obtain a $s$ stage Runge--Kutta method, we let

$$[4] \quad k_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{s} a_{i,j} k_j)$$

Here, $c_i$ $(i = 1,2, \dots, s)$ is defined as $\sum_{j=1}^{s} a_{i,j}$. The value of $c_i$ indicates the point where within the interval $h$ $x_n = x_o + hc_i$ for which $y_n$ is a good approximation to $y(x_n)$

If the method is explicit, the table [2] takes the simplified form

[5]

$$\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{2,1} & & & & \\
\dots & & \dots & & & \\
c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \dots & b_{s-1} & b_s
\end{array}$$

For explicit methods the $a_{i,j}$ values in the diagonal and above are zero (0). Entries at or above the diagonal will cause the right-hand side of equation [1] to involve $y_{n+1}$, and so yielding an implicit method.

The minimum number of stages $s$ necessary for an explicit method to attain order $p$ is still an open problem. Recall the order of a method ($p$) is how the order of magnitude of the global error (e.g., $\vartheta(h^p)$) changes when varying h. Calling this $s_{\min}(p)$, the present knowledge [Butcher 1987, Lambert 1991] is [6]:

| p | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s_{\min}(p)$ | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 11 | $12 \leq s_{min} \leq 17$ | $13 \leq s_{min} \leq 17$ |

(for a reference of the above table see: http://www.iact.ugr-csic.es/personal/julyan_cartwright/papers/rkpaper/node2.html)

Example.
The Butcher table for RK-4, which has 4 stages (i.e., s=4) is:
[7]

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \qquad \begin{array}{c|cccc} c_1 & a_{1,1} & a_{1,2} & \dots & a_{1,s} \\ c_2 & a_{2,1} & a_{2,2} & \dots & a_{2,s} \\ \dots & \dots & \dots & \dots & \dots \\ c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s} \\ \hline . & b_1 & b_2 & \dots & b_s \end{array}$$

*Comparing both tables*. From the table to the left extract the parameters: $c_i$, $a_{i,j}$, and $b_i$ in the same position of elements of the table to the right. Then, expand the equation [3] for $s = 4$:

$$[3] \quad y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i$$

$$[8] \; y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4$$

where $b_1 = \frac{1}{6}$; $b_2 = \frac{1}{3}$; $b_3 = \frac{1}{3}$; $b_4 = \frac{1}{6}$, (extracted from the table) then

$$[9.a] \; y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4$$

Which easily simplifies to the more common

$$[9.b] \quad y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Develop the expression for each $k_i$ by expanding [4] for $i = 1$:

$$[4] \; k_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{s} a_{i,j} k_j)$$

$$[10.a] \; k_1 = hf(x_n + c_1 h, y_n + a_{1,1}k_1 + a_{1,2}k_2 + a_{1,3}k_3 + a_{1,4}k_4)$$

For $i = 1$, $c_1 = 0$, $a_{1,1} = 0$, $a_{1,2} = 0$, $a_{1,3} = 0$, $a_{1,4} = 0$, (values extracted from the table) the equation [10] yields:

$$[10.b] \; k_1 = hf(x_n, y_n)$$

Similar expansions yield,
For $i = 2$, $c_2 = \frac{1}{2}$, $a_{2,1} = \frac{1}{2}$, $a_{2,2} = 0$, $a_{2,3} = 0$, $a_{2,4} = 0$, yields:

$$[11] \quad k_2 = hf(x_n + \left(\frac{1}{2}\right)h, y_n + \left(\frac{1}{2}\right)k_1)$$

For $i = 3$, $c_3 = \frac{1}{2}$, $a_{3,1} = 0$, $a_{3,2} = \frac{1}{2}$, $a_{3,3} = 0$, $a_{3,4} = 0$, yields:

$$[12]\ \ k_3 = hf(x_n + \left(\frac{1}{2}\right)h, y_n + \left(\frac{1}{2}\right)k_2)$$

For $i = 4$, $c_4 = 1$, $a_{4,1} = 0$, $a_{4,2} = 0$, $a_{4,3} = 1$, $a_{4,4} = 0$, yields:

$$[13]\ \ k_4 = hf(x_n + h, y_n + k_3)$$

Summarizing, for RK-4 method:

| Butcher table: | Usual Presentation: |
|---|---|
| $\begin{array}{c\|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$ | $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$<br>$k_1 = hf(x_n, y_n)$<br>$k_2 = hf(x_n + \left(\frac{1}{2}\right)h, y_n + \left(\frac{1}{2}\right)k_1)$<br>$k_3 = hf(x_n + \left(\frac{1}{2}\right)h, y_n + \left(\frac{1}{2}\right)k_2)$<br>$k_4 = hf(x_n + h, y_n + k_3)$ |

<u>Work out some more examples</u>:

1. Euler's explicit method:

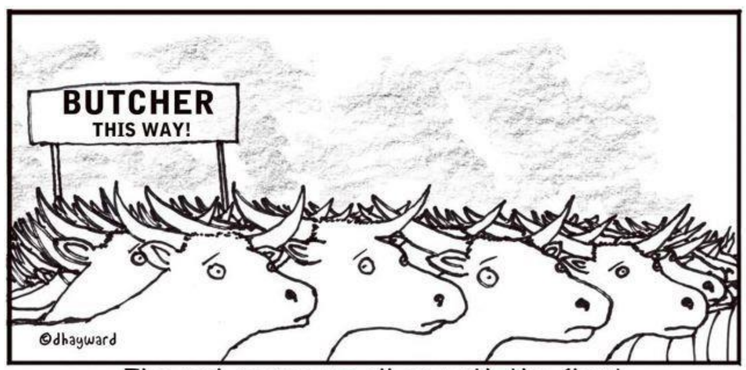
$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

2. Heun's method (also called RK-2)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

3. Euler's implicit method (backward Euler's)

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$



The real reason we all go with the flow!

4. 3/8-RK fourth order method (3/8-RK-4). This method doesn't have as much notoriety as the "classical" method, but is just as classical because it was proposed in the same paper (Kutta, 1901)

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \\ 2/3 & -1/3 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ \hline & 1/8 & 3/8 & 3/8 & 1/8 \end{array}$$

For a rich list of RK methods and more examples of Butcher tables, see:

https://en.wikipedia.org/wiki/List_of_Runge%E2%80%93Kutta_methods
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