Strength of Material

National Institute of Transport-NIT

Delivered by:

By Mr. Lucas Chacha Mwita
Mechanical and Environmental Engineer
Email: classeshomeworks@gmail.com
Phone: +255 752 024 320

Center of Gravity and Moment of Inertia

The Center of Gravity

The Center of gravity of a body is a point through which the whole weight of the body acts, or s the average location of the weight of an object.

Similarly, the center of gravity of a body or system of particles is the resultant of the weights of the individual particles that make up the body or system.

The centre of gravity locates the resultant weight of a system of particles.

The Centroid

The centroid is the geometry centre of a figure, also called the central point of a figure.

It is a point that matches the centre of gravity of a particular shape

It is the point which corresponds to the figure mean position of all the points

In other words, a lamina is a flat object with negligible thickness, the center of gravity is abbreviated as $C.G$

Lamina Body

In mechanics, a 2D rigid body is referred to as a lamina.

Lamina is so thin that it can be viewed as 2D dimensional regions.

Lamina has mass and weight even though their thicknesses are negligible.

The centroid is the term used for two-dimensional shapes or 2D rigid bodies. The centre of mass is the term used for three-dimensional shapes or 3D objects

A uniform Body

A uniform body is a body whose density is the same throughout the body.

If the body has a line of symmetry, the centre of mass will lie on this line.

For example, the centre of mass of circular lamina will be at the centre of the circle, since the centre of mass coincides with the geometric centre for the circular shape.

A uniform Body

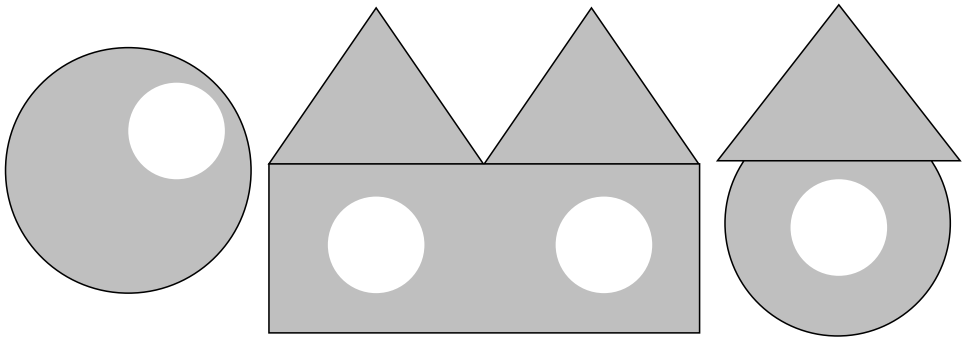
A uniform body is a body whose density is the same throughout the body.

If the body has a line of symmetry, the centre of mass will lie on this line.

For example, the centre of mass of circular lamina will be at the centre of the circle, since the centre of mass coincides with the geometric centre for the circular shape.

For a complex shaped- lamina, the center of mass is obtained by dividing the complex-shaped lamina into simple shape for which center of mass are know to us, i.e., circle, rectangular, triangle etc.

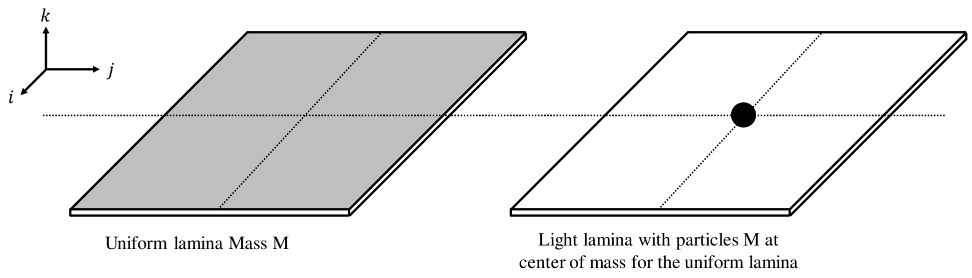
Complex shaped Lamina



Techniques to be adopted when calculating $C.G$

- Replace the uniform lamina by a light lamina for which a single particle of mass equal to the mass of the uniform lamina attached to the light lamina.

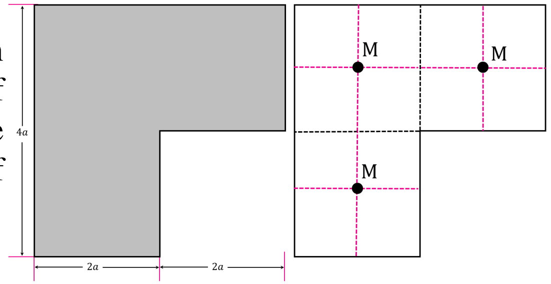
- Reduce the complex shaped lamina into simple shapes for which the centre of masses is known and determine the centre of mass for each set of particles positioned on these simpler shapes.



Example:

For the square lamina missing quadrant, the lamina could be broken into square shapes as shown

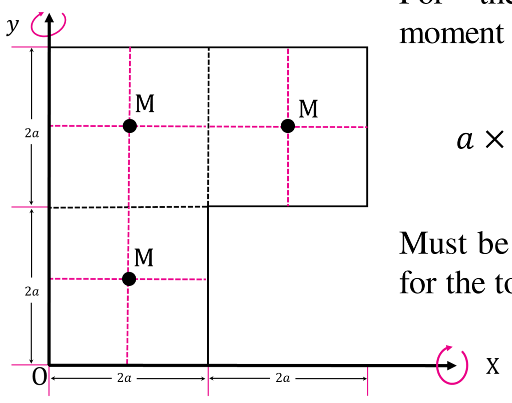
The CG of each square is then determined by geometric of symmetry, and the particles are replaced by particles of equivalent mass M



After reducing the complex shaped lamina into particle of known mass and spatial separation,

The CG = summing up all the moments for each particles about any reference of our choice

The moment of the particles of the total mass is equal to the sum of the moments for all particles.



For the component particles, the moment about y-axis is,

$$a \times Mg + a \times Mg + 3a \times Mg$$

Must be equivalent to the total moment for the total mass particle.

$$3Mg \times \bar{x}$$

Therefore, equating the two moments equation, we get:

$$a \times Mg + 3a \times Mg + 3a \times Mg$$

$$\bar{x} = \frac{5a \times Mg}{3Mg} = \frac{5a}{3}$$

Whereas, $\bar{x}$ is the distance of the centre of mass from the y-axis

Similarly, the moment about x-axis for the component particle is,

$$a \times Mg + 3a \times Mg + 3a \times Mg$$

Must be equivalent to the total moment for the total mass particle.

$$3Mg \times \bar{y}$$

Whereas, $\bar{y}$ is the centre of mass from x −axis

Therefore, equating the two moments equation, we get:

$$3Mg \times \bar{y} = a \times Mg + 3a \times Mg + 3a \times Mg$$

$$\bar{y} = \frac{7a \times Mg}{3Mg} = \frac{7a}{3}$$

Generally, we have,

$$(m_1 g + m_2 g + m_3 g \ldots \ldots m_n g)\bar{x} = m_1 g \times x_1 + m_2 g \times x_2 + \cdots m_n g \times x_n$$

$$\bar{x} = \frac{m_1 g \times x_1 + m_2 g \times x_2 + m_3 g \times x_3 + \cdots m_n g \times x_n}{m_1 g + m_2 g + m_3 g \ldots \ldots m_n g}$$

$$\bar{x} = \frac{m_1 \times x_1 + m_2 \times x_2 + m_3 \times x_3 + \cdots m_n \times x_n}{m_1 + m_2 + m_3 \ldots \ldots m_n}$$

$$m_1 \times x_1 + m_2 \times x_2 + m_3 \times x_3 + \cdots m_n \times x_n = \sum_{i=1}^{n} m_i x_i$$

$$M = m_1 + m_2 + m_3 \ldots \ldots m_n = \sum_{i=1}^{n} m_i$$

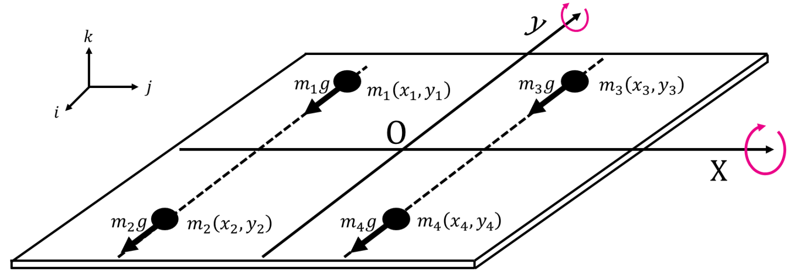
Therefore,

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

and

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

Depending on the chosen coordinates for the particular shape, the particles coordinate may be positive or negative.



If O is taken as the origin, then the coordinate for particle m_1 and m_2 are both negative.

When the CG can't be found using the axes of symmetry, it can then be found using an integral approach.

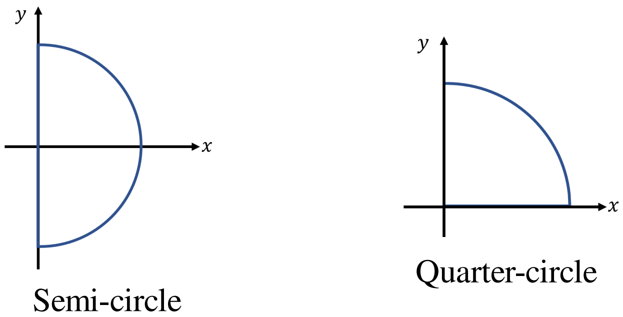
Another useful information,

$$Mass = Density \times Volume = \rho \times V = \rho \times A$$

For the uniform lamina, i.e., constant density, the volume of the strip is just its area since it is lamina with negligible thickness.

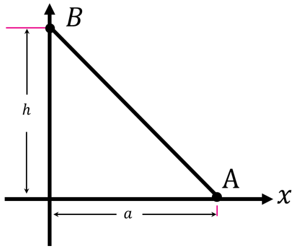
Example

Find the CG for shown figures

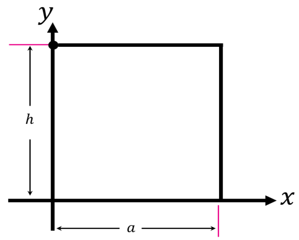


Example

Find the CG for shown figures



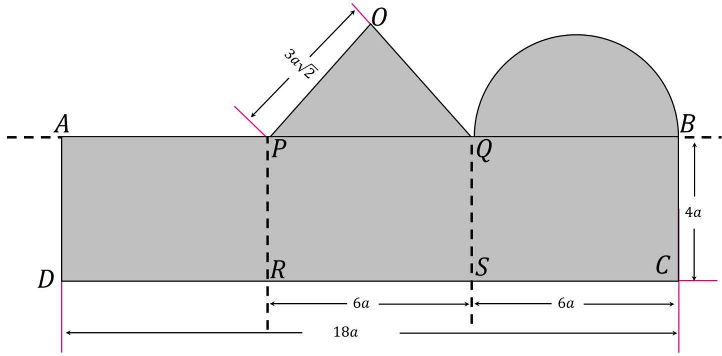
Isosceles-circle



Rectangular shape

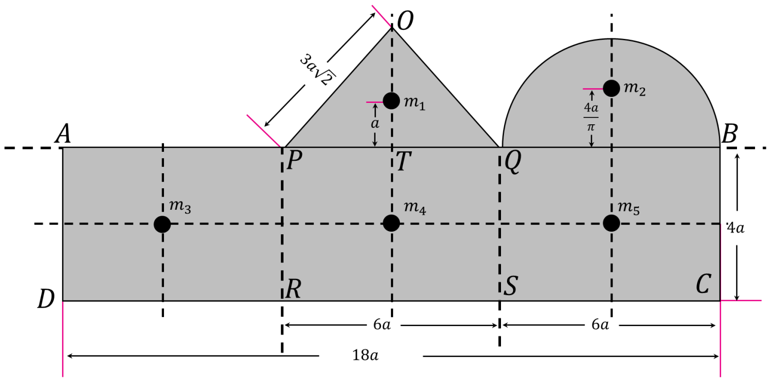
Example

Find the CG for the lamina shown below line AB



Solution

The complex shaped lamina is reduced to five particles located at the CG for each component part.



We know that,

$$OT = \sqrt{OP^2 - PT^2} = \sqrt{\left(3a\sqrt{2}\right)^2 - (3a)^2} = \sqrt{18a^2 - 9a^2} = 3a$$

Now, the centre mass of m_1 from reference line AB is

$$= \frac{1}{3} \times OT = \frac{1}{3} \times 3a = a$$

Similarly, for the semi−circle, centre mass of m_2 from reference line AB is

$$= \frac{4r}{3\pi} = \frac{4 \times 3a}{3\pi} = \frac{4}{\pi}$$

The total mass for the lamina is,

$$M = m_1 + m_2 + m_3 + m_4 + m_5$$

$$M = a_1 \times \rho + a_2 \times \rho + a_3 \times \rho + a_4 \times \rho + a_5 \times \rho$$

The area are given by,

$$a_1 = \frac{1}{2} \times 6a \times 3a = 9a^2$$

$$a_2 = \frac{\pi}{2} \times (3a)^2 = \frac{9a^2\pi}{2} = 4.5a^2\pi$$

$$a_3 = a_4 = a_5 = 6a \times 4a = 24a^2$$

The total mass for the lamina is,

$$M = 9a^2 \times \rho + 4.5a^2\pi \times \rho + (24a^2 \times 3) \times \rho$$

$$M = 9a^2\rho + 4.5a^2\pi\rho + 72a^2\rho$$

Thus,

$$\bar{y} \times M = m_1 \times y_1 + m_2 \times y_2 + m_3 \times y_3 + m_4 \times y_4 + m_5 \times y_5$$

$$\bar{y} = \frac{m_1 \times y_1 + m_2 \times y_2 + m_3 \times y_3 + m_4 \times y_4 + m_5 \times y_5}{M}$$

$$= \frac{m_1 \times y_1 + m_2 \times y_2 + m_3 \times y_3 + m_4 \times y_4 + m_5 \times y_5}{m_1 + m_2 + m_3 + m_4 + m_5}$$

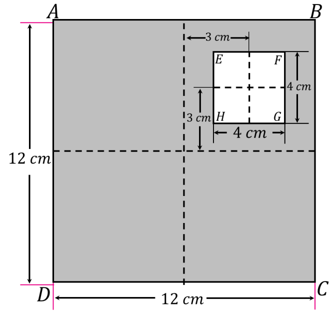
$$= \frac{9a^2 \times \rho \times a + 4.5a^2\pi \times \rho \times \frac{4a}{\pi} - 3 \times 24a^2\rho \times 2}{9a^2\rho + 4.5a^2\pi\rho + 72a^2\rho}$$

$$= \frac{9a^3\rho + 18a^3\rho - 144a^3\rho}{9a^2\rho + 4.5a^2\pi\rho + 72a^2\rho} = \frac{-117a^3\rho}{81a^2\rho + 4.5a^2\pi\rho}$$

$$\bar{y} = \frac{-117a}{81 + 4.5\pi} = -1.2298a$$

Example

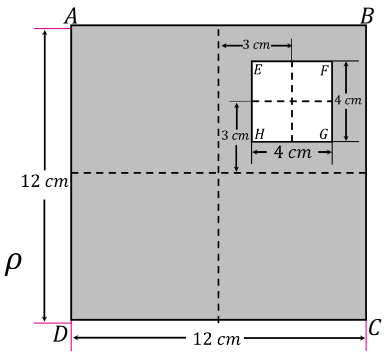
Find the CG for the lamina shown above line DC. If the bracket is suspended from A and hang at rest, find the size of angle between AB and the vertical



Solution:

Section $ABCD$ is having a hole $EFGH$

- ❑ Determine the main section as a complete one, and then subtract the area of a cut-out hole



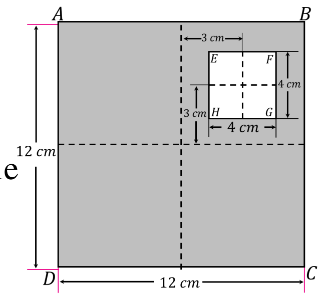
m_1 = mass of the rectangular $ABCD = A \times \rho$

$$= 12 \times 12 \times \rho = 144\rho$$

y_1 = distance of the CG of the rectangular $ABCD$

$$y_1 = \frac{12}{2} = 6\ cm$$

Solution:



m_2 = mass of the cut-out hole $EFGH = A \times \rho$

$$= 4 \times 4 \times \rho = 16\rho$$

y_2 = distance of the CG of the cut-out hole $ABCD$ from the bottom DC

$$y_2 = 6 + 3 = 9\ cm$$

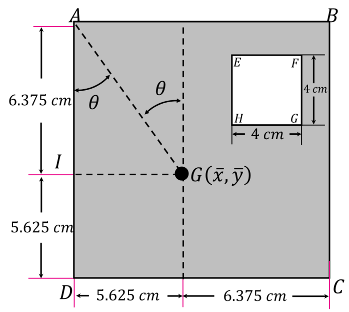
The moment about x-axis for the fully square is,

$$\bar{y} \times M = m_1 \times y_1 - m_2 \times y_2$$

$$\bar{y} = \frac{m_1 \times y_1 - m_2 \times y_2}{M} = \frac{m_1 \times y_1 - m_2 \times y_2}{m_1 - m_2} = \frac{144\rho \times 6 - 16\rho \times 9}{144\rho - 16\rho} = \frac{720}{128} = 5.625\ cm$$

Solution:

To get a solution, when a bracket is suspended at the rest from corner A, draw a line through A that passes through the CG,

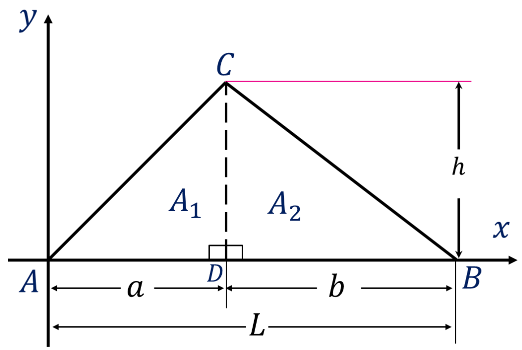


$$\tan\theta = \frac{IG}{IA} = \frac{5.625}{6.375}$$

$$41.4237\,°$$

Example

Find the centroid of a triangular lamina shown below



Solution:

$$A\bar{x} = A_1\bar{x}_1 + A_2\bar{x}_2$$

$$\frac{1}{2}hL\bar{x} = \frac{1}{2}ah \times \frac{2a}{3} + \left(a + \frac{b}{3}\right) \times \frac{1}{2}bh = \frac{a^2h}{3} + \frac{1}{2}abh + \frac{1}{6}b^2h = \frac{2a^2h + 3abh + b^2h}{6}$$

$$\bar{x}L = \frac{2a^2 + 3ab + b^2}{3} = \frac{2a^2 + 2ab + ab + b^2}{3} = \frac{2a(a+b) + b(a+b)}{3}$$

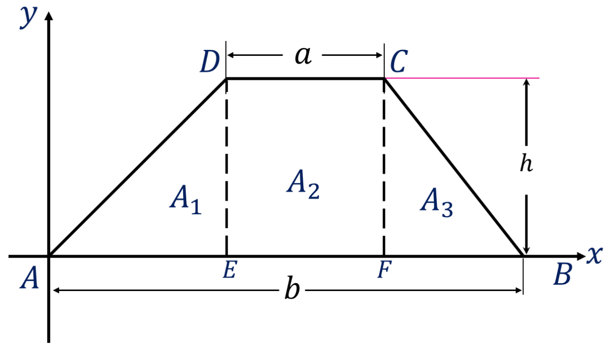
$$\bar{x}L = \frac{(a+b)(2a+b)}{3} = \frac{L(2a+b)}{3}$$

The centroidal distance from the left end is, $\bar{x} = \dfrac{(a+L)}{3}$

The centroidal distance from the left end is, $\bar{x} = \dfrac{(b+L)}{3}$

Example

Find the centroid of a trapezium lamina shown below



Solution:

$$A\bar{y} = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3$$

$$\frac{h}{2} \times (a+b) \times \bar{y} = \left(\frac{1}{2} \times AE \times h \times \frac{h}{3}\right) + \left(a \times h \times \frac{h}{2}\right) + \left(\frac{1}{2} \times FB \times h \times \frac{h}{3}\right)$$

$$\frac{h}{2} \times (a+b) \times \bar{y} = \frac{h^2}{6} \times (AE + FB) + \frac{ah^2}{2}$$

$$(a+b) \times \bar{y} = \frac{h(AE + FB) + 2ah}{3} = \frac{h(b-a) + 3ah}{3} = \frac{h(b-a+3a)}{3} = \frac{h(2a+b)}{3}$$

Hence, the centroidal distance from AB is, $\bar{y} = \dfrac{(2a+b)}{(a+b)}\dfrac{h}{3}$

From DC, the centroidal is, $\bar{y} = \dfrac{(a+2b)}{(a+b)}\dfrac{h}{3}$

Centroidal Axes

The centroidal axis is any line that passes through the centroid of the cross-section. These axes may include,

- Major Principal Axis

The major principal axis is the centroidal axis having the largest moment of inertia as compared to the other axes (the strongest axis)

- Minor Principal Axis

The minor principal axis is the centroidal axis having the smallest moment of inertia as compared to the other axes.

❑ Neutral axis

The neutral axis, is any axis having zero strain under the application of bending force on a bending element.

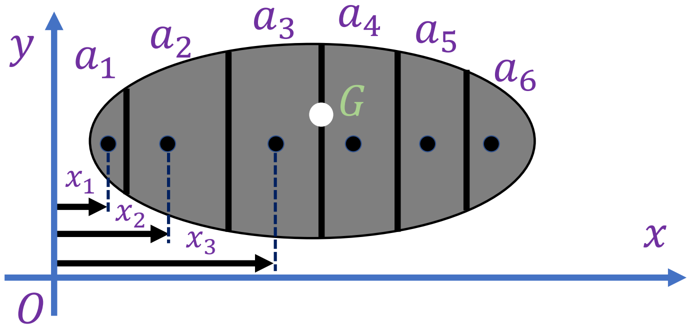
If the deformation is within the elastic limit or within a yield limit and linear in nature, i.e., the stress at any layer is proportional to the distance of the layer from the neutral axis, then the neutral axis will coincide with a centroidal axis.

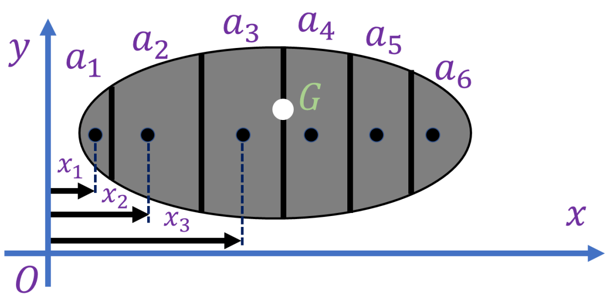
But if the stresses are more than the yield stresses, then the neutral axis will not coincide with a centroidal axis

The centroidal axis always doesn't coincide with the neutral axis, because the centroidal line varies with shapes.

Center of Gravity by Area Method

It is similar to the method of finding centroid as it is coincides with center of gravity





$$\bar{x} = \frac{x_1a_1 + x_2a_2 + x_3a_3 \dots\dots\dots\dots\dots}{a_1 + a_2 + a_3 \dots\dots\dots\dots} = \frac{\sum a_i x_i}{\sum a_i}$$

$$\bar{y} = \frac{y_1a_1 + y_2a_2 + y_3a_3 \dots\dots\dots\dots\dots}{a_1 + a_2 + a_3 \dots\dots\dots\dots} = \frac{\sum a_i y_i}{\sum a_i}$$

Let

a_i = Small area represented by dA

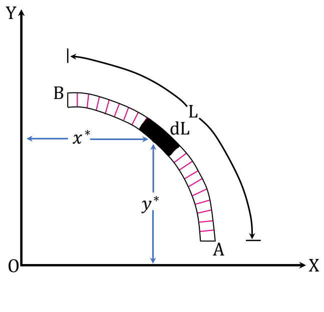
x^* = Distance of C.G of area dA from axis OY, and

y^* = Distance of C.G of area dA from axis OX, and

$$\bar{x} = \frac{\int x^* dA}{\int dA}$$

$$\bar{y} = \frac{\int y^* dA}{\int dA}$$

Center of Gravity of a Line



The center of gravity of a straight or curved line is obtained by dividing the given line into number of small lengths

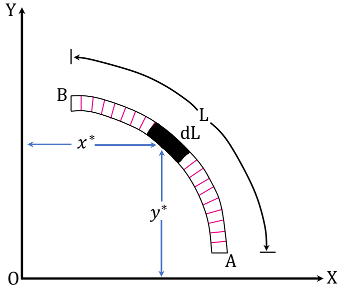
The C.G is obtained by replacing dA with dL

$$\bar{x} = \frac{\int x^* dL}{\int dL} \quad and \quad \bar{y} = \frac{\int y^* dL}{\int dL}$$

x^* = Distance of C.G of length dL from y-axis

y^* = Distance of C.G of length dL from x-axis

Center of Gravity of a Line



If the lines are straight, then the equation modified to:

$$\bar{x} = \frac{L_1 x_1 + L_2 x_2 + L_3 x_3 \ldots\ldots\ldots\ldots\ldots}{L_1 + L_2 + L_3 \ldots\ldots\ldots\ldots.}$$

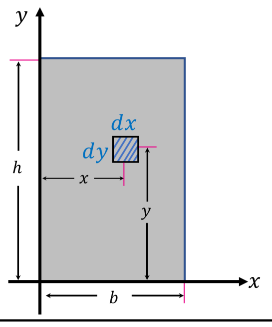
$$\bar{y} = \frac{L_1 y_1 + L_2 y_2 + L_3 y_3 \ldots\ldots\ldots\ldots\ldots}{L_1 + L_2 + L_3 \ldots\ldots\ldots\ldots.}$$

Centroid using a double integral

Sometimes, we find the body centroid using double integration

Example

Find the CG of a rectangular area using double integration.



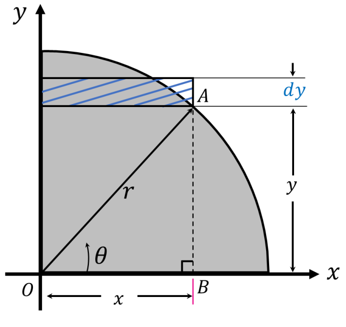
Example

Find the CG of a quarter circle using double integration

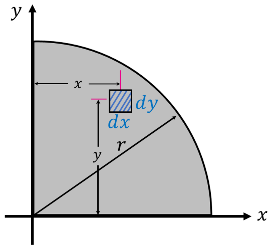
Solution:

First, find the area of a circle

$$\int dA = \frac{\pi r^2}{4}$$



Taking the moment about y-axis, we get,



$$\int x\,dA = \int_0^r \int_0^{\sqrt{r^2-x^2}} x\,dx\,dy$$

$$\int x\,dA = \int_0^r x\,dx\,[y]_0^{\sqrt{r^2-x^2}} = \int_0^r \sqrt{r^2-x^2}\,.x\,dx$$

We solve the integral by substitution technique,

Let,

$$u^2 = r^2 - x^2, \qquad 2u.du = -2x.dx$$

$$\int x\,dA = -\int_0^r u\,.udu = -\int_0^r u^2\,du = -\left|\frac{u^3}{3}\right|_0^r = \frac{r^3}{3}$$

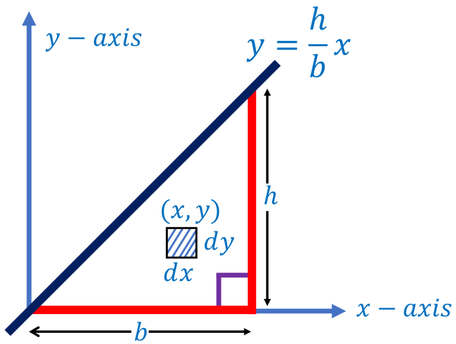
$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\frac{r^3}{3}}{\frac{\pi r^2}{4}} = \frac{4r^3}{3\pi r^2} = \frac{4r}{3\pi}$$

Similarly, taking a moment area along the x−axis, we get,

$$\bar{y} = \frac{4r}{3\pi}$$

Example

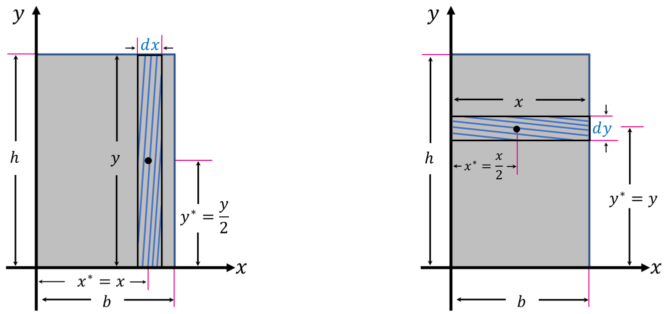
Find the CG of triangle using double integration.



The double integration to find the first moment may be avoided by defining dA as a thin rectangle or strip

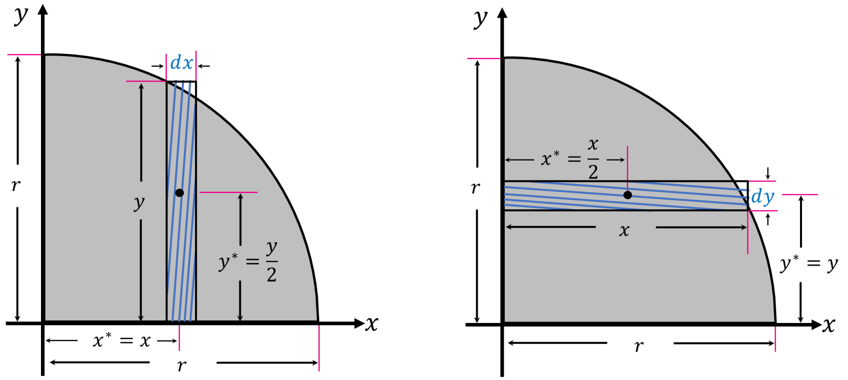
Example

Find the CG of a rectangular area using a single integral.



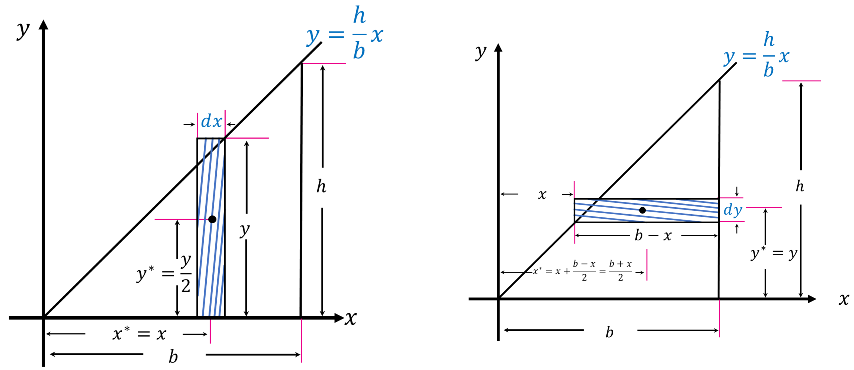
Example

Find the CG of a quarter circle using a single integral



Example

Find the CG of a triangle using a single integral



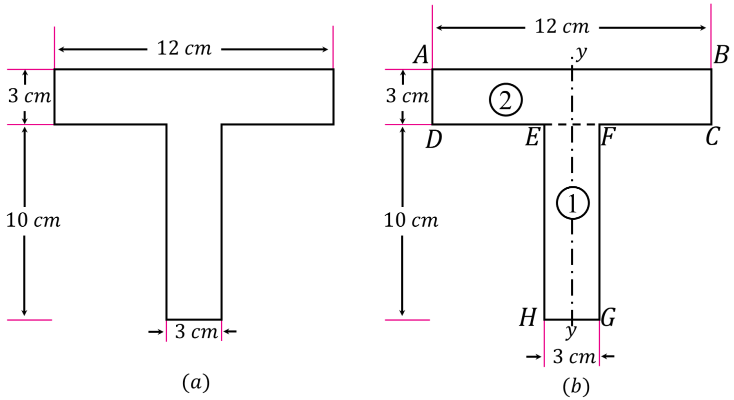
The CG of a Composite Geometric Object

For a composite geometry,

- ❑ Divide the body or composite lamina into simpler shapes
- ❑ If the complex lamina has a hole or geometric shape region having no material, then consider the composite lamina without the hole and find the hole as addition composite part having a negative value.
- ❑ Establish the coordinate axes on the given sketch and determine x^*, y^* and z^* of the centroid of each part
- ❑ Appy the centre of gravity equations to determine $\bar{x}$, $\bar{y}$ and $\bar{z}$.
- ❑ If an object is symmetrical about an axis, then the centre of gravity lies on this axis

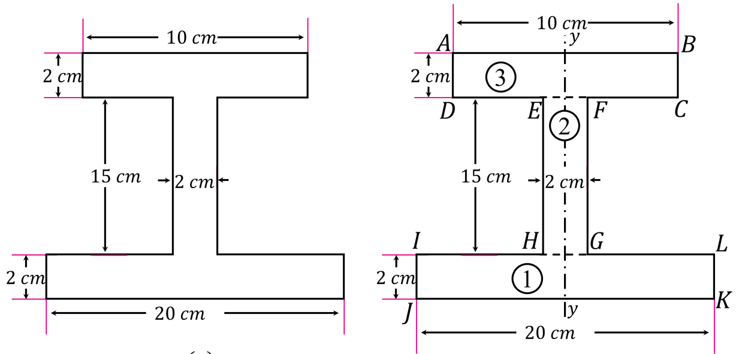
Example

Find the CG of the T-section shown



Example

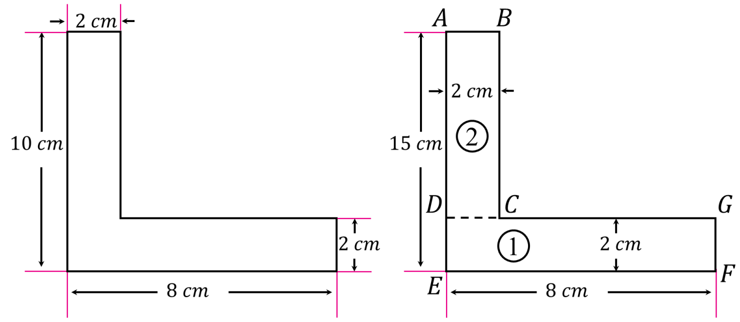
Find the CG of the I-section shown



(a) (b)

Example

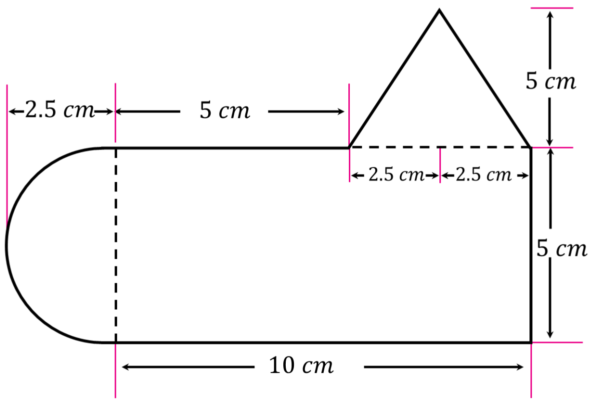
Find the CG of the L-section shown



(a) (b)

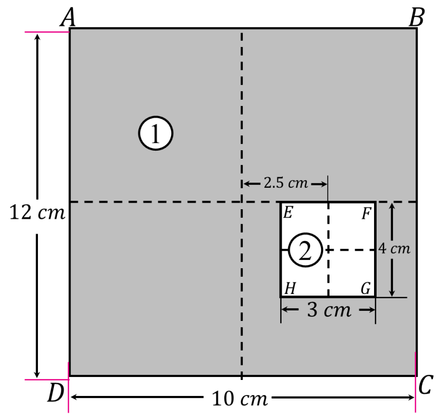
Example

Find the CG of the plane uniform lamina shown



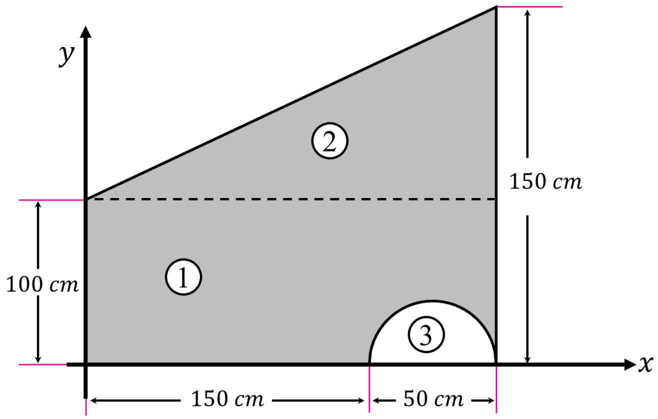
Example

Find the CG of the rectangular lamina shown



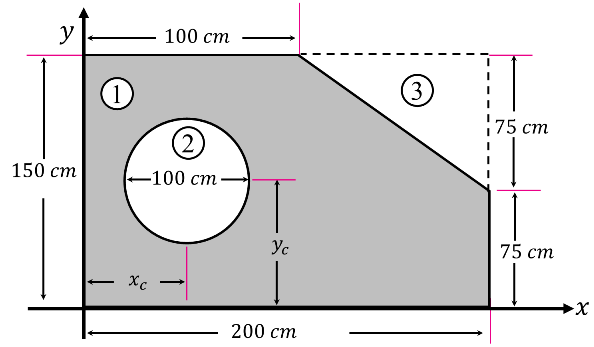
Example

Find the CG of the lamina shown



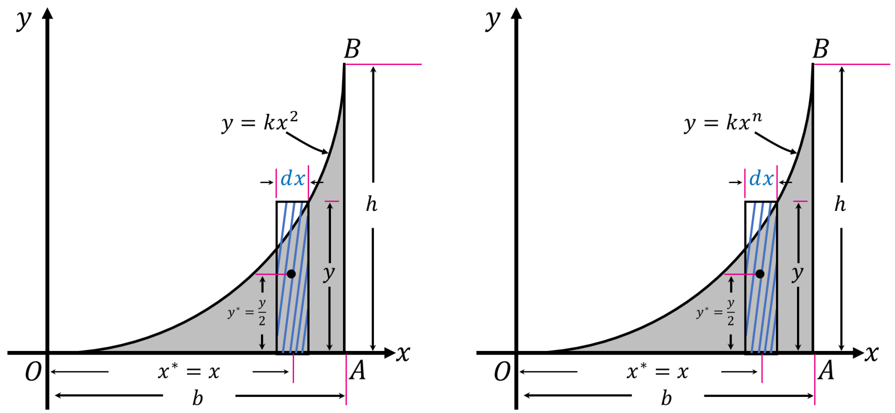
Example

Determine the coordinate x_c and y_c of the centre of a 100 mm diameter circular hole cut in a thin plate so that the point will be the centroid of the remaining shaded area,



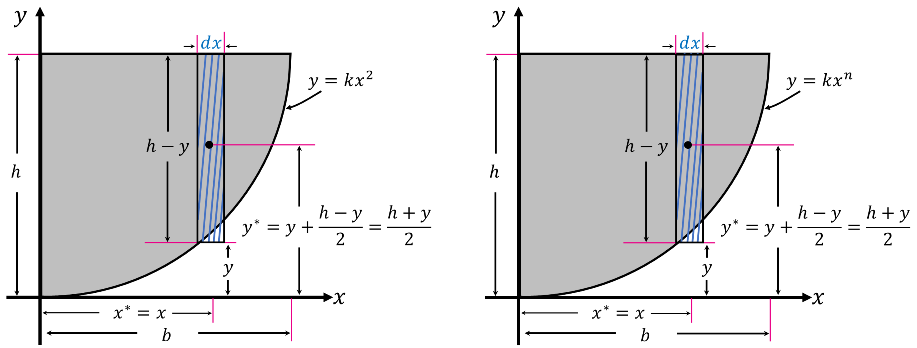
Example

Find the CG of the spandrel area AOB



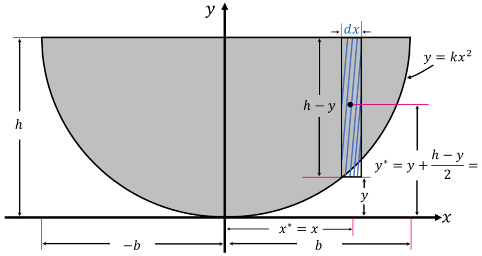
Example

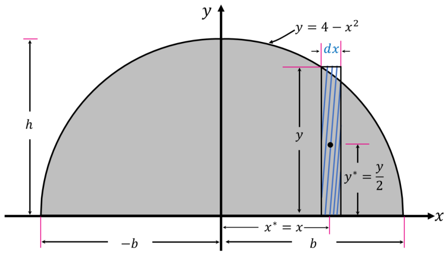
Find the CG of the semi-parabolic area shown below



Example

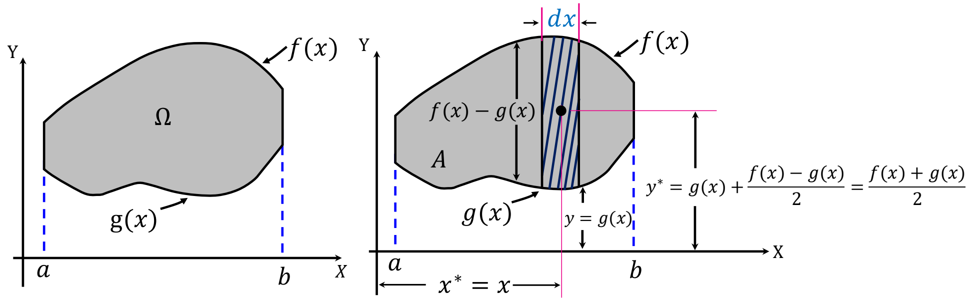
Find the CG of the parabolic area shown below





The CG of a region between the f and g graph

The figure below shows a region Ω bonded between $f(x)$ and $g(x)$ with an area A



Consider a height strip $y = f(x) - g(x)$ and width $d(x)$

The strip area is, dA $\qquad dA = [f(x) - g(x)].dx$

Integrating, we have, $\qquad \int dA = \int_a^b [f(x) - g(x)].dx$

Taking the moment about y-axis, we get,

$$\int x^* \, dA = \int x \, dA = \int_a^b x.[f(x) - g(x)].dx$$

$$\int x^* \, dA = \int_a^b x.[f(x) - g(x)].dx \qquad \bar{x} = \frac{\int_a^b x.[f(x) - g(x)].dx}{\int_a^b [f(x) - g(x)].dx}$$

Similarly, taking the moment about x−axis, we get,

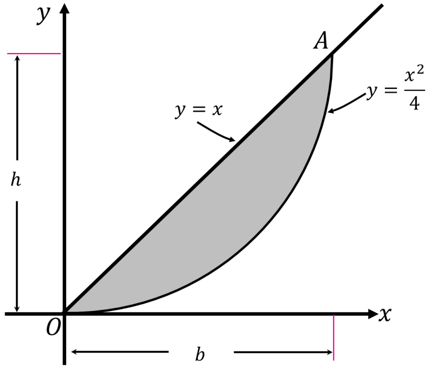
$$\int y^* \, dA = \int_a^b \left(\frac{f(x) + g(x)}{2}\right).\big(f(x) - g(x)\big)dx$$

$$\int y^* \, dA = \frac{1}{2}\int_0^b \left([f(x)]^2 - [g(x)]^2\right)dx$$

$$\bar{y} = \frac{\int y^* \, dA}{\int dA} = \frac{0.5 \int_0^b \left([f(x)]^2 - [g(x)]^2\right)dx}{\int_a^b [f(x) - g(x)].dx}$$

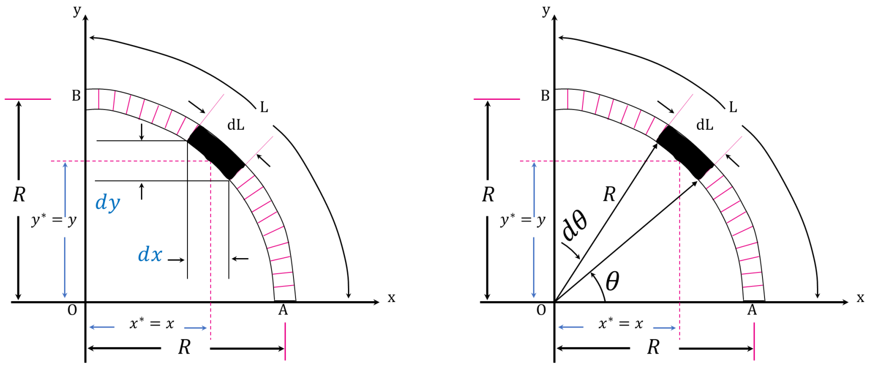
Example

Find the CG of the shaded area between the parabola $y = \frac{x^2}{4}$ and the straight line $y = x$



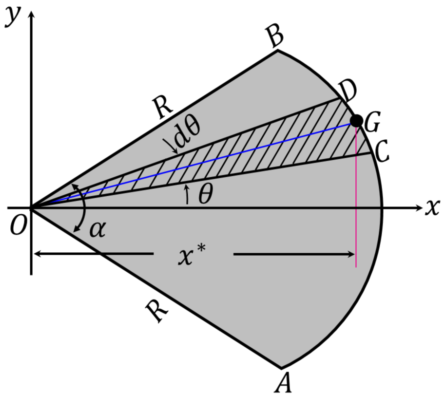
Example

Determine the C.G of gravity of quadrant AB of the arc of a circle of radius R as shown below



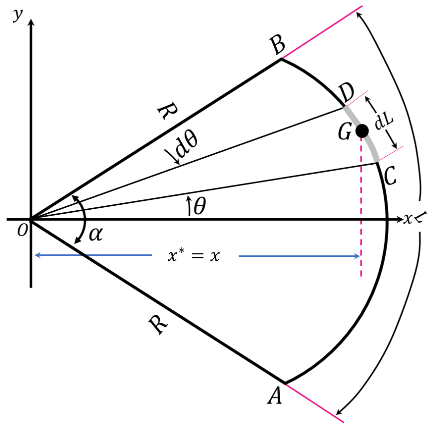
Example

Find the CG of the area of the circular sector AOB of radius R



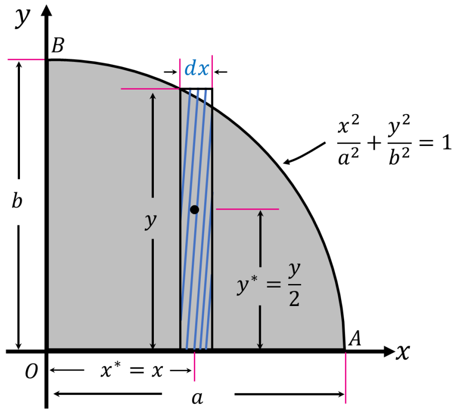
Example

Find the CG of a circular arc segment AOB of radius R



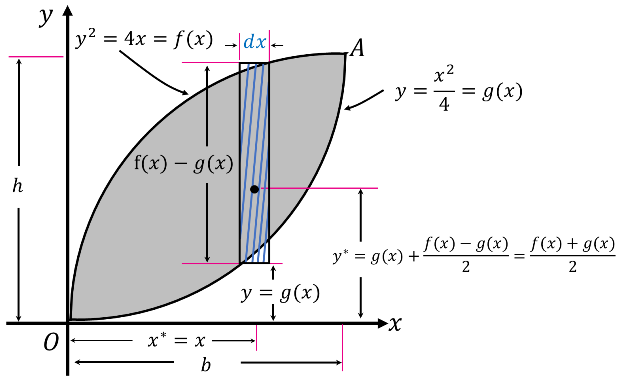
Example

Find the CG of the elliptical area shown below



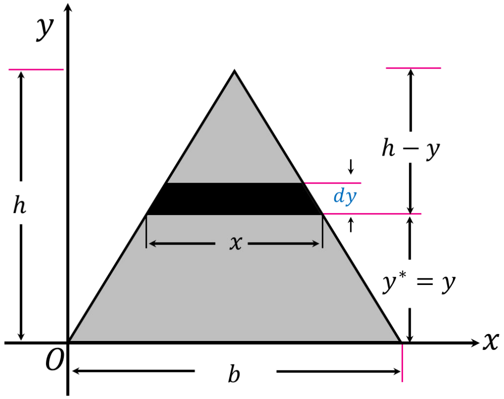
Example

Find the CG coordinates between the curve shown below



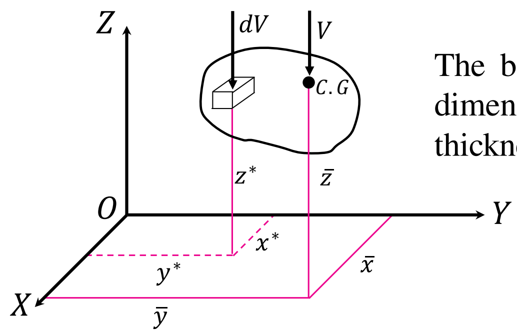
Example

Find the CG of the triangle along h from its base



The CG of Volume

The centroid of the volume is geometrically the centre of a body or simply the point at which the total volume of the body is assumed to concentrate



The body volume is expressed in 3-dimensions, volume, length and thickness

For the volume, the CG is given as a ratio of integral over the volume of the whole body,

$$\bar{x} = \frac{\int x^* \, dV}{\int dV}, \qquad \bar{y} = \frac{\int y^* \, dV}{\int dV}, \qquad \bar{x} = \frac{\int y^* \, dV}{\int dV}$$

For plane figure, the CG is determined by area method,

$$\bar{x} = \frac{x_1 A_1 + x_2 A_2 + x_3 A_3 \ldots \ldots \ldots \ldots \ldots}{A_1 + A_2 + A_3 \ldots \ldots \ldots \ldots .}$$

$$\bar{y} = \frac{y_1 A_1 + y_2 A_2 + y_3 A_3 \ldots \ldots \ldots \ldots \ldots}{A_1 + A_2 + A_3 \ldots \ldots \ldots \ldots .}$$

In the same way, we can find the CG of the solid part by taking the volume in place of areas, suitably for the part made of the same material

$$\bar{x} = \frac{x_1 V_1 + x_2 V_2 + x_3 V_3 \ldots \ldots \ldots \ldots \ldots}{V_1 + V_2 + V_3 \ldots \ldots \ldots \ldots .}$$

$$\bar{y} = \frac{y_1 V_1 + y_2 V_2 + y_3 V_3 \ldots \ldots \ldots \ldots \ldots}{V_1 + V_2 + V_3 \ldots \ldots \ldots \ldots .}$$

If the body is consisting of two different materials with two different shapes, we should consider the weights in place of volumes

$$\bar{x} = \frac{x_1 W_1 + x_2 W_2 + x_3 W_3 \ldots \ldots \ldots \ldots \ldots}{W_1 + W_2 + W_3 \ldots \ldots \ldots \ldots .}$$

$$\bar{y} = \frac{y_1 W_1 + y_2 W_2 + y_3 W_3 \ldots \ldots \ldots \ldots \ldots}{W_1 + W_2 + W_3 \ldots \ldots \ldots \ldots .}$$

In terms of cartesian coordinates, the integral is given as:

$$\int dV = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx \, dy \, dz$$

In terms of cylindrical coordinate, the integral is given as:

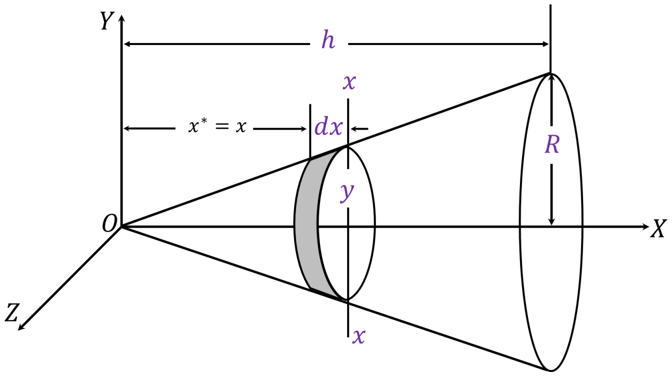
$$\int dV = \int_{z_1}^{z_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r dr \, d\theta \, dz$$

Similarly, in spherical coordinates, the integral is given as:

$$\int dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r^2 \sin\phi \, dr \, d\theta \, d\phi$$

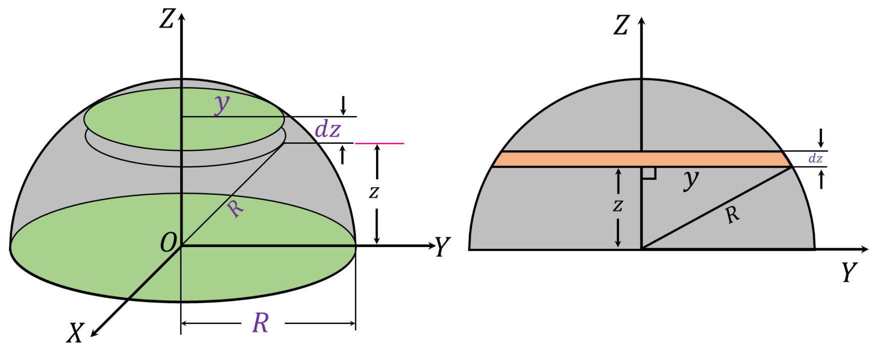
Example

Find the CG of the volume of the cone having radius R at the base and height h as shown,



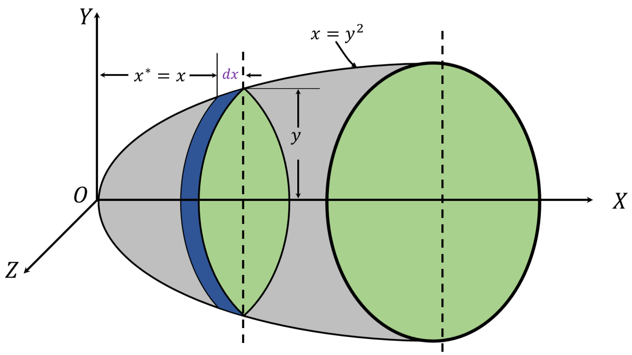
Example

Find the CG of the volume of hemisphere having radius R and placed along z-axis as shown below.



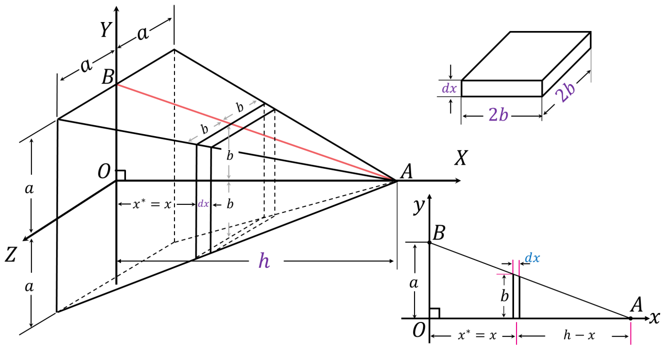
Example

Find the CG of the volume of the paraboloid shown below, the axis of symmetry is along x-axis.



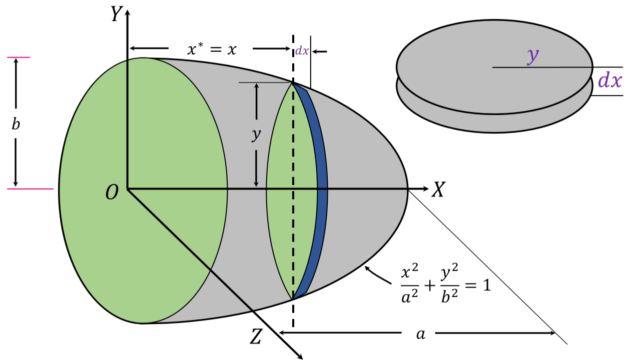
Example

Find the CG of the volume of the pyramid shown below, the axis of symmetry is along x-axis.



Example

Find the CG of the volume of the semi-ellipsoid, the axis of symmetry is along the x-axis.



Pappus's and Guldinus Centroid theorem

In mathematics, Pappus's centroid theorem also known as the Guldinus theorem is the method used to calculate *surface areas* generated by revolving a plane curve about a non-intersecting axis in the plane of the curve.

The theory is also used to calculate *volume* generated by revolving an area about a non-intersecting axis in the plane of the area

Pappus's Centroid Theorem

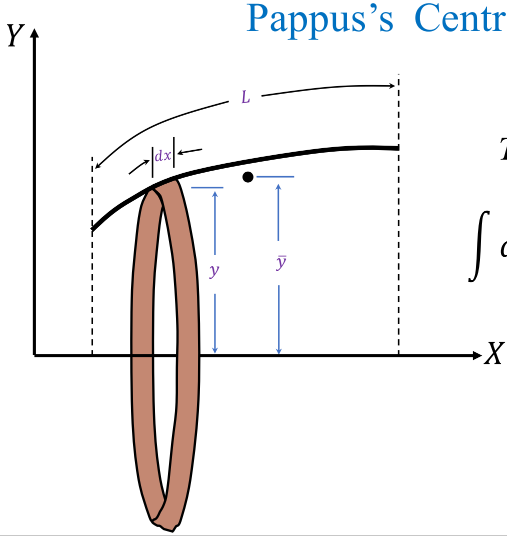
The First theorem of Pappus's – Guldinus

Surface area of revolution is generated by rotating a plane curve about a fixed axis.

The area of surface of revolution is equal to the length of the generating curve times the distance travelled by the centroid through the rotation

$$A = 2\pi\bar{y}L$$

Pappus's Centroid Theorem



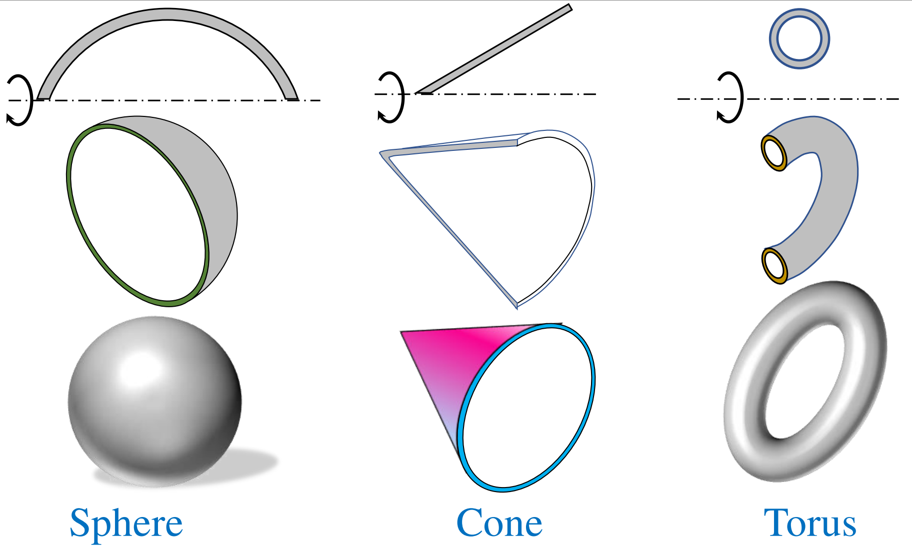
$$dA = 2\pi y \times dL$$

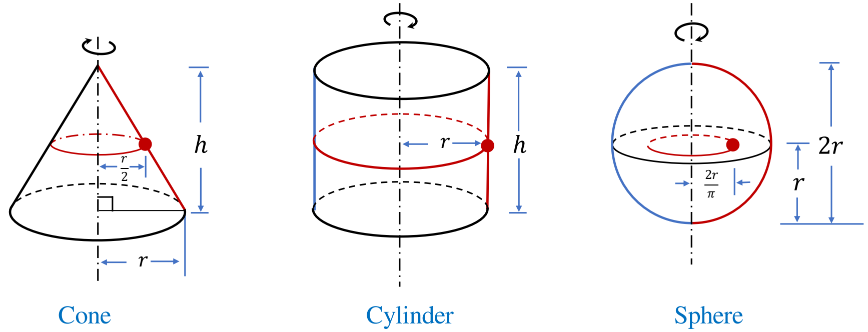
The total area,

$$\int dA = \int 2\pi y \times dL = 2\pi \int y\, dL$$

$$A = 2\pi \int y\, dL = 2\pi\bar{y}L$$

$$A = 2\pi\bar{y}L$$





In summary, the surface areas calculated using Pappus's centroid theorem are shown in the table below.

	Generated Lamina Length (L)	Centroid ($\bar{y}$)	Surface Area (A)
Cone	$\sqrt{h^2 + r^2}$	$r/2$	$\pi r\sqrt{h^2 + r^2}$
Cylinder	h	r	$2\pi rh$
Sphere	πr	$2r/\pi$	$4\pi r^2$

Pappus's Centroid Theorem

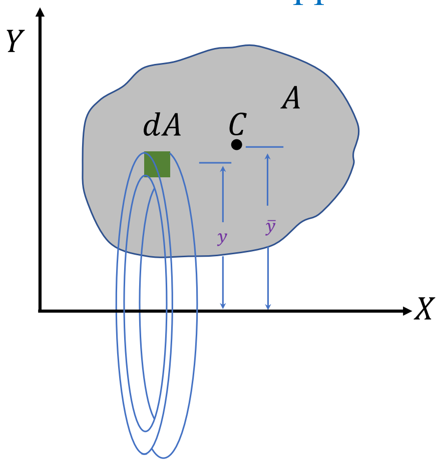
The Second Theorem of Pappus's – Guldinus

The body of revolution is generated by rotating a plane area about a fixed axis.

Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$V = 2\pi\bar{y}A$$

Pappus's Centroid Theorem



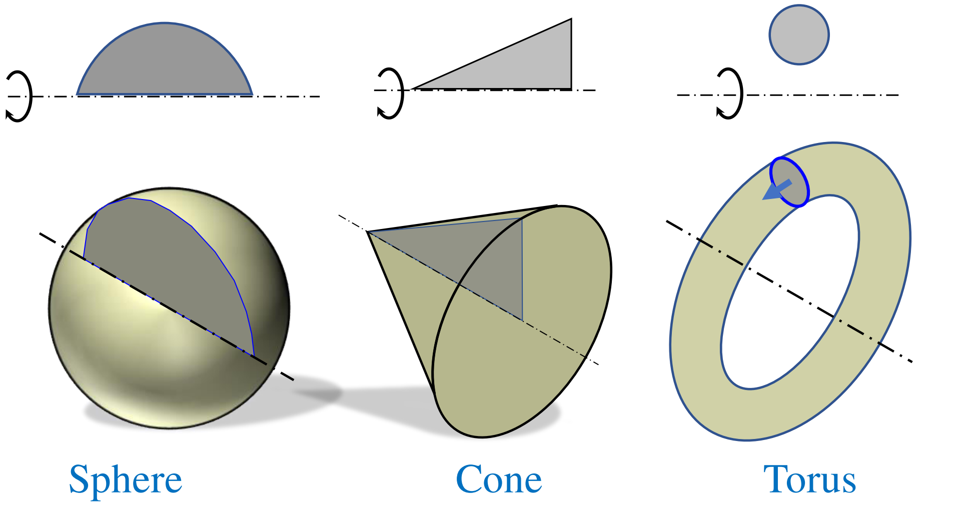
$$dV = 2\pi y \times dA$$

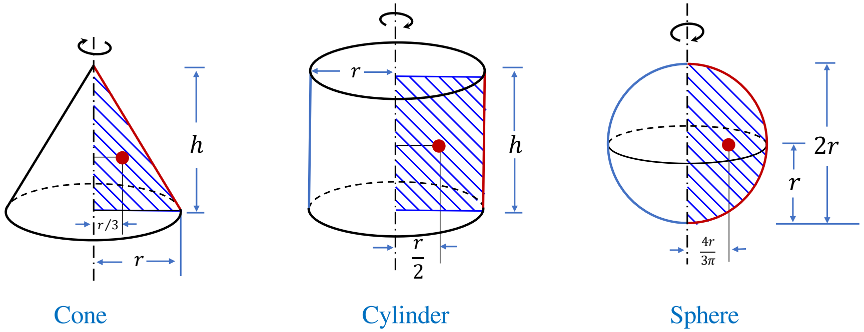
The total volume,

$$\int dV = \int 2\pi y \times dA = 2\pi \int y\, dA$$

$$V = 2\pi \int y\, dA = 2\pi \bar{y} A$$

$$V = 2\pi \bar{y} A$$



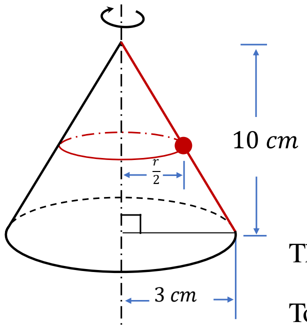


In summary, the surface volume calculated using Pappus's centroid theorem are shown in the table below.

	Area	Centroid ($\bar{y}$)	Surface Volume (V)
Cone (Right triangle)	$hr/2$	$r/3$	$\pi r^2 h/3$
Cylinder (Rectangle)	hr	$r/2$	$\pi r^2 h$
Sphere (Semicircle)	$\pi r^2/2$	$4r/3\pi$	$4\pi r^3/3$

Example

Find the surface area of the cone, and find the amount of paint required paint the cone, i.e., inside and outside the cone if one gallon of paint covers 150 cm^2



$$\bar{y} = \frac{1}{2}r = \frac{1}{2} \times 3 = 1.5\ cm$$

$$L^2 = h^2 + r^2 = \sqrt{h^2 + r^2} = \sqrt{10^2 + 3^2} = \sqrt{109}, cm$$

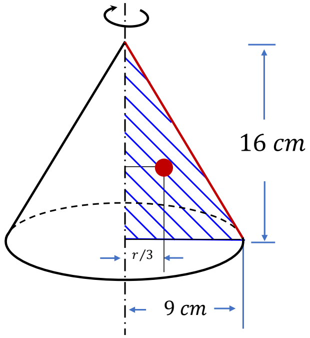
$$A = 2\pi\bar{y}L = 2\pi \times 1.5 \times \sqrt{109} = 98.3976\ cm^2$$

The total area of the paint $= 2 \times 98.3976 = 196.7951\ cm^2$

Total number of gallons $= \frac{196.7951}{250} = 0.7872\ \cong 1\ gallon$

Example

Find the volume of the Cone



$$\bar{y} = \frac{1}{3}r = \frac{1}{3} \times 9 = 3\ cm$$

$$A = \frac{1}{2}hr = \frac{1}{2} \times 16 \times 9 = 72\ cm^2$$

$$V = 2\pi\bar{y}A = 2\pi \times 3 \times 72 = 1{,}357.168\ cm^3$$

Example

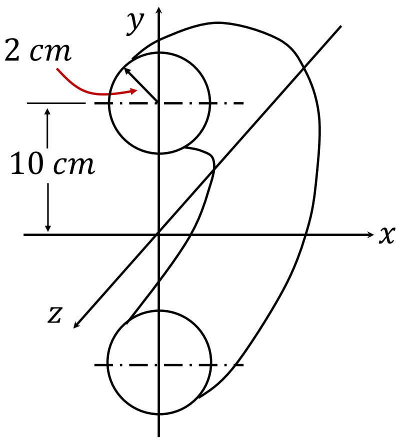
Find the area of the half-torus

$$\bar{y} = 10\ cm$$

$$L = \frac{\pi d}{2} = \frac{2\pi r}{2} = \pi r = 2\pi$$

$$A = 2\pi\bar{y}L = 2\pi \times 10 \times 2\pi = 40\ \pi^2 cm^2$$

$$A = 40\ \pi^2 cm^2$$



Example

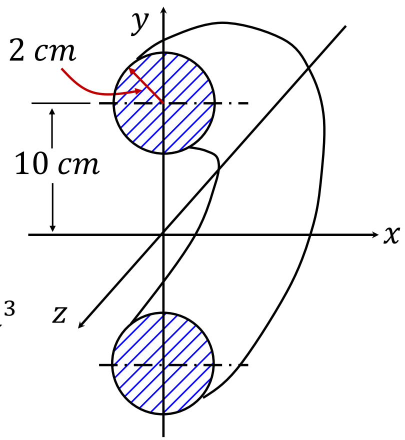
Find the volume of the half-torus

$$\bar{y} = 10\ cm$$

$$A = \frac{\pi r^2}{2} = \frac{1}{2} \times \pi \times 2^2 = 2\pi\ cm^2$$

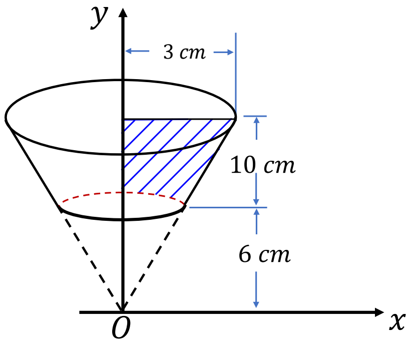
$$V = 2\pi\bar{y}A = 2\pi \times 10 \times 2\pi = 40\ \pi^2\ cm^3$$

$$V = 40\ \pi^2\ cm^3$$



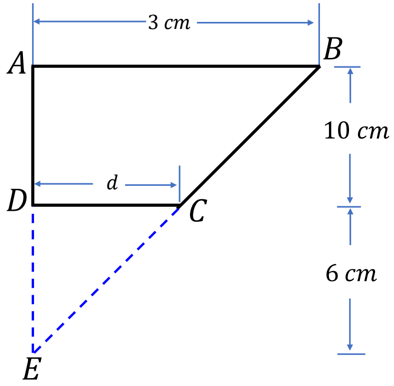
Example

Find the volume of frustrum shown



Solution

The axis of symmetry is along y-axis, the area of concern is trapezoidal

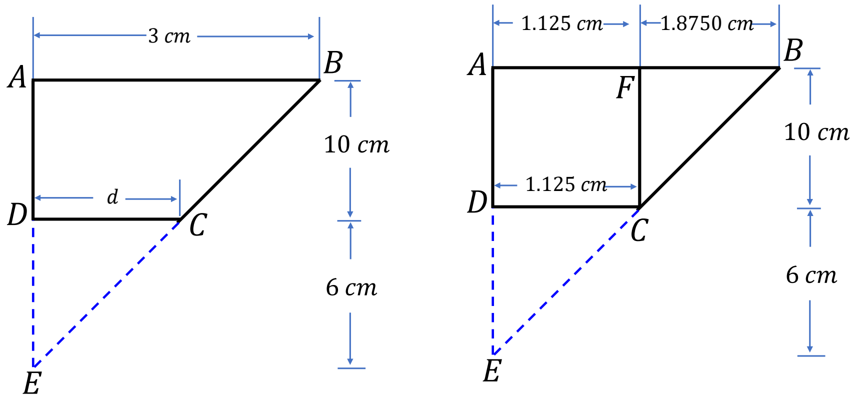


Using similarities of triangle,

$$\frac{DC}{DE} = \frac{AB}{AE}$$

$$DC = d = \frac{AB}{AE} \times DE = \frac{3}{16} \times 6 = 1.125\ cm$$

We can divide the trapezoidal into rectangular $AFCD$ and triangular FBC



The area of a rectangle $AFCD$ is,

$$= 1.125 \times 10 = 11.25\ cm^2$$

The centroid of rectangle $AFCD$

$$\bar{x} = \frac{1}{2} \times 1.125 = 0.5625\ cm$$

For the triangle FBC the area is

$$= \frac{1}{2} \times 1.8750 \times 10 = 9.3750\ cm^2$$

and the corresponding centroid is

$$\bar{x} = 1.125 + \frac{1}{3} \times 1.8750 = 1.75\ cm$$

Now, the total volume is given by,

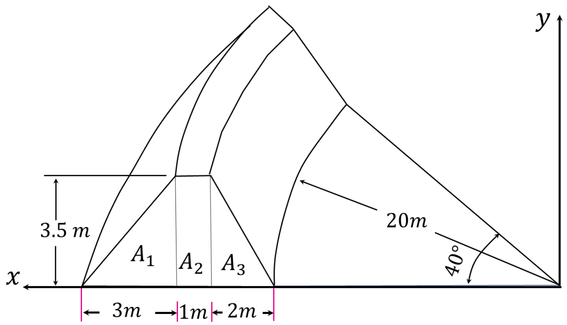
$$V_{ABCD} = V_{AFCD} + V_{FBC} = 2\pi \times \bar{x}_{AFCD} \times A_{ADCD} + 2\pi \times \bar{x}_{FBC} \times A_{FBC}$$

$$V_{ABCD} = 2\pi(\bar{x}_{AFCD} \times A_{ADCD} + \bar{x}_{FBC} \times A_{FBC})$$

$$= 2\pi \times (0.5625 \times 11.25 + 1.75 \times 9.3750) = 142.8443\ cm^3$$

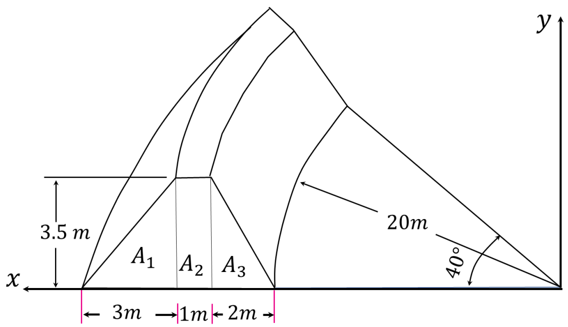
Example

Find the amount of paint required to paint the steps for the concrete for the concrete dam shown below



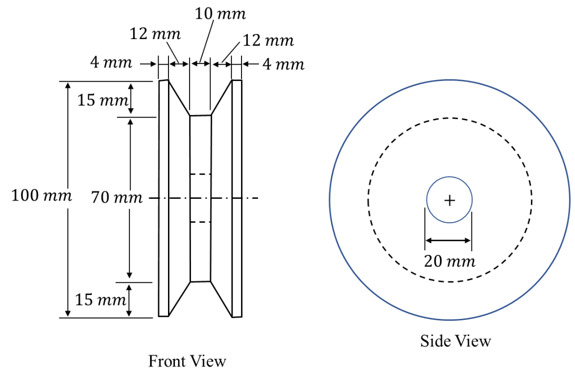
Example

Find the total number of cubic meters required to construct the steps required of the dam shown below,



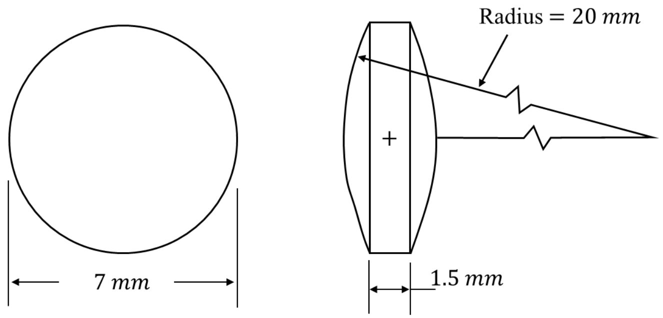
Example

Determine the steel V-belt pulley mass shown below



Example

A pharmaceutical firm plans to place a 0.01 mm thick coating on the outside pills as shown. Determine the amount of material required for coating



solution

Region	Length, (mm)	Centroid (mm), $\tilde{y}$	$y_i L_i$, (mm^2)
Line, L_1	$\frac{1.5}{2} = 0.75$	3.5	2.6250
Line, L_2	3.5181	1.7478	6.1489
Σ			8.7739

Now, the area of the solid is given by,

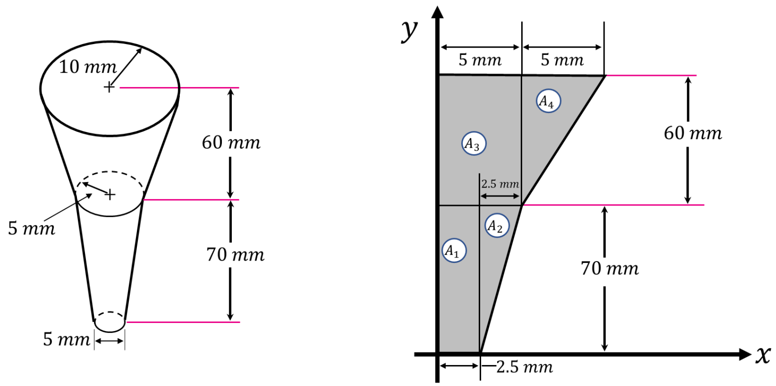
$$A = 2\pi \sum_{i}^{n} y_i L_i = 2\pi \times 8.7739 = 55.1283\ mm^2$$

The total area $= 110.2565\ mm^2$, and the total surface of coating will be,

$$= 110.2565\ m^2 \times 0.01\ mm = 1.1026\ mm^3$$

Example

Identify the funnel volume as shown below,



solution

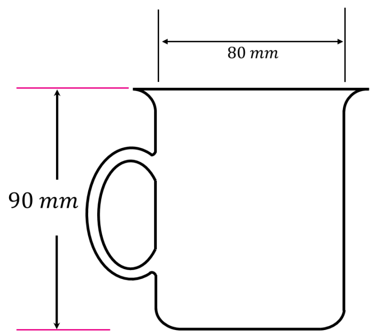
Region	Area, (mm^2)	Centroid (mm), $\tilde{x}$	$x_i A_i$, (mm^3)
Rectangle, A_1	$2.5 \times 70 = 175$	$\frac{2.5}{2} = 1.25$	218.75
Triangle, A_2	$\frac{1}{2} \times 2.5 \times 70 = 87.5$	$2.5 + \frac{2.5}{3} = 3.3333$	291.6667
Rectangle, A_3	$5 \times 60 = 300$	$\frac{5}{2} = 2.5$	750
Triangle, A_4	$\frac{1}{2} \times 5 \times 60 = 150$	$5 + \frac{5}{3} = 6.6667$	1,000
Σ	712.5		2,260.4167

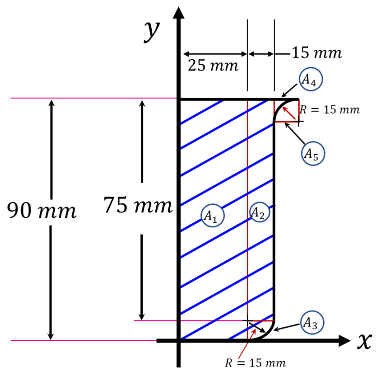
Now, the volume of the solid is given by,

$$V = 2\pi \sum_{i}^{n} x_i A_i = 2\pi \times 2{,}260.4167 = 14{,}202.6170\ mm^3$$

Example

Determine how much coffee, the coffee mug is carrying when it is full to the brim





solution

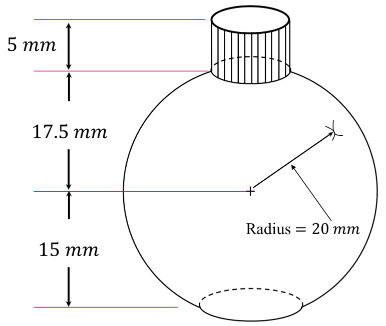
Region	Area, (mm^2)	Centroid (mm), $\tilde{x}$	x_iA_i, (mm^3)
Rectangle, A_1	$90 \times 25 = 2{,}250$	$\frac{25}{2} = 12.5$	28,125
Rectangle, A_2	$15 \times 75 = 1{,}125$	$25 + \frac{15}{2} = 32.5$	36,562.5
Quarter Circle, A_3	$\frac{\pi \times 15^2}{4} = 56.25\pi$	$25 + \frac{4 \times 15}{3 \times \pi} = 31.3662$	5,542.8647
Rectangle, A_4	$15 \times 15 = 225$	$40 + \frac{15}{2} = 47.5$	10,687.5
Quarter Circle, A_5	$\frac{\pi \times 15^2}{4} = -56.25\pi$	$40 + \frac{4 \times 15}{3 \times \pi} = 46.3662$	−8,193.5835
Σ	3,401.7146		72,724.2808

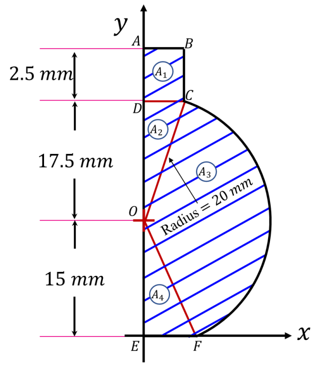
Now, the volume of the solid is given by,

$$V = 2\pi \sum_{i}^{n} x_iA_i = 2\pi \times 72{,}724.2808 = 456{,}940.1328\ mm^3$$

Example

Determine the capacity of the small bottle of lotion if the bottle is filled halfway up to the neck





solution

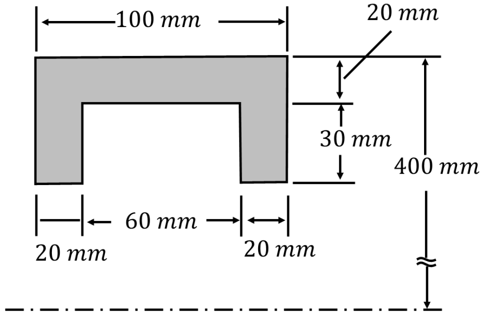
Region	Area, (mm^2)	Centroid (mm), $\tilde{x}$	x_iA_i, (mm^3)
Rectangle, A_1	$2.5 \times 9.6825 = 24.2063$	$\frac{9.6825}{2} = 4.8412$	117.1873
Triangle, A_2	$\frac{1}{2} \times 9.6825 \times 17.5 = 84.7215$	$\frac{9.6825}{3} = 3.2275$	273.4375
Triangle, A_4	$\frac{1}{2} \times 13.2288 \times 15 = 99.2157$	$\frac{13.2288}{3}$	437.5001
Area of the sector, A_3	$\frac{1}{2} \times 20^2 \times \frac{109.6354° \times \pi}{180°}$	11.3903	4,359.0644
Σ	615.0492		5,187.1893

Now, the volume of the solid is given by,

$$V = 2\pi \sum_{i}^{n} x_iA_i = 2\pi \times 5{,}187.1893 = 32{,}592{,}0716\ mm^3$$

Example

Pulley's outer diameter is $0.8\,m$, and its cross-sectional as shown. Knowing the pulley is made of steel, $\rho = 7{,}850\ kg/m^3$. Find the mass and weight of the rim



Solution

Region	Area, (mm^2)	Centroid (mm), $\tilde{y}$	y_iA_i, (mm^3)
Rectangle, A_1	$100 \times 50 = 5{,}000$	$350 + \frac{50}{2} = 375$	1,875,000
Rectangle, A_2	$60 \times 30 = 1{,}800$	$350 + \frac{30}{2} = -365$	−657,000
Σ	6,800		1,218,000

Now, the volume of the solid is given by,

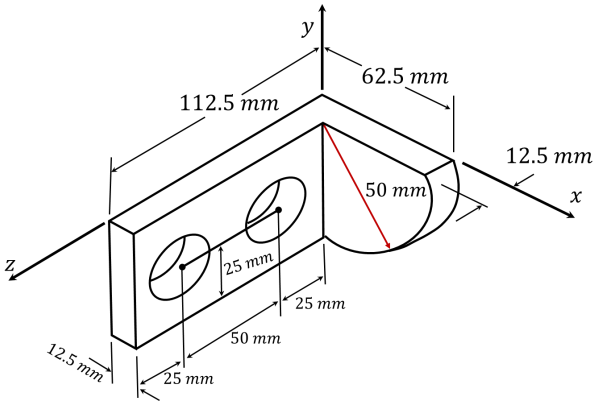
$$V = 2\pi \sum_{i}^{n} y_iA_i = 2\pi \times 1{,}218{,}000 = 7{,}652{,}919.704\ mm^3 \cong 0.0077\ m^3$$

Mass of the rim $m = V \times \rho = 0.0077 \times 7{,}850 = 60.0754\ kg$

Weight of the rim, $W = m \times g = 60.0754 \times 9.81 = 589.3399\ N$

Example

Locate the steel machine element's CG, each hole is $25\ mm$



Solution:

For the $\tilde{x}$ we have:

Region	Volume, (mm^3)	$\tilde{x}$ (mm)	$\tilde{x}_iV_i$, (mm^3)
Rectangle, A_1	$12.5 \times 112.5 \times 50$	$\frac{12.5}{2} = 6.25$	439,453.125
Circle, A_2	$\pi \times 12.5^2 \times 12.5$	$\frac{12.5}{2} = 6.25$	−38,349.5197
Circle, A_3	$\pi \times 25^2 \times 12.5$	$\frac{12.5}{2} = 6.25$	−38,349.5197
Area of a quarter circle, A_4	$\frac{\pi}{4} \times 50^2 \times 12.5 = 7{,}812.5\,\pi$	$12.5 + \frac{4 \times 50}{3\pi} = 33.7207$	827,629.4909
Σ	82,584.3463		1,190,383.576

$$\bar{x} = \frac{\sum \tilde{x}_iV_i}{V} = \frac{1{,}190{,}383.576}{82{,}584.3463} = 14.4142\ mm$$

For the $\bar{y}$ we have:

Region	Volume, (mm^3)	$\tilde{y}\ (mm)$	$\tilde{y}_i V_i,\ (mm^3)$
Rectangle, A_1	$12.5 \times 112.5 \times 50 = 70{,}312.5$	$\frac{50}{2} = -25$	$-1{,}757{,}812.5$
Circle, A_2	$\pi \times 12.5^2 \times 12.5 = -1{,}953.125\,\pi$	$\frac{50}{2} = -25$	$153{,}398.0788$
Circle, A_3	$\pi \times 25^2 \times 12.5 = -1{,}953.125\,\pi$	$\frac{50}{2} = -25$	$153{,}398.0788$
Area of a quarter circle, A_4	$\frac{\pi}{4} \times 50^2 \times 12.5 = 7{,}812.5\,\pi$	$\frac{4 \times 50}{3\pi} = -21.2207$	$-520{,}833.3333$
Σ	$82{,}584.3463$		$-1{,}971{,}849.675$

$$\bar{y} = \frac{\sum \tilde{y}_i V_i}{V} = \frac{-1{,}971{,}8490.675}{82{,}584.3463} = -23.8768\ mm$$

For the $\bar{z}$ we have:

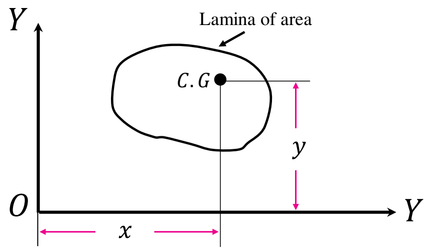
Region	Volume, (mm^3)	$\tilde{z}\ (mm)$	$\tilde{z}_i V_i,\ (mm^3)$
Rectangle, A_1	$12.5 \times 112.5 \times 50 = 70{,}312.5$	$\frac{112.5}{2} = 56.25$	$3{,}955{,}078.125$
Circle, A_2	$\pi \times 12.5^2 \times 12.5 = -1{,}953.125\,\pi$	$12.5 + 25 + 50 = 87.5$	$-536{,}893.2758$
Circle, A_3	$\pi \times 25^2 \times 12.5 = -1{,}953.125\,\pi$	$12.5 + 25 = 37.5$	$-230{,}097.1182$
Area of a quarter circle, A_4	$\frac{\pi}{4} \times 50^2 \times 12.5 = 7{,}812.5\,\pi$	$\frac{12.5}{2} = 6.25$	$153{,}398.0788$
Σ	$82{,}584.3463$		$3{,}341{,}485.809$

$$\bar{z} = \frac{\sum \tilde{z}_i V_i}{V} = \frac{3{,}341{,}485.809}{82{,}584.3463} = 40.4615\ mm$$

The Area Moment of Inertia (MOI)

For the lamina of area A, the moment of inertia about OY is

$= \text{Area} \times$ Perpendicular distance of CG of the area from axis OY



$$= A.x$$

$A.x =$ is defined as the *first moment of the area* along OY

The first moment area is used to determine the area's center of gravity.

If the first moment area is again multiplied by the perpendicular distance between the area's C.G and the OY axis, then we get,

$$= (Ax).x = A.x^2$$

$A.x^2$ is known as *the moment of the moment area* or *the second moment of area or area moment of inertia*.

The second moment of area measures body resistance to bending or deflection and forms a basis for the strength of the material (statics)

The second moment of area its also used to determine the shear stress due to shear, moment, and torsional on a given section

Similarly, *the first moment of area* about OX is $= A.y$

And the equivalent *second moment of area* about $OX = A.y^2$

If the area is replaced by the mass, the second moment of area is known as the *second moment of mass* or *mass moment of inertia*.

For the OX axis, the mass moment of inertia $= m.y^2$, whereas for the $OY = m.x^2$

Definition:

The product of the area or mass and the square distance of the C.G of the area or mass from an axis is known as the moment of inertia of the area or mass about the axis.

The mass moment of inertia measures the résistance of the physical object:

- ❑ To any change in state of motion or velocity, i.e., resistance to angular acceleration or resistance of the angular acceleration to the applied torque (dynamic)
- ❑ Measures resistance to rotation, and forms the basis of dynamic of the rigid bodies.

The moment of inertia is represented by the letter I. For the OX the moment of inertia is replaced by I_{yy}, while for the OY is I_{xx}

The product of area or mass and the square distance of the CG of the area or mass perpendicular to the plane of the area is known as the polar moment of inertia or the second polar moment of inertia.

- ❑ It is represented by the letter $J \text{ or } I_p$

The polar moment of inertia measures resistance to twisting in a beam or the measure of the body resistance to shear and forms a basis of shear strength or rigidity of the material.

The polar moment of inertia is the moment of inertia in the third axis, i.e., z-axis.

Mathematically,

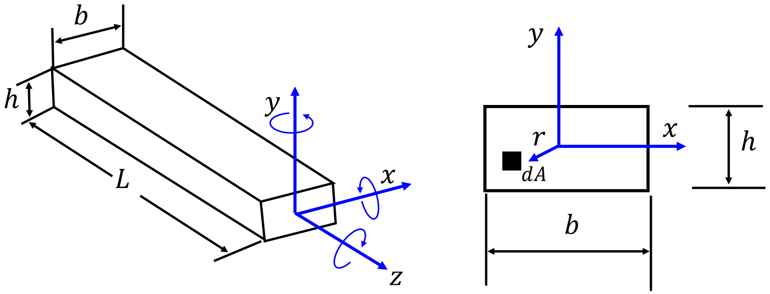
$$J = I_{xx} + I_{yy}$$

Definition

The *polar moment of inertia* is the distance that the material is off that axis or the measure of the shaft to the twisting.

The polar moment of inertia is required to calculate the twist of the shaft when subjected to the torque or twisting moment.

For the beam of length L, x, y and z are the beam axes



The moment of inertia along x-axis and y-axis represents the bending moment, and the moment about z-axis represents the polar moment of inertia, J

r = distance off z-axis of dA

$$r^2 = x^2 + y^2$$

Multiplying dA on both sides, we have,

$$\int r^2\, dA = \int x^2\, dA + \int y^2\, dA$$

$$J = I_{yy} + I_{xx}$$

The polar moment of inertia J is defined as,

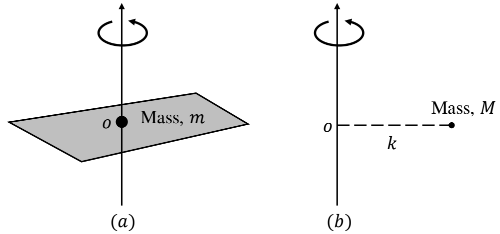
$$J = \int r^2\, dA$$

The polar moment of inertia is used only on a circular body to determine the angular body displacement

- ❑ Not appropriate for the study of non-circular cross-sections shafts and beams

This is because when the torque is applied to it, the body with non-circular cross-sections tends to wrap, and it also leads out of plane deformation.

The Radius of Gyration



For the figure (a), the mass moment of inertia is $= mr^2$

For the same axis of rotation in figure (b), the same is concentrated at the same point but away from the axis of rotation, and this time the moment of inertia is $= Mk^2$

If the two values of the moment of inertia is the same, then k is referred to as the *radius of gyration,* denoted by a letter k

$$Mk^2 = mr^2 = I$$

$$k = \sqrt{\frac{I}{M}}$$

The radius of gyration is the distance from the axis of rotation where a body's total mass is believed to be concentrated and at which the moment of inertia is equal to the body's actual moment of inertia.

The radius of gyration describes an average distance travelled by particles.

Suppose one body consists of n particles of mass m each, and let r_1, r_2, r_3,….r_n be their perpendicular distance from the axis of rotation

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots m_n r_n^2$$

For the same masses, the moment of inertia reduces to,

$$I = m(r_1^2 + r_2^2 + r_3^2 + \cdots r_n^2)$$

The total mass of the body, $M = n.m$

The mass of a particles, $m = \frac{M}{n}$

Then, the moment of inertia reduces further to,

$$I = \frac{M}{n}(r_1^2 + r_2^2 + r_3^2 + \cdots r_n^2)$$

When the same mass is concentrated on the same point, $I = Mk^2$

Equating the two values of the moment of inertia, we get,

$$Mk^2 = \frac{M}{n}(r_1^2 + r_2^2 + r_3^2 + \cdots r_n^2)$$

$$k^2 = \frac{(r_1^2 + r_2^2 + r_3^2 + \cdots r_n^2)}{n}$$

$$k = \sqrt{\frac{(r_1^2 + r_2^2 + r_3^2 + \cdots r_n^2)}{n}}$$

Whereas k represents, the radius of gyration.

The radius of gyration may also be defined as the root mean square distance from the axis of rotation between the various body particles, and often regarded as a measure of how the mass of rotating rigid body is distributed along its axis of rotational axis.

The gyration radius is also used in structural engineering to defined the distribution of cross-sectional areas in a column along its centroidal axis with the body mass.

$$k = \sqrt{\frac{I}{A}}$$

The gyration radius is useful in estimating the column stiffness

The Theorem of Perpendicular Axis.

In the plane object, the moment of inertia about an axis perpendicular to the plane,

$$I_z = I_x + I_y$$

The theory is a valuable tool in building or constructing the moment of inertia of three-dimensional objects such as cylinders.

For the plane section shown below,

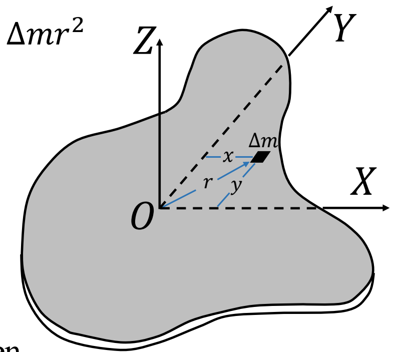
$\Delta I_x = \Delta m y^2 \qquad \Delta I_y = \Delta m x^2 \qquad \Delta I_z = \Delta m r^2$

Then, $\Delta I_y + \Delta I_x = \Delta m(x^2 + y^2)$

But since, $r^2 = x^2 + y^2$

An it follows that,

$$\Delta I_y + \Delta I_x = \Delta m r^2 = \Delta I_z$$



Since this applies to any mass element, then,

$$I_z = I_x + I_y = J$$

The moment of inertia along z-axis, I_{zz} is called the polar moment of inertia, J

The Parallel axes Theorem

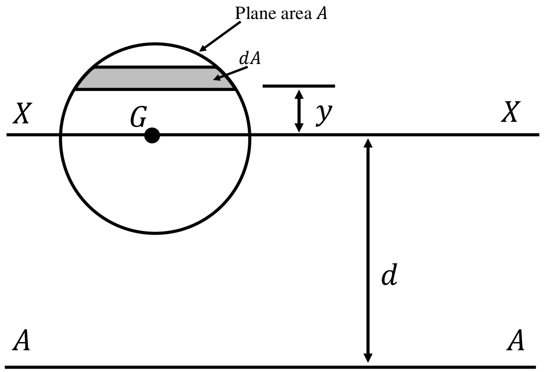
The parallel axis theorem states that the moment of a body's inertia around an axis parallel to an axis passing through the centre of the mass is equal to the sum of the moment of body's inertia around an axis passing through the centre of mass and the product of mass and the square of the distance between the two axes

The parallel axis theorem is very useful when identifying the moment of inertia of composite areas.

Mathematically,

$$I_{AA} = I_{GG} + Ad^2$$

Parallel axis theorem



The moment of the entire area about the axis AA is

$$I_{AA} = \int (d+y)^2 . dA = \int (d^2 + 2.d.y + y^2) . dA$$

$$= \int y^2 . dA + \int d^2 . dA + \int 2.d.y.dA$$

$$= I_{GG} + d^2 \int dA + 2.d \int y.dA$$

$y.dA$ represents the moment of the strip about the XX, and

$\int y.dA$ represents the moments of the total area about the XX axis.

The moment of the total area about XX axis is equal to the product of the total area, A and the distance of the center of gravity of the total area from the XX axis of the total area.

As the center of gravity of the total area from the XX axis is zero, hence

$\int y.dA = 0$, i.e., y measured from the centroidal axis is always zero.

Upon substitution, we get,

$$I_{AA} = I_{GG} + Ad^2$$

Whereas,

A = the total area of the section

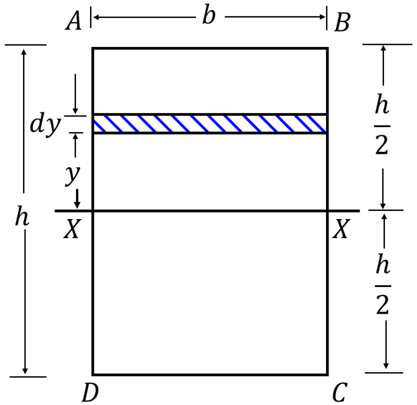
$I_{XX} = I_{GG}$ = the moment of inertia of the total area about the $X-X$ axis.

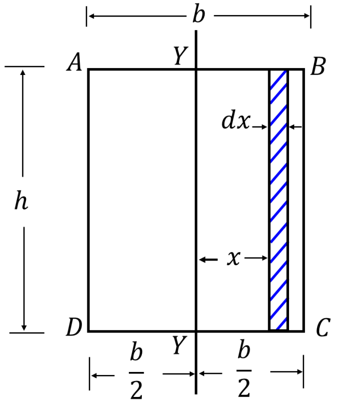
I_{AA} = the moment of inertia of the total area about AA

d = perpendicular distance between the XX and AA axes.

Example

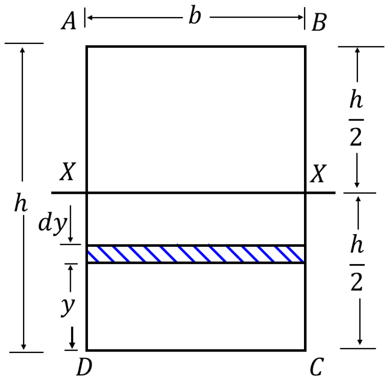
Find the moment of inertia of a rectangles shown below.





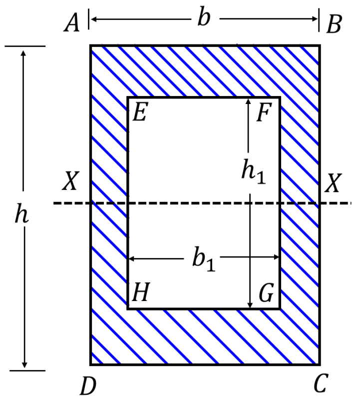
Example

Find the moment of inertia of a rectangle section about its base DC axis and verify your answer using the parallel axis theorem



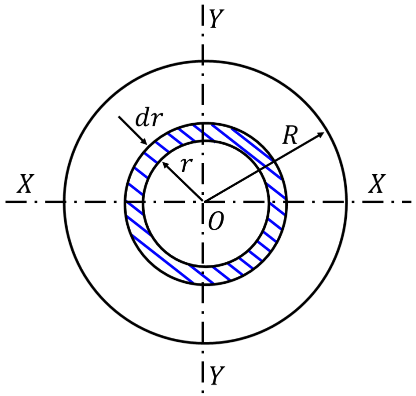
Example

Find the moment of inertia of a hollow rectangular section



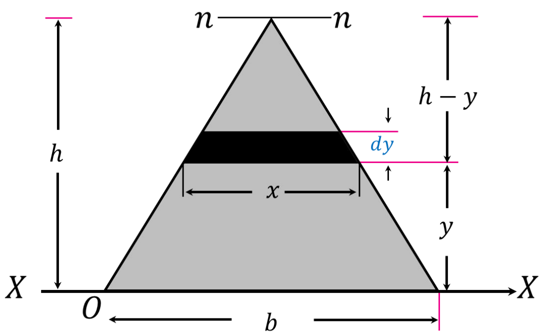
Example

Find the moment of inertia of a circle



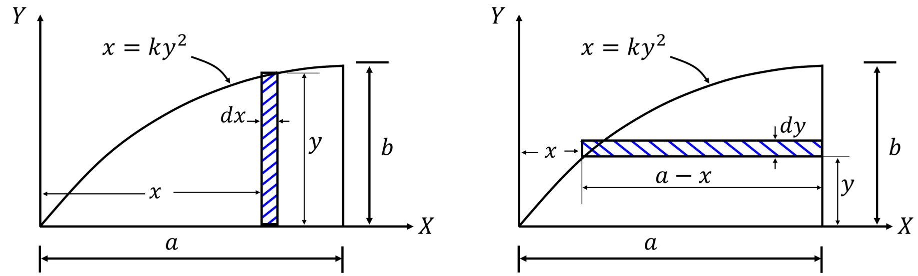
Example

Find the moment of inertia of a triangle



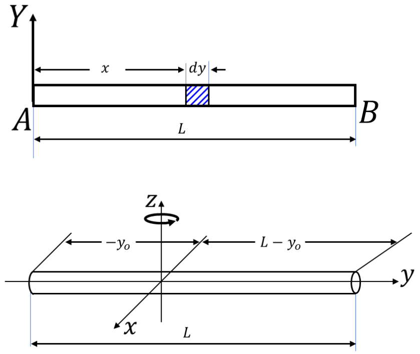
Example

Find the moment of inertia of a parabolic area



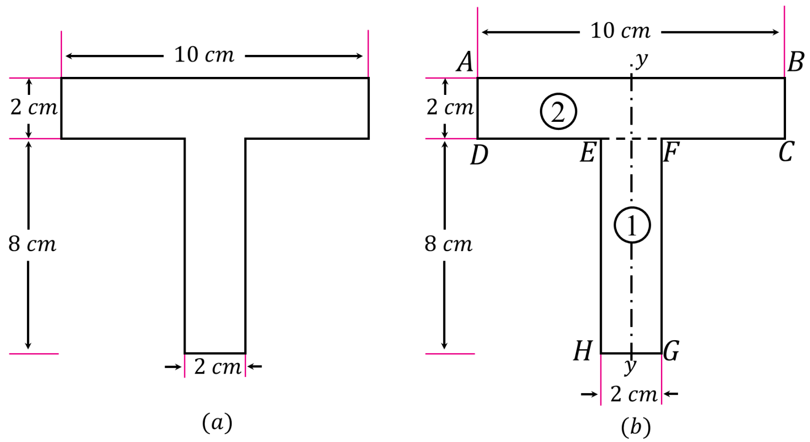
Example

Find the moment of inertia of a thin uniform rod



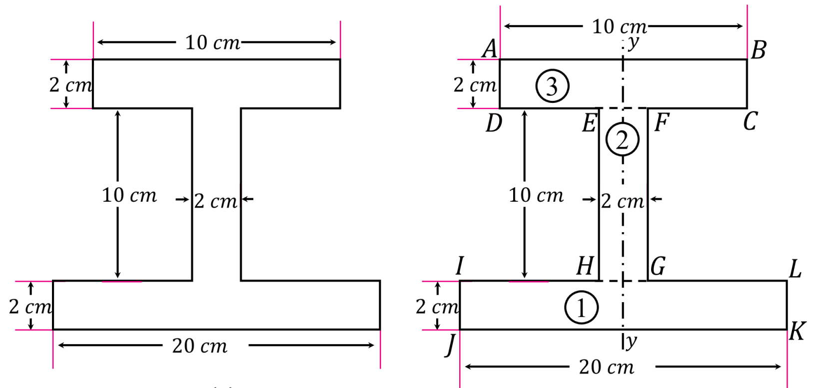
Example

Find the moment of inertia of T- section shown



Example

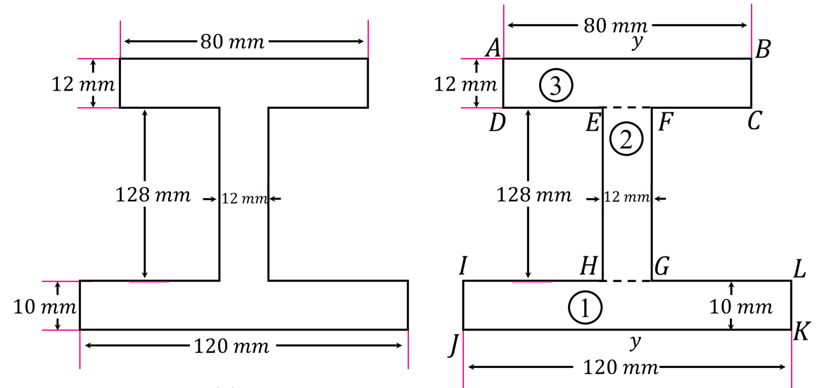
Find the moment of inertia of I- section shown



(a) (b)

Example

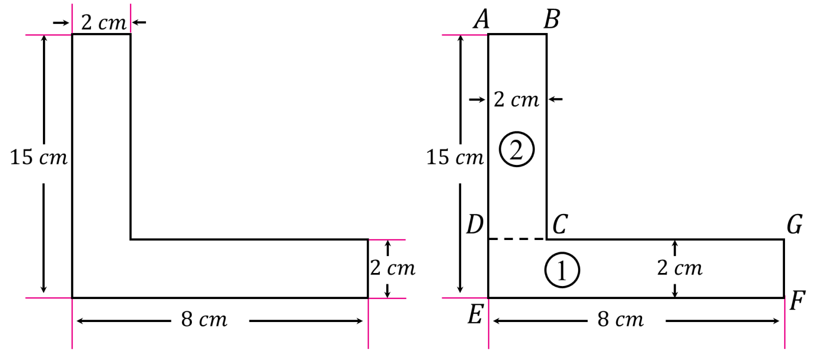
Find the moment of inertia of I- section shown



(a) (b)

Example

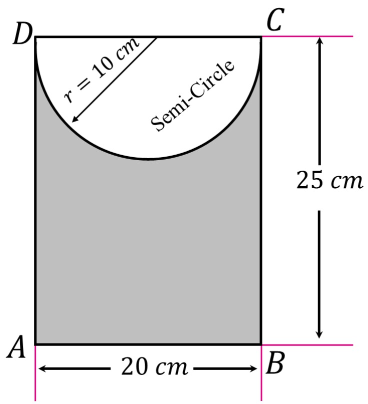
Find the moment of inertia of L- section shown below



(a) (b)

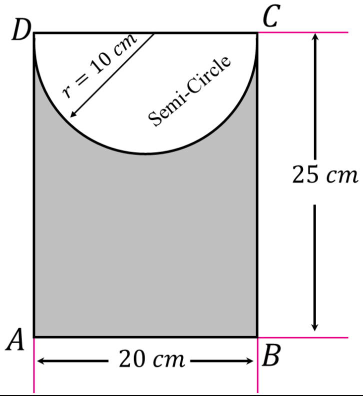
Example

Find the moment of inertia of the shaded area about edge AB



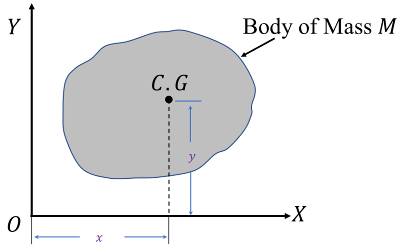
Example

Find the moment of inertia of the shaded area about its centroid, and the polar moment of inertia.



The Mass Moment of Inertia

The moment mass of inertia is the measure of a moving body's resistance to a change in motion.



The moment of mass about $OY = x.M$

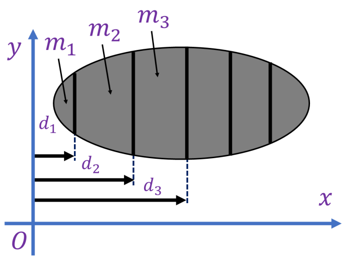
The second moment of mass about $OY = x^2.M$

The moment of mass about $OX = y.M$

The second moment of mass about $OX = y^2.M$

The mass moment of inertia is represented by I_m. For the OX and OY axes, the mass moment of inertia will be $(I_m)_{xx}$ and $(I_m)_{yy}$

The mass moment of inertia of the particles with masses m_1, m_2, m_3, and m_n, rotating a fixed point, i.e., O with distances d_1, d_2, d_3, and d_n



The mass moment of inertia

$$I_m = m_1 d_1^2 + m_2 d_2^2 + m_3 d_3^2 + \cdots m_n d_n^2$$

If masses are positioned in the same point of rotation, i.e., $d_1 = d_1 = d_n = R$

$$I_m = (m_1 + m_2 + m_3 + \cdots m_n)R^2$$

$$m_1 + m_2 + m_3 + \cdots m_n = \sum_i^n m_i = M$$

$$R = \sqrt{\frac{I}{M}}$$

$R =$ is called the radius of gyration

The radius of gyration describes an average distance traveled by particles.

In integral form, the mass moment of inertia, $I_m = \int r^2 \, dm$

$$I_m = (m_1 + m_2 + m_3 + \cdots m_n)R^2$$

$$m_1 + m_2 + m_3 + \cdots m_n = \sum_{i}^{n} m_i = M$$

$$R = \sqrt{\frac{I}{M}}$$

$R =$ is called the radius of gyration

The radius of gyration describes an average distance traveled by particles.

In integral form, the mass moment of inertia, $I_m = \int r^2\, dm$

Example

Find the moment of inertia and the radius of gyration with respect to the origin, (0,0) of a system which has masses at the points given:

Mass	6	5	9	2
Points	$(-3, 0)$	$(-2, 0)$	$(1, 0)$	$(8, 0)$

Solution:

The moment of inertia is given by:

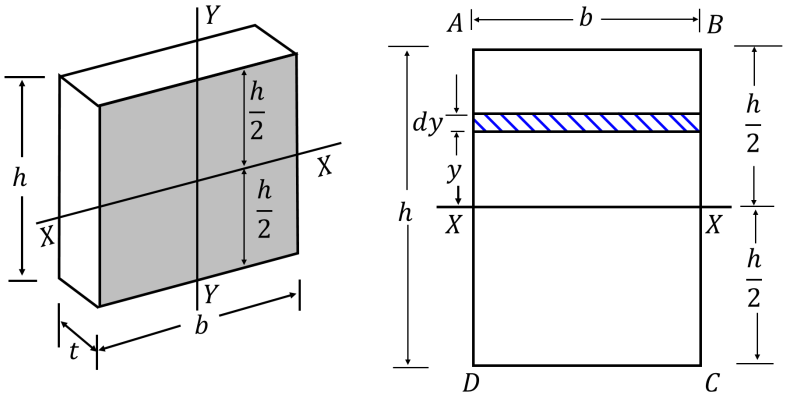
$$I_m = 6 \times (-3)^2 + 5 \times (-2)^2 + 9 \times (1)^2 + 2 \times (8)^2 = 211$$

The radius of gyration

$$R = \sqrt{\frac{I}{M}} = \sqrt{\frac{I}{m_1 + m_2 + m_3 + \cdots m_n}} = \sqrt{\frac{211}{(6 + 5 + 9 + 2)}} = 3.0969$$

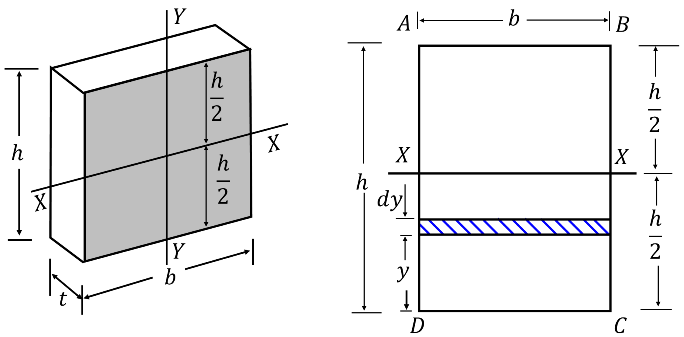
Example

Find the mass moment of inertia of a rectangular plate shown below



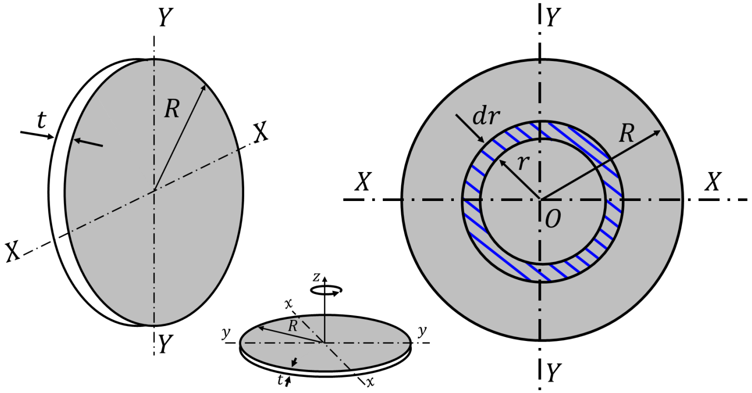
Example

Find the mass moment of inertia of a rectangular plate about its base DC axis.



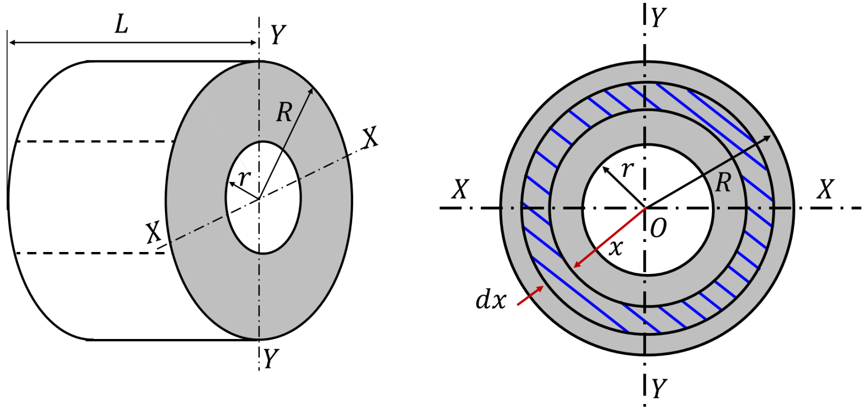
Example

Find the mass moment of inertia of a circular plate.



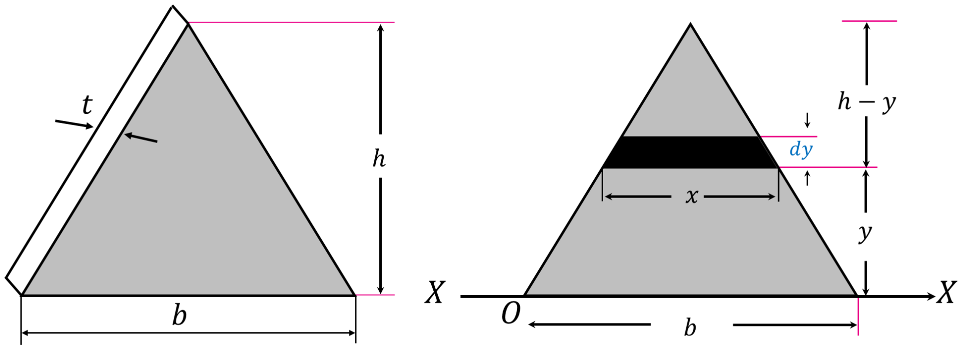
Example

Find the mass moment of inertia of a hollow shaft



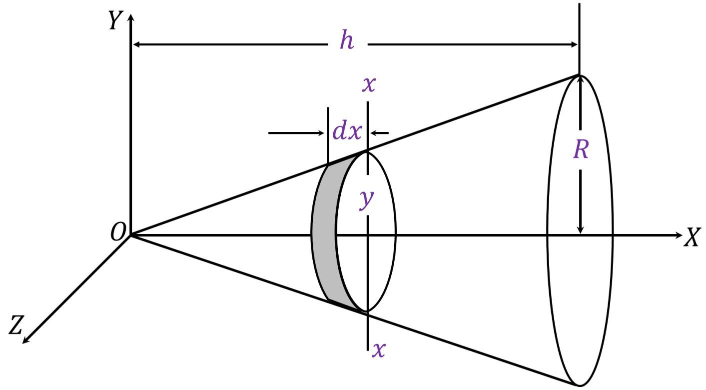
Example

Find the mass moment of inertia of a triangle



Example

Find the mass moment of inertia of a solid cone



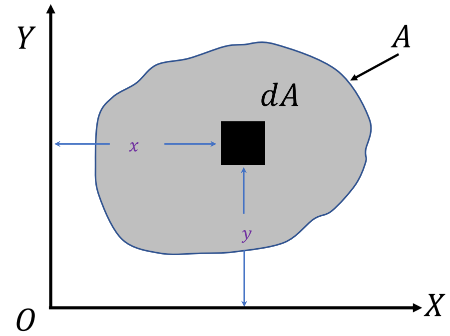
The Product of Inertia

The integral $\int xy\, dA$ is called product moment of inertia of area A with respect to x and y axes.

$$I_{xy} = \int xy\, dA$$

The product moment of inertia is the summation of all areas multiplied by their x and y coordiantes.

$$I_{xy} = \sum_{i}^{n} x_i y_i A_i = x_1 y_1 A_1 + x_2 y_2 A_2 + x_3 y_3 A_3$$



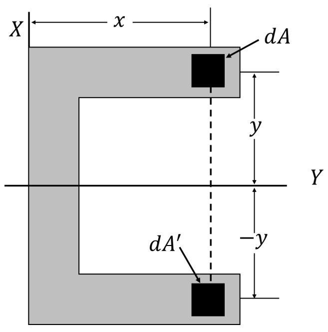
The product moment of inertia can be positive or negative, and have the same unit as compared to the moment of inertia.

When a body is rotated a bout its fixed axis, its second moment of inertia shifts also shifts to a new position.

We can't use the integral method to locate a newly rotated second moment of inertia after a body have been rotated.

The product moment of inertia, is therefore useful method to determine the second moment of inertia a newly rotated body axes.

When x or y or both axes are an axis of symmetry, then the product moment of inertia is zero.



For the upper part, the product-moment of inertia is,

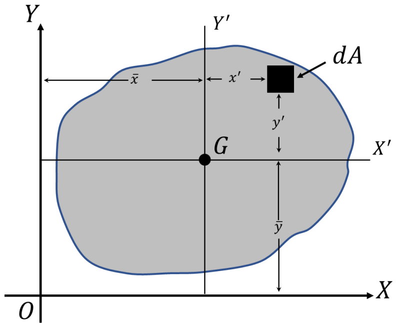
$$I_{xy} = \int xy\, dA$$

And for the lower part, the product-moment of inertia is,

$$I_{xy'} = -\int xy\, dA'$$

Then, the summation will be zero, i.e., $I_{xy} + I_{xy'} = 0$

The parallel axis theory for the product of inertia

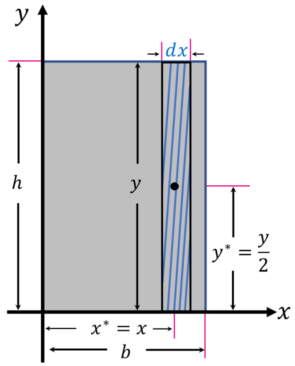


The parallel axis theorem for the product of inertia is

$$I_{XY} = I_G + \bar{x}\,\bar{y}\,A$$

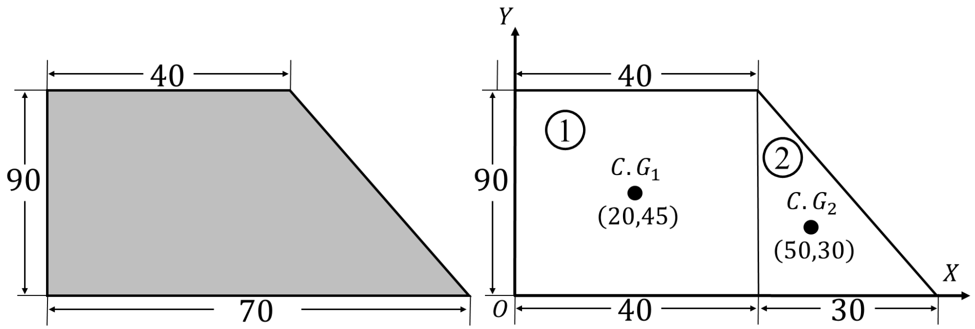
Example

Find the product moment of inertia of a rectangular section about its centroid axis.



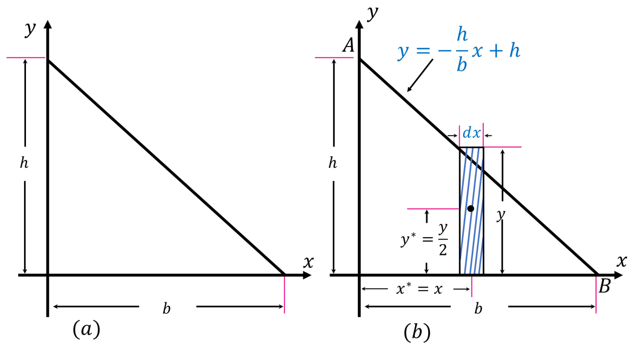
Example

Find the product moment of inertia of a given area shown below



Example

Find the product moment of inertia of a right triangle with respect to the x and y, and with respect to the centroidal axes.



Principal Axes

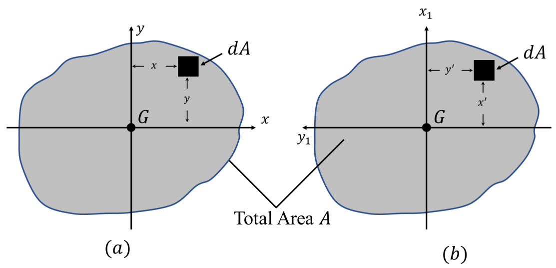
The principal axes are the axes about which the product of inertia is zero.

$$I_{xy} = \int xy\, dA$$

And the moment of inertia of the plane area A about x and y axes is,

$$I_{xx} = \int y^2\, dA \text{ and } I_{yy} = \int x^2\, dA$$

The product moment of inertia may be positive if both x and y are positive, or negative if one of the coordinates is positive and the other is negative.



Rotates the axes CCW about 90° (shown in (b)), maintaining the total area in the same position.

Let x_1 and y_1 be newly rotated axes, and x' and y' be the newly rotated coordinate of the small area dA corresponding to the new axes.

For an old axes, the product moment of inertia $I_{xy} = \int xy\, dA$

For a newly axes, the product moment of inertia $I_{x_1y_1} = \int x'y'\, dA$

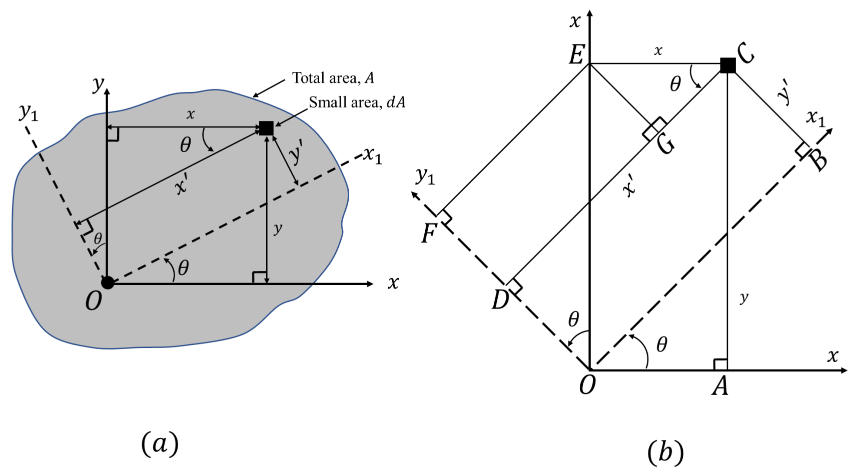
The axes for the new coordinates, $x = -y'$ or $y' = -x$ and $y = x'$

From the newly axes after rotation, we have,

$$I_{x_1y_1} = \int x'y'\, dA = \int (-x)(y)\, dA = -\int xy\, dA = -I_{xy}$$

The product of inertia become negative when axes have been rotated about 90° CCW, i.e., the product of inertia has changed the signs

The Principal Moment of Inertia



Equation for the newly axes, after axes have been rotated at an angle θ

$$I_{x_1x_1} = \frac{I_{xx}+I_{yy}}{2} + \frac{(I_{xx}-I_{yy})}{2}\cos 2\theta - \sin 2\theta \,.\, I_{xy}$$

$$I_{y_1y_1} = \frac{I_{xx}+I_{yy}}{2} - \frac{(I_{xx}-I_{yy})}{2}\cos 2\theta + \sin 2\theta \,.\, I_{xy}$$

$$I_{x_1y_1} = \frac{(I_{xx}-I_{yy})}{2}\,.\sin 2\theta + \cos 2\theta \,.\, I_{xy}$$

The directional of the principal axes

Now, the two-principal moments of inertia are,

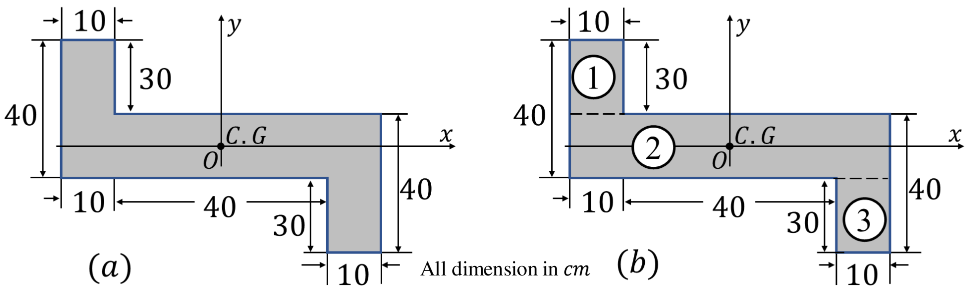
$$I_{\substack{max \\ min}} = \frac{I_{xx}+I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

The directions of the principal axes are,

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy}-I_{xx})}$$

Example

Find the moment of inertia about is centroidal axis, the moment of inertia about the new axes which is turned to an angle of 30° CCW to the old axes, the principal moments of inertia about its centroid.



Solution

Region	Area, (cm^2)	$\tilde{y}\ (cm)$	$\tilde{x}\ (cm)$	$\tilde{y}_iA_i,\ (cm^3)$	$\tilde{x}_iA_i,\ (cm^3)$
Rectangle, A_1	300	$40+\frac{30}{2}=55$	$\frac{10}{2}=5$	16,500	1,500
Rectangle, A_2	600	$30+\frac{10}{2}=35$	$\frac{60}{2}=30$	21,000	18,000
Rectangle, A_3	300	$\frac{30}{2}=15$	$50+\frac{10}{2}=55$	4,500	16,500
Σ	1,200			42,000	36,000

For $\bar{x}$, we have, $\bar{x} = \frac{\sum \tilde{x}_iA_i}{\sum A_i} = \frac{\sum \tilde{x}_iA_i}{A} = \frac{36{,}000}{1{,}200} = 30\ cm$

For $\bar{y}$, we have, $\bar{y} = \frac{\sum \tilde{y}_iA_i}{\sum A_i} = \frac{\sum \tilde{y}_iA_i}{A} = \frac{42{,}000}{1{,}200} = 35\ cm$

The moment of inertia about new axes which is turned through $30° \; CCW$

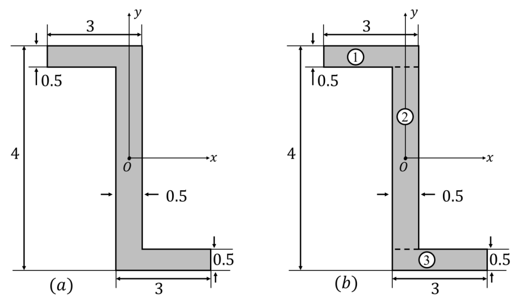
Region	Area, (cm^2)	$y_1 = \tilde{y} - \bar{y}, (cm)$	$x_1 = \tilde{x} - \bar{x}, (cm)$	$x_1 y_1 A_i, (cm^4)$
Rectangle, A_1	300	$55 - 35 = 20$	$5 - 30 = -25$	$-150{,}000$
Rectangle, A_2	600	$35 - 35 = 0$	$30 - 30 = 0$	0
Rectangle, A_3	300	$15 - 35 = -20$	$55 - 30 = 25$	$-150{,}000$
Σ	1,200			$-300{,}000$

Now, the product of inertia of the whole body is,

$$I_{xy} = -300{,}000 \; cm^4$$

Example

Find the moment of inertia about is centroidal axis, the moment of inertia about the new axes which is turned to an angle of $30° \; CCW$ to the old axes, the principal moments of inertia about its centroid.



All dimension in cm

Solution

Region	Area, (cm^2)	$\tilde{y}\;(cm)$	$\tilde{x}\;(cm)$	$\tilde{y}_i A_i, (cm^3)$	$\tilde{x}_i A_i, (cm^3)$
Rectangle, A_1	1.5	$3.5 + \frac{0.5}{2} = 3.75$	$\frac{3}{2} = 1.5$	5.625	2.25
Rectangle, A_2	1.5	$0.5 + \frac{3}{2} = 2$	$2.5 + \frac{0.5}{2} = 2.75$	3	4.125
Rectangle, A_3	1.5	$\frac{0.5}{2} = 0.25$	$2.5 + \frac{3}{2} = 4$	0.3750	6
Σ	4.5			9	12.375

For $\bar{x}$, we have, $\bar{x} = \frac{\sum \tilde{x}_i A_i}{\sum A_i} = \frac{\sum \tilde{x}_i A_i}{A} = \frac{12.375}{4.5} = 2.75 \; cm$

For $\bar{y}$, we have, $\bar{y} = \frac{\sum \tilde{y}_i A_i}{\sum A_i} = \frac{\sum \tilde{y}_i A_i}{A} = \frac{9.0}{4.5} = 2 \; cm$

The moment of inertia about new axes which is turned through $30° \; CCW$

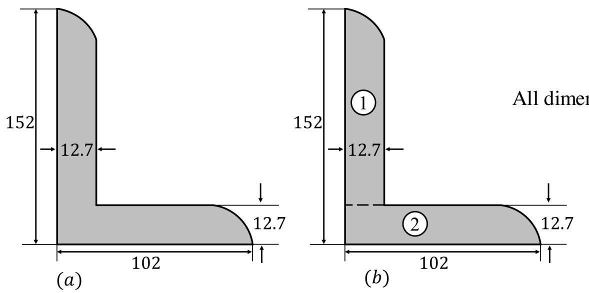
Region	Area, (cm^2)	$y_1 = \tilde{y} - \bar{y}, (cm)$	$x_1 = \tilde{x} - \bar{x}, (cm)$	$x_1 y_1 A_i, (cm^4)$
Rectangle, A_1	1.5	$3.75 - 2 = 1.75$	$1.5 - 2.75 = -1.25$	-3.2813
Rectangle, A_2	1.5	$2 - 2 = 0$	$2.75 - 2.75 = 0$	0
Rectangle, A_3	1.5	$0.25 - 2 = -1.75$	$4 - 2.75 = 1.25$	-3.2813
Σ	4.5			-6.5625

Now, the product of inertia of the whole body is,

$$I_{xy} = -6.5625 \; cm^4$$

Example

Find the moment of inertia about is centroidal axis, the moment of inertia about the new axes which is turned to an angle of 30° CCW to the old axes, the principal moments of inertia about its centroid. Draw the Mohr Circle



All dimension in cm

Solution

Region	Area, (mm^2)	$\tilde{y}\ (mm)$	$\tilde{x}\ (mm)$	$\tilde{y}_i A_i,\ (mm^3)$	$\tilde{x}_i A_i,\ (mm^3)$
Rectangle, A_1	1,769.11	$12.7+\frac{139.3}{2} = 82.35$	$\frac{12.7}{2} = 6.35$	145,686.2085	11,233.8485
Rectangle, A_2	1,295.4	$\frac{12.7}{2} = 6.35$	$\frac{102}{2} = 51$	8,225.79	66,065.4
Σ	3,064.51			153,911.9985	77,299.2485

For $\bar{x}$, we have, $\bar{x} = \frac{\sum \tilde{x}_i A_i}{\sum A_i} = \frac{\sum \tilde{x}_i A_i}{A} = \frac{77{,}299.2485}{3{,}064.51} = 25.2240\ cm$

For $\bar{y}$, we have, $\bar{y} = \frac{\sum \tilde{y}_i A_i}{\sum A_i} = \frac{\sum \tilde{y}_i A_i}{A} = \frac{153{,}911.9985}{3{,}064.51} = 50.2240\ cm$

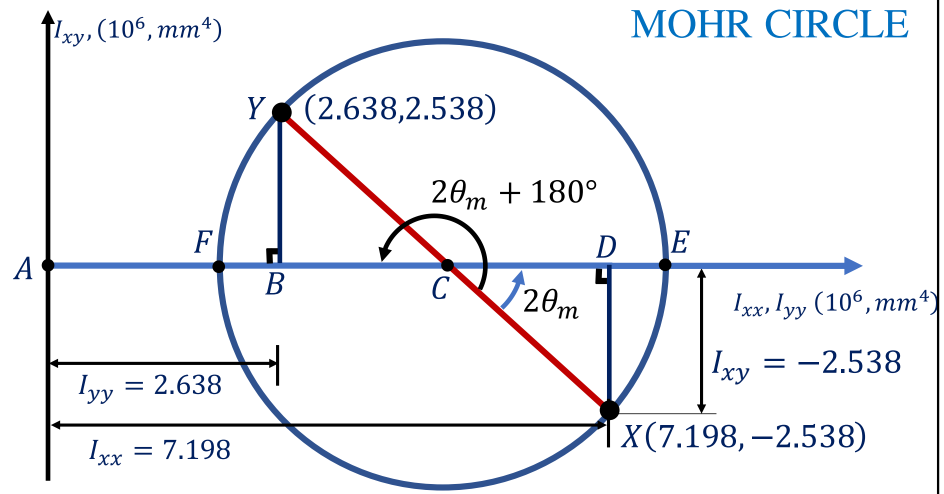
The moment of inertia about new axes which is turned through 60° CCW

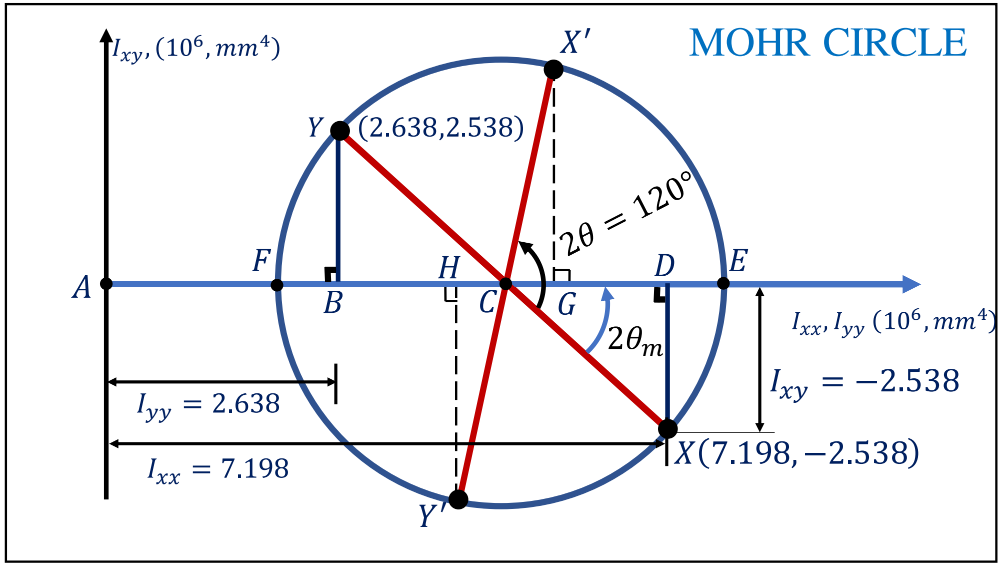
Region	Area, (mm^2)	$y_1 = \tilde{y} - \bar{y},\ (mm)$	$x_1 = \tilde{x} - \bar{x},\ (mm)$	$x_1 y_1 A_i,\ (mm^4)$
Rectangle, A_1	1,769.11	$82.35 - 50.224 = 32.126$	$6.35 - 25.224 = -18.874$	$-1{,}072{,}692.991$
Rectangle, A_2	1,295.4	$6.35 - 50.224 = -43.874$	$51 - 25.224 = 25.7760$	$-1{,}464{,}962.968$
Σ	3,064.51			$-2{,}537{,}655.959$

Now, the product of inertia of the whole body is,

$$I_{xy} = -2{,}537{,}655.959\ cm^4$$

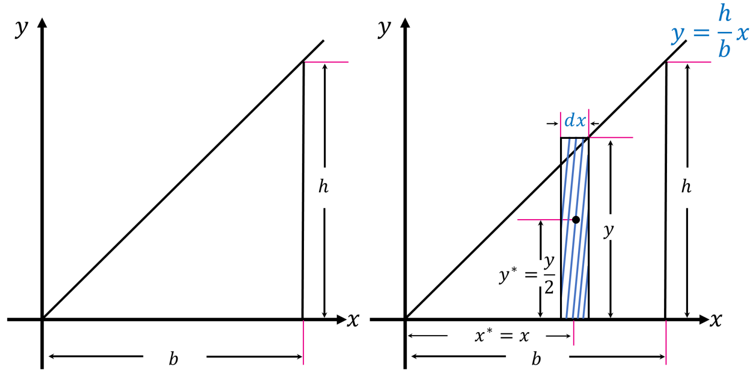
MOHR CIRCLE





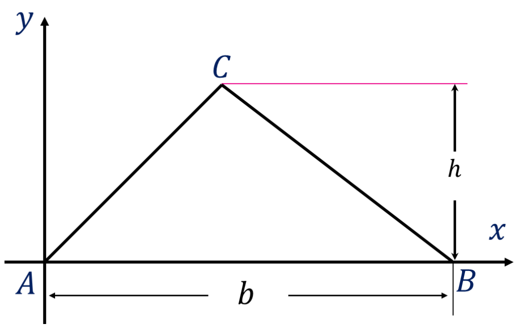
Example

Find the product moment of inertia of a right triangle



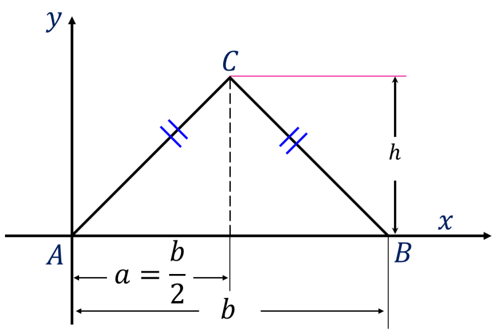
Example

For the triangle shown, find moment of inertia about x, y and its centroidal axes, find the radius of gyration about x, y and its centroidal axes



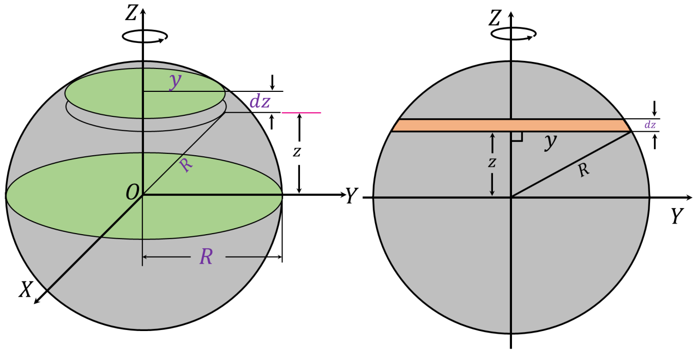
Example

Assume ABC is an isosceles triangle, find the moment along x, y axes, the radius of gyration, and the moment of inertia along its CG.



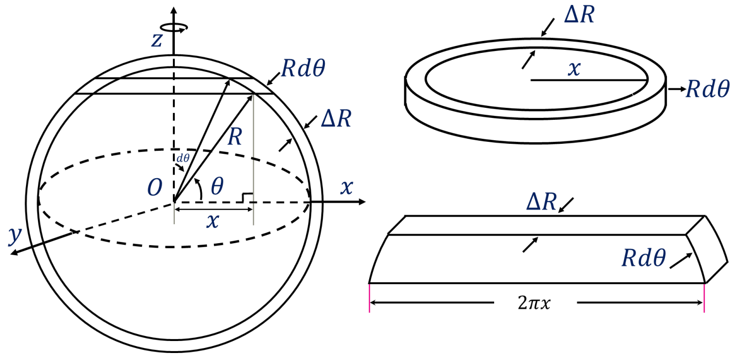
Example

Find the mass moment of inertia of a solid sphere



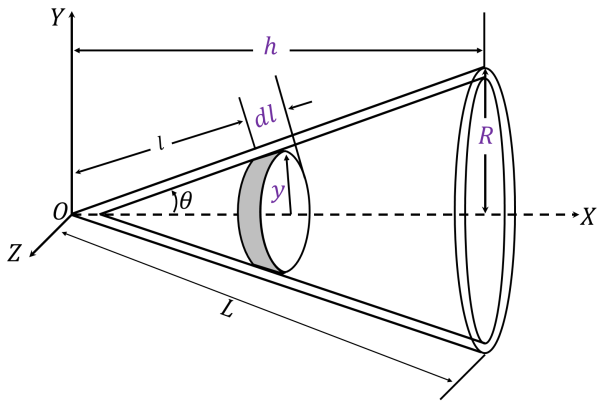
Example

Find the mass moment of inertia of a hollow sphere



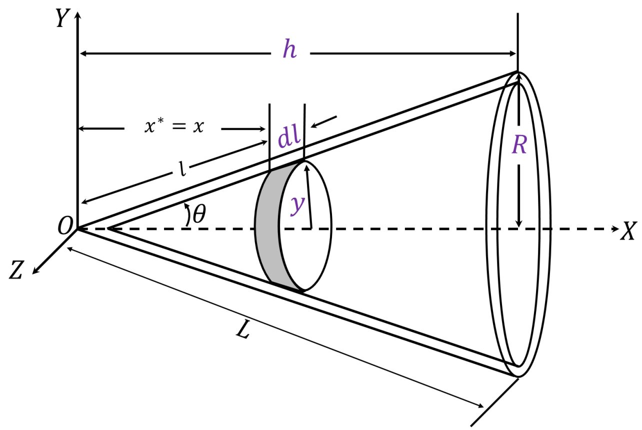
Example

Find the mass moment of inertia of a hollow cone



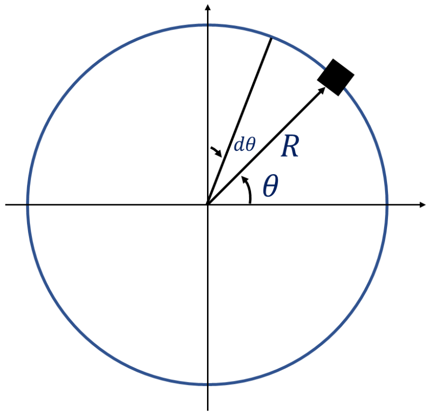
Example

Find the center mass or centroidal of a hollow cone about its axis



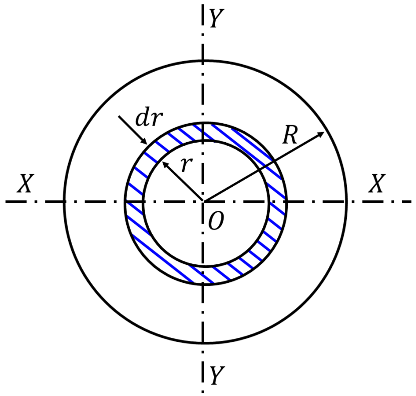
Example

Find the mass moment of inertia of a ring about its axis.



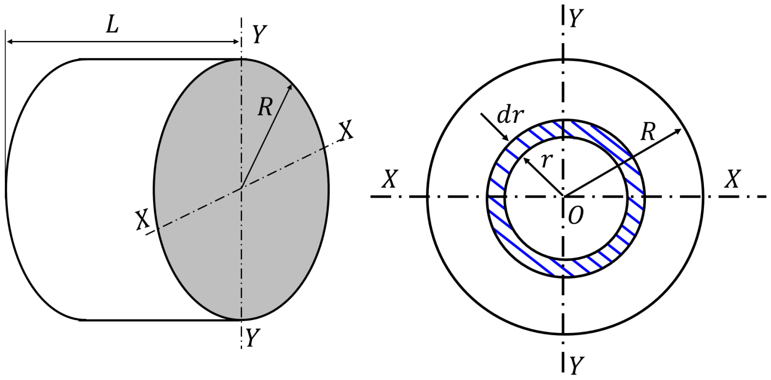
Example

Find the mass moment of inertia of a disk (no thickness)



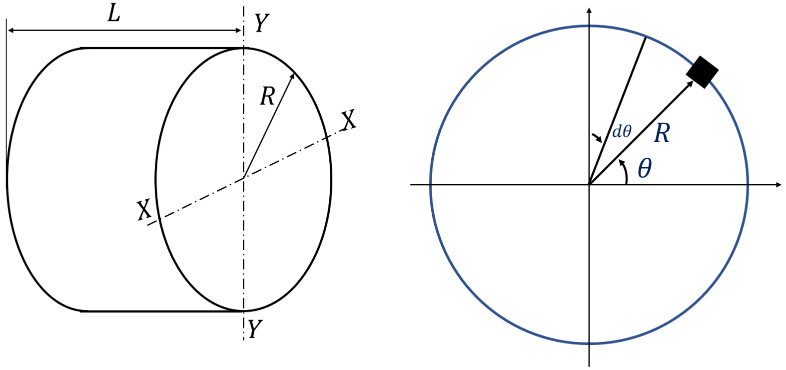
Example

Find the mass moment of inertia of a solid cylinder about its axis.



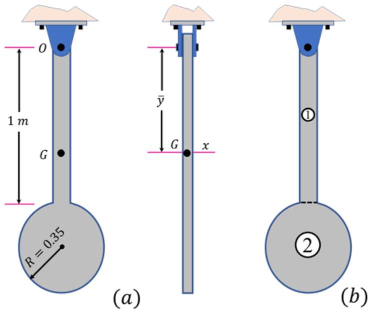
Example

Find the mass moment of inertia of a hollow cylinder about its axis.



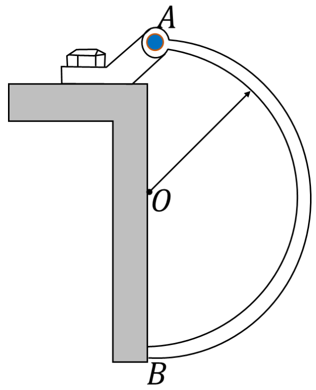
Example

For a composite pendulum shown in made of a uniform slender rod ($12\ kg$) and a uniform disk (8 kg), determine the mass moment of inertia about x axis passing through its center of gravity, as well as the radius of gyration about the x axis.



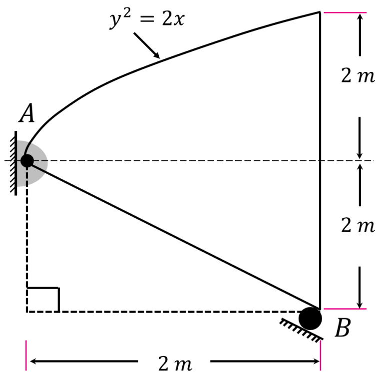
Example

The figure shows a uniform semicircular rod of weight W and radius r attached to a pin and rest against a frictionless surface B. Find the reaction at A and B



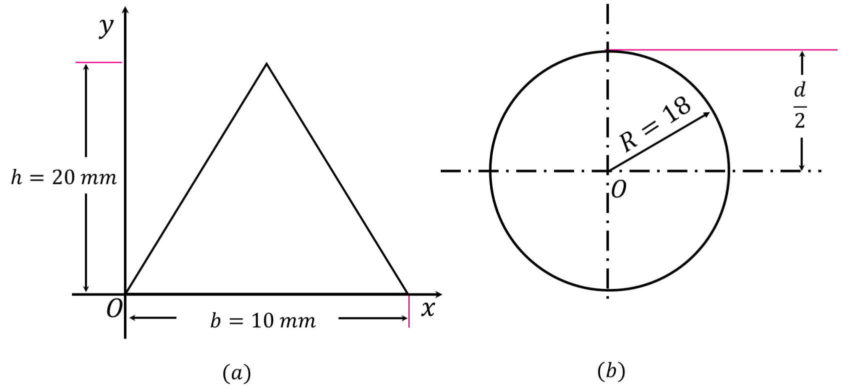
Example

The figure shows the plate having a density of 7,850 kg/m^3 and a thickness of 0.3 m. Find the plate CG, and the reactions at A and B.



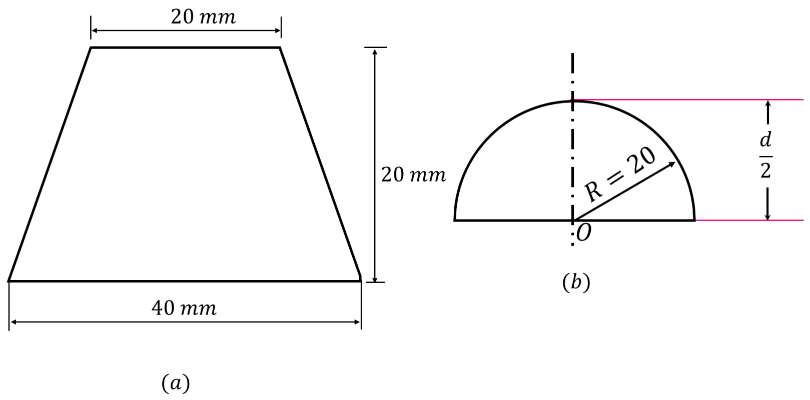
Example

Find the section modulus of the figure shown below



Example

Find the section modulus of the figure shown below



Example

Find the section modulus of I section