Fast Computing of the Grounded Extension in Acyclic Probabilistic Argumentation Frameworks

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Abstract

We propose a faster algorithm to compute the (exact) probability of acceptance of an argument in the grounded extension when using the constellations approach. We show that for singly connected graphs, the problem can be solved in linearithmic time instead of exponential. Moreover, in the case of acyclic graphs, the problem is exponential in the number of dependent arguments instead of exponential in the number of attacks, thus significantly improving the performance in the practice. Our approach is then compared against i) computing the whole constellations (an exact method), and ii) a Monte Carlo approach (an approximate method).

1 Introduction

The pioneering article in the field of abstract argumentation comes from P.M. Dung (Dung 1995), who defined the notion of abstract argumentation framework (*AF*). AFs can be seen as directed graphs where the nodes are arguments and the edges represent conflict relations (called attacks) between two arguments. Since Dung, many extensions have been proposed, e.g. the addition of a support relation (Cohen et al. 2014), the addition of a similarity relation (Amgoud and David 2018; Amgoud and David 2021) the addition of weights (Bistarelli, Rossi, and Santini 2018), or support relations (Amgoud and Ben-Naim 2018).

In this paper, we focus on probabilistic AFs (or *PrAF*) and more in particular on the *constellations* approach (Li, Oren, and Norman 2011): probability values determine the likelihood of both arguments and attacks to be part of an AF, thus generating different frameworks with a different existence probability. Hence, the uncertainty resides in the topology of the considered PrAF.[1] Thus, when we want to study the acceptance probability of an argument, we need to compute this value in all the possible AFs induced from a PrAF, and then aggregate. However, the number of induced AFs is in general exponential (Li, Oren, and Norman 2011), consequently reducing the research attractiveness in this field for practical reasons. This paper is the first step to showing that it is possible to optimize the computation of the acceptance probability without having to build the entire constellations.

[1]The epistemic approach (Hunter 2013) uses instead probability theory to represent degrees of belief in arguments, given a fixed AF.

Citing some related work, in (Fazzinga, Flesca, and Parisi 2015), the complexity of computing the acceptance probability of an argument has been done for different semantics and the result for the grounded is $FP^{\#P}$-$complete$. Having such a high complexity, the work in (Sun and Liao 2015) proposed some restrictions on the value of the probability to improve the complexity. If the probability is binary 0 or 1, then the probability of acceptance is polynomial in time in the case of grounded semantics. If the probability is ternary 0, 0.5, or 1, then the acceptance probability with the grounded is P-hard. Finally, in (Nofal, Atkinson, and Dunne 2021) a new fast algorithm to compute the ground extension has been proposed for classic AF. It could be interesting to study how this algorithm can be extended to PrAF.

In this paper, the key idea is to compute the acceptance probability of an argument in the ground extension of a PrAF by using local propagation, i.e., according to the acceptance probability of its direct attackers and the probability of incoming attacks. We do it very efficiently for AFs that follow the topology of *Singly Connected Network*s (*SCN*s) or "polytrees" (Chow and Liu 1968; Kim and Pearl 1983; Henrion 1988; Thomas, Howie, and Smith 2005). In these (acyclic) graphs, a node may have several parents but at most one underlying path exists between any pair of nodes. In this case, the propagation procedure is linearithmic (*O(n) log(n)*) in the number of arguments. However, we can also apply propagation to *Multi Connected Networks* (*MCN*s) (Henrion 1988), i.e., acyclic graphs in general. Even if such an exact approach is exponential in the number of dependent arguments (when more than one path exists between a pair of nodes), on random graphs we show that it still performs much better than computing all the induced AFs, and, by stopping the computation at the same time, the Monte Carlo approach used in (Li, Oren, and Norman 2011) often returns a significant approximation error.

All supplementary materials (codes, data sets, proofs) are available at this link: https://github.com/Vict0r-David/UNIPG/tree/main/Proba_Arg.

2 Background

Dung's Argumentation Frameworks. Following (Dung 1995), an argumentation framework (AF) is a pair $\langle \mathcal{A}, \mathcal{R} \rangle$, where $\mathcal{A}$ is a set of elements called arguments and $\mathcal{R}$ is a binary relation on $\mathcal{A}$, called the attack relation. For $a, b \in$

$\mathcal{A}$, if $(a, b) \in \mathcal{R}$, then we say that a attacks b and that a is an attacker of b. If for $a \in \mathcal{A}$ there is no $b \in \mathcal{A}$ with $(b, a) \in \mathcal{R}$, then a is unattacked. For a set of arguments $E \subseteq \mathcal{A}$ and an argument $a \in \mathcal{A}$, E defends a if $\forall (b, a) \in \mathcal{R}$, $\exists c \in E$ such that $(c, b) \in \mathcal{R}$. We say that a is defended if for each last argument (unattacked) b_n for each path to a (i.e. $\{(b_n, b_{n-1}), \ldots, (b_1, a)\}$), all the b_n arguments defend a, i.e. n is even. Let the set of attackers of a denoted by $\texttt{Att}(a) = \{b \in \mathcal{A} \mid (b, a) \in \mathcal{R}\}$.

An AF provides means to represent conflicting information. Reasoning with that information is done by means of argumentation semantics. A semantics provides a characterization of acceptable arguments in an AF. A set of acceptable arguments according to a semantics is called an extension and is taken as a reasoning outcome. Many semantics have been proposed, see e.g. (Charwat et al. 2015) for overviews. In this work, we will consider the very well-established grounded semantics: the grounded extension of $\langle \mathcal{A}, \mathcal{R} \rangle$ can be constructed as $\texttt{gr} = \bigcup_{i \geq 0} G_i$, where G_0 is the set of all unattacked arguments, and $\forall i \geq 0$, G_{i+1} is the set of all arguments that G_i defends. For any $\langle \mathcal{A}, \mathcal{R} \rangle$, the grounded extension $\texttt{gr}$ always exists and is unique.

Probabilistic Frameworks and Constellations. There exist different ways to extend the classic AF with probability into a probabilistic argumentation framework (PrAF). For example, we can label arguments and/or attacks with a probability. In (Mantadelis and Bistarelli 2020) the authors proposed a way to transform any PrAF having probability on arguments and attacks to PrAF with probability only on attacks (or only on arguments) thanks to the probabilistic attack normal forms (or probabilistic argument normal form). They showed that all these forms are equivalent, i.e. same probabilistic distribution on their extensions.

Definition 1 (PrAF)**.** *A probabilistic argumentation framework (PrAF) is a tuple* $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ *such that:* $\mathcal{A} \subseteq_f \texttt{Arg}$ [2]*,* $\mathcal{R} \subseteq_f \mathcal{A} \times \mathcal{A}$*,* $P_R : \mathcal{R} \rightarrow]0, 1]$.

We call SCN PrAF every PrAF respecting the constraints of a Singly Connected Network.

The constellation of a graph is composed of all its possible subgraphs (worlds), and we compute the probability of one subgraph as follows.

Definition 2 (Probability of a world)**.** *Let* $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ *and* $\omega = \langle \mathcal{A}', \mathcal{R}', P_R \rangle$ *be probabilistic argumentation graphs such that* $\omega \sqsubseteq AF$ [3]*. The probability of subgraph* ω*, denoted* $p(\omega) = \left(\prod_{att \in \mathcal{R}'} P_R(att) \right) \times \left(\prod_{att \in \mathcal{R} \setminus \mathcal{R}'} (1 - P_R(att)) \right)$

Example 1. *Let see the SCN PrAF* $\mathbf{G} = \langle \{a, b, c, d\}, \{(a, b), (c, d), (d, b)\}, P_R \rangle$ *such that* $P_R((a, b)) = 0.4$*,* $P_R((c, d)) = 0.7$*,* $P_R((d, b)) = 0.2$.

Recall that it was shown in (Hunter 2013) that the sum of the probability of any subgraph is equal to 1. Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ be a PrAF, then $\sum_{\omega \sqsubseteq AF} p(\omega) = 1$.

[2] $\mathcal{A} \subseteq_f \texttt{Arg}$ stands for: $\mathcal{A}$ is a finite subset of all arguments.

[3] The notation $\omega \sqsubseteq AF$ stands for: $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{R}' = \{(a, b) \in \mathcal{R} \mid a \in \mathcal{A}' \text{ and } b \in \mathcal{A}'\}$: ω is a subgraph of an AF.

Let us recall now how to compute the probability of an argument or a set of arguments belonging to the extensions of extension-based semantics.

Definition 3 (Acceptance Probability)**.** *Let* $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ *be a PrAF,* $X \subseteq \mathcal{A}$ *and* $\mathcal{S}$ *an extension-based semantics, we denote by* $P^{\mathcal{S}}(X) = \sum_{\omega \sqsubseteq AF} p(\omega) \times \texttt{In}^{\mathcal{S}}(\omega, X)$*, where* $\texttt{In}^{\mathcal{S}}(\omega, X) = 1$ *if* X *is a subset of each extension of* $\mathcal{S}$ *in* ω*, otherwise equal to 0.*

In a previous workshop paper (Bistarelli et al. 2022) we discussed the possibility of computing the probabilities based on the grounded semantics by using the next function.

Definition 4 ($\texttt{Fast}^{\texttt{gr}}$)**.** *Let* $\texttt{Fast}^{\texttt{gr}}$ *be the function from any PrAF* $\langle \mathcal{A}, \mathcal{R}, P_R \rangle$ *which computes the acceptance probability of any argument to be in the grounded extension* $(\mathcal{A} \rightarrow [0, 1])$*, s.t.* $\texttt{Fast}^{\texttt{gr}}(a) =$

$$\begin{cases} 1 & \text{if } \texttt{Att}(a) = \emptyset \\ \prod\limits_{b \in \texttt{Att}(a)} 1 - \left(\texttt{Fast}^{\texttt{gr}}(b) \times P_R(b, a) \right) & \text{otherwise} \end{cases}$$

Example 1 (Continued). The acceptance probabilities of the arguments in the grounded semantics are: $P^{\texttt{gr}}(a) = 1$, $P^{\texttt{gr}}(b) = \texttt{Fast}^{\texttt{gr}}(b) = 0.084 + 0.336 + 0.144 = 0.564$, $P^{\texttt{gr}}(c) = \texttt{Fast}^{\texttt{gr}}(c) = 1$, $P^{\texttt{gr}}(d) = \texttt{Fast}^{\texttt{gr}}(d) = 0.024 + 0.096 + 0.036 + 0.144 = 0.3$.

3 Fast Computation in Acyclic PrAF

We propose a method for computing the probability of an argument in a PrAF to be accepted with respect to the grounded semantics. In this paper, we restrict to acyclic (MCN) PrAF. Our approach is based on the $\texttt{Fast}^{\texttt{gr}}$ function of Def. 4. Recall that an argument is in the grounded extension if it is defended. Trivially, an unattacked argument is always defended, so its probability of being accepted is 1. On the other hand, when an argument a is attacked by b, $\texttt{Fast}^{\texttt{gr}}(b) \times P_R(b, a)$ computes the joint probability that argument b is acceptable and the attack (b, a) exists. Thus $1 - \texttt{Fast}^{\texttt{gr}}(b) \times P_R(b, a)$ gives the probability that either argument b is not acceptable or attack (b, a) does not exist. Finally, the product of this computation for each attack ensures that a is defended.

In the following, we present the two algorithms that constitute the proposed method. For more details and optimizations, the program implemented in Python is available on GitHub[4] and to deal with the symbolic computation we used the symPy library (Meurer et al. 2017).

The function $\texttt{Fast\_MCN}^{gr}$ of **Algorithm 1** takes as input an argument "goal" that we want to compute its probability and a PrAF, to give as output the final probability of the "goal". The idea of this first algorithm is to apply the right order to the second algorithm to calculate step by step the final probability of each attacker until the goal. We first use (line 1) the function `order` which traverses the graph back through the attackers from the "goal" to the roots. This function saves the maximum distance between each argument and the "goal" (find the longest path in a directed acyclic graph is linear in time (Eppstein 1998)). Then the `order`

[4] https://github.com/Vict0r-David/UNIPG/tree/main/Proba_Arg.

Algorithm 1 `Fast_MCN`^{gr}(goal, PrAF)

```
1:  goal_lvl = order(goal,PrAF);
2:  for arg ∈ goal_lvl do
3:      prob[arg], dep, arg_lvl = DevFast(arg,PrAF,prob);
4:      if len(dep) > 0 then
5:          arg_lvl.keepDep&Reverse(dep);
6:      for a ∈ arg_lvl do
7:          a* = 1;
8:          for b ∈ Att(a) do
9:              a* = a* × (1 - symb(b) × P_R(b,a)) ;
10:         prob[arg].subs(a,a*);
11:         prob[arg] = expand&deletePow();
12: return prob[goal];
```

Algorithm 2 `DevFast`(goal, PrAF, final_prob)

```
1:  goal_lvl = order(goal,PrAF) ;
2:  dep = dependant_arg(goal,PrAF);
3:  prob = {}, symbol = False ;
4:  for a ∈ goal_lvl do
5:      a* = 1;
6:      for b ∈ Att(a) do
7:          if b ∈ dep then
8:              a* = a* × (1 - symb(b) × P_R(b,a)) ;
9:              symbol = True;
10:         else if len(prob[b].vars) > 0 then
11:             a* = a* × (1 - prob[b] × P_R(b,a));
12:         else
13:             a* = a* × (1 - final_prob[b] × P_R(b,a));
14:     prob[a] = a* ;
15: if symbol then
16:     prob[a].expand&deletePow();
17: return prob[goal], dep, goal_lvl;
```

function applies a sorting algorithm (e.g., Timsort (Auger et al. 2018) or Heapsort (Schaffer and Sedgewick 1993) with a worst-case complexity in $O(nlogn)$ where n is the number of arguments) to obtain the arguments in the decreasing order with respect to the maximum distance of the "goal". Then (line 3) we apply for each argument the second algorithm `DevFast` which calculates the expanded expression of the argument according to its dependent arguments. If the argument has some dependency (line 4) we use the function `keepDep&Reverse` (line 5), deleting every non-dependent argument, then reverse the order (increasing way). Then for each dependent argument (line 6), we develop its expression (line 9). Finally, after the development, we substitute the symbol of the dependent argument (line 10) and solve the problem by expanding the expression followed by the deleting power (line 11).

In the **Algorithm 2** `DevFast`, the detection of the dependent arguments (line 2), is done by looking at all paths back from the goal argument and if an argument belongs to at least two paths it is dependent. Then we proceed to a symbolic development starting from the unattacked arguments towards the argument goal (thanks to the function `order` line 1). For this iterative process we will develop the probability of each argument by replacing its attacker according to 3 situations: i) (lines 7-9) the attacker is a dependent argument then we keep the symbol; ii) (lines 10-11) the attacker is not dependant to the "goal" but it has dependent attacker then we use its development iii) (lines 12-13) otherwise the attacker is neither a dependent argument nor have dependency hence we can replace its symbol by its final probability (computed at the previous steps). When we find dependent arguments (i.e. having conjunction with itself) we need to solve the development (line 16), by expanding the expression and then deleting each power (resulting from the product with itself) in the symbol of the dependent argument.

Note that when the context of the graph is clear, it will be omitted in the call of the algorithms. In the following theorem, we show that this algorithm characterizes the constellation approach in the case of acyclic PrAFs.

Theorem 1. *If* $\mathbf{G}$ *be an acyclic graph (MCN) then* $\forall a \in \mathcal{A}$, $P^{\mathtt{gr}}(a) = \mathtt{Fast\_MCN}^{\mathtt{gr}}(a)$.

Proof. Let $\mathbf{G}$ is an acyclic graph (i.e. MCN). As we explain in the proof of Theorem 2, we can characterise by local propagation (Algorithm 3) the constellation approach when there is no multiple connected (dependent) argument, i.e. for the case of SCN PrAF. Let illustrate the problem of multiple connected arguments, i.e. a pair of arguments having different paths, by building the PrAF $\mathbf{G}'$ which extand the graph $\mathbf{G}$ (of our our running example 1) by adding an attack from d to a ($P_R(d, a) = 0.6$). Visually we obtain:

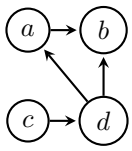

Figure 1: $\mathbf{G}'$

The probability of b from the Algorithm 3 is computed as follows (to be short, we use $F(b)$ to say the probability of b and $P(ab)$ to say the probability of the attack from a to b):
$F(b) = (1 - F(a) \times P(ab)) \times (1 - F(d) \times P(db))$, where $F(a) = 1 - F(d) \times P(da)$,
$F(d) = 1 - F(c) \times P(cd)$ and $F(c) = 1$. Now compute symbolically the probability of b until its dependant argument d:
$F(b) = (1 - F(a) \times P(ab)) \times (1 - F(d) \times P(db))$
To simpllify the computation we can replace the probability of attacks by their value (it is just a constant):
$F(b) = (1 - F(a) \times 0.4) \times (1 - F(d) \times 0.2)$
$= (1 - (1 - F(d) \times 0.6) \times 0.4) \times (1 - F(d) \times 0.6)$
$= (0.6 + 0.24 \times F(d)) \times (1 - 0.6 \times F(d))$
$= 0.6 - 0.144 \times F(d)^2 - 0.12 \times F(d)$
$= 0.6 - 0.144 \times (1 - F(c) \times 0.7)^2 - 0.12 \times (1 - F(c) \times 0.7)$
Given that c is unattacked we can replace $F(c)$ by 1:
$= 0.6 - 0.144 \times (1 - 0.7)^2 - 0.12 \times (1 - 0.7) = 0.55104.$

However, when applying a conjunction on the same argument (here $F(d)^2$), it makes no sense, each event is intrinsic,

either it exists or it does not. It is also clear that when we expand the argument d (with $1 - F(c) \times P(cd)$) we theoretically create a double attack from c to d, which also makes no sense.

Making conjunctions on the same argument is in fact understood as if there were a multitude (like clone) of arguments (the number of clones is equal to the power, i.e. according to the number of paths, here 2). Thus, we can show that by local propagation we compute in the case of MCN graphs their transformation using a clone for each path. In this example, here is the transforming graph in terms of cloning is the following $\mathbf{G}'_2$, such that $P_R(d_1, b) = 0.6$ and $P_R(d_2, a) = 0.6$:

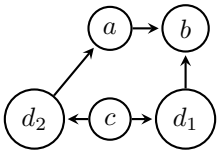

Figure 2: $\mathbf{G}'_2$

If we compute now on this graph $\mathbf{G}'_2$, we obtain: $F(b) = (1 - F(a) \times P(ab)) \times (1 - F(d_1) \times P(db))$
To simplify the computation we can replace the probability of attacks by their value (it is just a constant):
$F(b) = (1 - F(a) \times 0.4) \times (1 - F(d_1) \times 0.2)$
$= (1 - (1 - F(d_2) \times 0.6) \times 0.4) \times (1 - F(d_1) \times 0.6)$
$= (0.6 + 0.24 \times F(d_2)) \times (1 - 0.6 \times F(d_1))$
$= (0.6{+}0.24{\times}(1{-}F(c){\times}0.7)){\times}(1{-}0.6{\times}(1{-}F(c){\times}0.7))$
Given that c is unattacked we can replace $F(c)$ by 1:
$= (0.6{+}0.24{\times}(1{-}0.7)){\times}(1{-}0.6{\times}(1{-}0.7)) = 0.55104.$

In order to correct the computation of the method by local propagation in MCN graphs, a solution is to symbolically expand the computation of an argument and to eliminate any power on the arguments (to suppress bad independence/clones created).

Since not all arguments in a graph have dependencies on others and this expansion step is exponential, in order to reduce the complexity, in Algorithm 1 ($\texttt{Fast\_MCN}^{gr}$), line 3, we determine the dependent arguments of an argument goal, by using our Algorithm 2 (`DevFast`). Then line 5 we keep each dependent argument which we expand on lines 9 and 10. Finally, to solve the problem we expand the expression followed by deleting the powers on line 11.

Notice that in Algorithm 2 we also optimise the symbolic expansion on lines 8 (if the argument is a dependent argument) and 11 (if the argument is not a dependent argument for the goal but it is also itself dependent on a dependent argument of the goal), and the numerical expansion on line 13 (if the argument is neither dependent nor indirectly contains a dependency, then it can be replaced by its probability, calculated previously in Algorithm 1).

Therefore $\forall a \in \mathcal{A}$ (MCN PrAF), $P^{\texttt{gr}}(a) = \texttt{Fast\_MCN}^{\texttt{gr}}(a)$.

□

We next compare experimentally our algorithm $\texttt{Fast\_MCN}^{gr}$ with the constellation approach on 150 random acyclic graphs with 50 arguments and between 10 to 90 attacks (10 graphs by category, see Fig. 4).

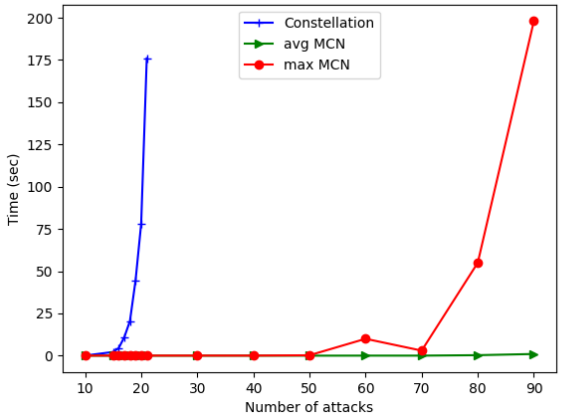

Figure 3: Comparison of run-times between the $\texttt{Fast\_MCN}^{\texttt{gr}}$ algorithm and the world's method (P^{gr}) in random acyclic graphs.

First of all, we can see that the constellation approach is exponential in the number of attacks (around 20) regardless of the graph structure. As for the algorithm $\texttt{Fast\_MCN}^{\texttt{gr}}$, we become exponential later because we depend on the structure of the graph and not on the attacks. These graphs are randomly generated, and in some cases we get a significant number of dependent arguments: for example, in a random AF with 50 arguments and 90 attacks, the max dependency for an argument is 11, and some of these arguments are dependent in turn.

Furthermore, since the computation time depends on the structure of the attackers, for all arguments without dependencies or close to the roots, the computation time is extremely fast. Thus, we can see that the average time on all arguments is very low. In order to have a better indication of the complexity of the problem we will discuss the max time.

4 Linearithmic Computation in SCN PrAF

In the case of acyclic graphs (MCN), to compute the probability of an argument we need to go through all its attackers and compute their probability by also going through all their attackers. This is why, in the case of SCN graphs, the acyclic Fast algorithm is polynomial (in the number of n attackers, i.e. $O(\frac{n(n-1)}{2})$) because without dependent argument there is no expansion and each computation is linear in time.

However, if we know that we will work on SCN graphs we can compute very quickly the exact probability of an argument, just by one path doing an ordering local propagation. We present next the new algorithm $\texttt{SCN\_Fast}^{\texttt{gr}}$, which is able to compute the probability of an argument to be accepted in the grounded extension.

We show next that $\texttt{Fast\_SCN}^{\texttt{gr}}$ algorithm characterizes $P^{\texttt{gr}}$ for any SCN PrAF.

Algorithm 3 `Fast_SCN`$^{\mathtt{gr}}$(goal, PrAF)

```
1: decreasing_lvl = order(goal,PrAF);
2: prob = {};
3: for a ∈ decreasing_lvl do
4:     prob[a] = 1;
5:     for b ∈ Att(a) do
6:         prob[a] = prob[a] × (1 - prob[b] × P_R(b,a));
7: return prob[goal];
```

Theorem 2. *If* $\mathbf{G}$ *be a SCN PrAF then* $\forall a \in \mathcal{A}$, $P^{\mathtt{gr}}(a) =$ `Fast_SCN`$^{\mathtt{gr}}(a)$.

Proof. Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ be a SCN PrAF, i.e. $\mathbf{G}$ is an acyclic graph where there not exist any pair of arguments with more than one path from each other.

To compute the probability of an argument, we can identify two cases, the first one is about the unattacked arguments and the second case is the opposite situation where an argument is attacked. Let us compare the two approaches: using the constellation (P^{gr}) and using the algorithm `Fast_SCN`$^{\mathtt{gr}}$.

Before going into the study of these two cases, note that for the algorithm `Fast_SCN`$^{\mathtt{gr}}$ we have a preprocessing step line 1, then in a second step we have the calculation of probability for each argument up to the goal (line 3 to 6).

Once we did the ordering (line 1), we obtain the list of attackers according to their maximum distance from the goal argument. As it is done in line 3, we can go through arguments from the farthest to the closest to goal and since we don't have any cycle this ensures that at each computation of an argument we have already calculated (previous iterations) the probability of its direct attackers.

Let us now look first at the case of unattacked arguments.

Unattacked arguments.

- **Constellation approach.** Start by recall how the grounded extension is computed: $\mathtt{Gr} = \bigcup_{i \geq 0} G_i$, where G_0 is the set of all unattacked arguments, and $\forall i \geq 0$, G_{i+1} is the set of all arguments that G_i defends. Moreover for any AF, the grounded extension `Gr` always exists and is unique.

  From the definition of the grounded semantics any unattacked argument belong to the grounded extension. From the Definition 3 of probability of an argument using the constellation, $P^{gr}(X) = \sum_{\omega \sqsubseteq AF} p(\omega) \times \mathtt{In}^{gr}(\omega, X)$. Given that for any unattacked argument $a \in \mathcal{A}$, $\forall \omega \sqsubseteq AF$, a will stay unattacked, then $\mathtt{In}^{gr}(\omega, a) = 1$. From the proposition shown in (Hunter 2013) such that the sum of the probability of any subgraph is equal to 1, we can conclude then any unattacked argument a have a probability of 1, i.e. $P^{gr}(a) = 1$.
- **Fast_SCN approach.** In Algorithm 3 (`Fast_SCN`$^{\mathtt{gr}}$), from line 4 (prob[a] = 1), any argument has a probability initialized to 1, then line 5 updates its value according to the probability of its attackers. However, if the goal argument is unattaked, the final value of an unattacked argument do not change and is therefore equal to 1 (`Fast_SCN`$^{\mathtt{gr}}(a) = P^{gr}(a) = 1$).

Attacked arguments.

- **Constellation approach.** From the definition of the grounded semantics any defended argument belong to the grounded extension. Therefore from Definition 3, the probability of an argument is the sum of the probability of each subgraph (world) such that the argument is defended. Therefore, for any argument a:

$$P^{gr}(a) = \sum_{\{\omega \sqsubseteq PrAF | a \text{ is defended in } \omega\}} p(\omega)$$

  We can notice that some attacks (such as those coming out of the argument we are computing) are useless to determine the acceptability of an argument (defended or not, i.e. in the grounded or not). We will start by showing that in a graph having only attacks relevant to the elaboration of the acceptability of an argument the Algorithm 3 (`Fast_SCN`$^{\mathtt{gr}}$) characterizes the constellation approach. Then we will explain why the irrelevant attacks do not change this probability.
- **Fast_SCN approach.** Similar to the definition of grounded semantics, we can calculate in the order of the defended arguments (G_i) from the unattacked arguments (G_0) ($\mathtt{Gr} = \bigcup_{i \geq 0} G_i$, where G_0 is the set of all unattacked arguments, and $\forall i \geq 0$, G_{i+1} is the set of all arguments that G_i defends). In the case of an attacked argument, as we explained in the pre-processing phase, having no cycle we can guarantee an order of arguments to be computed from the unattacked arguments to the goal argument. The question is therefore whether the computation of the Algorithm 3 (`Fast_SCN`$^{\mathtt{gr}}$) in line 6 characterises the constellation method.

  Let breakdown the computation of line 6 to understand the probability of an attacked argument a:

  – From the previous point, we know that at the first group of defended argument G_0 which are not attacked, `Fast_SCN`$^{\mathtt{gr}}(a_0) = P^{gr}(a_0) = 1$. Let us now show that for each group of defended arguments G_i it can be calculated according to its direct attacks.

  – Let us see now why lines 5 and 6 characterise all worlds of the constellation approach where a is defended (from all attackers b): prob[a] = prob[a] $\times$ (1 - prob[b] $\times$ $P_R(b, a)$).

  ∗ Given that $P_R(b, a)$ is the probability that the attack from b to a exists, prob[b] $\times$ $P_R(b, a)$ correspond to the conjunction of the two events, the attacker b is In and the attack (b, a) exist, i.e. it is a successful attack.

  ∗ Hence, 1 - prob[b] $\times$ $P_R(b, a)$ is the probability that b is Out or the attack (b, a) doesn't exist, i.e. the attack fail.

  ∗ Finally, lines 5 and 6 is equivalent to $prob$[a] $= \prod_{b \in \mathtt{Att}(a)} 1 - (prob[b] \times P_R(b, a))$; the probability of a correspond to the conjunction of the events such that

each attack fails (the attacker is Out or the attacks not exists). Therefore prob[a] is the probability that a is defended.

Let us show how works this computation (of our algorithm) iteratively in comparison to the worlds computation.

From Definition 2, recall that we compute the probability of a world as follow: $p(\omega) = \Big(\prod_{att\in\mathcal{R}'} P_R(att)\Big) \times \Big(\prod_{att\in\mathcal{R}\setminus\mathcal{R}'} \big(1-P_R(att)\big)\Big)$.

Assume for now, we have only relevant attacks.

Then, first case, trivially if an argument a is attacked only by unttacked arguments (acyclic graph, as in SCN), its probability is equal to the sum of worlds where it is not attacked: $prob[\text{a}] = \prod_{b\in\texttt{Att}(a)} 1 - (prob[b] \times P_R(b,a)) = \prod_{b\in\texttt{Att}(a)} 1 - (1 \times P_R(b,a)) = \prod_{b\in\texttt{Att}(a)} 1 - P_R(b,a)$. In this case of having only relevant attacks for this computation, then is it equivalent to the probability of the world without these attacks.

Second step, this argument a attacks an argument x. The corresponding worlds where x is defending is when the attack from a fails. As we said before x is defending if not(a is In and the attack from a to x exists), i.e. $1 - (prob[a] \times P_R(a,x))$.

Finally to generalize this computation for every computation with relevant attack we need to introduce argument having several attacks. So let add an argument a_2 which we know its probability (the sum of the worlds where it is defended) such that its attackers are different from a (if not, the common attackers are dependent and that must be take into account, see algorithm for MCN) which is the case because we are in a SCN PrAF. Given that the two (or any number) subgraphs from a and a_2 are independant to the attacks against x, we can combine them by the classic conjunction in probability, the product; then we obtain that $prob[x] = \prod_{a_i\in\texttt{Att}(x)} 1 - (prob[a_i] \times P_R(a_i,x)$.

Let us now see, why the irrelevant attacks, i.e. those not related to the acceptability of an argument do not change its probability. Let see for one irrelevant attack before to generalize. The reason is simple, when we add an attack (with a probability x) which not influence the acceptability of an argument a, the probability of a do not change because the sum of the worlds where a is defended will get the two cases: with and without this new attack, i.e. $(prob[a]\times x)+(prob[a]\times(1-x)) = (prob[a]\times x)+(prob[a])-(prob[a]\times x) = prob[a]$. Therefore with the same reasoning, any irrelevant attacks for an argument, have no impact on the probability of this argument.

□

We show further that this algorithm is linearithmic according to the number of arguments due to its ordering (as explained in section 3) and linear for the computation of the probability according to the number of attackers.

Theorem 3. *Let* $\mathbf{G}$ *be a SCN PrAF. For any argument* $a \in \mathcal{A}$*, the time complexity of* $\texttt{Fast}^{\texttt{gr}}(a)$ *is linearithmic on the number* n *of attackers (direct and indirect), i.e.* $O(nlogn)$.

Proof. Let $\mathbf{G} \in \mathcal{U}$ is a Singly Connected Network, i.e. a directed acyclic graph. The function `order` (line 1) traverses the graph back through the attackers from the "goal" to the roots. This function save the maximum distance between each argument and the "goal", and we know from (Eppstein 1998) that finding the longest path in directed acyclic graph is linear in time.

The function `order` then applies a sorting algorithm (e.g. Timsort (Auger et al. 2018) or Heapsort (Schaffer and Sedgewick 1993)) with a worst case complexity in $O(nlogn)$ where n is the number of arguments. Hence the function `order` is linearithmic in the number of arguments.

Finally, the computation of the probability of all argument (line 3-6) is lineare in the the number of arguments, because the number of attacks (line 3 times line 5) in a SCN graph cannot be greater than the number of arguments (otherwise it exists more than one path between at least two arguments), i.e. line 6 work at most n time (number of arguments).

Therefore, for any argument $a \in \mathcal{A}$, the time complexity of $\texttt{Fast\_SCN}^{\texttt{gr}}(a)$ is linearithmic on the number n of arguments (involved in the computation of a), i.e. $O(nlogn)$. □

Note that the intuition of $\texttt{SCN\_Fast}^{\texttt{gr}}$ cannot be used with MCN graphs in general, because in that case, we need to deal with dependent arguments. Indeed, if an argument attacks another one from several paths, it will be taken into account redundantly on each path by the conjunction (i.e. the product) of its attackers.

Let us take an example to illustrate this problem. Assume we add in the graph $\mathbf{G}$ of the example 1 an attack from d to a. Since the argument d has two paths, we will indirectly, in the calculation, conjunct this event with itself, as if it were a different argument, but it is not the case. To accept b, from the attack of a, we have a case where d is "In" (accepted), so it defends b against a; another case is that of the attack of d when it is "Out" (not accepted) and so b could be "In". Of course, d cannot be both "In" and "Out" at the same time, but since local propagation does not detect that the probability of a and b depend on d, the result is that the probability of b is evaluated as if we had two d arguments, one d_1 attacking a and another d_2 attacking b.

Finally, we compare experimentally on 9 random SCN graphs between 200 and 1000 arguments/attacks, the runtimes of our two new algorithms, and apply the approximate Monte-Carlo method (Li, Oren, and Norman 2011), to investigate what error rate this method produces using the execution time of our algorithms, see Fig. 4. We used only one graph and took the argument with a maximum time in order to compare the methods as well as possible. Indeed, if we make several executions and we average, it biases the comparison with the Monte-Carlo method.

We observe that the $\texttt{Fast\_MCN}^{\texttt{gr}}$ algorithm is polynomial and the $\texttt{Fast\_SCN}^{\texttt{gr}}$ is linear. When running the Monte-

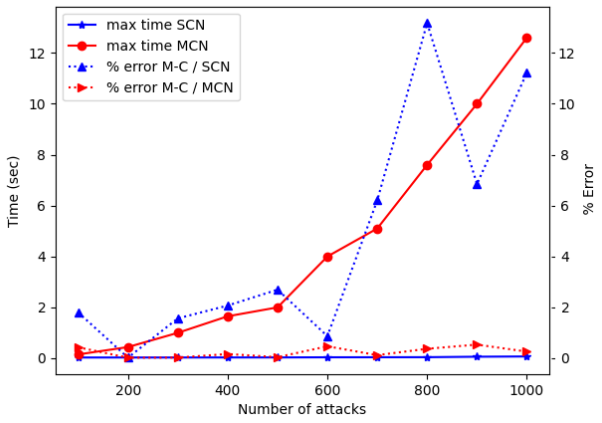

Figure 4: Run-times of $\texttt{Fast\_MCN}^{gr}$ vs $\texttt{Fast\_SCN}^{gr}$, and % of errors of Monte-Carlo approach in these times (in random SCN graphs).

Carlo method, we can notice that due to the longer time produced by $\texttt{Fast\_MCN}^{gr}$, the approximation is good. However, when applied to the time of the $\texttt{Fast\_SCN}^{gr}$, we see that the larger the graph, the more the approximation deteriorates, as the method decreases in the number of iterations and therefore in accuracy. Note also a non-regular error curve when using the $\texttt{Fast\_SCN}^{gr}$ time. Since there are few iterations, we can have randomly a good or bad approximation.

5 Conclusion

The constellations approach suffers from a high complexity due to the exponential number of generated worlds. In order to tackle this, we propose to compute the acceptance probability of an argument with two new algorithms, which are able to give the same score. In acyclic graphs, the time of our algorithm is related to the structure of the graph, we are not bound by the number of attackers but by the number of dependent arguments which is better. For the specific case of SCNs, we show that we can work in linearithmic time with an exact and faster solution than the Monte-Carlo approach.

In future work, we will investigate how to extend this algorithm to cyclic PrAFs, then how to compute the probability of acceptance of a set of arguments, and finally how extend it to other semantics.

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