What is Aperiodic Order?

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1 Introduction

Surely one of the most miraculous aspects of Nature is its self-organizing ability of creating solid substances with corresponding well-defined macroscopic properties (namely material objects of the world around us) using vast numbers of submicroscopic building blocks (namely atoms and molecules). Underlying this is the mystery of long-range order. Even putting aside the difficult kinematic questions about crystal growth, there remains a host of profound geometric problems: what do we mean by long-range order, how is it characterized, and how can we model it mathematically?

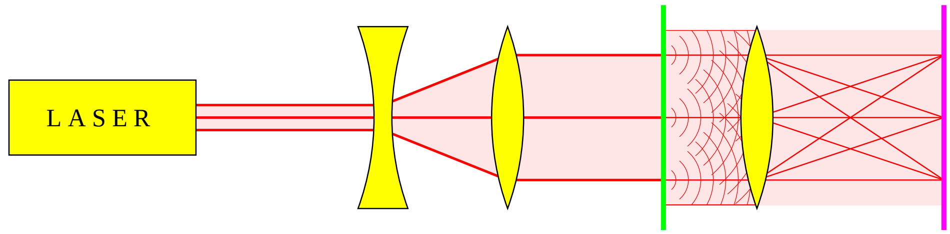
In crystals, like ice, sugar, and salt, many of the extraordinarily exact macroscopic features derive from a very simple geometric idea: the endless repetition of a (relatively) small pattern. A small arrangement of atoms forms a fundamental cell that constitutes a building block, copies of which are stacked together like bricks to fill out space by periodic repetition. Simple as this model is, it is still difficult to analyze in full mathematical detail: there are 230 possible symmetry classes (called space groups) theoretically available for such periodic cell arrangements, each of which is now also known to actually exist in Nature. However, it took almost 100 years from the theoretical classification of the 230 space groups to the experimental discovery of the last examples. Nonetheless, the underlying feature of all crystals, which appear ubiquitously in the natural world, is their pure periodic structure in three independent directions — their so-called lattice symmetry. The interesting thing is that there is striking long-range order in Nature that does not fit into this scheme, and one important example of this has only been discovered recently.

Early in the last century, the wonderful tool of X-ray diffraction was introduced, based on much older ideas of optical scattering (which is what we will use to explain its essence). Initially, diffraction pictures provided powerful evidence of the truth of the atomic theory of matter. Over the years, they have become a standard tool for analyzing crystals, and to detect long-rang order through the appearance of sharp reflection spots in the diffraction image. The basic idea can be visualized with an optical bench which is driven by a small laser as source for the coherent light (Box 1), see [3] for details on this, with many instructive examples.

Diffraction pictures of crystals display beautiful point-patterns that are symptomatic of the long-range repetitive lattice nature of the crystal. Sometimes these pictures seem so crystal-like themselves that, at first sight, they might lead one to think that they rather directly mark the atomic positions. In fact, however, they display the symmetry of another lattice that is dual (or reciprocal) to the one underlying the crystal structure. (See Boxes 8 and 12 for more on this).

For almost 80 years, the point-like feature of the diffraction image seemed to be the characterizing property of crystals; so much so that the three concepts of lattice symmetry, crystal structure, and pure point diffraction were considered as synonymous. Thus it was a minor crisis for the field of crystallography when in 1982 certain materials were found [1] with diffraction patterns that were as point-like as those of crystals, but showed other symmetries that are not commensurate with lattice symmetry! So, these new substances, which were definitely not crystals in the classical sense, were quickly dubbed *quasi-crystals*, and opened a new branch of crystallography. At the same time, they brought forth a surge of new mathematics with which to model the new geometry involved.

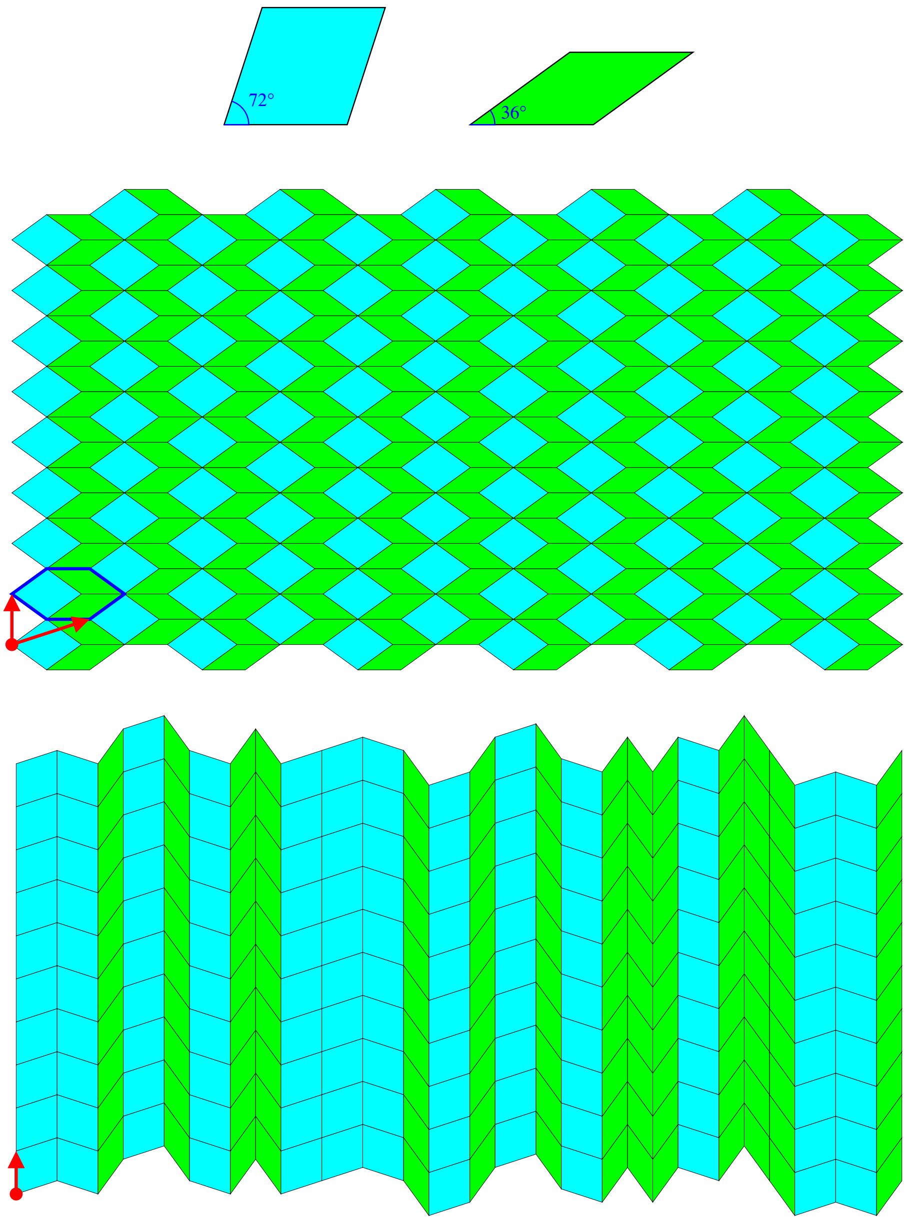
It is to this mathematical side that we turn in this article. For beyond the many physical questions raised by these new quasicrystals, there is a bundle of mathematical questions. What do we mean by 'order', in particular by 'aperiodic order', how do we detect or quantify it, what do we mean by repetition of patterns, what are the underlying symmetry concepts involved, how can one construct well-ordered aperiodic patterns? Beyond this, as one quickly realizes, is the general question of how the new class of quasicrystals and their geometric models are to be placed between the perfect world of ideal crystals and the random world of amorphous or stochastic disorder or, in other words, how can we characterize the level of 'disorder' that we may have reached?



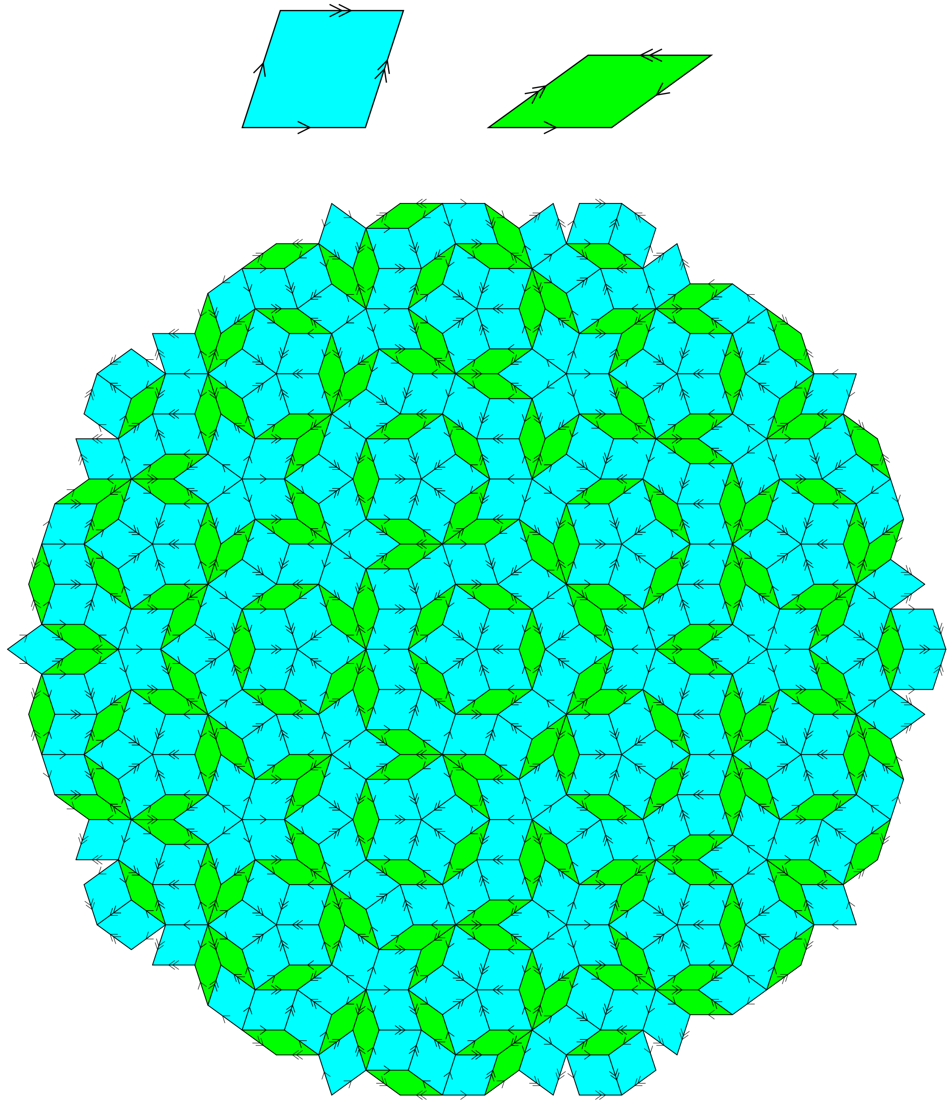
Box 1 *Experimental setup for optical diffraction*
The laser beam is widened by an arrangement of lenses and orthogonally illuminates the object located at the green plane. The light that emanates from the object plane then interferes, and the diffraction pattern is given by the distribution of light that one would observe at an infinite distance from the object. By another lens, this pattern is mapped onto the pink plane. Whereas for a picture of the object, as for instance in a camera, light rays emanating from one point of the object ideally are focused again into a single point of the picture, the situation is different in diffraction — light emanating from different regions within the object make up a single point of the diffraction pattern, as schematically indicated by the red lines in the right part of the figure. Therefore the diffraction pattern carries information about the entire illuminated part of the object. It provides some kind of measure of the correlations, and thus an account of the degree of order, in the structure of the object.

2 Planar tilings

A very instructive and also very attractive way to get a feeling for the ideas involved is to look at two-dimensional tiling models. The two rhombi (the so-called proto-tiles) shown in Box 2 are clearly capable of periodic stacking and so of lattice symmetry, the symmetry lattice being generated by the two translational shifts shown. Another possibility is shown below, which gives a tiling that is periodic in one direction and arbitrary (in particular, possibly aperiodic) in the other. On the other hand, the rhombi can also be used to tile the plane in the form of the famous Penrose tiling, see Box 3.



Box 2 *The undecorated Penrose tiles and some of their assemblies*
The prototiles are two rhombi, a fat one with opening angle 72° and a skinny one with 36°. They admit periodic arrangements like the one shown in the middle. The fundamental periods are indicated by arrows, and a fundamental domain in form of a hexagon is highlighted. It contains one fat and two skinny rhombi. Below, another arrangement is shown, which is periodic in the vertical direction, but admits an arbitrary 'worm' of rhombi in the horizontal direction.



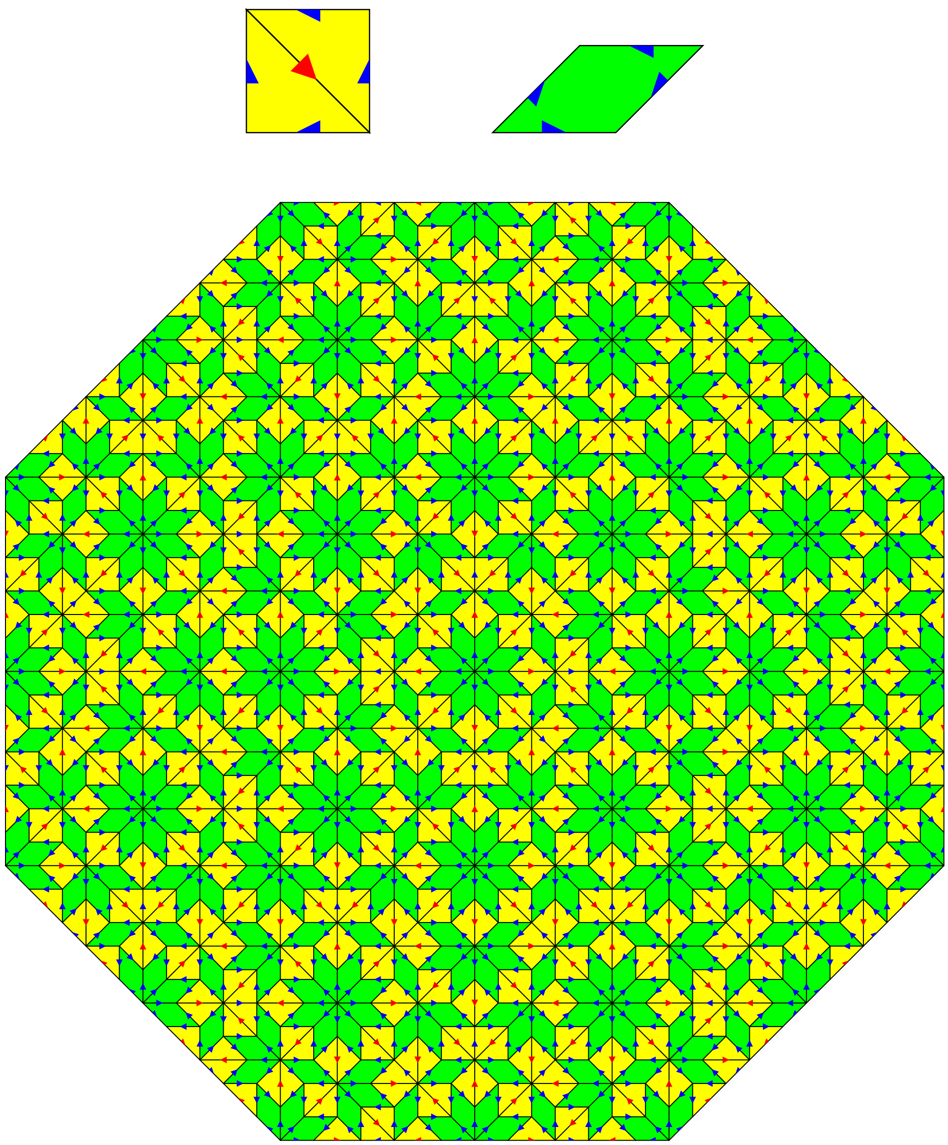
Box 3 *A central patch of Penrose's aperiodic tiling*
The two rhombi of Box 2 received a decoration of their edges by single and double arrows. If one now requires a perfect matching of all arrows on adjacent edges, the possible arrangements are highly restricted. In fact, the only permissible tilings of the entire plane are the so-called Penrose tilings. The different (global) possibilities cannot be distinguished by any local inspection. A fivefold symmetric patch of such a tiling is shown above.

Part of the intriguing nature of the Penrose tiling, of which just a circular fragment is shown in Box 3, is the obvious question of what exactly the rules might be for assembling these tiles. A properly constructed Penrose tiling has several marvellous properties of which the two most important at this point are:

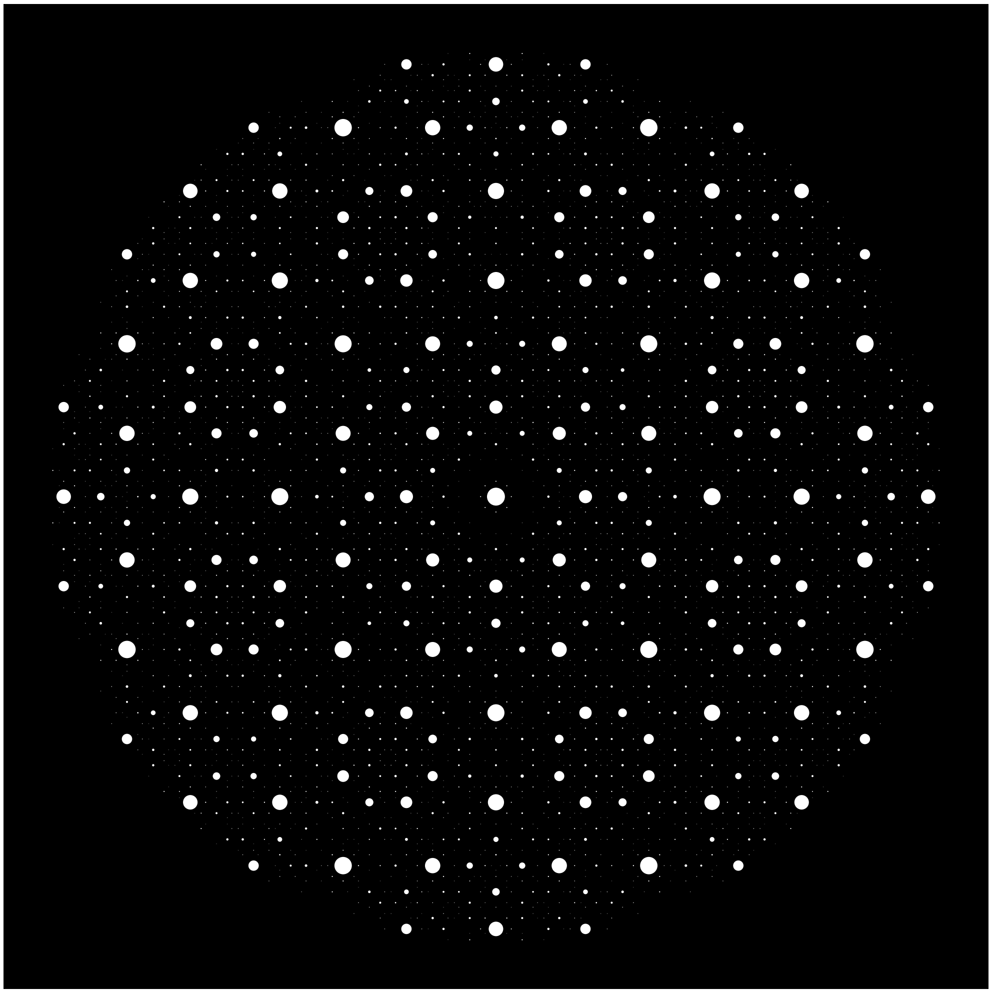
- A complete Penrose tiling of the plane is strictly *aperiodic* (in the sense of being totally without translational symmetries). Our particular example shows striking five-fold symmetry.

- If we ignore the tiles and just look at their vertices instead (we might think of the resulting point set as a toy model of an atomic layer) then, remarkably, this set of points is itself pure point diffractive, i.e. in the optical bench of Box 1, it produces a diffraction image on the screen with sharp spots only.

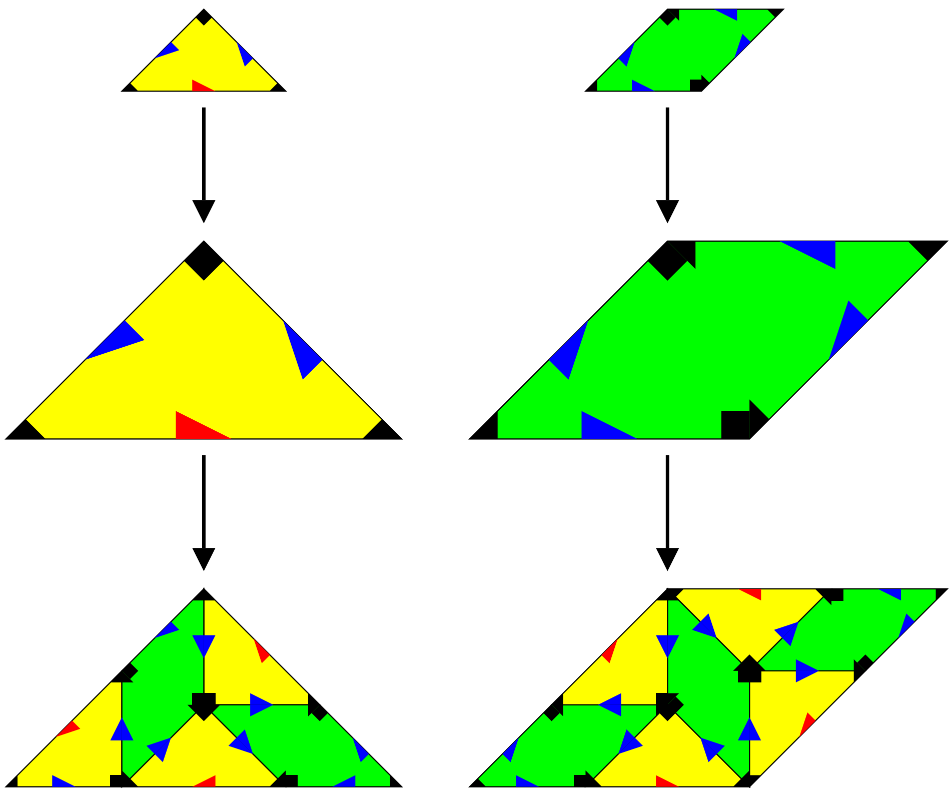
In Box 4, we see another aperiodic tiling, this time made out of two very simple tile types, a square (which we actually dissect into two isosceles triangles) and a rhombus. Its set of vertex points shows the same type of diffraction image as the Penrose tiling, namely sharp spots only, this time with eightfold symmetry (Box 5). In Box 6, we see the beautiful idea that is the secret behind many of the most interesting tilings (including the Penrose tiles): the idea of inflating and subdividing. To apply the idea here, we directly work with triangle and rhombus.



Box 4 *A central patch of the octagonal Ammann-Beenker tiling*
The original prototiles are a square and a 45° rhombus, decorated with blue arrows on the edges. For later use, the square is cut into two congruent isosceles triangles, carrying a red arrow on their common base. The orientation of arrows within each triangle is circular. Unlike the situation in the Penrose tiling, even with these arrows periodic tilings are still possible, for instance by repeating the square periodically. The octagonal patch shown belongs to the eightfold symmetric relative of the Penrose tiling, which is non-periodic and usually called the octagonal or the Ammann-Beenker tiling.



Box 5 *Diffraction pattern*
Diffraction pattern of the octagonal Ammann-Beenker tiling. The diffraction spots are indicated by circles whose area is proportional to the intensity of the diffraction peak. Spots with an intensity of less than 0.05% of the intensity of the central spot have been discarded.



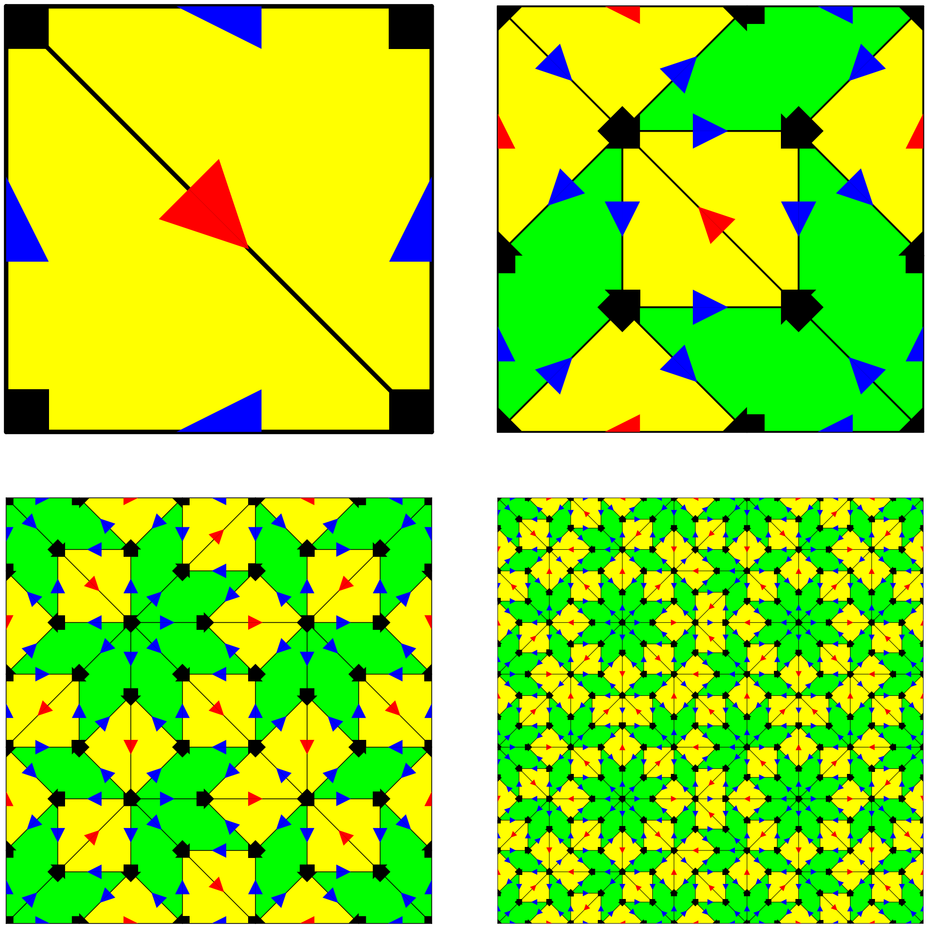
Box 6 *Inflation rule for the octagonal Ammann-Beenker tiling*
The inflation procedure consists of two steps, a rescaling by a factor of $\alpha = 1 + \sqrt{2}$, followed by a dissection into tiles of the original size. In comparison to Box 4, corner markings have been added which break the reflection symmetry of the rhombus. The patch shown in Box 4 can be obtained by applying this inflation rule (ignoring the corner markings) to an initial patch that coincides with the central octagon, filled by eight squares and sixteen rhombi. The corner markings are vital for obtaining matching rules. A sequence of inflation steps starting from a single square is shown in Box 7. Unlike the edge markings, and hence unlike the situation of the Penrose tiling, the corner markings cannot be reconstructed by local inspection of the undecorated tiling.

The inflation scheme in Box 6 shows us how to inflate each tile by a factor of $\alpha = 1 + \sqrt{2}$ and then how to decompose the resulting tile into triangles and rhombi of the original size. With this new device, we have a way of filling the whole plane with tiles. In comparison to Box 4, we added some markers in the corners of the tiles which will play some magic tricks for us later. Starting from a single tile, or from the combination of two triangles, and inflating repeatedly, we build up the sequence as shown in Box 7. Since there is no need to stop, we may go on and do this forever.

It is now easy to see that the resulting octagonal tiling has an amazing property: whatever finite pattern of tiles we see, that same pattern will be repeated infinitely often, in fact we can even specify the maximum distance we will have to search to find it again! A pattern with such a property is called repetitive. A perfect crystal is an example of a repetitive structure, of course, but the inflation procedure produces interesting new cases.

How does this happen? Imagine the partial tiling obtained after n inflations of an original patch P that consists of two triangles which build a square. It is composed of triangle pairs and rhombi. If we choose from it a patch P' which is a copy of P, then n steps after this patch was created, another patch P'' will show up which is a copy of P'. Furthermore, the position and orientation of P'' relative to P' will be the same as that of P' relative to the original P. Thus the pattern P, or a similar copy thereof, is bound to appear over and over again. In our example, P is just made of two tiles, but this idea works for any patch P that occurs somewhere in the inflation process, no matter how big it is.

The reason behind this is that the square, centred at the origin, is the seed of a fixed point under even numbers of inflation, as can be seen from the sequence in Box 7. The term 'fixed point' means that the sequence tends towards a global covering of the plane which is then left invariant (hence fixed) by further pairwise inflation steps, i.e., we have reached a stable pattern this way.



Box 7 *Repeated inflation steps of the octagonal tiling*
The sequence shows a square as an initial patch and three successive applications of the inflation rule of Box 6. (For the sake of presentability, we ignored the proper relative scale.) The inflation rule ensures that the corner markings always assemble a complete 'house'. Alternatively, assembling patches tile by tile, all complete tilings of the plane with this property and matching arrows on all edges are locally indistinguishable from the fixed point tiling created by inflation. Thus, arrows and houses together establish perfect matching rules.

So our pattern is *repetitive*, but in fact it has no periodic component at all! This is not self-evident yet, but it will become more so later. The main point right now is that the tiling has the strange and seemingly paradoxical property of having repetitivity on all scales, no matter how large, but with no periodic repetition. All patches repeat, but not periodically!

The Penrose tilings can also be built through substitution and likewise are repetitive without periodic repetition, see [2]. Thus they too have the striking property that you cannot really know where you are in the tiling by looking at any finite region around you. It follows that it is not possible to build such a tiling by any finite set of rules which tell you what to do next by looking at some finite neighbourhood of your position! To see why, imagine that this were possible. Then every time the same pattern appeared, the rules for continuing from it would be the same as those used for building at its previous occurrence. The result is that the pattern would globally repeat.

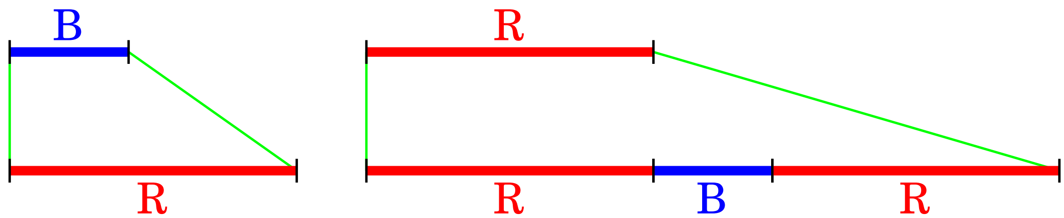
Having said this, the next reaction is going to be that our next assertion says the opposite. In fact there are assignments of marks — so-called matching rules — to the edges of the Penrose rhombi (Box 3), or to the edges and corners of the Ammann-Beenker tiles (Boxes 6 and 7), such that, if they are match everywhere in the tiling, the result is a perfect Penrose or a perfect Ammann-Beenker tiling, respectively. What is the catch?

The problem is that these matching rules guarantee that what you are getting is a Penrose tiling *as long as you never get stuck*. The trouble is that to not get stuck requires knowledge of the entire tiling to that point — it is not derivable from local information only!

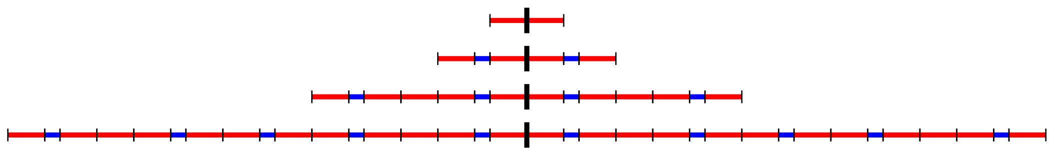
3 Cut and project sets

In view of these difficulties, one might ask what other possibilities exist to systematically create arbitrarily large faultless patches of these tilings. The idea of what is going on is more easily understood by first considering an even simpler object, namely a one-dimensional inflation tiling. This time we begin with two tiles

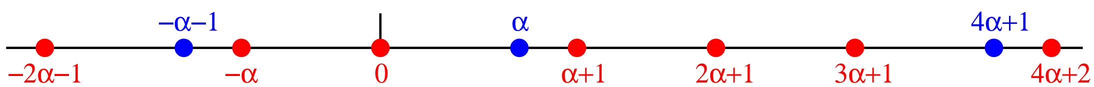
which we call B (for blue) and R (for red), respectively. We give the short tile B the length 1 and the long tile R the length $\alpha = 1 + \sqrt{2}$ (the same number also appears in the octagonal tiling). Inflation is stretching by a factor of α, followed by a subdivision which is consistent with $\alpha \cdot 1 = \alpha$ and $\alpha \cdot \alpha = 2\alpha + 1$. The final result is



Starting from a pair of R-tiles, centred at the origin, we have successively



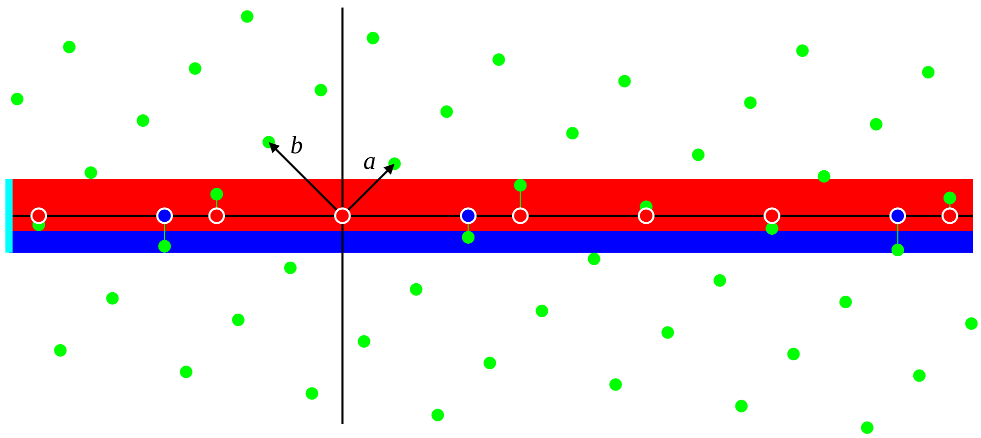
Using coordinates to label the left end point of each tile we have



The corresponding points form an infinite set $A = \{\ldots - \alpha - 1, -\alpha, 0, \alpha, \alpha + 1, 2\alpha + 1, \ldots\}$.

What is striking about the points of A is that they are all of the form $u+v\sqrt{2}$. How can we see which points $u + v\sqrt{2}$ are present and which not? Everyone knows that it is a good idea in dealing with square roots to see what happens if you change the sign of the square root. (Think of the high school exercises in rationalizing expressions of the form $\frac{1}{1+\sqrt{2}}$.)

Let us use this trick of replacing each appearance of $\sqrt{2}$ by its conjugate, $-\sqrt{2}$. This conjugation is called the star map, the image of a point $x = u + v\sqrt{2}$ is $x^* = u-v\sqrt{2}$. Box 8 shows a plot of our points. We make a new picture in which each point x is "lifted" to the point (x, x^*) in the plane. Our points of interest are shown against a backdrop consisting of all possible points $(u+v\sqrt{2}, u-v\sqrt{2})$ where u, v are integers.



Box 8 *An alternative way to construct the point set A*
The green points form the lattice $\{(u+v\sqrt{2}, u-v\sqrt{2}) \mid u, v \text{ integer}\}$ which is spanned by the basis vectors a and b. The orientation of the strip is irrational with respect to the lattice, i.e., the black line at its centre hits the origin, but no further lattice point. The green points within the strip are orthogonally projected onto the horizontal black line and are coloured according to their vertical position in the strip. The resulting set of red and blue points coincides with the point set constructed above by inflation.

The effect is striking. The entire set of points, including the backdrop, produces a lattice (a mathematical crystal). The B and R points now appear in a band that runs from height $-\frac{1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}$. Furthermore, the B points come from the bottom portion of the band, from $-\frac{1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}} - 1$, and the R points from the remaining top portion of the band. The actual points labelling our tiling, i.e. the set A, can be obtained just by dropping the second coordinate of each lattice point that lies in the band — in other words by projecting it onto the horizontal axis.

Now one sees that it is incredibly easy to compute the left hand end points of our 1D tiling, and hence to get hold of the tiling itself. On a computer, generate, in some ordered way, points of the type $u+v\sqrt{2}$. For each one look at its conjugate $u - v\sqrt{2}$. Test whether this number lies in either of the intervals corresponding to B and R points (e.g., $-\frac{1}{\sqrt{2}} < u - v\sqrt{2} < \frac{1}{\sqrt{2}}$ for B points) and choose the point and its colour accordingly. What we have accomplished here, apart from the visual clarity, is a remarkable way of connecting the geometry of our tiling with an algebraic method of calculating it.

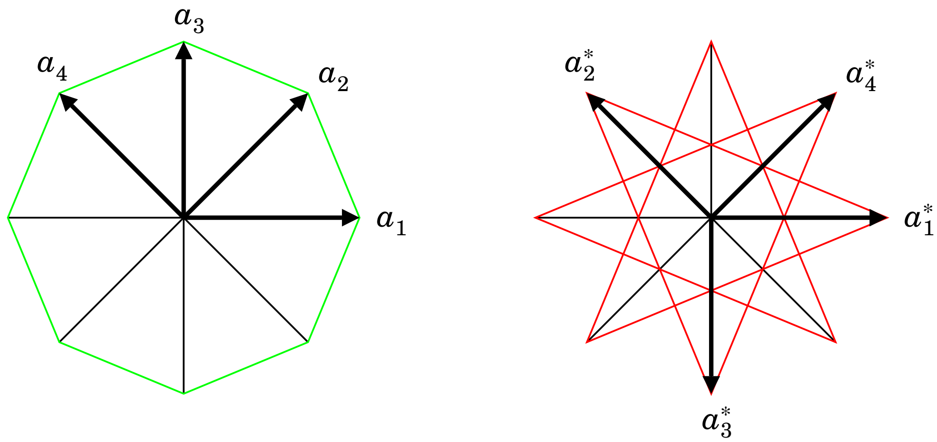
A point set that can be described in this way (by cutting through a lattice and projecting the selected points) is called, not surprisingly, a cut and project set. In this case the object that is used to cut (or to sweep out) the correct band is the vertical line segment indicated in black in Box 8. It is called the *window* of the projection method.

Another benefit of the cut and project view is that it shows immediately why the resulting point sets are aperiodic. For example, a period of our set of red and blue points is a shift t (to the left or right) that moves the set on top of itself. Necessarily it would be of the form $r + s\sqrt{2}$ since all our points have this form. However, after our lift into 2-space, we would then find that shifting by $(r + s\sqrt{2}, r - s\sqrt{2})$ takes the strip onto itself! This is impossible unless $r - s\sqrt{2} = 0$, i.e., $r = s\sqrt{2}$. However, $\sqrt{2}$ is irrational, while s, r are integers, so the only solution is $r = s = 0$, and the only period is 0.

4 The projection approach to planar tilings

The octagonal tiling, or more precisely the positions of its vertices, can also be described as a cut and project set. This goes via the projection of the points of a certain lattice in four dimensions, swept out by an octagon. We explain this in more detail.

The initial pool of points from which we select is given by the set M of all integer linear combinations $\{u_1a_1+u_2a_2+u_3a_3+u_4a_4 \mid u_1, u_2, u_3, u_4 \text{ integer}\}$ of the four unit vectors shown in left diagram of Box 9. This is a dense point set in the plane, and it is the two-dimensional analogue of the set $\{u + v\sqrt{2} \mid u, v \text{ integer}\}$ used above. Since the octagonal tiling consists of squares and rhombi (with unit edge length, say), the distance between any two vertex points is of this form, i.e. an element of M. Also the star map has an analogue, and it comes about simply by replacing the four vectors of the left diagram by those of the right diagram of Box 9; that is, $x = u_1a_1 + u_2a_2 + u_3a_3 + u_4a_4$ is mapped to $x^* = u_1a_1^* + u_2a_2^* + u_3a_3^* + u_4a_4^*$. As before, the set of pairs (x, x^*) forms a lattice, this time in four dimensions.



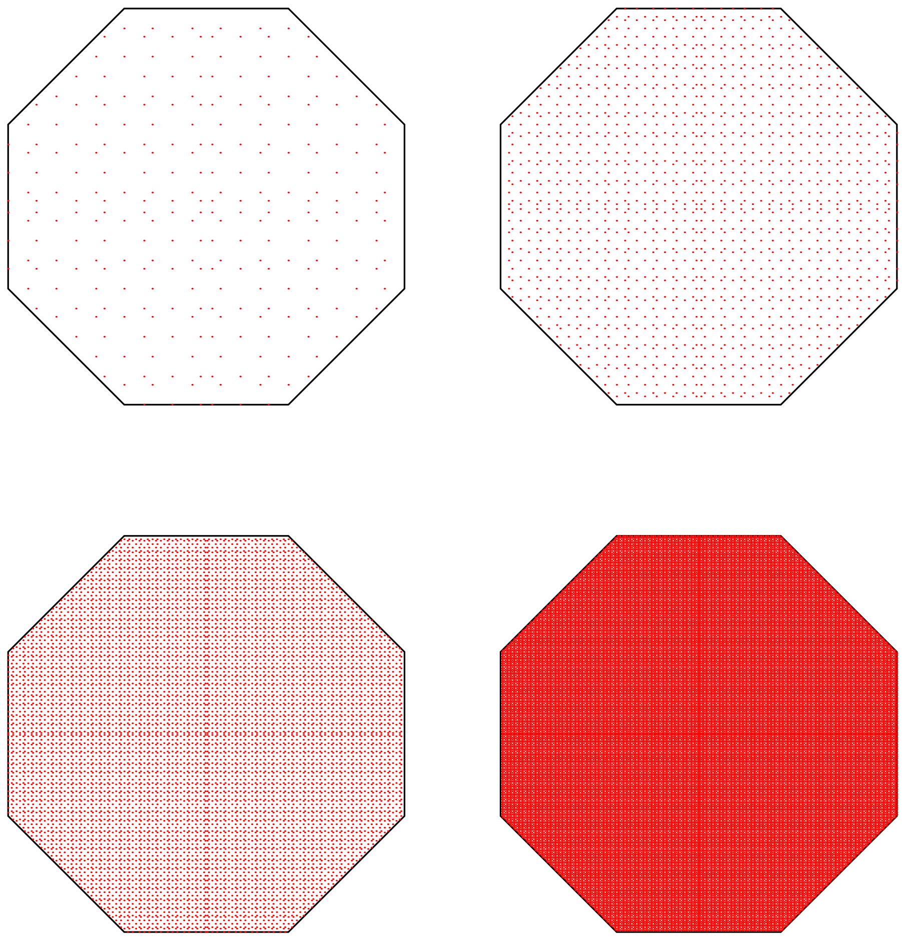
Box 9 *The two ways to count to four (and hence to eight)*
The two sets of vectors used to construct the octagonal tiling, a_i (left, for tiling space) and a_i^* (right, for internal space), $i = 1, 2, 3, 4$. The change from a_i to a_i^* demonstrates the action of the $*$-map in this case.

The vertex set of the Ammann-Beenker tiling can now be given as the set of points x whose image x^* under the star map lies inside a regular octagon of unit edge length. We can now link this back to our previous approach via inflation. If we start from a unit square and keep on inflating, as shown in Box 7, the images of the vertex points under the star map will densely populate this octagon in a uniform way, see Box 10.

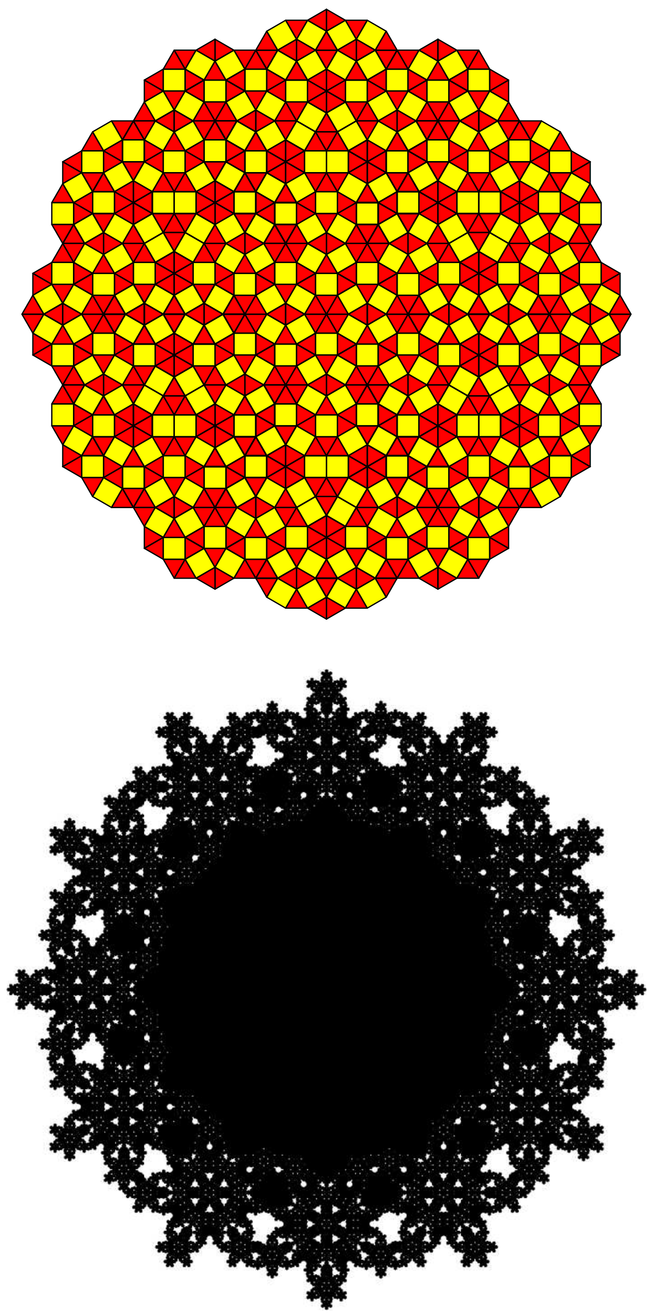
Needless to say, the additional visual clarity obtained from a 4D description is debatable! Still, the conceptual idea is very powerful, providing the essential link between geometry, algebra, and analysis that is at the heart of much of our understanding of aperiodic order.

Likewise the points of the Penrose tiling can be given a cut and project interpretation, as do many other similar pointsets. In both cases, the aperiodicity can be shown in the same way as for our one-dimensional example.

Another tiling of physical interest is built from a square and an equilateral triangle. The example shown in Box 11 can be created by a slightly more complicated inflation rule, or alternatively once again by the cut and project method. In this case, however, the corresponding window shows a new feature: it is a compact set with fractal boundary. An approximation is also shown in Box 11.



Box 10 *Filling the octagon in internal space*
The image points x^* under the star map of the vertex points are shown for larger and larger patches of the octagonal tiling, obtained by inflation of a square as shown in Box 7. Eventually, the points populate the regular octagon with uniform density. Here, the first picture of the sequence corresponds to the largest patch of Box 7.



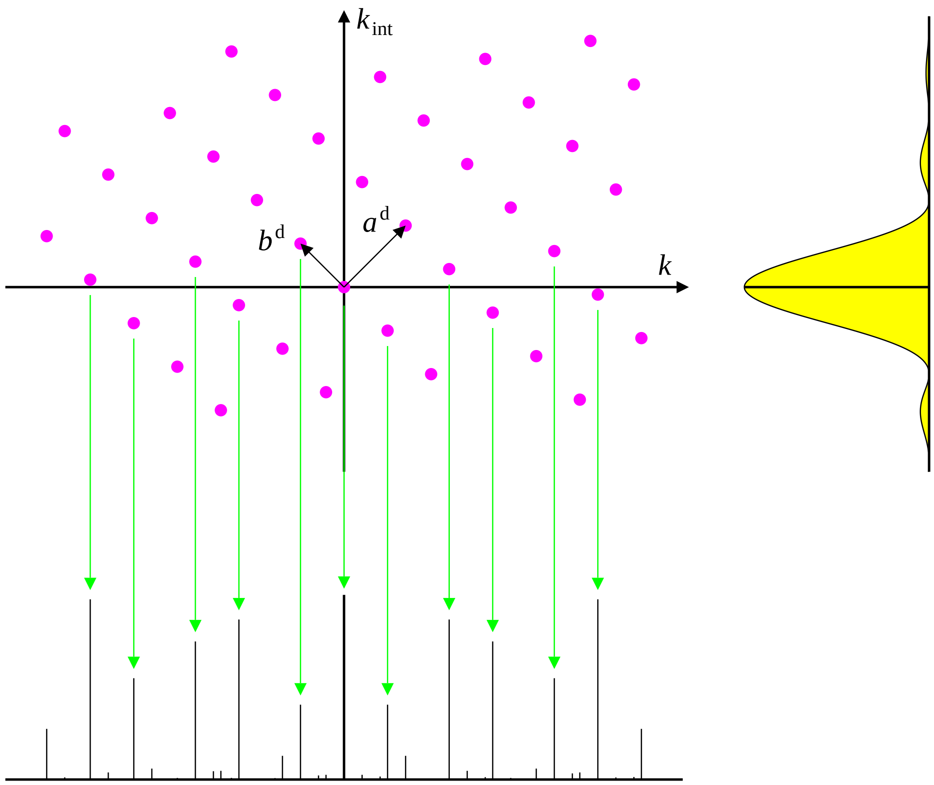
Box 11 *Quasiperiodic square triangle tiling*
This example of a square-triangle tiling can either be obtained by an inflation rule or by projection from a lattice in four dimensions. The points selected for projection lie in a generalized 'strip' whose cross section is a twelvefold symmetric object with fractal boundary.

5 The origin of diffraction

The picture that we see in Box 8 offers us considerable insight into the diffractive nature of sets that can be described as cut and project sets. The background is a lattice (crystal) and this, from the classical theory of crystals, is supposed to have perfect diffraction, i.e., the entire diffraction image is composed of sharp peaks only. The trick is how to restrict this down to the points in the band and ultimately to our line of points. Box 12 shows a picture of what happens. The bottom figure, which looks like an irregular comb, shows the diffraction of the points A of our 1D tiling. The diffraction intensity is shown here not by the size of the dots, but rather by the length of the teeth of the comb.

Above it is the diffraction picture of the background lattice, another lattice, that, as we mentioned before, is called the dual lattice. The points that carry the teeth of the comb (i.e. the spots of the diffraction) are nothing other than the projections of the points of the dual lattice — and this time *all* of them. The lengths of the teeth are provided by the profile on the right hand side. Where that profile comes from is a longer story. (Engineers may recognize its similarity to the Fourier transform of a single square pulse. It is, in fact, the square of the Fourier transform of the characteristic function of the interval defining the band.)

The teeth of the comb lie actually dense on the line. However, due to the damping nature of the profile, most of them are so small that, no matter what finite resolution we may use, we can see only a small fraction of them, and hence only an effectively discrete set of teeth, or spots, as in Box 5.



Box 12 *Explanation of the diffraction pattern*
The pink points indicate the lattice dual to the lattice of Box 8. It is explicitly given by $\{(\frac{m}{2} + \frac{n\sqrt{2}}{4}, \frac{m}{2} - \frac{n\sqrt{2}}{4}) \mid m, n \text{ integer}\}$. The lattice is spanned by the vectors a^{d} and b^{d} which satisfy the scalar product relations $a^{\text{d}} \cdot a = b^{\text{d}} \cdot b = 1$ and $a^{\text{d}} \cdot b = b^{\text{d}} \cdot a = 0$. In this case, all points of the lattice are projected, resulting in a dense set of positions on the horizontal line at the bottom. At each such position, a diffraction peak is located. Its height, i.e., the intensity seen in an experiment, is determined by the vertical coordinate k_{int} of the unique corresponding point of the dual lattice. The explicit value is given by the function $I(k_{\text{int}}) \sim \left(\frac{\sin(\sqrt{2}\pi k_{\text{int}})}{\sqrt{2}\pi k_{\text{int}}}\right)^2$ which is displayed on the right hand side.

6 What are cut and project sets?

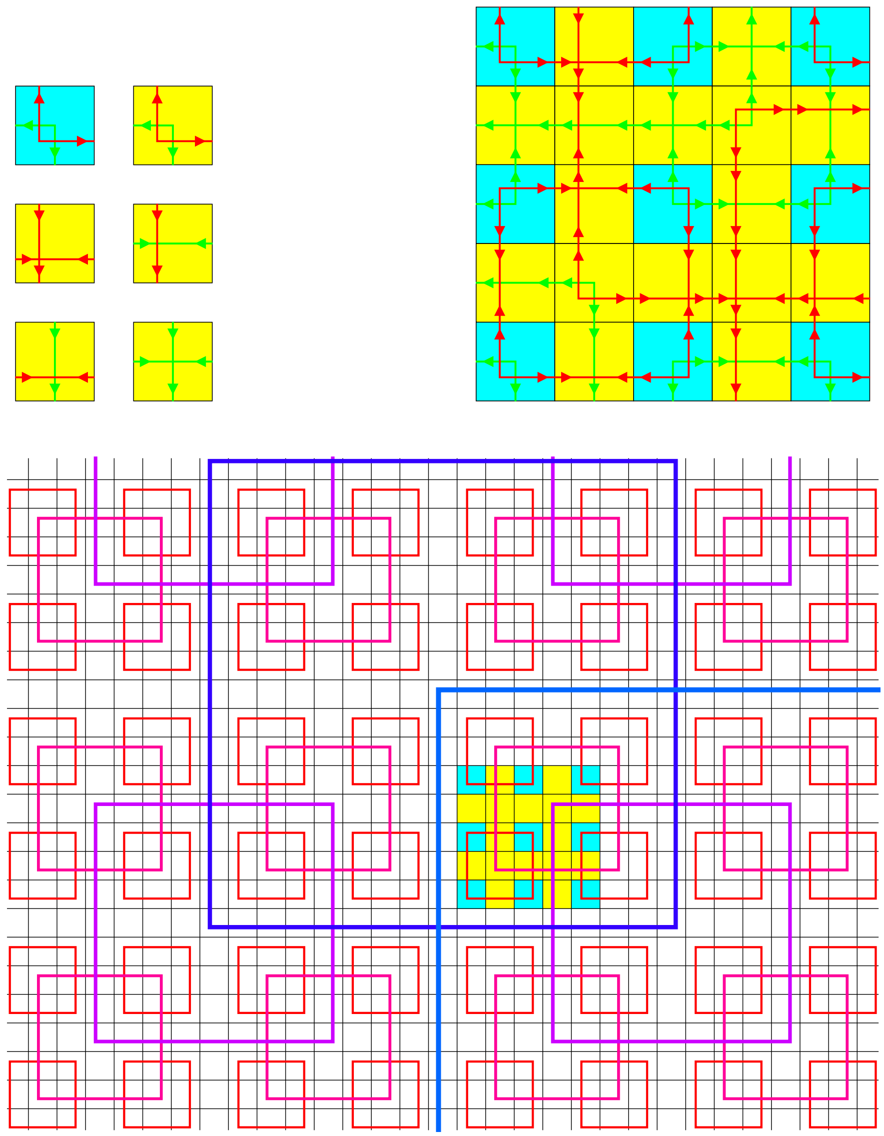
The realization of our point sets as lingering images of lattices in higher dimensional spaces is both visually appealing and sheds light on diffraction. However, the use of conjugation as we used it appears as a miracle and one is left wondering why it worked and when we might expect it to work again. In fact, the answer to this is not really known. We do not know when a given aperiodic point set, even if it is pure point diffractive, may be realized in the cut and project formalism. We do know that they are not restricted to sets involving irrationalities like $\sqrt{2}$. One of the most interesting and earliest examples of this is the one based on the Robinson square tiles.

These tiles arose out of another one of the streams whose confluence produced the subject of aperiodic order, namely the decision problem for tilings. Given a finite number of tile types, is there an algorithm for determining whether or not the plane can be tiled (covered without gaps and overlaps) by translated copies of these tiles? This problem had been raised and later brought to a negative conclusion by logicians. Tiles that only can tile aperiodically lie at the heart of this undecidability, and the hunt was on for the smallest collections of such tiles.

Raphael Robinson made a very interesting contribution to this by first linking the problem of tiling a plane with marked square tiles to Turing machines and the famous Halting Problem, and also coming up with a simple set of 6 square tiles with markings (shown in Box 13 — actually 28 tiles since all rotated and reflected images are also to be included) that only tile aperiodically. A rather dramatic proof of this can be glimpsed from the subsequent pictures where it is seen that legal arrangements of the tiles lead to a family of interlocking squares of increasing (by factors of 2) sizes. The aperiodicity is obvious: no finite translation could take the squares of all sizes into themselves.

If we mark the centre of each tile by a coloured point (to indicate its type) then we get 6 (or 28) families of points which are subsets of a square lattice. These point sets are in fact cut and project sets, but now the 'higher dimensional' space is far more exotic: it is the product of a Euclidean plane and an arithmetical-topological space that is based on the so-called 2-adic numbers. In spite of being very different from a Euclidean space, the diffraction results are provable much as before. Each of these point sets is pure point diffractive!

There remains though, the difficult problem of characterizing cut and project sets.



Box 13 *Robinson tiling*
The six Robinson tiles (upper left) given as squares of two different colours that are labeled by two types of oriented lines. Together with their images under rotation and reflection they make up an aperiodic set of tiles, if one requires that the oriented lines match at the edges, and that exactly three yellow squares meet at each corner (upper right). Disregarding the green lines, the red lines make up a pattern of interlocking larger and larger squares, indicated by different colours in the lower picture. The region tiled by coloured squares corresponds to the patch shown above.

7 Probabilistic ideas

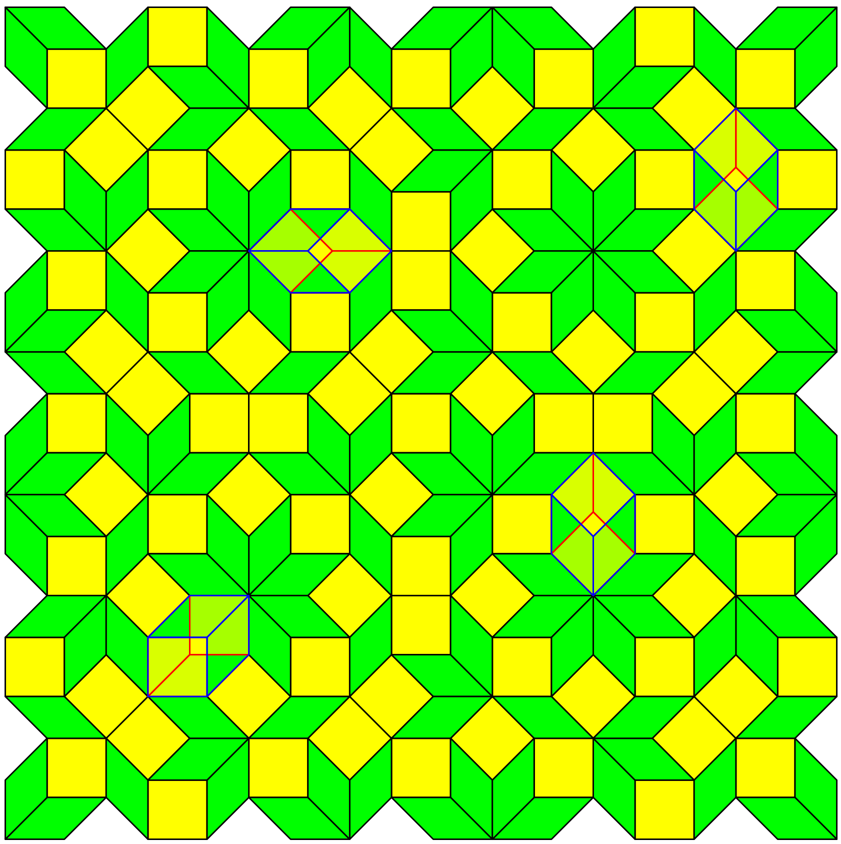
As was briefly mentioned in the beginning, quasicrystals can also be seen as a stepping stone for bridging the gap between perfect crystals on the one extreme and amorphous solids on the other. It can clearly only be a first step, as we have seen how close they are to crystals in so many properties.

Indeed, as all constructions above have shown, quasicrystals are completely deterministic, and what is still missing here is a source for some kind of randomness, or stochastic disorder. This would be an entire story in itself, but we can at least indicate one way to use crystallographic and quasicrystallographic tilings to make some steps into this new direction. The new facet here is that the underlying mechanism is *statistical* in origin, both for the reason of existence and for the appearance of symmetries, which are also statistical now.

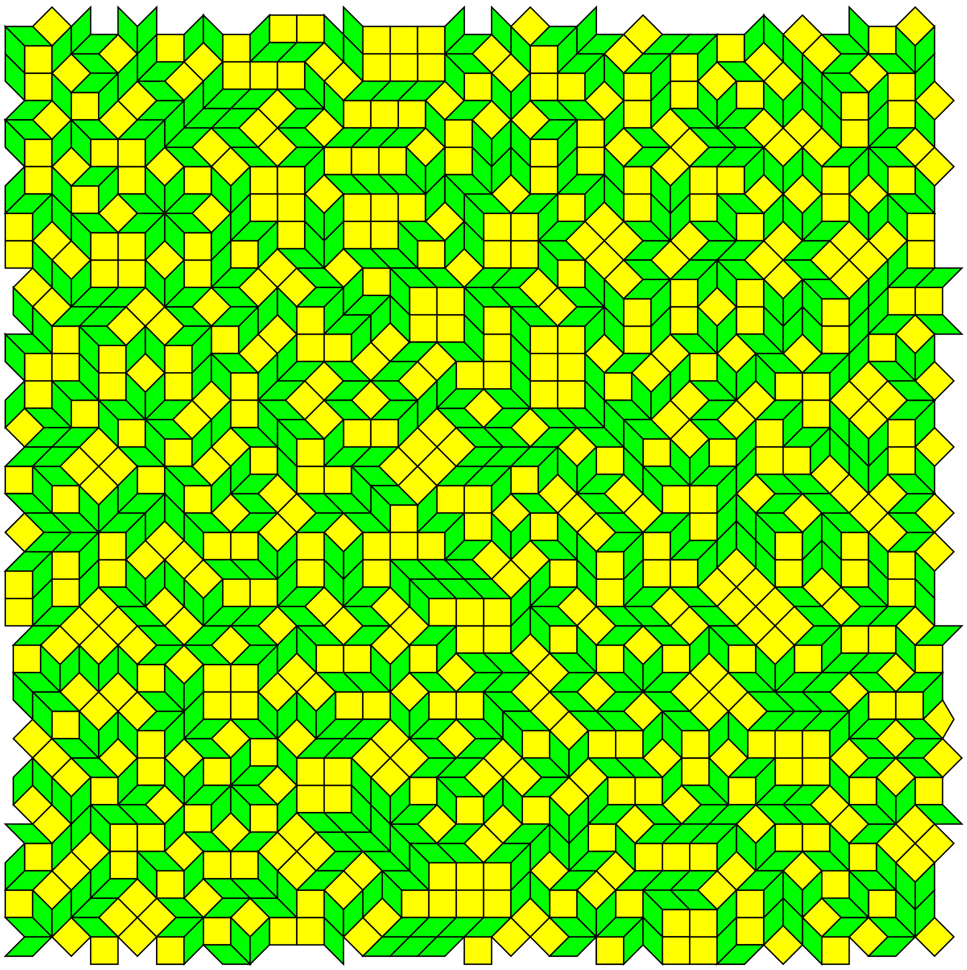
Inspecting Box 4 again, we now remove all markings, and also the long edges of the triangles. We obtain a square-rhombus tiling, with many "simpletons". By these we mean little (irregular) hexagons built from one square and two rhombi, as shown in Box 14. They can now be flipped as indicated, without affecting any face-to-face condition. If we randomly pick such simpletons and flip them, and continue doing so for a while (in fact, for eternity), we arrive at what is called the square-rhombus random tiling ensemble. A snapshot is shown in Box 15.

In this way, we have introduced an element of randomness into our tiling, but without destroying the basic building blocks (the square and the rhombus) and their face-to-face arrangements. Also, this does not change the ratio of squares to rhombi. Nevertheless, there are many such tilings now, in fact even exponentially many, i.e. the number of different patches of a given size grows exponentially with the size! This means that the ensemble even has positive entropy density, which opens the door for a completely different explanation of why we see them in nature: they are, given the building blocks (e.g. in the form of rather stable atomic clusters that can agglomerate), "very likely". Recent evidence seems to point into this direction, and a more detailed investigation of these random tilings is desirable.

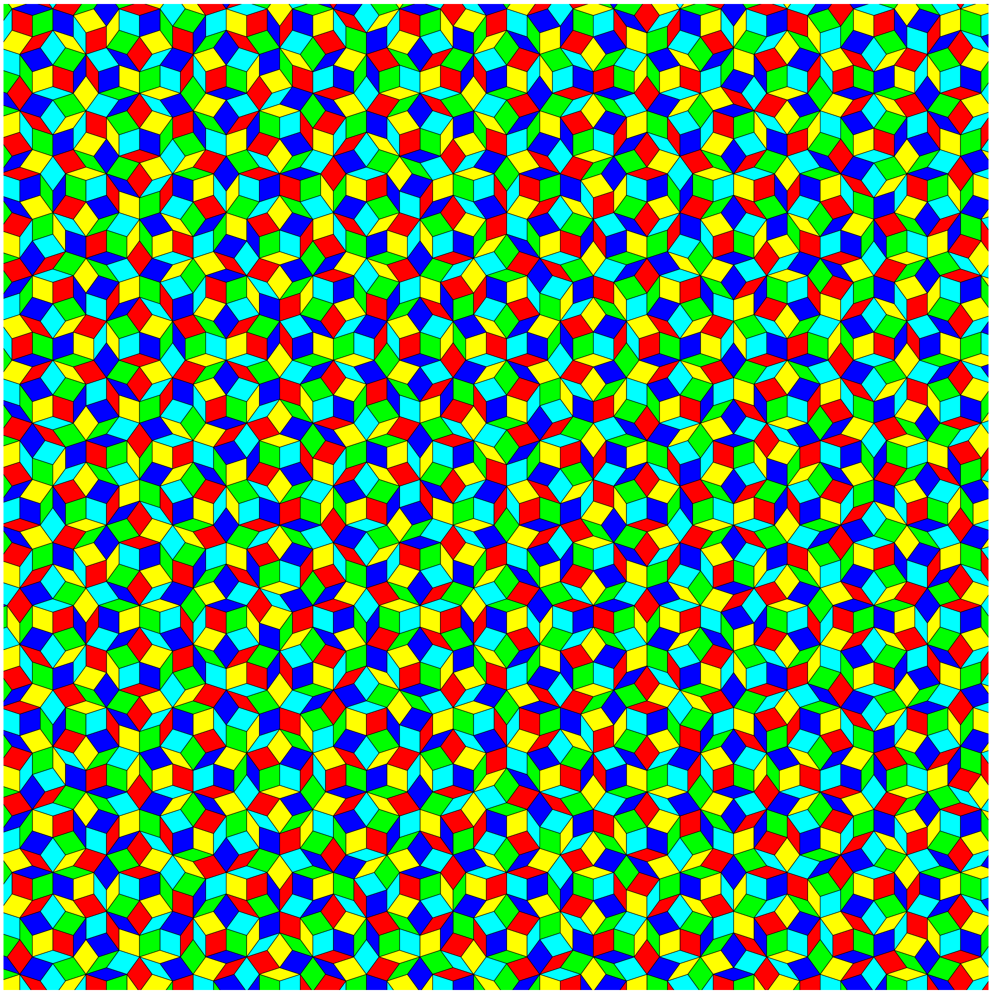
In fact, one could even start from just a pool of tiles of both types and admit all assemblies that cover the plane without gaps or overlaps, and without violating the face-to-face condition of the tiles. This way, one gets an even larger class of tilings, called the unrestricted square-rhombus random tiling ensemble, where arbitrary ratios of squares to rhombi are realizable. Among them, we also find the ones constructed by randomization of perfect tilings as explained above, and one can show that the tilings of maximal entropy (which basically means the most likely ones of this enlarged ensemble) have the square-rhombi ratio of the perfect Ammann-Beenker pattern and show eightfold, hence maximal, symmetry! The latter has to be interpreted in the statistical sense, meaning that each patch one can find occurs in all 8 orientations with the same frequency. This brings about a totally different symmetry concept which is statistical rather than deterministic in origin, a somewhat puzzling thought perhaps. Nevertheless, this is sufficient to make the corresponding diffraction image exactly eightfold symmetric!



Box 14 *Simpleton flips*
Four examples of simpleton flips in a patch of the perfect Ammann-Beenker tiling. The hexagons and their original dissection into a square and two rhombi are marked by the blue lines, whereas the red lines indicate the flipped arrangement. Note that only the three internal lines in the hexagon are affected by the flip, the outer shape stays the same. One can view the patch, and all variants obtained by such elementary simpleton flips, also as the projection of a (fairly rugged) roof in 3-space — the two versions of the simpleton fillings then correspond to the projection of two different half surfaces of a cube.



Box 15 *Square-rhombus random tiling*
A patch of a square-rhombus random tiling obtained by randomly rearranging a large approximating patch of the perfect Ammann-Beenker tiling. In fact, we started from a square-shaped patch as those shown in Box 7, whose translated copies, when glued together along its boundaries, generate a periodic pattern that violates the perfect matching rules only in the corners where the pieces are glued together. The same procedure could be applied to the disordered patch shown here, resulting in a periodic pattern which simply has an enormously large building block, namely the one shown above!



Box 16 *A colour-symmetric Penrose tiling*
The picture shows a colouring of the Penrose tiling with five different colours. The colours are chosen such that they permute in a definite way under rotation of the tiling. Figure courtesy of Max Scheffer (Chemnitz).

8 Summing up

One fascinating thing about the type of order exemplified in this discussion is how very close it comes to being periodic without admitting any actual periods.

So, let us ask again: 'what is aperiodic order?'. At present, we have a reasonable qualitative and a partial quantitative understanding, some aspects of which we have tried to explain above. However, we still don't have a complete answer, and such an answer might lie well into the future.

But what we do know is that there is a universe of beautiful questions out there, with unexpected results to be found, and with many cross-connections between seemingly disjoint disciplines. On top of that, it is definitely a lot of fun, for example, when producing new variants of Penrose tilings with colour symmetries, such as the example shown in Box 16 below! For a recent bibliographical review of the literature, we refer the reader to [4].

References

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