VNiVERSiDAD D SALAMANCA

FINAL DEGREE PROJECT

The Lippmann-Schwinger equation in Dimensional Regularization

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Chapter 1

Abstract

Keywords: Quantum Chromodynamics (QCD), strong interaction, low-energy regime, Effective Field Theory (EFT), NN interaction, chiral symmetry, Chiral Effective Field Theory (χEFT), Lippmann-Schwinger Equation (LSE), dimensional regularization (DR), Distorted Wave Theory (DWT), One-Pion Exchange Potential (OPE).[1]

Quantum Chromodynamics (QCD) is the fundamental theory of the *strong interaction*, responsible for keeping quarks bound inside nucleons, and nucleons bound together within the atomic nucleus. Asymptotic freedom is a property of this interaction: This property implies that the force between quarks becomes weaker as the distance between them decreases, behaving almost like free particles at very short distances.
In the *low-energy regime*, a perturbative treatment of the theory is not possible as it is at high energies. From the need to study phenomena at these scales arise the usefulness of *Effective Field Theories* (EFT).
The NN interaction is a residual interaction of the strong force. Nucleons are colorless particles affected by the nuclear force, a situation analogous to Van der Waals forces acting on atoms that are electrically neutral.

Chirality is an intrinsic property of massless fermions. They can be classified as right-handed if their spin is aligned with their motion, or left-handed in the opposite case. *Chiral symmetry* manifests in the limit of zero masses for the light quarks u and d. This symmetry can undergo spontaneous breaking, giving rise to the so-called *Nambu-Goldstone bosons* of small mass, which are none other than the pions (π^+, π^-, π^0). In order to respect this symmetry, *Chiral Effective Field Theories* (χEFT) arise to describe the interactions between nucleons (protons and neutrons) and pions, establishing hadronic mass scales.
Problems related to the NN interaction require non-perturbative tools to be described. *The Lippmann-Schwinger Equation* is an integral formulation analogous to *The Schrödinger equation* for solving scattering problems.

Regularization refers to all those methods that allow us to isolate divergences in certain integrals. The treatment of integrals in an arbitrary dimension d, later taking the limit to the original dimension, is called *dimensional regularization*, and can thus be understood as an *analytic continuation* of the dimension.
Using techniques such as DR and *Distorted Wave Theory* (DWT) we will find the solution to the LSE for our particular problem: Analyzing the *NN interaction* in the ${}^1S_0$ wave using as the long-range interaction the *regular one-pion exchange potential* (OPE) and as short-range the *singular contact terms* [15].

[1] These keywords are essential for understanding the work.

Resumen

Palabras clave: Cromodinámica cuántica (QCD), interacción fuerte, régimen de bajas energías, Teoría de Campo Efectiva (EFT), interacción NN, simetría quiral, Teoría de Campo Efectiva Quiral (χEFT), Ecuación de Lippmann-Schwinger (LSE), regularización dimensional (DR), Teoría de Ondas Distorsionadas (DWT), Potencial de intercambio de un pion (OPE).[2]

La *cromodinámica cuántica* (QCD) es la teoría fundamental de la *interacción fuerte*, y es la encargada de mantener unidos a los quarks dentro de los nucleones, y a los propios nucleones entre sí en el núcleo atómico. La libertad asintótica es una propiedad de dicha interacción: Esta propiedad implica que la fuerza entre los quarks se debilita a medida que la distancia entre ellos disminuye, comportándose casi como partículas libres a distancias muy cortas.
En el *régimen de bajas energías* no es posible un tratamiento perturbativo de la teoría como lo es a energías grandes. De la necesidad de estudiar los fenómenos en torno a estas escalas es son útiles las *Teorías de Campo Efectivas* (EFT).
La interacción NN se trata de una interacción residual de la interacción fuerte. Son los nucleones partículas sin color que se ven afectadas por la fuerza nuclear, situación análoga al caso de las fuerzas de Van der Walls actuando sobre átomos que son neutros.

La *quiralidad* es una propiedad intrínseca a los fermiones sin masa. Pueden ser clasificados como dextrógiros si su espín está alineado a su movimiento, o levógiro en el caso contrario. La *simetría quiral* se manifiesta en el límite de masas nulas para los quarks ligeros u y d. Esta puede sufrir una ruptura espontánea, surgiendo así los llamados *bosones de Nambu-Goldstone* de masa pequeña, que no son otros que los piones (π^+, π^-, π^0). Buscando respetar esta simetría surgen las *Teorías de Campo Efectivas Quirales* (χEFT) para describir las interacciones entre nucleones (protones y neutrones) y piones estableciendo escalas de masas hadrónicas.
Los problemas relacionados con la interacción NN precisan de herramientas no perturbativas para ser descritas. *La Ecuación de Lippmann-Schwinger* es una formulación integral análoga a *La Ecuación de Schrödinger* para resolver problemas de dispersión.

Regularización son todos aquellos métodos que nos permiten aislar divergencias de ciertas integrales. El tratamiento de integrales en una dimension d arbitraria para posteriormente tomar el límite a la dimensión original se denomina *regularización dimensional*, y puede entenderse así como una *continuación analítica* de la dimensión.
Usando técnicas como la DR y *Teoría de Ondas Distorsionadas* (DWT) hallaremos la solución a la LSE para nuestro problema en particular: Analizar la *interacción NN* en onda ${}^1S_0$ utilizando como interacción de largo alcance el *potencial de intercambio de un pion regular* (OPE) y como corto alcance *términos de contacto* singulares [15].

[2]Estas palabras son claves para la comprensión del trabajo.

Contents

Chapter 2

Introduction

The purpose of this work is to analyze the nucleon–nucleon (NN) interaction problem in the ${}^1S_0$ wave using, as the long-range interaction, the one-pion exchange potential (regular), and as the short-range interaction, contact terms (singular) containing up to two derivatives, reproducing the calculations presented in the original paper [15].

Using techniques such as dimensional regularization (DR) and distorted wave theory (DWT), we will obtain a solution to the Lippmann–Schwinger equation (LSE). In order to reproduce the experimental phase-shift data, the coefficients of the short-distance potential will be properly fixed, which are initially unknown quantities.

Once the scattering probability amplitude has been obtained after fixing the constants, we will focus on determining the values of the low-energy parameters that describe the physics of the process at those energy scales. Later, the so-called effective range expansion will be presented in detail, clarifying the origin of these parameters.

Before entering into the calculations, it is convenient to introduce the physical preliminaries, exploring the motivation behind the models, the theoretical foundations, and the strength of the method employed. I will therefore begin with a brief historical overview of nuclear theory for the NN interaction [12].

In 1935, Yukawa postulated the existence of a pseudoscalar meson mediating the strong interaction between nucleons through particle exchange [23]. These particles were the pions, spin-zero mediator bosons.

Around the 1950s, the one-pion exchange theory successfully described experimental NN scattering data, particularly the properties of the deuteron. However, the model proved insufficient for processes involving multipion exchange, ultimately leading to its failure.

During the 1960s, the discovery of heavier mesons resolved this limitation by considering possible pion resonances (ρ, σ, ...) and introducing a refined one-boson-exchange (OBE) model [1]. Its main weakness was the σ meson itself, associated with two-boson exchange resonance, whose empirical evidence remains incomplete.

One decade later, during the 1970s, several approaches were developed to explain this double exchange mechanism. Examples include the Stony Brook potentials [9], the Paris potential within dispersion theory, and the Partovi–Lomon model [16], among other field-theoretical formulations.

Finally, in the 1980s, the fundamental theory of the strong interaction was established: Quantum Chromodynamics (QCD). With the emergence of the quark model, meson theories were relegated to phenomenological models. Nevertheless, they regained importance with the development of Effective Field Theories (EFT), since at low energies the effective degrees of freedom once again become pions, consistent with Yukawa's original idea.

The following sections are devoted to introducing several essential ideas and theoretical concepts required for a deep understanding of this work, beyond the purely mathematical procedures and calculations.

From this point onward, it is natural to dedicate space to the aforementioned effective theories, discussing their motivation, purpose, and the development of chiral effective theories applied to the NN problem.

Afterward, given the necessity of non-perturbative tools to describe scattering processes associated with the NN system, we introduce the Lippmann–Schwinger Equation (LSE), which gives its name to this undergraduate thesis, as an integral formulation alternative to solving the Schrödinger equation.

Throughout the work we will inevitably encounter divergences arising in certain integrals. To isolate these divergences, regularization procedures are required. Among the various possible methods, we will employ dimensional regularization exclusively. Therefore, the final introductory section explains this method and presents the evaluation of a first general integral within this framework. This result is particularly important since most integrals appearing later can be reduced to it.

Once all preliminary concepts have been introduced, we proceed to the resolution of the main problem: the renormalization of the NN interaction ($^1S_0$ channel) through one-pion exchange in the presence of contact terms. The calculations are presented explicitly, step by step, following the order in which the required integrals appear.

The work concludes with the numerical results obtained after discretizing the integral equations. Finally, the main conclusions are presented, emphasizing the most relevant outcomes and providing a global summary of the objectives and achievements of this thesis, reinforcing the theoretical framework after its practical application.

2.1 Effective Field Theories (EFT)

Quantum Chromodynamics (QCD) [14] is the fundamental theory describing the strong interaction. At high energies, the theory can be treated perturbatively. However, this approach fails at low energies (the hadronic scale relevant to our problem) due to the loss of asymptotic freedom and the strong coupling between quarks and gluons. This property implies that the interaction between quarks weakens as their separation decreases, behaving almost as free particles at very short distances.

As an alternative to the absence of a solvable fundamental theory at low energies, the so-called *Effective Field Theories* (EFT) were developed. Their purpose can be summarized by the statement [13]: *The basic premise of effective theories is that low-energy (long-distance) dynamics do not depend on the details of high-energy (short-distance) dynamics.*

Consequently, low-energy physics can be described by an effective Lagrangian containing only a limited number of degrees of freedom, ignoring additional degrees of freedom relevant at higher energies. This does not imply the absence of a connection between both energy scales; rather, the relation is implicit. The tool that enables the separation between both regimes is renormalization [18].

An important feature of these theories is their model independence, since the main objective is to describe long-distance physics without making assumptions about short-distance dynamics. This distinguishes them from phenomenological models, where short-distance physics is explicitly introduced.

Based on this idea, a more realistic treatment of scattering processes avoids describing particles as simple plane waves and instead introduces contact terms correcting the interaction to account for wave distortion during scattering. The method that starts from an asymptotic wave function rather than a plane wave is known as distorted wave theory (DWT) [2]. The effective potential is written as

$$V = V_I + V_{II}.$$

In this formulation, the separation between long- and short-distance scales is not explicitly performed. As will be shown later, in our case the short-range potential is conveniently expressed as a polynomial expansion in momenta, whose coupling coefficients must be determined by fitting experimental data.

A second-order truncation corresponds to the distorted-wave Born approximation (DWBA) [19], one of the most useful approximations in nuclear physics. The long-range interaction considered here corresponds to one-pion exchange (OPE), commonly represented by the Yukawa potential.

To further illustrate the functioning of EFTs, we introduce Bethe's effective range expansion [3], which will later be used to calculate low-energy parameters:

$$f(k) = k \cot \delta(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + ...$$

This expansion of the phase shift is valid for $ka \ll 1$, where a represents the finite range of the potential $U(r)$. The important conclusion is that for large wavelengths λ, the system is described only by a few parameters determined by the potential, rather than by the

explicit form of $U(r)$.

In summary, although the potential must reproduce the correct long-distance physics, short-distance interactions can be modeled through a set of parameters without affecting the underlying physical description.

2.2 Chiral Effective Field Theories (χEFT)

Chirality is defined as a fundamental property in particle physics describing the behavior of massless fermions under symmetry transformations, particularly within relativistic theories such as the Standard Model. Fermions are classified as left-handed or right-handed depending on whether their spin is antiparallel or parallel to their momentum. From this concept arises *chiral symmetry* [10] of the Lagrangian, which becomes exact in the massless quark limit ($m_u = m_d = 0$). Without entering into excessive detail, spontaneous symmetry breaking leads to the appearance of light bosons known as *Nambu–Goldstone bosons* [4], which in χEFT correspond precisely to the pions originally proposed by Yukawa.

Respecting this symmetry motivates the development of *chiral effective field theories* [12], which describe nucleon–pion interactions while introducing hadronic mass scales. Weinberg first proposed applying EFT methods to nuclear forces [22]. The initial strategy consisted of deriving NN potentials from χEFT principles and subsequently solving the Lippmann–Schwinger equation to obtain scattering amplitudes.

The *soft* scale is given by the pion mass, $Q \sim m_\pi$, whereas the *hard* scale corresponds to $\Lambda_\chi \sim 1$ GeV (the chiral symmetry breaking scale), representing the energy threshold beyond which QCD becomes non-perturbative [12]. According to Weinberg [21], the most general hadronic Lagrangian is constructed by including all interaction terms consistent with the relevant low-energy degrees of freedom and symmetries [17]. These infinitely many interactions are organized according to the expansion parameter Q/Λ_χ:

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}^{(0)} + \mathcal{L}_{eff}^{(1)} + \mathcal{L}_{eff}^{(2)} + ...$$

where $\mathcal{L}_{eff}^{(n)} \propto (Q/\Lambda_\chi)^n$. The integer n defines the *power counting* scheme [5].

Although highly successful for $\pi\pi$ and $N\pi$ interactions, it became clear that non-perturbative methods were required to properly describe the NN system. For this reason, the next section introduces the equation central to this work: the Lippmann–Schwinger equation.

2.3 The Lippmann–Schwinger Equation (LSE)

The Lippmann–Schwinger equation provides an integral formulation for solving quantum mechanical scattering problems, equivalent to solving the Schrödinger equation. Numerous references derive the equation following similar steps (see, e.g., [8]).
Consider a system described by

$$H = H_0 + V,$$

where H_0 has eigenstate $|\phi\rangle$ and eigenvalue E:

$$H_0 |\phi\rangle = E |\phi\rangle .$$

We now construct a state with the same energy:

$$(H_0 + V) |\varphi\rangle = E |\varphi\rangle .$$

A possible solution is

$$|\varphi\rangle = |\phi\rangle + [E - H_0]^{-1} V |\varphi\rangle .$$

Because the operator is not invertible when its eigenvalue vanishes, an infinitesimal energy shift is introduced:

$$|\varphi^{\pm}\rangle = |\phi\rangle + [E - H_0 \pm i\epsilon]^{-1} V |\varphi^{\pm}\rangle .$$

This expression corresponds to the Lippmann–Schwinger equation in its simplest form. Acting with V on both sides gives

$$V |\varphi^{\pm}\rangle = V |\phi\rangle + V[E - H_0 \pm i\epsilon]^{-1} V |\varphi^{\pm}\rangle .$$

Defining the transition operator by $T |\phi\rangle = V |\varphi^{+}\rangle$, we obtain

$$T |\phi\rangle = V |\phi\rangle + VGT |\phi\rangle ,$$

where G denotes the propagator or Green's function. Finally,

$$T = V + VGT.$$

We denote by $T(k, k'; k)$ the matrix element

$$\langle \phi_k | T | \phi_{k'} \rangle = \langle \phi_k | V | \varphi^{+} \rangle ,$$

which is directly related to the form factor and therefore to the scattering cross section.

Its relation to the scattering matrix can be readily obtained through the identity

$$|\varphi_k^{+}\rangle - |\varphi_k^{-}\rangle = -2\pi i \delta(E_k - H) V |\phi_k\rangle .$$

The S matrix takes the form

$$\langle \phi_{k'} | S | \phi_k \rangle = \langle \varphi_{k'}^{-} | \varphi_k^{+} \rangle = \langle \phi_{k'} | \phi_k \rangle - 2\pi i \delta(E_{k'} - E_k) \langle \phi_{k'} | \varphi_k^{+} \rangle = \langle \phi_{k'} | 1 - 2\pi i \delta(E_{k'} - E_k) T | \phi_k \rangle .$$

Therefore,

$$S = 1 - 2\pi i \delta(E_{k'} - E_k) T.$$

This matrix satisfies the unitarity condition ($SS^{\dagger} = 1$). From this property one derives the *optical theorem* for the T matrix, which will later allow us to relate its modulus to its imaginary part.

2.4 Dimensional Regularization (DR)

Regularization refers to all methods that allow us to isolate divergences appearing in certain integrals. The behavior of the theory must be modified at large momenta so that it remains well defined at the energy scales of interest. Regularization consists of obtaining a finite result from an integral through intermediate steps, which are later followed by renormalization (absorbing divergences into a redefinition of physical quantities or by adding counterterms to the Lagrangian).

Several regularization methods exist: Pauli–Villars, cutoff regularization, lattice regularization, among others. In this work we focus on *dimensional regularization* [11]. The idea is to evaluate integrals in a d-dimensional space and later take the limit in which d approaches the physical dimension, commonly $d \to 4$. The advantages of this method over the others include computational efficiency and compatibility with a larger set of symmetries.

$$\int_0^\infty \frac{d^4p}{(p^2+m^2)^n} \to \int_0^\infty \frac{d^dp}{(p^2+m^2)^n}.$$

Integrals that diverge for integer d may yield finite results for non-integer d. In other words, dimensional regularization may be understood as an *analytic continuation in the dimension.*

$$\boxed{\int d^dq \frac{q^v}{(q^2+s)^n} = \int d\Omega_d \int_0^\infty dq \frac{q^{d-1+v}}{(q^2+s)^n} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{1}{2} s^{\frac{v+d}{2}-n} \frac{\Gamma(\frac{v+d}{2})\Gamma(n-\frac{v+d}{2})}{\Gamma(n)}.} \tag{2.1}$$

Expression (2.1) will serve as the basis for solving all integrals appearing throughout the calculations of the main problem. Its full derivation can be found in Sections (2.4.1) and (2.4.2) below.

Note that if $n - \frac{v+d}{2}$ is a negative non-integer number, the expression converges because the Gamma function $\Gamma(x)$ is defined. If this number is also an integer, the result diverges, just as the original integral does.
In such cases the divergence manifests itself as a first-order pole through the expansion

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left\{\frac{1}{\epsilon} + \psi_1(n+1) + O(\epsilon)\right\}.$$

Here $\psi_1(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma$, where $\gamma = \psi_1(1) = 0.577$ is the Euler–Mascheroni constant. If we take $v = 0$ and $d = 4 + \epsilon$ with $\epsilon \to 0$:

$$\Gamma(n - d/2) = \Gamma(n - 2 + \epsilon/2) = \frac{(-)^{2-n}}{(2-n)!}\left\{\frac{2}{\epsilon} + \psi(3-n) + O(\epsilon)\right\},$$

where ψ denotes the logarithmic derivative of the Gamma function, i.e. the digamma function.

We now compute the general integrals leading to (2.1), which constitute the foundation of all subsequent integrals appearing in this work.

2.4.1 Integral over the Euclidean norm

$$\int_0^\infty \frac{l^v}{(l^2+s)^n} dl. \tag{2.2}$$

Let's make the following variable change:

$$l^2 = xs \ , \ 2ldl = sdx.$$

$$dl = \frac{sdx}{2l} = \frac{dx}{2}\sqrt{\frac{s}{x}}.$$

The integral takes the following form:

$$\int_0^\infty \frac{(sx)^{\frac{v}{2}}}{(sx+s)^n} \frac{1}{2}\sqrt{\frac{s}{x}} dx = \frac{1}{2} s^{\frac{v+1}{2}-n} \int_0^\infty \frac{x^{\frac{v-1}{2}}}{(x+1)^n}.$$

Working with the complex function f(z):

$$f(z) = \frac{e^{\frac{v-1}{2} log(z)}}{(z+1)^n}.$$

The function has a pole of order n at $z = -1$. An integration region is chosen that allows us to use the residue theorem.

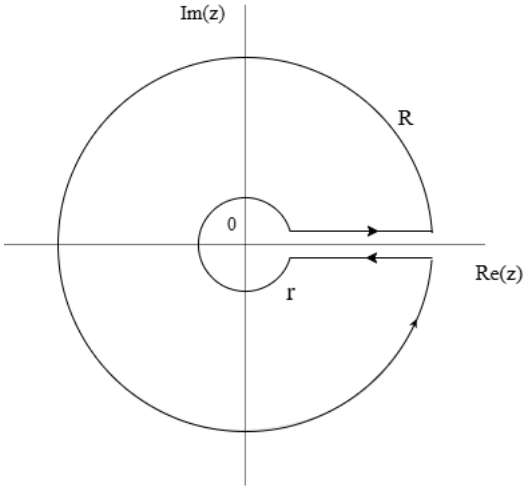

Figure 2.1: Recinto de integración.

$$\int_{C_+} f(z)dz = 2\pi i \sum Res_{z_k} f(z) = I_{1_+} + I_{2_-} + \int_{\gamma R} f(z)dz + \int_{\gamma r} f(z)dz$$

The result of the integral that interests us will be recovered once we take the limit of $R \to \infty$ and $r \to 0$ of I_1, in addition to taking the real part.

$$I_{1_+} + I_{2_-} = \int_{0+r}^{R} \frac{e^{\frac{v-1}{2}(log|z|+i0)}}{(z+1)^n} dz + \int_R^{0+r} \frac{e^{\frac{v-1}{2}(log|z|+i2\pi)}}{(z+1)^n} dz = (1 - e^{i2\pi\frac{v-1}{2}}) \int_{0+r}^{R} \frac{e^{\frac{v-1}{2}(log|z|+i0)}}{(z+1)^n} dz.$$

Given that $e^{i\pi} = -1$, we arrive at the $I_{1_+} + I_{2_-} = (1 + e^{i\pi v}) I_{1_+}$.

Using the appropriate integration lemmas, we can see that the contribution of the integrals of the circles is zero.

$$I = \lim_{R\to\infty} [\lim_{r\to 0} I_1] = \frac{1}{(1+e^{i\pi v})} 2\pi i Res_{-1} f(z).$$

The general expression for calculating a pole of order m is:

$$Res_{z_0} f(z) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}[(z-z_0)^m f(z)]}{dz^{m-1}}.$$

We need the $(n-1)_{th}$ derivative of $g(z) = (z-z_0)^m f(z) = z^{\frac{v-1}{2}}$. Denotando $\alpha = \frac{v-1}{2}$:

$\frac{dz^\alpha}{dz} = \alpha z^{\alpha-1}$; $\frac{d^2z^\alpha}{dz^2} = \alpha(\alpha-1)z^{\alpha-2}... \rightarrow \frac{d^kz^\alpha}{dz^k} = \alpha(\alpha-1)(\alpha-2)...(\alpha-(k-1))z^{\alpha-k}$

In terms of $\Gamma(x) = (x-1)! \rightarrow \frac{d^kz^\alpha}{dz^k} = \frac{\Gamma(\alpha+1)z^{\alpha-k}}{\Gamma(\alpha-k+1)}$.

Substituting in our case with $k = n-1$ and taking into account that $log(z) = ln|z| + iArg_z$:

$$Res_{-1}f(z) = \lim_{z\to-1} \frac{1}{(n-1)!}\frac{\Gamma(\alpha+1)e^{(\alpha-n+1)log(z)}}{\Gamma(\alpha-n+2)} = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(n)\Gamma(\frac{v+3}{2}-n)}e^{i\pi(\frac{v+1}{2}-n)}.$$

Using this result:

$$I = Re\left(2\pi i\frac{\Gamma(\frac{v+1}{2})}{\Gamma(n)\Gamma(\frac{v+3}{2}-n)}\frac{e^{i\pi(\frac{v+1}{2}-n)}}{(1+e^{i\pi v})}\right).$$

That can be simplified by $e^{i\pi/2} = i$ y $e^{i\pi n} = (-1)^n$, as $n \in \mathbb{Z}$.

$$2\pi i^2\frac{\Gamma(\frac{v+1}{2})}{\Gamma(n)\Gamma(\frac{v+3}{2}-n)}\frac{e^{i\frac{v\pi}{2}(-1)^n}}{(1+e^{i\pi v})} = 2\pi\frac{\Gamma(\frac{v+1}{2})}{\Gamma(n)\Gamma(\frac{v+3}{2}-n)}\frac{e^{i\frac{v\pi}{2}(-1)^{n+1}}}{(1+e^{i\pi v})} = 2\pi(-1)^{n+1}\frac{\Gamma(\frac{v+1}{2})}{\Gamma(n)\Gamma(\frac{v+3}{2}-n)}\frac{\cos(\pi v/2)+i\sin(\pi v/2)}{(1+\cos(\pi v))+i\sin(\pi v)}.$$

Let's elaborate on the terms sines and cosines:

$$\begin{aligned}\frac{\cos(\pi v/2)+i\sin(\pi v/2)}{(1+\cos(\pi v))+i\sin(\pi v)} &= \frac{\cos(\pi v/2)+i\sin(\pi v/2)}{(1+\cos(\pi v))+i\sin(\pi v)}\frac{(1+\cos(\pi v))-i\sin(\pi v)}{(1+\cos(\pi v))-i\sin(\pi v)}\\ &= \frac{\cos(\pi v/2)+\cos(\pi v/2)\cos(\pi v)+\sin(\pi v/2)\sin(\pi v)}{(1+\cos(\pi v))^2+\sin(\pi v)^2}\\ &+ \frac{i(\sin(\pi v/2)(1+\cos(\pi v)-\sin(\pi v/2)\cos(\pi v/2))}{(1+\cos(\pi v))^2+\sin(\pi v)^2}.\end{aligned}$$

Taking only the real part, and recognizing the cosine of the difference angle:
$\cos(a-b) = \cos(a)\cos(b)+\sin(a)\sin(b)$, we come to that:

$$Re\left(\frac{e^{i\frac{v\pi}{2}}}{1+e^{i\pi v}}\right) = \frac{2\cos(\pi v/2)}{1+\cos(\pi v)^2+\sin(\pi v)^2+2\cos(\pi v)} = \frac{\cos(\pi v/2)}{1+\cos(\pi v)}.$$

Using the following property of the gamma function: $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$, and identifying $a = n-\frac{v+1}{2}$ we can expand the denominator. Since n is an integer:

$$\sin(\pi(n-\frac{v}{2})-\frac{\pi}{2}) = -\cos(\pi n-\pi\frac{v}{2}) = (-1)^{n+1}\cos(\pi\frac{v}{2}).$$

The π cancel each other out, and the $(-)$ is always raised to an even power regardless of the value of n.

$$I = (-1)^{2n+2}2\frac{\Gamma(\frac{v+1}{2})\Gamma(n-\frac{v+1}{2})}{\Gamma(n)}\frac{\cos(\pi v/2)\cos(\pi v/2)}{1+\cos(\pi v)} = \frac{\Gamma(\frac{v+1}{2})\Gamma(n-\frac{v+1}{2})}{\Gamma(n)}\frac{2\cos(\pi v/2)\cos(\pi v/2)}{1+\cos(\pi v)}.$$

Note that : $\cos(2\theta) = 2\cos(\theta)^2-1$ so, $\frac{\cos(\theta)^2}{1+\cos(2\theta)} = 1/2$.

$$I = \int_0^\infty \frac{x^{\frac{v-1}{2}}}{(x+1)^n} = \frac{\Gamma(\frac{v+1}{2})\Gamma(n-\frac{v+1}{2})}{\Gamma(n)}.$$

The final value is:

$$\boxed{\int_0^\infty \frac{l^v}{(l^2+s)^n}dl = \frac{1}{2}s^{\frac{v+1}{2}-n}\int_0^\infty \frac{x^{\frac{v-1}{2}}}{(x+1)^n} = \frac{1}{2}s^{\frac{v+1}{2}-n}\frac{\Gamma(\frac{v+1}{2})\Gamma(n-\frac{v+1}{2})}{\Gamma(n)}.} \tag{2.3}$$

2.4.2 Angular Integral

Let's start with the generalization of spherical coordinates for n dimensions.

$$\begin{aligned} x_1 &= r\cos(\varphi_1), \\ x_2 &= r\sin(\varphi_1)\cos(\varphi_2), \\ x_3 &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3), \end{aligned}$$

$$\begin{aligned} &... \\ &... \end{aligned}$$

$$\begin{aligned} x_{n-1} &= r\sin(\varphi_1)\sin(\varphi_2)...\sin(\varphi_{n-2})\cos(\varphi_{n-1}), \\ x_n &= r\sin(\varphi_1)\sin(\varphi_2)...\sin(\varphi_{n-2})\sin(\varphi_{n-1}). \end{aligned}$$

The goal is to find a relationship between the determinant of the Jacobian for dimension n, $|J_n|$, and that of dimension $n-1$, $|J_{n-1}|$. It is important to note that in the case of dimension $n-1$, the coordinate x_{n-1} is different from that of the case shown above.

$$x_{n-1} = r\sin(\varphi_1)\sin(\varphi_2)...\sin(\varphi_{n-3})\sin(\varphi_{n-2}).$$

This will be important later, when it is necessary to relate the determinant of two matrices to that of J_{n-1}. Introducing the following notation: $\cos(\varphi_i) = c_i$ y $\sin(\varphi_i) = s_i$:

$$J_n = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_{n-1}}{\partial r} & \cdots & \frac{\partial x_{n-1}}{\partial \varphi_{n-1}} \\ \frac{\partial x_n}{\partial r} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{pmatrix}, \; J_{n-1} = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-2}} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_{n-2}}{\partial r} & \cdots & \frac{\partial x_{n-2}}{\partial \varphi_{n-2}} \\ \frac{\partial x_{n-1}}{\partial r} & \cdots & \frac{\partial x_{n-1}}{\partial \varphi_{n-2}} \end{pmatrix}.$$

We will expand $|J_n|$ by its last column. $|J_n| = \sum_{i=n,n-1} a_{in}(-1)^{i+n} det B_{in}$. Where B_{in} is the matrix element after removing the i-th row and n-th column. The summation will only have two terms, because the rest of the coefficients are zero.

$$\begin{pmatrix} 0 \\ 0 \\ ... \\ -rs_1...s_{n-1} \\ rs_1...c_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ... \\ a_{n-1,n} \\ a_{n,n} \end{pmatrix}.$$

On the other hand, the reduced matrices are as follows:

$$B_{n,n} = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-2}} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_{n-1}}{\partial r} & \cdots & \frac{\partial x_{n-1}}{\partial \varphi_{n-2}} \end{pmatrix}, \; B_{n-1,n} = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-2}} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial r} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-2}} \end{pmatrix}.$$

The expression of the determinant is

$$\begin{aligned} |J_n| &= rs_1...s_{n-2}c_{n-1}|B_{n,n}| + (-1)(-rs_1...s_{n-2}s_{n-1}|B_{n-1,n}| \\ &= rs_1...s_{n-2}(|B_{n,n}|c_{n-1} + |B_{n-1,n}|s_{n-1}). \end{aligned}$$

As mentioned at the beginning of this section, it is now time to relate the determinant of the matrices B to that of J_{n-1}. Given that

$$\begin{aligned}(x_{n-1})_{d=n} &= rs_1s_2...s_{n-2}c_{n-1} = (x_{n-1})_{d=n-1} * c_{n-1},\\ (x_n)_{d=n} &= rs_1s_2...s_{n-2}s_{n-1} = (x_{n-1})_{d=n} * t_{n-1} = (x_{n-1})_{d=n-1} * s_{n-1},\end{aligned}$$

and that then, the relationship between the partial derivatives of the matrix elements is related as

$$(\frac{\partial x_n}{\partial \varphi_j})_{d=n} = rs_1s_2...c_js_{n-1} = (\frac{\partial x_{n-1}}{\partial \varphi_j})_{d=n} * t_{n-1} = (\frac{\partial x_{n-1}}{\partial \varphi_j})_{d=n-1} * s_{n-1},$$

We obtained the following resulting expressions:

$$\begin{aligned}|B_{n,n}| &= c_{n-1}|J_{n-1}|,\\ |B_{n-1,n}| &= s_{n-1}|J_{n-1}|.\end{aligned}$$

Substituting the results into the previous equation, the relationship between the Jacobian for $d = n$ and that for $d = n - 1$ is:

$$|J_n| = |J_{n-1}|(c_{n-1}^2 + s_{n-1}^2)r\prod_{i=1}^{n-2} s_i = |J_{n-1}|r\prod_{i=1}^{n-2} s_i. \tag{2.4}$$

If we calculate the Jacobian in two dimensions (polar coordinates); $x = r\cos(\theta)$, $y = r\sin(\theta)$, the result will be $J = |r\cos^2(\theta) - (-r\sin^2(\theta))| = r$. Working in spherical coordinates with three dimensions; $x = r\sin(\theta)\cos(\varphi), y = r\sin(\theta)\sin(\varphi), z = r\cos(\theta)$, the Jacobian takes the form $J = r^2\sin(\theta)$.

Let's propose the expression:

$$|J_n| = r^{n-1}\prod_{i=1}^{n-2} s_i^{n-1-i}. \tag{2.5}$$

For both two-dimensional and three-dimensional coordinates (the previous cases), we recover the true result. $n = 1$, returns $J = 1$, as expected. If we assume this is correct for a dimension k, let's see what happens when we work in dimension $k + 1$ and substitute into (7):

$$|J_{k+1}| = |J_k|r\prod_{i=1}^{n-1} s_i = rr^{k-1}\prod_{i=1}^{k-1} s_i\prod_{i=1}^{k-2} s_i^{k-1-i} = r^k\prod_{i=1}^{k-1} s_i^{k-i}.$$

We retrieve the proposed result.

Now the goal will be to find the integral in angles of a differential volume in n dimensions:

$$dV = r^{n-1}\prod_{i=1}^{n-2} s_i^{n-1-i}drd\varphi_1...d\varphi_{n-2}d\varphi_{n-1}.$$

By integrating the differential volume over our entire space, we will find the volume we are looking for.

$$V = \int_0^R r^{n-1}dr\int_0^{2\pi} d\varphi_{n-1}\int_0^{\pi}\prod_{i=1}^{n-2} sen(\varphi_i)^{n-1-i}d\varphi_i.$$

The integral over $d\varphi_{n-1}$ will contribute a factor of 2π. To solve the integrals of the sines, we will divide the interval, obtaining two integrals from 0 to $\frac{\pi}{2}$. Furthermore, since the

integration variables are independent, the product of the sines becomes the product of the integrals. I will return to the usual notation to avoid confusion when solving them.

$$\int_0^{\pi} sen(\varphi_i)^{n-1-i} d\varphi_i = 2\int_0^{\pi/2} sen(\varphi_i)^{n-1-i} d\varphi_i.$$

Through the following change of variable: $t = sen^2(\varphi_i) \rightarrow d\varphi_i = \frac{dt}{2\sqrt{t}\sqrt{1-t}}$, we will recover the expression of the beta function:

$$I_i = \int_0^1 t^{\frac{n-i}{2}-1}(1-t)^{-1/2} dt = B(\frac{n-i}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{n-i}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-i}{2}+\frac{1}{2})}.$$

When contributing to the integral, we will have a factor $\pi^{\frac{n-2}{2}}$ due to the $\Gamma(1/2)$. As for the product, the terms in the numerator will cancel out with those in the denominator at the next value of i, which moves from 1 to $(n-2)$. Only one $\Gamma(n/2)$ remains in the denominator when $i = 1$ and one $\Gamma(1) = 1$ in the numerator when $i = n-2$. The angular integral is therefore:

$$\boxed{\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}.} \tag{2.6}$$

We can finish calculating the volume of a sphere of radius R in dimension n by simply solving for the additional radial term. This allows us to verify that we recover the known results for dimensions two and three, and guarantee that our result is correct.

$$V = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^R r^{n-1} dr = \frac{R^n 2\pi^{n/2}}{n\Gamma(\frac{n}{2})} = \frac{R^n \pi^{n/2}}{\Gamma(\frac{n+2}{2})}.$$

If $n = 2$, then $V = \pi R^2$, which is clearly the area of a circle. If $n = 3$, then $V = \frac{4\pi R^3}{3}$, which is the known volume of a three-dimensional sphere.

Chapter 3

Renormalization of the NN ($^1S_0$) interaction by pion exchange in the presence of contact interactions

The following potential employees are mentioned in the original paper:

$$V_\pi(p,p') = \frac{-2m\alpha_\pi}{\pi}\int_{-1}^{1}\frac{dx}{\vec{p^2}+\vec{p'^2}-2pp'x+m_\pi^2} = \frac{m\alpha_\pi}{\pi}\frac{1}{pp'}\log\frac{p_-^2+m_\pi^2}{p_+^2+m_\pi^2}, \quad (3.1)$$

$$V_s(p,p') = g_0 + g_1(p^2+p'^2). \quad (3.2)$$

where $p\pm = p \pm p'$ and $\alpha_\pi = \frac{g_A^2 m_\pi^2}{16\pi f_\pi^2}$. Here, $g_A = 1.25$ is the axial coupling constant, $m_\pi = 138$ MeV is the pion mass, and $f_\pi = 93$ MeV is its weak decay constant.

The exchange potential constants of a pion are assumed to be known, given by other experiments. We will thus assume α_π to be a known parameter, and $m = 469.46$ MeV to be the reduced mass of the NN system in question.

On the other hand, the short-range constants are not initially determined. Note that V_π is not expressed in the usual form for the Yukawa potential in momentum space, since the potential has been redefined after the angular contribution has been integrated. This simplifies the integral over the entire space, as only the radial term is needed.

The Lippmann-Schwinger equation for the total scattering matrix is given by:

$$T(E;p,p') = V(p,p') + \int_\infty^0 dqq^2\frac{V(p,q)T(E;q,p')}{2mE-q^2+i\epsilon}. \quad (3.3)$$

Where $V = V_\pi + V_s$. To clarify, most calculations use the notation $T(p,p';k) \equiv T(p,p')$. The normalization of the probability amplitude is such that for $p = p' = \sqrt{2mE} = k$ (on-shell), it is related to the phase shift δ(k) by:

$$T(k,k;k) = -\frac{2}{\pi}\frac{e^{2i\delta(k)}-1}{2ik}. \quad (3.4)$$

The scattering matrix for V_π alone will be called T_π. This follows the same LSE as in the total case but without the presence of the potential at short distances. T_s describes the scattering between distorted waves.

$$T_\pi = V_\pi + V_\pi G_0 T_\pi, \quad (3.5)$$

$$T_s = V_s + V_s G_\pi T_s = V_s + V_s G_0 T_s + V_s G_0 T_\pi G_0 T_s. \quad (3.6)$$

Operator notation will be used for this first part of the development. G_0 refers to the free Green's function for a nucleon, represented by

$$\langle q'|G_0(E+i\epsilon)|q\rangle = \frac{1}{(k^2-q^2+i\epsilon)}\delta(q-q').$$

Since the angular projection has already been taken into account when projecting ${}^1S_0$ onto the wave.

The first step will be to propose a solution for T that satisfies its corresponding equation.

$$T = T_\pi + (I+T_\pi G_0)T_s(I+G_0T_\pi) = T_\pi + T_s + T_\pi G_0 T_s + T_s G_0 T_\pi + T_\pi G_0 T_s G_0 T_\pi.$$

Inserting in (3.3):

$$\begin{aligned} T &= V + VG_0T = (V_\pi+V_s) + (V_\pi+V_s)G_0(T_\pi + (I+T_\pi G_0)T_s(I+G_0T_\pi)) \\ &= (V_\pi+V_s) + (V_\pi+V_s)G_0(T_\pi + T_s + T_\pi G_0 T_s + T_s G_0 T_\pi + T_\pi G_0 T_s G_0 T_\pi) \\ &= V_\pi + V_s + V_\pi G_0 T_\pi + V_\pi G_0 T_s + V_\pi G_0 T_\pi G_0 T_s + V_\pi G_0 T_s G_0 T_\pi + V_\pi G_0 T_\pi G_0 T_s G_0 T_\pi \\ &+ V_s G_0 T_\pi + V_s G_0 T_s + V_s G_0 T_\pi G_0 T_s + V_s G_0 T_s G_0 T\pi + V_s G_0 T_\pi G_0 T_s G_0 T_\pi. \end{aligned}$$

Factoring out the common factor G_0T_i:

$$\begin{aligned} T &= (V_\pi + V_\pi G_0 T_\pi) + (V_s + V_s G_0 T_s + V_s G_0 T_\pi G_0 T_s) + (V_\pi + V_\pi G_0 T_\pi) G_0 T_s \\ &+ (V_s + V_s G_0 T_s + V_s G_0 T_\pi G_0 T_s) G_0 T_\pi + (V_\pi + V_\pi G_0 T_\pi) G_0 T_s G_0 T_\pi \\ &= T_\pi + T_s + T_\pi G_0 T_s + T_s G_0 T_\pi + T_\pi G_0 T_s G_0 T_\pi. \end{aligned}$$

We verify that this is a possible solution since (3.5) and (3.6) are satisfied simultaneously.

Starting from the definition of T_s.

$$T_s(p,p') = V_s(p,p') + \int_0^\infty dqq^2 \frac{V_s(p,q)T_s(q,p')}{(k^2-q^2+i\epsilon)} + \int_0^\infty dqdq' \frac{q^2q'^2V_s(p,q)T_\pi(q,q')T_s(q',p')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)},$$

It is possible to reduce it to an expression of the type:

$$T_s(p,p') = \alpha + \beta(p^2+p'^2) + \gamma p^2 p'^2.$$

Let's look at it in detail, but first, both for the solution of the coefficients α, β, γ, as well as for the rest of the calculations, it is convenient to introduce at this point three important integrals.

$$I_n = \int_0^\infty dq \frac{q^{n+2}}{(k^2-q^2+i\epsilon)}, \quad J_n = \int_0^\infty \frac{dqdq'q^aq'^bT_\pi(q,q')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)}, \quad L_n = \int_0^\infty \frac{dqq^{n+2}T_\pi(q,k)}{(k^2-q^2+i\epsilon)}.$$

Inserting T_s into the Lippman-Schwinger equation:

$$\begin{aligned} T_s(p,p') &= g_0 + g_1p^2 + g_1p'^2 + g_0\int_0^\infty \frac{dqq^2(\alpha+\beta p'^2+\beta q^2+\gamma p'^2q^2)}{(k^2-q^2+i\epsilon} \\ &+ g_1p^2\int_0^\infty \frac{dqq^2(\alpha+\beta p'^2+\beta q^2+\gamma p'^2q^2)}{(k^2-q^2+i\epsilon} + g_1\int_0^\infty \frac{dqq^4(\alpha+\beta p'^2+\beta q^2+\gamma p'^2q^2)}{(k^2-q^2+i\epsilon} \\ &+ g_0\int_0^\infty \frac{dqdq'q^2q'^2T_\pi(q,q')(\alpha+\beta q'^2+\beta p'^2+\gamma q'^2p'^2)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} \\ &+ g_1p^2\int_0^\infty \frac{dqdq'q^2q'^2T_\pi(q,q')(\alpha+\beta q'^2+\beta p'^2+\gamma q'^2p'^2)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} \\ &+ g_1\int_0^\infty \frac{dqdq'q^4q'^2T_\pi(q,q')(\alpha+\beta q'^2+\beta p'^2+\gamma q'^2p'^2)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)}, \end{aligned}$$

we can recognize the expressions of the integrals I_n and J_n recently presented.

$$\begin{aligned} T_s &= (g_0 + \alpha g_0 K_0 + \beta g_0 K_2 + g_1 \alpha K_2 + g_1 \beta K_4) + p^2 p'^2 (g_1 \beta K_0 + g_1 \gamma K_2) \\ &+ p^2 (g_1 + g_1 \alpha K_0 + g_1 \beta K_2) + p'^2 (g_1 + g_0 \beta K_0 + g_0 \gamma K_2 + g_1 \beta K_2 + g_1 \gamma K_4). \end{aligned}$$

We have used the fact that $K_n = I_n + J_n$. Equating this expression to the proposed one, we find a system of three equations for three unknowns:

$$\begin{aligned} g_0 + \alpha g_0 K_0 + \beta g_0 K_2 + g_1 \alpha K_2 + g_1 \beta K_4 &= \alpha. \\ g_1 \beta K_0 + g_1 \gamma K_2 &= \gamma. \\ g_1 + g_0 \beta K_0 + g_0 \gamma K_2 + g_1 \beta K_2 + g_1 \gamma K_4 = g_1 + g_1 \alpha K_0 + g_1 \beta K_2 &= \beta. \end{aligned}$$

The solution of the system is:

$$\begin{aligned} \alpha &= \frac{g_0 + g_1^2 K_4}{\Delta}, \\ \beta &= \frac{g_1 - g_1^2 K_2}{\Delta}, \\ \gamma &= \frac{g_1^2 K_0}{\Delta}. \end{aligned}$$

Being

$$\Delta = 1 - g_0 K_0 - 2 g_1 K_2 + g_1^2 K_2^2 - g_1^2 K_0 K_4.$$

I will begin by solving the first of the integrals.

$$I_n = \int_0^\infty dq \frac{q^{n+2}}{k^2 - q^2 + i\epsilon} = -\int_0^\infty dq \frac{q^{n+2}}{q^2 + (-(k^2 + i\epsilon))}. \tag{3.7}$$

We will be interested in even powers ($n = 0, 2, 4...$), so I will rewrite the integral as follows:

$$I_{2n} = -\int_0^\infty dq \frac{q^{2n+2}}{q^2 + (-(k^2 + i\epsilon))},$$

where now n can be any integer. Applying the result (2.3):

$$I_{2n} = (-1) \frac{(-k^2 - i\epsilon)^{n+\frac{1}{2}}}{2} \frac{\Gamma(\frac{2n+3}{2}) \Gamma(-(\frac{2n-1}{2}))}{\Gamma(1)} = (-1)^{n+1} k^{2n} \frac{\pi}{\sin(\pi(n + \frac{3}{2}))} \frac{(-k^2 - i\epsilon)^{\frac{1}{2}}}{2}.$$

We encounter the square root of a negative number, which will result in $\pm ik$, giving two possibilities. Our function exhibits singularities when the denominator becomes zero. By definition, $q^2 \geq 0$, and $0 < \epsilon \ll 1$, so it will only vanish when we have the negative root. The branching cut will be a left cut. That is, $(-k^2 - i\epsilon)^{\frac{1}{2}} = -ik$ because it approaches from below the cut.

Realizing that $\sin(\pi(n + \frac{3}{2})) = (-1)^{n+1}$:

$$I_{2n} = (-1)^{n+1} k^{2n} (-1)^{n+1} \frac{(-1) i\pi k}{2} = -ik^{2n} \frac{k\pi}{2}.$$

For convenience, it is renamed back to its original name.

$$\boxed{I_n = -i\frac{k^{n+1}\pi}{2}.} \tag{3.8}$$

Not forgetting that n can only be even. This result will be necessary when finding recurrence rules for the other two integrals, which will be presented in due course.

We will be interested in knowing T for the particular on-shell case.

$$\begin{aligned} T(k,k) &= T_\pi(k,k) + T_s(k,k) + T_\pi(k,q)G_0(q)T_s(q,k) + T_s(k,q)G_0(q)T_\pi(q,k) \\ &+ T_\pi(k,q)G_0(q)T_s(q,q')G_0(q')T_\pi(q',k). \end{aligned}$$

I will work out the last terms individually. The L_n integrals are easily identifiable, and we can express the result in terms of them.

$$\begin{aligned} T_\pi(k,q)G_0(q)T_s(q,k) &= \int_0^\infty dqq^2 \frac{T_\pi(k,q)(\alpha+\beta(q^2+k^2)+\gamma k^2q^2)}{(k^2-q^2+i\epsilon)} \\ &= \alpha L_0 + \beta k^2 L_0 + \beta L_2 + \gamma k^2 L_2, \\ T_s(k,q)G_0(q)T_\pi(q,k) &= \int_0^\infty dqq^2 \frac{(\alpha+\beta(k^2+q^2)+\gamma q'^2k^2)T_\pi(k,q)}{(k^2-q^2+i\epsilon)} \\ &= \alpha L_0 + \beta k^2 L_0 + \beta L_2 + \gamma k^2 L_2, \\ T_\pi(k,q)G_0(q)T_s(q,q')G_0(q')T_\pi(q',k) &= \int_0^\infty dqdq'q^2q'^2 \frac{T_\pi(k,q)(\alpha+\beta(q^2+q'^2)+\gamma q^2q'^2)T_\pi(k,p')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} \\ &= \alpha L_0^2 + 2\beta L_0 L_2 + \gamma L_2^2. \end{aligned}$$

Putting everything together and rewriting T:

$$T(k) = T_\pi(k) + T_s(k) + 2\{(\alpha+\beta k^2)L_0 + (\beta+\gamma k^2)L_2\} + \alpha L_0^2 + 2\beta L_0 L_2 + \gamma L_2^2.$$

From this point forward, I will adopt the notation $T_\pi(k,k;k) \equiv T_\pi(k)$.

Next, I will write the expression for the Yukawa potential in momentum space.

$$V_\pi(p,p') = -g'^2 \frac{1}{(\vec{p}-\vec{p'})^2 + m_\pi^2}.$$

The integral over the entire space is defined by:

$$\begin{aligned} \int d^3r &= 2\pi \int_0^\pi \sin\theta d\theta \int_0^\infty r^2 dr \\ &= 2\pi \int_{-1}^1 dx \int_0^\infty r^2 dr, \end{aligned}$$

with the clear replacement of $x = \cos\theta$.

If we now perform the integral over the entire potential space, and redefine V_π as the contribution of the angular integral itself. (Note that $\vec{p}\cdot\vec{p'} = pp'\cos\theta$)

$$V_\pi = \frac{-2m\alpha_\pi}{\pi} \int_{-1}^1 \frac{dx}{\vec{p^2} + \vec{p'^2} - 2pp'x + m_\pi^2}.$$

It remains to be seen

$$\int_0^\infty dpp^2 V_\pi(p,p').$$

To do this, we will perform the following integral in dimensional regularization:

$$\int \frac{1}{(2\pi)^3} \frac{d^3p}{(\vec{p}-\vec{p'})^2 + m_\pi^2}.$$

We are working across the entire space, therefore the integral cannot ultimately depend on p'. I will perform a translation such that we call $\vec{q} = \vec{p} - \vec{p'}$.

$$\frac{1}{(2\pi)^3}\int_0^\infty \frac{d^3q}{\vec{q}^2+m_\pi^2} = \frac{1}{(2\pi)^3}\int d\Omega_3 \int_0^\infty \frac{dq\cdot q^2}{\vec{q}^2+m_\pi^2} = \frac{1}{(2\pi)^3}\frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})}\frac{m_\pi\Gamma(\frac{3}{2})\Gamma(\frac{-1}{2})}{2\Gamma(1)}.$$

Under the same procedure:

$$I_\pi = \frac{-2m\alpha_\pi}{\pi}\int dpp^2\int_{-1}^1 \frac{dx}{(\vec{p}-\vec{p'})^2+m_\pi^2} = \frac{-2m\alpha_\pi}{\pi}\cdot\frac{m_\pi\Gamma(\frac{3}{2})\Gamma(\frac{-1}{2})}{2}\int_{-1}^1 dx.$$

Substituting $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2$ y $\Gamma(\frac{-1}{2}) = -2\sqrt{\pi}$,

$$\boxed{\int_0^\infty dqq^2V_\pi(q,q') = 2m\alpha_\pi m_\pi.} \tag{3.9}$$

As mentioned earlier, it is independent of p' and depends only on m_π.

Similarly, we can calculate the integral

$$\int_0^\infty dqq^4V_\pi(q,p').$$

When performing the above translation, we will have two additional terms due to the square of the numerator.

$$\begin{aligned}
\int_0^\infty dqq^4V_\pi(q,p') &= -\frac{2m\alpha_\pi}{\pi}\int_{-1}^1 dx\int_0^\infty dq\frac{q^2(\vec{q}+\vec{p'})^2}{\vec{q}^2+m_\pi^2}\\
&= -\frac{2m\alpha_\pi}{\pi}\int_{-1}^1 dx\left(\int_0^\infty dq\frac{q^4}{q^2+m_\pi^2} + p'^2\int_0^\infty dq\frac{q^2}{q^2+m_\pi^2} + 2\int_0^\infty dq\frac{q^2\vec{q}\cdot\vec{p'}}{q^2+m_\pi^2}\right)\\
&= -\frac{2m\alpha_\pi}{\pi}2\left(\frac{\Gamma(\frac{5}{2})\Gamma(\frac{-3}{2})m_\pi^3}{2} + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{-1}{2})p'^2m_\pi}{2}\right)\\
&= -\frac{2m\alpha_\pi}{\pi}(\pi m_\pi^3 - \pi m_\pi p'^2)\\
&= 2m\alpha_\pi m_\pi(p'^2 - m_\pi^2).
\end{aligned}$$

$$\boxed{\int_0^\infty dqq^4V_\pi(q,q') = 2m\alpha_\pi m_\pi(q'^2 - m_\pi^2).} \tag{3.10}$$

Note that the third integral is zero. Consider, for example, the case where $\vec{p'}$ has direction $\hat{z}$. In this case, we will have an odd integrand integrated over the entire space, resulting in a zero contribution to the integral.

To compile results, we can define:

$$I_\pi^n(q'^2) = \int dqq^{2+2n}V_\pi(q,q').$$

Where (3.9) corresponds to the case $n = 0$ and (3.10) to that of $n = 1$. (Note that $I_\pi^1 = I_\pi^0(q'^2 - m_\pi^2)$).

We can go further and relate the integral L_2 to L_0 by taking advantage of this last result. Now representing $G_0(p) = \int_0^\infty dp\frac{p^2}{k^2-p^2+i\epsilon}$:

$$L_2 = \int_0^\infty \frac{dqq^4V_\pi(k,q)}{(k^2-q^2+i\epsilon)} + \int_0^\infty \frac{dqq^4V_\pi(k,p)}{(k^2-q^2+i\epsilon)}\int_0^\infty \frac{dpp^2T_\pi(p,q)}{(k^2-p^2+i\epsilon)},$$

and applying the equality $\frac{q^4}{(k^2-q^2)} = \frac{k^2q^2}{(k^2-q^2)} - q^2$, we rewrite the two contributions as:

$$\begin{aligned}
\int_0^\infty \frac{dqq^4V_\pi(k,q)}{(k^2-q^2+i\epsilon)} &= \int_0^\infty k^2\frac{dqq^2V_\pi(k,q)}{(k^2-q^2+i\epsilon)} - \int_0^\infty dqq^2V_\pi(k,q), \\
\int_0^\infty \frac{dqq^4V_\pi(k,p)}{(k^2-q^2+i\epsilon)}\int_0^\infty \frac{dpp^2T_\pi(p,q)}{(k^2-p^2+i\epsilon)} &= \int_0^\infty k^2\frac{dqq^2V_\pi(k,p)}{(k^2-q^2+i\epsilon)}\frac{dpp^2T_\pi(p,q)}{(k^2-q^2+i\epsilon)} \\
&- \int_0^\infty dqq^2V_\pi(k,p)\int_0^\infty \frac{dpp^2T_\pi(p,q)}{(k^2-p^2+i\epsilon)}.
\end{aligned}$$

The final expression results:

$$\begin{aligned}
L_2(k) &= k^2\int_0^\infty \frac{dqq^2(V_\pi(k,q)+V_\pi(k,p)G_0(p)T_\pi(p,q))}{(k^2-q^2+i\epsilon)} - \int_0^\infty dqq^2V_\pi(k,q) \\
&- \int_0^\infty dqq^2V_\pi(k,p)\int_0^\infty \frac{dpp^2T_\pi(p,q)}{(k^2-p^2+i\epsilon)}.
\end{aligned}$$

Writing $k^2 = 2mE$ and identifying (3.9).

$$\boxed{L_2(k) = (2mE - 2m\alpha_\pi m_\pi)L_0(k) - 2m\alpha_\pi m_\pi.} \tag{3.11}$$

An analogous development can be made for integrals.

$$J_n = \int_0^\infty \frac{dqdq'q^aq'^bT_\pi(q,q')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)},$$

where (a,b) corresponds to $(4,4)$, $(4,2)$, $(2,2)$ for $n = 0,\ 2,\ 4$ respectively.

$$J_2 = \int_0^\infty \frac{dqdq'q^4q'^2V_\pi(q,q')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} + \int_0^\infty \frac{dqdq'q^4q'^2V_\pi(q,p)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)}\int_0^\infty \frac{dpp^2}{(k^2-p^2+i\epsilon)}T_\pi(p,q').$$

Expressing again $\frac{q^4}{(k^2-q^2)} = \frac{k^2q^2}{(k^2-q^2)} - q^2$, we obtain a total of four terms.

Looking at the first integral:

$$\int_0^\infty \frac{dqdq'q^4q'^2V_\pi(q,q')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} = \int_0^\infty k^2\frac{dqdq'q^2q'^2V_\pi(q,q')}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} - \int_0^\infty \frac{dqdq'q^2q'^2V_\pi(q,q')}{(k^2-q'^2+i\epsilon)}.$$

and similarly for the second:

$$\begin{aligned}
&\int_0^\infty \frac{dqdq'q^4q'^2V_\pi(q,p)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)}\int_0^\infty \frac{dpp^2}{(k^2-p^2+i\epsilon)}T_\pi(p,q') = \\
&\int_0^\infty k^2\frac{dqdq'q^2q'^2V_\pi(q,p)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)}\int_0^\infty \frac{dpp^2}{(k^2-p^2+i\epsilon)}T_\pi(p,q') \\
&- \int_0^\infty \frac{dqdq'q^2q'^2V_\pi(q,p)}{(k^2-q'^2+i\epsilon)}\int_0^\infty \frac{dpp^2}{(k^2-p^2+i\epsilon)}T_\pi(p,q').
\end{aligned}$$

$G_0(p)$, Let's remember, it acts as $\int_0^\infty \frac{dpp^2}{(k^2-p^2+i\epsilon)}$. Factoring out the common factor will give us:

$$\begin{aligned}
J_2 &= k^2\int_0^\infty \frac{dqdq'q^2q'^2(V_\pi + V_\pi G_0T_\pi)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} - \int_0^\infty \frac{dqdq'q^2q'^2V_\pi}{(k^2-q'^2+i\epsilon)} \\
&- \int_0^\infty dqq^2V_\pi\int_0^\infty \frac{dpdq'p^2q'^2T_\pi}{(k^2-p^2+i\epsilon)(k^2-q'^2+i\epsilon)} \\
&= \left(k^2 - \int_0^\infty dqq^2V_\pi\right)\cdot J_0 - \int_0^\infty dqq^2V_\pi\cdot I_2
\end{aligned}$$

Substituting (3.9) and taking into account that $k^2 = 2mE$, we arrive at the final expression. The development is analogous with J_4 and J_2, allowing us to relate these integrals to J_0, which has a logarithmic divergence and can be solved numerically by subtraction.

$$\boxed{\frac{J_n(E)}{2m} = (E - m_\pi\alpha_\pi)J_{n-2}(E) + i\frac{\pi}{2}m_\pi\alpha_\pi k^{n-1}.} \tag{3.12}$$

Applying this result, we get:

$$\begin{aligned} T(k) &= T_\pi(k) + T_s(k) + 2\alpha L_0 + 2\beta k^2 L_0 + 2\beta(k^2 - 2m\alpha_\pi m_\pi)L_0 - 2\beta m\alpha_\pi m_\pi \\ &+ 2\gamma k^2(k^2 - 2m\alpha_\pi m_\pi)L_0 - 2\gamma k^2 2m\alpha_\pi m_\pi + \alpha L_0^2 + 2\beta L_0^2(k^2 - 2m\alpha_\pi m_\pi) \\ &- 2\beta L_0 2m\alpha_\pi m_\pi + \gamma(k^2 - 2m\alpha_\pi m_\pi)L_0^2 + \gamma(2m\alpha_\pi m_\pi)^2 - 2\gamma L_0(k^2 - 2m\alpha_\pi m_\pi)2m\alpha_\pi m_\pi. \end{aligned}$$

Factoring out $2L_0$ and L_0^2:

$$\begin{aligned} T(k) &= T_\pi(k) + (\alpha + \beta(k^2 + k^2) + \gamma k^2 k^2 + \gamma(2m\alpha_\pi m_\pi)^2 - 2\gamma k^2 2m\alpha_\pi m_\pi - 2\beta 2m\alpha_\pi m_\pi) \\ &+ 2L_0(\alpha + \beta(k^2 - 2m\alpha_\pi m_\pi + k^2 - 2m\alpha_\pi m_\pi) + \gamma(k^2 - 2m\alpha_\pi m_\pi)(k^2 - 2m\alpha_\pi m_\pi)) \\ &+ L_0^2(\alpha + \beta(k^2 - 2m\alpha_\pi m_\pi + k^2 - 2m\alpha_\pi m_\pi) + \gamma(k^2 - 2m\alpha_\pi m_\pi)^2), \end{aligned}$$

and defining $\hat{k}^2 = k^2 - 2m\alpha_\pi m_\pi$:

$$\begin{aligned} T(k) &= T_\pi(k) + (\alpha + \beta(\hat{k}^2 + \hat{k}^2) + \gamma\hat{k}^2\hat{k}^2) + 2L_0(\alpha + \beta(\hat{k}^2 + \hat{k}^2) + \gamma\hat{k}^2\hat{k}^2) \\ &+ L_0^2(\alpha + \beta(\hat{k}^2 + \hat{k}^2) + \gamma\hat{k}^2\hat{k}^2). \end{aligned}$$

We can recognize $T_s(E;\hat{k},\hat{k})$ in each factor.

$$T(k) = T_\pi(k) + T_s(E;\hat{k},\hat{k}) + T_s(E;\hat{k},\hat{k})\cdot 2L_0 + T_s(E;\hat{k},\hat{k})\cdot L_0^2 = T_\pi(k) + (1 + L_0(k))^2 T_s(E;\hat{k},\hat{k}).$$

The final expression for the complete scattering matrix:

$$\boxed{T(k) = T_\pi(k) + (1 + L_0(k))^2 T_s(E;\hat{k},\hat{k}) = T_\pi(k) + \frac{(1+L_0(k))^2}{V_s^{-1}(\hat{k},\hat{k}) + i\pi k/2 - (\bar{J}_0(E) - J_0(0))}.} \tag{3.13}$$

Where it has been used:

$$\begin{aligned} T_s(\hat{k}) &= (1 - V_s(\hat{k})G_0(q) - V_s(\hat{k})G_0(q)T_\pi(q,q')G_0(q'))^{-1}V_s(\hat{k}) \\ &= (V_s(\hat{k})^{-1} - G_0(q) - G_0(q)T_\pi(q,q')G_0(q'))^{-1} = (V_s(\hat{k})^{-1} - I_0 - J_0)^{-1}. \end{aligned}$$

The integral J_0 is logarithmically divergent and requires only one subtraction: $\bar{J}_0(E) = J_0(E) - J_0(0)$ is finite and can be solved numerically.

From the equation (**??**), we can obtain the relationship between T and the scattering matrix. Working on-shell, $S(k)$ is simply the phase-off, and its modulus is 1.

$$S = 1 - i\pi kT.$$

Note that this does not correspond exactly to the expression shown at the beginning of the work in the theoretical introduction. The definition of T depends on conventions and can be written in different ways.

From the unitarity condition

$$SS^\dagger = 1,$$

we can derivate the Optical Theorem that satisfies T_π:

$$(1-ik\pi T)(1+ik\pi T^\dagger)=1=1-i\pi kT^\dagger+i\pi kT+(k\pi)^2TT^\dagger\rightarrow 2ImT=-\pi kTT^\dagger.$$

And returning to our usual notation, the optical of-shell theorem is:

$$\frac{T(p_1,p_2)-T^*(p_1,p_2)}{i}=2ImT_\pi(E;p_1,p_2)=-\pi kT_\pi(p_1,k)T^*_\pi(k,p_2). \tag{3.14}$$

Let's recall our expression for L_0:

$$L_0(k)=\int_0^\infty dq\frac{q^2T_\pi(k,q)}{(k^2-q^2+i\epsilon)},$$

which rewriting it in terms of the Cauchy principal value:

$$\lim_{\epsilon\to 0}\int_0^\infty dq\frac{q^2T_\pi(k,q)}{(k^2-q^2+i\epsilon)}=\mathcal{P}\int_0^\infty dq\frac{q^2T_\pi(k,q)}{k^2-q^2}-i\pi k/2T_\pi(k).$$

Denoting $p_1=k$:

$$2ImT_\pi(E;k,p_2)=-\pi kT_\pi(k,k)T^*_\pi(k,p_2),$$

and know $p_2=k$;

$$2ImT_\pi(E;p_1,k)=-\pi kT_\pi(k,p_1)T^*_\pi(k,k).$$

Since T is a symmetric matrix, we can integrate and equate both expressions obtained.

$$T_\pi(k)\mathcal{P}\int_0^\infty dq\frac{q^2T^*_\pi(k,q)}{(k^2-q^2)}=T^*_\pi(k)\mathcal{P}\int_0^\infty dq\frac{q^2T_\pi(k,q)}{(k^2-q^2)}.$$

We reach the case where a complex number is equal to its conjugate. That is, it is a real number.

$$l_0'(k)=T^*_\pi(k)\mathcal{P}\int_0^\infty dq\frac{q^2T_\pi(k,q)}{(k^2-q^2)}.$$

Multiplying both sides of the equation by $T_\pi(k)$:

$$T_\pi(k)l_0'=|T_\pi(k)|^2\mathcal{P}\int_0^\infty dq\frac{q^2T_\pi(k,q)}{(k^2-q^2)}\rightarrow\mathcal{P}\int_0^\infty dq\frac{q^2T_\pi(k,q)}{(k^2-q^2)}=T_\pi(k)l_0, \tag{3.15}$$

we find that

$$L_0(k)=(l_0(k)-i\pi k/2)T(k). \tag{3.16}$$

We will need to apply the optical theorem to the on-shell case. This will allow us to arrive at a relationship between the real and imaginary parts of $T_\pi(k,k;k)$. Generalizing for $p_1=p_2=k$:

$$2ImT_\pi(k)=-\pi k|T_\pi(k)|^2\rightarrow ImT_\pi(k)=-\pi k|T_\pi(k)|^2/2.$$

The ratio of the real and imaginary parts can therefore be rewritten as:

$$\frac{-\pi kReT_\pi(k)}{ImT_\pi(k)}=\frac{2ReT_\pi(k)}{|T_\pi(k)|^2}=2Re\frac{T_\pi(k)}{T_\pi(k)\cdot T^*_\pi(k)}=2ReT_\pi(k)^{-1}. \tag{3.17}$$

It is also possible to deduce that

□

$$Im\bar{J}_0(k) = -\{(\pi k/2)^2 - l_0(l_0 + 2ReT_\pi^{-1}(k))\}ImT_\pi(k), \tag{3.18}$$

where l_0 is a real number.
Let's rewrite our equation (3.13) so that unitarity is readily apparent using (3.18) and (3.16).

$$\begin{aligned} T(k) &= T_\pi(k) + \frac{(1+L_0(k))^2}{V_s^{-1}(\hat{k},\hat{k}) + i\pi k/2 - (J_0(\bar{E}) - J_0(0))} \\ &= \frac{T_\pi(k)(V_s^{-1}(\hat{k},\hat{k}) + i\pi k/2 - ReJ_0(k) - iImT_\pi(k)\{l_0^2 + 2l_0ReT_\pi(k) - (\pi k/2)^2\})}{V_s^{-1}(\hat{k},\hat{k}) + i\pi k/2 - ReJ_0(k) - iImT_\pi(k)\{l_0^2 + 2l_0ReT_\pi(k) - (\pi k/2)^2\}} \\ &+ \frac{T_\pi(k)\{\frac{1}{T_\pi(k)} + 2l_0 - i\pi k + T_\pi(l_0^2 - (\pi k/2)^2 - i\pi k l_0)\}}{V_s^{-1}(\hat{k},\hat{k}) + i\pi k/2 - ReJ_0(k) - iImT_\pi(k)\{l_0^2 + 2l_0ReT_\pi(k) - (\pi k/2)^2\}}. \end{aligned}$$

By definition, the matrix T_π is unitary, so if we can express $T^{-1} = T_\pi^{-1} + \mathbb{R}$, where $\mathbb{R}$ is a real number, we guarantee that T is also a real number.

For convenience, I will omit the on-shell notation for now, although the calculations refer to that specific case. We have obtained an expression of the type $T = \frac{T_\pi \alpha}{\beta}$:

$$\frac{1}{T} = \frac{1}{T_\pi} + \frac{\beta}{T_\pi \alpha} - \frac{1}{T_\pi} = \frac{1}{T_\pi} - \frac{\alpha - \beta}{T_\pi \alpha}.$$

Being

$$\begin{aligned} \alpha &= V_s^{-1} + i\pi k/2 - ReJ_0 - iImT_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2) + \frac{1}{T_\pi} \\ &+ 2l_0 - i\pi k + T_\pi(l_0^2 - (\pi k/2)^2 - i\pi k l_0), \\ \beta &= V_s^{-1} + i\pi k/2 - ReJ_0 - iImT_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2). \end{aligned}$$

Thanks to the optical theorem, we can find the imaginary part of T_π^{-1}:

$$\frac{1}{T_\pi} = \frac{ReT_\pi - iImT_\pi}{|T_\pi|^2} = ReT_\pi^{-1} + i\pi k/2.$$

Furthermore, it will be convenient to express the following term in an alternative way.

$$\begin{aligned} T_\pi(l_0^2 - (\pi k/2)^2 - i\pi k l_0) &= T_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2) - 2l_0T_\pi(ReT_\pi^{-1} + i\pi k/2) \\ &= T_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2) - 2l_0. \end{aligned}$$

Several terms will cancel in the numerator and denominator.

$$\begin{aligned} \alpha - \beta &= ReT_\pi^{-1} - i\pi k/2 + T_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2), \\ \alpha &= V_s^{-1} - ReJ_0 + ReT_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2). \end{aligned}$$

Regrouping everything:

$$\frac{1}{T} = \frac{1}{T_\pi} - \frac{ReT_\pi^{-1} - i\pi k/2 + T_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2)}{T_\pi(V_s^{-1} - ReJ_0 + ReT_\pi(l_0^2 + 2l_0ReT_\pi - (\pi k/2)^2))}.$$

If we now multiply the top and bottom by T_π^*, and recognize T_π accompanying both terms of the numerator.

$$ReT_\pi^{-1} - i\pi k/2 = \frac{T_\pi^*}{|T_\pi|^2}.$$

□

We obtain:

$$\begin{aligned}\frac{1}{T} &= \frac{1}{T_\pi} - \frac{\frac{T_\pi^*}{|T_\pi|^2}(\frac{T_\pi}{|T_\pi|^2} + T_\pi(l_0^2 + 2l_0 ReT_\pi - (\pi k/2)^2)}{V_s^{-1} - ReJ_0 + ReT_\pi(l_0^2 + 2l_0 ReT_\pi - (\pi k/2)^2)} \\ &= \frac{1}{T_\pi} - \frac{\frac{1}{|T_\pi|^2} - (\pi k/2)^2 + l_0^2 + 2l_0 ReT_\pi^{-1}}{V_s^{-1} - ReJ_0 + ReT_\pi(l_0^2 + 2l_0 ReT_\pi - (\pi k/2)^2)} \\ &= \frac{1}{T_\pi} - \frac{(ReT_\pi)^2 + l_0^2 + 2l_0 ReT_\pi^{-1}}{V_s^{-1} - ReJ_0 + ReT_\pi(l_0^2 + 2l_0 ReT_\pi - (\pi k/2)^2)}.\end{aligned}$$

Calling the real quantity $g(k) = V_s^{-1}(\hat{k}, \hat{k}) - ReJ_0(k) + \frac{Im\bar{J}_0(k)}{\tan\delta_\pi(k)}$ and recognizing $\tan\delta_\pi(k) = -\frac{\pi k}{2ReT_\pi^{-1}}$ by the relations (3.17), (3.18):

$$\boxed{\frac{1}{T(k)} = \frac{1}{T_\pi(k)} - \frac{(l_0(k) + ReT_\pi^{-1}(k))^2}{ReT_\pi^{-1}(k) + g(k)}.} \tag{3.19}$$

Up to this point, dimensional regularization has led to a drastic reduction in the UV integrals. Our only problem now is $J_0(0)$, which should be absorbed by redefining the potential constants V_s, provided that such changes are consistent with the symmetry principles.

For our particular case, this is not possible since we are not working at orders higher than two. Looking at the equation (3.19) and the expression for $g(k)$, it seems natural to define a normalized contact potential:

$$(V_s^R)^{-1}(\hat{k}, \hat{k}) = V_s^{-1}(\hat{k}, \hat{k}) - J_0(0),$$

leading to an infinite power series of $\hat{k}^2$. The LSE is based on the non-perturbative resolution of the probability amplitude, so in principle, infinite counterterms should also be included in the bare amplitude so that the sum of both is finite.

At each order of $\hat{k}^2$, the coefficient of the divergent counterterm series is fully defined, but leaves its finite part as an arbitrary constant. This results in the function $g(k)$ being completely unknown due to the appearance of divergences.

The solution adopted is as follows: For small moments, operators with more derivatives have less influence, so to avoid the predictive loss of the model based on free coefficients, relationships are imposed between the higher-order counterterms of the counterterm series and those of the potential expansion, so that we can recover the model's behavior at low energies without them having too much of an impact.

They are related in such a way that the amplitude can be written following the equation (3.19) and satisfies classical unitarity. In practice, this is equivalent to considering $J_0(0)$ as a renormalized and unknown parameter. I will denote the renormalized and finite quantity as $J_0^R(0)$ to differentiate it from the divergent integral.

Treating the divergent parameter as a renormalized constant now leaves us with three indeterminate parameters to be fixed using experimental data: g_0, g_1, and $J_0^R(0)$.

Chapter 4

Numerical resolution

Let's begin with the solution of $T_\pi(p,k;k)$ and the on-shell phase shift obtained for different energies. Recall that if $p=k$, the scattering and transfer matrices are related by the following expression:

$$S_\pi(k,k;k) = 1 - i\pi k T_\pi(k,k;k) = e^{2i\delta_\pi(k)}. \tag{4.1}$$

For the numerical solution, points p_j and weights w_j obtained with the GAUSS-L subroutine and matrix inversion using the ZGESV subroutine of LAPACK will be used. The theoretical treatment prior to matrix declaration will be shown.

$$T_\pi(p',p;k) = V_\pi(p',p) + \int dq \frac{q^2 V_\pi(p',q)}{k^2 - q^2 + i\epsilon} T_\pi(q,p;k).$$

Separating the integral into the principal Cauchy part and the constant term:

$$\lim_{\epsilon\to 0^+} \frac{f(x)}{x - x_0 + i\epsilon} = \mathcal{P}\frac{f(x)}{x-x_0} - i\pi f(x)\delta(x-x_0),$$

It results in:

$$T_\pi(p',p;k) = V_\pi(p',p) - i\frac{\pi k}{2} V_\pi(p',k) T_\pi(k,p;k) + \mathcal{P}\int dq \frac{q^2 V_\pi(p',q)}{k^2-q^2} T_\pi(q,p;k).$$

Taking advantage of the fact that the following integral has a zero value for $k^2 > 0$:

$$\mathcal{P}\int \frac{dq}{k^2-q^2} = 0.$$

And generalizing for the first case of interest where $p=k$.

$$T_\pi(p',k;k) = V_\pi(p',k) - i\frac{\pi k}{2} V_\pi(p',k) T_\pi(k,p;k) + \mathcal{P}\int dq \frac{q^2 V_\pi(p',q) T_\pi(q,k,k) - k^2 V_\pi(p',k) T_\pi(k,p;k)}{k^2-q^2}.$$

We can calculate these integrals numerically and transform them into a summation.

$$\int f(p)dp \to \sum_{j=1}^{N} f(p_j) w_j.$$

Finally,

$$T_\pi(p',k;k) = V_\pi(p',k) + \sum_{j=1}^{N} \frac{w_j p_j^2 V_\pi(p',p_j)}{k^2-p_j^2} T_\pi(p_j,k;k) - \left(i\frac{\pi k}{2} + \sum_{j=1}^{N} \frac{w_j k^2}{k^2-p_j^2} \right) V_\pi(p',k) T_\pi(k,k;k).$$

Defining the vectors

$$\begin{aligned} T_\pi(i) &= T_\pi(p_i,k;k), T_\pi(N+1) = T_\pi(k,k;k), \\ V_\pi(i) &= V_\pi(p_i,k), V_\pi(N+1) = V_\pi(k,k), \end{aligned}$$

We turned our initial problem into a problem of eigenvalues.

$$\sum_{j=1}^{N+1} a_{ij} T_{\pi_j} = V_{\pi_i}.$$

Here the elements of matrix a_{ij} are:

$$\begin{aligned} a_{ij} &= \delta_{ij} - \frac{p_j^2 V(p_i,p_j)}{k^2-p_j^2} w_j. \\ a_{iN+1} &= \left(i\frac{\pi k}{2} + \bar{w} \right) V(p_i,k). \\ a_{N+1j} &= -\frac{p_j^2 V(k,p_j)}{k^2-p_j^2} w_j. \\ a_{N+1N+1} &= 1 + \left(i\frac{\pi k}{2} + \bar{w} \right) V(k,k). \\ \bar{w} &= \sum_{j=1}^{N} \frac{k^2}{k^2-p_j^2} w_j. \end{aligned}$$

It will also be necessary to calculate the off-shell matrix. The method is the same, except for the final expression of the matrices to be processed.

$$\sum_{j=1}^{N} a_{ij} T_{\pi_{i,j}} = b_{ij},$$

where A and B are two matrices $N \times N$:

$$\begin{aligned} a_{ij} &= \delta_{ij} - \frac{p_j^2 V(p_i,p_j)}{k^2-p_j^2} w_j. \\ b_{ij} &= V(p_i,p_j) - \left(\frac{i\pi k}{2} + \bar{w} \right) V(p_i,k) T_\pi(p_j,k;k). \end{aligned}$$

Note that calculating the half-on-shell matrix is necessary for the full-off-shell matrix.

Once we have our complete scattering matrix T_π, the next step is to numerically solve the functions $L_0(k)$ and $J_0(k)$. For the former, I will follow the same procedure as for T_π, taking advantage of the fact that the integral

$$\mathcal{P} \int dq \frac{k^2 T_\pi(k,k;k)}{k^2-q^2} = 0 \tag{4.2}$$

is zero.
Differentiating again between main part and independent term:

$$L_0(k) = \sum_{j=1}^{N} w_j \frac{p_j^2 T_\pi(k,p_j;k) - k^2 T_\pi(k,k;k)}{k^2-p_j^2} - \frac{i\pi k}{2} T_\pi(k,k;k).$$

It can be seen that in the case of $k = p_j$, the numerator also becomes zero and the pole disappears. The next step is to discretize $J_0(k)$. The procedure is more laborious but follows the same idea.

$$\begin{aligned}
J_0(k) &= \int_0^\infty \int_0^\infty \frac{dqdq' q^2 q'^2 T_\pi(q,q';k)}{(k^2-q^2+i\epsilon)(k^2-q'^2+i\epsilon)} - \int_0^\infty dqdq' T_\pi(q,q';0) \\
&= \int_0^\infty \frac{dqq^2}{k^2-q^2+i\epsilon}\left(\mathcal{P}\int_0^\infty \frac{dq'q'^2 T_\pi(q,q';k)}{k^2-q'^2} - \frac{i\pi k}{2}T_\pi(q,k;k)\right) - \int_0^\infty dqdq' T_\pi(q,q';0) \\
&= \mathcal{P}\int_0^\infty \frac{dqq^2}{k^2-q^2}\left(\mathcal{P}\int_0^\infty \frac{dq'q'^2 T_\pi(q,q';k)}{k^2-q'^2} - \frac{i\pi k}{2}T_\pi(q,k;k)\right) - \frac{i\pi k}{2}\mathcal{P}\int_0^\infty \frac{dq'q'^2 T_\pi(k,q';k)}{k^2-q'^2} \\
&- \left(\frac{i\pi k}{2}\right)^2 T_\pi(k,k;k) - \mathcal{P}\int_0^\infty \frac{dqdq'(k^2-q^2)(k^2-q'^2)T_\pi(q,q';0)}{(k^2-q^2)(k^2-q'^2)} \\
&= \mathcal{P}\int_0^\infty \frac{dqdq' q^2 q'^2 T_\pi(q,q';k)}{(k^2-q^2)(k^2-q'^2)} - i\pi k L_0(k) + \frac{\pi^2 k^2}{4}T_\pi(k,k;k) \\
&- \mathcal{P}\int_0^\infty \frac{dqdq'(k^2-q^2)(k^2-q'^2)T_\pi(q,q';0)}{(k^2-q^2)(k^2-q'^2)}.
\end{aligned}$$

To solve the first term numerically, I will again use the result (4.2). Our integral will exhibit divergences in the cases where $q = k$, $q' = k$, and $q = q' = k$. Adding the following terms:

$$\mathcal{P}\int_0^\infty \frac{dqdq'}{(k^2-q^2)(k^2-q'^2)}(k^4 T_\pi(k,k;k) - k^2 q^2 T_\pi(q,k,k) - k^2 q'^2 T_\pi(q',k,k)),$$

We avoid the poles by also making the numerator zero. The resulting expression is:

$$\begin{aligned}
J_0(k) &= \sum_{j=0}^{N}\sum_{z=0}^{N}\frac{w(j)w(z)}{(k^2-p_j^2)(k^2-p_z^2)}((p_j^2 p_z^2 T_\pi(p_j,p_z,k) + k^4 T_\pi(k,k,k)) \\
&- (k^2(p_j^2 T_\pi(p_j,k,k) + p_z^2 T_\pi(p_z,k,k)) - ((k^2-p_j^2)(k^2-p_z^2)T_\pi(p_j,p_z,0))) \\
&- i\pi k L_0(k) + \frac{\pi^2 k^2}{4}T_\pi(k,k,k).
\end{aligned}$$

We will now have everything ready to calculate the on-shell matrix $T(k,k;k)$ using the formula (3.13), and with it the total phase shift that interests us. If we look at the expression for our potential (3.1), we will see how, for small moments, the argument of our logarithm tends to 1. By performing a series expansion, it is easy to see that the expression to consider when the moment $p_j \to 0$ is close to zero is:

$$\begin{aligned}
V_\pi(p_i, p_j \to 0) &= \frac{m\alpha_\pi}{\pi}\frac{1}{p_i p_j}\log\frac{p_i^2 + \cancel{p_j^2} - 2p_i p_j + m_\pi^2}{p_i^2 + \cancel{p_j^2} + 2p_i p_j + m_\pi^2} \\
&\approx \frac{m\alpha_\pi}{\pi}\frac{1}{p_i p_j}\frac{-4p_i p_j}{p_i^2 + m_\pi^2}.
\end{aligned}$$

Where have I used that $\log(\frac{x-\epsilon}{x+\epsilon}) \approx \frac{-2\epsilon}{x}$, where $x = p_i^2 + m_\pi^2$ y $\epsilon = 2p_i p_j \to 0$.

$$V_\pi(p_i, p_j) \approx -\frac{4m_\pi\alpha_\pi}{\pi(m_\pi^2 + p_i^2)}.$$

We can go even further and describe low-energy scattering in terms of parameters. Our

matrix $T_\pi(p, p'; k)$ does not allow a series expansion around zero because it has a right cut. The function that does allow such an expansion is:

$$f(k) = k \cot \delta(k) = -1/a + \frac{1}{2} r_0 k^2 + v k^4 ... \tag{4.3}$$

This expression has been presented previously and is known as the effective range expansion. It is considered for the elastic scattering of two spinless, non-relativistic particles by a central potential with a finite range. The condition is that $k < 1/d_{car}$, this being the aforementioned finite range. Developed by Bethe in the 1950s [3], where a is the scattering length and r_0 is the so-called effective range.

To calculate these parameters, we will use the residue theorem. The trick is to numerically solve the proposed integral using a closed contour that includes zero. This will give us poles of order n that we can relate to the value of the $(n-1)th$ derivative at that point, and then collect the parameter values as we increase the order. Let's recall how to calculate a pole at a complex number z_0.

$$Res_{z_0} f(z) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}[(z - z_0)^m f(z)]}{dz^{m-1}}.$$

And also considering $f(k) = f_0 + f_1 k^2 + f_2 k^4 + \dots$

$$\oint \frac{f(k)}{k^{2n+1}} dk = 2\pi i f_n = I(n). \tag{4.4}$$

Let us also recall the expression we obtained for $\tan(\delta_\pi)$, which allows us to express $f(k)$ as:

$$f(k) = -\frac{2}{\pi} T_\pi^{-1}(k, k) + ik.$$

By making a simple change of variable $k = k_0 e^{i\phi}$, we can solve this integral with points between 0 and 2π, whose weights we will also obtain with the GAUSS_AB subroutine. It is important that our auxiliary variable $k_0^2 < \frac{-m_\pi^2}{4}$.

The fact that the matrix T is unitary translates into the presence of a right-hand cut on the positive real axis (*right-hand cut*) starting from a certain threshold value. This opens the possibility of intermediate states in the scattering.

On the other hand, our pion exchange potential brings with it the existence of another cut, but this time a left-hand cut (*left-hand cut*) on the negative real axis. The limit of this cut is determined by[6]:

$$0 < \omega_L < -\frac{m_\pi^2}{4}.$$

We can find the low-energy parameters as follows:

$$a = -\frac{-2\pi i}{I(0)},$$

$$r_0 = \frac{I(0)}{i\pi}.$$

And so on for higher orders of k accompanied by the coefficients v_i.

Chapter 5

Results

As I mentioned, to find the solution for the total transfer matrix, it is necessary to know separately the matrix of the exchange potential of a pion in the absence of a short-range potential V_s. The graph obtained for δ_π along with the low-energy parameters a_π and r_π is shown below.

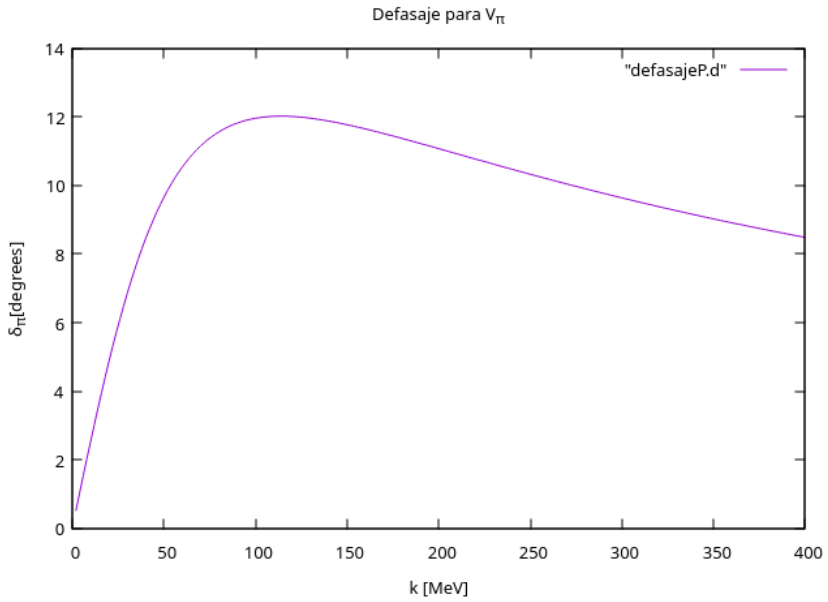

Figure 5.1: Defasaje para V_π en función de k.

$$a_\pi = -0.8783 \text{ fm.} \tag{5.1}$$

$$r_\pi = 12.3766 \text{ fm.} \tag{5.2}$$

Whose values are conveniently expressed in fermis for later comparison with values from other works, with the only manipulation being to multiply by $\hbar c = 197,326$ MeV fm.

Following the structure of the original work [15], the following representation is obtained for $l_0(k)$ y $\mathrm{Re}\bar{J}_0(k)$.

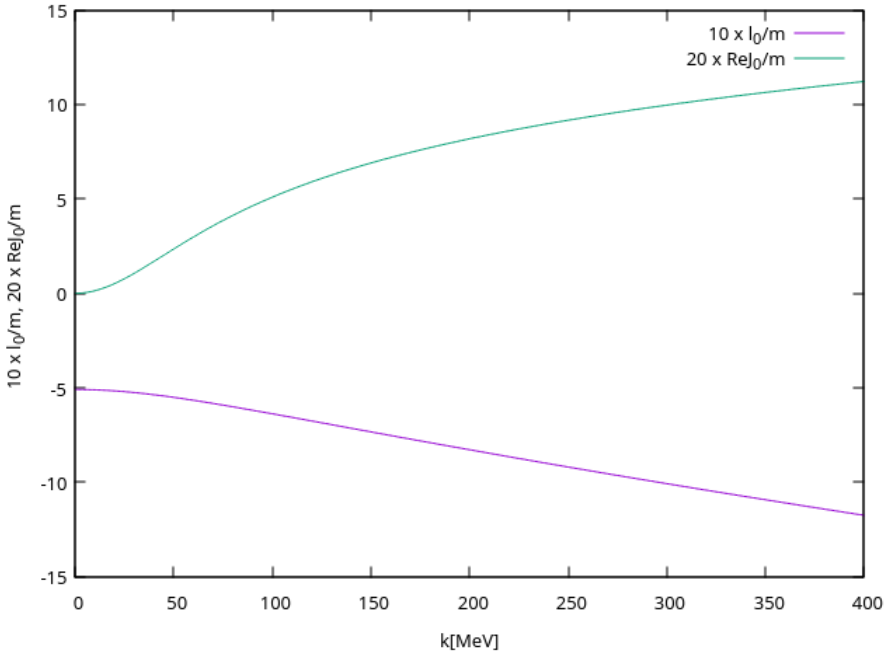

Figure 5.2: Representation of $10 \times l_0(k)/m$ and $20 \times \bar{J}_0(k)/m$. The multiplicative factors make both results dimensionless and on the same scale of values.

For the same reason, I will give the phase shift both on a normal scale and on a logarithmic scale applied to the x-axis of energies.

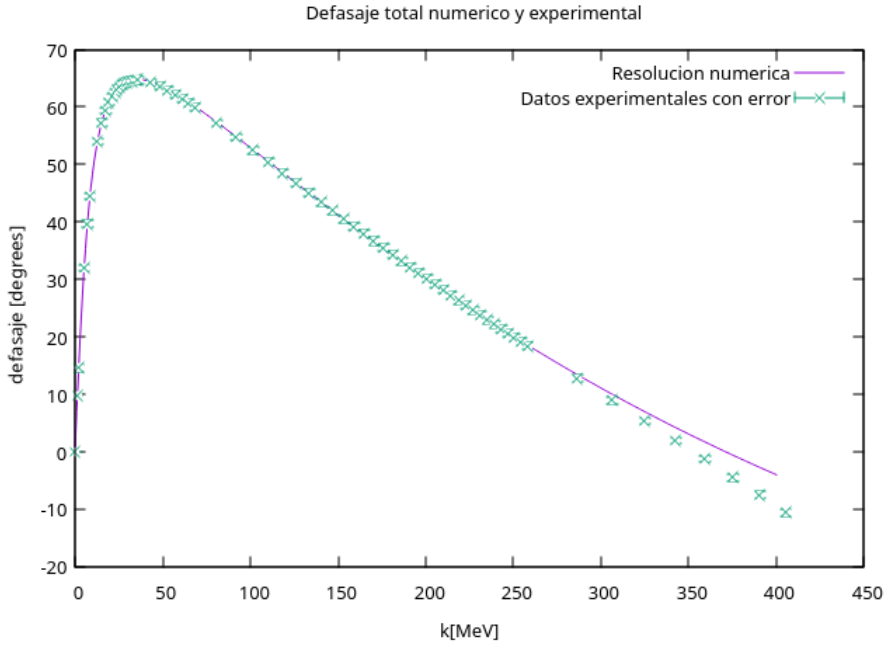

Figure 5.3: Total phase shift for $V = V_\pi + V_s$. To adjust the data to the Nijmegen phase shift, the following values have been set: $g_0 m = -0.276$, $g_1 m^3 = 0.347$ and $\frac{J_0^R(0)}{m} = -3.21$.

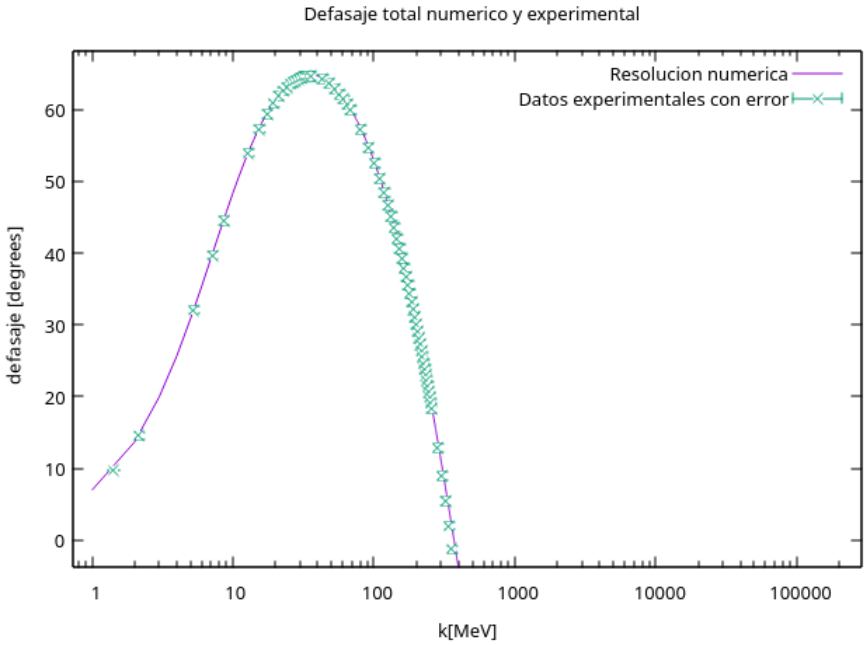

Figure 5.4: Same phase shift with the x-axis on a logarithmic scale.

Before moving on to the short-range parameters, I will give one last graph for δ for different values of g_0, g_1 and $J_0^R(0)$.

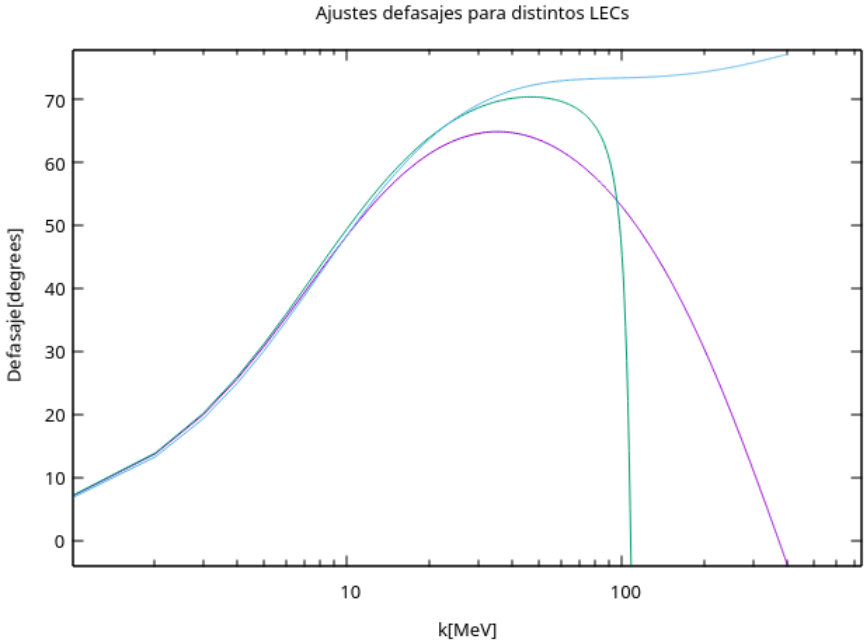

Figure 5.5: In purple we have the previous example for g_0, g_1, $J_0^R(0) \neq 0$, in blue the case where g_1, $J_0^R(0) = 0$, and in green the last particular case with only $J_0^R(0) = 0$.

The values obtained for a and r_0 are:

$$a = -23.9975 \text{ fm.} \tag{5.3}$$

$$r_0 = 2.6736 \text{ fm.} \tag{5.4}$$

Conclusions

To fix the value of the short-range constants: g_0, g_1 and $J_0^R(0)$, the author of the original work [15] carried out a χ^2 fit to reproduce as faithfully as possible the phase shift using the Nijmegen method [20]. The best-fit parameters are:

$$\begin{aligned} g_0 m &= -0.276 \\ g_1 m^3 &= 0.347 \\ \frac{J_0^R(0)}{m} &= -3.21 \end{aligned}$$

It can be seen in figure [5.3] how our data fits perfectly to the curve up to a certain value $k_{max} \approx 260$ MeV for the nucleon CM momentum. If larger momenta were included, the value of χ^2/dof would almost double [15].

It was also noted that the correlation between the coefficients is very low. That is, there is really only one independent parameter. If $g_1 = 0$, we can absorb the ultraviolet (UV) divergence into g_0 and tune the last parameter to achieve an adequate fit.

In figure [5.5] different cases are shown. It can be seen how these methods achieve a good fit up to much smaller momenta than k_{max}.

Recalling the effective range expansion at low energies:

$$T^{-1}(k) = \frac{-\pi}{2}\left(\frac{1}{a} + \frac{1}{2}r_0^2 k^2 + ... - ik\right)$$

We obtained the following values for the low-energy parameters after the numerical solution:

$$\begin{aligned} a_\pi &= -0.8783 fm \\ r_\pi &= 12.3766 fm \\ a &= -23.9975 fm \\ r_0 &= 2.6736 fm \end{aligned}$$

Note that these results are very sensitive to the values chosen for the short-range parameters.

By using the residue theorem we have ensured that at least the nth-order derivative appearing in equation (4.4) evaluated at zero is very precise. This precision cannot be guaranteed with an alternative method such as numerical differentiation.

The original paper [15] ultimately gives the following values: $a = -23.65$ fm, $r_0 = 2.63$ fm, $a_\pi = -0.88$ fm and $r_\pi = 12.38$ fm. Although not exactly identical, they are very close. Clearly, since this work aims to reproduce the same calculations, the discrepancy

in the values comes solely from the method each author used to calculate the derivative of $f(k)$. In turn, the work on which the fit was based [20] gives results: $a = -23.7$ fm and $r_0 = 2.73$ fm.

Finally, I would like to compare with the values given in [7]. In this more recent work, the parameters are calculated analogously using the residue theorem. The results in this case were: $a = -23.7588$ fm and $r_0 = 2.6728$. These are only slightly different due solely to a small discrepancy between the values of g_0, g_1 and $J_0^R(0)$ used in [15] and [7].

I would like to end this work by summarizing a bit of everything that has been discussed so far, to clearly conclude the reason, motivation, and physics behind all the calculation.

We began by briefly introducing the reader to the history of the strong nuclear interaction, from Yukawa's discovery of the pion in 1935 to the development of quantum chromodynamics and the subsequent need for effective field theories (EFT) to explain physics at low energies. At these energy scales, there is no perturbative resolution of the fundamental theory, and pions once again become the relevant degrees of freedom of the interactions.

We saw how the Lagrangian could be written ignoring the degrees of freedom present at higher energies without this meaning there is no correlation between the physics at high and low energies. An example of this is thermodynamics: One can study the macroscopic properties of a system without having information about its microscopic structure.

Based on this idea, we treated our problem by separating two well-differentiated scales in our potential. The first term corresponds to one-pion exchange, and the second to contact terms expressed in powers of k^2 up to second order. This is known as the Distorted Wave Theory (DWT).

As an illustrative example I used Bethe's Effective Range Expansion, and we concluded that *effective theories are based on the premise that the dynamics at low energies (large distances) do not depend on the details of the dynamics at high energies (short distances).*

It was Weinberg who first adapted these theories to nuclear physics. One last ingredient was necessary for the formulation of a consistent theory: chiral symmetry. This gives its name to chiral effective field theories (χEFT) on which the NN interactions are based. They were successful for the study of $\pi\pi$ as well as $N\pi$, but to correctly describe the NN problem, a non-perturbative tool was needed for its solution. This is where the Lippmann-Schwinger equation (LSE) comes into play.

Both DR and DWT techniques have been used to solve the LSE of the NN system in the ${}^1S_0$ wave. For a contact potential up to order k^2 we achieved a very good approximation to experimental data for momenta below 260 MeV with only the inclusion of one additional parameter $J_0^R(0)$.

This additional parameter arose from the impossibility of absorbing the divergence in $J_0(0)$ by redefining the coefficients of our potential at short distances, once it reaches order less than or equal to two.

The normalized contact potential $(V_s^R)^{-1}(\hat{k}, \hat{k})$ was conveniently redefined, leading to an infinite series of powers of $\hat{k}^2$, and the need for infinitely many counter-terms for the

non-perturbative solution of the scattering amplitude.

At each order in $\hat{k}^2$, the divergent part of these counter-terms is determined, but not the constant finite part.

The solution chosen was not to consider the coefficients as free and independent, but to impose relations between those of higher orders in the counter-term series and those in the expansion of the potential, so as not to lose the predictability of our model at low energies (where the influence of these higher orders is small).

In other words: We have treated J_0 as a renormalized and independent parameter; one more constant to be fixed to reproduce the experimental phase shift data. In this way, the amplitude can be expressed as in the form deduced in (3.19), satisfying classical unitarity.

For the numerical solution we had to take into account the left and right branch cuts present in the $T(k, k)$ matrix due to it being unitary and having used the one-pion exchange (OPE) potential to describe the physics at long distances. The discretization of the functions $L_0(k)$ and $J_0(k)$ was carried out in detail in the corresponding section.

Once the scattering amplitude was obtained with the short-range parameters adjusted to reproduce the phase shift up to an energy of $k_{max} \approx 260$ MeV, the low-energy parameters were found with the help of the residue theorem.

Finally, the expected results were achieved after reproducing the original paper [15], arriving at the same results and representations as can be found in that work.

Chapter 6

Conclusiones

Para fijar el valor de las constantes de corto alcance: g_0, g_1 y $J_0^R(0)$, el autor del trabajo original [15] llevó a cabo un ajuste - χ^2 para reproducir lo más fielmente posible el desfasaje por medio del método de Nijmegen [20]. Los parámetros que mejor se ajustan son:

$$\begin{aligned} g_0 m &= -0.276 \\ g_1 m^3 &= 0.347 \\ \frac{J_0^R(0)}{m} &= -3.21 \end{aligned}$$

Puedes verse en la figura [5.3] como nuestros datos se ajustan perfectamente a la gráfica hasta un cierto valor $k_{max} \approx 260$ MeV para el momento en CM del nucleón. Si se incluyesen momentos más grandes el valor de χ^2/dof aumentaría hasta casi hasta el doble [15].

También se percató de que la correlación entre los coeficientes es muy baja. Es decir, verdaderamente solo hay un parámetro independiente. En caso de hacer $g_1 = 0$, podemos absorber la divergencia ultravioleta (UV) en g_0 y jugar con el último parámetro para conseguir un ajuste adecuado.

En la gráfica [5.5] se muestran diferentes casos. Puede comprobarse como dichos métodos consiguen un ajuste bueno hasta momentos bastantes más pequeños que k_{max}.

Recordando la expansión de rango efectivo para bajas energías:

$$T^{-1}(k) = \frac{-\pi}{2}(\frac{1}{a} + \frac{1}{2}r_0^2 k^2 + ... - ik)$$

Hemos obtenido los siguientes valores de los parámetros de baja energía tras la resolución numérica:

$$\begin{aligned} a_\pi &= -0.8783 fm \\ r_\pi &= 12.3766 fm \\ a &= -23.9975 fm \\ r_0 &= 2.6736 fm \end{aligned}$$

Destacar que estos resultados son muy sensibles al valor que se han fijado los parámetros de corto alcance.

Al haber utilizado el teorema de los residuos hemos garantizado que al menos la derivada

de orden n que aparece en la ecuación (4.4) evaluada en cero es muy precisa. Esta precisión no puede garantizarse con una resolución alternativa como la derivada numérica.

El paper original [15] termina por dar los siguientes valores: $a = -23.65$ fm, $r_0 = 2.63$ fm, $a_\pi = -0.88$ fm y $r_\pi = 12.38$ fm. A pesar de no ser exactamente iguales se acercan mucho. Evidentemente, al tratarse este trabajo de reproducir sus mismos cálculos, la discrepancia en los valores viene únicamente del método con que cada quién haya calculado la derivada de $f(k)$. A su vez, el trabajo bajo el cual se ha hecho el ajuste [20] da por resultados: $a = -23.7$ fm y $r_0 = 2.73$ fm.

Por último, quisiera comparar con los valores dados en [7]. En este trabajo más reciente los parámetros son calculados de manera análoga con el teorema de los residuos. Los resultados en este caso han sido: $a = -23.7588$ fm y $r_0 = 2.6728$. Que son ligeramente distintos únicamente por una pequeña discrepancia entre los valores de g_0, g_1 y $J_0^R(0)$ utilizados en [15] y [7].

Me gustaría terminar este trabajo resumiendo un poco todo lo que se ha visto hasta ahora, y así dejar bien concluida la razón, motivación y física tras todo el cálculo.

Se ha comenzado por introducir al lector brevemente a la historia de la interacción nuclear fuerte, desde el descubrimiento del pion por parte de Yukawa en 1935 hasta el desarrollo de la cromodinámica cuántica y la posterior necesidad de las teorías efectivas (EFT) para explicar la física a bajas energías. En dichas escalas de energía, no hay resolución perturbativa de la teoría fundamental, y los piones vuelven a ser los grados de libertad propios de las interacciones.

Vimos como el lagrangiano podría ser escrito ignorando los grados de libertad presentes a energías más altas sin que esto significase que no hay una correlación entre la física de altas y bajas energías. Un ejemplo de esto es la termodinámica: Uno puede estudiar las propiedades macroscópicas de su sistema sin tener información de la estrucutra microscópica.

Sobre esta idea hemos tratado nuestro problema separando dos escalas bien diferenciadas en nuestro potencial. El primer término correspondiente al intercambio de un pion, y el segundo términos de contacto expresados en potencias de k^2 hasta segundo orden. Esto recibe el nombre de Teoría de Ondas Distorsionadas (DWT).

Como ejemplo ilustrativo me ayudé de la Expansión de Rango Efectivo de Bethe, y llegamos a la conclusión de que *las teorías efectivas parten de la premisa de que la dinámica a bajas energías (distancias grandes) no depende de los detalles de la dinámica a altas energías (distancias cortas).*

Fue Weinberg el primero es adaptar estas teorías en la física nuclear. Un último ingrediente era necesario para la formulación de una teoría consistente: La simetría quiral. esta da nombre a las teorias efectivas quirales (χEFT) de las cuales se basan las interacciones NN. Fueron un éxito para el estudio de $\pi\pi$ junto con $N\pi$, pero para describir correctamente el problema NN hacía falta una herramienta no perturbativa para su resolución. Ahí entra en juego la ecuación de Lippmann-Schwinger (LSE).

Tanto el uso de técnicas de DR como DWT han sido empleadas para solucionar la LSE del sistema NN en onda ${}^1S_0$. Para un potencial de contacto de hasta orden k^2 hemos dado

con una muy buena aproximación de los datos experimentales para momentos inferiores a 260 MeV con tan solo la inclusión de un parámetro adicional $J_0^R(0)$.

Este parámetro adicional venía de la imposibilidad de absorber la divergencia en $J_0(0)$ redefiniendo los coeficientes de nuestro potencial a cortas distancias, al llegar este a orden menor o igual a dos.

Convenientemente se redefinió el potencial de contacto normalizado $(V_s^R)^{-1}(\hat{k}, \hat{k})$, conduciendo a una serie infinita de potencias de $\hat{k}^2$, y la necesidad de infinitos contra-términos para la resolución no perturbativa de la amplitud de probabilidad.

A cada orden de $\hat{k^2}$, la parte divergente de dichos contra-términos está determinada, pero no así la parte finita constante.

Se optó por la solución de no considerar los coeficientes libres e independientes, e imponer relaciones entre aquellos de órdenes superiores de la serie de contra-términos a los de la expansión del potencial, de forma que evitemos perder la predictivilidad de nuestro modelo a bajas energías (escala donde la influencia de dichos órdenes altos es pequeña).

En otras palabras: Hemos considerado J_0 como un parámetro renormalizado e independiente; una constante más que fijar para reproducir los datos experimentales del defasaje experimental. De esta manera, la amplitud puede expresarse tal como la expresión deducida en (3.19), satisfaciendo la unitariedad clásica.

Para la resolución numérica hemos tenido que tener en cuenta los cortes de rama izquierdo y derecho que presenta la matriz $T(k, k)$ por el hecho de ser unitaria y de haber usado el potencial de intercambio de un pion (OPE) para describir la física a largas distancias. La discretización de las funciones $L_0(k)$ y $J_0(k)$ se ha seguido con detalle en la sección correspondiente

Una vez obtenida la amplitud de probabilidad con el ajuste de parámetros de corto alcance para reproducir el defasaje hasta una energía de $k_{max} \approx 260$ MeV, se hallaron los parámetros de baja energía con apoyo del teorema de los residuos.

Finalmente, se lograron los resultados esperados tras la reproducción del paper original [15], llegando a los mismos resultados y representaciones que pueden encontrarse en el trabajo.

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