Topic 6:
Multiple Regression

Major Topics:

- Matrix Notation
- Assumptions
- Estimators
- Output
- ANOVA
- Adjusted R^2

Multiple Regression
Introduction

- Expanded model
  - Our model now must account for several factors

$$Y_i = b_0 + b_1 X_{i1} + b_2 X_{i2} + \ldots + b_p X_{ip} + e_i$$

  - Now have *i* equations with $p > 1$ independent variables plus one constant
    - Have $p+1$ unknown parameters
    - Still have *n* observations

Multiple Regression
Notation

- To economize on writing equations, use matrix notation
  - Model is now written as

$$\boxed{\mathbf{Y} = \mathrm{X}b + e}$$

where

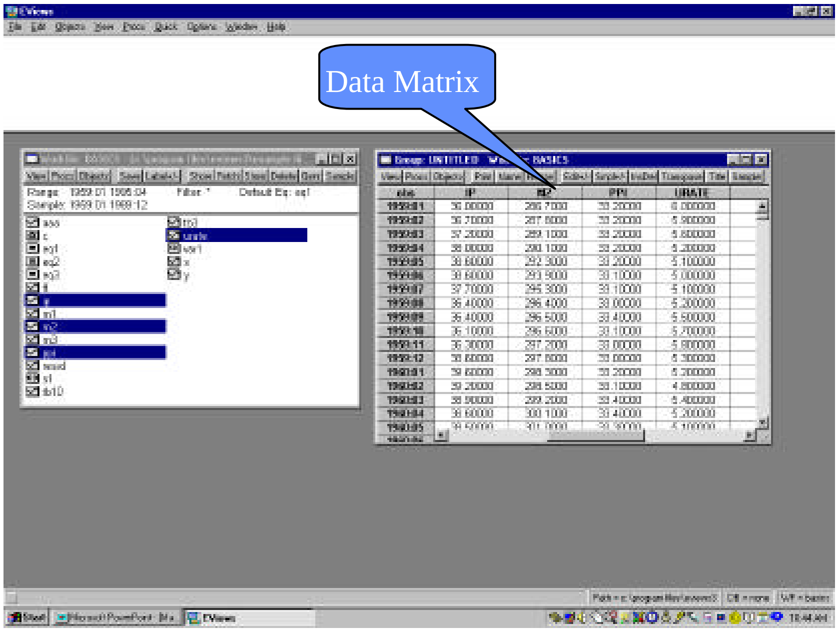
$$\mathbf{Y} = \begin{vmatrix} Y_1 \\ Y_2 \\ \ldots \\ Y_n \end{vmatrix} \qquad \mathbf{X} = \begin{vmatrix} 1 & X_{11} & X_{12} & \ldots & X_{1p} \\ 1 & X_{21} & X_{22} & \ldots & X_{2p} \\ & & \ldots & & \\ 1 & X_{n1} & X_{n2} & \ldots & X_{np} \end{vmatrix}$$

n X 1 n X (p+1)

Multiple Regression
Notation (Continued)

$$b = \begin{vmatrix} b_0 \\ b_1 \\ b_2 \\ \ldots \\ b_p \end{vmatrix} \qquad e = \begin{vmatrix} e_1 \\ e_2 \\ \ldots \\ e_n \end{vmatrix}$$

(p+1) X 1 nX1



Multiple Regression
Classical Assumptions

- Same classical assumptions as for simple regression model, but...
  - New assumption:
    - The *p* independent variables are linearly independent

Multiple Regression
Classical Assumptions (Continued)

- New assumption interpretation
  - Cannot write one independent variable as a linear combination of the other *p - 1* variables
    - Example: cannot write

$$\mathbf{X}_1 = a\mathbf{X}_2 + g\mathbf{X}_3$$

  - If could write one variable as linear combination, then that variable is redundant
  - Example

$$\mathrm{RU_t} = b_0 + b_1\mathrm{GDP_t} + b_2\mathrm{C_t} + b_3\mathrm{I_t} + b_4\mathrm{G_t} + b_5\mathrm{X_t} + e_t$$

but

$$\mathrm{GDP} = \mathrm{C} + \mathrm{I} + \mathrm{G} + \mathrm{X}$$

Multiple Regression
Classical Assumptions (Continued)

- Writing any independent variable as linear combination of one or more of the other independent variables is called ***multicollinearity***
  - Serious problems arise if multicollinearity exists
  - Yet this is a common problem with economic data
  - We will discuss this extensively in a separate lecture

Multiple Regression
Digression on Inverses

- Suppose we have the two simultaneous equations

$$-2 = 4b_1 - 10b_2$$
$$13 = 3b_1 + 7b_2$$

- In matrix form, this is

$$\begin{bmatrix} -2 \\ 13 \end{bmatrix} = \begin{bmatrix} 4 & -10 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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Multiple Regression
Digression on Inverses (Continued)

- Solve these two simultaneous equations for b_1

$$(-2)(7) = (4)(7)b_1 - (10)(7)b_2$$
$$(13)(-10) = (3)(-10)b_1 + (7)(-10)b_2$$
$$or$$
$$-14 = 28b_1 - 70b_2$$
$$-130 = -30b_1 - 70b_2$$

- So $$b_1 = \frac{116}{58} = 2$$

Multiple Regression
Classical Assumptions (Continued)

Key Classical Assumptions

Normality	$e_i \approx N \quad \forall i$
Zero Mean	$E(e_i) = 0 \quad \forall i$
Homoskedasticity	$s_e^2 = s^2 \quad \forall i$
Zero Autocorrelation	$COV(e_i, e_j) = 0 \quad \forall i, j \quad i \neq j$
Independence of X_i, e_i	$COV(e_i, X_i) = 0 \quad \forall i$
Linearity	Linear in Parameters
No Exact Linear Relationship	No Multicollinearity
Non-stochastic X_i	Fixed X

Multiple Regression
Estimators

- Derive estimators in same manner as before
  - Minimize the error sum of squares
  - Solve normal equations for unknown parameter estimators
    - Have *p+1* normal equations
- Estimator written in matrix notation

$$\hat{b} = (\mathbf{X'}\,\mathbf{X})^{-1}\,\mathbf{X'}\,\mathbf{Y}$$

Multiple Regression
Estimators (Continued)

- Note that estimator form is no different than previously derived estimator for simple OLS
  - Multiple regression

$$\hat{b} = (\mathbf{X'X})^{-1}\mathbf{X'Y}$$

  - Simple regression

$$\hat{b}_1 = \frac{S_{XY}}{S_{XX}}$$
$$= S_{XX}^{-1} S_{XY}$$

Multiple Regression
Estimators (Continued)

- Other estimator formulas

$$s_b^2 = s^2(\mathbf{X'X})^{-1}$$

$$\hat{\mathbf{Y}} = \mathrm{X}\hat{b} = \mathbf{HY} \qquad \mathrm{H} = (\mathrm{X'X})^{-1}\mathrm{X'}$$

$$s_{\hat{\mathbf{Y}}}^2 = s^2\mathbf{H}$$

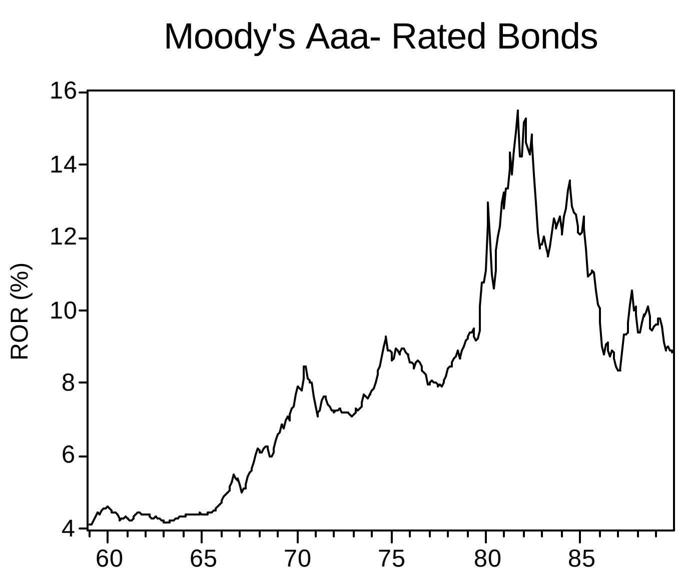
$$\mathbf{e} = \mathbf{Y} - \hat{\mathbf{Y}}$$

$$s_{\mathbf{e}}^2 = s^2(\mathbf{I} - \mathbf{H}) = s^2\mathbf{M}$$

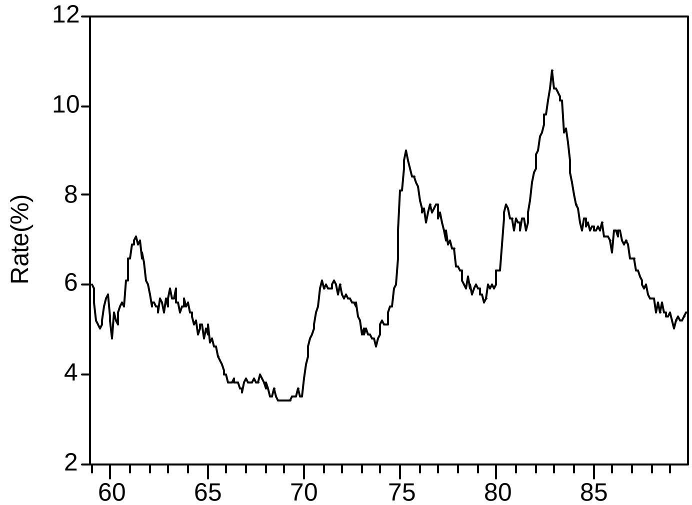
$$\mathrm{s}^2 = \frac{\mathbf{e'e}}{\mathrm{n - p - 1}}$$

Multiple Regression
Estimators (Continued)

- Properties
  - Desirable small sample properties still hold for general case of multiple regression
    - Linearity
    - Unbiasedness
    - Minimum variance
  - Under expanded classical assumptions, OLS estimators are still BLUE

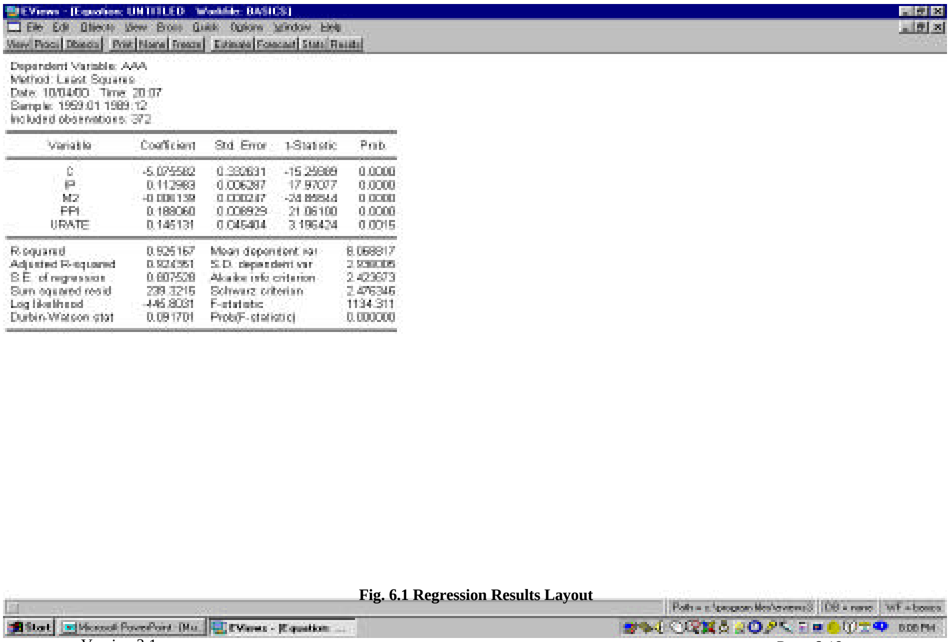


Unemployment Rate



Variable Coding Sheet

Variable	Mnemonic	Source	Possible Values
Moody's AA Rated Corporate Bonds	AAA	Moody's	% per Annum
Industrial Production Index	IP	Federal Reserve	Index, 1987=100, SA
Money Supply	M2	Federal Reserve	$Billion
Producer Price Index, Finished Goods	PPI	BLS	Index, 1982=100, NSA
Unemploy. Rate	URATE	BLS	% for 16+ older



Dependent Variable: AAA
Method: Least Squares
Date: 10/04/00 Time: 20:07
Sample: 1959:01 1989:12
Included observations: 372

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-5.075582	0.332631	-15.25889	0.0000
IP	0.112983	0.006287	17.97077	0.0000
M2	-0.006139	0.000247	-24.85844	0.0000
PPI	0.188060	0.008929	21.06100	0.0000
URATE	0.145131	0.045404	3.196424	0.0015

R-squared	0.925167	Mean dependent var	8.068817
Adjusted R-squared	0.924351	S.D. dependent var	2.938006
S.E. of regression	0.807528	Akaike info criterion	2.423673
Sum squared resid	239.3215	Schwarz criterion	2.476346
Log likelihood	-445.8031	F-statistic	1134.311
Durbin-Watson stat	0.091701	Prob(F-statistic)	0.000000

Fig. 6.1 Regression Results Layout

Multiple Regression
Goodness of Fit

ANOVA Table

Source of Variation	DF	Sum of Squares	Mean Square	F-Ratio
Regression	p	SSR	MSR= SSR/p	MSR/MSE
Residual	n - p - 1	SSE	MSE= SSE/n-p-1	
Total	n - 1	SST	MST= SST/n-1	

Multiple Regression
New R^2

- Have to adjust conventional R^2
  - R^2 can be made arbitrarily large simple by adding independent variables to model
  - New R^2 is called ***Adjusted R²*** or $\overline{R}^2$

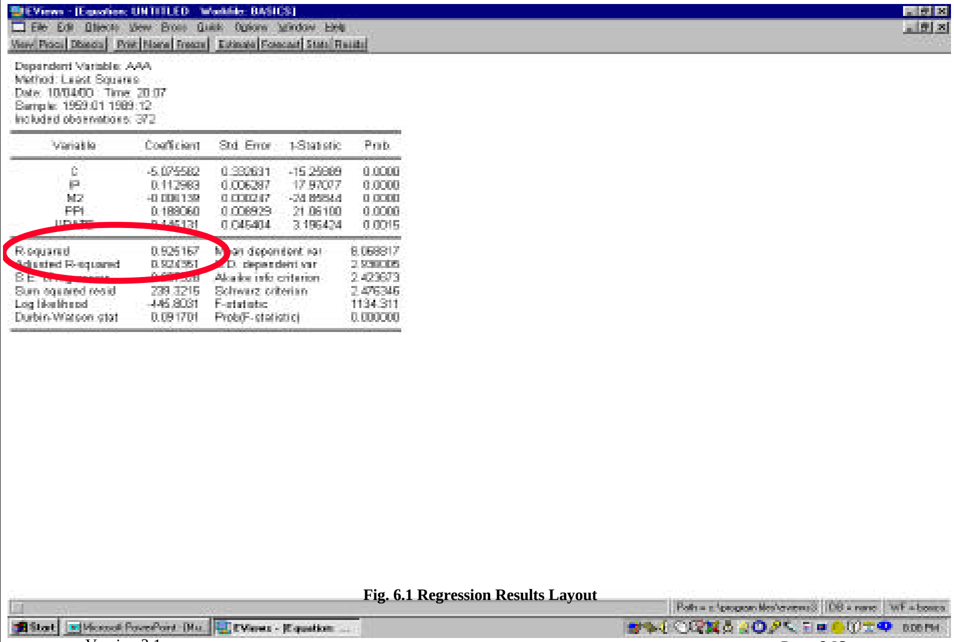
$$\overline{R}^2 = 1 - \left(\frac{n-1}{n-p-1}\right)(1-R^2)$$
$$= 1 - \frac{SSE/(n-p-1)}{SST/(n-1)}$$
$$= 1 - \frac{s^2}{s_Y^2}$$

Multiple Regression
New R^2 (Continued)

- Notice that

$$\overline{R}^2 \le R^2$$

One modeling objective
  - Maximize $\overline{R}^2$
- Property of $\overline{R}^2$
  - Can decline if added independent variable has no or little explanatory power
  - Reason
    - Increase n - p - 1



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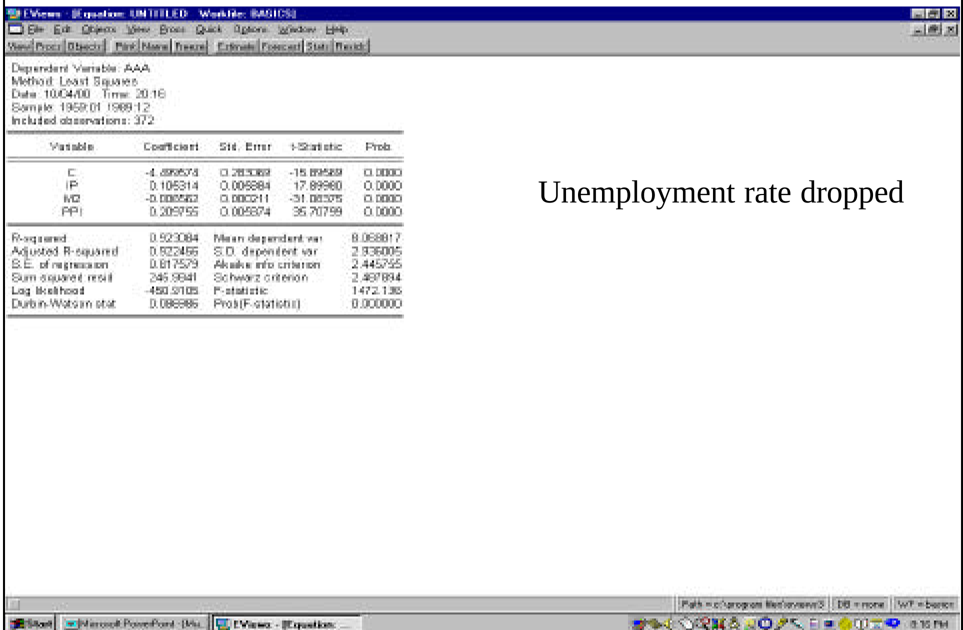
Durbin-Watson stat	0.091701	Prob(F-statistic)	0.000000

Fig. 6.1 Regression Results Layout

Multiple Regression
Alternative Models

- Can now develop alternative models by adding or deleting variables
  - Each model is a new regression run
  - Criteria for selecting variables will be discussed later

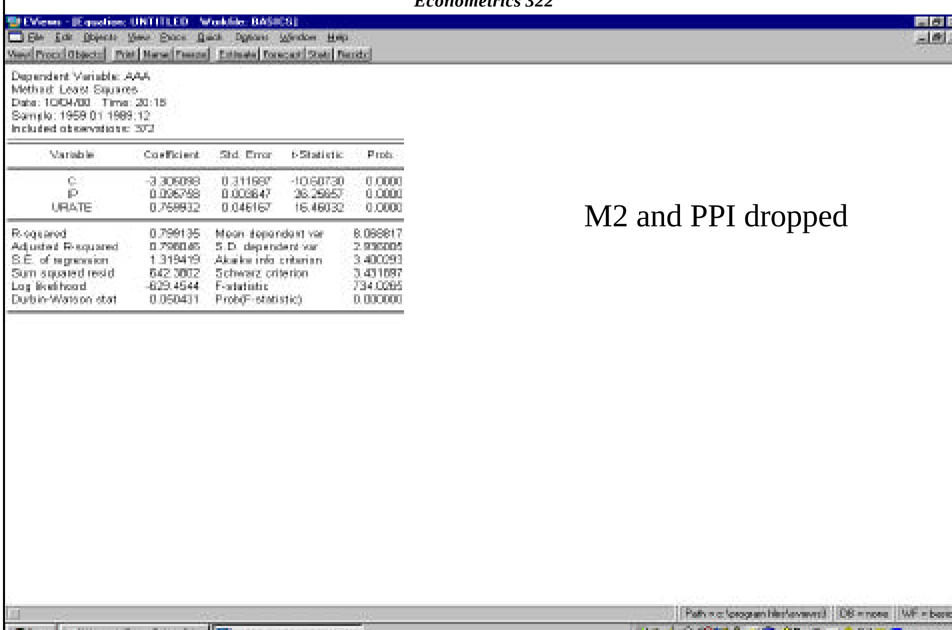


Dependent Variable: AAA
Method: Least Squares
Date: 10/04/00 Time: 20:16
Sample: 1959:01 1989:12
Included observations: 372

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-4.499574	0.283089	-15.89569	0.0000
IP	0.105314	0.005884	17.89980	0.0000
M2	-0.006562	0.000211	-31.08375	0.0000
PPI	0.209755	0.005874	35.70759	0.0000

R-squared	0.923084	Mean dependent var	8.068817
Adjusted R-squared	0.922456	S.D. dependent var	2.936005
S.E. of regression	0.817579	Akaike info criterion	2.445755
Sum squared resid	245.9841	Schwarz criterion	2.487894
Log likelihood	-450.9105	F-statistic	1472.136
Durbin-Watson stat	0.088986	Prob(F-statistic)	0.000000

Unemployment rate dropped



Dependent Variable: AAA
Method: Least Squares
Date: 10/04/00 Time: 20:15
Sample: 1959:01 1989:12
Included observations: 372

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-3.306098	0.311697	-10.60730	0.0000
IP	0.096798	0.003647	26.25657	0.0000
URATE	0.759932	0.046167	16.46032	0.0000

R-squared	0.799135	Mean dependent var	8.068817
Adjusted R-squared	0.798046	S.D. dependent var	2.936005
S.E. of regression	1.319419	Akaike info criterion	3.400293
Sum squared resid	642.3802	Schwarz criterion	3.431897
Log likelihood	-629.4544	F-statistic	734.0265
Durbin-Watson stat	0.050431	Prob(F-statistic)	0.000000

M2 and PPI dropped

Model Portfolio

	Models		
Independent Variables	1	2	3
Constant	-5.075* (0.0000)	-4.499* (0.0000)	-3.305* (0.0000)
IP	0.112* (0.0000)	0.105* (0.0000)	0.095* (0.0000)
M2	-0.006* (0.0000)	-0.006* (0.0000)	
PPI	0.188* (0.0000)	0.209* (0.0000)	
URATE	0.145* (0.0000)		0.759* (0.0000)
R^2	0.925	0.923	0.799
R^2 Adjusted	0.924	0.922	0.798
F-stat	1134.311* (0.0000)	1472.136* (0.0000)	734.026* (0.0000)
N	372	372	372

Notes: p-values in parentheses; * = significant

Multiple Regression
Model Selection

- Develop ***Model Portfolio***
- Very important topic
  - Will discuss shortly

Multiple Regression
Knowledge Checks

- Outline the procedure for deriving the estimator vector for the parameters. How does this compare to the procedure used in Topic 5?
- What is the final form in matrix notation for the parameter vector estimator? For the standard linear model, what is the first element of the estimator?

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Multiple Regression
Knowledge Checks (Continued)

- What is the problem with R^2?
- How does $\bar{R}^2$ correct for this problem?
- What is an upper bound on $\bar{R}^2$? Can it ever decrease or be negative?