Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Priority Queues

Priority Queues

- Define priority queue ADT
- Implement a priority queue with an unsorted and sorted lists
- Explain Heap data structure
- Implement a priority queue with a heap
- Analyze heap-based priority queues
- Sorting with a priority queue

Trees Review

- Hierarchical data structure that consists of **nodes** with a **parent-child relation**
- **Root**: node **without parent** (A)
- **Internal node**: node with **at least one child** (A, B, C, F)
- **External node** (a.k.a. **leaf**): node without children (E, I, J, K, G, H, D)
- **Ancestors** of a node: parent, grandparent, grand-grandparent, etc.
- **Depth of a node**: number of ancestors
- **Height of a tree**: maximum depth of any node (3)
- **Descendant of a node**: child, grandchild, grand-grandchild, etc.
- **Subtree**: tree consisting of a node and its descendants
- A tree is ***ordered*** if there is a meaningful linear order among the children of each node
- Tree ADT
  - We **use positions** to abstract nodes
  - **Tree interface** for trees where nodes can have any number of children
    - parent
    - children
    - numCHildren

Trees Review

- isInternal
- isExternal
- isRoot
- isEmpty
- Iterator
- position

- **AbstractTree Base Class implements:**
  - isInternal
  - isExternal
  - isRoot
  - isEmpty

- **Tree Traversal**
  - a **preorder** traversal of a tree T, the root of T is visited first and then the subtrees rooted at its children are traversed recursively
  - **postorder** traversal of a tree it recursively traverses the subtrees rooted at the children of the root first, and then visits the root

Trees Review

- **Binary Trees**
  - Each internal node has **at most two children** (exactly two for proper binary trees)
  - The children of a node are an ordered pair
  - We call the children of an internal node left child and right child
  - Applications
    - arithmetic expressions
    - decision processes
    - searching
- **BinaryTree ADT**
- **BinaryTree** interface extends the Tree interface and adds three methods:
  - position left(p)
  - position right(p)
  - position sibling(p)
- **AbstractBinaryTree** Base Class extends **AbstractTree** and implement **BinaryTree**
- Implements:
  - sibling
  - numChildren
  - children

Trees Review

- **Binary Trees**
  - **inorder traversal** a node is visited after its left subtree and before its right subtree
- **Implementing Trees**
  - Using a **linked structure**
  - A node is represented by an object storing
    - Element
    - Parent node
    - Sequence of children nodes
  - Node objects implements the Position ADT
- **Linked Structure for Binary Trees**
  - A node is represented by an object storing
    - **Element**
    - **Parent node**
    - **Left child node**
    - **Right child node**
  - Node objects implements the Position ADT
- **LinkedBinaryTree class**
  - Inner **Node** class
  - Two instance variables: ***root*** and ***size***
  - Updating operations

Priority Queue ADT

- FIFO principle used by queues does not suffice – priorities must come into play
- A **priority queue** stores a collection of entries
- Each entry is a pair **(key, value)**
- Main methods of the Priority Queue ADT
  - insert(k, v) - inserts an entry with key k and value v
  - removeMin() - removes and returns the **entry with smallest key**, or null if the the priority queue is empty
- Additional methods
  - min() - returns, but does not remove, an entry with smallest key, or null if the the priority queue is empty
  - size()
  - isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Example

- A sequence of priority queue methods:

Method	Return Value	Priority Queue Contents
insert(5,A)		{ (5,A) }
insert(9,C)		{ (5,A), (9,C) }
insert(3,B)		{ (3,B), (5,A), (9,C) }
min( )	(3,B)	{ (3,B), (5,A), (9,C) }
removeMin( )	(3,B)	{ (5,A), (9,C) }
insert(7,D)		{ (5,A), (7,D), (9,C) }
removeMin( )	(5,A)	{ (7,D), (9,C) }
removeMin( )	(7,D)	{ (9,C) }
removeMin( )	(9,C)	{ }
removeMin( )	null	{ }
isEmpty( )	true	{ }

Total Order Relations

- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct entries in a priority queue **can have the same key**

- Mathematical concept of total order relation $\leq$
  - **Comparability** property: either $x \leq y$ or $y \leq x$
  - **Antisymmetric** property: $x \leq y$ and $y \leq x \Rightarrow x = y$
  - **Transitive** property: $x \leq y$ and $y \leq z \Rightarrow x \leq z$

Entry ADT

- An entry in a priority queue is simply a **key-value pair**
- Priority queues store entries to allow for efficient insertion and removal based on keys
- Methods:
  - getKey: returns the key for this entry
  - getValue: returns the value associated with this entry

- As a Java interface:

```
/**
 * Interface for a key-value
 * pair entry
 **/
public interface Entry<K,V>
  {
    K getKey();
    V getValue();
  }
```

Comparator ADT

- A **comparator** encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses an auxiliary comparator
- The comparator is external to the keys being compared
- When the priority queue needs **to compare two keys**, it **uses its comparator**

- Primary method of the **Comparator** ADT
- compare(x, y): returns an integer i such that
  - $i < 0$ if $a < b$,
  - $i = 0$ if $a = b$
  - $i > 0$ if $a > b$
  - An error occurs if a and b cannot be compared.

Example Comparator

- As a concrete example, Code Fragment 9.3 defines a comparator that evaluates strings based on their length (rather than their natural lexicographic order):

```
public class StringLengthComparator implements Comparator<String> {
  /** Compares two strings according to their lengths. */
  public int compare(String a, String b) {
    if (a.length() < b.length()) return −1;
    else if (a.length() == b.length()) return 0;
    else return 1;
  }
}
```

Code Fragment 9.3: A comparator that evaluates strings based on their lengths.

The AbstractPriorityQueue Base Class

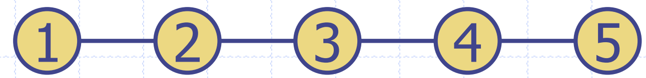
- The base class provides four means of support:
  1. a **PQEntry** class as a concrete implementation of the entry interface
  2. an instance variable ***comp*** for a general Comparator and a protected method, ***compare*(a, b)**, that makes use of the comparator.
  3. a boolean ***checkKey*** method that verifies that a given key is appropriate for use with the comparator
  4. an ***isEmpty*** implementation based upon the abstract ***size()*** method.

Sequence-based Priority Queue

- Implementation with an unsorted list

- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list



- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Unsorted List Implementation

```
/** An implementation of a priority queue with an unsorted list. */
public class UnsortedPriorityQueue<K,V> extends AbstractPriorityQueue<K,V> {
  /** primary collection of priority queue entries */
  private PositionalList<Entry<K,V>> list = new LinkedPositionalList<>();

  /** Creates an empty priority queue based on the natural ordering of its keys. */
  public UnsortedPriorityQueue() { super(); }
  /** Creates an empty priority queue using the given comparator to order keys. */
  public UnsortedPriorityQueue(Comparator<K> comp) { super(comp); }

  /** Returns the Position of an entry having minimal key. */
  private Position<Entry<K,V>> findMin() {    // only called when nonempty
    Position<Entry<K,V>> small = list.first();
    for (Position<Entry<K,V>> walk : list.positions())
      if (compare(walk.getElement(), small.getElement()) < 0)
        small = walk;         // found an even smaller key
    return small;
  }

```

Unsorted List Implementation, 2

```
/** Inserts a key-value pair and returns the entry created. */
public Entry<K,V> insert(K key, V value) throws IllegalArgumentException {
  checkKey(key);      // auxiliary key-checking method (could throw exception)
  Entry<K,V> newest = new PQEntry<>(key, value);
  list.addLast(newest);
  return newest;
}

/** Returns (but does not remove) an entry with minimal key. */
public Entry<K,V> min() {
  if (list.isEmpty()) return null;
  return findMin().getElement();
}

/** Removes and returns an entry with minimal key. */
public Entry<K,V> removeMin() {
  if (list.isEmpty()) return null;
  return list.remove(findMin());
}

/** Returns the number of items in the priority queue. */
public int size() { return list.size(); }
}
```

Sorted List Implementation

```
/** An implementation of a priority queue with a sorted list. */
public class SortedPriorityQueue<K,V> extends AbstractPriorityQueue<K,V> {
  /** primary collection of priority queue entries */
  private PositionalList<Entry<K,V>> list = new LinkedPositionalList<>();

  /** Creates an empty priority queue based on the natural ordering of its keys. */
  public SortedPriorityQueue() { super(); }
  /** Creates an empty priority queue using the given comparator to order keys. */
  public SortedPriorityQueue(Comparator<K> comp) { super(comp); }

  /** Inserts a key-value pair and returns the entry created. */
  public Entry<K,V> insert(K key, V value) throws IllegalArgumentException {
    checkKey(key);      // auxiliary key-checking method (could throw exception)
    Entry<K,V> newest = new PQEntry<>(key, value);
    Position<Entry<K,V>> walk = list.last();
    // walk backward, looking for smaller key
    while (walk != null && compare(newest, walk.getElement()) < 0)
      walk = list.before(walk);
    if (walk == null)
      list.addFirst(newest);                          // new key is smallest
    else
      list.addAfter(walk, newest);                    // newest goes after walk
    return newest;
  }

```

Sorted List Implementation, 2

```
  /** Returns (but does not remove) an entry with minimal key. */
  public Entry<K,V> min() {
    if (list.isEmpty()) return null;
    return list.first().getElement();
  }

  /** Removes and returns an entry with minimal key. */
  public Entry<K,V> removeMin() {
    if (list.isEmpty()) return null;
    return list.remove(list.first());
  }

  /** Returns the number of items in the priority queue. */
  public int size() { return list.size(); }
}
```

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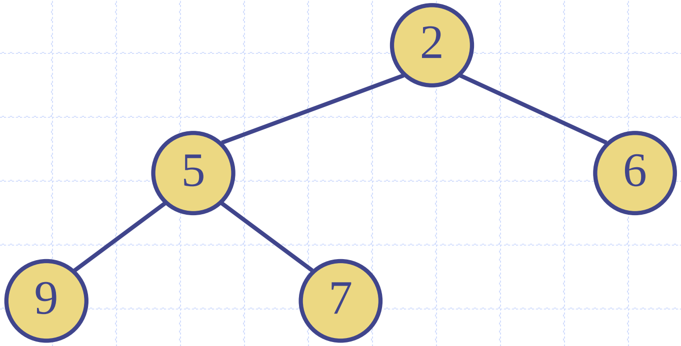
Heaps

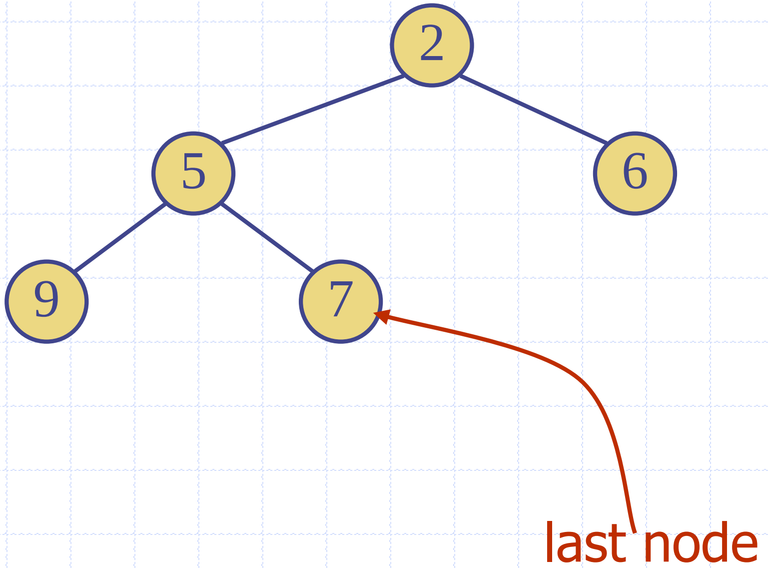


Recall Priority Queue ADT

- A priority queue stores a collection of entries
- Each entry is a pair (key, value)
- Main methods of the Priority Queue ADT
  - insert(k, v) - inserts an entry with key k and value v
  - removeMin() - removes and returns the entry with smallest key
- Additional methods
  - min() - returns, but does not remove, an entry with smallest key
  - size()
  - isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Heaps

- A **heap is a binary tree storing keys at its nodes** and satisfying the following properties:
- Heap-Order: for every internal node v other than the root, $key(v) \geq key(parent(v))$
- Complete Binary Tree: let h be the height of the heap
  - for $i = 0, \dots, h - 1$, there are 2^i nodes of depth i
  - at depth $h - 1$, the internal nodes are to the left of the external nodes
- The two nodes in level 2 are in the two leftmost possible positions at that level.
- The last node of a heap is the rightmost node of maximum depth

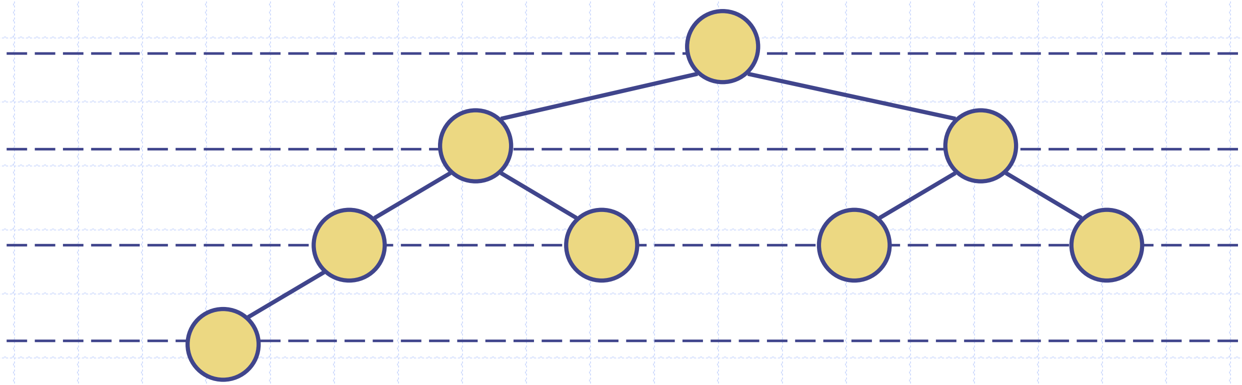


Height of a Heap

- **Theorem:** A heap storing n keys has height $O(\log n)$

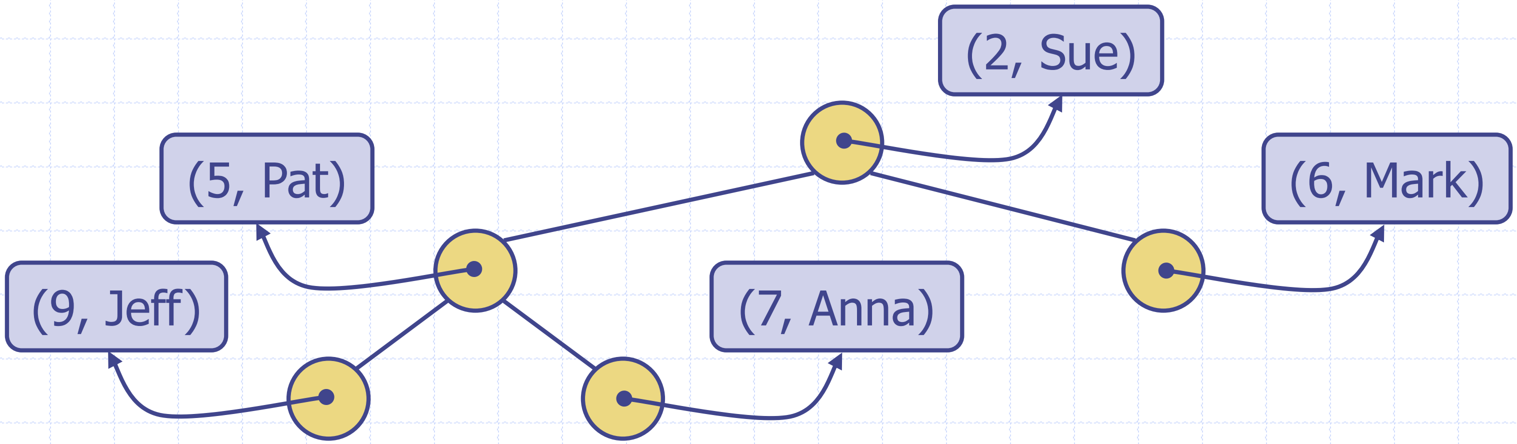
  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \dots, h-1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \dots + 2^{h-1} + 1$
  - Thus, $n \geq 2^h$, i.e., $h \leq \log n$

depth	keys
0	1
1	2
$h-1$	2^{h-1}
h	1



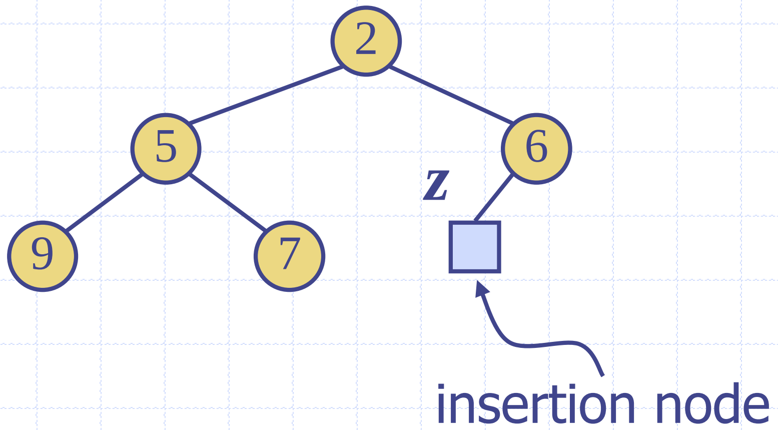
Heaps and Priority Queues

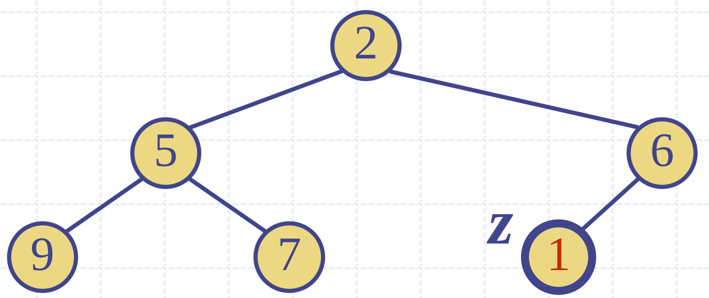
- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node



Insertion into a Heap

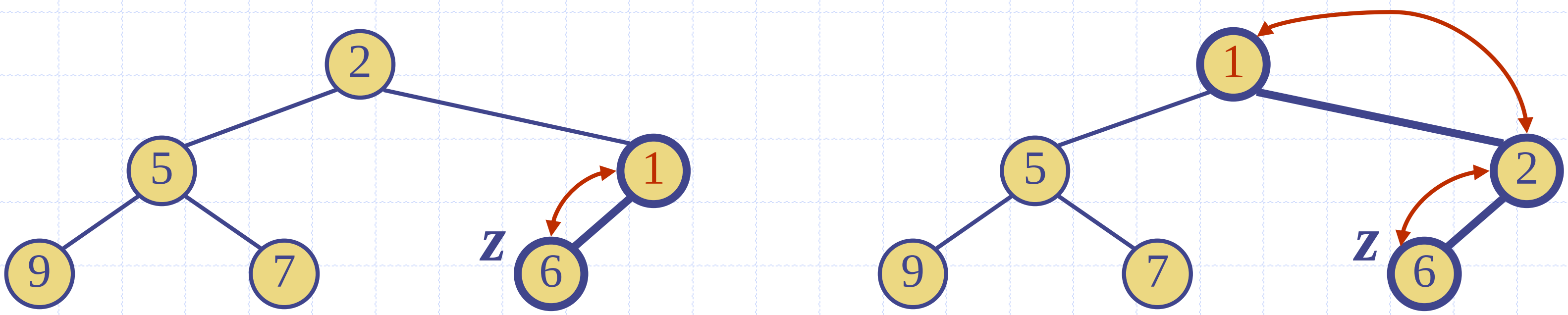
- Method *insertItem* of the priority queue ADT corresponds to the insertion of a key k to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z
  - Restore the heap-order property (discussed next)





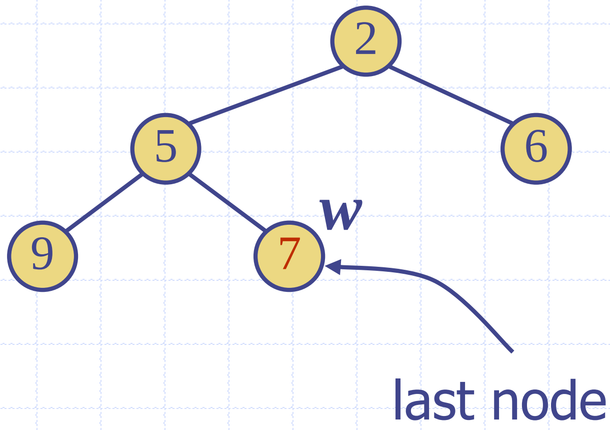
Upheap

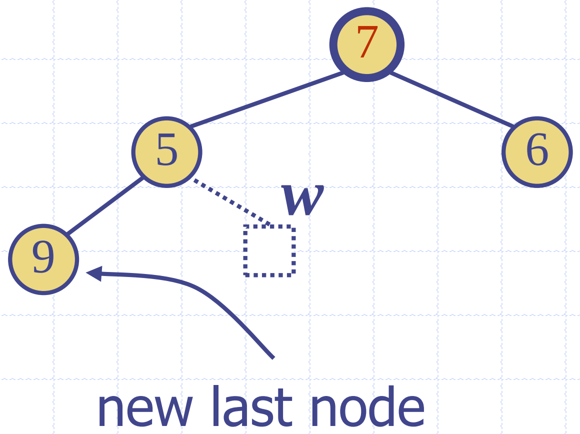
- After the insertion of a new key k, the heap-order property may be violated
- Algorithm **upheap** restores the heap-order property by **swapping k along an upward path from the insertion node**
- **Upheap** terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, **upheap** runs in $O(\log n)$ time



Removal from a Heap

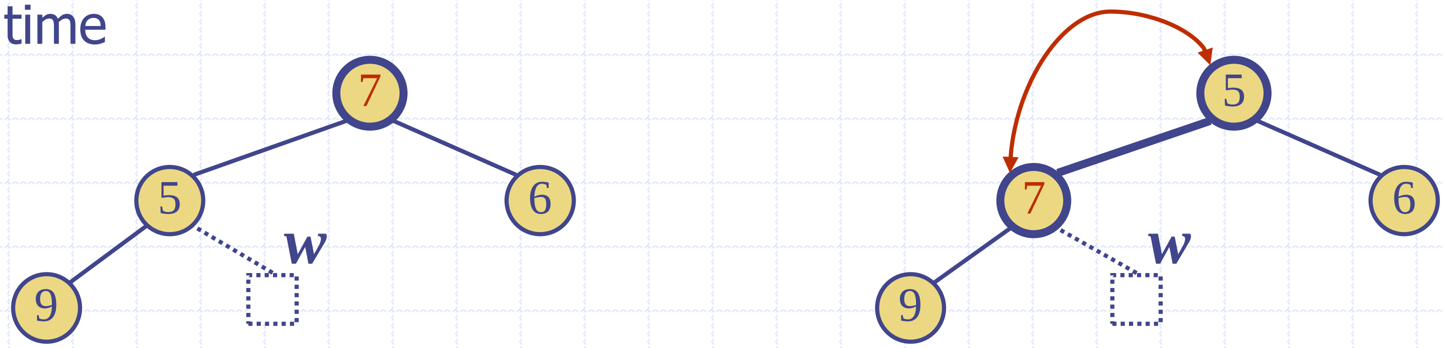
- Method *removeMin* of the priority queue ADT corresponds to the **removal of the root key from the heap**
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Remove w
  - Restore the heap-order property (discussed next)





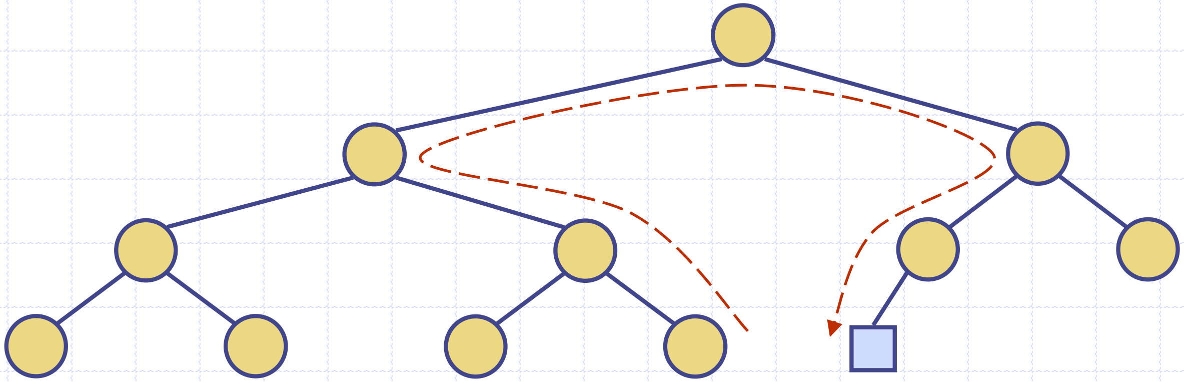
Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm **downheap** restores the heap-order property by **swapping key k along a downward path from the root**
- **Downheap** terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, **downheap** runs in $O(\log n)$ time



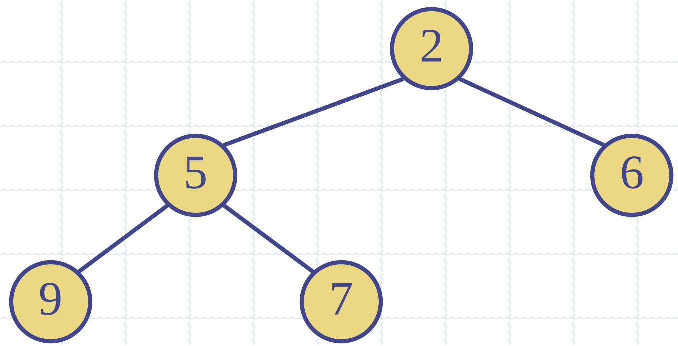
Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal



Array-based Heap Implementation

- We can represent a heap with n keys by means of an array of length n
- For the node at rank i
  - the left child is at rank $2i + 1$
  - the right child is at rank $2i + 2$
- Links between nodes are not explicitly stored
- Operation *add* corresponds to inserting at rank $n + 1$
- Operation *remove_min* corresponds to removing at rank n
- Yields in-place heap-sort



2	5	6	9	7
0	1	2	3	4

Java Implementation

```
/** An implementation of a priority queue using an array-based heap. */
public class HeapPriorityQueue<K,V> extends AbstractPriorityQueue<K,V> {
  /** primary collection of priority queue entries */
  protected ArrayList<Entry<K,V>> heap = new ArrayList<>();
  /** Creates an empty priority queue based on the natural ordering of its keys. */
  public HeapPriorityQueue() { super(); }
  /** Creates an empty priority queue using the given comparator to order keys. */
  public HeapPriorityQueue(Comparator<K> comp) { super(comp); }
  // protected utilities
  protected int parent(int j) { return (j-1) / 2; }          // truncating division
  protected int left(int j) { return 2*j + 1; }
  protected int right(int j) { return 2*j + 2; }
  protected boolean hasLeft(int j) { return left(j) < heap.size(); }
  protected boolean hasRight(int j) { return right(j) < heap.size(); }
  /** Exchanges the entries at indices i and j of the array list. */
  protected void swap(int i, int j) {
    Entry<K,V> temp = heap.get(i);
    heap.set(i, heap.get(j));
    heap.set(j, temp);
  }
  /** Moves the entry at index j higher, if necessary, to restore the heap property. */
  protected void upheap(int j) {
    while (j > 0) {              // continue until reaching root (or break statement)
      int p = parent(j);
      if (compare(heap.get(j), heap.get(p)) >= 0) break;   // heap property verified
      swap(j, p);
      j = p;                                    // continue from the parent's location
    }
  }
```

Java Implementation, 2

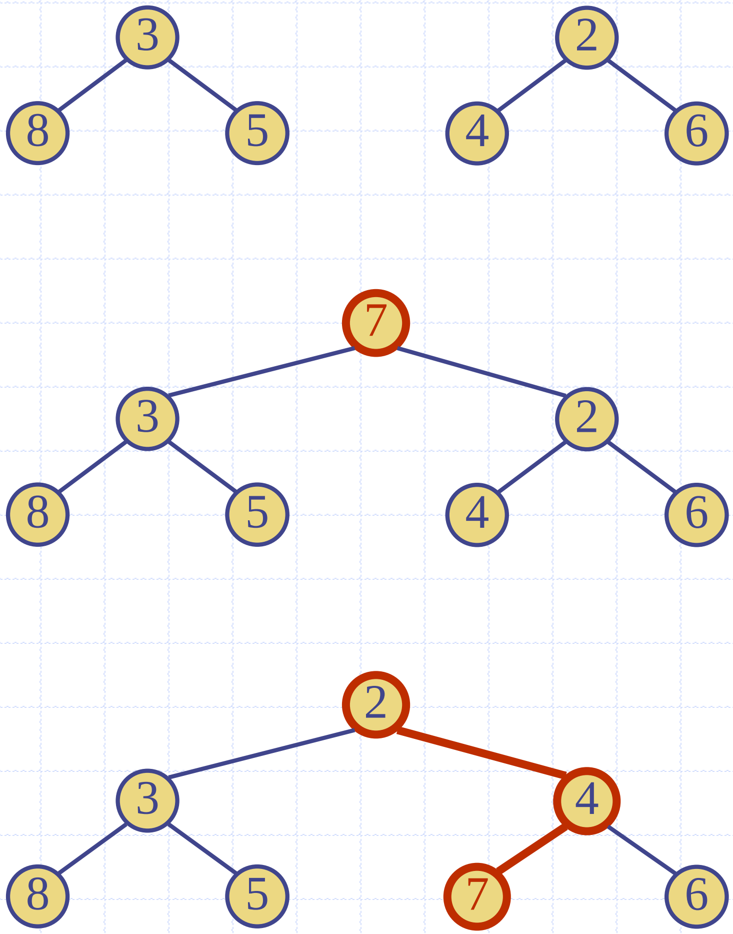
```
/** Moves the entry at index j lower, if necessary, to restore the heap property. */
protected void downheap(int j) {
  while (hasLeft(j)) {                    // continue to bottom (or break statement)
    int leftIndex = left(j);
    int smallChildIndex = leftIndex;          // although right may be smaller
    if (hasRight(j)) {
        int rightIndex = right(j);
        if (compare(heap.get(leftIndex), heap.get(rightIndex)) > 0)
          smallChildIndex = rightIndex;       // right child is smaller
    }
    if (compare(heap.get(smallChildIndex), heap.get(j)) >= 0)
      break;                                  // heap property has been restored
    swap(j, smallChildIndex);
    j = smallChildIndex;                      // continue at position of the child
  }
}

// public methods
/** Returns the number of items in the priority queue. */
public int size() { return heap.size(); }
/** Returns (but does not remove) an entry with minimal key (if any). */
public Entry<K,V> min() {
  if (heap.isEmpty()) return null;
  return heap.get(0);
}
```

Java Implementation, 3

```java
/** Inserts a key-value pair and returns the entry created. */
public Entry<K,V> insert(K key, V value) throws IllegalArgumentException {
  checkKey(key);      // auxiliary key-checking method (could throw exception)
  Entry<K,V> newest = new PQEntry<>(key, value);
  heap.add(newest);                           // add to the end of the list
  upheap(heap.size() - 1);                    // upheap newly added entry
  return newest;
}
/** Removes and returns an entry with minimal key (if any). */
public Entry<K,V> removeMin() {
  if (heap.isEmpty()) return null;
  Entry<K,V> answer = heap.get(0);
  swap(0, heap.size() - 1);                   // put minimum item at the end
  heap.remove(heap.size() - 1);               // and remove it from the list;
  downheap(0);                                // then fix new root
  return answer;
}
}
```

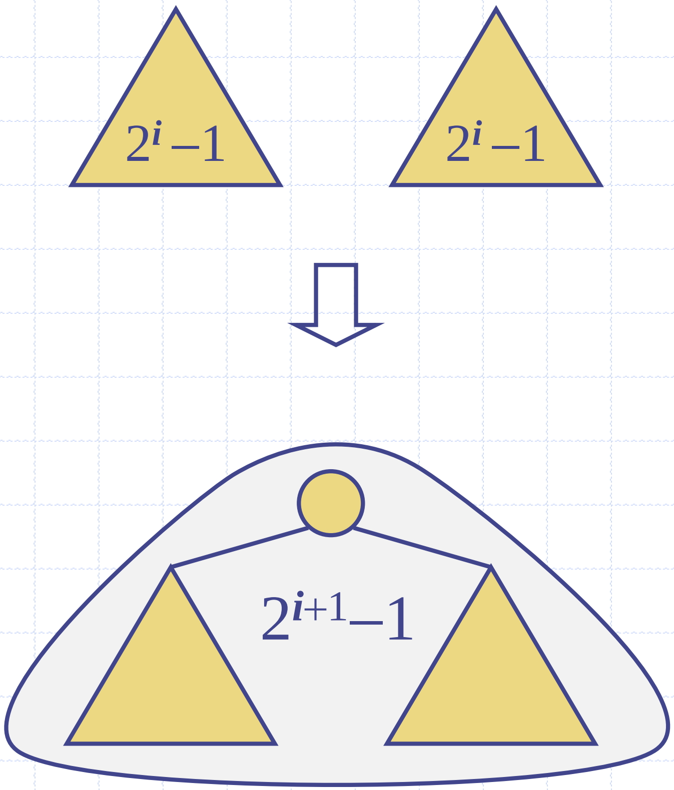
Merging Two Heaps

- We are given two two heaps and a key ***k***
- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We perform *downheap* to restore the heap-order property

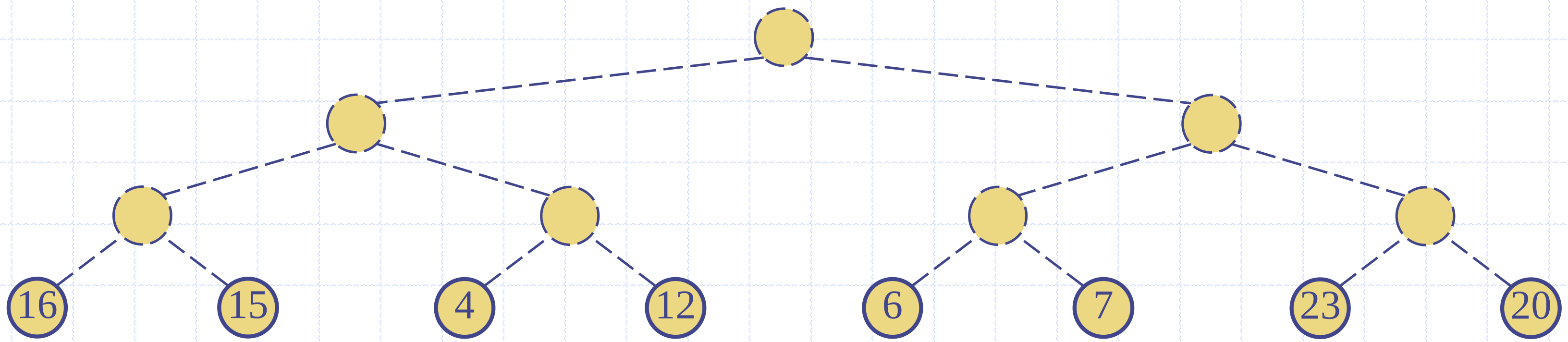


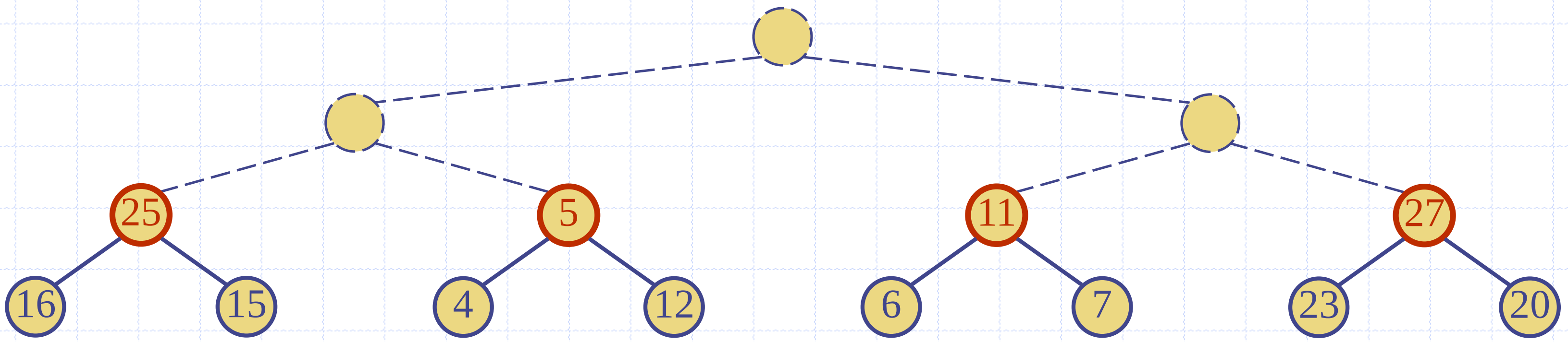
Bottom-up Heap Construction

- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^{i}-1$ keys are merged into heaps with $2^{i+1}-1$ keys ($2^{i}-1 + 2^{i}-1 + 1$)

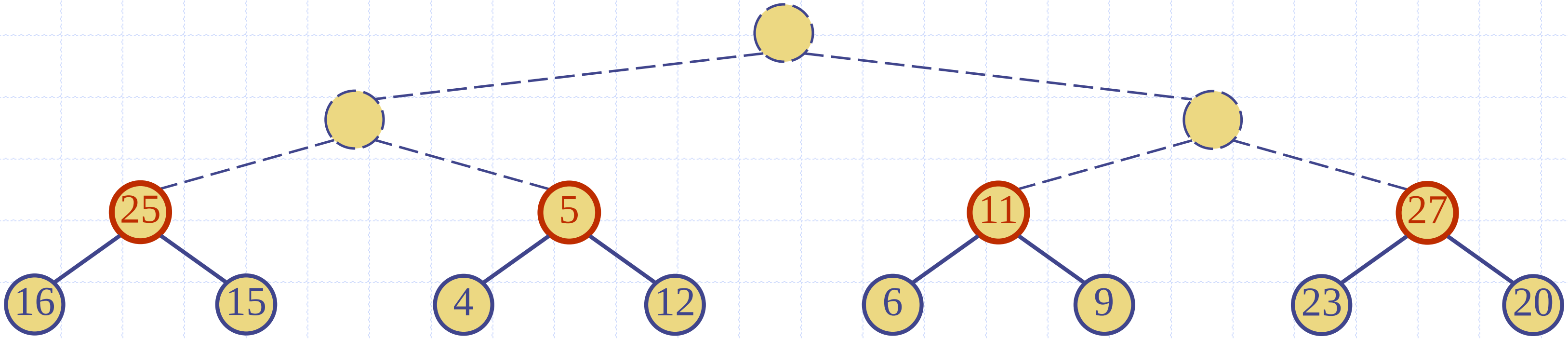


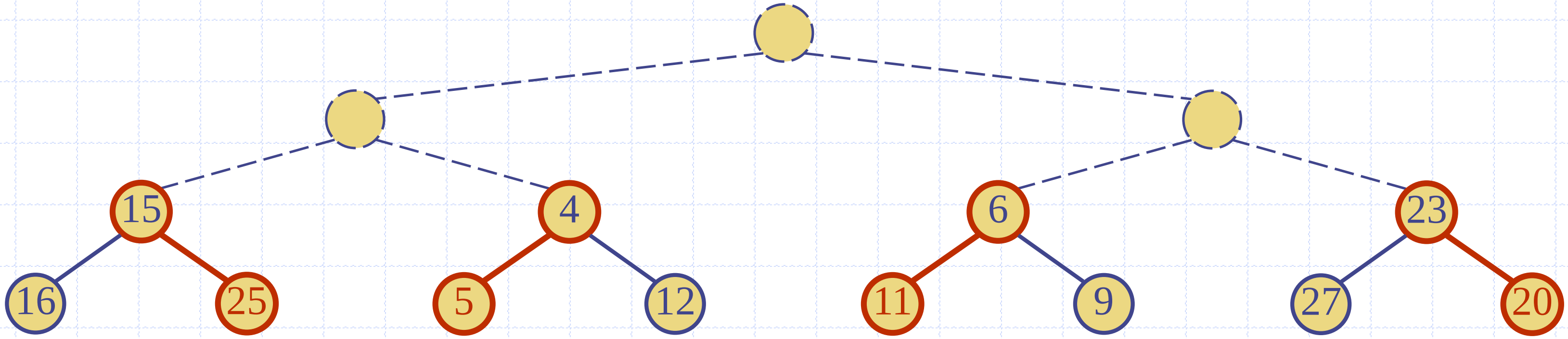
Example



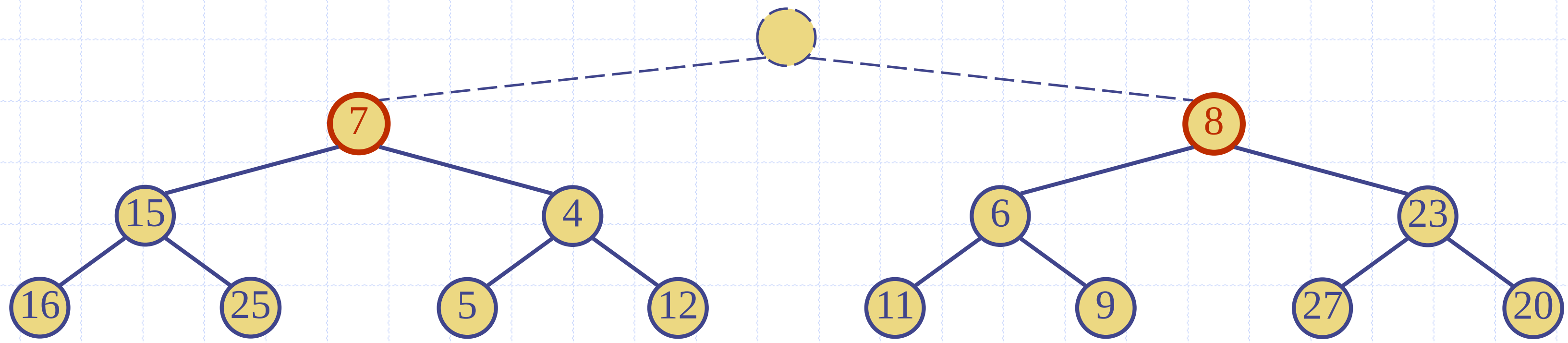


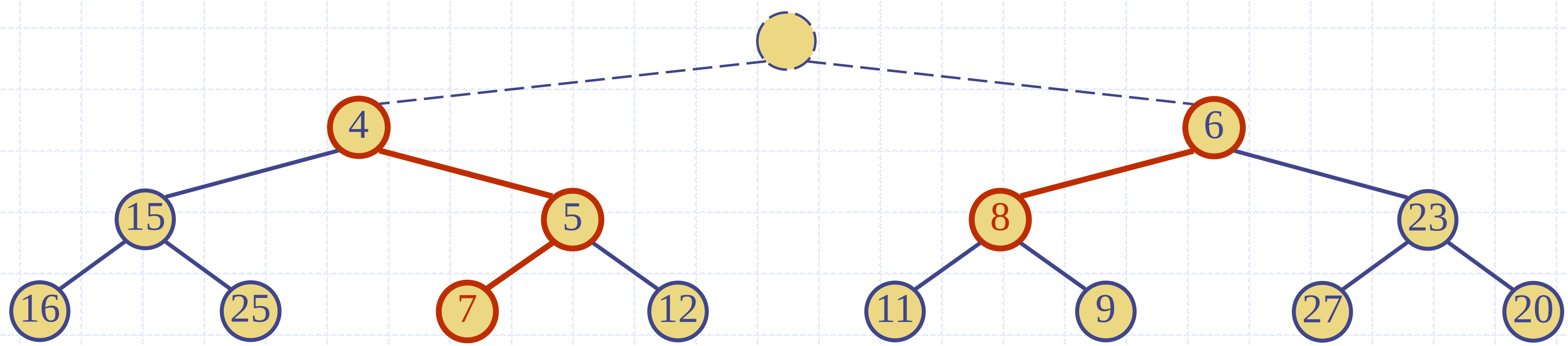
Example (contd.)



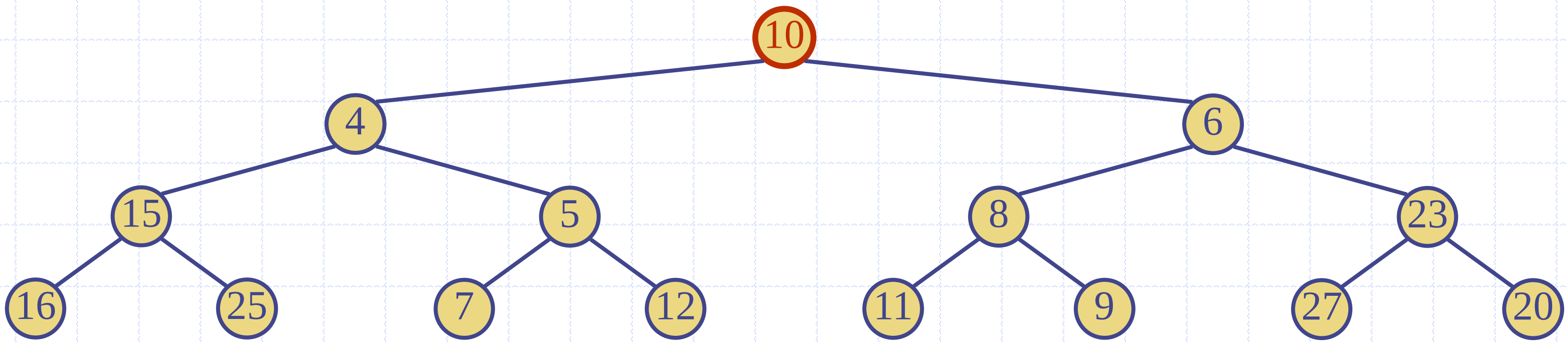


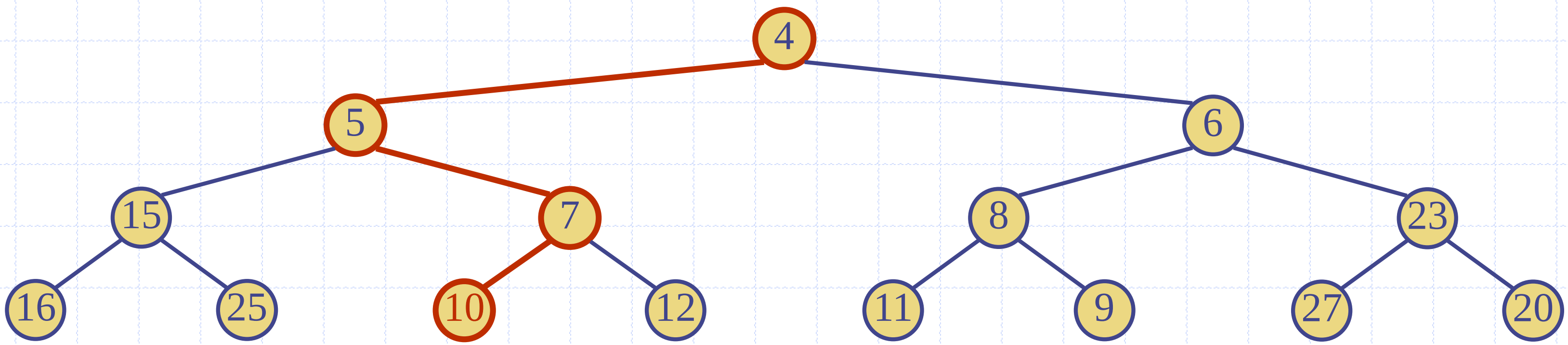
Example (contd.)



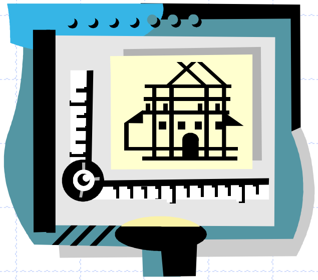


Example (end)

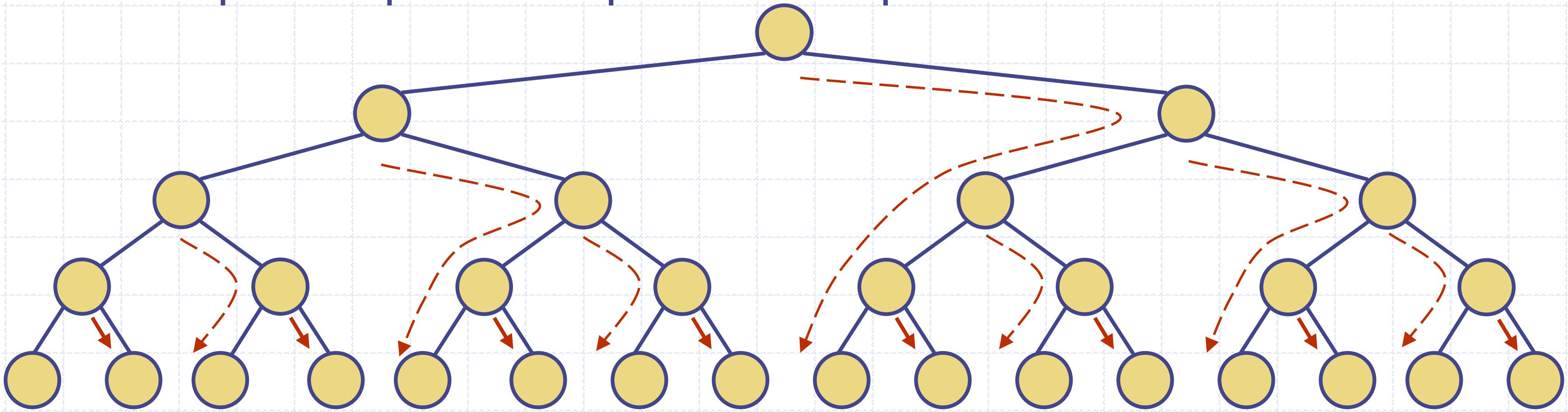




Analysis



- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



Priority Queue Sorting

- We can **use a priority queue to sort a list of comparable elements**
  1. Insert the elements one by one with a series of insert operations
  2. Remove the elements in sorted order with a series of removeMin operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm *PQ-Sort*(*S*, *C*)
Input list *S*, comparator *C* for the elements of *S*
Output list *S* sorted in increasing order according to *C*
$P \leftarrow$ priority queue with comparator C
while $\neg$*S.isEmpty* ()
  $e \leftarrow$ *S.remove*(*S.first* ())
  P.insert (*e*, $\emptyset$)
while $\neg$*P.isEmpty*()
  $e \leftarrow$ *P.removeMin*().*getKey*()
  S.addLast(*e*)

Selection-Sort

- Selection-sort is the variation of PQ-sort where the **priority queue is implemented with an unsorted sequence**
- Running time of Selection-sort:
  1. Inserting the elements into the priority queue with n insert operations takes $O(n)$ time
  2. Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

Selection-Sort Example

	Sequence S	Priority Queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(7,4)
..	.. ..	
(g)	()	(7,4,8,2,5,3,9)
Phase 2		
(a)	(2)	(7,4,8,5,3,9)
(b)	(2,3)	(7,4,8,5,9)
(c)	(2,3,4)	(7,8,5,9)
(d)	(2,3,4,5)	(7,8,9)
(e)	(2,3,4,5,7)	(8,9)
(f)	(2,3,4,5,7,8)	(9)
(g)	(2,3,4,5,7,8,9)	()

Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the **priority queue is implemented with a sorted sequence**
- Running time of Insertion-sort:
  1. Inserting the elements into the priority queue with n insert operations takes time proportional to

     $$1 + 2 + \ldots + n$$

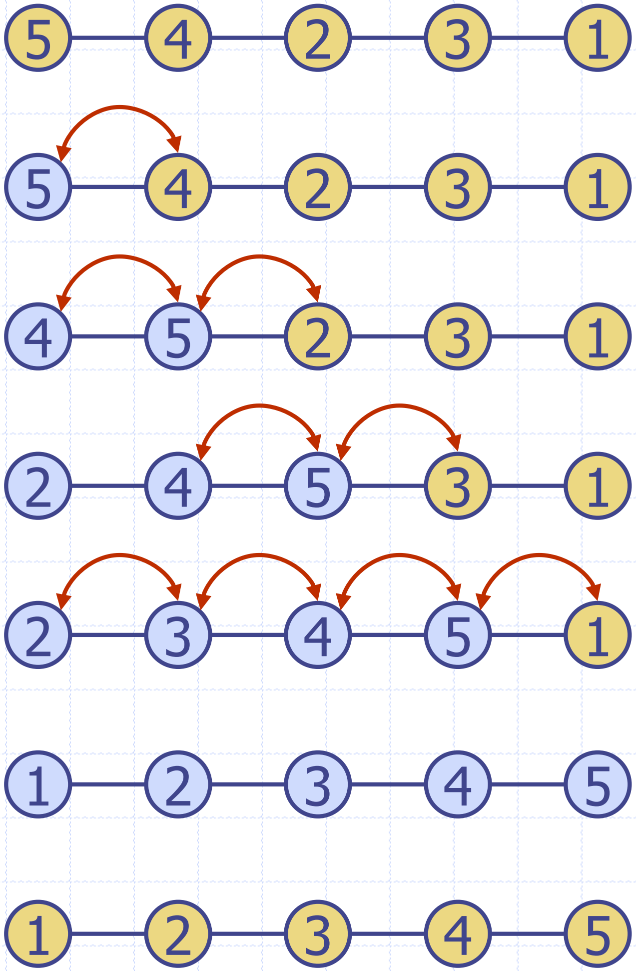
  2. Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

Insertion-Sort Example

	Sequence S	Priority queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(4,7)
(c)	(2,5,3,9)	(4,7,8)
(d)	(5,3,9)	(2,4,7,8)
(e)	(3,9)	(2,4,5,7,8)
(f)	(9)	(2,3,4,5,7,8)
(g)	()	(2,3,4,5,7,8,9)
Phase 2		
(a)	(2)	(3,4,5,7,8,9)
(b)	(2,3)	(4,5,7,8,9)
..	..	..
(g)	(2,3,4,5,7,8,9)	()

In-place Insertion-Sort

- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place
- A portion of the input sequence itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the sequence
  - We can use swaps instead of modifying the sequence

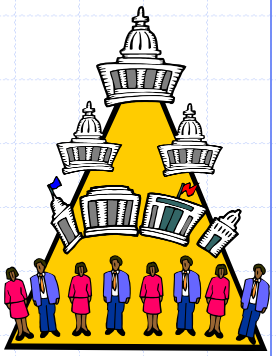


Recall PQ Sorting

- We use a priority queue
  - Insert the elements with a series of insert operations
  - Remove the elements in sorted order with a series of removeMin operations
- The running time depends on the priority queue implementation:
  - Unsorted sequence gives selection-sort: $O(n^2)$ time
  - Sorted sequence gives insertion-sort: $O(n^2)$ time
- Can we do better?

Algorithm ***PQ-Sort***(***S***, ***C***)
    Input sequence ***S***, comparator ***C*** for the elements of ***S***
    Output sequence ***S*** sorted in increasing order according to ***C***
    P ← priority queue with
        comparator ***C***
    while ¬***S.isEmpty*** ()
        e ← ***S.remove*** (***S. first*** ())
        P.insert (***e***, ***e***)
    while ¬***P.isEmpty***()
        e ← ***P.removeMin***().***getKey***()
        S.addLast(***e***)

Heap-Sort



- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, isEmpty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called **heap-sort**
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort