Proofs

Proof for Lemma 2

Lemma (0 Eigenvalue)**.** *0 is an eigenvalue of* $\mathbf{A}$ *and the corresponding eigenvector is* $[\boldsymbol{\upsilon}^\top\ \boldsymbol{\xi}^\top]^\top = [\mathbf{1}^\top\ \mathbf{0}^\top]^\top$.

Proof. Consider the following quadratic eigenvalue problem:

$$\lambda^2 \mathbf{M}\boldsymbol{\nu} + \lambda \mathbf{D}\boldsymbol{\nu} + \mathbf{L}\boldsymbol{\nu} = \mathbf{0} \tag{1}$$

Since $\mathbf{L}$ has an eigenvalue of 0 with the eigenvector $\mathbf{1}$, we have $\lambda = 0$ and $\boldsymbol{\nu} = \mathbf{1}$ as one of the solutions to Eq.(1).

Now, let $\boldsymbol{\xi}$ and $\boldsymbol{\upsilon}$ be the eigenvector and eigenvalue of the matrix $\mathbf{A}$ such that:

$$\boldsymbol{\xi} = \lambda\boldsymbol{\upsilon}, \quad -\mathbf{L}\boldsymbol{\upsilon} - \mathbf{D}\boldsymbol{\xi} = \lambda\mathbf{M}\boldsymbol{\xi} \tag{2}$$

Hence, $\lambda = 0$ and $[\boldsymbol{\upsilon}^\top\ \boldsymbol{\xi}^\top]^\top = [\mathbf{1}^\top\ \mathbf{0}^\top]^\top$. are the eigenvalue and eigenvector of matrix $\mathbf{A}$ that satisfy Eq.(2), completing the proof. □

Proof for Lemma 2

Lemma (Mode Decay)**.** *The modes will all be within* $\lambda \leqslant -\beta$ *except for the eigenvalue* 0 *when there exists a positive variable* r *that makes the following conditions satisfied:*

$$\mathbf{D} - 2\beta\mathbf{M} \succcurlyeq 0 \tag{3}$$

$$\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} + r\mathbf{1}\mathbf{1}^\top \succcurlyeq 0 \tag{4}$$

Proof. The shifted eigenvalue solution problem is represented as follows:

$$\hat{\lambda}^2\mathbf{M}\boldsymbol{\nu} + \hat{\lambda}\left(\mathbf{D} - 2\beta\mathbf{M}\right)\boldsymbol{\nu} + \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\boldsymbol{\nu} = \mathbf{0} \tag{5}$$

The quadratic equation about the eigenvalue λ can be expressed as:

$$\hat{\lambda}^2\left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu}\right\rangle + \hat{\lambda}\left\langle \boldsymbol{\nu}, \left(\mathbf{D} - 2\beta\mathbf{M}\right)\boldsymbol{\nu}\right\rangle + \left\langle \boldsymbol{\nu}, \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\boldsymbol{\nu}\right\rangle = 0 \tag{6}$$

Given a fixed vector $\boldsymbol{\nu}$, the roots of Eq.(6) satisfy:

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{-\boldsymbol{\nu}^{\mathrm{H}}\left(\mathbf{D} - 2\beta\mathbf{M}\right)\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}}, \quad \hat{\lambda}_1\hat{\lambda}_2 = \frac{\boldsymbol{\nu}^{\mathrm{H}}\left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}} \tag{7}$$

To ensure that all the eigenvalues of the shifted system (except for the eigenvalue $0 + \beta$) lie in the left half plane, we need the following conditions:

- For eigenvalue pairs that do not contain the eigenvalue $0 + \beta$, the necessary and sufficient condition is $\hat{\lambda}_1 + \hat{\lambda}_2 \leqslant 0$ and $\hat{\lambda}_1\hat{\lambda}_2 \geqslant 0$.
- For the eigenvalue pair that contains the eigenvalue $0 + \beta$, the sufficient condition is $\hat{\lambda}_1 + \hat{\lambda}_2 \leqslant 0$.

Since the eigenvectors corresponding to different eigenvalues are linearly dependent, any other eigenvector $\boldsymbol{\nu}$ can be decomposed as the sum of $\boldsymbol{\nu} = a\mathbf{1} + b\mathbf{1}_\perp$ where $\mathbf{1}_\perp$ is the vector in the orthogonal space of $\mathbf{1}$ in $\mathbb{C}_{n\times 1}$. To ensure the above conditions are met, the following conditions are sufficient:

$$\boldsymbol{\nu}^{\mathrm{H}}\left(\mathbf{D} - 2\beta\mathbf{M}\right)\boldsymbol{\nu} \geqslant 0 \tag{8}$$

$$\boldsymbol{\nu}^{\mathrm{H}}\left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\boldsymbol{\nu} \geqslant 0,\ \forall\boldsymbol{\nu}\perp\mathbf{1} \tag{9}$$

Inequality (9) can be transformed into (4) by setting r large enough to make $\boldsymbol{\nu}^{\mathrm{H}}\left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} + r\mathbf{1}\mathbf{1}^\top\right)\boldsymbol{\nu} \geqslant 0$ when $\boldsymbol{\nu} = \mathbf{1}$ for possible $\mathbf{L}, \mathbf{M}$, and $\mathbf{D}$. □

Proof for Lemma 4

Lemma (Mode Damping)**.** *All the eigenvalues lie in the conic region* $\sin\zeta \cdot \mathrm{Re}\left(\lambda\right) + \cos\zeta \cdot \left|\mathrm{Im}\left(\lambda\right)\right| \leqslant 0$ *if and only if the following condition is satisfied:*

$$\left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu}\right\rangle^2 - 4\left(\cos\zeta\right)^2\left\langle \boldsymbol{\nu}, \mathbf{L}\boldsymbol{\nu}\right\rangle\left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu}\right\rangle \geqslant 0 \tag{10}$$

Proof. For real eigenvalue pairs, the condition is naturally satisfied because the discriminant $\left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu}\right\rangle^2 - 4\left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu}\right\rangle\left\langle \boldsymbol{\nu}, \mathbf{L}\boldsymbol{\nu}\right\rangle \geqslant 0$.

For complex conjugate eigenvalue pairs λ and λ^*, the necessary and sufficient condition is:

$$\begin{cases} \sin\zeta \cdot \left(\lambda + \lambda^*\right) < 0 \\ \left(\sin\zeta\right)^2\left(\lambda + \lambda^*\right)^2 + \left(\cos\zeta\right)^2\left(\lambda - \lambda^*\right)^2 \geqslant 0 \end{cases} \tag{11}$$

By using

$$\left(\lambda + \lambda^*\right)^2 = \left(\frac{-\boldsymbol{\nu}^{\mathrm{H}}\mathbf{D}\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}}\right)^2, \quad \left(\lambda - \lambda^*\right)^2 = \left(\frac{-\boldsymbol{\nu}^{\mathrm{H}}\mathbf{D}\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}}\right)^2 - 4\frac{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{L}\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}} \tag{12}$$

we can easily get the condition. □

Proof for Theorem 1

Theorem (Mode Placement Constraints)**.** *When the following condition is satisfied, all the eigenvalues are in the left plane* $\lambda \leqslant -\beta$ *and the conic region* $\sin\zeta \cdot \mathrm{Re}(\lambda) + \cos\zeta \cdot |\mathrm{Im}(\lambda)| \leqslant 0$ *simultaneously:*

$$\mathbf{D} - 2\beta\mathbf{M} \succcurlyeq 0 \tag{13}$$

$$\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} + r\mathbf{1}\mathbf{1}^\top \succcurlyeq 0 \tag{14}$$

$$\beta\mathbf{D} - 2(\cos\zeta)^2\,\mathbf{L} \succcurlyeq 0 \tag{15}$$

Proof. The first two inequalities are from Lemma 2. From the first inequality, we have:

$$\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \rangle \geqslant 2\beta \langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \rangle \tag{16}$$

As a result, we obtain:

$$\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \rangle^2 \geqslant 2\beta \langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \rangle \tag{17}$$

Therefore, in order to make the constraint of Lemma 4 hold, the following condition is sufficient:

$$\begin{aligned} &\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \rangle^2 - 4(\cos\zeta)^2 \langle \boldsymbol{\nu}, \mathbf{L}\boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \rangle \\ &\geqslant 2\beta \langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \rangle - 4(\cos\zeta)^2 \langle \boldsymbol{\nu}, \mathbf{L}\boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \rangle \geqslant 0 \end{aligned} \tag{18}$$

Since $\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu} > 0$, (15) implies (18). □

Proof for Inequalities (31-32)

The inequalities are:

$$\begin{bmatrix} 1^\top\mathbf{M}11^\top\mathbf{R}1 & x \\ x & 1 \end{bmatrix} \succcurlyeq 0 \tag{19}$$

$$x \geqslant \varepsilon_{\mathrm{nadir}}^{-1} \left|\mathbf{1}^\top\boldsymbol{p}_{\mathrm{dis}}\right| - \frac{\mathrm{e}-1}{2} 1^\top\mathbf{D}1, \quad x \geqslant 0 \tag{20}$$

Proof. The original inequality is:

$$\left|\mathbf{1}^\top\boldsymbol{p}_{\mathrm{dis}}\right| \leqslant \varepsilon_{\mathrm{nadir}} \left(\frac{\mathrm{e}-1}{2}\mathbf{1}^\top\mathbf{D}\mathbf{1} + \sqrt{\mathbf{1}^\top\mathbf{M}\mathbf{1}\mathbf{1}^\top\mathbf{R}\mathbf{1}} \right) \tag{21}$$

where the right-hand side is a concave function. Since the constraint is in the form 'concave function $\geqslant$ constant', the feasible set constrained by the inequality must be convex. It must be able to be transformed into the canonical form of convex constraints by lifting its dimension.

Break the inequality into two parts and lift the dimension of the inequality by an extra variable x:

$$\mathbf{1}^\top\mathbf{M}\mathbf{1}\mathbf{1}^\top\mathbf{R}\mathbf{1} \geqslant x^2 \tag{22}$$

$$x \geqslant 0, x \geqslant \varepsilon_{\mathrm{nadir}}^{-1} \left|\mathbf{1}^\top\boldsymbol{p}_{\mathrm{dis}}\right| - \frac{\mathrm{e}-1}{2}\mathbf{1}^\top\mathbf{D}\mathbf{1} \tag{23}$$

Inequality (22) ('a convex function $\leqslant$ an affine function') is also a convex constraint. It can be transformed into Eq.(19) via the following lemma:

Lemma (Positive-Definiteness of Schur Complement)**.** *When* $\mathbf{A}$ *is a block Hermitian positive definite matrix with the following block form:*

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{D} \end{bmatrix}$$

where $\mathbf{B}$ *and* $\mathbf{D}$ *are square matrices of appropriate sizes, and* $\mathbf{C}$ *is a rectangular matrix with compatible dimensions. If* $\mathbf{D} \succeq 0$*, then* $\mathbf{A} \succeq 0$ *if and only if the Schur complement* $\mathbf{A}/\mathbf{D} = \mathbf{A} - \mathbf{C}^\top\mathbf{D}^{-1}\mathbf{C} \succeq 0$.

□

Proof for Lemma 6

Lemma. *VCG mechanism satisfies the following properties:*

- *Efficiency Maximization: The allocation of inertia and damping coefficients obtained through the VCG mechanism minimizes the total cost required to provide them.*
- *Individual Rationality: The payment made to every generator under the VCG mechanism is non-negative, meaning that each generator receives compensation for its contribution to the system stability.*

- *Incentive Compatible: For each generator, truthfully reporting its cost function (bidding curve) is a dominant strategy.*

$$u_k\left(\mathcal{C}_k, \mathcal{B}_{-k}\right) \geqslant u_k\left(\mathcal{B}_k, \mathcal{B}_{-k}\right), \forall \mathcal{B}_k, \forall k \tag{24}$$

Proof. Each of the properties is reasoned as follows:

1) Efficiency Maximization: This is obvious because the allocation function aims to minimize the total cost.

2) Individual Rationality: According to the payment function:

$$q_{\hat{k}} = \min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) - \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k^\star, d_k^\star\right) \tag{25}$$

If the optimal solution exists for $\min \sum_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right)$, denoted as:

$$\hat{\boldsymbol{m}}_{-\hat{k}}^\star, \hat{\boldsymbol{d}}_{-\hat{k}}^\star = \min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) \tag{26}$$

Clearly, $\boldsymbol{m}_{-\hat{k}} = \hat{\boldsymbol{m}}_{-\hat{k}}^\star, \boldsymbol{d}_{-\hat{k}} = \hat{\boldsymbol{d}}_{-\hat{k}}^\star, m_{\hat{k}} = 0, d_{\hat{k}} = 0$ is one of the feasible solutions to the original cost minimization problem. Thus, the corresponding cost must be larger than the cost of the original optimal solution $\boldsymbol{m} = \boldsymbol{m}^\star, \boldsymbol{d} = \boldsymbol{d}^\star$, which is larger than $\sum_{-\hat{k}} \mathcal{B}_k\left(m_k^\star, d_k^\star\right)$.

3) Incentive Compatible: When generator $\hat{k}$ bid truthfully, the allocation is:

$$\boldsymbol{m}^\star, \boldsymbol{d}^\star = \arg\min\left(\mathcal{C}_{\hat{k}}\left(m_{\hat{k}}, d_{\hat{k}}\right) + \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right)\right) \tag{27}$$

Now, suppose generator $\hat{k}$ intentionally misreports its cost and bids $\mathcal{C}'_{\hat{k}}\left(m_{\hat{k}}, d_{\hat{k}}\right)$, then the new allocation and payment will be:

$$\begin{aligned} \tilde{\boldsymbol{m}}^\star, \tilde{\boldsymbol{d}}^\star &= \arg\min\left(\mathcal{C}'_{\hat{k}}\left(m_{\hat{k}}, d_{\hat{k}}\right) + \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right)\right) \\ \tilde{q}_{\hat{k}} &= \min \sum\nolimits_{-\hat{k}} \mathcal{C}_k\left(m_k, d_k\right) - \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(\tilde{m}_k^\star, \tilde{d}_k^\star\right) \end{aligned} \tag{28}$$

The total profit for generator $\hat{k}$ should be:

$$\tilde{u}_{\hat{k}}^\star = \min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) - \left(\sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(\tilde{m}_k^\star, \tilde{d}_k^\star\right) + \mathcal{C}_{\hat{k}}\left(\tilde{m}_{\hat{k}}^\star, \tilde{d}_{\hat{k}}^\star\right)\right) \tag{29}$$

where $\sum_{-\hat{k}} \mathcal{B}_k\left(\tilde{m}_k^\star, \tilde{d}_k^\star\right) + \mathcal{C}_{\hat{k}}(\tilde{m}_{\hat{k}}^\star, \tilde{d}_{\hat{k}}^\star)$ must be larger than $\sum_{-\hat{k}} \mathcal{B}_k\left(m_k^\star, d_k^\star\right) + \mathcal{C}_{\hat{k}}(m_{\hat{k}}^\star, d_{\hat{k}}^\star)$. Thus, for a rational generator seeking to maximize its profit, its dominant strategy is to submit its true costs. $\square$