Basic Linear Algebra and its Geometry

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Usually Start with

- Vector: a list of scalars.
- Matrix: a 2D grid of scalars.
- Operation: a strange multiplication.
- ...

We Start from

Linear geometry:

- Vector: Angle and Length
- Transformation
- Mapping, SVD, Determinant and Eigen-decomposition
- Generalization.

Vector

Vector

An oriented line segment (arrow) $\vec{a}$, but can be parallel translated:

- Length and the orientation, nothing else.

$$\|\vec{a}\|, \vec{a}/\|\vec{a}\|.$$

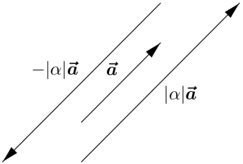
Vector

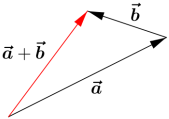
An oriented line segment (arrow) $\vec{a}$, but can be parallel translated:

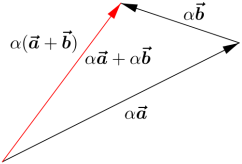
- Length and the orientation, nothing else.

$$\|\vec{a}\|, \vec{a}/\|\vec{a}\|.$$

- $\alpha\vec{a}$: keep orientation, but scale the length.
- $\vec{a}+\vec{b}$: parallelogram rule.







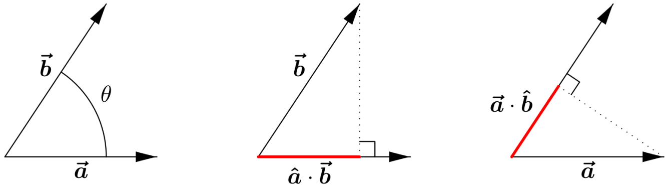
Dot Product

Relationship between two vectors $\vec{a}, \vec{b}$:

- The angle θ between them.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta)$$

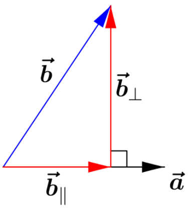
$$||\vec{b}|| \cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} = \vec{b} \cdot \hat{a}$$



Parallel and Perpendicular Components

- Orthogonal: $\cos(\theta) = 0$.
- Parallel: $\cos(\theta) = 1$.

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}.$$



$$\vec{b}_\parallel = (\vec{b} \cdot \hat{a})\,\hat{a} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a}$$

$$\vec{b}_\perp = \vec{b} - \vec{b}_\parallel = \vec{b} - (\vec{b} \cdot \hat{a})\hat{a} = \vec{b} - \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a}$$

But How Describe a Vector

For a scalar, we way $0, 0.6, -1.9, \pi, \cdots \in \mathbb{R}$.

For a vector?

- A list of scalars
  - Where do they come from?

But How Describe a Vector

For a scalar, we way $0, 0.6, -1.9, \pi, \cdots \in \mathbb{R}$.

For a vector?

- A list of scalars
  - Where do they come from?
- Measure!

Measure a Geometric Entity

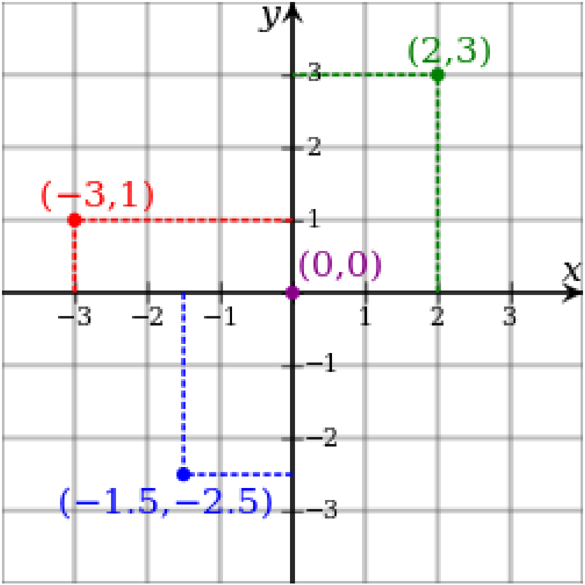
Build a coordinate system

- $X - Y$: Cartesian coordinates.
- $\rho - \theta$: polar coordinates.
- ...

Then

- Moving the starting point to origin
- Describe the position of the end.

Usually, Cartesian Coordinates



More Precisely

A set of special "vectors": unit orthogonal basis

- 3D: X, Y, Z.

$$X \cdot X = 1, X \cdot Y = 0, \cdots$$

- n dimensional: $e_1, e_2, \cdots, e_n$.

$$e_i \cdot e_i = 1, \quad e_i \cdot e_j = 0, i \neq j.$$

Then a vector $\vec{v}$ is described by a list of scalars:

$$\vec{v}_i = \vec{v} \cdot e_i = \|\vec{v}\| \cos(\theta_i).$$

Projection of $\vec{v}$ on the bases e_i.

The Matrix Form

$$\begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix} = \begin{pmatrix} e_1^T \\ \vdots \\ e_n^T \end{pmatrix} \vec{v} = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}^T \vec{v}$$

An example

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Mother’s Mother?

Mother's Mother?

We get geometrical interpretation.

- Row of matrix is "basis".
- (Column) vector is a vector.
- Element of vector is projection: measured coordinate.

From Measurement back to Geometry

$$\vec{v} = \vec{v}_1 e_1 + \cdots + \vec{v}_n e_n = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix}$$

Check

$$\vec{v} \cdot e_k = (\vec{v}_k e_k) \cdot e_k = \vec{v}_k.$$

Or

$$\begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} \left(\begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}^T \vec{v} \right) = \vec{v}.$$

Thus, orthogonal matrix

$$\begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}^T = \mathrm{Id}\,.$$

Frame

A set of unit orthogonal bases, coordinate system:

$$F = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}$$

- Geometry to Algebra, vector to coordinates:

$$(\vec{v}_i) = F^T \vec{v}$$

- Algebra to Geometry, coordinates to vector:

$$\vec{v} = F(\vec{v}_i)$$

Algebra of Dot Product

$$
\begin{aligned}
\vec{a} \cdot \vec{b} &= \left(\sum_i \vec{a}_i e_i \right) \cdot \left(\sum_i \vec{b}_i e_i \right) \\
&= \sum_i (\vec{a}_i e_i) \cdot (\vec{b}_i e_i) \\
&= \sum_i \vec{a}_i \vec{b}_i .
\end{aligned}
$$

Or

$$\vec{a} \cdot \vec{b} = (F(\vec{a}_i)) \cdot (F(\vec{b}_i)) = (\vec{a}_i)^T F^T F(\vec{b}_i) = (\vec{a}_i)^T (\vec{b}_i).$$

Transformation

Use Different Frame

The coordinates $(\vec{v}_i)$ from frame F to F'.

$$\vec{v} = \sum_i \vec{v}_i e_i = F(\vec{v}_i)$$
$$\vec{v} = \sum_i \vec{v}'_i e'_i = F'(\vec{v}'_i)$$

$$F(\vec{v}_i) = F'(\vec{v}'_i) \Longrightarrow (\vec{v}'_i) = F^{'-1} F(\vec{v}_i)$$

Thus, transformation M is:

$$M_F^{F'} = F^{'T} F.$$

Element in M

$$
\begin{aligned}
M_F^{F'} &= F'^T F \\
&= \begin{pmatrix} e'_1 & \cdots & e'_n \end{pmatrix}^T \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}
\end{aligned}
$$

M_{ij} is the projection of e'_i on e_j.

Back to Angle and Length

Under different frames:

$$(\vec{v} \Longrightarrow \vec{v}' = M\vec{v}).$$

Because M is orthogonal $M^T M = \mathrm{Id}$.

- The angle is same for ANY frame.

$$\vec{a} \cdot \vec{b} = \sum_i \vec{a}_i \vec{b}_i = \sum_i \vec{a}'_i \vec{b}'_i.$$

- The length is same for ANY frame.

$$\|\vec{a}\| = \sqrt{\sum_i \vec{a}_i^2} = \sqrt{\sum_i \vec{a}_i'^2}.$$

General Bases

$$u_1, \cdots, u_n$$

- Not orthogonal
- Not unit

$$\vec{v} = \sum_i \vec{v}_i^u u_i = (u_1, \cdots, u_n)(\vec{v}_i) = U(\vec{v}_i^u)$$

What is $\vec{v}_i$?

- Choosing an frame to measure $\vec{v}$ and u_i with $(\vec{v}_i), (u_i)$, then

$$(\vec{v}^u) = U^{-1}(\vec{v}).$$

Linearly Independent

U^{-1} exists.

...

Mapping

Vector Function

A continuous function

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

What is the meaning of the Jacobian?

$$\begin{pmatrix} \Delta f_1 \\ \vdots \\ \Delta f_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}$$

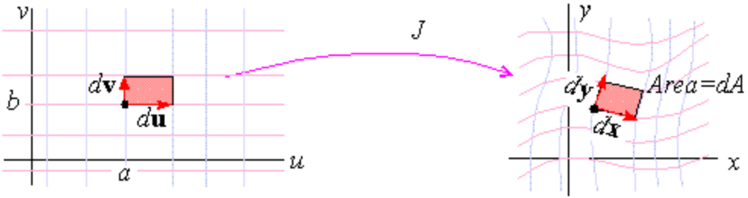
Vector to Vector

On the side of x:

$$\Delta\vec{x} = F_x(\Delta\vec{x}_i)$$

On the side of f:

$$\Delta\vec{f} = F_f(\Delta\vec{f}_i)$$



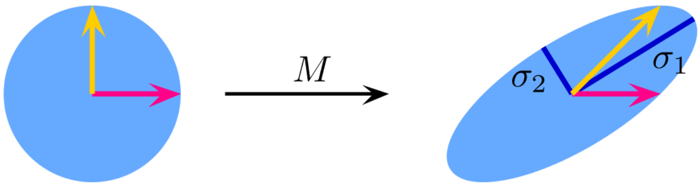
Local Behavior

A mapping $M : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$.

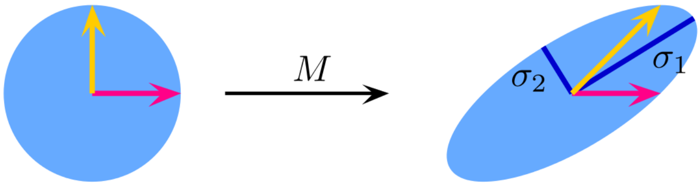
- Matrix-vector product.
- Or a function.

If $\Delta\vec{x}$ has unit length:

- A sphere to an ellipsoid



The Ellipsoid



Unit orthogonal axes u_i with different lengths σ_i.

How to find them?

- Stretch them back, still orthogonal, but in the same length.
- A set of unit orthogonal bases v_i, or, frame for $\vec{x}$.

Singular Value Decomposition

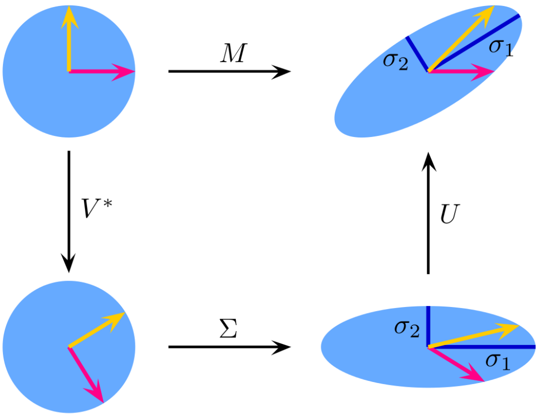
$$
\begin{aligned}
M &= U\Sigma V^T \\
&= \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{pmatrix} \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}.
\end{aligned}
$$

Thus

$$\Delta \vec{f} = M \Delta \vec{x} = U\Sigma V^T \Delta \vec{x}.$$

Singular Value Decomposition

View U, V as transformation respect to Id.



$$M = U \cdot \Sigma \cdot V^*$$

What is Determinant

$$
\begin{aligned}
|M| &= |U\Sigma V^T| = |U||\Sigma||V^T| = |\Sigma| \\
&= \sigma_1 \cdots \sigma_n
\end{aligned}
$$

Scale of volume!

When the $|M| = 0$

- Not a one-one mapping
- No inverse

Eigen Decomposition

If $U = V$, $M = U\Sigma U^T$.

View U^T as a transformation respect to Id.

The effect of matrix multiplication

$$\begin{aligned}
\vec{y} &= M\vec{x} = U\Sigma U^T\vec{x} \\
(U^T\vec{y}) &= \Sigma(U^T\vec{x}) \\
\vec{y}' &= \Sigma\vec{x}' \\
\begin{pmatrix} \vec{y}'_1 \\ \vdots \\ \vec{y}'_n \end{pmatrix} &= \begin{pmatrix} \sigma_1\vec{x}'_1 \\ \vdots \\ \sigma_n\vec{x}'_n \end{pmatrix}
\end{aligned}$$

The coordinates of $\vec{x}$ under U^T scaled by Σ.

Generalization

Dot Product, Integral

$$\sum_i f_i g_i \Longrightarrow \int_a^b f(x)g(x)dx$$

For $f : [0, 2\pi] \rightarrow \mathbb{R}$, Fourier analysis is:

- Unit orthogonal bases:

$$u_0(x) = \frac{1}{2\pi}$$
$$u_1(x) = \frac{\cos(kx)}{\sqrt{\pi}}$$
$$u_2(x) = \frac{\sin(kx)}{\sqrt{\pi}}, k = 1, 2, \cdots$$

Fourier Analysis

Giving a function $f : [0, 2\pi] \to \mathbb{R}$.

- Projection, decomposition:

$$f_i = \int_0^{2\pi} f(x) u_i(x) dx.$$

- Reconstruction, composition:

$$f(x) = f_i u_i(x).$$

Pythagorean theorem (Parsevaals theore)

$$\int_0^{2\pi} f(x)^2 dx = \sum_i f_i^2$$

General Fourier Transform for $f : \mathbb{R} \to \mathbb{C}$

Bases:

$$u(\omega, x) = e^{-2\pi i \omega x}, \omega, x \in (-\infty, +\infty)$$

Projection

$$f_\omega = F(\omega) = \int_{-\infty}^{+\infty} f(x) u(\omega, x) dx$$

Composition

$$f(x) = \int_{-\infty}^{+\infty} F(\omega) \frac{1}{u(\omega, x)} d\omega$$