## 4.2 Error Correction Coding

## Motivation

In the set of all multisymbol sequences $\mathcal{S}$.

- $\mathbf{w}_{m_a}, \mathbf{w}_{m_b}, m_a, m_b \in \mathcal{S}, a \neq b$ may be similar.

## Sample

- $L = 3, |\mathcal{A}| = 4, \mathcal{W}_{\mathcal{AL}}[i,j] \cdot \mathcal{W}_{\mathcal{AL}}[i,j] = N$
  - $\mathbf{w}_{312} = \mathcal{W}_{\mathcal{AL}}[1,3] + \mathcal{W}_{\mathcal{AL}}[2,1] + \mathcal{W}_{\mathcal{AL}}[3,2]$.
  - $\mathbf{w}_{314} = \mathcal{W}_{\mathcal{AL}}[1,3] + \mathcal{W}_{\mathcal{AL}}[2,1] + \mathcal{W}_{\mathcal{AL}}[3,4]$.
  - Inner product:

$$\begin{aligned} &\mathcal{W}_{\mathcal{AL}}[i,a] \cdot \mathcal{W}_{\mathcal{AL}}[j,b] = 0, \quad i \neq j \\ \Longrightarrow \quad & \mathbf{w}_{312} \cdot \mathbf{w}_{314} = \mathcal{W}_{\mathcal{AL}}[1,3] \cdot \mathcal{W}_{\mathcal{AL}}[1,3] \\ &\quad + \mathcal{W}_{\mathcal{AL}}[2,1] \cdot \mathcal{W}_{\mathcal{AL}}[2,1] \\ &\quad + \mathcal{W}_{\mathcal{AL}}[3,2] \cdot \mathcal{W}_{\mathcal{AL}}[3,4] \\ &\quad \geq N + N - N = N \end{aligned}$$

- $h$ different symbols in a length $L$ sequence

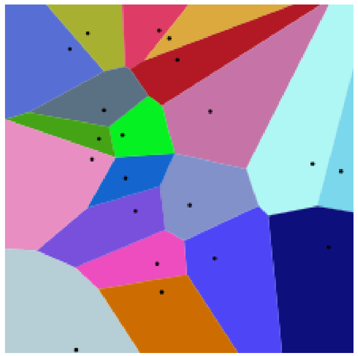
$$(L - 2h)N.$$

## The Idea of Error Correction Codes

Decompose all possible sequences $\mathcal{S}$ into $\mathcal{S}_c \cup \bar{\mathcal{S}}_c$.

- $\mathcal{S}_c$: Code words
  - Messages to encode.
  - Well separate to each other.
- $\bar{\mathcal{S}}_c$: Corrupted code words
  - Polluted messages.
  - Associated with the closest code word.

## $\mathcal{S}_c \cup \bar{\mathcal{S}}_c$



## Error Correction Code (ECC)

To preserve the capacity

- Increase the length of sequence.
- Expand the alphabet.

## Increase the Length of Sequence

Sample

- $4$-bits message set $\mathcal{M}$
  - Length $4$ binary sequence, $16$ messages.
- $7$-bits word space $\mathcal{S}$
  - Length $7$ binary sequence, $128$ words.
  - $|\mathcal{S}_c| = |\mathcal{M}| = 16$.
  - $a, b \in \mathcal{S}_c, a \neq b$ have at less $3$ different bits.
    - Why $3$? Flip one bit for each of the two.
  - Decode $s \in \mathcal{S}$: find $c \in \mathcal{S}_c$ has at most one different bit.

## Question: ECC

What is the minimal length of a sequence that can be used as ECC for a length $L$ binary sequence with $h$ bit error tolerance.

## Performance

- Without ECC
  - Length $4$, $1$ bit difference for different message.
  - Min inner product: $N(4 - 2 \times 1) = 2N$.
- With ECC
  - Length $7$, at least $3$ bit differences for different message.
  - Min inner product: $N(7 - 2 \times 3) = N$.

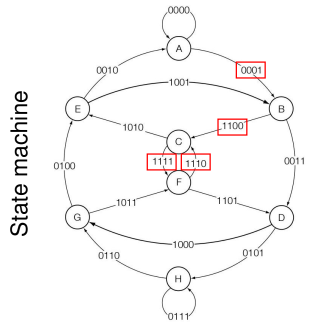
## Expand the Alphabet

From $|\mathcal{A}| = 2$ to $|\mathcal{A}'| = 4$.

- Less typical.
- Equivalent to increase length in capacity.
- But different in modulation.

## Trellis Codes

$$1010 \Rightarrow 0001\ 1100\ 1111\ 1110$$

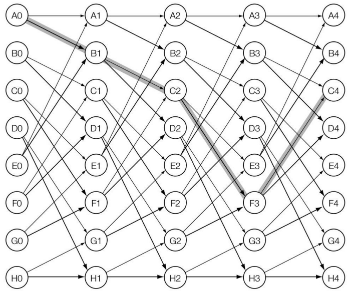



## Modulation

- Increase length to $4L$.
- Expand the alphabet to contain $2^4 = 16$ symbols.

## Viterbi Decoding

- A greedy method to find most closest code.
- Based on Trellis diagram.




## Performance of E_TRELLIS_8/D_TRELLIS_8

The same to E_SIMPLE_8/D_SIMPLE_8:

- $8$-bit message instead of $4$-bit.
  - Pad two more zero at the end: $10$-bit indeed.
  - More redundancy: a priory for accuracy.
- $6$ integers in each of $2000$ images.

Much better accuracy

- $1$ out of $12000$ is wrong.

## 4.3 Detecting Multisymbol Watermarks

## False Positive

If there is no watermark

- Direct message encoding
  - The most likely one is still poor in correction.
- Multisymbol system:
  - The corrections for all the symbols are not good enough.
  - How to define "good".

## Valid Messages

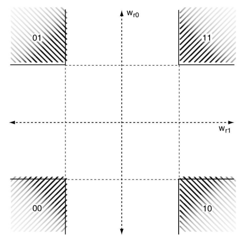
An intelligible message or a garbage.

- Checksum for verification
  - $16$-bits message: $m$.
  - $9$-bits checksum: $c = m[1:8] + m[9:16]$.
  - $25$-bits watermarking: $(m, c)$.
- Detector
  - Extractor $25$-bits watermarking $(m, c)$.
  - Compare $c$ and $m[1:8] + m[9:16]$.
- False positive probability: $P_{fp} = \frac{1}{2^9}$.

## Individual Symbols 1

All symbols are reliable (high correlated).

- Watermark presence.




*$2$-bit system in linear correlation.*

## Individual Symbols 2

False positive probability

- Single reference mark: $P_{fp0}$.
- In each index/position/order
  - If one mark in $\mathcal{A}$

$$P_{fp1} \approx |\mathcal{A}| P_{fp0}.$$

- For the whole length $L$ sequence.
  - All of them is high

$$P_{fp} = (P_{fp1})^L \approx (|\mathcal{A}| P_{fp0})^L.$$

## Normalized Correlation 1

- Multiple-symbol embedding
  - $\mathbf{w}_{ri}$ orthogonal to each other and unit.

$$\mathbf{v}_L = \mathbf{v}_o + \sum_{i=1}^{L} \mathbf{w}_{ri}, \quad \|\mathbf{v}_L\| \approx \sqrt{L}.$$

  - Linear correlation: independent of $L$

$$z_{lc}(\mathbf{v}_L, \mathbf{w}_{r1}) = \mathbf{v}_o \cdot \mathbf{w}_{r1} + \mathbf{w}_{r1} \cdot \mathbf{w}_{r1} = \varepsilon + 1.$$

  - Normalized correlation: difficult for larger $L$

$$z_{nc}(\mathbf{v}_L, \mathbf{w}_{r1}) = \frac{\mathbf{v}_L}{\|\mathbf{v}_L\|} \cdot \mathbf{w}_{r1} = \frac{\varepsilon + 1}{\sqrt{L}}.$$

## Normalized Correlation 2

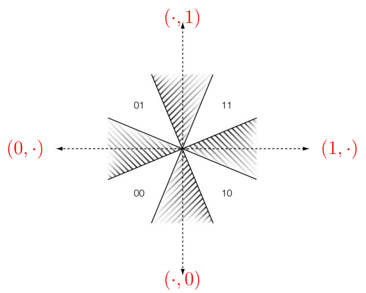
Less distinguishable.

- Large threshold: none is correlated enough, no symbol found.
- Small threshold: High false positive probability.

## Geometric Interpretation

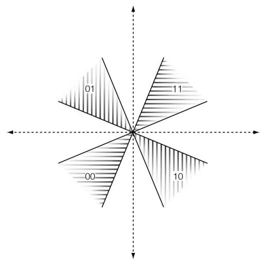
Large threshold: no overlap for the cones.

- No detectable $2$-bit message.




## Reencode

1. Extract message $m$.
2. Reencode $m$ into mark $\mathbf{v}_m$.
3. Test the presence of $\mathbf{v}_m$




## False Positive Probability

When the detection regions for the different messages do not overlap,

$$P_{fp} = |\mathcal{M}| P_{fp0}.$$

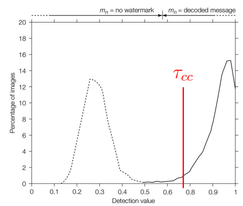
## E_BLK_8/D_BLK_8

$8$-bit message:

- Trellis code with two padding $0$ at the end.
  - A sequence of $10$ symbols drawn from a $16$-symbol alphabet.
- Reference marks:
  - $8 \times 8$ (block): low dimensional mark space.
    - So choose seed to reduce max correlation $(0.73)$.
- Embedding strength $\alpha = 2$.
- $\tau_{cc} = 0.65$: false positive probability $10^{-6}$.

## Performance

- $2000$ unwatermarked images (dashed line).
  - No false positive found.
- $12000$ unwatermarked images (solid line).
  - $6$ messages $\times 2000$ images.
  - $109$ fail: effectiveness $99\%$.




## Project: System 6

E_BLK_8/D_BLK_8

- marking space: $8 \times 8$ block
- $8$-bit message.
- ECC: hamming or optional.
- Reencode check.

## Presentation: 7.6 Analysis of Normalized Correlation

Approximate Gaussian Method

- False Positive Analysis
- False Negative Analysis