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CONVEX HULL MADE EASY

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Given a finite set S of n points and a point q in R^d, deciding if $q \in \operatorname{conv} S$ is a matter of one linear program of size n, after a very simple linear-time preprocessing of S.

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For a set S and a point q in R^d, let conv S denote the convex hull of S, and int S the interior of S.

Suppose q is a point and S is a d-dimensional set of n points in R^d. The following is a reduction of the decision if $q \in \operatorname{conv} S$ to one linear program. It is so irresistably easy to program, that perhaps it deserves the attention of every owner of a Simplex program in working order.

Preprocessing—given S

Find a point s_0 in int conv S. ($s_0 = (1/n)\Sigma_{s \in S} s$ will do. Finding an affine independent subset T of S, and then $s = [1/(d+1)]\Sigma_{s \in T} s$ may be faster when n is large.)

Compute the list $S' = \{s - s_0 : s \in S\}$.

The decision—given q

Let $q' = q - s_0$. Solve the linear program:

$$m = \max\langle q', x\rangle$$

$$\text{s.t. } \langle s', x\rangle \leqslant 1 \quad \text{for all } s' \in S'.$$

$$q \in \operatorname{conv} S \text{ iff } m \leqslant 1.$$

Proof. The set

$$P^* = \{x : \langle s', x\rangle \leqslant 1 \text{ for all } s' \in S'\}$$

is called the *polar polytope* of $P = \operatorname{conv} S'$, and the hyperplane $Q^* = \{x : \langle q', x\rangle = 1\}$ is *dual by polarity* to the point q' (see [1, p. 25]). Since int P includes the origin O, it is well known that $Q^* \cap \operatorname{int} P^* = 0$ iff $q' \in P$, and, clearly, $q' \in P$ iff $q \in \operatorname{conv} S$. But O is a point in int P^*, and $\langle q', O\rangle = 0 < 1$, hence $Q^* \cap \operatorname{int} P^* = 0$ iff $m \leqslant 1$.

Remarks. (1) If V is a vertex of P for which $\langle q', v\rangle = m$, then the hyperplane $H = \{x : \langle v, x\rangle = m\}$ separates q' from P, and it supports a facet of P. It follows that the hyperplane $s_0 + H$ separates q from conv S, and it supports a facet of conv S.

(2) If $m = 1$, then q lies on the boundary of conv S.

(3) If many points q are to be tested versus one and the same S, some time can be saved by deleting all non-vertices from S during the preprocessing.

Reference

[1] H.G. Eggleston, Convexity (Cambridge University Press, London, 1969).