AHMADU BELLO UNIVERSITY DISTANCE LEARNING CENTRE

COURSE: ALGEBRA

COURSE CODE: MATH102

LECTURE NOTE 002: REMAINDER THEOREM AND FACTOR THEOREM

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2.1 Learning Outcomes

After studying this session, I expect you to be able to:

1. State and apply the remainder theorem
2. comprehend the factor theorem and its applications

2.2 Remainder Theorem

The remainder theorem is a powerful tool for the roots of any polynomial equation.

Remainder theorem: States that if a polynomial $p(x)$ is divided by $(x-a)$ then the remainder, R is given by $R = p(a)$

Proof

Let $p(x)$ be the given polynomial in (x),

Let $Q(x)$ by any quotient when $p(x)$ is divided by $(x-a)$ and let R be the remainder.

Thus $\frac{p(x)}{(x-a)} = Q(x) + \frac{R}{x-a}$ where R is the remainder

$p(x) = Q(x)(x-a) + R$

Putting $x = a$, then we have:

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$p(a) = Q(a)(a - a) + R$

$p(a) = R$ (Q.E.D)

Example 2.1: Find the remainder when $4x^3 - 2x^2 + 4x - 1$ is divided by

(i) $(x + 2)$

(ii) $(x - 4)$

(iii) $(x + 1)$

(iv) x

Solution

Let $y(x) = 4x^3 - 2x^2 + 4x - 1$. By remainder theorem, when a polynomial is divided by a factor, the remainder is $y(a)$.

(i) Comparing $x - a$ with $x + 2$

Implies $a = -2$,

Hence the remainder is

$y(-2) = 4(-2)^3 - 2(-2)^2 + 4(-2) - 1$

$= -32 - 8 - 8 - 1$

$= -49$

So that the remainder when $4x^3 - 2x^2 + 4x - 1$ is divided by $(x + 2)$ is:

$y(-2) = -49$

(ii) Comparing $x - a$ with $x - 4$

Implies $a = 4$,

Hence the remainder is

$y(4) = 4(4)^3 - 2(4)^2 + 4(4) - 1$

$= 256 - 32 + 16 - 1$

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So that the remainder when $4x^3 - 2x^2 + 4x - 1$ is divided by $(x - 4)$ is:

$y(4) = 239$

(iii) Comparing $x - a$ with $x + 1$

Implies $a = -1$,

Hence the remainder is

$y(-1) = 4(-1)^3 - 2(-1)^2 + 4(-1) - 1$

$= -4 - 2 - 4 - 1$

So that the remainder when $4x^3 - 2x^2 + 4x - 1$ is divided by $(x + 1)$ is:

$y(-1) = -11$

(iv) Comparing $x - a$ with x

Implies $a = 0$,

Hence the remainder is

$y(0) = 4(0)^3 - 2(0)^2 + 4(0) - 1$

$= -1$

So that the remainder when $4x^3 - 2x^2 + 4x - 1$ is divided by (x) is:

$y(0) = -1$

Example 2.2: Find the remainder when $3x^2 - 4x + 2$ is divided by

(a) $(x - 2)$

(b) $(x + 1)$

Solution

(a) Let $p(x) = 3x^2 - 4x + 2$, the remainder when $p(x)$ is divided by $(x - 2)$ is $p(2)$

$p(2) = 3(2)^2 - 4(2) + 2$

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$p(2) = 12 - 8 + 2$

So that $p(2) = 6$

Hence the remainder when $3x^2 - 4x + 2$ is divided by $(x - 2)$ is 6

(b) Let $p(x) = 3x^2 - 4x + 2$, the remainder when $p(x)$ is divided by $(x + 1)$ is $p(-1)$

$p(-1) = 3(-1)^2 - 4(-1) + 2$

$p(-1) = 3 + 4 + 2$

So that $p(-1) = 9$

Hence the remainder when $3x^2 - 4x + 2$ is divided by $(x + 1)$ is 9

2.3 Factor Theorem

Factor Theorem: if $p(x)$ is a polynomial in x and remainder $R = p(a) = 0$, when divided by $(x - a)$, then $(x - a)$ is a factor of $p(x)$.

In fact, if $p(a) = 0$, then a is a root of the polynomial equation $p(x) = 0$.

Proof:

Let $p(x)$ be the given polynomial in (x),

Let $Q(x)$ by any quotient when $p(x)$ is divided by $(x - a)$ and let R be the remainder.

Thus $\frac{p(x)}{(x-a)} = Q(x) + \frac{R}{x-a}$ where R is the remainder

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If $R = 0$, then

$p(x) = Q(x)(x - a) + 0$

Putting $x = a$, then we have

$p(a) = Q(a)(a - a)$

$p(a) = 0$

Example 2.3: Determine whether (i) $(x - 1)$ and (ii) $(x + 1)$ is a factor of the cubic equation $x^3 - 2x^2 - 5x + 6 = 0$.

Solution

Let $p(x) = x^3 - 2x^2 - 5x + 6$, by factor theorem, if $(x - a)$ is factor of the polynomial $p(x)$, then $p(a) = 0$.

(i) So that $a = 1$, hence, $p(1)$

$p(1) = 1^3 - 2(1)^2 - 5(1) + 6$

$p(1) = 1 - 2 - 5 + 6$

$p(1) = 0$,

Hence $(x - 1)$ is a factor of the equation $x^3 - 2x^2 - 5x + 6 = 0$.

(ii) Also, $a = -1$, hence, $p(-1)$

$p(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6$

$p(-1) = -1 - 2 + 5 + 6$

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$p(-1) = 8$,

Since $p(-1) \neq 0$, hence $(x+1)$ is not a factor of the equation $x^3 - 2x^2 - 5x + 6 = 0$.

Example 2.4: Solve $x^4 - 15x^2 - 10x + 24 = 0$

Solution

Using factor theorem,

Let $p(x) = x^4 - 15x^2 - 10x + 24$

By inspection, $p(1) = 0$ which means $(x-1)$ is a factor,

Next, $p(2) \neq 0$, which means $(x-2)$ is not a factor

By further inspection, $p(-2) = 0$ which means $(x+2)$ is a factor

Similarly, $p(-3) = 0$ which means $(x+3)$ is a factor

And $p(4) = 0$ which means $(x-4)$ is a factor

So that $x^4 - 15x^2 - 10x + 24 = (x-1)(x+2)(x+3)(x-4)$

So that the roots of the quartic equation $x^4 - 15x^2 - 10x + 24 = 0$ are $x = 1, -2, -3, 4$

Example 2.4: Factorize $x^4 - 5x^2 + 4$, and hence find the roots of $x^4 - 5x^2 + 4 = 0$

Solution

Let $p(x) = x^4 - 5x^2 + 4$, By inspection $p(-1) = p(1) = p(2) = p(-2) = 0$

Hence $(x-1)(x+1)(x-2)(x+2)$ are factors

So that $x^4 - 5x^2 + 4 = (x-1)(x+1)(x-2)(x+2)$

And the roots of $x^4 - 5x^2 + 4 = 0$ are $x = -1, 1, -2, 2$

2.4 Principle of undetermined coefficient

If two polynomial expression in x are equal for all values of x, we can equate the like powers of x. i.e If

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$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$. Then $a_n = 0$, $a_{n-1} = b_{n-1}, \ldots . a_1 = b_1$, $a_0 = b_0$. The principle stated is called the principle of undetermined coefficient e.g

Example 2.6: Determine the values of s, t and u .If

$$2x^2 + 3x + 8 = s(x+1)(x-2) + t(x+3) + u,$$

Solution

By the principle of undetermined coefficient,

$2x^2 + 3x + 8 = s(x^2 - x - 2) + t(x+3) + u,$

Implies, $2x^2 + 3x + 8 = sx^2 + x(t-s) + 3t - 2s + u,$

By principle of undetermined coefficient,

$s = 2,$

Then $(t-s) = 3$ which implies $t = 5$

and $3t - 2s + u = 8,$

which makes $u = -3$

In-text Question 1: If $p(a) = -1$ then $(x-a)$ is a factor of the polynomial (TRUE/ FALSE)

2. If $p(a) = 0$ then $(x-a)$ is a factor of the polynomial (TRUE/ FALSE)

Exercise

1. Determine the remainder when $x^3 - 6x^2 + x - 5$ is divided by

(a) $(x+2)$ (b) $(x-3)$

2. Solve the cubic equation $x^3 - 2x^2 - 5x + 6 = 0$ by using the factor theorem

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3. Find the remainder when $2x^5 - 2x^3 + 4x^2 - 12x + 3$ is divided by

(i) $(x - 2)$ (ii) $(x - 4)$ (iii) $(x - 1/2)$ (iv) $(x - 10)$

4. If $(x - 1)$ and $(x + 2)$ are factors of $p(x) = x^4 + bx^3 + 4x^2 + cx - 4$, find the value of b and c. show that with these value of b and c, $(x + 1)$ is also a factor of $p(x)$ and find the fourth factor.

5. Solve the equation$x^3 - 2x^2 - x + 2 = 0$.

6. Determine the value of ‘a’ if $(x + 2)$ is a factor of $x^3 - ax^2 + 14^x + 8 = 0$.

7. Determine the remainder when $x^3 - 6x^2 + x - 5$ is divided by

(a) $(x + 2)$ (b) $(x - 3)$

8. Solve the cubic equation $x^3 - 2x^2 - 5x + 6 = 0$ by using the factor theorem

9. Find the remainder when $2x^5 - 2x^3 + 4x^2 - 12x + 3$ is divided by

(i) $(x - 2)$ (ii) $(x - 4)$ (iii) $(x - 1/2)$ (iv) $(x - 10)$

10. Solve (i) $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

(ii) $x^4 - 41x^2 + 400 = 0$

2.5 Conclusion/Summary

In this section, the remainder theorem as well as the factor theorem have been presented. We have established that for any given polynomial $p(x)$, the remainder when divided by $(x - a)$ is $p(a)$. In addition, we showed that if the remainder is zero, i.e $p(a) = 0$, then $(x - a)$ is a factor of the polynomial $p(x)$. In fact, $x = a$ is a root of the polynomial $p(x)$.

References/Further Readings

“Course code: Course Title, National Open University of Nigeria, 2014 at http://www.nou.edu.ng/index.htm#

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