Application of Newton-Raphson Homotopy Analysis Method for Solving SEIRS Epidemic Model

Kewei Zhang 1064198

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- Homotopy Analysis Method
- Newton-Raphson HAM
- SEIRS Epidemic Model
- Application

Nonlinear Differential Equation

Perturbation: ε perturbation quantity or Non-perturbation

$\downarrow$?

convergence region

+

base function

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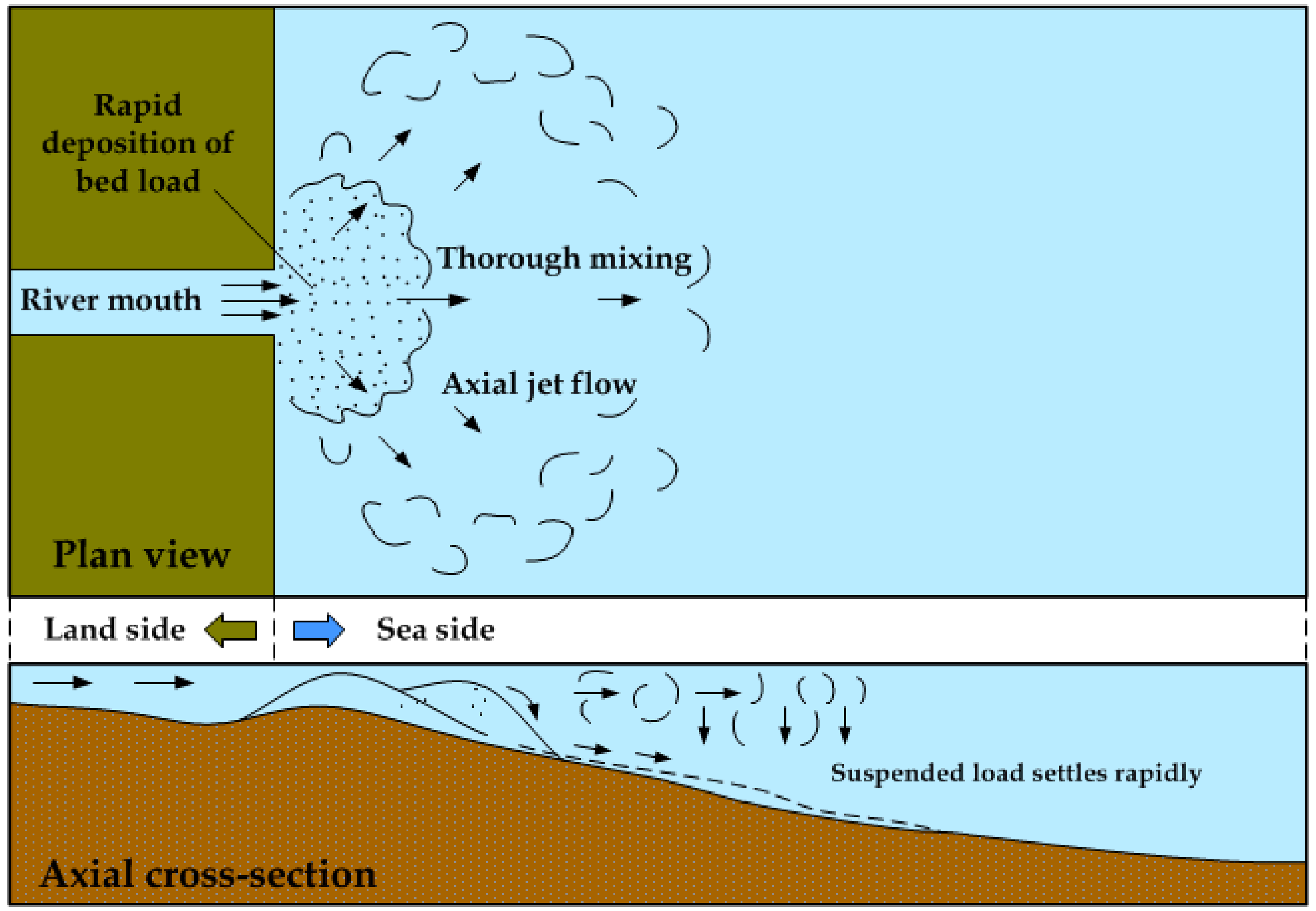
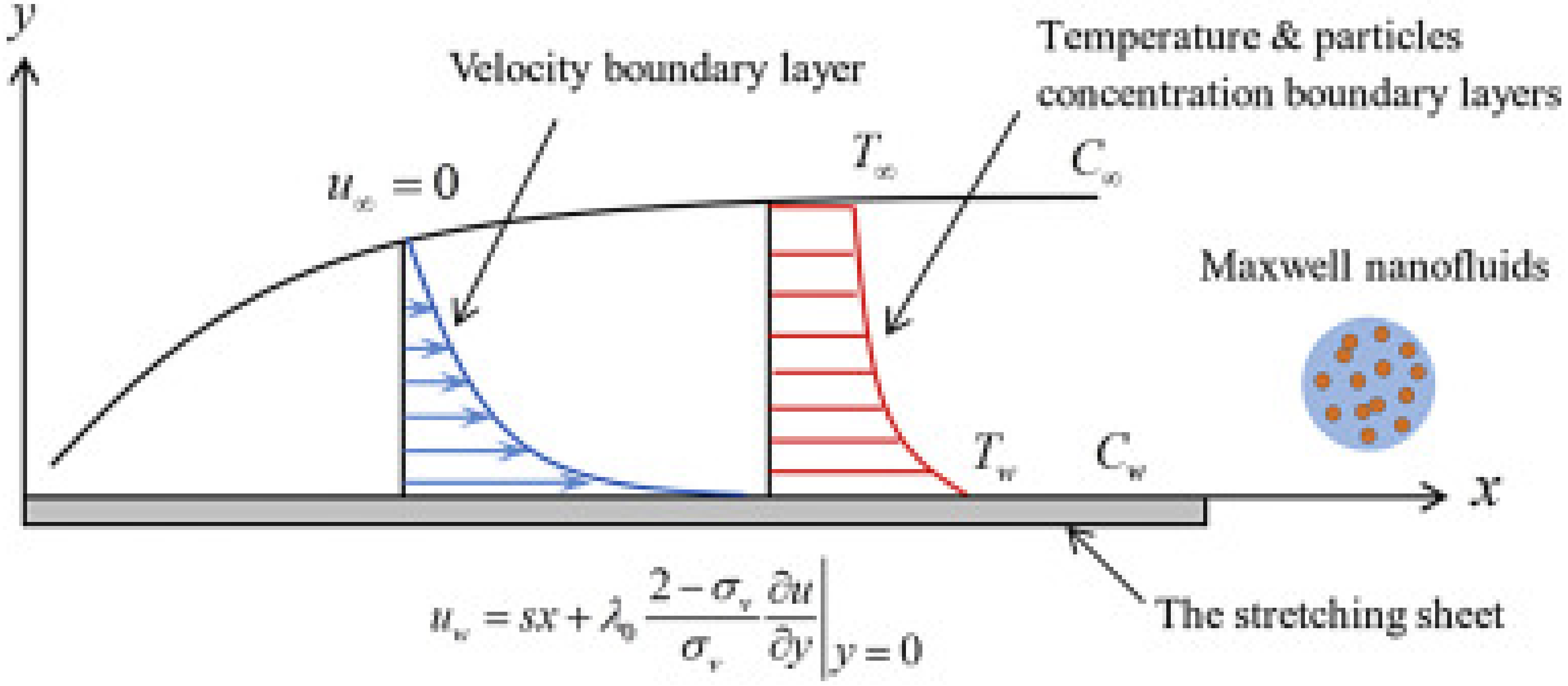
Homotopy Analysis Method

- Independent with small parameters. For any nonlinear equation contains any parameters or not, HAM can be applied to get the approximate solution.
- Convergence. HAM can adjust and control the convergence region and rate of approximation series.
- Freely Choose. For any system contains parameters or not, choice of base function can be arbitrary so that the nonlinear problem can be transferred into infinite linear sub problems to provide a more convenient approximation method.



Wave Current Interaction

Heat and mass transfer
with a boundary layer flow



Homotopy

f(x)=0 → x0

embedding parameter t, t∈ [0; 1]

$$H(x;t) = tg(x) + (1-t)f(x) \rightarrow H(x;t) : f(x) \sim g(x)$$

one known solution x0, unknown parameter φ

$$(1-t)[\phi(t) - x_0] + tf[\phi(t)] = 0 \rightarrow \phi(t) : x_0 \sim x$$

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Homotopy Analysis Method

zeroth-order deformation equation

$$\mathcal{N}[u(t)] = 0$$

$$(1-p)\mathcal{L}[\phi(t;p)] + p\mathcal{N}[\phi(t;p)] = 0, p \in [0,1]$$

$$\phi(t;p) : u_0(t) \sim u(t)$$

deriving with Taylor Series near p

$$\phi(t;p) = u_0(t) + \sum_{\infty}^{k=1} u_k(t)p^k \qquad u_k(t) = \frac{1}{k!}\frac{\partial^k \phi(t;p)}{\partial p^k}\bigg|_{p=0}$$

kth-order deformation equation

Differentiating the zeroth-order deformation equation k times with respect to the embedding parameter p, and setting p = 0, dividing the final equation with k!,

$$\mathcal{L}[u_k(t) - \chi_k u_{k-1}(t)] = R_k(t),$$

$$\chi_k = \begin{cases} 0, k \leq 1 \\ 1, k > 1 \end{cases}$$

$$R_k(t) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \mathcal{N}[\phi(t;p)]}{\partial p^{k-1}} \bigg|_{p=0}$$

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Modified Homotopy Analysis Method

$$(1-p)\mathcal{L}[\phi - u_0(t)] = p\hbar H(t)\mathcal{N}[\phi], p \in [0,1]$$

$$\mathcal{L}[u_k(t) - \chi_k u_{k-1}(t)] = \hbar H(t) R_k(t)$$

nonzero auxiliary parameter hbar: To be easier get the initial solution u0(t)
nonzero auxiliary function H(t): better control the convergence of series solution

Newton-Raphson Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

reconsider f(x)=0, deriving the series centered at x:

$$f(x-\delta) = f(x) - \delta f'(x) + \frac{\delta^2}{x} f''(x) + o(\delta^3)$$

find δ:

$$f(x-\delta) = 0 \approx f(x) - \delta f'(x) + \frac{\delta^2}{2} f''(x)$$

$$\delta = \frac{f(x)}{f'(x)} + \frac{\delta^2 f''(x)}{2f'(x)}$$

$$A(\delta) = L(\delta) + N(\delta) = c$$

$$L(\delta) = \delta, N(\delta) = \gamma\delta^2, \gamma = -\frac{f''(x)}{2f'(x)}, c = \frac{f(x)}{f'(x)}$$

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Newton-Raphson HAM

zeroth-order deformation function

$$(1-p)\mathcal{L}[\phi(p)-\delta_0] = p\hbar H(\delta)\{\mathcal{N}[\phi(p)]-c\}$$

Mth-order deformation function $\Delta M = \delta_0 + \delta_1 + ... + \delta_M$

M=0, Newton-Raphson Method

M=1, Householder iterative method

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M=2

$$\delta \approx \delta_0 + \delta_1 + \delta_2$$

$$x_0 = x - \delta = x - \frac{f(x)}{f'(x)} + (2+\hbar)\hbar \frac{f^2(x) f''(x)}{2f'^3(x)} - \hbar^2 \frac{f^3(x) f''^2(x)}{2f'^5(x)}$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)} + (2+\hbar)\hbar \frac{f^2(x_n) f''(x_n)}{2f'^3(x_n)} - \hbar^2 \frac{f^3(x_n) f''^2(x_n)}{2f'^5(x_n)}$$

Choice of hbar

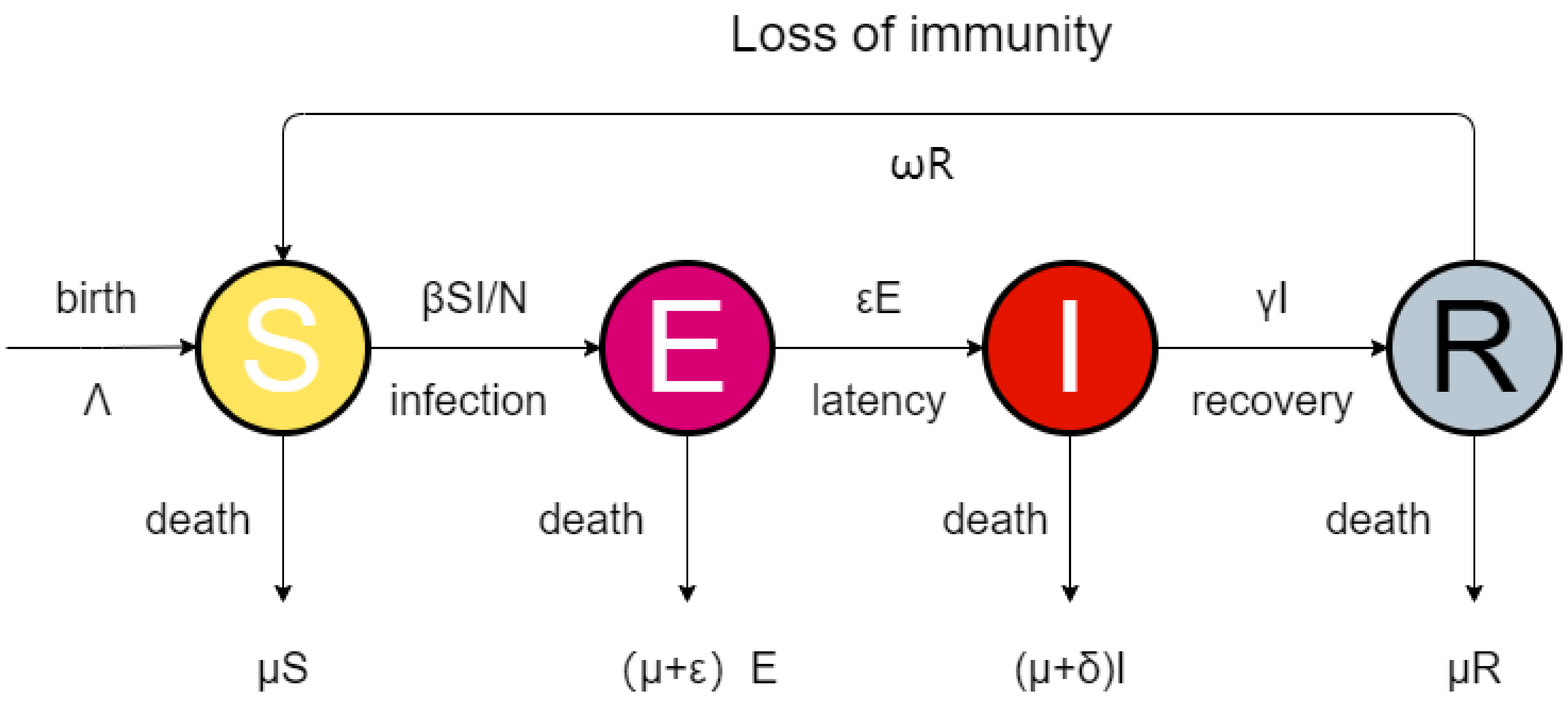
$$x_{n+1} = a_n + \hbar_n b_n + \hbar_n^2 c_n$$

$$g(\hbar) = f(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})$$

$$\hbar_{n+1} = \hbar_n - \frac{g(\hbar_n)}{g'(\hbar_n)} = \hbar_n - \frac{f(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})}{f'(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})[2\hbar_n c_{n+1}]}$$

$$h_0 = -\frac{f(a_0)}{b_0 f'(a_0)}$$

SEIRS Epidemic Model



Assumptions:

1) The immunity is not permanent, the recovered individuals can still be susceptible again, presented as omega(t)

2) The population is not closed. People can enter the population through birth or immigration, also they can leave by death or infection.

3) Due to the complex incidence situation practically, to simple the question the rate of birth and immigration is presented as one constant parameter, presented by Λ(t)

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Mathematical Model Formulation

Variables and Parameters	Description
S(t)	The number of susceptible people
I(t)	The number of infected people
E(t)	The number of exposed people
R(t)	The number of recovered people
N(t)	Population
$\gamma(t)$	The recovering rate of infected individuals
$\omega(t)$	The rate of recovered individuals become susceptible
$\Lambda(t)$	Recruitment rate
$\beta(t)$	The effective contact rate
$\epsilon(t)$	The rate of exposed individuals become infectious
$\mu(t)$	Natural death rate

Variables0	Values
S(0)	40
I(0)	20
E(0)	10
R(0)	10
λ	10
μ	0.2
ϵ	1.2
γ	0.4
β	0.05
δ	0.2

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Nonlinear Differential Equations

$$\frac{dS(t)}{dt} = \Lambda + \omega R - \frac{\beta SI}{N} - \mu S,$$

Λ : birth, ωR : lost immunity, $\frac{\beta SI}{N}$: infection, , μS : death

$$\frac{dE(t)}{dt} = \frac{\beta SI}{N} - (\mu + \epsilon)E$$

ϵE :latency

$$\frac{dI(t)}{dt} = \epsilon E - (\mu + \delta + \gamma)I$$

γI :recovery

$$\frac{dR(t)}{dt} = \gamma I - (\mu + \omega)R$$

$$N(t) = S(t) + E(t) + I(t) + R(t)$$

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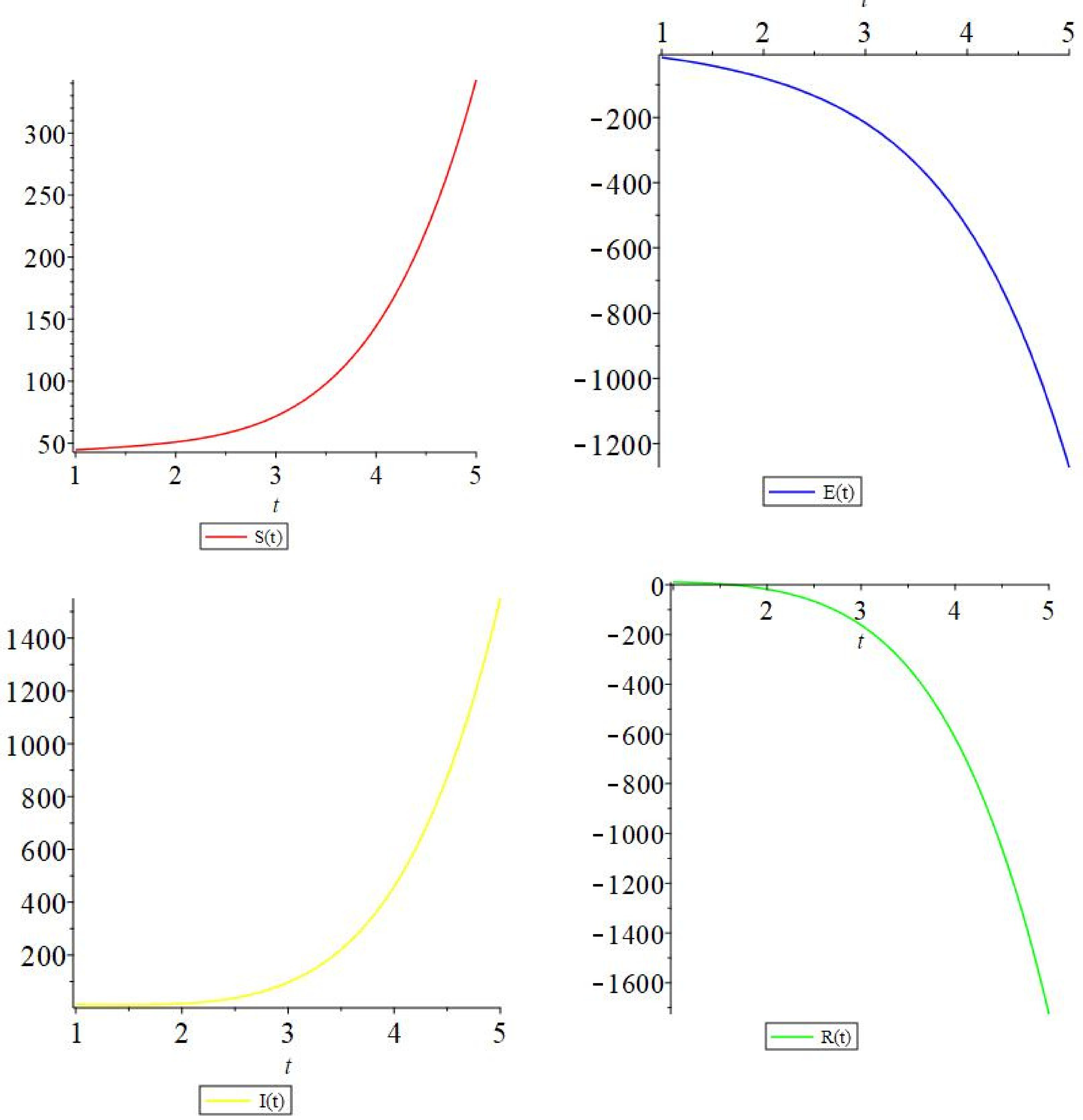
Graph Results

(of the fifth iteration)

The Susceptible and Recovered go up. These indicated that one infected individual will produce less than one new infected individual, the final trend of epidemic model will presents locally asymptotically stable.



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