PROOF FOR LEMMA 1

Proof. Consider the following quadratic eigenvalue problem:

$$\lambda^2 \mathbf{M}\boldsymbol{\nu} + \lambda \mathbf{D}\boldsymbol{\nu} + \mathbf{L}\boldsymbol{\nu} = \mathbf{0} \tag{1}$$

Since $\mathbf{L}$ has an eigenvalue of 0 with the eigenvector $\mathbf{1}$, we have $\lambda = 0$ and $\boldsymbol{\nu} = \mathbf{1}$ as one of the solutions to Eq.(1).

Now, let $\boldsymbol{\xi}$ and $\boldsymbol{\nu}$ be the eigenvector and eigenvalue of the matrix $\mathbf{A}$ such that:

$$\boldsymbol{\xi} = \lambda\boldsymbol{\nu}, \quad -\mathbf{L}\boldsymbol{\nu} - \mathbf{D}\boldsymbol{\xi} = \lambda \mathbf{M}\boldsymbol{\xi} \tag{2}$$

Hence, $\lambda = 0$ and $[\boldsymbol{\nu}^\top \ \boldsymbol{\xi}^\top]^\top = [\mathbf{1}^\top \ \mathbf{0}^\top]^\top$ are one of the solution to Eq.(2). □

PROOF FOR PROPOSITION 1

We will employ the subsequent definitions and lemmas. The dimension will be represented as N.

Definition 1 (Orthogonal Complement Subspace $\mathbb{I}_\perp$). The orthogonal complement subspace of $\mathbf{1}$ in $\mathbb{R}^N$ is denoted by $\mathbb{I}_\perp = \left\{\boldsymbol{x} \in \mathbb{R}^N | \mathbf{1}^\top \boldsymbol{x} = 0\right\}$.

Lemma 1 (Orthogonality of Eigenvectors for Symmetric Matrices). Given a real symmetric matrix $\mathbf{A}$, the eigenvectors associated with different eigenvalues are orthogonal.

Proof. Consider a symmetric real matrix $\mathbf{A}$ and suppose σ_1 and σ_2 are two distinct eigenvalues of $\mathbf{A}$, with corresponding eigenvectors $\mathbf{y}_1$ and $\mathbf{y}_2$. By definition of an eigenvector and eigenvalue, we have:

$$\mathbf{A}\mathbf{y}_1 = \sigma_1 \mathbf{y}_1, \ \mathbf{A}\mathbf{y}_2 = \sigma_2 \mathbf{y}_2 \tag{3}$$

Let's take the dot product of the first equation with $\mathbf{y}_2$:

$$\mathbf{y}_2^\top \mathbf{A}\mathbf{y}_1 = \sigma_1 \mathbf{y}_2^\top \mathbf{y}_1$$

We can rewrite the left-hand side as:

$$\mathbf{y}_2^\top \mathbf{A}\mathbf{y}_1 = \mathbf{y}_1^\top \mathbf{A}^\top \mathbf{y}_2$$

Since $\mathbf{A}$ is symmetric, $\mathbf{A}^\top = \mathbf{A}$. So, $\mathbf{y}_1^\top \mathbf{A}\mathbf{y}_2 = \mathbf{y}_1^\top \sigma_2 \mathbf{y}_2 = \sigma_2 \mathbf{y}_1^\top \mathbf{y}_2$. Combining the two equations, we get:

$$\sigma_1 \mathbf{y}_2^\top \mathbf{y}_1 = \sigma_2 \mathbf{y}_1^\top \mathbf{y}_2$$

Rearranging:

$$\sigma_1 \mathbf{y}_2^\top \mathbf{y}_1 - \sigma_2 \mathbf{y}_1^\top \mathbf{y}_2 = 0$$

Since σ_1 and σ_2 are distinct and $\mathbf{y}_2^\top \mathbf{y}_1 = \mathbf{y}_1^\top \mathbf{y}_2$, the only way the above equation can hold true is if $\mathbf{y}_2^\top \mathbf{y}_1 = 0$, which means the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ are orthogonal. □

Lemma 2 (Subspace Positive Semi-definiteness). If $\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} + v\mathbf{1}\mathbf{1}^\top \succcurlyeq 0$, then the matrix $\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}$ is positive semi-definite on $\mathbb{I}_\perp$.

Proof. Because $\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} + v\mathbf{1}\mathbf{1}^\top \succcurlyeq 0$:

$$\boldsymbol{x}^\top \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} + v\mathbf{1}\mathbf{1}^\top\right) \boldsymbol{x} \geqslant 0$$

for all $\boldsymbol{x} \in \mathbb{I}_\perp$. Besides, due to $\mathbf{1}^\top \boldsymbol{x} = 0$, the equation above suggests:

$$\boldsymbol{x}^\top \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right) \boldsymbol{x} \geqslant 0$$

for all $\boldsymbol{x} \in \mathbb{I}_\perp$. □

Lemma 3 (Eigenvalues of Subspace Positive Semi-definite Matrix). If a real symmetric matrix $\mathbf{A}$ is positive definite $\mathbf{A} \succ 0$ on $\mathbb{I}_\perp$, then $\mathbf{A}$ possesses at most one eigenvalue $\leqslant 0$.

Proof. We first prove that there is at most one 0 eigenvalue:

Suppose a vector $\boldsymbol{x} \in \mathbb{I}_\perp$ is the solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, which leads to $\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x} = 0$, which is a contradiction with $\mathbf{A} \succ 0$ on $\mathbb{I}_\perp$. Consequently, all the vectors $\boldsymbol{x} \in \mathbb{I}_\perp$ cannot lie in the null space of $\mathbf{A}$. The dimension of the null space for $\mathbf{A}$ must be less or equal to 1: $\dim(\mathrm{null}(\mathbf{A})) \leqslant N - \dim(\mathbb{I}_\perp) = 1$. By the rank-nullity relationship, we know the rank of $\mathrm{rank}(\mathbf{A}) = N - \dim(\mathrm{null}(\mathbf{A})) \geqslant N - 1$, which indicates there at least $N - 1$ non-zero eigenvalues.

Next, we prove there cannot be multiple non-positive eigenvalues:

By contradiction, suppose there are two different eigenvalues $\sigma_1 \leqslant 0$ and $\sigma_2 \leqslant 0$, whose eigenvectors are $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, respectively. Given that $\mathrm{rank}(\mathbf{A}) \geqslant N - 1$, σ_1 and σ_2 cannot be 0 simultaneously. Since $\mathbf{A}$ is a real symmetric matrix, according to lemma 1, $\boldsymbol{y}_1 \perp \boldsymbol{y}_2$. Because of the linear independence of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, their span has at least a dimension of 2. We have $\mathrm{span}\{\boldsymbol{y}_1, \boldsymbol{y}_2\} \cap \mathbb{I}_\perp \neq \varnothing$.

Thereby, there is a nonzero real vector $\boldsymbol{z} \in \mathrm{span}\{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ such that $\boldsymbol{z} \in \mathbb{I}_\perp$. Let $\boldsymbol{z} = a\boldsymbol{y}_1 + b\boldsymbol{y}_2$ where at least one of a, b is not zero. We have:

$$\begin{aligned}
&\boldsymbol{z}^\top \mathbf{A}\boldsymbol{z} = (a\boldsymbol{y}_1 + b\boldsymbol{y}_2)^\top \mathbf{A}(a\boldsymbol{y}_1 + b\boldsymbol{y}_2) \\
&= a^2 \boldsymbol{y}_1^\top \mathbf{A}\boldsymbol{y}_1 + ab\boldsymbol{y}_1^\top \mathbf{A}\boldsymbol{y}_2 + ab\boldsymbol{y}_2^\top \mathbf{A}\boldsymbol{y}_1 + b^2 \boldsymbol{y}_2^\top \mathbf{A}\boldsymbol{y}_2 \\
&= a^2 \sigma_1 \boldsymbol{y}_1^\top \boldsymbol{y}_1 + ab\sigma_2 \boldsymbol{y}_1^\top \boldsymbol{y}_2 + ab\sigma_1 \boldsymbol{y}_2^\top \boldsymbol{y}_1 + b^2 \sigma_2 \boldsymbol{y}_2^\top \boldsymbol{y}_2 \\
&= \sigma_1 a^2 \|\boldsymbol{y}_1\|^2 + \sigma_2 b^2 \|\boldsymbol{y}_2\|^2 \leqslant 0
\end{aligned} \tag{4}$$

which leads to a contradiction to the condition of $\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x} > 0$ for all vectors $\boldsymbol{x} \in \mathbb{I}_\perp$. □

Now we are ready to prove the proposition 1.

Proof. The shifted eigenvalue solution problem is represented as follows:

$$\hat{\lambda}^2 \mathbf{M}\boldsymbol{\nu} + \hat{\lambda}(\mathbf{D} - 2\beta\mathbf{M})\boldsymbol{\nu} + \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\boldsymbol{\nu} = \mathbf{0} \tag{5}$$

The quadratic equation about $\hat{\lambda}$ can be expressed as:

$$\begin{aligned}
&\hat{\lambda}^2 \langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu}\rangle + \hat{\lambda}\langle \boldsymbol{\nu}, (\mathbf{D} - 2\beta\mathbf{M})\boldsymbol{\nu}\rangle \\
&+ \left\langle \boldsymbol{\nu}, \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\boldsymbol{\nu}\right\rangle = 0
\end{aligned} \tag{6}$$

- For complex eigenvalue pairs $\hat{\lambda}$ and its conjugate $\hat{\lambda}^*$, notice the two complex roots of (6) satisfy:

  $$\hat{\lambda} + \hat{\lambda}^* = \frac{-\boldsymbol{\nu}^{\mathrm{H}}(\mathbf{D} - 2\beta\mathbf{M})\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}} \tag{7}$$

  Thereby, if the first inequality of this proposition is satisfied, we have

  $$\hat{\lambda} + \hat{\lambda}^* = \frac{-\boldsymbol{\nu}^{\mathrm{H}}(\mathbf{D} - 2\beta\mathbf{M})\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}} \leqslant 0 \tag{8}$$

  which further indicates the real part of $\hat{\lambda}$ is less than or equal to ero, and is sufficient to guarantee the complex eigenvalue pairs both have non-positivereal parts.
- For a real eigenvalue $\hat{\lambda}$ other than eigenvalue $0 + \beta$, by contradictions, suppose there is a real eigenvalue larger than

zero $0 < \hat{\lambda} < \beta$ (it must hold that $\hat{\lambda} < 0 + \beta$ because all the eigenvalues in the original system strictly lie in the left half plane). Notice (5) can be written as:

$$\underbrace{\left(\hat{\lambda}^2\mathbf{M} + \hat{\lambda}\left(\mathbf{D} - 2\beta\mathbf{M}\right) + \left(\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M}\right)\right)}_{\mathbf{A}(\hat{\lambda})}\boldsymbol{\nu} = \mathbf{0} \tag{9}$$

Given that $\hat{\lambda} > 0$, the following relations hold: $\hat{\lambda}^2\mathbf{M} \succ 0$ and $\hat{\lambda}\left(\mathbf{D} - 2\beta\mathbf{M}\right) \succcurlyeq 0$, which is valid because $\mathbf{D} - 2\beta\mathbf{M} \succcurlyeq 0$. Based on Lemma 2, the matrix $\mathbf{L} - \beta\mathbf{D} + \beta^2\mathbf{M} \succcurlyeq 0$ is necessarily positive semi-definite over $\mathbb{I}_\perp$. When considering the summation of these matrices, the matrix $\mathbf{A}(\hat{\lambda})$ must be positive definite on $\mathbb{I}_\perp$. Referring to Lemma 3, this matrix can have, at most, a single eigenvalue $\leqslant 0$. The other $N - 1$ eigenvalues must be positive. Our subsequent task is to establish that this eigenvalue must be negative rather than zero.

For $\mathbf{A}(\hat{\lambda})$, we have:

$$\mathbf{A}(\hat{\lambda}) = (\hat{\lambda} - \beta)^2\mathbf{M} + (\hat{\lambda} - \beta)\mathbf{D} + \mathbf{L} \tag{10}$$

$$\preccurlyeq \frac{(\hat{\lambda} - \beta)^2}{2\beta}\mathbf{D} + (\hat{\lambda} - \beta)\mathbf{D} + \mathbf{L} \tag{11}$$

$$= \frac{(\hat{\lambda} - \beta)(\hat{\lambda} + \beta)}{2\beta}\mathbf{D} + \mathbf{L} \tag{12}$$

$$\prec \mathbf{L} \tag{13}$$

where (11) is due to $\mathbf{D} - 2\beta\mathbf{M} \succcurlyeq 0$; (13) is due to $\hat{\lambda} - \beta < 0$. The above inequality implies:

$$\mathbf{1}^\top\mathbf{A}(\hat{\lambda})\mathbf{1} < \mathbf{1}^\top\mathbf{L}\mathbf{1} = 0 \tag{14}$$

From this, it's evident that the matrix $\mathbf{A}(\hat{\lambda})$ has at least one negative eigenvalue. Consequently, the last eigenvalue must be negative. Given $\mathbf{A}(\hat{\lambda})$ possesses one negative eigenvalue and $N - 1$ positive eigenvalues, $\mathbf{A}(\hat{\lambda})$ is full rank. The only feasible solution to the equation $\mathbf{A}(\hat{\lambda})\boldsymbol{\nu} = \mathbf{0}$ is $\boldsymbol{\nu} = \mathbf{A}(\hat{\lambda})^{-1}\mathbf{0} = \mathbf{0}$. This presents an inconsistency, implying that no real solution exists in the range $0 < \hat{\lambda} < \beta$ to solve the quadratic eigenvalue problem.

In summary, all the solutions to $\hat{\lambda}$ except the $0+\beta$ of the shifted eigenvalue problem are in the left half plane. Therefore, the solutions λ except 0 of the original eigenvalue problem are in the plane $\mathrm{Re}\left(\lambda\right) \leqslant -\beta$. □

Proof for Proposition 2

Proof. For real eigenvalue pairs, the condition is naturally satisfied because the discriminant $\left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \right\rangle^2 - 4\left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \right\rangle \left\langle \boldsymbol{\nu}, \mathbf{L}\boldsymbol{\nu} \right\rangle \geqslant 0$.

For complex conjugate eigenvalue pairs λ and λ^*, the necessary and sufficient condition is:

$$\begin{cases} \sin\zeta \cdot (\lambda + \lambda^*) \leqslant 0 \\ \left(\sin\zeta\right)^2 (\lambda + \lambda^*)^2 + \left(\cos\zeta\right)^2 (\lambda - \lambda^*)^2 \geqslant 0 \end{cases} \tag{15}$$

The first inequality is guaranteed because all the eigenvalues except 0 strictly lies in the left-half plane. For the second inequality, by taking

$$(\lambda + \lambda^*)^2 = \left(\frac{-\boldsymbol{\nu}^{\mathrm{H}}\mathbf{D}\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}}\right)^2 \tag{16}$$

$$(\lambda - \lambda^*)^2 = \left(\frac{-\boldsymbol{\nu}^{\mathrm{H}}\mathbf{D}\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}}\right)^2 - 4\frac{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{L}\boldsymbol{\nu}}{\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu}} \tag{17}$$

into the (15), we can easily get the condition. □

Proof for Theorem 1

Proof. From the first inequality of proposition 1, we have:

$$\left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \right\rangle \geqslant 2\beta \left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \right\rangle > 0 \tag{18}$$

As a result, we obtain:

$$\left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \right\rangle^2 \geqslant 2\beta \left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \right\rangle \left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \right\rangle \tag{19}$$

Therefore, in order to make the constraint of Proposition 2 hold, the following condition is sufficient:

$$2\beta \left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \right\rangle \left\langle \boldsymbol{\nu}, \mathbf{D}\boldsymbol{\nu} \right\rangle - 4\left(\cos\zeta\right)^2 \left\langle \boldsymbol{\nu}, \mathbf{L}\boldsymbol{\nu} \right\rangle \left\langle \boldsymbol{\nu}, \mathbf{M}\boldsymbol{\nu} \right\rangle \geqslant 0 \tag{20}$$

Since $\boldsymbol{\nu}^{\mathrm{H}}\mathbf{M}\boldsymbol{\nu} > 0$, $\beta\mathbf{D} - 2\left(\cos\zeta\right)^2\mathbf{L} \succcurlyeq 0$ implies (20). □

Proof for Proposition 3

Proof. Each of the properties is reasoned as follows:

1) Efficiency Maximization: This is obvious because the allocation function aims to minimize the total cost.

2) Individual Rationality: According to the payment function:

$$q_{\hat{k}} = \min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) - \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k^\star, d_k^\star\right) \tag{21}$$

If the optimal solution exists for $\min \sum_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right)$, denoted as:

$$\hat{\boldsymbol{m}}_{-\hat{k}}^\star, \hat{\boldsymbol{d}}_{-\hat{k}}^\star = \min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) \tag{22}$$

Clearly, $\boldsymbol{m}_{-\hat{k}} = \hat{\boldsymbol{m}}_{-\hat{k}}^\star, \boldsymbol{d}_{-\hat{k}} = \hat{\boldsymbol{d}}_{-\hat{k}}^\star, m_{\hat{k}} = 0, d_{\hat{k}} = 0$ is one of the feasible solutions to the original cost minimization problem. Thus, the corresponding cost must be larger than the cost of the original optimal solution $\boldsymbol{m} = \boldsymbol{m}^\star, \boldsymbol{d} = \boldsymbol{d}^\star$.

3) Incentive Compatible: When generator $\hat{k}$ bid truthfully, the allocation is:

$$\boldsymbol{m}^\star, \boldsymbol{d}^\star = \arg\min \left(\mathcal{C}_{\hat{k}}\left(m_{\hat{k}}, d_{\hat{k}}\right) + \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right)\right) \tag{23}$$

Now, suppose generator $\hat{k}$ intentionally misreports its cost and bids $\mathcal{C}'_{\hat{k}}\left(m_{\hat{k}}, d_{\hat{k}}\right)$, then the new allocation and payment will be:

$$\begin{aligned} \tilde{\boldsymbol{m}}^\star, \tilde{\boldsymbol{d}}^\star &= \arg\min \left(\mathcal{C}'_{\hat{k}}\left(m_{\hat{k}}, d_{\hat{k}}\right) + \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right)\right) \\ \tilde{q}_{\hat{k}} &= \min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) - \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(\tilde{m}_k^\star, \tilde{d}_k^\star\right) \end{aligned} \tag{24}$$

The total profit for generator $\hat{k}$ should be:

$$\begin{aligned} \tilde{u}_{\hat{k}}^\star = &\min \sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(m_k, d_k\right) - \\ &\left(\sum\nolimits_{-\hat{k}} \mathcal{B}_k\left(\tilde{m}_k^\star, \tilde{d}_k^\star\right) + \mathcal{C}_{\hat{k}}\left(\tilde{m}_{\hat{k}}^\star, \tilde{d}_{\hat{k}}^\star\right)\right) \end{aligned} \tag{25}$$

where $\sum_{-\hat{k}} \mathcal{B}_k\left(\tilde{m}_k^\star, \tilde{d}_k^\star\right) + \mathcal{C}_{\hat{k}}(\tilde{m}_{\hat{k}}^\star, \tilde{d}_{\hat{k}}^\star)$ must be larger than $\sum_{-\hat{k}} \mathcal{B}_k\left(m_k^\star, d_k^\star\right) + \mathcal{C}_{\hat{k}}(m_{\hat{k}}^\star, d_{\hat{k}}^\star)$ because of (23). Thus, for a rational generator seeking to maximize its profit, its dominant strategy is to submit its true costs. □