① – ⑥ KOMBA ##

I AREA MOMENTS of INERTIA

- Also called second moment of area.
- Units: m^4 [Note: mass moment of inertia has units kgm^2]

Now consider area 'A' in the x-y plane (fig. below). An elemental area 'dA' will have second moments of area about x-axis, y-axis and z-axis given by

$d\bar{I}_x = y^2 dA$, $d\bar{I}_y = x^2 dA$ and $d\bar{I}_z = r^2 dA$ respectively.

∴ Moments of inertia of whole area 'A' about x, y and z axes are

$$\bar{I}_x = \int d\bar{I}_x = \int y^2 dA, \quad \bar{I}_y = \int x^2 dA \quad \text{and}$$

$$\bar{I}_z = \int r^2 dA.$$

Now $\hat{I}_z$ is usually denoted by $\bar{J}_z$ – is called polar moment of inertia. Is the moment of inertia of area 'A' about pole axis (z-axis).

Note $r^2 = x^2 + y^2$

$$\therefore \bar{J}_z = \int (x^2+y^2)\, dA = \bar{I}_x + \bar{I}_y$$

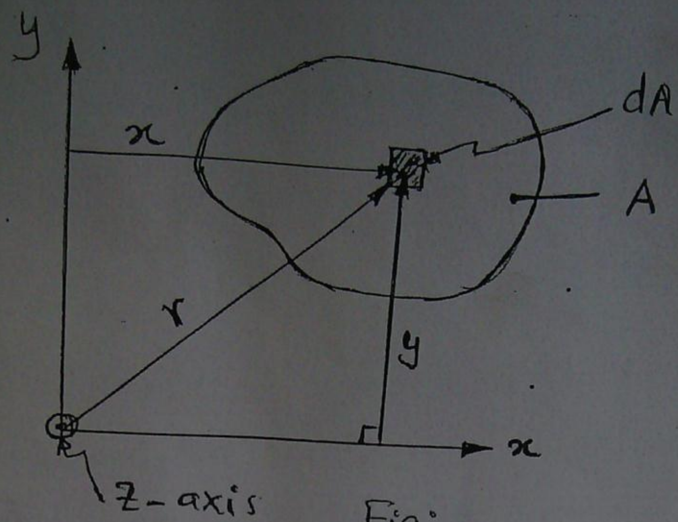
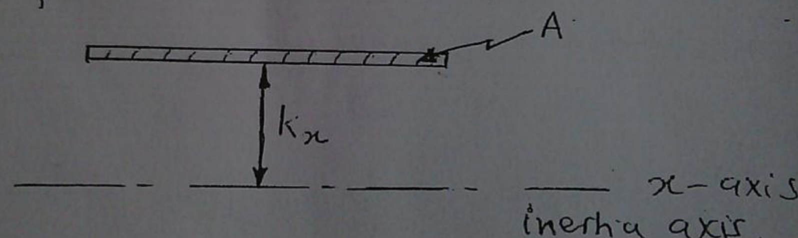


Fig:

Radius of gyration

— Is a measure of the distribution of the area from the inertia axis.

e.g.



That is the area is assumed to be concentrated into a thin strip at a distance 'k' from the axis of inertia.

Thus $\underline{I = k^2 A \quad \text{or} \quad k = \sqrt{I/A}}$

k = radius of gyration

Note: $\underline{k_z^2 = k_x^2 + k_y^2}$ for pole axis.

Parallel axis theorem

Let x', y' be c.g. axes of area 'A'.
Then consider moments of inertia of area at some parallel axes x and y say (fig. below).

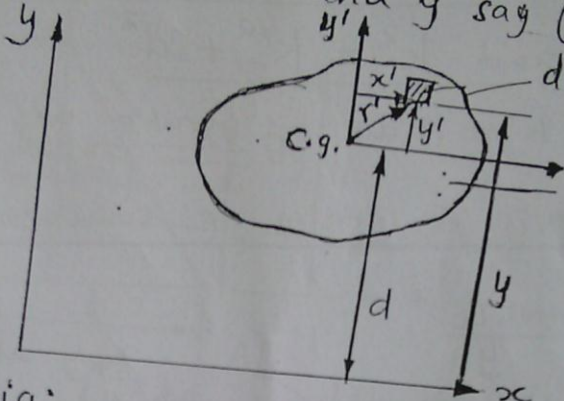


Fig:

Now $I_{x'} = \int y'^2 dA$ and $I_{y'} = \int x'^2 dA$
about c.g. axes say are known.

Now $I_x = ?$

$$I_x = \int y^2 dA, \quad y = y' + d$$

$$\therefore I_x = \int (y'+d)^2 dA$$

$$= \int y'^2 dA + 2d\int y' dA + d^2 \int dA$$

$$= I_{x'} + 0 + d^2 \cdot A$$

i.e. $\int y' dA = 0$ Varignon's theorem.
(First moment of area about c.g. axis, $-$ve cancel with $+$ve y)

Note $\int dA = A$

$$\therefore \bar{I}_x = \bar{I}_{x'} + d^2 \cdot A$$

parallel axes theorem

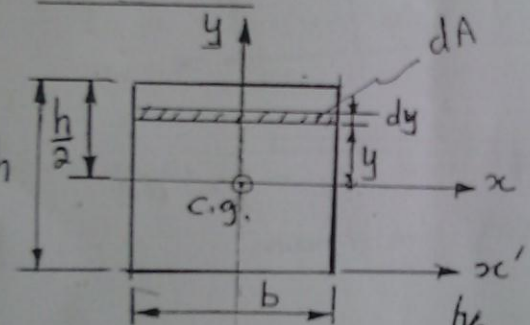
Similarly $$J_z = J_{z'} + d^2 A$$

and $$k^2 = k'^2 + d^2$$

Where k' = radius of gyration at c.g. axis.

MOMENTS OF INERTIA OF SOME COMMON SHAPES

1. RECTANGLE



x, y axes throug c.g.

$\therefore dA = b\,dy$

$$\bar{I}_x = \int y^2 dA = \int_{-h/2}^{h/2} b y^2 dy = 2b \int_0^{h/2} y^2 dy$$

$$\therefore \bar{I}_x = \frac{2b}{3}\left[y^3\right]_0^{h/2} = \frac{2b}{3} \times \frac{h^3}{8} = \frac{1}{12} bh^3$$

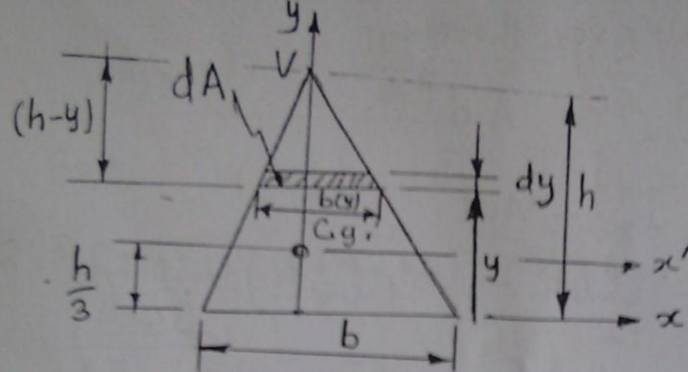
$$\therefore \bar{I}_x = \frac{1}{12} bh^3 \quad /// \quad \bar{I}_y = \frac{1}{12} hb^3$$

Note: $I_{x'} = \bar{I}_x + A\left(\frac{h}{2}\right)^2$, $A = bh$

by parallel axes theorem.

$$\therefore \quad I_{x'} = \frac{1}{3} b h^3$$

2. TRIANGLE



Now $I_x = \int y^2 dA$; $dA = b(y) \cdot dy$

By similar Δ_s $\quad \dfrac{b(y)}{b} = \dfrac{(h-y)}{h}$

$$\therefore \quad b(y) = \frac{(h-y)b}{h}$$

$$\therefore \quad dA = \frac{b}{h}(h-y)\, dy$$

$$\therefore \quad I_x = \int \frac{b}{h}(h-y)\, y^2\, dy = \frac{b}{h}\int_0^h y^2 (h-y)\,dy$$

$$\therefore \quad I_x = \frac{b}{h}\left[\frac{y^3 h}{3} - \frac{y^4}{4}\right]_0^h = \frac{b}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right)$$

$$\therefore \quad I_x = \frac{1}{12} b h^3$$

$I_{x'} = ?$ i.e. about c.g. axis.

c.g. $= \frac{1}{3}h$ from the base

Area of $\Delta = \frac{1}{2}bh$

By parallel axes theorem

$$I_x = I_{x'} + Ad^2$$

$$\therefore I_{x'} = I_x - Ad^2$$

$$= \frac{1}{12}bh^3 - \frac{1}{2}bh\left(\frac{1}{3}h\right)^2 = \underline{\frac{1}{36}bh^3}$$

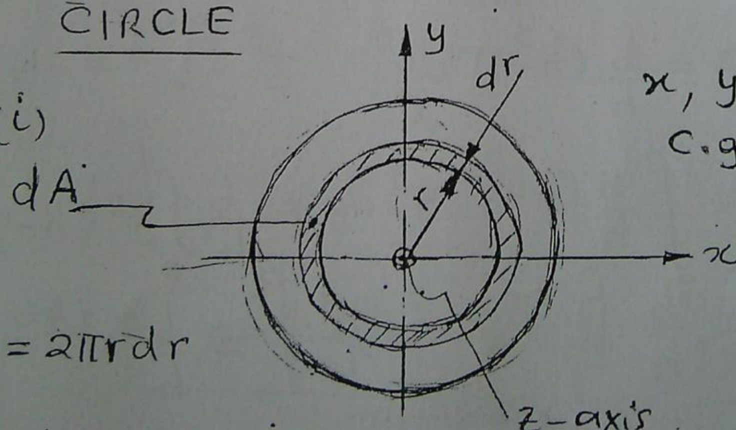
$$\therefore \underline{I_{x'} = \frac{1}{36}bh^3}$$ i.e. I_G

Through vertex V by parallel axes theorem

$$\underline{I_V = \frac{1}{4}bh^3}$$ prove.

3. <u>CIRCLE</u>

(i)



x, y axes through c.g. of circle

$dA = 2\pi r\,dr$

Now $J_z = \int r^2 dA$

$\therefore J_z = \int_0^r 2\pi r^3 dr = 2\pi\left[\frac{r^4}{4}\right]_0^r$

$\therefore J_z = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$, $d =$ dia.

But $J_z = I_x + I_y$, $I_x = I_y$

$\therefore J_z = 2I_x = 2I_y$

$\therefore I_x = I_y = \frac{J_z}{2}$

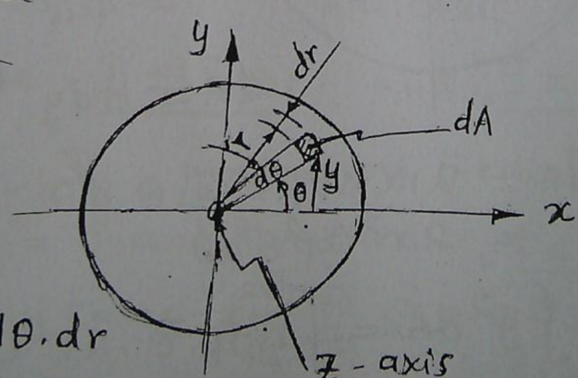
$\therefore I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$

Hollow sections d_i, d_o, r_i, r_o

Then $J = J_o - J_i$

and $I = I_o - I_i$

(ii)



$dA = r d\theta . dr$

$y = r\sin\theta$

$$\therefore \quad I_x = \int y^2 dA = \int r^2 \sin^2\theta \, . \, r\,d\theta \, . \, dr$$

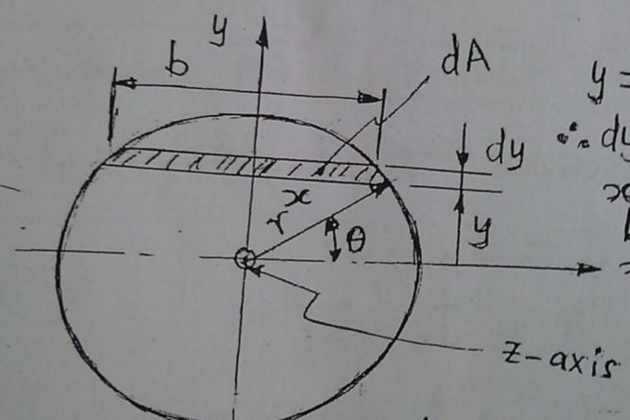
$$\therefore \quad I_x = \int_0^{2\pi}\int_0^{r} r^3 \sin^2\theta \, d\theta \, dr$$

$$= \int_0^{2\pi} \frac{r^4}{4} \sin^2\theta \, d\theta$$

$$= \frac{r^4}{4}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$\therefore \quad I_x = I_y = \frac{\pi r^4}{4} \quad \& \therefore J_z = \frac{\pi r^4}{2}$$

(iii)



$y = r\sin\theta$

$\therefore dy = r\cos\theta \, d\theta$

$x = r\cos\theta$

$b = 2x$

$$dA = b\,dy$$

$$\therefore \quad dA = 2r\cos\theta \, . \, r\cos\theta \, d\theta$$

$$= 2r^2 \cos^2\theta \, d\theta$$

$$I_x = \int y^2 dA = \int_{-\frac{\pi}{2}}^{\pi/2} r^2 \sin^2\theta \, . \, 2r^2\cos^2\theta \, d\theta$$

$$\therefore \bar{I}_x = 2\int_0^{\pi/2} 2r^4 \sin^2\theta \cos^2\theta \, d\theta$$

$$= 4r^4 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta$$

$$= 4r^4 \left[\frac{\theta}{8} - \frac{\sin 4\theta}{32} \right]_0^{\pi/2}$$

$$\therefore \bar{I}_x = \bar{I}_y = 4r^4 \times \frac{\pi}{16} = \frac{\pi r^4}{4}$$

$$\text{and } \bar{J}_z = 2\bar{I}_x = 2\bar{I}_y = \frac{\pi r^4}{2}$$

Integral formula

$$\int \sin^2 ax \cos^2 ax \, dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

II <u>AREA MOMENTS OF INERTIA PER UNIT LENGTH</u>

i.e $\bar{I}_u$, $\bar{J}_u$ APPLICABLE TO WELD JOINTS.

Hence $I = \bar{I}_u \times t$ and $J = \bar{J}_u \times t$

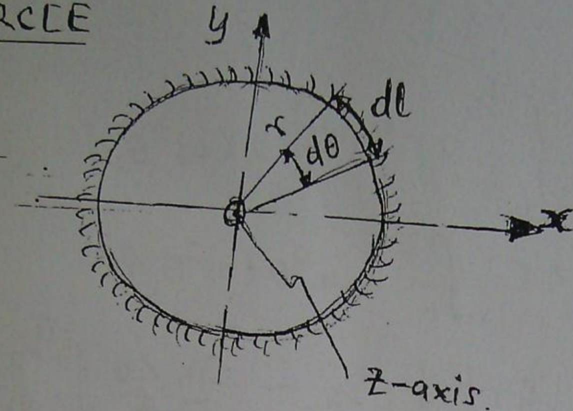
$\bar{I}_u$, $\bar{J}_u$ moments of inertia per unit length

t = throat thickness $t = k.S$

Many cases $t = \frac{\sqrt{2}}{2} S \simeq 0.7S$

S = size of weld.

1. CIRCLE

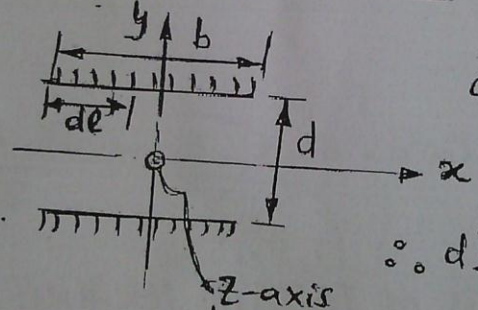


$$\therefore \; dJ_{uz} = r^2 dl \quad , \quad dl = r\,d\theta$$

$$\therefore \; J_{uz} = r^3 \int_0^{2\pi} d\theta = \underline{2\pi r^3}$$

$$\& \therefore \; \underline{I_{ux} = I_{uy} = \pi r^3}$$

2.



$$dI_{ux} = 2y^2 dl$$

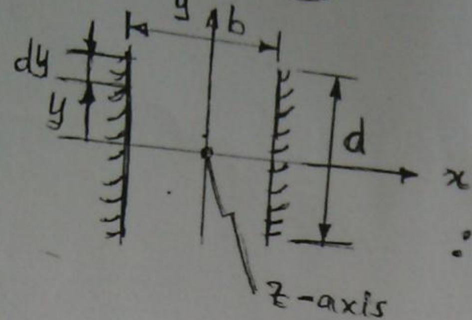
$$y = \frac{d}{2} \; , \; dl = dx$$

$$\therefore \; dI_{ux} = 2 \cdot \frac{d^2}{4} \cdot dx$$

$$\therefore \; I_{ux} = \frac{d^2}{2} \int_{-b/2}^{b/2} dx = d^2 * \frac{b}{2} = \frac{bd^2}{2}$$

$$\therefore \; \underline{I_{ux} = \frac{bd^2}{2}}$$

3.



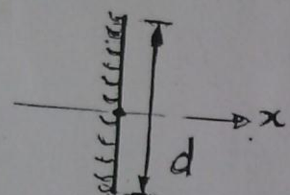
$$dI_{ux} = 2 \cdot y^2 dy$$

$$\therefore I_{ux} = 2 \int_{-d/2}^{d/2} y^2 dy$$

$$\therefore I_{ux} = \frac{4}{3}\left[y^3\right]_0^{d/2} = \frac{4}{3} \times \frac{d^3}{8} = \frac{d^3}{6}$$

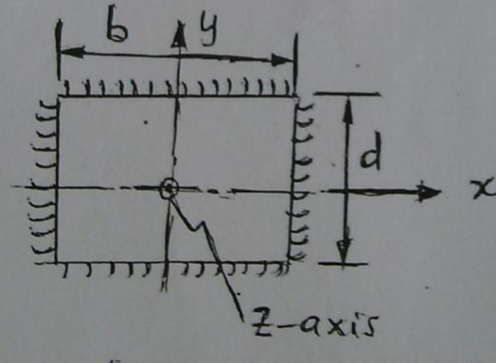
$$\therefore \underline{I_{ux} = d^3/6}$$

4.



$$\underline{I_{ux} = \frac{d^3}{12}}$$

5.



Then using above results we have

$$I_{ux} = \frac{bd^2}{2} + \frac{d^3}{6} = \underline{\frac{d^2}{6}(3b + d)}$$

and $I_{uy} = \frac{db^2}{2} + \frac{b^3}{6}$

$= \frac{b^2}{6}(3d + b)$

$\therefore J_{uz} = I_{ux} + I_{uy}$

$= \frac{bd^2}{2} + \frac{d^3}{6} + \frac{db^2}{2} + \frac{b^3}{6}$

$= (3bd^2 + d^3 + 3db^2 + b^3)/6$

$= (b+d)^3/6$

$\therefore J_{uz} = (b+d)^3/6$