Matrix Factorization of graph potentials and Mirror Symmetry for Character varieties

Victor Ibrahim Santos[1]

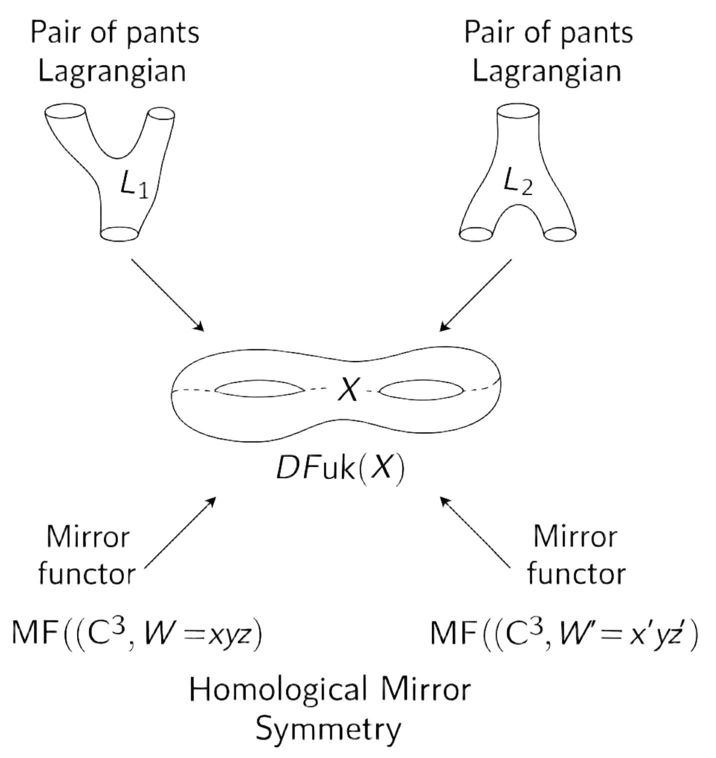
[1]PhD student at PUC-Rio

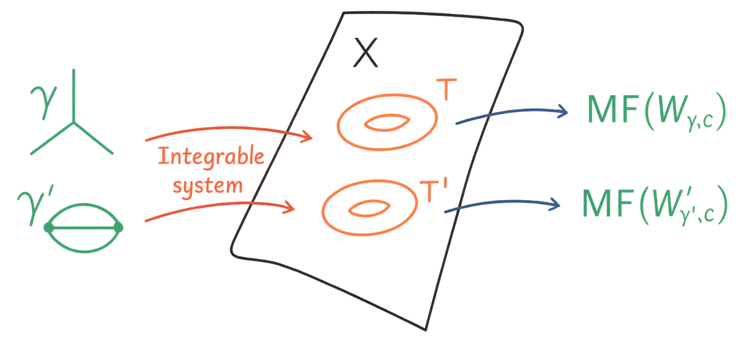
Motivation

Homological Mirror Symmetry (HMS) predicts deep relationships between symplectic and algebraic geometry, formulated as equivalences between derived categories associated to mirror pairs. For a Fano variety X, HMS posits the existence of a mirror Landau–Ginzburg model (Y, W), where $W : Y \to \mathbb{A}^1$, such that

$$D^{\pi}\mathrm{Fuk}(X) \simeq \overline{MF(Y, W)},$$

with $D^{\pi}\mathrm{Fuk}(X)$ the split-closed derived Fukaya category of X, and $\overline{MF(Y, W)}$ the (graded) category of matrix factorizations of the potential W.





Matrix Factorization category

Let X be an algebraic variety over $\mathbb{C}$ with superpotential $W : X \to \mathbb{A}^1_{\mathbb{C}}$ flat and regular and $w_0 \in \mathbb{C}$. Define differential $\mathbb{Z}/2$-graded category $DG_{w_0}(X, W)$ such that

- **Objects:**

$$E := E_1 \underset{e_0}{\overset{e_1}{\rightleftarrows}} E_0$$

  where E_0, E_1 vector bundles such that $e_1 \circ e_0 = (W - w_0)I_{E_0}$ and $e_0 \circ e_1 = (W - w_0)I_{E_1}$
- **Morphism:**

$$\mathrm{Hom}_{DG_{w_0}(X,W)}(E, F) = \bigoplus_{0 \le i,j \le 1} \mathrm{Hom}(E_i, F_j)$$

  grading $(i - j) \bmod 2$ and D a differential defined this way. Given $p \in \mathrm{Hom}(E_i, F_j)$

$$D(p) = f_j \circ p + (-1)^{\deg p} p \circ e_i$$

Define the triangulated category of matrix factorizations $MF_{w_0}(X, W)$ at the fibre of w_0 as the Verdier quotient $H^0 DG_{w_0}(X, W)/Ac_{w_0}(X, W)$. Finally, the graded matrix factorization category of X is given by

$$\bigoplus_{w_0 \in \mathbb{C}} MF_{w_0}(X, W) := MF(X, W),$$

- **Example:** Consider $\mathbb{A}^3_{x,y,z}$ and $W = x^3 + y^3 + z^3 - 3xyz$. Suppose that $(a, b, c) \in (\mathbb{C}^*)^3$ is a zero of W. Then, the matrix

$$e_0 = \begin{pmatrix} ax & cy & bz \\ cz & bx & ay \\ by & az & cx \end{pmatrix}$$

  satisfies $det(e_0) = abcW$. Thus setting $e_1 = \frac{1}{abc} e_0^{\#}$, where $\_^{\#}$ denotes the matrix of cofactors, we obtain a family of rank 3 factorizations parameterized by $V(W) \cap (\mathbb{C}^*)^3$.

Derived Structures in Matrix Factorization Categories

In the matrix factorization category, we have algebraic-geometric dualities, such as:

- **Fourier-Mukai functor**
  For every given two matrix factorization categories $MF_{w_0}(X, W)$ and $MF_{w_0'}(X', W')$, we have a tensor product defined as $MF_{w_0}(X, W) \otimes_k MF_{w_0'}(X', W') \cong MF_{w_0 + w_0'}(X \times X', W \boxplus W')$, where $W \boxplus W' := W(x) + W(x')$. Moreover, we have

$$Fun(MF_{w_0}(X, W), MF_{w_0'}(X', W')) \cong MF_{w_0' - w_0}(X \times X', W' \boxminus W)$$

  which is defined as follows

$$\begin{tikzcd}
& MF_{w_0'-w_0}(X \times X', W' \boxminus W) \arrow[dr, "\pi'_*"] & \\
MF_{w_0}(X, W) \arrow[ur, "\pi^*"] \arrow[rr, "\Phi_{\mathcal{K}}(_) := \pi'_*(\pi^*(_) \otimes \mathcal{K})"'] & & MF_{w_0'}(X', W')
\end{tikzcd}$$

- **McKay correspondence**
  Let G be a finite group, a crepant resolution of X/G is denoted Y implies that

$$D^b(X)^G \cong D^b(Y)$$

  For $\dim X \le 3$ and $G < SL_3$, we have $Y = \mathrm{Hilb}^G X$.
  Thus, there is a generalization for matrix factorization given by

$$MF^G_{w_0}(X, W) \cong MF_{w_0}(Y, \widehat{W})$$

  where $\widehat{W}$ is defined as this composition $Y \xrightarrow{C} X \xrightarrow{W} \mathbb{A}^1_{\mathbb{C}}$.
- **Knörrer periodicity** Let X be a quasi-projective variety and $W : X \to \mathbb{A}^1$.

$$\mathrm{MF}_{w_0}(X, W) \cong \mathrm{MF}_{w_0}(X \times \mathbb{A}^2, W + xy).$$

  More generally, we have that, given $Z(g) = D \subset X$ a divisor.

$$\mathrm{MF}_{w_0}(D, W_{|D}) \cong \mathrm{MF}_{w_0}(X \times \mathbb{A}^1, W + xg).$$

Graph Potentials

Let $\gamma = (V, E)$ be an undirected trivalent connected graph (possibly containing loops) of genus g with vertex set V and edge set E.

$$\#V = 2g - 2 \quad \text{and} \quad \#E = 3g - 3$$

We have a color map $c : V \to \{\pm 1\}$, defined this way:

$$c(v) = \begin{cases} 1, & \text{if } v \text{ is } \circ, \\ -1, & \text{if } v \text{ is } \bullet. \end{cases}$$

For example, the colored Theta graph has two vertices: one black and another white.

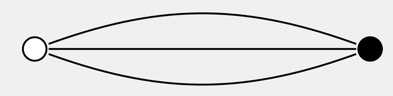


Figure 1. Theta graph

Let x_i be the coordinate functions in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_{3g-3}^{\pm 1}]$ corresponding to each oriented edge $e_i \in E$.

- The ***vertex potential*** for a vertex $v \in V$ adjacent to the edges e_i, e_j, e_k is defined as the Laurent polynomial

$$\widetilde{W}_{v,c} := \sum_{\substack{s_i,\ s_j,\ s_k \in \{\pm 1\} \\ s_i \cdot s_j \cdot s_k = c(v)}} x_i^{s_i} x_j^{s_j} x_k^{s_k} \in \mathbb{Z}[x_i^{\pm 1}, x_j^{\pm 1}, x_k^{\pm 1}].$$

- The ***graph potential*** of (γ, c) is as the Laurent polynomial

$$\widetilde{W}_{\gamma,c} := \sum_{v \in V} W_{v,c} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{3g-3}^{\pm 1}]$$

example if γ is the Theta graph

$$\widetilde{W}_{\gamma,c} = xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} + \frac{1}{xyz} + \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} = \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(z + \frac{1}{z}\right).$$

Matrix factorization on graph potentials

Given two trivalent connected graphs, we gluing rule is as follows

Specifically, when we identify a with a', it is obtained a new graph. This relation have its matrix factorization counterpart

$$\begin{aligned} MF(U\gamma_1, \widetilde{W}_{\gamma_1,a}) \times_{Spec(\mathbb{C}[a])} MF(U\gamma_2, \widetilde{W}_{\gamma_2,c}) &:= MF(U\gamma_1 \times \times_{Spec(\mathbb{C}[a])} U\gamma_2, \widetilde{W}_{\gamma_1,c} + \widetilde{W}_{\gamma_2,c}) \\ &= MF(U\gamma_3, \widetilde{W}_{\gamma_3,c}) \end{aligned}$$

For every graph potential,locally we have the following:

$$\begin{tikzcd}
\boxed{\text{Integrable System}} \arrow[r] & \boxed{\text{Lagragian Torus Charts } T_{\gamma,c}} \arrow[d] \\
\boxed{\text{Graph Potential } \gamma, c} \arrow[u] \arrow[r] & \boxed{\text{Matrix Factorization } MF(U_\gamma, \widetilde{W}_{\gamma,c})}
\end{tikzcd}$$

where $U_\gamma = Spec(\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}])$.

Character Varieties

Let C be a curve of genus $g \ge 2$. We can define the **Moduli space of stable rank 2 vector bundles with fixed determinant $\mathcal{L}$ of odd degree over C**. Denoted as $M_C(2, \mathcal{L})$. This character is a Fano variety of index 2 and dimension $3g - 3$. For example for hyperelliptic curve

$$M_C(2, \mathcal{L}) \cong \{v \in \mathrm{Gr}(g - 2, \mathbb{P}^{2g+1}) : \ v \subset Q_1 \cap Q_2\}$$

Where $Q_i \subset \mathbb{P}^{2g+1}$ is a quadric hypersurface.

Furthermore, let $\mathcal{M} \in \mathrm{Pic}(C)[2]$ and $\mathcal{E} \in M_C(2, \mathcal{L})$. Therefore, we have

$$\det(\mathcal{E} \otimes \mathcal{M}) = \det(\mathcal{E}) \otimes \mathcal{M}^{\otimes 2} = \mathcal{L}$$

Thus, we have that $\mathrm{Pic}(C)[2]$ acts on $M_C(2, \mathcal{L})$.

Mirror Symmetry in $M_C(2, \mathcal{L})$

The expected mirror partners of $M_C(2, \mathcal{L})$ and $[M_C(2, \mathcal{L}/\mathrm{Pic}(C)[2]]$ are the following:

Symplectic manifold	Algebraic Variety
$M_C(2, \mathcal{L})$	$W = (xyz)^{-1}(1 + yz)(1 + xz)(1 + xy)$
$[M_C(2, \mathcal{L})/\mathrm{Pic}(C)[2]]$	$\widetilde{W} = (x + x^{-1})(y + y^{-1})(z + z^{-1})$

This means we must have

$$D^{\pi}\mathcal{F}uk(M_C(2, \mathcal{L})) \cong \overline{\mathrm{MF}(\widetilde{X}, \widetilde{f})} \quad \text{and} \quad D^{\pi}\mathcal{F}uk([M_C(2, \mathcal{L}/\mathrm{Pic}(C)[2]]) \cong \overline{\mathrm{MF}(X, f)},$$

where X and $\widetilde{X}$ are compactification such that

$$\begin{tikzcd}
X \arrow[dr, "f"] & \\
(\mathbb{C}^*)^3 \arrow[u] \arrow[r, "W"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{X} \arrow[dr, "\tilde{f}"] & \\
(\mathbb{C}^*)^3 \arrow[u] \arrow[r, "\widetilde{W}"'] & \mathbb{C}
\end{tikzcd}$$

The evidence is that the eigenvalues of $QH(M_C(2, \mathcal{L}))$ are equal to the singular fiber of W.

Scan QR code for references

victoribrahim@aluno.puc-rio.br