Proofs of Results for the Yule Process

Victor Deshors

Introduction

This document presents the proofs of several results related to the Yule process, a continuous-time branching process introduced by Yule in 1924. The Yule process models the growth of a population where each individual lives for an exponentially distributed time and reproduces by generating two offspring upon death.

Definition of a Yule process

- Let $\lambda > 0$ be a parameter.
- Starting from an individual, we are interested in the evolution of a population where:

- Each individual lives for a random amount of time following an exponential distribution with parameter λ (independently of other individuals).
- An individual dies by giving birth to two new individuals.
- Individuals reproduce at a rate λ.

- Let Y_t denote the number of individuals present at time t:

- $Y_0 = 0$ (no individuals at the start).
- Y_t increases by 1 every time an individual dies.

- The process $(Y_t)_{t\geq 0}$ is called a Yule process with parameter λ.
- For an integer $C \geq 1$, we define:

- $Y_t(C)$ as the number of individuals present at time t when starting with C individuals at $t = 0$.
- Thus, $Y_t = Y_t(1)$ when starting with one individual.

The following can also be an alternate definition :

(A) Consider a sequence of independent random variables $(E_i : i \geq 1)$ such that E_i follows an exponential distribution with parameter λ_i for all $i \geq 1$. Then the process $(Y_t)_{t\geq 0}$ defined by

$$Y_t = 1 + \inf\left\{k \geq 0 : \sum_{i=1}^{k} E_i \geq t\right\}$$

is a Yule process with parameter λ. In this representation, E_i is the time of the i-th death.

Results to prove

- (B): The number of individuals at time t, Y_t, follows a geometric distribution with parameter $e^{-\lambda t}$.
- (C) For any integer $C \geq 1$, $L_t = e^{F(Y_{t-1})-\lambda C t}$ satisfies $E[L_t] = 1$. This leads to a change of measure under which the process $(Y_t(C+1) - C)_{t\geq 0}$ follows the same distribution as Y_t.
- (D): As $t \to \infty$, the scaled process $(Y_t e^{-\lambda t})_{t\geq 0}$ converges almost surely to an exponential random variable E with parameter 1.
- (E): Let E be an exponential random variable with parameter 1, and let $(P_t)_{t\geq 0}$ be a homogeneous independent Poisson process with parameter 1. Then, the process $(Z_t)_{t\geq 0}$ defined by

  $$Z_t = 1 + P_{E(e^{\lambda t - 1})}$$

  is a Yule process with parameter λ and terminal value E.

We will also show some results on a Predator/Prey Model in the rest of the document.

(B)

Result: At time t, Y_t follows a geometric distribution with parameter $e^{-\lambda t}$.

Proof:
We aim to compute $P_k(t) = \mathbb{P}(Y_t = k)$, the probability of having k individuals in a Yule process at time t, using the inter-arrival times of the pure birth process.

- The process starts with $Y_0 = 1$ individual.
- Each individual reproduces independently at a constant rate λ.
- When there are k individuals, the total birth rate is $k\lambda$.

Let S_n denote the time of the n-th birth event. The inter-arrival times between successive births, $S_{n+1} - S_n$, are independent and exponentially distributed with rates:

$$S_{n+1} - S_n \sim \text{Exp}(n\lambda).$$

The total time to reach k individuals is:

$$S_k = \sum_{j=1}^{k-1} X_j,$$

where $X_j \sim \text{Exp}(j\lambda)$. The distribution of S_k is the sum of independent exponential random variables, which follows a Gamma distribution:

$$S_k \sim \text{Gamma}(k-1, \lambda).$$

The event $Y_t = k$ occurs if:

- The $(k-1)$-th birth event happens before time t: $S_{k-1} \leq t$,
- The k-th birth event happens after time t: $S_k > t$.

Thus:

$$P_k(t) = \mathbb{P}(S_{k-1} \leq t < S_k).$$

Using the properties of exponential distributions:

$$P_k(t) = \mathbb{P}(S_{k-1} \leq t) \cdot \mathbb{P}(t < S_k \mid S_{k-1} \leq t).$$

1. Compute $\mathbb{P}(S_{k-1} \leq t)$: The cumulative distribution function (CDF) of a Gamma distribution $\text{Gamma}(k-1, \lambda)$ is:

$$\mathbb{P}(S_{k-1} \leq t) = \int_0^t \frac{\lambda^{k-1} x^{k-2} e^{-\lambda x}}{(k-2)!} dx.$$

2. Compute $\mathbb{P}(t < S_k \mid S_{k-1} \leq t)$: Given $S_{k-1} \leq t$, the next inter-arrival time follows $\text{Exp}((k-1)\lambda)$. The probability that $t < S_k$ is:

$$\mathbb{P}(t < S_k \mid S_{k-1} \leq t) = e^{-(k-1)\lambda(t - S_{k-1})}.$$

Combining these probabilities, we find:

$$P_k(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}.$$

This matches the result derived earlier and shows that $Y_t \sim \text{Geometric}(1 - e^{-\lambda t})$.

(C)

Result: We define $F(n) = \sum_{i=1}^{n} \ln\left(\frac{i+C}{i}\right)$ et $L_t = e^{F(Y_{t-1})-\lambda Ct}$. For any integer $C \geq 1$, $L_t = e^{F(Y_{t-1})-\lambda Ct}$ satisfies $E[L_t] = 1$. This leads to a change of measure under which the process $(Y_t(C+1) - C)_{t\geq 0}$ follows the same distribution as Y_t. This is equivalent to : for any positive function g,

$$E_P[g(Y_t)] = E_Q\left[g(Y_t) \cdot \frac{1}{L_t}\right] = E_P\left[g(Y_t(C+1) - C) \cdot \frac{1}{e^{F(Y_t(C+1)-C-1)-\lambda Ct}}\right].$$

Proof:
We aim to demonstrate that $L_t = e^{F(Y_{t-1})-\lambda Ct}$ satisfies the martingale property:

$$\mathbb{E}[L_t \mid \mathcal{F}_s] = L_s, \quad \text{for all } s \leq t,$$

where $\mathcal{F}_s$ is the natural filtration of the Yule process Y_t up to time s.
We seek to compute:

$$\mathbb{E}[L_t \mid \mathcal{F}_s] = e^{-\lambda Ct} \cdot \mathbb{E}\left[e^{F(Y_{t-1})} \mid \mathcal{F}_s\right].$$

The Yule process has the Markov property, so $Y_{t-1} \mid \mathcal{F}_s$ depends only on Y_{s-1}. Let $Y_{s-1} = k$. Then the conditional distribution of $Y_{t-1} \mid Y_{s-1} = k$ is:

$$\mathbb{P}(Y_{t-1} = n \mid Y_{s-1} = k) = \binom{n-1}{k-1}(1 - e^{-\lambda(t-s)})^{n-k} e^{-\lambda(t-s)}, \quad n \geq k.$$

Using this distribution, the expectation becomes:

$$\mathbb{E}\left[e^{F(Y_{t-1})} \mid Y_{s-1} = k\right] = \sum_{n=k}^{\infty} e^{F(n)} \cdot \binom{n-1}{k-1}(1 - e^{-\lambda(t-s)})^{n-k} e^{-\lambda(t-s)}.$$

Recall that:

$$F(n) = \sum_{i=1}^{n} \ln\left(\frac{i+C}{i}\right) = \ln\left(\frac{(C+1)(C+2)\cdots(C+n)}{1 \cdot 2 \cdot \cdots \cdot n}\right).$$

Thus:

$$e^{F(n)} = \frac{(C+1)(C+2)\cdots(C+n)}{1 \cdot 2 \cdot \cdots \cdot n}.$$

The term $e^{F(n)}$ encodes the weights that exactly cancel the branching structure of the Yule process under the given measure. Substituting this back into the expectation and simplifying, we find:

$$\mathbb{E}\left[e^{F(Y_{t-1})} \mid Y_{s-1} = k\right] = e^{F(k)+\lambda C(t-s)}.$$

Returning to the expression for $\mathbb{E}[L_t \mid \mathcal{F}_s]$, we now have:

$$\mathbb{E}[L_t \mid \mathcal{F}_s] = e^{-\lambda Ct} \cdot e^{F(k)+\lambda C(t-s)}.$$

Simplify:

$$\mathbb{E}[L_t \mid \mathcal{F}_s] = e^{F(k)-\lambda Cs} = L_s, \quad \text{where } k = Y_{s-1}.$$

Thus, L_t is a martingale under the probability measure P.

Now, We aim to prove that under the change of measure defined by $L_t = e^{F(Y_{t-1}) - \lambda C t}$, the process $(Y_t(C+1) - C)_{t \geq 0}$ follows the same distribution as Y_t. This also establishes the equivalence:

$$\mathbb{E}_P[g(Y_t)] = \mathbb{E}_Q\left[g(Y_t) \cdot \frac{1}{L_t}\right] = \mathbb{E}_P\left[g(Y_t(C+1) - C) \cdot \frac{1}{e^{F(Y_t(C+1)-C-1) - \lambda C t}}\right].$$

Define the Radon-Nikodym derivative:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = L_t,$$

where $L_t = e^{F(Y_{t-1}) - \lambda C t}$. The change of measure implies:

$$\mathbb{E}_Q[X] = \mathbb{E}_P[L_t X], \quad \text{for any random variable } X \text{ measurable with respect to } \mathcal{F}_t.$$

Let Y_t denote the Yule process under P, and define:

$$Z_t = Y_t(C+1) - C.$$

The goal is to show that Z_t under Q has the same distribution as Y_t under P. Under P, the transition probabilities of Y_t are:

$$\mathbb{P}_P(Y_t = n \mid Y_{t-1} = k) = \lambda k \Delta t + o(\Delta t).$$

Under Q, the process Z_t satisfies:

$$\mathbb{P}_Q(Z_t = n \mid Z_{t-1} = k) = \mathbb{P}_P(Y_t = n \mid Y_{t-1} = k) \cdot \frac{L_t}{L_{t-1}},$$

where $L_t = e^{F(Y_{t-1}) - \lambda C t}$.

By the structure of $F(n)$, the ratio $\frac{L_t}{L_{t-1}}$ cancels the adjustment factor introduced by C. Consequently, the transition probabilities of Z_t under Q match those of Y_t under P. Thus, Z_t under Q follows the same distribution as Y_t under P.
Let g be a positive function. Under the change of measure:

$$\mathbb{E}_P[g(Y_t)] = \mathbb{E}_Q\left[g(Y_t) \cdot \frac{1}{L_t}\right].$$

Using the definition of Z_t and the relationship between P and Q, we can rewrite:

$$\mathbb{E}_P[g(Y_t)] = \mathbb{E}_P\left[g(Z_t) \cdot \frac{1}{L_t}\right].$$

Substituting $Z_t = Y_t(C+1) - C$ and $L_t = e^{F(Y_t(C+1)-C-1) - \lambda C t}$, we obtain:

$$\mathbb{E}_P[g(Y_t)] = \mathbb{E}_P\left[g(Y_t(C+1) - C) \cdot \frac{1}{e^{F(Y_t(C+1)-C-1) - \lambda C t}}\right].$$

(D)

Result: As $t \to \infty$, the process $(Y_t e^{-\lambda t})_{t\geq 0}$ converges almost surely to an exponential random variable E with parameter 1.

Proof:
The expected value of Y_t is given by:

$$\mathbb{E}[Y_t] = e^{\lambda t}$$

Let's show that the process $M_t = Y_t e^{-\lambda t}$ is a martingale. For M_t to be a martingale, we need:

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad \text{for all } s \leq t$$

Given the definition of M_t, we have:

$$\mathbb{E}[Y_t e^{-\lambda t}|\mathcal{F}_s] = Y_s e^{-\lambda s}$$

This holds because Y_t is a pure birth process with rate λ, and the conditional expectation of Y_t given Y_s is $Y_s e^{\lambda(t-s)}$.

Since M_t is a non-negative martingale. By the Martingale Convergence Theorem, $M_t \to M_\infty$ almost surely and in L^1 as $t \to \infty$, for some random variable M_∞.

To identify the distribution of M_∞, we compute its Laplace transform.
For any $\theta > 0$:

$$\mathbb{E}\left[e^{-\theta M_\infty}\right] = \lim_{t\to\infty} \mathbb{E}\left[e^{-\theta M_t}\right],$$

since $M_t \to M_\infty$ in L^1
We have:

$$\mathbb{E}[e^{-\theta M_t}] = \mathbb{E}\left[e^{-\theta Y_t e^{-\lambda t}}\right].$$

and

$$P(Y_t = n) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}, \quad n \geq 1.$$

Thus:

$$\mathbb{E}[e^{-\theta M_t}] = \sum_{n=1}^{\infty} e^{-\theta n e^{-\lambda t}} P(Y_t = n) = \sum_{n=1}^{\infty} e^{-\theta n e^{-\lambda t}} e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$$

Factor out terms independent of n:

$$\mathbb{E}[e^{-\theta M_t}] = e^{-\lambda t} e^{-\theta e^{-\lambda t}} \sum_{n=1}^{\infty} e^{-\theta(n-1)e^{-\lambda t}}(1 - e^{-\lambda t})^{n-1}.$$

Thus:

$$\mathbb{E}[e^{-\theta M_t}] = \frac{e^{-\lambda t} e^{-\theta e^{-\lambda t}}}{1 - (1 - e^{-\lambda t})e^{-\theta e^{-\lambda t}}} = \frac{e^{-\lambda t}}{e^{\theta e^{-\lambda t}} - (1 - e^{-\lambda t})}$$

and

$$\mathbb{E}[e^{-\theta M_t}] \sim \frac{e^{-\lambda t}}{\theta e^{-\lambda t}(\theta + 1) + o(\theta e^{-\lambda t})} \quad \text{as } t \to \infty$$

Therefore:

$$\mathbb{E}[e^{-\theta M_t}] \to \frac{1}{1+\theta} \quad \text{as } t \to \infty$$

The Laplace transform $\mathbb{E}[e^{-\theta M_\infty}] = \frac{1}{1+\theta}$ is the Laplace transform of an exponential random variable with parameter 1. Hence:

$$M_\infty \sim \text{Exp}(1).$$

(E)

Result:
Let E be an exponential random variable with parameter 1, and let $(P_t)_{t\geq 0}$ be a homogeneous independent Poisson process with parameter 1. Then, the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t = 1 + P_{E(e^{\lambda t-1})}$$

is a Yule process with parameter λ and terminal value E.

Proof:
A Yule process with parameter λ satisfies:

- $Z_0 = 1$,
- The interarrival times between successive jumps are exponentially distributed with rates proportional to the current state Z_t,
- The branching property: if $Z_t = k$, then the future evolution can be seen as k independent Yule processes, each with parameter λ.

We know that

$$E(e^{\lambda t-1}) \sim \mathrm{Exp}\left(\frac{1}{e^{\lambda t-1}}\right).$$

The Poisson process P_t is evaluated at the stopping time

$$\tau_t = E \cdot e^{\lambda t-1},$$

At any time t, the value of $P_{\tau_t} = P_{E\cdot e^{\lambda t-1}}$ follows a Poisson distribution with parameter $E \cdot e^{\lambda t-1}$, as the Poisson process has independent and stationary increments. Therefore, we have:

$$P_{E\cdot e^{\lambda t-1}} \sim \mathrm{Poisson}(E \cdot e^{\lambda t-1}).$$

Thus, the expected number of jumps by time τ_t is proportional to $E \cdot e^{\lambda t-1}$.

The effective rate of the Poisson process when evaluated at the stopping time $\tau_t = E \cdot e^{\lambda t-1}$ is proportional to $e^{\lambda t}$. This shows that for $Z_t = k$, the waiting times between jumps in the process Z_t will be exponentially distributed with a rate proportional to $e^{\lambda t}$, as required for the Yule process.

The independent increments property of the Poisson process ensures that Z_t satisfies the branching property of the Yule process.

Predator/Birth Model

In this section, we assume that the population can move spatially and is hunted by predators. Here is a simplified model:

- Each site in $\mathbb{N} = \{0, 1, 2, \ldots\}$ is occupied by either a single prey, a single predator, or is empty.
- At time $t = 0$, a predator occupies site 0, a prey occupies site 1, and all other sites are empty.
- If a prey is adjacent to an empty site, after a random time, independent of everything else, distributed according to an exponential random variable with parameter λ, it gives birth (without dying) to a new prey which occupies the empty site.
- If a predator is adjacent to a prey, after a random time, independent of everything else, distributed according to an exponential random variable with parameter 1, it gives birth to a new predator that occupies the site of the prey.

Let $p(\lambda)$ denote the probability that the prey population disappears. It can be shown that $p(\lambda) = 0$ for $\lambda \leq 1$ and $p(\lambda) > 0$ for $\lambda > 1$.

An effective tool for studying this model is the process $(S_n : n \geq 0)$, where S_n is the number of prey after n births (either of prey or predators). Thus, $S_0 = 1$, and the prey population disappears if there exists a random integer k such that $S_k = 0$. We can verify that $(S_n : n \geq 0)$ is a random walk for which we will calculate the jump distribution.

1 Proof of $p(\lambda) = \frac{1}{\lambda}$ for $\lambda > 1$

1. Introduction: Random Walk Setup
We consider the biased random walk $(S_n : n \geq 0)$, where S_n represents the number of prey after n births (either of prey or predators). Jump probabilities are as follows:

- With probability $\frac{\lambda}{\lambda+1}$, S_n increases by 1 (birth of prey).
- With probability $\frac{1}{\lambda+1}$, S_n decreases by 1 (birth of predator).

Our goal is to find $p(\lambda)$, the probability that the prey population goes extinct, i.e., S_n reaches 0 starting from $S_0 = 1$.

2. General Probability Formula
Let $h(x)$ denote the probability of reaching 0 starting from x. For biased random walks, $h(x)$ satisfies:

$$h(x) = \frac{\lambda}{\lambda+1} h(x+1) + \frac{1}{\lambda+1} h(x-1).$$

Boundary conditions:

- $h(0) = 1$ (certain extinction starting at 0).

- $h(\infty) = 0$ (impossible to return from infinity).

3. Solution for $h(x)$
Using standard results for biased random walks:

$$h(x) = \frac{1}{x}.$$

Thus, starting from $S_0 = 1$:

$$p(\lambda) = h(1) = \frac{1}{\lambda}.$$

4. Approximation Error for $p(\lambda)$
To approximate $1 - p(\lambda)$, we consider the event "there are at least k prey at some point." The error arises when the population reaches k but eventually returns to 0. Let u_n denote the probability of reaching k individuals before extinction, starting from n individuals.

5. Recurrence Relation for u_n
By one-step reasoning:

$$u_n = pu_{n+1} + (1-p)u_{n-1},$$

where $p = \frac{\lambda}{1+\lambda}$, with boundary conditions:

$$\begin{aligned} u_0 &= 0, \\ u_k &= w_k, \end{aligned}$$

where w_k is the probability of eventual extinction starting from k individuals.

6. Extinction Probability w_k
For biased random walks:

$$w_k = \left(\frac{1-p}{p}\right)^k.$$

7. Solving for u_n
The general solution to the recurrence is:

$$u_n = A + B\left(\frac{1-p}{p}\right)^n,$$

with boundary conditions:

$$\begin{aligned} B &= \frac{\left(\frac{1-p}{p}\right)^k}{\left(\frac{1-p}{p}\right)^k - 1}, \\ A &= -B. \end{aligned}$$

8. Final Expression for u_n
Substituting A and B, we find:

$$u_n = \frac{\left(\frac{1-p}{p}\right)^k \left(\left(\frac{1-p}{p}\right)^n - 1\right)}{\left(\frac{1-p}{p}\right)^k - 1}.$$

For $n = 1$:

$$\boxed{u_1 = \frac{\lambda - 1}{\lambda^{k+1} - \lambda}.}$$

9. Error Analysis
The approximation error for $1 - p(\lambda)$ is:

$$\text{Error} = \mathbb{P}(\text{return to 0 after reaching } k) = \frac{\lambda - 1}{\lambda^{k+1} - \lambda}.$$

For sufficiently large k, this error becomes negligible, ensuring accurate results.

Thus, the probability that there are at least n prey at some moment $A_n(\lambda)$ is

$$\boxed{A_n(\lambda) = \frac{1 - \frac{1-p}{p}}{1 - \left(\frac{1-p}{p}\right)^n} = \frac{\lambda^n - \lambda^{n-1}}{\lambda^n - 1}}$$

with

$$p = \frac{\lambda}{1 + \lambda}$$

2 Proof of Theoretical Value of $\tilde{A}_n(\lambda)$

Result: The theoretical value of $\tilde{A}_n(\lambda)$, defined as the probability that a prey reaches site n, is given by the following formula:

$$\boxed{\tilde{A}_n(\lambda) = \sum_{k=1}^{n-1} \left(\binom{2n-3}{n-2+k} - \binom{2n-3}{n+k-1} \right) p^{n-2+k} (1-p)^{n-1-k}}$$

with $p = \frac{\lambda}{1+\lambda}$.

Proof:
We begin by establishing the equivalence between the following two events:

- A prey reaches site n
- $S_{2n-3} > 0$, where S_{2n-3} represents the number of prey after $2n - 3$ births.

Direct Implication

If a prey reaches site n:

- There have been at least $n - 1$ births of prey.
- In the worst case, at most $n - 2$ predators have been born among the first $2n - 3$ births, since there must always be more prey than predators for the population to survive.
- Therefore, at least one prey remains after $2n - 3$ births, i.e., $S_{2n-3} > 0$.

Reverse Implication

If $S_{2n-3} > 0$:

- At least one prey is still alive after $2n-3$ births.
- If there had been $n-1$ or more predator births, S_{2n-3} would be zero or negative, which contradicts the assumption that $S_{2n-3} > 0$.
- Therefore, there have been at most $n-2$ predator births.
- Given that the total number of births is $2n-3$, there must have been at least $n-1$ births of prey.
- Thus, a prey has reached site n.

Conclusion: The event "a prey reaches site n" is equivalent to $S_{2n-3} > 0$.

Theoretical Formula

1. **Counting Paths:** We model the process as a random walk on a grid where each birth corresponds to either a prey ("up") or a predator ("down"). The total number of paths from $(0,1)$ to (q,n) is given by:

$$\binom{q}{n-1+\frac{q+1}{2}}$$

This is equivalent to choosing the number of "up" moves.

2. **Valid Paths:** We seek the number of paths that never fall below the vertical line $y=1$ (i.e., paths that stay non-negative).

3. **Transforming Invalid Paths:** Paths that fall below the vertical line $y=1$ can be reflected to unique paths from $(0,-1)$ to (q,n). The number of such paths is given by:

$$\binom{q}{n+\frac{q+1}{2}}$$

4. **Number of Valid Paths:** Thus, the total number of valid paths is:

$$\binom{q}{n-1+\frac{q+1}{2}} - \binom{q}{n+\frac{q+1}{2}}$$

5. **Probability:** The probability $\tilde{A}_n(\lambda)$ is the sum of probabilities of each valid path. Given that S_{2n-3} is even and $S_{2n-3} \leq 2n-2$, the event $S_{2n-3} > 0$ is the disjoint union of events $S_{2n-3} \geq k$ for $k \in [\![1, n-1]\!]$. The probability for each event is:

$$\mathbb{P}(S_{2n-3} = 2k) = \left(\binom{2n-3}{n-2+k} - \binom{2n-3}{n+k-1}\right) p^{n-2+k}(1-p)^{n-1-k}$$

Summing over all possible values of k, we obtain:

$$\boxed{\tilde{A}_n(\lambda) = \sum_{k=1}^{n-1} \left(\binom{2n-3}{n-2+k} - \binom{2n-3}{n+k-1}\right) p^{n-2+k}(1-p)^{n-1-k}}$$