Demonstration of the Frisch-Waugh-Lovell (FWL) theorem

Authors
Alexander Gómez
Joseph Amaya
Julio Trejo
Víctor Arica

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1 Multiple Linear Regression Model

Consider the linear regression model:

$$y = X_1\beta_1 + X_2\beta_2 + u \tag{1}$$

Where y is the vector of outcomes, X_1 is a matrix of regressors of interest, X_2 is a matrix of control variables, β_1 and β_2 are the coefficient vectors, and u is the vector of errors.

2 Solve for $\hat{\beta}_2$ Estimator

We start by finding the expression for $\hat{\beta}_2$ in terms of $\hat{\beta}_1$, which is derived from multiplying both sides of the equation by $(X_2'X_2)^{-1}X_2'$. Solving for $\hat{\beta}_2$, we get:

$$\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'y - (X_2'X_2)^{-1}X_2'X_1\hat{\beta}_1 + (X_2'X_2)^{-1}X_2'u \tag{2}$$

Given that the vector of errors is orthogonal to the control variables ($X_2'u = 0$), the last term becomes zero. Therefore, the estimator simplifies to:

$$\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'y - (X_2'X_2)^{-1}(X_2'X_1)\hat{\beta}_1 \tag{3}$$

Note that we use $\hat{\beta}_1$ instead of β_1, as we are working with the estimators.

3 Derivation of the $\hat{\beta}_1$ Estimator

By multiplying both sides of the equation by X_1', we have:

$$(X_1'X_1)\hat{\beta}_1 + (X_1'X_2)\hat{\beta}_2 = X_1'y \tag{4}$$

We substitute the expression for $\hat{\beta}_2$ into this equation:

$$(X_1'X_1)\hat{\beta}_1 + (X_1'X_2)\left[(X_2'X_2)^{-1}X_2'y - (X_2'X_2)^{-1}(X_2'X_1)\hat{\beta}_1\right] = X_1'y \tag{5}$$

Expanding and grouping the terms containing $\hat{\beta}_1$ and y:

$$(X_1'X_1)\hat{\beta}_1 - (X_1'X_2)(X_2'X_2)^{-1}(X_2'X_1)\hat{\beta}_1 = X_1'y - (X_1'X_2)(X_2'X_2)^{-1}X_2'y \tag{6}$$

Factoring out $\hat{\beta}_1$ and y:

$$\left[(X_1'X_1) - (X_1'X_2)(X_2'X_2)^{-1}(X_2'X_1)\right]\hat{\beta}_1 = \left[X_1' - (X_1'X_2)(X_2'X_2)^{-1}X_2'\right]y \tag{7}$$

Now we can factor out X_1' from both sides of the equation and X_1 from the left side. The equation is rewritten as:

$$\left[X_1'(I - X_2(X_2'X_2)^{-1}X_2')X_1\right]\hat{\beta}_1 = X_1'(I - X_2(X_2'X_2)^{-1}X_2')y \tag{8}$$

We introduce the matrix $M_{X_2} = I - X_2(X_2'X_2)^{-1}X_2'$. Substituting M_{X_2}:

$$(X_1'M_{X_2}X_1)\hat{\beta}_1 = X_1'M_{X_2}y \tag{9}$$

Solving for $\hat{\beta}_1$, we get the OLS (Ordinary Least Squares) estimator for β_1:

$$\hat{\beta}_1 = (X_1'M_{X_2}X_1)^{-1}X_1'M_{X_2}y \tag{10}$$

Given that M_{X_2} is idempotent we have:

$$\hat{\beta}_1 = (X_1'M_{X_2}'M_{X_2}X_1)^{-1}X_1'M_{X_2}'M_{X_2}y \tag{11}$$

Finally we have:

$$\hat{\beta}_1 = (\tilde{X}_1{}'\tilde{X}_1)^{-1}\tilde{X}_1{}'\tilde{y} \tag{12}$$

where $\tilde{y}$=$M_{X_2}y$ and $\tilde{X}_1$=$M_{X_2}X_1$