Mathematics Q2

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Question – Consider the real sequence p_n defined by $p_0 = 0, p_1 = 1, and\ p_{n+2} = 3p_{n+1} - 2p_n$ for $n = 0, 1, \ldots$

Another sequence (y_n) is defined by $y_n = {p_n}^2 + 2^{n+2}$. Prove that for every n > 0, y_n is the square of an odd integer.

Solution

First off, let's find $p_2\ to\ p_6$

$p_2 = 3p_1 - 2p_0 = 3 * 1 - 2 * 0 = 3$

$p_3 = 3p_2 - 2p_1 = 3 * 3 - 2 * 1 = 9 - 2 = 7$

$p_4 = 3p_3 - 2p_2 = 3 * 7 - 2 * 3 = 21 - 6 = 15$

$p_5 = 3p_4 - 2p_3 = 3 * 15 - 2 * 7 = 45 - 14 = 31$

$p_6 = 3p_5 - 2p_4 = 3 * 31 - 2 * 15 = 93 - 30 = 63.$

Also, finding $y_1\ to\ y_6$,

$y_1 = {p_1}^2 + 2^{1+2} = 1^2 + 2^3 = 9 = 3^2$

$y_2 = {p_2}^2 + 2^{2+2} = 3^2 + 2^4 = 25 = 5^2$

$y_3 = {p_3}^2 + 2^{3+2} = 7^2 + 2^5 = 81 = 9^2$

$y_4 = {p_4}^2 + 2^{4+2} = 15^2 + 2^6 = 289 = 17^2$

$y_5 = {p_5}^2 + 2^{5+2} = 31^2 + 2^7 = 1089 = 33^2$

$y_6 = {p_6}^2 + 2^{6+2} = 63^2 + 2^8 = 4225 = 65^2$

By inspection,

$y_1 = (3)^2 = (2^1 + 1)^2$

$y_2 = (5)^2 = (2^2 + 1)^2$

$y_3 = (9)^2 = (2^3 + 1)^2$

$y_4 = (17)^2 = (2^4 + 1)^2$

In general, we discover that

$y_n = {p_n}^2 + 2^{n+2} = (2^n + 1)^2$

$\therefore\ y_n = (2^n + 1)^2$_______________________$(i)$

We need to proof that (i) holds by induction.

$$\text{Let } v_n = 2^n + 1.\ This\ implies\ that\ y_n = {v_n}^2$$

Before we prove by induction, let see if we can get a different recursive formula for y_n

$When\ n = 1,\ \ v_1 = 2^1 + 1 = 3$

$When\ \ n = 2,\ \ v_2 = 2^2 + 1 = 5 = 2 * 3 - 1$

$When\ n = 3,\ \ v_3 = 2^3 + 1 = 9 = 2 * 5 - 1$

$When\ n = 4,\ \ v_4 = 2^4 + 1 = 17 = 2 * 9 - 1$

We notice a new pattern,

$$v_n = 2v_{n-1} - 1 \quad \forall\, n > 1$$

$\therefore\ y_n = (2^n + 1)^2 = (2v_{n-1} - 1)^2$__________$(ii)$

<u>Proof by induction</u>

1) Base case:
   When $n = 1$, $\ y_n = (2^1 + 1)^2 = (2 + 1)^2 = 3^2 = 9$. This is true.
2) Induction step:
   When $n = k,\ let's\ assume\ that\ y_k = (2^k + 1)^2\ is\ true.$
   Next, we need to show that it's true when $n = k + 1$.
   $y_{k+1} = (2^{k+1} + 1)^2 = {v_{k+1}}^2 = \left(2v_{(k+1)-1} - 1\right)^2$
   $y_{k+1} = (2v_k - 1)^2$ ____________(iii)
   $Recall\ that\ v_n = 2^n + 1, which\ is\ the\ same\ as\ v_k = 2^k + 1.$

   $Replacing\ it\ in\ (iii), we\ have$
   $y_{k+1} = (2(2^k + 1) - 1)^2 = (2^k * 2 + 2 - 1)^2 = (2^{k+1} + 1)^2$

   $$\therefore\ y_{k+1} = (2^{k+1} + 1)^2$$
   Since it holds for $n = k + 1, it\ means\ that\ it\ holds\ for\ all\ n \geq 1$

Hence, we have proved by induction that

$y_n = (2^n + 1)^2$ is true $\forall\, n \geq 1$.

Therefore, $y_n\ is\ the\ square\ of\ an\ odd\ integer. -QED.$

Q1

Find all functions $\mathbb{R} \rightarrow \mathbb{R}$ such that $f(x)f(y) + f(x+y) = xy$, for all real numbers $x \text{ and } y$.

Answer

Let $f(x)f(y) + f(x+y) = xy$___________________________(i)

Let $x = 0, and\ y = y$, we have that

$f(0)f(y) + f(y) = 0 * y$

$f(y)[f(0) + 1] = 0$

$\Rightarrow f(y) = 0\ or\ f(0) = -1$

<u>Check</u>

$if\ f(y) = 0, then\ f(x) = f(x+y) = 0$, replace into (i), we get

$0 * 0 + 0 = xy$

$\Longrightarrow x = 0\ or\ y = 0.\ Therefore, f(x) = 0\ is\ a\ conditional\ solution$

$If\ f(0) = -1$, from Left hand side (LHS),
$LHS = f(0)f(0) + f(x+y) = (-1^2) - 1 = 1 - 1 = 0$

$From\ the\ right\ hand\ (RHS),$

$RHS = xy = 0 * 0 = 0. Therefore\ f(0) = -1\ is\ a\ fixed\ point.$

Let $x = y = 0, substitute\ into\ (i)$

$LHS: f(0)f(0) + f(0+0) = (f(0))^2 + f(0)$

$RHS \coloneqq 0 * 0 = 0.$

$Therefore,\ \left(f(0)\right)^2 + f(0) = 0, \Rightarrow f(0) = 0\ or\ f(0) = -1.$

Testing that $f(0) = 0\ is\ a\ fixed\ point, it\ works\ well\ to$

Let $f(x) = x,\ this\ implies\ that\ f(y) = y\ , and\ f(x+y) = x + y$. Substitute into (i),

$LHS: xy + x + y$

$RHS: xy$

Since $LHS\ \neq RHS, then\ f(x) = x\ is\ not\ a\ solution, same\ with\ f(x) = -x.$

Let $x = 1, y = 0, substitute\ into\ (i)$

$f(1)f(0) + f(0+1) = 1 * 0$
$f(1)f(0) + f(1) = 0$
$f(1)[f(0) + 1] = 0$
$\Longrightarrow f(0) = \ -1\ or\ f(1) = 0$

Let $x = y = 1, substitute\ into\ (i)$

$f(1)f(1) + f(1+1) = 1 * 1$
$\left(f(1)\right)^2 + f(2) = 1. Recall\ that\ f(1) = 0. Therefore, f(2) = 1$

Let $x = 1, y = 2, substitute\ into\ (i)$

$f(1)f(2) + f(1+2) = 1 * 2$
$f(1)f(2) + f(3) = 2.\ Recall\ that\ f(1) = 0. Therefore, f(3) = 2$

Let $x = x, y = 1, substitute\ into\ (i)$

$f(x)f(1) + f(x+1) = x * 1$
$f(x)f(1) + f(x+1) = x, \qquad recall\ that\ f(1) = 0$
$\therefore f(x+1) = x, which\ can\ be\ rewritten\ as\ f(x) = x - 1$

Check

let $f(x) = x - 1\ then\ f(y) = y - 1\ and\ f(x + y\ ) = x + y - 1$, substitute into (i)

$LHS: (x-1)(y-1) + x + y - 1 = xy = RHS$

$\therefore f(x) = x - 1\ is\ a\ solution\ to\ our\ functional\ equation$

Let $x = 0, y = \ -1, substitute\ into\ (i)$

$f(0)f(-1) + f(0-1) = 0 * -1$
$f(0)f(-1) + f(-1) = 0$
$f(-1)[f(0) + 1] = 0$

$\Longrightarrow f(0) = \ -1\ or\ f(-1) = 0$

Let $x = y = \ -1, substitute\ into\ (i)$

$f(-1)f(-1) + f(-1-1) = \ -1 * \ -1$
$f(-1)f(-1) + f(-2) = 1, recall\ f(-1) = 0$, therefore $f(-2) = 1$

Let $x = x, y = -1, substitute\ into\ (i)$

$f(x)f(-1) + f(x-1) = x * -1$
$f(x)f(-1) + f(x-1) = -x, recall\ that\ f(-1) = 0$

$\therefore f(x-1) = -x, which\ can\ be\ rewritten\ as\ f(x) = -(x+1)$

<u>Check</u>

Let $f(x) = -(x+1), f(y) = -(y+1), f(x+y) = -(x+y+1)$

$LHS\colon -(x+1) * -(y+1) - (x+y+1) = xy + y + x + 1 - x - y - 1 = xy = xy$
$= RHS$

$\therefore f(x) = -(x+1)\ is\ a\ solution\ to\ the\ functional\ equation$

$\therefore The\ solution\ to\ the\ functional\ equation\ are$
1. $f(x) = -(x+1)\ \forall x \in \mathbb{R}$

2. $f(x) = x - 1\ \forall x \in \mathbb{R}$

3. $f(x) = 0\ iff\ x = 0$