Similarity Measures between Order-Sorted Logical Arguments

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Submission Id: 821

ABSTRACT

The notion of similarity in formal argumentation has received some attention recently, since one can argue that, in some context, using similar arguments to reach a conclusion is not the same as using dissimilar ones. In this work, we adapt the notion of similarity measures to arguments built from Order-Sorted First Order Logic, an extension of First Order Logic which allows to easily represent complex information, taking into account the type of the data. We study and evaluate our approach with respect to an adaptation of axioms from the literature. This paves the way to new reasoning modes for agents taking into account similarity between arguments in complex settings like ontologies.

KEYWORDS

Logic-based Argumentation, Similarity Measure, First Order Logic

ACM Reference Format:
Anonymous Author(s). 2023. Similarity Measures between Order-Sorted Logical Arguments. In *Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), London, United Kingdom, May 29 – June 2, 2023*, IFAAMAS, 15 pages.

1 INTRODUCTION

Formal argumentation has become a major topic in Knowledge Representation and Reasoning, with various applications like decision making [40], defeasible reasoning [22], as well as dealing with inconsistent knowledge bases [12]. Argumentation can also be applied in multi-agent systems [31] (see for instance automated negotiation based on argumentation frameworks [18, 19]). So, when agents use logic-based information for reasoning, it is possible to build arguments from this information, where typically an argument is a pair made of a set of formulae (called support) and a single formula (called conclusion). The conclusion should be a logical consequence of the support. Examples of arguments are $A = \langle \{p \wedge q \wedge r\}, p \wedge q \rangle$, $B = \langle \{p \wedge q\}, p \wedge q \rangle$ and $C = \langle \{p, q\}, p \wedge q \rangle$. From the definition of arguments, one can identify attacks between them, and then use a semantics to evaluate the arguments. Finally, conclusions of the "strong" arguments are inferred from the base. In the literature, there exist several families of semantics (e.g. extension-based, ranking-based or gradual semantics) to determine which arguments are "strong". We refer the reader to [1] for a complete overview of the existing families of semantics in abstract argumentation and the differences between these approaches (e.g., definition, outcome, application). Among the existing gradual semantics, like h-Categorizer [12], some of them satisfy the Counting (or Strict Monotony) principle defined in [2]. This principle states that each attacker of an argument contributes to weakening the argument. For instance, if the argument $D = \langle \{\neg p \vee \neg q\}, \neg p \vee \neg q \rangle$ is attacked by A, B, C, then each of the three arguments will decrease the strength of D. However, the three attackers are somehow similar, thus D will lose more than necessary. Consequently, the authors in [4] have motivated the need for investigating the notion of similarity between pairs of such logical arguments. They introduced a set of principles that a reasonable similarity measure should satisfy, and provided several measures that satisfy them. In [3, 5, 6] several extensions of h-Categorizer that take into account similarities between arguments have been proposed. All these works consider propositional logic. In this paper, we suggest to adapt the principles behind similarity measures for logical arguments, to a much more expressive framework, namely Order-Sorted First Order Logic ($\mathtt{OS}-\mathtt{FOL}$). Fragments of $\mathtt{OS}-\mathtt{FOL}$ have been used for reasoning in multi-agent systems (*e.g.* [23] uses FOL for reasoning about policies, and [30] proposes an architecture for building cognitive agents able of deduction on facts and rules inferred directly from natural language). Also, negotiation approaches based on fragments of $\mathtt{OS}-\mathtt{FOL}$ have been proposed in [33, 35]. So we focus on $\mathtt{OS}-\mathtt{FOL}$, a formalism which generalizes (standard) First Order Logic ($\mathtt{FOL}$). While $\mathtt{FOL}$ has already interesting modelling capabilities, $\mathtt{OS}-\mathtt{FOL}$ allows to naturally model situations where variables belong to a given domain, and there can be relations between the domains of the variables (for instance, the domains made of all the penguins is a subset of the domain containing all the birds). So, by studying logical arguments built from $\mathtt{OS}-\mathtt{FOL}$, we are able to apply our work to existing argumentation frameworks based on $\mathtt{FOL}$ [10, 13], but also other rich frameworks like description logic [11], which can be translated into (Order-Sorted) $\mathtt{FOL}$. This paves to way to applications of argumentation (and similarity measures) to inconsistent knowledge expressed in these rich structured frameworks. Proofs are available in the supplementary material.

2 BACKGROUND

2.1 Logic and Arguments

We assume that the reader is familiar with propositional logic. First Order Logic (FOL) is a rich framework for expressing knowledge about objects, including relations between them (using predicates). An example is "Tweety is a penguin, all penguins are birds and all birds have wings, so Tweety has wings" which can be expressed as $penguin(Tweety) \wedge (\forall x, penguin(x) \rightarrow bird(x)) \wedge (\forall x, bird(x) \rightarrow haveWings(x))$ for the premises, and $haveWings(Tweety)$ as the consequence. However, this framework does not allow to distinguish between various types of objects. This means that it would be possible to write a FOL formula like $hasRoots(Tweety)$, which does not not make sense since Tweety is a bird, not a plant. Since we want to apply our method to contexts where data can have a specific type, we use Order-Sorted FOL, a generalization of (standard) FOL where all the variables are associated with a *sort* (as well as the parameters of the predicates).[1] Then, when interpreting a

Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), A. Ricci, W. Yeoh, N. Agmon, B. An (eds.), May 29 – June 2, 2023, London, United Kingdom.

[1] In this paper, we restrict ourselves to formulae without functions.

formula, the domain of variables is constrained by its sort. An additional constraint can be added to these sorts, as a partial order over them, corresponding to inclusion relations between the domains associated to the sorts.

Definition 2.1 (Order-Sorted FOL). Let $\mathbf{So} = \{s_1, \ldots, s_n\}$ be a set of sorts, and $\prec \subseteq \mathbf{So} \times \mathbf{So}$ a partial order over $\mathbf{So}$. An *Order-Sorted First Order Language* $\mathtt{OS-FOL}$, is a set of formulae built up by induction from:
– a set $\mathbf{C}$ of constants ($\mathbf{C} = \{a_1, \ldots, a_l\}$),
– a set $\mathbf{V}$ of variables ($\mathbf{V} = \{x^s, y^s, z^s, \ldots \mid s \in \mathbf{So}\}$),
– a set $\mathbf{P}$ of predicates ($\mathbf{P} = \{P_1, \ldots, P_m\}$),
– a function $\mathtt{ar} : \mathbf{P} \longrightarrow \mathbb{N}$ which tells the arity of any predicate,
– a function $\mathtt{sort}$ s.t. for $P \in \mathbf{P}$, $\mathtt{sort}(P) \in \mathbf{So}^{\mathtt{ar}(P)}$, and for $c \in \mathbf{C}$, $\mathtt{sort}(c) \in \mathbf{So}$,
– the usual connectives ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$), Boolean constants $\top$ (true) and $\bot$ (false) and quantifier symbols ($\forall, \exists$).
A *grounded formula* is a formula without any variable.

We use lowercase greek letters (*e.g.* ϕ, ψ) to denote formulae, and uppercase ones (*e.g.* Φ, Ψ) to denote sets of formulae. The set of all formulae is denoted by $\mathtt{OS-FOL}$. We assume formulae to be *prenex*, *i.e.* written as $Q_1 x_1, \ldots, Q_k x_k \phi$ where Q_i is a quantifier (for each $i \in \{1, \ldots, k\}$) and ϕ is a non-quantified formula. A formula ϕ is in negative normal form (NNF) if and only if it does not contain implication or equivalence symbols, and every negation symbol occurs directly in front of an atom. Following [29], we slightly abuse words and denote by $\mathtt{NNF}(\phi)$ the formula in NNF obtained from ϕ by "pushing down" every occurrence of $\neg$ (using De Morgan's law) and eliminating double negations. For instance, $\mathtt{NNF}(\neg((P(a) \rightarrow Q(a)) \vee \neg Q(b))) = P(a) \wedge \neg Q(a) \wedge Q(b)$. In that case, we call *literal* either an atom (*i.e.* a predicate with its parameters) or the negation of an atom. We denote by $\mathtt{Lit}(\phi)$ the set of literals occurring in $\mathtt{NNF}(\phi)$, hence $\mathtt{Lit}(\neg((P(a) \rightarrow Q(a)) \vee \neg Q(b))) = \{P(a), \neg Q(a), Q(b)\}$. For a given set of predicates $\mathbf{P}$, we define $\mathbf{L} = \{P(x_1^{s_1}, \ldots, x_k^{s_k}), \neg P(x_1^{s_1}, \ldots, x_k^{s_k}) \mid P \in \mathbf{P}, \mathtt{sort}(P) = (s_1, \ldots, s_k)\}$ the set of literals. We say that a literal is *negative* when it starts with a negation, denoted by $\mathtt{Pol}(L) = -$. Otherwise we say that it is *positive*, denoted by $\mathtt{Pol}(L) = +$. And we say that two literals have the same *polarity* if they are either both positive or both negative.

Let $\phi \in \mathtt{OS-FOL}$, ϕ is in a conjunctive normal form (CNF) if it is a conjunction of clauses $\bigwedge_i c_i$ where each clause c_i is a disjunction of literals $\bigvee_j l_j$. For instance $P(a) \wedge (Q(a) \vee Q(b))$ is in a CNF while $(P(a) \wedge Q(a)) \vee Q(b)$ is not. CNF formulae are particular NNF formulae. Clauses are also usually represented as sets of literals.

In $\mathtt{OS-FOL}$, the partial order $\prec$ represents "sub-type" relations between groups of entities. For instance, the fact that dogs are a special type of mammals can be represented by such a sub-type relation. In the case where $s_1 \prec s_2$, a predicate which expects a parameter of type s_2 can be applied to a constant or variable of type s_1 (for instance, a predicate about mammals can be applied to dogs).

Example 2.2. $\mathtt{OS-FOL}$ formulae can be used to reason about ontological information. Assume that we have the following information: mammals and birds are animals, dogs and cats are mammals, penguins and chickens are birds. Moreover, Zazu is a bird, Tweety is a penguin, and Dogmatix is a dog. Finally, animals are living beings, as well as plants. This can be represented by the following sorts and constants:
– $\mathbf{So} = \{m, b, a, d, c, p, ch, l, pl\}$ with $m \prec a$, $b \prec a$, $d \prec m$, $c \prec m$, $p \prec b$, $ch \prec b$, $a \prec l$, $pl \prec l$ (see Figure 1),
– $Z \in \mathbf{C}$ with $\mathtt{sort}(Z) = b$ is a constant representing Zazu,
– $T \in \mathbf{C}$ with $\mathtt{sort}(T) = p$ is a constant representing Tweety,
– $D \in \mathbf{C}$ with $\mathtt{sort}(D) = d$ is a constant representing Dogmatix.

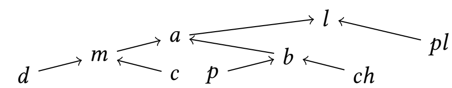

Figure 1: Hierarchy of sorts from Example 2.2. An arrow from s_1 to s_2 means $s_1 \prec s_2$.

We know that all birds have wings, and both mammals and birds are warm-blooded. Also, some birds and some mammals fly, but not all of them. If a bird is wounded, then it cannot fly. If a bird is penguin, then it cannot fly. Some birds are wounded. Finally, Tweety is a penguin. This information can be represented by the following predicates:

- $\mathbf{P} = \{hW, wB, f, w, p\}$, standing respectively for "haveWings", "warmBlooded", "fly", "wounded" and "penguin" s.t. $\mathtt{ar}(P_i) = 1$ and $\mathtt{sort}(P_i) = a$ for each $P_i \in \mathbf{P}$.

We can build, *e.g.* the formula $\forall x^b, hW(x^b)$ meaning that all birds have wings (because the variable x^b has the sort b). The other pieces of information are represented by

$\forall x^b wB(x^b)$	$\forall x^m wB(x^m)$
$\exists x_1^b, x_2^b f(x_1^b) \wedge \neg f(x_2^b)$	$\exists x_1^m, x_2^m f(x_1^m) \wedge \neg f(x_2^m)$
$\forall x^b w(x^b) \rightarrow \neg f(x^b)$	$\forall x^b p(x^b) \rightarrow \neg f(x^b)$
$\exists x^b w(x^b)$	$p(T)$

However formulae like $\exists x^l, f(x^l)$ or $\forall x^{pl}, wB(x^{pl})$ are not well-formed, since the predicates fly and wB cannot be applied to living beings or plants.

$\mathtt{OS-FOL}$ formulae are evaluated via a notion of structure:

Definition 2.3 (Structure). Given $n \in \mathbb{N}$, a *n-sorted structure* is $\mathbf{St} = (\{D_1, \ldots, D_n\}, \{R_1, \ldots, R_m\}, \{c_1, \ldots, c_l\})$ where:
– $D_1, \ldots, D_n$ are the (non-empty) domains,
– $R_1, \ldots, R_m$ are relations between the elements of the domains,
– $c_1, \ldots, c_l$ are distinguished constants in the domains.

Example 2.4. An example of structure associated with the $\mathtt{OS-FOL}$ from Example 2.2 is $\mathbf{St} = (\{D_1, \ldots, D_9\}, \{R_1, \ldots, R_5\}, \{Zazu, Tweety, Dogmatix\})$ where
– $D_1, \ldots, D_9$ are the sets of all individuals of the various types (*e.g.* D_1 is the set of mammals, corresponding to the sort symbol m; D_2 is the set of birds, corresponding to the sort symbol b; etc),
– $R_1, \ldots, R_5$ are the relations corresponding to the predicate symbols (*e.g.* R_1 indicates which animals have wings,. . .),
– Zazu, Tweety and Dogmatix correspond respectively to a particular bird (an element of the domain D_2 associated with the sort b), a particular penguin (an element of the domain D_6 associated with the sort p) and a particular dog (an element of the domain D_4 associated with the sort d).

Classical first order logic formulae can be evaluated via 1-sorted structures. For this reason, any fragment of first order logic is captured by $\mathtt{OS-FOL}$. Now, we show how $\mathtt{OS-FOL}$ formulae are interpreted.

Definition 2.5 (Interpretation). An *interpretation* $\mathbf{I_{St}}$ *over a structure* $\mathbf{St}$ assigns to elements of the $\mathtt{OS-FOL}$ vocabulary some values in the structure $\mathbf{St}$. Formally,
– $\mathbf{I_{St}}(s_i) = D_i$, for $i \in \{1, \dots, n\}$ s.t. for each $s_i, s_j \in \mathrm{S}$, if $s_i \preceq s_j$ then $\mathbf{I_{St}}(s_i) \subseteq \mathbf{I_{St}}(s_j)$ (each sort symbol is assigned to a domain s.t. the sub-type relations are respected),
– $\mathbf{I_{St}}(P_i) = R_i$, for $i \in \{1, \dots, m\}$ (each predicate symbol is assigned to a relation),
– $\mathbf{I_{St}}(a_i) = c_i$, for $i \in \{1, \dots, l\}$ (each constant symbol is assigned to a constant value). As a shorthand, we write $\mathbf{I_{St}}((s_1, \dots, s_k)) = \mathbf{I_{St}}(s_1) \times \dots \times \mathbf{I_{St}}(s_k)$. Then satisfaction of formulae is recursively defined by:
– $\mathbf{I_{St}} \models P_i(x_1, \dots, x_k)$, where $(x_1, \dots, x_k) \in \mathbf{I_{St}}((s_1, \dots, s_k))$ with $\mathtt{sort}(x_i) = s_i$ for each $i \in \{1, \dots, k\}$, iff $(x_1, \dots, x_k) \in R_i$,
– $\mathbf{I_{St}} \models \exists x^{s_i}\phi$ iff $\mathbf{I}_{\mathbf{St}, x^{s_i} \leftarrow v} \models \phi$ for some $v \in D_i$,
– $\mathbf{I_{St}} \models \forall x^{s_i}\phi$ iff $\mathbf{I}_{\mathbf{St}, x^{s_i} \leftarrow v} \models \phi$ for each $v \in D_i$,
– $\mathbf{I_{St}} \models \phi \land \psi$ iff $\mathbf{I_{St}} \models \phi$ and $\mathbf{I_{St}} \models \psi$,
– $\mathbf{I_{St}} \models \phi \lor \psi$ iff $\mathbf{I_{St}} \models \phi$ or $\mathbf{I_{St}} \models \psi$,
– $\mathbf{I_{St}} \models \neg\phi$ iff $\mathbf{I_{St}} \not\models \phi$,
where $\mathbf{I}_{\mathbf{St}, x^{s_i} \leftarrow v}$ is a modified version of $\mathbf{I_{St}}$ s.t. the variable x^{s_i} is replaced by a value v in the domain D_i corresponding to the sort symbol s_i. Finally, if Φ is a set of formulae, then $\mathbf{I_{St}} \models \Phi$ iff $\mathbf{I_{St}} \models \phi$ for each $\phi \in \Phi$.

Observe that Definition 2.5 does not specify the satisfaction of implications and equivalences, but they can be defined as usual by $(\phi \to \psi) \equiv (\neg\phi \lor \psi)$, and $(\phi \leftrightarrow \psi) \equiv (\phi \to \psi) \land (\psi \to \phi)$. We use $\mathtt{Mod}(\Phi)$ to denote the set of interpretations satisfying a set of formulae Φ, and we call Φ *consistent* if $\mathtt{Mod}(\Phi) \neq \emptyset$.

Example 2.6. Continuing Example 2.2, we define $\mathbf{I_{St}}$ by:
– $\mathbf{I_{St}}(m) = D_1, \mathbf{I_{St}}(b) = D_2, \dots, \mathbf{I_{St}}(pl) = D_9$,
– $\mathbf{I_{St}}(hW) = R_1, \dots, \mathbf{I_{St}}(p) = R_5$,
– $\mathbf{I_{St}}(Z) = Zazu, \mathbf{I_{St}}(T) = Tweety, \mathbf{I_{St}}(D) = Dogmatix$.
The formula $\phi = \forall x^b hW(x^b)$ is satisfied by $\mathbf{I_{St}}$, since all elements of the domain D_2 associated with the sort symbol b actually have wings. On the contrary, consider the set of formulae $\Phi = \{\forall x^b f(x^b), \forall x^p \neg f(x^p)\}$. This set of formulae is not satisfied, because $p \prec b$, and so the domains satisfy $D_6 \subset D_2$, meaning that all penguins are birds. Then, from Φ we can deduce that any penguin can fly (because of the first formula) and cannot fly (because of the second formula) at the same time. So, this formula is not satisfied by $\mathbf{I_{St}}$. Notice that we could not define an interpretation $\mathbf{I'_{St}}$ s.t. $\mathbf{I'_{St}}(Z) = Tweety$ and $\mathbf{I'_{St}}(T) = Zazu$, since $Zazu$ is a bird, and T has the sort p (*i.e.* it can only be a penguin, not any kind of bird).

Now we introduce the concept of instantiation, *i.e.* grounded formulae which are compatible with a given $\mathtt{OS-FOL}$ formula.

Definition 2.7 (Instantiation). Given Φ a set of $\mathtt{OS-FOL}$ formulae and $\mathbf{I_{St}}$ an interpretation over a structure $\mathbf{St}$, the set of *instantiations* of Φ is defined recursively by:
– $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \{\Phi\}$ if $\Phi = \{\phi\}$, where ϕ is a grounded formula s.t. $\mathbf{I_{St}} \models \phi$,
– $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \{\{\phi_{x^s \leftarrow v} \mid \mathbf{I_{St}} \models \phi_{x^s \leftarrow v}, v \in \mathbf{I_{St}}(s)\}\}$ if $\Phi = \{\forall x^s \phi\}$,
– $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \{\{\phi_{x^s \leftarrow v} \mid \mathbf{I_{St}} \models \phi_{x^s \leftarrow v}, v \in V\} \mid \emptyset \subset V \subseteq \mathbf{I_{St}}(s)\}$ if $\Phi = \{\exists x^s \phi\}$,
– $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \{I_1 \cup I_2 \mid I_1 \in \mathtt{Inst}_{\mathbf{I_{St}}}(\{\phi_1\}), I_2 \in \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi_2), \mathbf{I_{St}} \models I_1 \cup I_2\}$ if $\Phi = \{\phi_1\} \cup \Phi_2$,
where $\phi_{x^s \leftarrow v}$ is the formula ϕ s.t. all the occurrences of the variable x^s are replaced by the value v (from the domain associated with the sort s).

The idea is that formulae with quantified variables may be instanciated in various ways. Assuming that the domain of a variable x is $\{A, B\}$, then the formula $\exists x P(x)$ means that either $P(A)$ is true, or $P(B)$, or both at the same time. And $\forall x P(x)$ means that $P(A)$ and $P(B)$ are both true. This is what is captured by the notion of instantiation. Moreover, an instantiation is consistent because of the constraint $\mathbf{I_{St}} \models I_1 \cup I_2$ in the last part of the definition. This constraint means that, if *e.g.* we consider the set of formulae $\{\exists x P(x), \exists x \neg P(x)\}$, then we keep the instantiations where $P(A)$ is true and $P(B)$ is false, or the opposite. But we exclude situations where $P(A)$ is both true (because of the first formula) and false (because of the second formula) at the same time.

Example 2.8. Consider the set of formulae $\Phi = \{\phi_1 = \exists x^b w(x^b), \phi_2 = \forall x^b w(x^b) \to \neg f(x^b)\}$. We assume here that the domain associated with the sort b is the set $\{Tweety, Zazu\}$. Applying Definition 2.7, $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \{I_1 \cup I_2 \mid I_1 \in \mathtt{Inst}_{\mathbf{I_{St}}}(\{\exists x^b w(x^b)\}), I_2 \in \mathtt{Inst}_{\mathbf{I_{St}}}(\{\forall x^b w(x^b) \to \neg f(x^b)\}), \mathbf{I_{St}} \models I_1 \cup I_2\}$.
We start with the first formula, *i.e.* $\phi_1 = \exists x^b w(x^b)$.
$\mathtt{Inst}_{\mathbf{I_{St}}}(\{\phi_1\}) = \{\{w(Tweety)\}, \{w(Zazu)\}, \{w(Tweety), w(Zazu)\}\}$. For $\phi_2 = \forall x^b w(x^b) \to \neg f(x^b)$, there is only one possible instantiation: $\mathtt{Inst}_{\mathbf{I_{St}}}(\{\phi_2\}) = \{\{w(Tweety) \to \neg f(Tweety), w(Zazu) \to \neg f(Zazu)\}\}$.
We conclude that $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) =$
$\{\{w(Tweety), w(Tweety) \to \neg f(Tweety),$
$w(Zazu) \to \neg f(Zazu)\}, \{w(Zazu), w(Tweety) \to \neg f(Tweety),$
$w(Zazu) \to \neg f(Zazu)\}, \{w(Tweety), w(Zazu),$
$w(Tweety) \to \neg f(Tweety), w(Zazu) \to \neg f(Zazu)\}\}$

From the notions of structure and interpretation, we can define the consequence relation over $\mathtt{OS-FOL}$ formulae.

Definition 2.9 (Consequence Relation). Let ϕ and ψ be two $\mathtt{OS-FOL}$ formulae. We say that ψ *is a consequence of* ϕ, denoted by $\phi \vdash \psi$, if for any structure $\mathbf{St}$, and any interpretation $\mathbf{I_{St}}$ over $\mathbf{St}$, $\mathbf{I_{St}} \models \phi$ implies $\mathbf{I_{St}} \models \psi$. Two formulae ϕ and ψ are *equivalent* (denoted $\phi \equiv \psi$) iff $\phi \vdash \psi$ and $\psi \vdash \phi$.

Classical logic can be used to define arguments, *i.e.* logic-based representation of reasons supporting a specific conclusion. Logical arguments usually need to satisfy some constraints [12]:

Definition 2.10 (Logical Argument). An *argument* built under a logic $(\mathcal{L}, \vdash)$ is a pair $\langle \Phi, \phi \rangle$, where $\Phi \subseteq_f \mathcal{L}$ [2] and $\phi \in \mathcal{L}$, s.t. Φ is consistent, $\Phi \vdash \phi$, and $\nexists \Phi' \subset \Phi$ s.t. $\Phi' \vdash \phi$. An argument $A = \langle \Phi, \phi \rangle$ is *trivial* iff $\Phi = \emptyset$ and $\phi \equiv \top$. Φ is called the support of the argument ($\mathtt{Supp}(A) = \Phi$) and ϕ its conclusion ($\mathtt{Conc}(A) = \phi$). The set of all arguments built under $(\mathcal{L}, \vdash)$ is denoted $\mathtt{Arg}(\mathcal{L})$.

In this paper, we will focus on the set of arguments $\mathtt{Arg}(\mathtt{OS-FOL})$ built under the logic $(\mathtt{OS-FOL}, \vdash)$, where $\vdash$ is the consequence relation from Definition 2.9.

[2] $X \subseteq_f Y$ means X is a finite subset of Y

Example 2.11. Let A_1 and A_2 are examples of arguments:
$A_1 = \langle\{\exists x^b w(x^b), \forall x^b w(x^b) \rightarrow \neg f(x^b)\}, \exists x^b \neg f(x^b)\rangle$
$A_2 = \langle\{p(Tweety), \forall x^b p(x^b) \rightarrow \neg f(x^b)\}, \neg f(Tweety)\rangle$

Note that two sets of formulae $\Phi, \Psi \subseteq_f \mathcal{L}$ are *equivalent*, denoted by $\Phi \cong \Psi$, iff there is a bijection $f : \Phi \rightarrow \Psi$ s.t. $\forall \phi \in \Phi$, $\phi \equiv f(\phi)$. However, we may want to consider that a set of formulae is equivalent with the conjunction of its elements (*e.g.* $\{P(a), Q(a)\}$ and $\{P(a) \wedge Q(a)\}$ are equivalent). For getting them equivalent, we borrow the method used in [7]. We transform every formula into a CNF, then we split it into a set containing its clauses. In our approach, we consider one CNF per formula. For that purpose, we will use a finite sub-language $\mathcal{F}$ that contains one formula per equivalent class and the formula should be in a CNF.

Definition 2.12 (Finite CNF Language $\mathcal{F}$). Let $\mathcal{F} \subset_f \mathcal{L}$ s.t. $\forall \phi \in \mathcal{L}$, there is a unique $\psi \in \mathcal{F}$ s.t. $\phi \equiv \psi$, $\mathtt{Lit}(\phi) = \mathtt{Lit}(\psi)$ and ψ is a CNF formula. We define $\mathtt{CNF}(\phi) = \psi$.

While we do not specify the elements of $\mathcal{F}$, we use concrete formulae in the examples, and they are assumed to belong to $\mathcal{F}$.

Now we introduce $\mathtt{UC}(\Phi)$ as the representation of the formulae in Φ as one set of clauses. Intuitively, recall that any formula can be seen as a set of clauses, associated with a sequence of quantifiers. A set of formulae can then be seen as set of clauses and a sequence of quantifiers, such that variables are renamed to avoid ambiguities. As an example, assume $\phi_1 = \exists x P(x) \wedge Q(x)$ and $\phi_2 = \exists x Q(x) \vee R(x)$. We have $\mathtt{UC}(\{\phi_1, \phi_2\}) = \exists x, x'\{P(x), Q(x), Q(x') \vee R(x')\}$. Formally, for $\Phi = \{\mathcal{Q}_{\phi_i}\phi_i \mid i \in \mathbb{N}\} \subseteq_f \mathcal{L}$, where ϕ_i is a non-quantified CNF formula (i.e. a set of clauses), and $\mathcal{Q}_{\phi_i}$ is the sequence of quantifiers associated with ϕ_i, we define $\mathtt{UC}(\Phi) = \mathcal{Q}^*_{\phi_1} \dots \mathcal{Q}^*_{\phi_n} \bigcup_{\phi\in\Phi} \bigcup_{\delta\in\phi} \delta^*$, where a renaming is applied to each clause (δ^*) and each sequence of quantifiers ($\mathcal{Q}^*_{\phi_i}$) in order to guarantee that no variable is shared between quantifiers $\mathcal{Q}^*_{\phi_i}$ and $\mathcal{Q}^*_{\phi_j}$ (with $i \neq j$) or between clauses coming from different formulae ϕ_i and ϕ_j (with $i \neq j$). We simply write $\mathtt{UC}(\phi)$ instead of $\mathtt{UC}(\{\phi\})$, for $\phi \in \mathcal{L}$.

Note that $\mathtt{UC}(\{P(a), Q(a)\}) \cong \mathtt{UC}(P(a) \wedge Q(a))$.
Let us now introduce the notion of compiled argument.

Definition 2.13 (Compiled Argument). The *compilation* of $A \in \mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})$ is $A^* = \langle \mathtt{UC}(\mathtt{Supp}(A)), \mathtt{Conc}(A)\rangle$.

Example 2.14. The three pairs $A = \langle\{P(a) \wedge Q(a) \wedge Q(b)\}, P(a) \wedge Q(a)\rangle$, $B = \langle\{P(a) \wedge Q(a)\}, P(a) \wedge Q(a)\rangle$ and $C = \langle\{P(a), Q(a)\}, P(a) \wedge Q(a)\rangle \in \mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})$. The compilations of the three arguments A, B, C are: $A^* = \langle\{P(a), Q(a), Q(b)\}, P(a) \wedge Q(a)\rangle$, $B^* = \langle\{P(a), Q(a)\}, P(a) \wedge Q(a)\rangle$ and $C^* = \langle\{P(a), Q(a)\}, P(a) \wedge Q(a)\rangle$.

We can see in the previous example that argument A is not concise, meaning that it has irrelevant information ($Q(b)$) for implying its conclusion. As it was shown in [7], using clausal arguments ensure that the arguments are concise.

Definition 2.15 (Equivalent Arguments). Two arguments $A, B \in \mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})$ are equivalent, denoted by $A \approx B$, iff $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(B))$ and $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$. We denote by $A \not\approx B$ when A and B are not equivalent.

We adapt the classical notion of sub-argument to our formalism.

Definition 2.16 (Sub-argument). Given two arguments $A = \langle\Phi, \phi\rangle$ and $B = \langle\Psi, \psi\rangle$, we say that A is a sub-argument of B if $\mathtt{UC}(\Phi) \subseteq \mathtt{UC}(\Psi)$.

2.2 Binary Similarity Measure between $\mathtt{OS}-\mathtt{FOL}$ Arguments

A similarity measure is used to indicate whether two arguments are similar or not, *i.e.* whether they share some parts of the reasoning mechanism used to build the arguments.

Definition 2.17 (Similarity Measure). Let $\mathbb{X}$ be a set of objects. A similarity measure on $\mathbb{X}$, denoted by $\mathtt{sim}^{\mathbb{X}}$, is a function from $\mathbb{X}\times\mathbb{X}$ to $[0, 1]$.

In this section, we focus on similarity measures over arguments, *i.e.* $\mathbb{X} = \mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})$. Intuitively, $\mathtt{sim}^{\mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})}(A, B)$ is close to 0 if the difference between A and B is important, while it is close to 1 if the arguments are similar. Several principles that similarity measures should satisfy have been discussed in the literature [4, 7, 8]. Some of the principles (Maximality, Symmetry, Substitution, Syntax Independence, and Non-Zero) can be stated exactly as in the literature [7], since they do not concern the internal structure of the arguments. For the other ones, we may need to adapt them to our $\mathtt{OS}-\mathtt{FOL}$-based arguments. Notice that some authors have argued against the fact that a similarity measures should absolutely satisfy symmetry [27, 38].

Now, we adapt the Minimality principle. It states that, if two arguments do not have anything in common in their content, then their degree of similarity should be minimal. While, in propositional logic, determining the set of common propositional variables is enough, here we need to consider (domains of) predicates and constants. We do not consider variables here since they are use in the context of quantifiers: there is no reason to assume that there is something common between $\forall x, P(x)$ and $\forall x, Q(x)$.

Before presenting the Minimality principle, let us introduce some useful notations. Given a formula ϕ, $\mathtt{Dom}(\phi) = \bigcup_{P\in\mathtt{Pred}(\phi)} \mathtt{sort}(P)$ represents the domains of the predicates in ϕ (or, more precisely, the sort symbols associated with these domains). We extend the notation to $\mathtt{Dom}(\Phi) = \bigcup_{\phi\in\Phi} \mathtt{Dom}(\phi)$ for Φ a set of formulae.

Principle 1 (Minimality). *A similarity measure* $\mathtt{sim}^{\mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})}$ *satisfies* Minimality *iff for all* $A, B \in \mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})$, *if 1) A and B are not trivial, 2)* $\forall s_i \in \mathtt{Dom}(\mathtt{Supp}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Supp}(B))$ *s.t.* $s_i \prec s_j$ *or* $s_j \prec s_i$ *or* $s_j = s_i$, *3)* $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Conc}(B))$ *s.t.* $s_i \prec s_j$ *or* $s_j \prec s_i$ *or* $s_j = s_i$, *then* $\mathtt{sim}^{\mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})}(A, B) = 0$.

The second (resp. third) principles states that the more an argument shares formulae in its support (resp. conclusion) with an another one, the higher is their similarity.

Principle 2 (Monotony – Strict Monotony). *A similarity measure* $\mathtt{sim}^{\mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})}$ *satisfies* Monotony *iff for all* $A, B, C, A^*, B^*, C^* \in \mathtt{Arg}(\mathtt{OS}-\mathtt{FOL})$, *if*
1. $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$ *or* $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Conc}(C))$ *s.t.* $s_i \prec s_j$ *or* $s_j \prec s_i$ *or* $s_j = s_i$,
2. $\mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(C)) \subseteq \mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(B))$,
3. for $B_A = \mathtt{UC}(\mathtt{Supp}(B)) \setminus \mathtt{UC}(\mathtt{Supp}(A))$ *and* $C_A = \mathtt{UC}(\mathtt{Supp}(C)) \setminus \mathtt{UC}(\mathtt{Supp}(A))$, $B_A \subseteq C_A$, $C_A \setminus B_A \subseteq \mathbb{C}$ *and* $\forall s_i \in \mathtt{Dom}(\mathtt{Supp}(A))$, $\nexists s_j \in \mathtt{Dom}(C_A \setminus B_A)$ *s.t.* $s_i \prec s_j$ *or* $s_j \prec s_i$ *or* $s_j = s_i$,

then the following hold:
– $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, C)$, ***(Monotony)***
– *If the inclusion in cond. 2. is strict or,* $\mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(C)) \neq \emptyset$ *and* $B_A \subset C_A$*, then* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) > \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, C)$. ***(Strict Monotony)***

Principle 3 (Dominance – Strict Dominance). *A similarity measure* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ *satisfies* Dominance *iff for all* $A, B, C, A^*, B^*, C^* \in \mathtt{Arg(OS-FOL)}$*, if*
1. $\mathtt{UC}(\mathtt{Supp}(B)) = \mathtt{UC}(\mathtt{Supp}(C))$,
2. $\mathtt{UC}(\mathtt{Conc}(A)) \cap \mathtt{UC}(\mathtt{Conc}(C)) \subseteq \mathtt{UC}(\mathtt{Conc}(A)) \cap \mathtt{UC}(\mathtt{Conc}(B))$,
3. for $B_A = \mathtt{UC}(\mathtt{Conc}(B)) \setminus \mathtt{UC}(\mathtt{Conc}(A))$ *and* $C_A = \mathtt{UC}(\mathtt{Conc}(C)) \setminus \mathtt{UC}(\mathtt{Conc}(A))$, $B_A \subseteq C_A$, $C_A \setminus B_A \subseteq \mathbb{C}$ *and* $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(C_A \setminus B_A)$ *s.t.* $s_i \prec s_j$ *or* $s_j \prec s_i$ *or* $s_j = s_i$,
then the following hold:
– $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, C)$. ***(Dominance)***
– *If the inclusion in cond. 2. is strict or,* $\mathtt{UC}(\mathtt{Conc}(A)) \cap \mathtt{UC}(\mathtt{Conc}(C)) \neq \emptyset$ *and* $B_A \subset C_A$*, then* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) > \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, C)$. ***(Strict Dominance)***

Notice that the first conditions allow to isolate the interesting behaviours on second and third conditions.

3 SIMILARITY MODELS

To define the similarity between two arguments, we will split the reasoning in several steps, corresponding to the different levels used in the construction of the arguments. At each level, different similarity measures can be used to compare the objects, and various aggregation functions can then be used to go from the comparison of objects to the comparison of sets of objects (leading to the next level). This level structure is based on the fact that our arguments are built from CNF formulae. More precisely,
Level 1: compute the similarity between two literals, by combining the similarity between their polarity, the predicate involved, and the predicates parameters (Section 3.1);
Level 2: then we use the previous level and aggregate the result of comparing literals in order to compare grounded clauses (Section 3.2);
Level 3: next, we aggregate the similarity between grounded clauses to obtain the similarity between sets of grounded clauses (Section 3.3);
Level 4: finally, we can define the similarity between sets of instantiations, since each instantiation is a set of grounded clauses (Section 3.4).
The similarity between two arguments is obtained by computing the similarity between the instantiations of their supports and the similarity between their conclusions, so Level 4 is the last level of abstraction that we need.

3.1 Similarity between literals

Recall that a literal is a predicate with or without a negation operator "¬". To know how similar are two literals, we compute the similarity between two atoms (*i.e.* without the literals' polarity) and combine these scores according to the polarity. At the level of atoms, we identify two parameters influencing the similarity: the value of the predicates and those of their vectors of parameters. Thus the similarity between two atoms can be seen as a combination of three functions: c to compute the similarity between two vectors of constants, p between two predicates and g to aggregate these scores.

Definition 3.1 (Similarity between Atoms). Let $\mathbf{c} : \bigcup_{j,k=1}^{+\infty} \mathbf{C}^j \times \mathbf{C}^k \rightarrow [0, 1]$ be a similarity measure between a pair of vectors of constants, $\mathbf{p} : \mathbf{P} \times \mathbf{P} \rightarrow [0, 1]$ be a similarity measure between a pair of predicates and $\mathbf{g} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be an aggregation function. Given $P_1, P_2 \in \mathbf{P}$ with two vectors of constants $A = \langle a_1, \dots, a_j \rangle$, $B = \langle b_1, \dots, b_k \rangle$ where $\forall a \in A, a \in \mathbf{C}$ and $\forall b \in B, b \in \mathbf{C}$. To compute the similarity score between two atoms we define $\mathtt{simA}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle} : \bigcup_{j,k=1}^{+\infty} \mathbf{P} \times \mathbf{C}^j \times \mathbf{P} \times \mathbf{C}^k \rightarrow [0, 1]$ s.t. $\mathtt{simA}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}(P_1, A, P_2, B) = \mathbf{g}\Big(\mathbf{p}(P_1, P_2), \mathbf{c}\big(A, B\big)\Big)$.

A possible $\mathbf{p}$ is the function returning 1 if the predicates are the same, 0 otherwise.

Definition 3.2 (Function Equal). Let x, y be two arbitrary objects. The function $\mathtt{eq} : \mathbb{X} \times \mathbb{X} \rightarrow \{0, 1\}$ is defined by $\mathtt{eq}(x, y) = 1$ if $x = y$; or $\mathtt{eq}(x, y) = 0$ otherwise.

We propose an instance of function $\mathbf{c}$ suited to vectors of objects.

Definition 3.3 (Pointwise Similarity). Let $X = \langle x_1, \dots, x_j \rangle$, $Y = \langle y_1, \dots, y_k \rangle$ be arbitrary vectors of objects. The *pointwise similarity* between X and Y is:

$$\mathtt{pws}(X, Y) = \begin{cases} 1 & \text{if X and Y are both empty} \\ \frac{\sum_{i=1}^{\mathtt{min}(j,k)} \mathtt{eq}(x_i, y_i)}{\mathtt{max}(j,k)} & \text{otherwise} \end{cases}$$

Having a similarity score between two atoms, we propose to use the polarities as binary factors of acceptance or not of the similarity between atoms.

Definition 3.4 (Similarity between Literals). Let two literals $l_1, l_2 \in \mathbf{L}$. We define $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle} : \mathbf{L} \times \mathbf{L} \rightarrow [0, 1]$, the similarity measure between two literals according to a similarity measure between atoms $\mathtt{simA}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}$ s.t.: $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}(l_1, l_2) =$

$$\begin{cases} \mathtt{simA}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}(\mathtt{Pred}(l_1), \mathtt{Param}(l_1), \\ \qquad \mathtt{Pred}(l_2), \mathtt{Param}(l_2)) & \text{if } \mathtt{Pol}(l_1) = \mathtt{Pol}(l_2) \\ 0 & \text{otherwise} \end{cases}$$

Example 3.5. $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws} \rangle}(P(A, B), \neg P(A, C)) = 0$ because the polarity is not the same. Conversely, we have $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws} \rangle}(P(A, B), P(A, C)) = \frac{1}{2}$ because $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws} \rangle}(P(A, B), P(A, C)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws} \rangle}(P, \langle A, B \rangle, P, \langle A, C \rangle) = \mathtt{min}(\mathtt{eq}(P, P), \mathtt{pws}(\langle A, B \rangle, \langle A, C \rangle)) = \mathtt{min}(1, \frac{\mathtt{eq}(A,A)+\mathtt{eq}(B,C)}{2}) = \mathtt{min}(1, \frac{1}{2}) = \frac{1}{2}$.

3.2 Similarity between grounded clauses

From the level two of the definition of our similarity measures on arguments, we will need several mathematical tools that can be defined in an abstract way. In this part, we apply these tools only for level 2 (the comparison of two CNF formulae), but they will be applicable also at the next levels. Let us start with the notion of aggregation function.

Definition 3.6 (Aggregation Function). We say that $\oplus$ is an aggregation function if $\forall k \in \mathbb{N}$, $\oplus$ is a mapping $[0, 1]^k \rightarrow [0, 1]$ s.t.

– if $x_i \geq x'_i$, then $\oplus(x_1,\ldots,x_i,\ldots,x_k) \geq \oplus(x_1,\ldots,x'_i,\ldots,x_k)$ **(non-decreasingness)**
– if $\forall i \in \{1,\ldots,k\}$, $x_i = 0$ then $\oplus(x_1,\ldots,x_k) = 0$ **(minimality)**
– $\oplus(x) = x$ **(identity)**

These properties are satisfied by *e.g.* $\mathtt{min}$, $\mathtt{max}$ and $\mathtt{avg}$.

Now we introduce the notion of *membership* function which expresses how much an object is similar to the elements of a set.

Definition 3.7 (Membership Function). Given $\mathbb{X}$ a set of objects, $x \in \mathbb{X}$ an object, $X \subseteq \mathbb{X}$, $\oplus$ an aggregation function and $\mathtt{sim}$ a similarity measure the *membership function* of x in X, $\varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}} : \mathbb{X} \times 2^{\mathbb{X}} \rightarrow [0,1]$ is defined by: $\varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}(x,X) = \oplus_{x' \in X}(\mathtt{sim}^{\mathbb{X}}(x,x'))$.

It is interesting to note that classical set-membership can be captured by $\varepsilon_{\mathtt{max},\mathtt{eq}}$ where $\mathtt{eq}$ is the equality function from Definition 3.2.Now we can evaluate how much a literal is similar to a clause, *i.e.* a set of literals: given $l \in \mathbf{L}$ a literal, $L \subseteq \mathbf{L}$ a set of literals and $\oplus^1$ an aggregation function, we define the function $\mathtt{s}^{\mathbf{L}} = \varepsilon^{\mathbf{L}}_{\oplus^1,\mathtt{simL}^{\langle \mathrm{g,p,c} \rangle}}$. Then, the similarity between two grounded clauses is computed by $\mathtt{simC}^{\mathtt{s}^{\mathbf{L}}}$.

Tversky [38] proposed the "ratio model", a general similarity measure which encompasses different well known similarity measure as the Jaccard measure [25], Dice measure [17], Sorensen one [37], Symmetric Anderberg [9] and Sokal and Sneath 2 [36]. We propose to extend it in two different ways. Firstly, instead of using the usual operators of membership of an element to a set, we propose to use our parameterisable membership function ε (see Definition 3.7). Then a new parameter γ is added allowing us to vary these scores in an increasing or decreasing way only in the cases where the sets of objects are partially similar.

Definition 3.8 (Extended Tversky Measure). Let $X, Y \subseteq \mathbb{X}$ be arbitrary sets of objects. Let $\varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}$ be a membership function with $\oplus$ an aggregation function and $\mathtt{sim}$ a similarity measure. We denote by $\mathtt{avg}$ the average function. Let $a = \mathtt{avg}\Big(\sum_{x \in X} \varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}(x,Y), \sum_{y \in Y} \varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}(y,X)\Big)$, $b = \sum_{x \in X} 1 - \varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}(x,Y)$, $c = \sum_{y \in Y} 1 - \varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}(y,X)$, and $\alpha, \beta \in [0,+\infty[$, $\gamma \in]0,+\infty[$. The *extended Tversky measure* between X and Y is:

$$\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}}(X,Y) = \begin{cases} 1 & \text{if } X = Y = \emptyset \\ \left(\frac{a}{a+\alpha\cdot b+\beta\cdot c}\right)^{\gamma} & \text{otherwise} \end{cases}$$

Classical similarity measures (see Table 1 in [4] for the definitions) can be obtained with $\alpha = \beta = 2^{-n}$ and the classical set-membership. In particular, the Jaccard measure (i.e. $\mathtt{jac}$) is obtained with $n = 0$, Dice (i.e. $\mathtt{dic}$) with $n = 1$, Sorensen (i.e. $\mathtt{sor}$) with $n = 2$, Anderberg (i.e. $\mathtt{adb}$) with $n = 3$, and Sokal and Sneah 2 (i.e. $\mathtt{ss}_2$) with $n = -1$.

Under some reasonable assumptions, Tversky measure s.t. $\alpha = \beta$ are symmetric.

Proposition 3.9. *For any $X, Y \subseteq \mathbb{X}$, any $\gamma \in]0,+\infty[$, any membership function $\varepsilon^{\mathbb{X}}_{max,\mathtt{sim}}$ s.t. $\mathtt{sim}$ is symmetric, we have $\mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{max,\mathtt{sim}}}(X,Y) = \mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{max,\mathtt{sim}}}(Y,X)$.*

Proof. Compute $T_1 = \mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{max,\mathtt{sim}}}(X,Y) = \frac{a_1}{a_1+\alpha.b_1+\alpha.c_1}$, and $T_2 = \mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{max,\mathtt{sim}}}(Y,X) = \frac{a_2}{a_2+\alpha.b_2+\alpha.c_2}$. From Definition 3.8, we deduce $a_1 = a_2$, $b_1 = c_2$, and $c_1 = b_2$. So it is easy to conclude that $T_1 = T_2$. □

Table 1: Set of parametric (non-)symmetric measures

Symmetric Measures	Non-Symmetric Measures
$\mathtt{Tve}^{1,1,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{jac}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$	$\mathtt{Tve}^{0,1,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{ns}-\mathtt{jac}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$
$\mathtt{Tve}^{\frac{1}{2},\frac{1}{2},\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{dic}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$	$\mathtt{Tve}^{0,\frac{1}{2},\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{ns}-\mathtt{dic}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$
$\mathtt{Tve}^{\frac{1}{4},\frac{1}{4},\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{sor}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$	$\mathtt{Tve}^{0,\frac{1}{4},\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{ns}-\mathtt{sor}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$
$\mathtt{Tve}^{\frac{1}{8},\frac{1}{8},\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{adb}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$	$\mathtt{Tve}^{0,\frac{1}{8},\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{ns}-\mathtt{adb}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$
$\mathtt{Tve}^{2,2,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{ss_2}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$	$\mathtt{Tve}^{0,2,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = \mathtt{ns}-\mathtt{ss_2}^{\gamma,\mathtt{sim}^{\mathbb{X}}}(X,Y)$

In the rest of the paper we will focus our study on membership function using the aggregator function $\mathtt{max}$. Table 1 denotes the set of parametric (non-)symmetric extended versions of the well known similarity measures, where fixing α and β corresponds to choosing among Jaccard, Dice, Sorensen, Anderberg, or Sokal and Sneah. The other parameters of the different similarity measures are only the coefficient γ and the similarity function $\mathtt{sim}^{\mathbb{X}}$. Please note that γ allows us to have a lower evaluation between a set of literals than a set of clauses (or instantiations), i.e. when sets of objects are interpreted disjunctively or conjunctively. Let us prove that any such measure satisfies some intuitive properties: two sets are maximally similar if they are identical (in the symmetric case), or at least included in one another (non-symmetric case).

Proposition 3.10. *If $\mathtt{sim}^{\mathbb{X}}$ satisfies Maximality [4], then, for any $\gamma \in]0,+\infty[$, $\alpha \neq 0$,*
– if $Y = X$, then $\mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = 1$ (symmetric measures),
– if $Y \subseteq X$, then $\mathtt{Tve}^{0,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = 1$ (non-symmetric measures).

Proof. First, we consider symmetric measures, *i.e.* $\mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}$ for any α. Assume $X = Y$. If $X = \emptyset$, then $\mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = 1$ by definition, so now we also assume that $X \neq \emptyset$. It is easy to see that $b = c = 0$, because for each $x \in X$, $\varepsilon_{\mathtt{max},\mathtt{sim}^{\mathbb{X}}}(x,X) = 1$ for any $\mathtt{sim}^{\mathbb{X}}$ satisfying Maximality, *i.e.* such that $\mathtt{sim}^{\mathbb{X}}(x,x) = 1$. For similar reasons, we are sure that $a = |X| > 0$. So $\mathtt{Tve}^{\alpha,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}(X,Y) = (\frac{a}{a})^{\gamma} = 1$.

Now, consider asymmetric measures, *i.e.* $\mathtt{Tve}^{0,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}$.Assume that $Y \subseteq X$. For the same reason as the symmetric case, we have $c = 0$. So $\mathtt{Tve}^{0,\alpha,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max,sim}}}$
$= (\frac{a}{a+0\times b+\alpha\times c})^{\gamma} = (\frac{a}{a})^{\gamma} = 1$.

□

Example 3.11. Let $P_1 = P(A,B)$, $P_2 = P(A,C)$ and $P_3 = P(C,B)$. Let $\mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}$. $\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}}(P_1, P_2 \vee P_3) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}}(P_1, P_2 \vee P_3) = \frac{a}{a+b+c} = \frac{1}{3}$ with :
– $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_1, P_2 \vee P_3), \varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_2,P_1) + \varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_3,P_1)) = \mathtt{avg}(\frac{1}{2}, 1) = \frac{3}{4}$,
– $b = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_1, P_2 \vee P_3) = \frac{1}{2}$,
– $c = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_2,P_1)) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_3,P_1)) = \frac{1}{2} + \frac{1}{2} = 1$, with $\varepsilon^{\mathbf{L}}_{\mathtt{max,s^L}}(P_1, P_2 \vee P_3) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(P_1,P_2), \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}$

$(P_1, P_3)) = \frac{1}{2}$ and $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(P_1, P_2) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(P_1, P_2)) = \frac{1}{2}$ (idem for $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(P_1, P_3)$).

3.3 Similarity between grounded clauses

We introduce $\mathbb{C}$ the set of all grounded clauses in $\mathtt{OS-FOL}$.

Definition 3.12 (Grounded clause membership). Let $\delta \in \mathbb{C}$ be a grounded clause and $\Delta \subseteq \mathbb{C}$ be a set of grounded clauses. Let an aggregation function $\oplus^{\mathtt{c}}$ and a similarity measure between a pair of clauses $\mathtt{s}^{\mathbb{C}} = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}$, with $\mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}$. The membership function of a grounded clause in a set of grounded clauses, denoted $\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}} : \mathbb{C} \times 2^{\mathbb{C}} \to [0,1]$, is $\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}(\delta, \Delta) = \oplus^{\mathtt{c}}_{\delta' \in \Delta}(\mathtt{s}^{\mathbb{C}}(\delta, \delta'))$.

Definition 3.13 (Similarity between sets of grounded clauses). Let $\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}$ be a membership function with $\mathtt{s}^{\mathbb{C}} = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}$ and $\mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}$. A similarity measure between two sets of grounded clauses is defined as $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}} : 2^{\mathbb{C}} \times 2^{\mathbb{C}} \to [0,1]$.

3.4 Similarity between instantiations

Now, define $\mathbb{I}$ the set of all instantiations in $\mathtt{OS-FOL}$.

Definition 3.14 (Instantiation membership). Let an instantiation $\Delta \in \mathbb{I}$ and a set of instantiations $I \subseteq \mathbb{I}$. Let an aggregation function $\oplus^{\mathtt{i}}$ and a similarity measure between a pair of set of clauses $\mathtt{s}^{\mathbb{I}} = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}}$ with $\mathtt{s}^{\mathbb{C}} = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}$ and $\mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}$. The membership function of an instantiation in a set of instantiations, $\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}} : \mathbb{I} \times 2^{\mathbb{I}} \to [0,1]$, is $\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}(\Delta, I) = \oplus^{\mathtt{i}}_{\Delta' \in I}(\mathtt{s}^{\mathbb{I}}(\Delta, \Delta'))$.

Definition 3.15 (Similarity between sets of instantiations). Let $\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}$ be a membership function with $\mathtt{s}^{\mathbb{I}} = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}}$, $\mathtt{s}^{\mathbb{C}} = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}$ and $\mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}$. The similarity measure between two set of instantiations is defined as $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}} : 2^{\mathbb{I}} \times 2^{\mathbb{I}} \to [0,1]$.

Let us now define a similarity measure between sets of formulae.

Definition 3.16 (Similarity Models). A *Similarity Model* ($\mathtt{SM}$) is a tuple $\mathbf{M} = \langle \mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}, \mathtt{s}^{\mathbb{C}} = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}, \mathtt{s}^{\mathbb{I}} = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}}, \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}} \rangle$. Let two sets of formulae $\Phi, \Psi \subseteq \mathtt{OS-FOL}$ and $\mathbf{I_{St}}$ an interpretation over a structure $\mathbf{St}$. The similarity between Φ and Ψ is $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\Phi, \Psi) = \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi), \mathtt{Inst}_{\mathbf{I_{St}}}(\Psi))$.

Finally, using the measure of similarity between sets of formulae, we can extend the definition from [4] to asses the similarity between two $\mathtt{OS-FOL}$ arguments.

Definition 3.17 (Similarity between OS-FOL Arguments). Let a coefficient $0 < \eta < 1$ and a $\mathtt{SM}$ $\mathbf{M}$ and $\mathbf{I_{St}}$ an interpretation over a structure $\mathbf{St}$. We define $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta} : \mathtt{Arg(OS-FOL)} \times \mathtt{Arg(OS-FOL)} \to [0,1]$ by
$\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, B) = \eta \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(B)))$
$+(1-\eta) \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B)))$.

Example 3.18. Let $\mathbf{M}_{\mathtt{jac}} = \langle \mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}, \mathtt{s}^{\mathbb{C}} = \mathtt{jac}^{2,\mathtt{s}^{\mathbf{L}}}, \mathtt{s}^{\mathbb{I}} = \mathtt{jac}^{1,\mathtt{s}^{\mathbb{C}}}, \mathtt{jac}^{1,\mathtt{s}^{\mathbb{I}}} \rangle$ be a similarity instantiation model and let A_1 and A_2 be the two OS-FOL arguments from Example 2.11. Their respective instantiations are given in Example 2.8 for the premises and the conclusions. Let us compute the similarity between A_1 and A_2 with $\eta = 0.5$.
$\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}},0.5}(A_1, A_2) = 0.5 \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Supp}(A_1), \mathtt{Supp}(A_2))$
$+0.5 \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Conc}(A_1), \mathtt{Conc}(A_2)) = 0.5 \cdot \frac{73}{1143} + 0.5 \cdot \frac{5}{11} \simeq 0.2592$
where $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Supp}(A_1), \mathtt{Supp}(A_2)) =$
$\mathtt{jac}^{1,\mathtt{s}^{\mathbb{I}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Supp}(A_1)), \mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Supp}(A_2))) = \frac{73}{1143} \simeq 0.064$
and $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Conc}(A_1), \mathtt{Conc}(A_2)) =$
$\mathtt{jac}^{1,\mathtt{s}^{\mathbb{I}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Conc}(A_1)), \mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Conc}(A_2))) = \frac{5}{11} \simeq 0.4545$

4 AXIOMATIC EVALUATION

Before determining the principles satisfied by our similarity measures, we introduce the notion of well-behaved $\mathtt{SM}$. It is a bridge between the (lower level) properties of the measures that we use (*e.g.* the Tversky measures) and the (higher level) properties of the similarity measure between arguments defined from such a $\mathtt{SM}$.

Definition 4.1 (Well-Behaved $\mathtt{SM}$). A $\mathtt{SM}$ $\mathbf{M} = \langle \mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c} \rangle}, \mathtt{s}^{\mathbb{C}} = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}, \mathtt{s}^{\mathbb{I}} = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}}, \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}} \rangle$ is well-behaved iff the following holds:

(1)(a)(i) $\mathbf{g}(1,1) = 1$,
(ii) $\mathbf{g}(0,0) = 0$,
(b)(i) $\mathbf{p}(P,P) = 1$,
(ii) $\mathbf{p}(P,Q) = 0$ iff $P \neq Q$,
(c)(i) $\mathbf{c}(\langle a_1, \ldots, a_k \rangle, \langle a_1, \ldots, a_k \rangle) = 1$,
(ii) if $\forall i \in \{1, \ldots, k\}$, $\nexists j \in \{1, \ldots, n\}$ s.t. $a_i = b_j$ then $\mathbf{c}(\langle a_1, \ldots, a_k \rangle, \langle b_1, \ldots, b_n \rangle) = 0$,

(2) Given $\mathbb{X}$ a set of objects,
(a) $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X, X) = 1$ for any set of objects $X \subseteq \mathbb{X}$,
(b) if $\forall x \in X, \forall x' \in X'$, $\mathtt{s}(x, x') = 0$ then $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X, X') = 0$,
(c) let $X_0, X_1, X_2 \subseteq \mathbb{X}$ s.t. $X_1 \subset X_2$ and $X_2 \setminus X_1 = \{x_2\}$. If $\exists x_0 \in X_0$ s.t. $\mathtt{s}(x_0, x_2) = \mathtt{s}(x_2, x_0) = 1$ then $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0, X_2) \geq \mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0, X_1)$,
(d) let $X_0, X_1, X_2 \subseteq \mathbb{X}$ s.t. $X_1 \subset X_2$ and $X_2 \setminus X_1 = \{x_2\}$. If $\forall x_0 \in X_0$, $\mathtt{s}(x_0, x_2) = \mathtt{s}(x_2, x_0) = 0$ then $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0, X_1) \geq \mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0, X_2)$.

In the last item, $\mathbb{X}$ can be the set of all literals (for characterizing $\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathtt{l}},\mathtt{s}^{\mathbf{L}}}}$), the set of all grounded clauses (for characterizing $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathtt{c}},\mathtt{s}^{\mathbb{C}}}}$) or the set of instantiations (for characterizing $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}}$). Now we can show that a well-behaved $\mathtt{SM}$ guarantees that the corresponding similarity measure satisfies some principles.

THEOREM 4.2. *For any $\mathbf{M} \in \mathtt{SM}$, if $\mathbf{M}$ is well-behaved then $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}$ satisfies the following principles: Maximality, Minimality, Monotony and Dominance.*

PROOF. Assume that $\mathbf{M}$ is a well-behaved $\mathtt{SM}$.

- Maximality is satisfied from Definition 4.1, item 2.(a).
  $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(A))$:
  Given that $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(A)) = \Phi$ then for any $\mathbf{I_{St}}$, $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi)$. Therefore from Definition 4.1, item 2.(a) when $\mathbb{X} = \mathbb{I}$, $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathtt{i}},\mathtt{s}^{\mathbb{I}}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi), \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi)) = 1$, i.e.

Table 2: Satisfaction of the principles of similarity measures. The symbol • (resp. ∘) means the measure satisfies (resp. violates) the principle. $\mathtt{sim}_x$ **is a shorthand for** $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$**.**

	$\mathtt{sim}_{\mathtt{jac}}$	$\mathtt{sim}_{\mathtt{dic}}$	$\mathtt{sim}_{\mathtt{sor}}$	$\mathtt{sim}_{\mathtt{adb}}$	$\mathtt{sim}_{\mathtt{ss_2}}$	$\mathtt{sim}_{\mathtt{ns-jac}}$	$\mathtt{sim}_{\mathtt{ns-dic}}$	$\mathtt{sim}_{\mathtt{ns-sor}}$	$\mathtt{sim}_{\mathtt{ns-adb}}$	$\mathtt{sim}_{\mathtt{ns-ss_2}}$
Maximality	•	•	•	•	•	•	•	•	•	•
Symmetry	•	•	•	•	•	∘	∘	∘	∘	∘
Substitution	•	•	•	•	•	∘	∘	∘	∘	∘
Syntax Independence	•	•	•	•	•	•	•	•	•	•
Minimality	•	•	•	•	•	•	•	•	•	•
Monotony	•	•	•	•	•	•	•	•	•	•
Strict Monotony	•	•	•	•	•	∘	∘	∘	∘	∘
Dominance	•	•	•	•	•	•	•	•	•	•
Strict Dominance	•	•	•	•	•	∘	∘	∘	∘	∘

$\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(A))) = 1$.

$\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$:
Given that $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B)) = \Phi$ then for any $\mathbf{I_{St}}$, $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi)$. Therefore from Definition 4.1, item 2.(a) when $\mathbb{X} = \mathbb{I}$, $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},s^{\mathbb{I}}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi), \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi)) = 1$, i.e.
$\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(A))) = 1$.
Hence, for any $\eta \in]0,1[$, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, A) = 1$.

- Minimality is satisfied (from 1.(a)ii. + 1(b)ii. + 1(c)ii. + Def 3.1 + Def 3.4 + 2.(b)). Let $A, B \in \mathtt{Arg(OS-FOL)}$, such that:
  (1) A and B are not trivial,
  (2) $\forall s_i \in \mathtt{Dom}(\mathtt{Supp}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Supp}(B))$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$,
  (3) $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Conc}(B))$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$.

  Item 1 ensure that there exist content to evaluate between the support, in order to measure a 0 similarity. In the case of both empty set (both trivial argument), it can be acceptable to assign similarity between the supports, that why we exclude this case.

  Item 2 ensure that there is no common predicates and constants between $\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)))$ and $\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(B)))$, i.e. for any computation in the level 1 by $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$:
  - for any $P_1 \in \mathtt{Pred}(\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A))))$, $P_2 \in \mathtt{Pred}(\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(B))))$, from Definition 4.1 item 1.(b)ii., $\mathbf{p}(P_1, P_2) = 0$.
  - for any c_1 set of all possible constant in $\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)))$ and c_2 set of all possible constant in $\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(B)))$, from Definition 4.1 item 1.(b)ii., for any vector of constants c_a from c_1 and any vectors c_b from c_2, $\mathbf{c}(c_a, c_b) = 0$.

  Therefore, for the computation of $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(B)))$, at the level 1 for computation of $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$ as we saw before $\mathbf{p}$ and $\mathbf{c}$ are always equal to 0. Hence from Definition 4.1 item 1.(a)ii. and from Definition 3.4, any $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$ are also equal to 0.
  Then, from Definition 4.1 item 2.(b), successively with $\mathbb{X} = \mathbf{L}$, $\mathbb{X} = \mathbb{C}$ and $\mathbb{X} = \mathbb{I}$, we obtain at each level that for each $\mathtt{simC} = 0$, $\mathtt{simI} = 0$ and $\mathtt{simSI} = 0$, i.e. $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(B))) = 0$.

  Item 3 ensure that there is no common predicates and constants between $\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)))$ and $\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(B)))$. Using the same reasoning as for item 2 but on the conclusion, we obtain that $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B))) = 0$.

  Therefore, for any $\eta \in]0,1[$, from Definition 3.17: $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, B) = \eta \cdot 0 + (1-\eta) \cdot 0 = 0$.

- Monotony:
  Let $A, B, C, A^*, B^*, C^* \in \mathtt{Arg(OS-FOL)}$ such that
  (1) $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$ or $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Conc}(C))$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$,
  (2) $\mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(C)) \subseteq \mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(B))$,
  (3) for $B_A = \mathtt{UC}(\mathtt{Supp}(B)) \setminus \mathtt{UC}(\mathtt{Supp}(A))$ and $C_A = \mathtt{UC}(\mathtt{Supp}(C)) \setminus \mathtt{UC}(\mathtt{Supp}(A))$, $B_A \subseteq C_A$, $C_A \setminus B_A \subseteq \mathbb{C}$ and $\forall s_i \in \mathtt{Dom}(\mathtt{Supp}(A))$, $\nexists s_j \in \mathtt{Dom}(C_A \setminus B_A)$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$,
  For any $A, B \in \mathtt{Arg(OS-FOL)}$, $\eta \in]0,1[$ and a well-behaved $\mathbf{M} \in \mathtt{SM}$, from Definition 3.17: $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, B) = \eta \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(B))) + (1-\eta) \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B)))$.

  From Definition 3.16, if $\Phi, \Psi \subseteq \mathtt{OS-FOL}$ then

  $$\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\Phi, \Psi) = \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},s^{\mathbb{I}}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi), \mathtt{Inst}_{\mathbf{I_{St}}}(\Psi))$$

  Item 1:

  Case 1: $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$**.**
  Given that $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B)) = \Phi$ then for any $\mathbf{I_{St}}$, $\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi) = \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi)$. Therefore from Definition 4.1, item 2.(a) when $\mathbb{X} = \mathbb{I}$, $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},s^{\mathbb{I}}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\Phi), \mathtt{Inst}_{\mathbf{I_{St}}}(\Phi)) = 1$, i.e. $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B))) = 1$.
  Case 2: $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(\mathtt{Conc}(B))$ **s.t.** $s_i \prec s_j$ **or** $s_j \prec s_i$ **or** $s_j = s_i$**.**

As showed for Item 3 in the proof of Minimality, this condition ensure that $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Conc}(A)), \mathsf{UC}(\mathsf{Conc}(B))) = 0$.

For these both cases, then for any $\eta \in]0,1[$, $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Conc}(A)), \mathsf{UC}(\mathsf{Conc}(B))) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Conc}(A)), \mathsf{UC}(\mathsf{Conc}(C)))$. Hence, if $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(B))) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(C)))$, then $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A,B) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A,C)$.

Item 2 and 3:

To show that $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(B))) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(C)))$ we breakdown the proof in two steps.

In step 1, from Item 3 we will show that: $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup C_A) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(C)))$

In step 2, from Item 2 we will show that: $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(B))) = \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup X) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A)$ where $X = \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(B)) \setminus \big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C))\big)$

Therefore, by transitivity we will have: $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(B))) = \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup X) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup C_A) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(C)))$, i.e. $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(B))) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathsf{UC}(\mathsf{Supp}(A)), \mathsf{UC}(\mathsf{Supp}(C)))$

Step 1:

Let us begin by show that:
$\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}}\Big(\mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A))\big), \mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A\big)\Big) \geq \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}}\Big(\mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A))\big), \mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup \{\delta'\}\big)\Big)$ Where $\delta' \in C_A \setminus B_A$.

From Definition 2.7, item 4, when we add a formula ϕ to a set of formulae Φ we create all the consistent combination of instantiation of ϕ with Φ. Given that a grounded clause have only one instantiation (itself), adding a consistent grounded clause to a set of formulae implies in term of instantiation to add this grounded clauses to each instantiation of the set of formulae.
From item 3 we know that $C_A \setminus B_A \subseteq \mathbb{C}$ then $\delta' \in \mathbb{C}$.
From Definition 2.10, we know that the support of an argument is consistent, then $\delta' \in \mathsf{Supp}(C)$ is consistent with $\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \subset \mathsf{Supp}(C)$.

Hence, $\mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup \{\delta'\}\big) =$

$\bigcup_{\Delta \in \mathtt{Inst}_{\mathbf{I_{St}}}\left(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A\right)} \Delta \cup \delta'$

Let us see now that we can apply item 2.(d) of Definition 4.1. We recall it at follow. Let $\Delta_0, \Delta_1, \Delta_2 \in \mathbb{I}$ s.t. $\Delta_1 \subset \Delta_2$ and $\Delta_2 \setminus \Delta_1 = \{\delta_2\}$. If $\forall \delta_0 \in \Delta_0$, $\mathrm{s}^{\mathbb{C}}(\delta_0, \delta_2) = \mathrm{s}^{\mathbb{C}}(\delta_2, \delta_0) = 0$ then $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathrm{c}},\mathrm{s}^{\mathbb{C}}}}(\Delta_0, \Delta_1) \geq \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathrm{c}},\mathrm{s}^{\mathbb{C}}}}(\Delta_0, \Delta_2)$.

We show that, $\forall \Delta_i \in \mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A))\big)$, $\forall \Delta_j \in \mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A\big)$, $\forall \delta' \in C_A \setminus B_A$, $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathrm{c}},\mathrm{s}^{\mathbb{C}}}}(\Delta_i, \Delta_j) \geq \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{\mathrm{c}},\mathrm{s}^{\mathbb{C}}}}(\Delta_i, \Delta_j \cup \delta')$ given that: $\forall \delta_i \in \Delta_i$, $\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathrm{l}},\mathrm{s}^{\mathbf{L}}}}(\delta_i, \delta') = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\oplus^{\mathrm{l}},\mathrm{s}^{\mathbf{L}}}}(\delta', \delta_i) = 0$ because:
$\forall l_i \in \delta_i, \forall l' \in \delta'$, $\mathtt{simL}^{\langle \mathrm{g,p,c} \rangle}(l_i, l') = \mathtt{simL}^{\langle \mathrm{g,p,c} \rangle}(l', l_i) = 0$ from item 1.(a)ii., 1(b)ii. and 1(c)ii of Definition 4.1, because:
From item 3 we know that $\forall s_i \in \mathsf{Dom}(\mathsf{Supp}(A))$, $\nexists s_j \in \mathsf{Dom}(C_A \setminus B_A)$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$, then there not exist common predicate and common constant between their sets of instantiations.

Let us see now, how to generalise the previous result with one additional different element (δ') to the addition of a set of different element ($C_A \setminus B_A$).

According to the item 2.(d) of Definition 4.1, assume that $\Delta_0 = \mathsf{UC}(\mathsf{Supp}(A))$ and successively (in n steps with n the size of $C_A \setminus B_A$) we will increase Δ_1^i and Δ_2^i (with $i \in \{1, \ldots, n\}$). Let us first take:
$\Delta_1^1 = \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A$ and $\Delta_2^1 = \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup \{\delta_1'\}$, then we take:
$\Delta_1^2 = \Delta_2^1$ and $\Delta_2^2 = \mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup \{\delta_1', \delta_2'\}$, until getting: $\Delta_1^n = \Delta_2^{n-1}$ and $\Delta_2^n = \mathsf{UC}(\mathsf{Supp}(C))$.
Therefore by transitivity and iteratively we obtain that
$\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}}\Big(\mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A))\big), \mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A\big)\Big) \geq \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}}\Big(\mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A))\big), \mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A)) \cap \mathsf{UC}(\mathsf{Supp}(C)) \cup B_A \cup \{\delta_1'\}\big)\Big) \geq \mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}}\Big(\mathtt{Inst}_{\mathbf{I_{St}}}\big(\mathsf{UC}(\mathsf{Supp}(A))\big),

$\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \delta'_2\}\big)\Big) \geq$
$\ldots \geq$
$\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

$\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \ldots, \delta'_n\}\big)\Big)$
where $\{\delta'_1, \ldots, \delta'_n\} = C_A \setminus B_A$.

Step 2:
For the second step we can apply the same reasoning at the first step but using item 2.(c) of Definition 4.1 instead of item 2.(d) and using item 1.(a)i., 1(b)i. and 1(c)i instead of item 1.(a)ii., 1(b)ii. and 1(c)ii of Definition 4.1.
Then we obtain:
$\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

$\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \ldots, \delta'_n\}\big)\Big) \geq$
$\ldots \geq$
$\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

$\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \delta'_2\}\big)\Big) \geq$
$\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

$\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1\}\big)\Big) \geq$
$\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

$\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A\big)\Big)$
where $\{\delta'_1, \ldots, \delta'_n\} = \text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(B)) \setminus \big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C))\big)$.

- Dominance: with the same reasoning as for the proof of Monotony, Dominance holds. Item 1 of Dominance ensure that for any $\eta \in]0, 1[$, $\text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Supp}(A)), \text{UC}(\text{Supp}(B))) = \text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Supp}(A)), \text{UC}(\text{Supp}(C)))$.
Hence, if $\text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Conc}(A)), \text{UC}(\text{Conc}(B))) \geq \text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Conc}(A)), \text{UC}(\text{Conc}(C)))$, then $\text{sim}^{\text{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, B) \geq \text{sim}^{\text{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, C)$.

  Item 2 and 3 are identical as in Monotony but applied between conclusion. Therefore we can conclude that $\text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Conc}(A)), \text{UC}(\text{Conc}(B))) \geq \text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Conc}(A)), \text{UC}(\text{Conc}(C)))$.

$\square$

To satisfy other principles we propose additional constraints.

Theorem 4.3. *Let a well-behaved* $\mathbf{M} \in \text{SM}$ *and* $\text{sim}^{\text{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}$ *a similarity based on* $\mathbf{M}$.
– $\text{sim}^{\text{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}$ *satisfies Symmetry (resp. Syntax Independence) if all the functions in* $\mathbf{M}$ *are symmetric (resp. syntax independent).*
– $\text{sim}^{\text{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}$ *satisfies Strict Monotony and Strict Dominance if it satisfies condition 2.(c'): let* $X_0, X_1, X_2 \subseteq \mathbb{X}$ *s.t.* $X_1 \subset X_2$ *and* $X_2 \setminus X_1 = \{x_2\}$. *If* $\text{sim}^{\varepsilon,\text{s}}(X_0, X_1) < 1$ *and* $\exists x_0 \in X_0$ *s.t.* $\text{s}(x_0, x_2) = \text{s}(x_2, x_0) = 1$ *then* $\text{sim}^{\varepsilon,\text{s}}(X_0, X_2) > \text{sim}^{\varepsilon,\text{s}}(X_0, X_1)$.

Proof. Assume that $\mathbf{M}$ is a well-behaved SM.

- Symmetry is satisfied if $\text{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$, $\text{simC}^{\varepsilon^{\mathbf{L}}}_{\oplus^{\mathrm{l}},\mathrm{s}^{\mathbf{L}}}$, $\text{simI}^{\varepsilon^{\mathbb{C}}}_{\oplus^{\mathrm{c}},\mathrm{s}^{\mathbb{C}}}$, $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}$ are all symmetric.
- Syntax Independence is satisfied if $\text{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$, $\text{simC}^{\varepsilon^{\mathbf{L}}}_{\oplus^{\mathrm{l}},\mathrm{s}^{\mathbf{L}}}$, $\text{simI}^{\varepsilon^{\mathbb{C}}}_{\oplus^{\mathrm{c}},\mathrm{s}^{\mathbb{C}}}$, $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}$ are all syntax independent.
- Strict Monotony: using the same reasoning as in the proof of monotony and thanks to the guarantee $\text{sim}^{\varepsilon,\text{s}}(X_0, X_1) < 1$, we obtain for the first step that:

  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A\big)\Big) >$
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1\}\big)\Big) >$
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \delta'_2\}\big)\Big) >$
  $\ldots >$
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \ldots, \delta'_n\}\big)\Big)$
  where $\{\delta'_1, \ldots, \delta'_n\} = C_A \setminus B_A$.
  Hence, $\text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Supp}(A)), \text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A) > \text{sim}^{\text{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\text{UC}(\text{Supp}(A)), \text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup C_A) = \text{sim}^{\text{Arg(OS-FOL)}}(\text{UC}(\text{Supp}(A)), \text{UC}(\text{Supp}(C)))$.

  Then at the second step:
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \ldots, \delta'_n\}\big)\Big) >$
  $\ldots >$
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1, \delta'_2\}\big)\Big) >$
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A \cup \{\delta'_1\}\big)\Big) >$
  $\text{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{\mathrm{i}},\mathrm{s}^{\mathbb{I}}}\Big(\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A))\big),$

  $\text{Inst}_{\mathbf{I_{St}}}\big(\text{UC}(\text{Supp}(A)) \cap \text{UC}(\text{Supp}(C)) \cup B_A\big)\Big)$

where $\{\delta'_1, \ldots, \delta'_n\} = \mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(B)) \setminus \big(\mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(C))\big)$.

Therefore when we combine the two steps we have: $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(B))) > \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(C)))$
Moreover, for both cases of item 1 in strict monotony, we have that for any $\eta \in]0, 1[$, $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B))) \geq \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M},\mathbf{I_{St}}}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(C)))$.

Finally we obtain $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, B) > \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, C)$.

- Strict Dominance: with the same reasoning as in Dominance and Strict Monotony we obtain that combining the two steps we have

$$\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, B) > \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A, C).$$

□

We extend also some previous results from [4].

Proposition 4.4. *Let* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ *a similarity measure.*
– Let $A, B \in \mathtt{Arg(OS-FOL)}$*, if* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ *satisfies Maximality, Monotony, Strict Monotony and Strict Dominance then* $A \approx B$ *iff* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$.
– If $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ *satisfies Symmetry, Maximality, Strict Monotony, Dominance, and Strict Dominance then* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ *satisfies Substitution.*

Proof.
- Let $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ be a similarity measure which satisfies Maximality and Monotony. Let $A, B \in \mathtt{Arg(OS-FOL)}$ be such that $A \approx B$. Let us show that $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$. From Definition 2.15, $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(B))$ and $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$. From Monotony, it follows that $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, A) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B)$ and $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, A)$. Therefore, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, A) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B)$. From Maximality, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, A) = 1$, so $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$.
- If $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ satisfies Maximality, Strict Monotony and Strict Dominance then if $A, B \in \mathtt{Arg(OS-FOL)}$ s.t. $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$ then $A \approx B$.

  Let $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ be a similarity measure which satisfies Maximality, Strict Monotony and Strict Dominance. Let $A, B \in \mathtt{Arg(OS-FOL)}$ be such that $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$. Let us show that $A \approx B$.
  Assume that $A \not\approx B$. By definition, $\mathtt{UC}(\mathtt{Supp}(A)) \neq \mathtt{UC}(\mathtt{Supp}(B))$ or $\mathtt{UC}(\mathtt{Conc}(A)) \neq \mathtt{UC}(\mathtt{Conc}(B))$. Let us study the two cases:
  i) Consider the case where $\mathtt{UC}(\mathtt{Supp}(A)) \neq \mathtt{UC}(\mathtt{Supp}(B))$. Clearly,
  - $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(A))$
  - $\mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(B)) \subset \mathtt{UC}(\mathtt{Supp}(A)) \cap \mathtt{UC}(\mathtt{Supp}(A))$ (this inclusion is strict since $\mathtt{UC}(\mathtt{Supp}(A)) \neq \emptyset$ and $\mathtt{UC}(\mathtt{Supp}(A)) \neq \mathtt{UC}(\mathtt{Supp}(B))$),
  - for $A_A = \mathtt{UC}(\mathtt{Supp}(A)) \setminus \mathtt{UC}(\mathtt{Supp}(A))$ and $B_A = \mathtt{UC}(\mathtt{Supp}(B)) \setminus \mathtt{UC}(\mathtt{Supp}(A))$, $A_A \subseteq B_A$, $B_A \setminus A_A \subseteq \mathbb{C}$ and $\forall s_i \in \mathtt{Dom}(\mathtt{Supp}(A))$, $\nexists s_j \in \mathtt{Dom}(B_A \setminus A_A)$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$,

  By applying Strict Monotony, we get $\mathtt{sim}(A, A) > \mathtt{sim}(A, B)$. From Maximality $\mathtt{sim}(A, A) = 1$, so $\mathtt{sim}(A, B) < 1$. This shows that $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(B))$.
  ii) Consider now the case where $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(B))$ and $\mathtt{UC}(\mathtt{Conc}(A)) \neq \mathtt{UC}(\mathtt{Conc}(B))$.
  The conditions of Strict Dominance are verified, indeed:
  - $\mathtt{UC}(\mathtt{Supp}(A)) = \mathtt{UC}(\mathtt{Supp}(B))$,
  - $\mathtt{UC}(\mathtt{Conc}(A)) \cap \mathtt{UC}(\mathtt{Conc}(B)) \subset \mathtt{UC}(\mathtt{Conc}(A)) \cap \mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(A))$ (The implication is strict since $\mathtt{UC}(\mathtt{Conc}(A)) \neq \mathtt{UC}(\mathtt{Conc}(B))$),
  - for $A_A = \mathtt{UC}(\mathtt{Conc}(A)) \setminus \mathtt{UC}(\mathtt{Conc}(A))$ and $B_A = \mathtt{UC}(\mathtt{Conc}(B)) \setminus \mathtt{UC}(\mathtt{Conc}(A))$, $A_A \subseteq B_A$, $B_A \setminus A_A \subseteq \mathbb{C}$ and $\forall s_i \in \mathtt{Dom}(\mathtt{Conc}(A))$, $\nexists s_j \in \mathtt{Dom}(B_A \setminus A_A)$ s.t. $s_i \prec s_j$ or $s_j \prec s_i$ or $s_j = s_i$,

  Strict Dominance ensures $\mathtt{sim}(A, A) > \mathtt{sim}(A, B)$ while Maximality ensures $\mathtt{sim}(A, A) = 1$, so $\mathtt{sim}(A, B) < 1$. This shows that $\mathtt{UC}(\mathtt{Conc}(A)) = \mathtt{UC}(\mathtt{Conc}(B))$.
- Substitution:
  Let $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ be a similarity measure which satisfies Maximality, Symmetry, Strict Monotony, Dominance, and Strict Dominance. Let $A, B, C \in \mathtt{Arg(OS-FOL)}$ such that $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$.
  From the previous proof, if $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}$ satisfies Maximality, Strict Monotony and Strict Dominance then if $A, B \in \mathtt{Arg(OS-FOL)}$ s.t. $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, B) = 1$ then $A \approx B$.
  By applying Dominance twice, we get $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, A) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, B)$ and $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, B) \geq \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, A)$. Hence, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, A) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, B)$. Symmetry implies $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, A) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(A, C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(C, B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}(B, C)$.

□

Let us prove that the functions **g**, **p** and **c** used in the paper satisfy the expected properties of a well-behaved SM.

Lemma 4.5. *For* $\mathbf{g} \in \{\mathtt{min}, \mathtt{avg}\}$, $\mathbf{p} = \mathtt{eq}$ *and* $\mathbf{c} = \mathtt{pws}$, $\langle \mathbf{g}, \mathbf{p}, \mathbf{c} \rangle$ *satisfies item 1. of Def.* 4.1.

Proof. The proof is trivial for **g** and **p**:
- For $\mathbf{g} = \mathtt{min}$,
  (1) $\mathtt{min}(1, 1) = 1$,
  (2) $\mathtt{min}(0, 0) = 0$,
- For $\mathbf{g} = \mathtt{avg}$,
  (1) $\mathtt{avg}(1, 1) = 1$,
  (2) $\mathtt{avg}(0, 0) = 0$,
- For $\mathbf{p} = \mathtt{eq}$,
  (1) $\mathtt{eq}(P, P) = 1$,
  (2) $\mathtt{eq}(P, Q) = 0$ iff $P \neq Q$.

Now, focus on $\mathbf{c} = \mathtt{pws}$.
1. $\mathtt{pws}(\langle a_1, \ldots, a_k \rangle, \langle a_1, \ldots, a_k \rangle) = 1$ is obvious, since $\mathtt{pws}(X, X) = \frac{\sum_{i=1}^{k} \mathtt{eq}(a_i, a_i)}{k}$, and $\mathtt{eq}(a_i, a_i) = 1$.
2. Assume $k \neq 0$ or $n \neq 0$ (otherwise the result is trivial). If $\forall i \in \{1, \ldots, k\}$, $\nexists j \in \{1, \ldots, n\}$ s.t. $a_i = b_j$ then in particular

for each $i \in \{1, \ldots, \mathtt{min}(k, n)\}$, $\mathtt{eq}(a_i, b_i) = 0$. So the sum is equal to 0, hence the result $\mathtt{pws}(\langle a_1, \ldots, a_k\rangle, \langle b_1, \ldots, b_n\rangle) = 0$.

□

We can show similar results for the Tversky measures that we use to define $\mathtt{simC}^{\varepsilon^{\mathbb{L}}_{\oplus^{l}, s^{\mathbb{L}}}}$, $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\oplus^{c}, s^{\mathbb{C}}}}$ and $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\oplus^{i}, s^{\mathbb{I}}}}$. We consider the measures described in Table 1.

Lemma 4.6. *If* $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}}$ *is a Tversky measure, with* $\oplus = \mathtt{max}$, *and* $\mathtt{sim}$ *is*
– either $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$ *(from Definition 3.4) s.t.* $\langle \mathbf{g}, \mathbf{p}, \mathbf{c}\rangle$ *satisfies item 1. of Definition 4.1,*
– or a similarity measure satisfying the item 2. of Definition 4.1,
then $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\oplus,\mathtt{sim}}}$ *satisfies the item 2. of Definition 4.1.*

Proof. (1) First, assume that $\mathtt{sim} = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$, *i.e.* we consider $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}$.

(a) $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}(X, X) = (\frac{a}{a+\alpha.b+\beta.c})^{\gamma}$. Since $\langle \mathbf{g}, \mathbf{p}, \mathbf{c}\rangle$ satisfies the properties of well-behaved SM, $\sum_{x \in X} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X) = |X|$, so $a = |X|$. For similar reasons, $b = c = 0$. So we obtain that the similarity between X and X is $(\frac{|X|}{|X|+0.\alpha+0.\beta})^{\gamma} = 1$.

(b) Let $X, X' \subseteq \mathbb{X}$ be two sets of objects s.t. $\forall x \in X, \forall x' \in X'$, $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x, x') = 0$. Then, for any $x \in X$, $\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X') = 0$ (and similarly, $\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x', X) = 0$), so $a = 0$. It also implies that $b \neq 0$ and $c \neq 0$. So we obtain $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}(X, X') = (\frac{0}{0+\alpha.b+\beta.c})^{\gamma} = 0$.

(c) Assume three sets of objects $X_0, X_1, X_2 \subseteq \mathbb{X}$ s.t. $X_1 \subset X_2$ and $X_2 \setminus X_1 = \{x_2\}$. Assume also $\exists x_0 \in X_0$ s.t. $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x_0, x_2) = 1$. Let us define, for $i \in \{1, 2\}$, a_i, b_i, c_i the numbers involved in the computation of $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}(X_0, X_i)$.
We start with studying the relation between a_1 and a_2:
- First, focus on $\sum_{x \in X_0} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2)$. We can easily show that $\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) \geq \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1)$ for each $x \in X_0$, since adding an object (x_2) to a set (X_1) can increase the maximal value of the objects (w.r.t. $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}$) which are in the set, but cannot decrease it. So we obtain $\sum_{x \in X_0} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) \geq \sum_{x \in X_0} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1)$.
- Then, $\sum_{x \in X_2} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_0)) = \sum_{x \in X_1} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_0)) + \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x_2, X_0))$ comes from the fact that $X_2 = X_1 \cup \{x_2\}$, and moreover $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x_2, x_0) = 1$, so we conclude $\sum_{x \in X_2} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_0)) > \sum_{x \in X_1} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_0))$.

From both facts above, we deduce that $a_2 \geq a_1$.
Now we study the relation between b_1 and b_2. We can state that, for each $x \in X_0$, $\varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) =$

$$\begin{cases} \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1) & \text{if } \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x, x_2) \leq \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1) \\ \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x, x_2) & \text{otherwise} \end{cases}$$

From that, we deduce:

$$\begin{array}{ll} & \forall x \in X_O, \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) \geq \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1) \\ \text{so} & \forall x \in X_0, 1 - \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) \leq 1 - \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1) \\ \text{so} & \sum_{x \in X_0} 1 - \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) \leq \sum_{x \in X_0} 1 - \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1) \end{array}$$

which means that $b_2 \leq b_1$.
Then, we focus on the relation between c_1 and c_2. Since $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x_2, x_0) = 1$, we easily show that $c_1 = c_2$, because $c_2 = c_1 + (1 - \varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x_2, X_0))$, and $\varepsilon_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x_2, X_0) = 1$.
Finally, what we need to prove is that $T_2 \geq T_1$, where

$$T_i = \frac{a_i}{a_i + \alpha.b_i + \beta.c_i}$$

First, assume that $b_2 = b_1$. Let $\rho = \alpha.b_1 + \beta.c_1 = \alpha.b_2 + \beta.c_2$.
So, verifying whether $T_2 \geq T_1$ is equivalent to:

$$\begin{array}{ll} & \frac{a_2}{a_2+\rho} \geq \frac{a_1}{a_1+\rho} \\ \text{iff} & \frac{a_2(a_1+\rho)}{(a_2+\rho)(a_1+\rho)} \geq \frac{a_1(a_2+\rho)}{(a_1+\rho)(a_2+\rho)} \\ \text{iff} & a_2(a_1 + \rho) \geq a_1(a_2 + \rho) \\ \text{iff} & a_2.a_1 + a_2.\rho \geq a_1.a_2 + a_1.\rho \\ \text{iff} & a_2.\rho \geq a_1.\rho \\ \text{iff} & a_2 \geq a_1 \end{array}$$

Since the last point holds, we deduce that $T_2 \geq T_1$.
Now, assume that $b_2 < b_1$. So we write $\rho_1 = \alpha.b_1 + \beta.c_1$ and $\rho_2 = \alpha.b_2 + \beta.c_2$. It is obvious that $\rho_2 < \rho_1$. Then, we deduce that $T_2 \geq T_1$ if and only if:

$$\begin{array}{ll} & \frac{a_2}{a_2+\rho_2} \geq \frac{a_1}{a_1+\rho_1} \\ \text{iff} & \frac{a_2(a_1+\rho_1)}{(a_2+\rho_2)(a_1+\rho_1)} \geq \frac{a_1(a_2+\rho_2)}{(a_1+\rho_1)(a_2+\rho_2)} \\ \text{iff} & a_2(a_1 + \rho_1) \geq a_1(a_2 + \rho_2) \\ \text{iff} & a_1.a_1 + a_2.\rho_1 \geq a_1.a_2 + a_1.\rho_2 \\ \text{iff} & a_2.\rho_1 \geq a_1.\rho_2 \end{array}$$

The last point holds since $a_2 \geq a_1$ and $\rho_1 \geq \rho_2$. So we can conclude that $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}(X_0, X_2) \geq \mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}(X_0, X_1)$.

(d) Let $X_0, X_1, X_2 \subseteq \mathbb{X}$ s.t. $X_1 \subset X_2$ and $X_2 \setminus X_1 = \{x_2\}$. Moreover, assume $\forall x_0 \in X_0$, $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x_0, x_2) = \mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x_2, x_0) = 0$. Similarly to the previous item, for $i \in \{1, 2\}$, let

$$T_i = \frac{a_i}{a_i + \alpha.b_i + \beta.c_i}$$

be the value of the Tversky measure, where a_i, b_i, c_i are the numbers involved in the computation of $\mathtt{Tve}^{\alpha,\beta,\gamma,\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}}(X_0, X_i)$. We need to show that $T_1 \geq T_2$.
We start with studying the relation between a_1 and a_2:
- First, focus on $\sum_{x \in X_0} \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2)$. We can easily show that $\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) = \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1)$ for each $x \in X_0$, since $\varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_2) = \mathtt{max}(\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x, x_2), \varepsilon^{\mathbb{X}}_{\mathtt{max},\mathtt{simL}^{\langle g,p,c\rangle}}(x, X_1))$, and it is already known that $\mathtt{simL}^{\langle \mathbf{g},\mathbf{p},\mathbf{c}\rangle}(x, X_0) = 0$.

- Then, $\sum_{x\in X_2} \varepsilon^{\mathbb{X}}_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_0)) = \sum_{x\in X_1} \varepsilon^{\mathbb{X}}_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_0)) + \varepsilon^{\mathbb{X}}_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x_2,X_0))$ comes from the fact that $X_2 = X_1 \cup \{x_2\}$, and moreover $\mathtt{simL}^{\langle \mathbf{g,p,c}\rangle}(x_2,x_0) = 0$, so we conclude $\sum_{x\in X_2} \varepsilon^{\mathbb{X}}_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_0)) = \sum_{x\in X_1} \varepsilon^{\mathbb{X}}_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_0))$.

This means that $a_1 = a_2$. Now, let us compare b_1 and b_2. For each $x \in X_0$, we have already stated that $\varepsilon_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_2) = \varepsilon_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_1)$, so it is easy to conclude that for each $x \in X_0$, $1-\varepsilon_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_2) = 1-\varepsilon_{\mathtt{max,simL}^{\langle g,p,c\rangle}}(x,X_1)$, and thus $b_1 = b_2$.
Finally, comparing c_1 and c_2, we have

$$\begin{aligned} c_2 &= \sum_{x\in X_2} 1-\varepsilon(x,X_0) \\ &= (1-\varepsilon(x_2,X_0)) + \sum_{x\in X_1} 1-\varepsilon(x,X_0) \\ &= 1 + c_1 \end{aligned}$$

So we can easily see that

$$\frac{a_1}{a_1+\alpha.b_1+\beta.c_1} \geq \frac{a_2}{a_2+\alpha.b_2+\beta.c_2} = \frac{a_1}{a_1+\alpha.b_1+\beta.(1+c_1)}$$

This concludes the proof.

(2) Now, if $\mathtt{sim}$ is a similarity measure satisfying the item 2. of Definition 4.1, the proof is analogous because we did not need to use an assumption which is not true in this case as well.

□

Proposition 4.7. *For* $x \in \{\mathtt{jac}, \mathtt{dic}, \mathtt{sor}, \mathtt{adb}, \mathtt{ss_2}, \mathtt{ns-jac}, \mathtt{ns-dic}, \mathtt{ns-sor}, \mathtt{ns-adb}, \mathtt{ns-ss_2}\}$, *define* $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$. *Then define the* SM $\mathbf{M}_x = \langle \mathtt{simL}^{\langle \mathtt{min,eq,pws}\rangle}, x^{2,\mathtt{sim}^{\mathbb{L}}}, x^{1,\mathtt{sim}^{\mathbb{C}}}, x^{1,\mathtt{sim}^{\mathbb{I}}}\rangle$. *The satisfaction of principles by the measures is given in Table* 2.

Proof. Let us start by the symmetric similarity measures $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$, i.e. where $x \in \{\mathtt{jac}, \mathtt{dic}, \mathtt{sor}, \mathtt{adb}, \mathtt{ss_2}\}$. From Lemma 1 and Lemma 2, we know that measures used in each similarity models (used for the $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$) are well-behaved (Definition 4.1). From Theorem 1, we know then that each of these measure satisfies the principles of Maximality, Minimality, Monotony and Dominance.

Given that for these 5 measures, their $\alpha = \beta$ then from Proposition 2, these measures are symmetric. For $\mathtt{simL}^{\langle \mathbf{g,p,c}\rangle}$ let see that $\mathtt{pws}$ is symmetric because the $\mathtt{eq}$ is symmetric, the sum is borned under the minimal size of the two vectors and the denominator is also symmetric using the function $\mathtt{max}$. Then it is trivial that $\mathtt{simL}^{\langle \mathbf{g,p,c}\rangle}$ is trivial since it use also the function $\mathtt{min}$ and $\mathtt{eq}$.

We can also observe that each similarity measure used in the definitions of these 5 similarity models are syntax independent. Therefore from item 1 of Theorem 2, we have that these 5 $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$ satisfy also Symmetry and Syntax Independence.

Let us show now that the condition 2.(c') defined in item 2 of Theorem 2 is satisfied by these measures. We recall first this condition: let $X_0, X_1, X_2 \subseteq \mathbb{X}$ s.t. $X_1 \subset X_2$ and $X_2 \setminus X_1 = \{x_2\}$. If $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0,X_1) < 1$ and $\exists x_0 \in X_0$ s.t. $\mathtt{s}(x_0,x_2) = \mathtt{s}(x_2,x_0) = 1$ then $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0,X_2) > \mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0,X_1)$.

We will show the satisfaction of this condition only on the Tversky measure given that the strict inclusion of element between two litteral (for $\mathtt{simL}^{\langle \mathbf{g,p,c}\rangle}$) is not possible.
From Definition 18 using $\alpha = \beta \neq 0$, we know that adding an element fully similar ($\mathtt{s}(x_0,x_2) = \mathtt{s}(x_2,x_0) = 1$) will strictly increase "a" and not change the score of "b" and "c". Moreover, given that $\alpha = \beta \neq 0$, the only way to obtain a similarity score of 1 is that "b" and "c" are equal to 0, i.e. there exist some difference between the sets. From the condition 2.(c') we know that $X_1 \subset X_2$ and $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0,X_1) < 1$ then $b+c > 0$. Hence, increasing the score of "a" will strictly increase the similarity between $\mathtt{sim}^{\varepsilon,\mathtt{s}}(X_0,X_2)$ given that $\frac{a+1}{a+1+x} > \frac{a}{a+x}$ when $x > 0$.
Therefore the condition 2.(c') is satisfied and the 5 $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$ satisfy the principles of Strict Monotony and Strict Dominance.

Now we can deduce from item 2 of Proposition 3 that these 5 $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$ satisfy Substitution.

Let us now study the non-symmetric similarity measures $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$, i.e. where $x \in \{\mathtt{ns-jac}, \mathtt{ns-dic}, \mathtt{ns-sor}, \mathtt{ns-adb}, \mathtt{ns-ss_2}\}$. As previous: From Lemma 1 and Lemma 2, we know that measures used in each similarity models (used for the $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$) are well-behaved (Definition 4.1). From Theorem 1, we know then that each of these measure satisfies the principles of Maximality, Minimality, Monotony and Dominance.

And, we can also observe that each similarity measure used in the definitions of these 5 similarity models are syntax independent. Therefore from item 1 of Theorem 2, we have that these 5 $\mathtt{sim}_x^{\mathtt{Arg(OS-FOL)}}$ satisfy also Syntax Independence.

For the violation of the 4 principles Symmetry, Substitution, Strict Monotony and Strict Dominance let us take some counter example.
Let $A = \langle\{P(a)\}, P(a)\rangle, B = \langle\{P(a), Q(b)\}, P(a) \wedge Q(b))\rangle$ and $C = \langle\{Q(b)\}, Q(b)\rangle \in \mathtt{Arg(OS-FOL)}$.

Violation of Symmetry:
$\mathtt{sim}_{\mathtt{ns-jac}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-dic}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-sor}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-adb}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-ss_2}}^{\mathtt{Arg(OS-FOL)}}(A,B) = 1$ while $\mathtt{sim}_{\mathtt{ns-jac}}^{\mathtt{Arg(OS-FOL)}}(B,A) < 1$, $\mathtt{sim}_{\mathtt{ns-dic}}^{\mathtt{Arg(OS-FOL)}}(B,A) < 1$, $\mathtt{sim}_{\mathtt{ns-sor}}^{\mathtt{Arg(OS-FOL)}}(B,A) < 1$, $\mathtt{sim}_{\mathtt{ns-adb}}^{\mathtt{Arg(OS-FOL)}}(B,A) < 1$ and $\mathtt{sim}_{\mathtt{ns-ss_2}}^{\mathtt{Arg(OS-FOL)}}(B,A) < 1$.

Violation of Substitution:
$\mathtt{sim}_{\mathtt{ns-jac}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-dic}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-sor}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-adb}}^{\mathtt{Arg(OS-FOL)}}(A,B) = \mathtt{sim}_{\mathtt{ns-ss_2}}^{\mathtt{Arg(OS-FOL)}}(A,B) = 1$.
$\mathtt{sim}_{\mathtt{ns-jac}}^{\mathtt{Arg(OS-FOL)}}(A,C) = \mathtt{sim}_{\mathtt{ns-dic}}^{\mathtt{Arg(OS-FOL)}}(A,C) = \mathtt{sim}_{\mathtt{ns-sor}}^{\mathtt{Arg(OS-FOL)}}(A,C)$

$= \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-adb}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-ss_2}}(A,C) = 0$ while $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-jac}}(B,C) > 0$, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-dic}}(B,C) > 0$, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-sor}}(B,C) > 0$, $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-adb}}(B,C) > 0$ and $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-ss_2}}(B,C) > 0$.

Let define new arguments: $A = \langle\{P(a), Q(b), (P(a) \wedge Q(b)) \to R(c)\}, R(c)\rangle$, $B = \langle\{P(a), Q(b)\}, P(a) \wedge Q(b))\rangle$ and $C = \langle\{P(a)\}, P(a)\rangle \in \mathtt{Arg(OS-FOL)}$.
Violation of Strict Monotony: $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-jac}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-dic}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-sor}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-adb}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-ss_2}}(A,B) = \eta$ while $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-jac}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-dic}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-sor}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-adb}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-ss_2}}(A,C) = \eta$ also.

Let define new arguments: $A = \langle\{P(a), P(a) \to Q(b)\}, P(a) \wedge Q(b))\rangle$, $B = \langle\{P(a), P(a) \to Q(b)\}, P(a) \wedge Q(b)\rangle$ and $C = \langle\{P(a), P(a) \to Q(b)\}, Q(b)\rangle \in \mathtt{Arg(OS-FOL)}$.
Violation of Strict Dominance: $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-jac}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-dic}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-sor}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-adb}}(A,B) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-ss_2}}(A,B) = 1$ while $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-jac}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-dic}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-sor}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-adb}}(A,C) = \mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathtt{ns-ss_2}}(A,C) = 1$ also.

□

Notice that Proposition 4.7 implies that all the principles are compatible. Moreover with the result of item 1 of Proposition 4.4, we can deduce that our 5 symmetric extended Tversky measures satisfying a stronger form of maximality, since equivalent arguments are maximally similar. For non-symmetric measures, we show that they can obtain full similarity in a particular case of sub-argument.

Proposition 4.8. *Let $A, B \in \mathtt{Arg(OS-FOL)}$ be two arguments. Assume that* $\mathbf{M}$ *is a* SM *s.t.* $\mathtt{simC}^{\varepsilon^{\mathbb{L}}}_{\oplus^{1},s^{\mathbb{L}}}$, $\mathtt{simI}^{\varepsilon^{\mathbb{C}}}_{\oplus^{c},s^{\mathbb{C}}}$ *and* $\mathtt{simSI}^{\varepsilon^{\mathbb{I}}}_{\oplus^{i},s^{\mathbb{I}}}$ *are Tversky measures s.t.* $\alpha \neq \beta$ *for at least one of them* (i.e. *it is non-symmetric). If B is a sub-argument of A, then* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A,B) \geq \eta$. *Moreover, if* $\mathtt{UC}(\mathtt{Conc}(B)) \subseteq \mathtt{UC}(\mathtt{Conc}(A))$*, then* $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A,B) = 1$.

Proof. Assume that B is a sub-argument of A. By definition, $\mathtt{UC}(\mathtt{Supp}(B)) \subseteq \mathtt{UC}(\mathtt{Supp}(A))$ and then, from Proposition 3.10, $\mathtt{sim}(\mathtt{UC}(\mathtt{Supp}(A)), \mathtt{UC}(\mathtt{Supp}(B))) = 1$. This means that $\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M},\mathbf{I_{St}},\eta}(A,B) = \eta + (1-\eta)\mathtt{sim}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B)))$, hence the result. Now, if we also assume that $\mathtt{UC}(\mathtt{Conc}(B)) \subseteq \mathtt{UC}(\mathtt{Conc}(A))$, then Proposition 3.10 implies that $\mathtt{sim}(\mathtt{UC}(\mathtt{Conc}(A)), \mathtt{UC}(\mathtt{Conc}(B))) = 1$, which allows to conclude the proof. □

5 RELATED WORK

Similarity Measure. In the literature of similarity measure between FOL (e.g. [14, 24, 34, 39]) or fragment of FOL as description logic (DL) (e.g. [15, 20, 21, 26]), it is common to combine different layers of similarity because the knowledge are structured in different levels (e.g. constants are in predicates in FOL or individuals are in concepts or roles in DL). However, our approach differs from existing ones in that it allows the manipulation of FOL with quantifiers and variables, sorted knowledge and parametric measure together. The originalities of our work are the evaluation of the similarity of formulas of a higher level (i.e. with quantifiers and variables) and the definition of similarity measures (Section 3) in a more general way. Indeed, rather than an ad hoc similarity measure, we propose a methodology that can be instantiated by existing similarity measures (like Tversky's for example) combining with a family of aggregation functions (and not a specific function as it is the case in the literature).

Logical Argumentation. In addition to the proposition of a similarity evaluation model between OS-FOLs, we also extend the study of similarity evaluation in logical argumentation. Indeed, we adapt the principles from [7] to define the similarity measures between OS-FOL arguments. We also generalize and extend different works defining similarity measure between propositional arguments [4, 7, 8] by a parametric model for OS-FOL arguments that can combine existing similarity measures [9, 17, 25, 36–38] and aggregation functions.

6 CONCLUSION

In this paper, we have proposed the rich methodology of similarity models which are able to express large families of similarity measures between Order-Sorted First Order Logic ($\mathtt{OS-FOL}$) arguments, thanks to various parameters which allow to define generalized versions of similarity measures from the literature. For the first time in the logical argumentation literature, we define non-symmetric similarity measures. A set of nine principles for these $\mathtt{OS-FOL}$ arguments has been proposed with a set of well-behaved properties ensuring some principles. We have shown that our symmetric measures satisfy all the principles, while their non-symmetric counterparts only satisfy a subset of the principles.

This work paves the way to several interesting research questions. First of all, we can consider additional measures (*e.g.* Ochiai [32], Kulczynski [28]) and principles (*e.g.* triangular inequality, non-zero, independent distribution [16]) to allow a more accurate comparison of similarity measures. Another research line could be to consider situations where different predicates are partially similar. For instance, one can consider that $greaterOrEqual(A,B)$ is somehow similar to $strictlyGreater(A,B)$. Following the same idea as in [6], we also plan to use our similarity measures as a parameter of acceptability semantics. Finally, we want to apply our work on real data expressed in fragments of $\mathtt{OS-FOL}$.

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A Detailed Example

Let $\mathbf{M}_{\mathtt{jac}} = \langle \mathtt{s}^{\mathbf{L}} = \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}, \mathtt{s}^{\mathbb{C}} = \mathtt{jac}^{2,\mathtt{s}^{\mathbf{L}}}, \mathtt{s}^{\mathbb{I}} = \mathtt{jac}^{1,\mathtt{s}^{\mathbb{C}}}, \mathtt{jac}^{1,\mathtt{s}^{\mathbb{I}}} \rangle$ be a similarity instantiation model and let A_1 and A_2 be the two OS-FOL arguments from Example 5.

- $A_1 = \langle \{\exists x^b w(x^b), \forall x^b w(x^b) \to \neg f(x^b)\}, \exists x^b \neg f(x^b) \rangle$
- $A_2 = \langle \{p(T), \forall x^b p(x^b) \to \neg f(x^b)\}, \neg f(T) \rangle$

Their respective instantiations are given in Example 4 for the premises and the conclusions. Let us compute the similarity between A_1 and A_2 with $\eta = 0.5$.

$\mathtt{sim}^{\mathtt{Arg(OS-FOL)}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}},0.5}(A_1, A_2) = 0.5 \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Supp}(A_1), \mathtt{Supp}(A_2))$
$\quad + 0.5 \cdot \mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Conc}(A_1), \mathtt{Conc}(A_2)) = 0.5 \cdot \frac{73}{1143} + 0.5 \cdot \frac{5}{11} \simeq 0.2592$

where $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Supp}(A_1), \mathtt{Supp}(A_2)) =$
$\quad \mathtt{jac}^{1,\mathtt{s}^{\mathbb{I}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Supp}(A_1)), \mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Supp}(A_2))) = \frac{73}{1143} \simeq 0.064$

and $\mathtt{sim}^{\mathtt{OS-FOL}}_{\mathbf{M}_{\mathtt{jac}},\mathbf{I_{St}}}(\mathtt{Conc}(A_1), \mathtt{Conc}(A_2)) =$
$\quad \mathtt{jac}^{1,\mathtt{s}^{\mathbb{I}}}(\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Conc}(A_1)), \mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Conc}(A_2))) = \frac{5}{11} \simeq 0.4545$

$\boxed{\text{PREMISES}}$

In order to facilitate the reading of the example concerning the premises, we have chosen to put in red the calculations returning a non-zero result for levels 2 and 3.

$\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Supp}(A_1)) = \{\Delta_1, \Delta_2, \Delta_3\} = I_1$ with

- $\Delta_1 = \{w(T), \neg w(T) \vee \neg f(T), \neg w(Z) \vee \neg f(Z)\}$
- $\Delta_2 = \{w(Z), \neg w(T) \vee \neg f(T), \neg w(Z) \vee \neg f(Z)\}$
- $\Delta_3 = \{w(T), w(Z), \neg w(T) \vee \neg f(T), \neg w(Z) \vee \neg f(Z)\}$

$\mathtt{Inst}_{\mathbf{I_{St}}}(\mathtt{Supp}(A_2)) = \{\Delta_4\} = I_2$ with

- $\Delta_4 = \{p(T), \neg p(T) \vee \neg f(T), \neg p(Z) \vee \neg f(Z)\}$

Level 4

$\mathtt{simSI}^{\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}}(I_1, I_2) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}}(I_1, I_2) = \frac{a}{a+b+c} = \frac{73}{1143} \simeq 0.064$ with :

$$a = \mathtt{avg}\Big(\sum_{x \in I_1} \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(x, I_2), \sum_{y \in I_2} \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(y, I_1)\Big)$$

$$= \mathtt{avg}\Big(\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_1, I_2) + \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_2, I_2) + \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_3, I_2), \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_4, I_1)\Big)$$

$$= \mathtt{avg}(\frac{1}{8} + \frac{1}{8} + \frac{2}{19}, \frac{1}{8}) = \mathtt{avg}(\frac{27}{76}, \frac{1}{8}) = \frac{73}{304}$$

$$b = \sum_{x \in I_1} 1 - \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(x, I_2)$$

$$= (1 - \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_1, I_2)) + (1 - \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_2, I_2)) + (1 - \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_3, I_2)) = \frac{7}{8} + \frac{7}{8} + \frac{17}{19} = \frac{201}{76}$$

$$c = \sum_{y \in I_2} 1 - \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(y, I_1)$$

$$= 1 - \varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_4, I_1) = \frac{7}{8}$$

$\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_1, I_2) = \mathtt{max}(\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_1, \Delta_4)) = \frac{1}{8}$
$\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_2, I_2) = \mathtt{max}(\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_2, \Delta_4)) = \frac{1}{8}$
$\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_3, I_2) = \mathtt{max}(\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_3, \Delta_4)) = \frac{2}{19}$
$\varepsilon^{\mathbb{I}}_{\mathtt{max},\mathtt{s}^{\mathbb{I}}}(\Delta_4, I_1) = \mathtt{max}(\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1), \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2), \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_3)) = \frac{1}{8}$

Level 3
$\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1) = \frac{a}{a+b+c} = \frac{1}{8}$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T), \Delta_1) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T) \vee \neg f(T), \Delta_1) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z) \vee \neg f(Z), \Delta_1),$
  $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(T), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(T) \vee \neg f(T), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(Z) \vee \neg f(Z), \Delta_4))$
  $= \mathtt{avg}(0 + \frac{1}{3} + \frac{1}{3}, 0 + \frac{1}{3} + \frac{1}{3}) = \frac{2}{3}$
- $b = (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T), \Delta_1)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T) \vee \neg f(T), \Delta_1)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z) \vee \neg f(Z), \Delta_1)) = 1 + \frac{2}{3} + \frac{2}{3} = \frac{7}{3}$
- $c = (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(T), \Delta_4)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(T) \vee \neg f(T), \Delta_4)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(Z) \vee \neg f(Z), \Delta_4)) = 1 + \frac{2}{3} + \frac{2}{3} = \frac{7}{3}$

Since $\mathtt{simI}$ is a symmetric function, we can say that $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_1, \Delta_4) = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1)$.

$\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2) = \frac{a}{a+b+c} = \frac{1}{8}$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T), \Delta_2) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T) \vee \neg f(T), \Delta_2) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z) \vee \neg f(Z), \Delta_2),$
  $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(Z), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(T) \vee \neg f(T), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(Z) \vee \neg f(Z), \Delta_4))$
  $= \mathtt{avg}(0 + \frac{1}{3} + \frac{1}{3}, 0 + \frac{1}{3} + \frac{1}{3}) = \frac{2}{3}$
- $b = (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T), \Delta_2)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T) \vee \neg f(T), \Delta_2)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z) \vee \neg f(Z), \Delta_2)) = 1 + \frac{2}{3} + \frac{2}{3} = \frac{7}{3}$
- $c = (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(Z), \Delta_4)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(T) \vee \neg f(T), \Delta_4)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(Z) \vee \neg f(Z), \Delta_4)) = 1 + \frac{2}{3} + \frac{2}{3} = \frac{7}{3}$

Since $\mathtt{simI}$ is a symmetric function, we can say that $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_2, \Delta_4) = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2)$.

$\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_3) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_3) = \frac{a}{a+b+c} = \frac{2}{19}$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T), \Delta_3) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T) \vee \neg f(T), \Delta_3) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z) \vee \neg f(Z), \Delta_3),$
  $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(T), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(Z), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(T) \vee \neg f(T), \Delta_4) +$
  $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(Z) \vee \neg f(Z), \Delta_4))$
  $= \mathtt{avg}(0 + \frac{1}{3} + \frac{1}{3}, 0 + 0 + \frac{1}{3} + \frac{1}{3}) = \frac{2}{3}$
- $b = (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T), \Delta_3)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T) \vee \neg f(T), \Delta_3)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z) \vee \neg f(Z), \Delta_3)) = 1 + \frac{2}{3} + \frac{2}{3} = \frac{7}{3}$

- $c = (1-\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(T),\Delta_4)) + (1-\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(Z),\Delta_4)) + (1-\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(T)\vee\neg f(T),\Delta_4)) + (1-\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg w(Z)\vee\neg f(Z),\Delta_4)) = 1+1+\frac{2}{3}+\frac{2}{3} = \frac{10}{3}$

Since simI is a symmetric function, we can say that $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_3,\Delta_4) = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}}(\Delta_4,\Delta_3)$.

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T),\Delta_1) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),w(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),\neg w(Z)\vee\neg f(Z))) = 0$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T)\vee\neg f(T),\Delta_1) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),w(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(Z)\vee\neg f(Z))) = \frac{1}{3}$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z)\vee\neg f(Z),\Delta_1) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),w(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),\neg w(Z)\vee\neg f(Z))) = \frac{1}{3}$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T),\Delta_2) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),w(Z)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),\neg w(Z)\vee\neg f(Z))) = 0$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T)\vee\neg f(T),\Delta_2) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),w(Z)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(Z)\vee\neg f(Z))) = \frac{1}{3}$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z)\vee\neg f(Z),\Delta_2) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),w(Z)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),\neg w(Z)\vee\neg f(Z))) = \frac{1}{3}$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(p(T),\Delta_3) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),w(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),w(Z)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T),\neg w(Z)\vee\neg f(Z))) = 0$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(T)\vee\neg f(T),\Delta_3) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),w(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),w(Z)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(Z)\vee\neg f(Z))) = \frac{1}{3}$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(\neg p(Z)\vee\neg f(Z),\Delta_3) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),w(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),w(Z)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),\neg w(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z)\vee\neg f(Z),\neg w(Z)\vee\neg f(Z))) = \frac{1}{3}$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathtt{s}^{\mathbb{C}}}(w(Z),\Delta_4) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(w(Z),p(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(w(Z),\neg p(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(w(Z),\neg p(Z)\vee\neg f(Z))) = 0$

- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathbf{s}^{\mathbb{C}}}(w(T),\Delta_4) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(w(T),p(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(w(T),\neg p(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(w(T),\neg p(Z)\vee \neg f(Z))) = 0$
- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathbf{s}^{\mathbb{C}}}(\neg w(T)\vee\neg f(T),\Delta_4) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg w(T)\vee\neg f(T),p(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg w(T)\vee \neg f(T),\neg p(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg w(T)\vee\neg f(T),\neg p(Z)\vee\neg f(Z))) = \frac{1}{3}$
- $\varepsilon^{\mathbb{C}}_{\mathtt{max},\mathbf{s}^{\mathbb{C}}}(\neg w(Z)\vee\neg f(Z),\Delta_4) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg w(Z)\vee\neg f(Z),p(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg w(Z)\vee \neg f(Z),\neg p(T)\vee\neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg w(Z)\vee\neg f(Z),\neg p(Z)\vee\neg f(Z))) = \frac{1}{3}$

Level 2

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(p(T),w(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(p(T),w(T)) = \frac{a}{a+b+c} = 0$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(p(T),w(T)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(w(T),p(T))) = 0$
- $b = 1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(p(T),w(T)) = 0$
- $c = 1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(w(T),p(T)) = 0$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(p(T),\neg w(T)\vee\neg f(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(p(T),\neg w(T)\vee\neg f(T)) = 0:$

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(p(T),\neg w(T)\vee\neg f(T)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg w(T),p(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(T),p(T))) = 0$
- $b = 1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(p(T),\neg w(T)\vee\neg f(T)) = 1$
- $c = (1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg w(T),p(T))) + (1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(T),p(T))) = 2$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(p(T),\neg w(Z)\vee\neg f(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(p(T),\neg w(Z)\vee\neg f(Z)) = 0$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(p(T),\neg w(Z)\vee\neg f(Z)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg w(Z),p(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(Z),p(T))) = 0$
- $b = 1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(p(T),\neg w(Z)\vee\neg f(Z)) = 1$
- $c = (1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg w(Z),p(T))) + (1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(Z),p(T))) = 2$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),w(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),w(T)) = 0:$

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg p(T),w(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(T),w(T)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(w(T),\neg p(T)\vee\neg f(T))) = 0$
- $b = (1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg p(T),w(T))) + (1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(T),w(T))) = 2$
- $c = 1-\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(w(T),\neg p(T)\vee\neg f(T)) = 1$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(T)\vee\neg f(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}}(\neg p(T)\vee\neg f(T),\neg w(T)\vee \neg f(T)) = \frac{1}{3}$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg p(T),\neg w(T)\vee\neg f(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(T),\neg w(T)\vee\neg f(T)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg w(T),\neg p(T)\vee \neg f(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathbf{s}^{\mathbf{L}}}(\neg f(T),\neg p(T)\vee\neg f(T))) = 1$

- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), \neg w(T) \vee \neg f(T))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg w(T) \vee \neg f(T))) = 1 + 0 = 1$
- $c = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(T), \neg p(T) \vee \neg f(T))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg p(T) \vee \neg f(T))) = 1 + 0 = 1$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T) \vee \neg f(T), \neg w(Z) \vee \neg f(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T) \vee \neg f(T), \neg w(Z) \vee \neg f(Z)) = 0:$

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), \neg w(Z) \vee \neg f(Z)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg w(Z) \vee \neg f(Z)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), \neg p(T) \vee \neg f(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg p(T) \vee \neg f(T))) = 0$
- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), \neg w(Z) \vee \neg f(Z))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg w(Z) \vee \neg f(Z))) = 2$
- $c = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), \neg p(T) \vee \neg f(T))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg p(T) \vee \neg f(T))) = 2$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), w(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), w(T)) = 0:$

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), w(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), w(T)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(T), \neg p(Z) \vee \neg f(Z))) = 0$
- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), w(T))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), w(T))) = 2$
- $c = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(T), \neg p(Z) \vee \neg f(Z)) = 1$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), \neg w(T) \vee \neg f(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), \neg w(T) \vee \neg f(T)) = 0$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), \neg w(T) \vee \neg f(T)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg w(T) \vee \neg f(T)),$
  $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(T), \neg p(Z) \vee \neg f(Z)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg p(Z) \vee \neg f(Z))) = 0$
- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), \neg w(T) \vee \neg f(T))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg w(T) \vee \neg f(T))) = 2$
- $c = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(T), \neg p(Z) \vee \neg f(Z))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg p(Z) \vee \neg f(Z))) = 2$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), \neg w(Z) \vee \neg f(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), \neg w(Z) \vee \neg f(Z)) = \frac{a}{a+b+c} = \frac{1}{3}$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), \neg w(Z) \vee \neg f(Z)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg w(Z) \vee \neg f(Z)),$
  $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), \neg p(Z) \vee \neg f(Z)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg p(Z) \vee \neg f(Z))) = 1$
- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), \neg w(Z) \vee \neg f(Z))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg w(Z) \vee \neg f(Z))) = 1 + 0 = 1$
- $c = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), \neg p(Z) \vee \neg f(Z))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg p(Z) \vee \neg f(Z))) = 1 + 0 = 1$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T), w(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(p(T), w(Z)) = \frac{a}{a+b+c} = 0$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(p(T), w(Z)), \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), p(T))) = 0$

- $b = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(p(T), w(Z)) = 1$
- $c = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), p(T)) = 1$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T) \vee \neg f(T), w(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(T) \vee \neg f(T), w(Z)) = 0$:

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), w(Z)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), w(Z)),$
  $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), \neg p(T) \vee \neg f(T))) = 0$
- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), w(Z))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), w(Z))) = 2$
- $c = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), \neg p(T) \vee \neg f(T)) = 1$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), w(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}}(\neg p(Z) \vee \neg f(Z), w(Z)) = 0$:

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), w(Z)) + \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), w(Z)),$
  $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), \neg p(Z) \vee \neg f(Z))) = 0$
- $b = (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), w(Z))) + (1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), w(Z))) = 2$
- $c = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), \neg p(Z) \vee \neg f(Z)) = 1$

So, we have :

- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(p(T), w(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), w(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(p(T), \neg w(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), \neg w(T)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(T), p(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(T), p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), p(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(p(T), \neg w(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), \neg w(Z)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), p(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), p(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), w(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), w(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), w(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), w(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(T), \neg p(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(T), \neg p(T)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(T), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), \neg w(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), \neg w(T)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg w(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg w(T)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg f(T))) = 1$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(T), \neg p(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(T), \neg p(T)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(T), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg p(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg p(T)),$
  $\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg f(T))) = 1$

- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), \neg w(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), \neg w(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg w(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg w(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), \neg p(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), \neg p(T)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), \neg p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg p(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg p(T)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), w(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), w(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), w(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), w(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(T), \neg p(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(T), \neg p(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(T), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), \neg w(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), \neg w(T)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg w(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg w(T)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(T), \neg p(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(T), \neg p(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(T), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), \neg p(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg p(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), \neg w(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), \neg w(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg w(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg w(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg f(Z))) = 1$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg w(Z), \neg p(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), \neg p(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), \neg f(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg p(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg p(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg f(Z))) = 1$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(p(T), w(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), w(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), p(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), p(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(T), w(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), w(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(T), w(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), w(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), \neg p(T) \vee \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(Z), \neg p(T)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(Z), \neg f(T))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg p(Z), w(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), w(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), w(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), w(Z))) = 0$
- $\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(w(Z), \neg p(Z) \vee \neg f(Z)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(Z), \neg p(Z)), \mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w(Z), \neg f(Z))) = 0$

Level 1

Obviously, the similarity between two identical atoms with the same polarity is 1.
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(Z), \neg f(Z)) = 1$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg f(T)) = 1$

If the atoms have the same polarity, then we use $\mathtt{simA}$:
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p(T), w(T)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p, \langle T\rangle, w, \langle T\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), \neg w(T)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p, \langle T\rangle, w, \langle T\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg w(T)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(f, \langle T\rangle, w, \langle T\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(T), \neg f(T)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p, \langle T\rangle, f, \langle T\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg f(T), \neg f(Z)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(f, \langle T\rangle, f, \langle Z\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), \neg f(Z)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p, \langle Z\rangle, f, \langle Z\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), \neg w(Z)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(p, \langle Z\rangle, w, \langle Z\rangle) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), \neg f(Z)) = \mathtt{simA}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(w, \langle Z\rangle, f, \langle Z\rangle) = 0$

If the atoms do not have the same polarity, then, by definition, the similarity measure is 0. For example :
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg w(Z), p(T)) = 0$
$\mathtt{simL}^{\langle \mathtt{min},\mathtt{eq},\mathtt{pws}\rangle}(\neg p(Z), w(T)) = 0$

$\boxed{\text{CONCLUSION}}$

$\texttt{Inst}_{\mathbf{I_{St}}}(\texttt{Conc}(A_1)) = \texttt{Inst}_{\mathbf{I_{St}}}(\{\exists x^b \neg f(x^b)\}) = \{\Delta_1, \Delta_2, \Delta_3\} = I_1$ with

- $\Delta_1 = \{\neg f(T)\}$
- $\Delta_2 = \{\neg f(Z)\}$
- $\Delta_3 = \{\neg f(T), \neg f(Z)\}$

$\texttt{Inst}_{\mathbf{I_{St}}}(\texttt{Conc}(A_2)) = \texttt{Inst}_{\mathbf{I_{St}}}(\{\neg f(T)\}) = \{\Delta_4\} = I_2$ with

- $\Delta_4 = \{\neg f(T)\}$

Level 4

$\texttt{simSI}^{\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}}(I_1, I_2) = \texttt{Tve}^{1,1,1,\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}}(I_1, I_2) = \frac{a}{a+b+c} = \frac{5}{11} \simeq 0.4545$ with :

$$
\begin{aligned}
a &= \texttt{avg}\Big(\sum_{x \in I_1} \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(x, I_2), \sum_{y \in I_2} \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(y, I_1)\Big)\\
&= \texttt{avg}\Big(\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_1, I_2) + \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_2, I_2) + \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_3, I_2), \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_4, I_1)\Big)\\
&= \texttt{avg}(1 + 0 + \frac{1}{2}, 1) = 1.25\\
b &= \sum_{x \in I_1} 1 - \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(x, I_2)\\
&= (1 - \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_1, I_2)) + (1 - \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_2, I_2)) + (1 - \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_3, I_2)) = 0 + 1 + 0.5 = 1.5\\
c &= \sum_{y \in I_2} 1 - \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(y, I_1)\\
&= 1 - \varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_4, I_1) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_1, I_2) = \texttt{max}(\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_1, \Delta_4)) = 1\\
&\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_2, I_2) = \texttt{max}(\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_2, \Delta_4)) = 0\\
&\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_3, I_2) = \texttt{max}(\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_3, \Delta_4)) = \frac{1}{2}\\
&\varepsilon^{\mathbb{I}}_{\texttt{max},\texttt{s}^{\mathbb{I}}}(\Delta_4, I_1) = \texttt{max}(\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1), \texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2), \texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_3)) = 1
\end{aligned}
$$

Level 3

$\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1) = \texttt{Tve}^{1,1,1,\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1) = \frac{a}{a+b+c} = 1$ with

- $a = \texttt{avg}(\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}(\neg f(T), \Delta_1), \varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}(\neg f(T), \Delta_4)) = 1$
- $b = 1 - \varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}(\neg f(T), \Delta_1) = 0$
- $c = 1 - \varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}(\neg f(T), \Delta_4) = 0$

Since $\texttt{simI}$ is a symmetric function, we can say that $\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_1, \Delta_4) = \texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_1)$.

$\texttt{simI}^{\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2) = \texttt{Tve}^{1,1,1,\varepsilon^{\mathbb{C}}_{\texttt{max},\texttt{s}^{\mathbb{C}}}}(\Delta_4, \Delta_2) = \frac{a}{a+b+c} = 0$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_2), \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(Z), \Delta_4)) = 0$
- $b = 1 - \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_2) = 1$
- $c = 1 - \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(Z), \Delta_4) = 1$

Since $\mathtt{simI}$ is a symmetric function, we can say that $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}}(\Delta_2, \Delta_4) = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}}(\Delta_4, \Delta_2)$.

$\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}}(\Delta_4, \Delta_3) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}}(\Delta_4, \Delta_3) = \frac{a}{a+b+c} = \frac{1}{2}$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_3), \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_4) + \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(Z), \Delta_4)) = 1$
- $b = 1 - \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_3) = 0$
- $c = (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_4)) + (1 - \varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(Z), \Delta_4)) = 1$

Since $\mathtt{simI}$ is a symmetric function, we can say that $\mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}}(\Delta_3, \Delta_4) = \mathtt{simI}^{\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}}(\Delta_4, \Delta_3)$.

$\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_1) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(T))) = 1$
$\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_4) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(T))) = 1$
$\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_2) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(Z))) = 0$
$\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(Z), \Delta_4) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(Z), \neg f(T))) = 0$
$\varepsilon^{\mathbb{C}}_{\mathtt{max,s}^{\mathbb{C}}}(\neg f(T), \Delta_3) = \mathtt{max}(\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(T)), \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(Z))) = 1$

Level 1 et 2
$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(T)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(T)) = \frac{a}{a+b+c} = 1$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(T)), \varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(T))) = 1$
- $b = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(T)) = 0$
- $c = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(T)) = 0$

$\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(Z)) = \mathtt{Tve}^{1,1,1,\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(Z)) = \frac{a}{a+b+c} = 0$ with

- $a = \mathtt{avg}(\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(Z)), \varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(Z), \neg f(T))) = 0$
- $b = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(Z)) = 1$
- $c = 1 - \varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(Z), \neg f(T)) = 1$

Since $\mathtt{simC}$ is a symmetric function, we have $\mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(Z), \neg f(T)) = \mathtt{simC}^{\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}}(\neg f(T), \neg f(Z))$.

$\varepsilon^{\mathbf{L}}_{\mathtt{max,s}^{\mathbf{L}}}(\neg f(T), \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(\neg f(T), \neg f(T)))$
$= \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(\neg f(T), \neg f(T))$

$= \mathtt{simA}^{\langle \mathtt{min,eq,pws} \rangle}(f, \langle T \rangle, f, \langle T \rangle)$
$= \mathtt{min}(\mathtt{eq}(f, f), \mathtt{pws}(\langle T \rangle, \langle T \rangle))$
$= \mathtt{min}(1, \frac{\mathtt{eq}(T,T)}{1}) = \mathtt{min}(1, 1) = 1$

$\varepsilon^{\mathbf{L}}_{\mathtt{max},\mathtt{s}^{\mathbf{L}}}(\neg f(Z), \neg f(T)) = \mathtt{max}(\mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(\neg f(T), \neg f(Z)))$
$= \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(\neg f(T), \neg f(Z))$
$= \mathtt{simA}^{\langle \mathtt{min,eq,pws} \rangle}(f, \langle T \rangle, f, \langle Z \rangle)$
$= \mathtt{min}(\mathtt{eq}(f, f), \mathtt{pws}(\langle T \rangle, \langle Z \rangle))$
$= \mathtt{min}(1, \frac{\mathtt{eq}(T,Z)}{1}) = \mathtt{min}(1, 0) = 0$

Since $\mathtt{simL}$ is a symmetric function, we have :
$\mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(\neg f(Z), \neg f(T)) = \mathtt{simL}^{\langle \mathtt{min,eq,pws} \rangle}(\neg f(T), \neg f(Z)) = 0.$