Application of Newton-Raphson Homotopy Analysis Method for Solving SEIRS Epidemic Model

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Abstract

In this paper, the basic idea of homotopy analysis method is introduced. Based on Newton-Raphson method, the modified version of HAM which is much more effective is proposed. The advanced algorithm is applied for solving the SEIRS Epidemic Mathematical Model, the effectiveness of the method is proved through numerical formulation.

Keywords: Homotopy analysis method; Nonlinear differential equation; Homotopy perturbation method; Analytically approximation; Newton-Raphson Method; SEIRS epidemic model

1. Introduction

Most of the engineering problems are nonlinear, and in most cases it is very difficult to solve them analytically, except for some specific cases it is possible to find the simple closed-form analytic solution[17, 18]. Actually there are no general methods which can be applied to find the exact solutions of nonlinear algebraic equations, they can only determine the approximate solutions of nonlinear equation systems.

For differential equations, the unknown is a function not a parameter[10]. One of the most famous and most commonly used techniques for determining the approximate solution of nonlinear differential equations is the perturbation techniques. Based on the existence of small parameters called perturbation quantity, the phenomena in vibrating systems cause by nonlinear effect can be described through applying perturbation methods. Expanding in terms of a small parameter, the approximate solutions can be determined. Assume the small parameter as ϵ, the traditional perturbation technique will express the solution of the nonlinear differential equation containing ϵ as the power series of ϵ, especially when $\epsilon = 0$, this nonlinear differential equation will become linear, and the solution of nonlinear equation will be solution of linear equation with vibrating[18].

Unfortunately, many nonlinear problems don't contain perturbation parameter, which make perturbation techniques are limited to apply on several classes of mathematical problems, like weakly nonlinear problems. Towards with these there are many non perturbation techniques are developed. However, both of the perturbation techniques and the current nonperturbative methods themselves cannot basically overcome the limitations of perturbation techniques, the most important limitation is how to adjust control the convergence region and rate of given approximate series and choose the base functions expressing solutions of the nonlinear equation freely[17]. It is necessary to determine one general method for solving nonlinear differential equations.

Based on the situation, the homotopy analysis method(HAM) proposed by Liao is one powerful analytic method providing a convenient way to solve all tricky cases above. This method employees the basic idea of homotopy in topology to propose one general analytical method for solving nonlinear differential equations[19]. It has following advantages[17]:

1) Independent with small parameters. For any nonlinear equation contains any parameters or not, HAM can be applied to get the approximate solution.

2) Convergence. HAM can adjust and control the convergence region and rate of approximation series.

3) Freely Choose. For any system contains parameters or not, choice of base function can be arbitrary so that the nonlinear problem can be transferred into infinite linear sub problems to provide a more convenient approximation method.

HAM has been applied in many different practical nonlinear questions and made some significant improvement, involving the question about heat and mass transfer with a boundary layer flow[7], wave-current interaction[11], traffic jamming problem[6] and so on. The successful applications of HAM on these important engineering problem have proven the immense potential of homotopic analysis method.

Still, there keeps room for HAM to improve. In HAM one auxiliary parameter $\hbar$ is introduced for keeping room for the convergence of the approximation, which ensures the convergence of series , greatly improve the efficiency. However, for some strong nonlinear questions, only under high level of series could get approximation solutions accurate enough. On the another hand, the freedom of choosing basic function, nonlinear calculator and initial solution is one significant advantage of HAM[8]. Based on different choice of these parameters, how to ensure the convergence of solution also require detailed discussion.

Many scholars have proposed different modified method to improve the efficiency and make for deficiency of the method. Yabushita[21] proposed one optimal HAM approach through minimizing the residual of governing equations. An iterative method using intermediate approximation to substitute the initial guess was suggested by Lin[22]. All these methods made excellent improvement.

In this paper, one simple and effective modified HAM —— Newton - HAM proposed by Zhang has been introduced[11]. It is very easy and can control the choice of $\hbar$ and initial solution, this is one unique method comparing with other modified HAM approach. And it will be applied to solve the SERIES Epidemic Mathematical model.

2. Homotopy Analysis Method

2.1. Introduction of Homotopy

In topology, to build connection between any two continuous real function $f(x)$ and $g(x)$, the idea of homotopy is introduced: assume $t \in [0, 1]$, named t as embedding parameter, a homotopy between f and g is defined as one continuous function:

$$H(x;t) = tg(x) + (1-t)f(x)$$

when $t = 0$, $H(x; 0) = f(x)$; when $t = 1$, $H(x; 1) = g(x)$. Therefore, when t is changing from 0 to 1, the function $H(x;t)$ is continuously changing from $f(x)$ to $g(x)$. Therefore, it can see the function $H(x;t)$ has build the connection between $f(x)$ and $g(x)$, this is called topology, presenting as:

$$H(x;t) : f(x) \sim g(x)$$

Applying with the nonlinear function

$$f(x) = 0 \tag{1}$$

with the homotopic idea, it can construct one single-parameter nonlinear family, assume x_0 is one known solution of x, ϕ is unknown parameter, it gets:

$$(1-t)(\phi - x_0) + tf(\phi) = 0 \tag{2}$$

It can see ϕ is changing by t, ϕ is one function of t, presenting it as $\phi(t)$. Representing the function as:

$$(1-t)[\phi(t) - x_0] + tf[\phi(t)] = 0 \tag{3}$$

when $t = 0$, we get a linear function:

$$\phi(0) = x_0$$

when $t = 1$, we get:

$$f[\phi(1)] = 0 = f(x)$$

so

$$\phi(1) = x$$

when the embedding parameter change from 0 to t, $\phi(t)$ changes from x_0 to the solution of $f(x)$, the homotopy has been constructed as:

$$\phi(t) : x_0 \sim x$$

Therefore, the connection between nonlinear function $f(x) = 0$ and solution x_0 has been built through constructing one parameter family (2). Dispersing the interval $t \in [0, 1]$ as:

$$t_k = \frac{k}{M}, k = 0, 1, 2....M$$

staring from $\phi(0) = x_0$, applying the iterative method to get the solution of the function (3) when $t = t_k$. The solution of function (3) at t-1 will be the solution of $f(x) - 0$. Since t_k is one dense interval, solution of $\phi(t)$ at $t = t_k$ will be close to $t = t_{k-1}$, thus the convergence of the iterative method is promised. Starting from x_0, we the solution of the original function $f(x) = 0$ can always be found[5].

2.2. Original Homotopy Analysis Method

Based on the idea of homotopy, Liao proposed the homotopy analysis method[17, 18] on the function . Assume $\mathcal{N}$ as one nonlinear calculator, t denotes time and $u(t)$ is an unknown function, constructing one nonlinear differential equation as

$$\mathcal{N}[u(t)] = 0 \tag{4}$$

Given $p \in [0, 1]$ as embedding parameter, with another linear calculator $\mathcal{L}$, generalizing the concept of traditional homotopy, the first zero-order deformation-equation as

$$(1-p)\mathcal{L}(\phi) + p\mathcal{N}(\phi) = 0, p \in [0, 1] \tag{5}$$

since ϕ is also the function of p, presenting as $\phi(p)$, so the equation can be represented as

$$(1-p)\mathcal{L}[\phi(t;p)] + p\mathcal{N}[\phi(t;p)] = 0, p \in [0, 1] \tag{6}$$

when $p = 0$, it has

$$\mathcal{L}[\phi(t;0)] = 0$$

assume the solution of the function as $u_0(t)$, which means

$$\phi(t;0) = u_0(t)$$

when $p = 1$, it has

$$\mathcal{N}[\phi(t;1)] = 0 = \mathcal{N}[u(t)]$$

which means

$$\phi(t;1) = u(t)$$

when the embedding parameter changes from 0 to 1, the solution of $\phi(t;p)$ changes from $u_0(t)$ to $u(t)$, the homotopy relation has been built as

$$\phi(t;p) : u_0(t) \sim u(t)$$

If the variation is smooth enough, it can construct the Taylor series of $\phi(t;p)$ at $p = 0$, the higher order deformation equations can provide with the coefficients of all higher terms. At first, deriving the function $\phi(t;p)$ within Taylor Series as

$$\phi(t;p) = u_0(t) + \sum_{\infty}^{k=1} u_k(t)p^k \tag{7}$$

with

$$u_k(t) = \frac{1}{k!} \frac{\partial^k \phi(t;p)}{\partial p^k}\bigg|_{p=0} \tag{8}$$

Choosing correct linear calculator $\mathcal{L}$ in order to make the Taylor series converges at $p = 1$, then we can get the series solution

$$u(t) = u_0(t) + \sum_{k=1}^{\infty} u_k(t) \tag{9}$$

For each question, with the same zero-order deformation question, the solution $\phi(t;p)$ will also be same, the corresponding function $u_k(t)$ will also be unique. Differentiating the zeroth-order deformation equation k times with respect to the embedding parameter p, and setting $p = 0$, dividing the final equation with $k!$, then it gets the kth-order deformation equations presented as

$$\mathcal{L}[u_k(t) - \chi_k u_{k-1}(t)] = R_k(t), \tag{10}$$

$$\chi_k = \begin{cases} 0, k \leq 1 \\ 1, k > 1 \end{cases} \tag{11}$$

$$R_k(t) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \mathcal{N}[\phi(t;p)]}{\partial p^{k-1}}\bigg|_{p=0} \tag{12}$$

Related to the Taylor series (9), the nonlinear question has been transformed into infinity linear question, and the choice of linear calculator $\mathcal{L}$ can be arbitrarily chose, this is one significant improvement comparing with other old methods. Still, the method needed to be improved. First sometimes it can be hard to find the answer of $u_0(t)$. At the same time, the convergence of the Taylor series (9) at $p = 1$ is related to the choice of linear calculator $\mathcal{L}$, if the choice of calculator is not correct enough, then the convergence of (9) cannot be guaranteed.

2.3. Modified Homotopy Analysis Method

To be easier get the initial solution $u_0(t)$ and better control the convergence of series solution (9), the equation was generalized and one nonzero auxiliary parameter $\hbar$ was introduced[12]. Further, one more generalized zeroth-order deformation equation was realized through introducing one nonzero auxiliary function $H(t)$ as[13]

$$(1-p)\mathcal{L}[\phi - u_0(t)] = p\hbar H(t)\mathcal{N}[\phi], p \in [0,1] \tag{13}$$

where ϕ is depends on t, embedding parameter p, auxiliary parameter $\hbar$ and auxiliary function $H(t)$. And the corresponding kth-order deformation equation is

$$\mathcal{L}[u_k(t) - \chi_k u_{k-1}(t)] = \hbar H(t) R_k(t), \tag{14}$$

By such method, one nonlinear method has been transformed into a series of linear equations, the exaction solution of $u_0(t)$ is approximated by the Taylor series expanding at $p = 1$. Mathematically, if

$$\mathcal{L}[f(t)] = 0$$

then to any $\hbar$ and $H(t)$, it will have

$$\hbar H(t)\mathcal{L}[f(t)]$$

Then the steps of the parameter choosing can be separated in the following steps: at first the linear calculator $\mathcal{L}$ should be chose arbitrarily, next it is the nonzero auxiliary function $H(t)$, at last the auxiliary parameter $\hbar$ should be chose based on the requirement of convergence.

Through applying the HAM method, solution of nonlinear equations can be expressed in a group of basic functions(Taylor Series), presenting as :

$$u(t) = \sum_{k=0}^{\infty} c_k e_k(t) \tag{15}$$

where $e_k(t)$ is the basic function and the set of basis is presented as

$$e_0(t), e_1(t), e_2(t), ..$$

To better understand the application of the method, Liao proposed three fundamental principles to make a better choices on the initial solution $u_0(t)$, auxiliary parameter $\hbar$ and auxiliary function $H(t)$[15] ,

1) The rule of solution expression. The choice of the initial solution $u_0(t)$, auxiliary parameter $\hbar$ and auxiliary function $H(t)$ should guarantee the expression of the high order deformation function follows the expression of (15).

2) The rule of solution existence. The choice of the initial solution $u_0(t)$, auxiliary parameter $\hbar$ and auxiliary function $H(t)$ should guarantee should promise the solution of all high order deformation function exists and is unique.

3) The rule of ergodicity for coefficients of homotopy series solution. The basis coefficients c_k can be improved in the deriving process.

2.4. Newton-Raphson Homotopy Analysis Method

HAM has been successfully applied in many different nonlinear ordinary and partial differential equations, involves Blasius viscous flow problems and the progressive waves in deep water problem[14, 16], and the room for the improvement of the method should also be paid attention.

One auxiliary parameter $\hbar$ is applied in the method to control the convergence of the solution. However, one appropriate $\hbar$ is difficult to define, especially for some strong nonlinear problems, the convergence interval is difficult to find, or the interval can be defined only under really high order of approximation. Therefore, this is one questions still need to be discussed[20].

To solve the questions, the advantage of the method, as freedom, can be took to guarantee the convergence, for instance, some new deformation or choosing arbitrary parameters can be introduced. In this paper, the Newton-Raphson method will be introduced to define the auxiliary parameter $\hbar$.

Newton- Raphson method is one classical algorithm used for approximating the roots of function[9, 1]. For a function f for a variable defined on real number, suppose the function is differential on R and the derivative is $f^{'}$, $f^{'} \neq 0$ Let one initial guess of root as x_0, then the approximation x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

will be a better approximation comparing withx_0. Generally, the iterative method can be simplified as[4]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Reconsidering the original nonlinear function (1), given the initial guess x_0, suppose $|f^{'}(x_0)| > 0$, deriving the series near x can get[20]:

$$f(x-\delta) = f(x) - \delta f^{'}(x) + \frac{\delta^2}{x} f^{''}(x) + o(\delta^3)$$

to find the parameter δ such as

$$f(x-\delta) = 0 \approx f(x) - \delta f^{'}(x) + \frac{\delta^2}{2} f^{''}(x)$$

giving

$$\delta = \frac{f(x)}{f'(x)} + \frac{\delta^2 f^{''}(x)}{2f'(x)}$$

suppose

$$A(\delta) = L(\delta) + N(\delta) = c$$

with

$$L(\delta) = \delta, N(\delta) = \gamma\delta^2, \gamma = -\frac{f^{''}(x)}{2f'(x)}, c = \frac{f(x)}{f'(x)}$$

The basic idea of Newton-Raphson HAM method is applying Newton-Raphson method, this classical technique to solve the original nonlinear equation $f(x) = 0$. Given the embedding parameter $p, p \in [0, 1]$, one nonzero auxiliary parameter $\hbar$, one nonzero auxiliary function $H(t)$, the linear operator $\mathcal{L}$. Then constructing the zeroth-order deformation function as

$$(1-p)\mathcal{L}[\phi(p) - \delta_0] = p\hbar H(\delta)\{\mathcal{N}[\phi(p)] - c\} \tag{16}$$

here ϕ is one unknown function, δ_0 is the initial approximation of δ

When $p = 0$ and $p = 1$, respectively it has

$$\phi(0) = \delta_0, \phi(1) = \delta$$

When p increases from 0 to 1, $\phi(p)$ varies from the initial guess δ_0 to the solution δ, the homotopy builds. Similarly, deriving $\phi(p)$ in Taylor Series near p it can get

$$\phi(p) = \delta_0 + \sum_{m=1}^{\infty} \delta_m q^m \tag{17}$$

where

$$\delta_m = \frac{1}{m!} \frac{\partial^m \phi(p)}{\partial p^m}\bigg|_{p=0}$$

Given the series converges at $p = 1$ by properly choosing the auxiliary parameter $\hbar$ and auxiliary function $H(t)$, it must have one of the solutions of (1) presented as

$$\delta = \delta_0 + \sum_{m=0}^{\infty} \delta_m$$

Setting $H(t) = 1$, Therefore we can have the higher-order deformation function as

$$\mathcal{L}[\delta_m - \chi_m \delta_{m-1}] = \hbar R_m(\delta_0, \delta_1, ...\delta_{m-1}) \tag{18}$$

where same as (11)

$$\chi_k = \begin{cases} 0, k \leq 1 \\ 1, k > 1 \end{cases}$$

and

$$R_m(\delta_0, \delta_1, ...\delta_{m-1}) = \delta_{m-1} + \gamma \sum_{j=0}^{m-1} \delta_j \delta_{m-1-j} - (1-\chi_m)c$$

when $k \geq 1$, it has

$$\delta_m = (\chi_m + \hbar)\delta_{m-1} + \hbar\gamma \sum_{k=0}^{m-1} \delta_k \delta_{m-1-k} - \hbar(1-\chi_m)c \tag{19}$$

Use $\Delta M = \delta_0 + \delta_1 + ... + \delta_M$ to present the Mth-order approximation of δ:

When M=0, this is the Newton-Raphson method,

$$\delta \approx c = \frac{f(x)}{f'(x)}$$
$$x_0 = x - \delta = x - \frac{f(x)}{f'(x)}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

When M =1, this is the Householder iterative method[9].

$$\delta \approx \delta_0 + \delta_1$$
$$x_0 = x - \delta = x - \frac{f(x)}{f'(x)} + \hbar \frac{f^2(x) f''(x)}{2f'^3(x)}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \hbar \frac{f^2(x_n) f''(x_n)}{2f'^3(x_n)}$$

When M=2, it has

$$\begin{aligned}
\delta &\approx \delta_0 + \delta_1 + \delta_2 \\
x_0 &= x - \delta = x - \frac{f(x)}{f'(x)} + (2+\hbar)\hbar \frac{f^2(x) f^{''}(x)}{2f'^3(x)} \\
&- \hbar^2 \frac{f^3(x) f^{''2}(x)}{2f'^5(x)} \\
x_{n+1} &= x_n - \frac{f(x)}{f'(x_n)} + (2+\hbar)\hbar \frac{f^2(x_n) f^{''}(x_n)}{2f'^3(x_n)} \\
&- \hbar^2 \frac{f^3(x_n) f^{''2}(x_n)}{2f'^5(x_n)}
\end{aligned} \tag{20}$$

Assume $\hbar_0$ is the initial value of $\hbar$, rewrite (18) in a simple form as

$$x_{n+1} = a_n + \hbar_n b_n + \hbar_n^2 c_n \tag{21}$$

where

$$a_n = x_n - \frac{\omega(x_n)}{u'(x_n)}, b_n = \frac{\omega^2(x_n) f''(x_n)}{u'^3(x_n)},$$
$$c_n = \frac{\omega^2(x_n) f''(x_n)}{2u'^3(x_n)} - \frac{\omega^3(x_n) f''^2(x_n)}{2u'^5(x_n)}$$

also, rewriting the $\hbar_n$ as:

$$g(\hbar) = f(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})$$

so we have

$$\begin{aligned}
\hbar_{n+1} &= \hbar_n - \frac{g(\hbar_n)}{g'(\hbar_n)} \\
&= \hbar_n - \frac{f(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})}{f'(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})[2\hbar_n c_{n+1}]}
\end{aligned} \tag{22}$$

and

$$h_0 = -\frac{f(a_0)}{b_0 f'(a_0)}$$

where a_0 and b_0 are initial value of a_n and b_n

Therefore, the result of (21) will be one solution of the original question, and the convergence "speed" can be controlled through the choice of parameter γ.

If the approximation of initial guess is good enough, then it will be possible to find the answer within several terms. It is also proved by literatures that even if the initial value x_0 is not good enough, comparing with other methods, still much fewer iterations are needed by HAM[20]. The convergence region and rate of solution can be controlled by the parameter $\hbar$. The value of $\hbar$ can be find through the $\hbar - curves$[18].

3. Application in SEIRS Epidemic Model

The pandemic of COVID-19 in the past year has brought incalculable losses to the whole world, created more than one million of deceased and enormous financial losses. The study on epidemiology is attached with significantly practical value under the continuing global virus crisis. And many problems in epidemiology, the modeling of spread of disease, has been constructed by nonlinear differential questions. Among variety of epidemiological models, one famous is the SIR model.

3.1. Model Construction

SIR model is one epidemiological model designed for calculating the number of infected population in one closed population over time. The independent variable of the function is time t, measured in days. In the function system, two related sets of dependent variables are considered[2]

$S = S(t)$it is the number of susceptible people at time t
$I = I(t)$it is the number of infected people at time t
$R = R(t)$it is the number of recovered people at time t

With the aim of easier calculating, another sets of ratio dependent variables is also considered, that the variable of

Variables and Parameters	Description
S(t)	The number of susceptible people
I(t)	The number of infected people
E(t)	The number of exposed people
R(t)	The number of recovered people
N(t)	Population
$\gamma(t)$	The recovering rate of infected individuals
$\omega(t)$	The rate of recovered individuals become susceptible
$\Lambda(t)$	Recruitment rate
$\beta(t)$	The effective contact rate
$\epsilon(t)$	The rate of exposed individuals become infectious
$\mu(t)$	Natural death rate

Table 1. Symbols and Description of Model Parameters

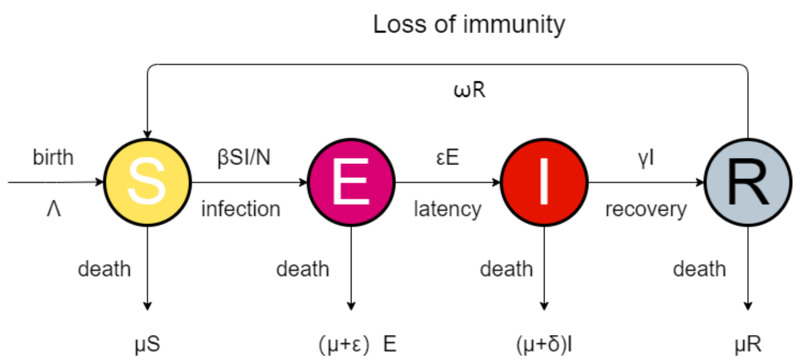

Figure 1. Diagram of Model

population N is introduced, $N = S + I + R$

$s(t) = S(t)/N$it is the susceptible fraction of the population

$i(t) = I(t)/N$ it is the infected fraction of the population

$r(t) = R(t)/N$ it is the recovered fraction of the population

Defined the rate of recovering of infected individuals as $\gamma(t)$, the rate of recovered individuals become susceptible again as $\omega(t)$. The progress of one individual is presented as[8]

$$S \to I \to R$$

To consider much more complex real life examples, one new group $E(t)$(exposed group at time t) is introduced. The exposed group is one latent period between being infected and becoming infectious[20]. And other following parameters are also introduced: the recruitment rate is defined as $\Lambda(t)$, the effective contact rate is defined as $\beta(t)$, the rate of exposed individuals become infectious is presented as $\epsilon(t)$, while the natural death rate is defined as (t). Thus, the important practical aspects including birth, death, loss of immunity all can be simulated in the system. The flow diagram of the proposed model can be presented as figure 1. And the table of variables with their description are mentioned in table 1.

3.2. Assumption of Models

To construct one real life model, the assumptions of the model are defined as[3]

1) The immunity is not permanent, the recovered individuals can still be susceptible again, presented as $\omega(t)$

2) The population is not closed. People can enter the population through birth or immigration, also they can leave by death or infection.

3) Due to the complex incidence situation practically, to simple the question the rate of birth and immigration is presented as one constant parameter, presented by $\Lambda(t)$

3.3. Mathematical Model Formulation

Based on the diagram above, the formulation of the SEIRS mathematical model can be illustrate as[2]:

$$\frac{dS(t)}{dt} = \Lambda + \omega R - \frac{\beta SI}{N} - \mu S,$$

Λ : birth, ωR : lost immunity, $\frac{\beta SI}{N}$: infection, , μS : death

$$\frac{dE(t)}{dt} = \frac{\beta SI}{N} - (\mu + \epsilon)E$$

ϵE :latency

$$\frac{dI(t)}{dt} = \epsilon E - (\mu + \delta + \gamma)I$$

γI :recovery

$$\frac{dR(t)}{dt} = \gamma I - (\mu + \omega)R$$

$$N(t) = S(t) + E(t) + I(t) + R(t) \tag{23}$$

3.4. Solution of SEIRS model by Homotopy Analysis Method

After the mode system is constructed, then it can be applied with the Newton-Raphson HAM method. At beginning, defining the initial guess of each nonlinear differential equation as

$$S_0(t) = NS, E_0(t) = NE, I_0(t) = NI, R_0(t) = NR$$

Given p as the embedding parameter, $p \in [0, 1]$ first it has

$$\begin{aligned} S(t) &\to \phi_1(t; p) \\ E(t) &\to \phi_2(t; p) \\ I(t) &\to \phi_3(t; p) \\ R(t) &\to \phi_4(t; p) \end{aligned}$$

then define the nonlinear parameter $\mathcal{L}$ as

$$\mathcal{L}_i[\phi_i(t; p)] = \frac{\partial u_i(t; p)}{\partial t}, i = 1, 2, 3, 4, 5$$

Given the nonzero auxiliary parameter $\hbar_i$ and the nonzero auxiliary function $H_i(t)$, use function (14) to construct one family of equations:

$$\begin{aligned} (1-p)\mathcal{L}[\phi_1(t;p) - S_0(t)] &= p\hbar_1 H_1(t)\mathcal{N}_1[\phi_1(t;p)] \\ (1-p)\mathcal{L}[\phi_2(t;p) - E_0(t)] &= p\hbar_2 H_2(t)\mathcal{N}_2[\phi_1(t;p)] \\ (1-p)\mathcal{L}[\phi_3(t;p) - I_0(t)] &= p\hbar_3 H_3(t)\mathcal{N}_3[\phi_1(t;p)] \\ (1-p)\mathcal{L}[\phi_4(t;p) - R_0(t)] &= p\hbar_4 H_4(t)\mathcal{N}_4[\phi_1(t;p)] \end{aligned}$$

Applying with function (7) and (8), with the given initial guess, expanding $\phi_i(t)$ using the Taylor's Theorem of the embedding parameter p as

Variables0	Values
S(0)	40
I(0)	20
E(0)	10
R(0)	10
λ	10
μ	0.2
ϵ	1.2
γ	0.4
β	0.05
δ	0.2

Table 2. Parameter Values

$$\phi_1(t; p) = S_0(t) + \sum_{k=1}^{\infty} S_k(t)p^k$$

$$\phi_2(t; p) = E_0(t) + \sum_{k=1}^{\infty} E_k(t)p^k$$

$$\phi_3(t; p) = I_0(t) + \sum_{k=1}^{\infty} I_k(t)p^k$$

$$\phi_4(t; p) = R_0(t) + \sum_{k=1}^{\infty} R_k(t)p^k$$

where

$$S_k(t) = \frac{1}{k!}\frac{\partial^k \phi_1(t; p)}{\partial p^k}\bigg|_{p=0}$$

$$E_k(t) = \frac{1}{k!}\frac{\partial^k \phi_2(t; p)}{\partial p^k}\bigg|_{p=0}$$

$$I_k(t) = \frac{1}{k!}\frac{\partial^k \phi_3(t; p)}{\partial p^k}\bigg|_{p=0}$$

$$R_k(t) = \frac{1}{k!}\frac{\partial^k \phi_4(t; p)}{\partial p^k}\bigg|_{p=0}$$

Following is the function (14), presented as

$$\begin{aligned} \mathcal{L}[S_k(t) - \chi_k S_{k-1}(t)] &= \hbar_1 H_1(t) R_k(t) \\ \mathcal{L}[E_k(t) - \chi_k E_{k-1}(t)] &= \hbar_2 H_2(t) R_k(t) \\ \mathcal{L}[I_k(t) - \chi_k I_{k-1}(t)] &= \hbar_3 H_3(t) R_k(t) \\ \mathcal{L}[R_k(t) - \chi_k R_{k-1}(t)] &= \hbar_4 H_4(t) R_k(t) \end{aligned}$$

3.5. Numerical Solutions

To test the efficiency of the method, following values of the parameters contained in the model are presented in table 2[3].

And the graphs results OF Calculating the approximation series solution by N-R HAM for the four equations, up to fifth iteration. The graph of the fifth iteration are showed as follow

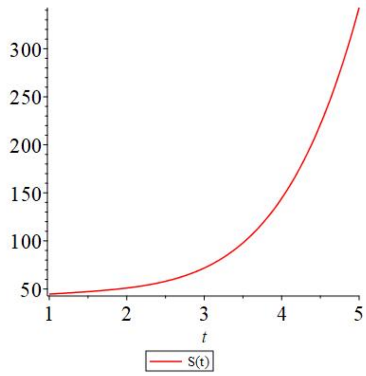

Figure 2. Graph of S(t)

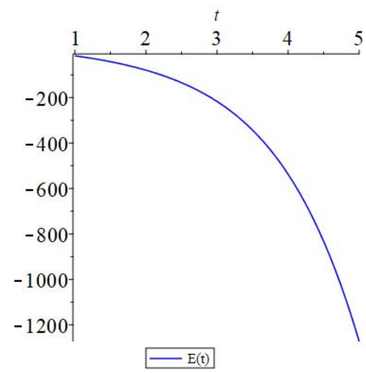

Figure 3. Graph of E(t)

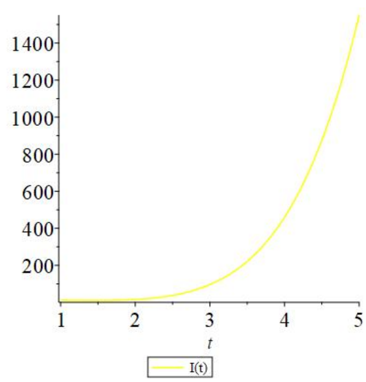

Figure 4. Graph of I(t)

4. Summary and Conclusion

In this paper, the basic idea of Homotopy analysis method with the modified version Newton-Raphson HAM were introduced. And the application of the methods to one classical epidemic model was realized. In the mathematical formulation part, the strong ability of the method to fast converging was proved. Only with several iterations, the series could converge. Thus the conclusion here is the HAM

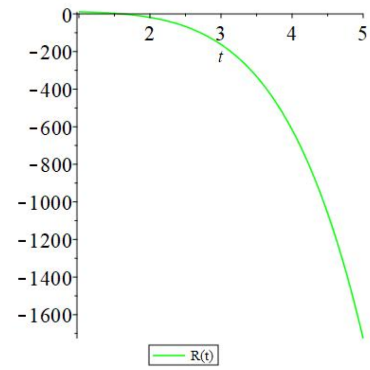

Figure 5. Graph of R(t)

is very powerful method for solving nonlinear differential equations. From graph, the Susceptible and Recovered go up. These indicated that one infected individual will produce less than one new infected individual, the final trend of epidemic model will presents locally asymptotically stable.

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