Robustness Against the Decision-Maker's Attitude to Risk in Problems With Conflicting Objectives

Lam T. Bui, Hussein A. Abbass, *Senior Member, IEEE,* Michael Barlow, and Axel Bender

***Abstract*—In multiobjective optimization problems (MOPs), the Pareto set consists of efficient solutions that represent the best trade-offs between the conflicting objectives. Many forms of uncertainty affect the MOP, including uncertainty in the decision variables, parameters or objectives. A source of uncertainty that is not studied in the evolutionary multiobjective optimization (EMO) literature is the decision-maker's attitude to risk (DMAR) even though it has great significance in real-world applications. Often the decision-makers change over the course of the decision-making process and thus, some relevant information about preferences of future decision-makers is unknown at the time a decision is made. This poses a major risk to organizations because a new decision-maker may simply reject a decision that has been made previously. When an EMO technique attempts to generate the set of nondominated solutions for a problem, then DMAR-related uncertainty needs to be reduced. Solutions generated by an EMO technique should be robust against perturbations caused by the DMAR. In this paper, we focus on the DMAR as a source of uncertainty and present two new types of robustness in MOP. In the first type, dominance robustness (DR), the robust Pareto solutions are those which, if perturbed, would have a high chance to move to another Pareto solution. In the second type, preference robustness (PR), the robust Pareto solutions are those that are close to each other in configuration space. Dominance robustness captures the ability of a solution to move along the Pareto optimal front under some perturbative variation in the decision space, while PR captures the ability of a solution to produce a smooth transition (in the decision variable space) to its neighbors (defined in the objective space). We propose methods to quantify these robustness concepts, modify existing EMO techniques to capture robustness against the DMAR, and present test problems to examine both DR and PR.**

***Index Terms*—Decision making, evolutionary, multi-objective, risk, robustness.**

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L. T. Bui is with the Department of Software Engineering, Le Quy Don University, Hanoi, Vietnam (e-mail: lam.bui07@gmail.com).

H. Abbass and M. Barlow are with the Defense and Security Applications Research Center, University of New South Wales, Australian Defense Force Academy, Canberra, ACT 2600, Australia (e-mail: h.abbass@adfa.edu.au; m.barlow@adfa.edu.au).

A. Bender is with the Land Operations Division, Defense, Science and Technology Organization, Edinburgh, SA 5111, Australia (e-mail: axel.bender@dsto.defence.gov.au).

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I. Introduction

REAL-WORLD optimization problems are characterized by conflicting objectives and high levels of uncertainty. Those who will use the optimization result—e.g., want to make the best possible decision given the optimization outputs—are faced with a situation where they need to trade off benefits and drawbacks amongst alternatives that are hard to compare. Picking a solution thus becomes a matter of *preference*. Given the high levels of uncertainty, this is significantly influenced by attitudes to risk with "risk" here being defined as the "effect of uncertainty on objectives" [37]. Even when two decision-makers have the same intent—namely to make the best possible decision—a risk taker is likely to select a different alternative to what a risk averse and cautious user would choose. For the analyst, it is therefore important to assess the impact of uncertainties on objectives and find ways of articulating how "robust" different optimization solutions are when circumstances—including risk attitudes—change.

Optimization under conflicting objectives defines the class of multiobjective optimization problems, while optimizing under vast uncertainty defines the class of robust optimization (RO) problems.

Generally, there are four main approaches in the multi-objective optimization literature [38]. The first approach does not use preference information and is called *no-preference*. No-preference methods solve a problem and directly pass on all solutions to the decision-maker. The second one (called *decision-making after search*, or *posterior*) finds all possible nondominated solutions and then applies the user's preference *a posteriori* to determine the most suitable one. The third approach is called *decision-making before search*, or *a priori*, and incorporates user preference prior to the optimization process; hence it results in but one solution. In this approach, the user preference bias [31] is imposed all the time. The fourth approach is called *decision-making during search*, or *interactive* [2], and hybridizes the posterior and *a priori* approaches. In this interactive method, human decision-making is used periodically to refine the obtained trade-off solutions and thus, guides the search. In general, the posterior method is mostly preferred within the research community because it is less subjective than the *a priori* and *interactive* approaches.

When considering trade-off solutions either during the optimization process or at the decision-making stage, it is important to take into account the robustness of the solutions [5], [14]. An RO problem takes many forms based on the source and nature of uncertainty. For example, when evaluating a decision alternative, there can be noise in the evaluation (not necessarily caused by noise in the decision variables or the parameters, but by noise in the function itself). This noise may mislead the optimization process [7], [8], [22]. Another source of noise can be in the parameters. For example, if the parameters are measures coming from a sensor, noise in the sensor measurements can generate perturbations in the evaluation process. A third source of noise resides in the decision variable space (also known as the configuration space). Every time a configuration is retrieved, the decision variables might get perturbed. These different types of uncertainty and the associated robustness (or lack thereof) of solutions attracted much attention in the literature because of the real-world significance and implications [20], [33], [34], [39], [40], [43], [49]. The robustness concept is rooted in some very practical concerns.

1) In *noisy environments*, the objective values of solutions are usually disturbed by noise and the final solutions need to be robust under noise effects.
2) In design, the *variation of parameter values* (in decision space) is inevitable, and the robustness of a solution is shown in the sense that the solution should be stable in objective space against all allowed parameter variations.

As already indicated earlier, in a number of applications that involve decision-making, another source of uncertainty plays a major role. It is uncertainty arising from user preferences, in particular the decision-maker's attitude to risk (DMAR). How to trade-off objectives varies markedly depending on whether the decision-maker is risk averse or a risk-taker; how forward thinking he or she is and what he or she typically includes in his or her planning horizon; what level of confidence he or she requires to make a decision; and how he or she can handle varying degrees of ambiguity. The effect of factors that influence the DMAR can by far outweigh the effect of uncertainties in other decision variables, parameters or objectives. For example, in future air-traffic management, the decision support system in the cockpit may offer the pilot a number of solutions to resolve a conflict with another aircraft. Each solution would be trading off different objectives such as the comfort of the passengers and the fuel cost of the maneuver. Which solution the pilot is going to select is very much dependent on his or her attitude to risk.

Another example relates to the rotation of military decision-makers who are involved in long-term planning and major equipment acquisitions. In Australia, for instance, officers who manage military capability development projects typically rotate into different positions every two years, although most of these projects span five to ten and often even more years. It is also almost certain that a number of the defense capability decision-makers (at the two and three-star level) will change throughout the life of these major projects, and it is fair to assume that different decision-makers will come with different attitudes to risk. Thus, in problems with some likelihood for decision-maker substitution, the set of nondominated solutions should account for possible changes in the DMAR, and therefore should be robust against DMAR variations.

In [3], the concept of the Pareto operating curve (POC) was introduced and defined as; "POC is a Pareto curve for a problem where the trade-off between the objectives to be optimized varies over time; thus, a solution selected along this curve at one point of time needs to move to a different solution at another point of time to minimize the impact of uncertainty on objectives (i.e., risk)." The author illustrated the POC by way of examples but did not provide any theoretical framework on how to account for the movement of solutions within an EMO context. In this present paper, we address this issue. We introduce two types of robustness to accommodate for the DMAR. The first type is DR and describes those efficient solutions that stay firmly in the efficient set when there is some variation in the decision variables. In other words, Pareto solutions which are dominance-robust are those efficient solutions that are surrounded by other efficient solutions in the decision space. The set of dominance-robust solutions has the characteristic that if any member of the set is perturbed, a solution is reached that is still efficient but shows a different level of trade-off. This type of robustness accommodates for uncertainty arising from noise in the configuration space which can be caused by variations in the DMAR (see Section V).

The second type of robustness is called *preference robustness* (PR) and describes the following circumstance: if a solution is selected from the Pareto set according to some preference information obtained from the decision-maker, how much variation in the configuration space will occur as the result of a change of the decision-maker's preference? Those solutions for which the variation in configuration space is small are more preference robust than those solutions for which variations are large.

In summary, the contributions of this paper are given below.

1) In the context of MOPs we introduce two new concepts of robustness that account for uncertainty caused by potential variations in the DMAR.
2) We propose evolutionary computation (EC) methods that can evaluate and select robust solutions according to the previous two concepts.

The paper is organized in seven sections. Section II is dedicated to common notations in MOP, while a summary of the robustness concept found in the literature is given in Section III. A description of robustness in MOP is presented in Section IV and followed by the formal definition of the newly proposed dominance and PR measures in Section V. A number of test problems are presented, solved and studied in Section VI to demonstrate the concept, and this paper concludes in Section VII.

II. Common Concepts

Real-life problems often have multiple conflicting objectives. For example, we would like to have a good quality car, but want to pay as little as possible for it. In this instance, we will probably have a number of options, each of them representing a different compromise between the two objectives. A subset of these options contains all *Pareto optimal* solutions, which in case of MOP with objectives, or criteria, to be minimized are defined as follows.

"A solution to a MOP is Pareto optimal if there exists no other feasible solution which would decrease some criterion without causing a simultaneous increase in at least one other criterion" [9].

Note that maximization problems (such as maximizing the quality of a car to be purchased) can be transformed to minimization problems by replacing the objective with its negative, its inverse, combinations thereof or other monotonically decreasing transformation functions. Without loss of generality we therefore can consider all MOP to be minimization problems. The set of solutions that satisfy the previous definition is known as the *Pareto optimal set*. Their projections in objective space is called Pareto optimal front (POF), or just Pareto front.

Mathematically, in a k-objective unconstrained optimization problem, a vector function $\vec{f}(\vec{x})$ of k objectives is defined as

$$\vec{f}(\vec{x}) = [f_1(\vec{x}), f_2(\vec{x}), ..., f_k(\vec{x})] \tag{1}$$

in which $\vec{x}$ is a vector of decision variables in the n-dimensional space $\mathbb{R}^n$; n and k are not necessarily the same. Each individual in the evolutionary process represents a vector $\vec{x}$ and therefore, is attached to its evaluation vector $\vec{f}$.

An individual $\vec{x}_1$ is said to dominate $\vec{x}_2$ if $\vec{x}_1$ is better than $\vec{x}_2$ when measured on all objectives. If $\vec{x}_1$ does not dominate $\vec{x}_2$ and $\vec{x}_2$ also does not dominate $\vec{x}_1$, then they are said to be *nondominated*. If we use the symbol "$\preceq$" to denote that $\vec{x}_1 \preceq \vec{x}_2$ means that $\vec{x}_1$ dominates $\vec{x}_2$, and the symbol "$\lhd$" between two scalars a and b such that $a \lhd b$ means a is better than b (and, similarly, $a \rhd b$ means that a is worse than b, and $a \not\rhd b$ that a is not worse than b), then the dominance concept is formally defined as follows [13].

Definition 1 (Dominance): $\vec{x}_1 \preceq \vec{x}_2$ if the following conditions are held:

1) $f_j(\vec{x}_1) \not\rhd f_j(\vec{x}_2) \forall j \in \{1, 2, \ldots, k\}$;
2) $\exists j \in \{1, 2, \ldots, k\} : f_j(\vec{x}_1) \lhd f_j(\vec{x}_2)$.

In general, if an individual in a population is not dominated by any other individual in the population, it is called a nondominated individual. All nondominated individuals in a population form the nondominated set (as formally described in Definition 2).

Definition 2 (Nondominated Set): A set S is said to be the nondominated set of a population P if the following conditions are met:

1) $S \subseteq P$;
2) $\forall \vec{s} \in S \nexists \vec{x} \in P : \vec{x} \preceq \vec{s}$.

When the population P represents the entire search space, the set of nondominated solutions S is called the *global Pareto optimal set*. Note that some authors call it the *efficient set*. Therefore, in this paper we use the two terms interchangeably. If P represents a sub-space, S is called the *local Pareto optimal set*. There is only one global Pareto optimal set, but there can be multiple local ones. In general, we refer to the global Pareto optimal set simply as the *Pareto optimal set* (POS). Furthermore, the image of a nondominated solution in objective space will be referred to as *nondominated point*; it is a point on the POF.

III. Robustness

The only certain fact about the world is that it is uncertain. Uncertainty characterizes the world surrounding us, both in the past, present and future. Decision-making thus has to come to terms with this fact. In the optimization literature managing uncertainty is known as RO. In this section, we will trace the roots of uncertainty in a number of fields and attempt to provide a summary of the vast literature on that topic.

Robustness is normally defined as a system's ability to withstand, i.e., *maintain its functionalities*, under conditions of varying internal or external parameters. There is a close relationship between robustness and adaptation. The former attempts to self-organize the system (potentially through structural changes) in order to maintain the system's functionalities, while the latter allows the introduction and deletion of functionalities. An example is given in Fig. 1 where a system is under both internal and external perturbations and even multiple perturbations. Over time, the robustness of the system helps it maintain the current functionalities while the system is evolving toward new functionalities.

Robustness can be seen as a concept that is broader than that of *stability*. As stated in [32], stability is defined as the system's ability to keep new solutions—generated by small perturbations to the original solution—staying close to the original solutions at all times. Both stability and robustness focus on the *persistence* of the system to *perturbations*. The common features of both concepts are: 1) they are defined in a given context; that is, they are defined for *specific features*, and *specific perturbations* applied to the system, and 2) they both focus on the concept of *persistence* of specified features or functionalities under specified perturbations. However, robustness is investigated in broader classes of systems, features and perturbations. While stability theory usually considers a single perturbation at a time, robustness deals with a system's response to multiple perturbations with different dimensions and scales. Ostensibly, a system with a high level of robustness may seem to be limited in its ability for adaptation or evolvability. However, in [48] it is shown that high *sequence or genotypic robustness* can only result in low *sequence or genotypic evolvability*, while high *structure or phenotypic robustness* can lead to high *structure or phenotypic evolvability*.

If the subject of study is an organization then the concept of robustness is more useful than that of stability and robustness theory is more appropriate than stability theory. An organization consists of a number of components. The reciprocal interaction of these components results in collective behavior at the system level. In these situations stability theory faces difficulties because of the cascading effect and the occurrence of second-order phase transition phenomena. Moreover, an organization can be hierarchically structured and might be subjected to perturbations that cross various hierarchical levels and scales. Robustness is concerned with aspects and features that are sometimes hard to quantify or parameterize; for example, robustness often takes into account a system's response to so-called "deep uncertainties" that are difficult to model [12]. In addition, robustness is not only concerned with a system's ability to maintain its functionalities

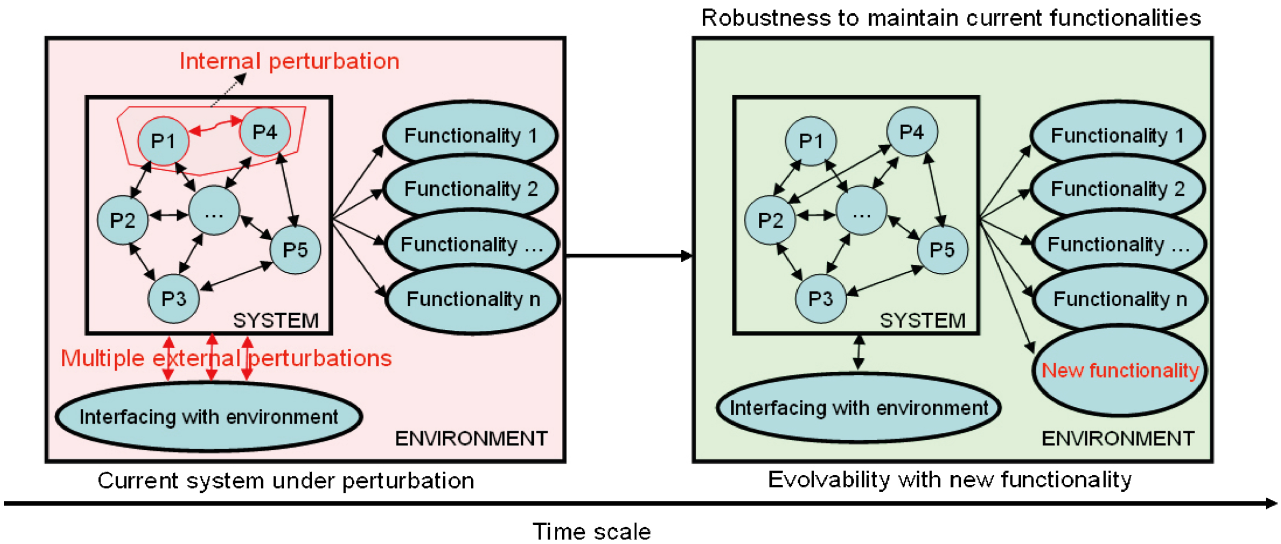


Fig. 1. Demonstration of the robustness concept. The system on the left is perturbed internally and externally. These perturbations result in a loss of connection between the two sub-systems P1 and P4 and introduce a new connection between P4 and P2 (right panel). The system is able to maintain its functionalities and even introduces a new one.

by applying predefined rules but also by learning new rules. The response to perturbations in some of these systems is to simply switch between options, or system states, without necessarily changing the system *per se*. Of course, in this case there is an obvious trade-off relationship between the cost and the benefit of generating options.

Some brief descriptions of the robustness concept in various fields of research are given in the subsequent subsections.

A. Robustness in Life Sciences

Robustness is a fundamental feature in biology and it appears everywhere in this area [35]. It is defined as the persistence of an organism's trait (organismal feature) under perturbation [18]. A trait is considered the proper fold or activity of a protein, a gene expression pattern produced by a regulatory gene network, the regular progression of a cell division cycle, the communication of a molecular signal from cell surface to nucleus, a cell interaction necessary for embryogenesis or the proper formation of a viable organ or organism. A perturbation can be stochastic noise, i.e., fluctuations that are inherent to any biological system; environmental change caused by variations of environmental parameters such as temperature, salinity or available nutrients; and genetic variations via mutation and recombination.

Robustness in biology is often loosely captured in other concepts such as buffering, canalization, developmental stability, efficiency, homeostasis, or tolerance [47]. It can be classified as either mutational robustness or environmental/noisy robustness that results in the phenomenon of phenotypical plasticity. There are several mechanisms that increase the robustness of biological systems. These include: the use of (positive and negative) feedback control to enable the perception of and response to perturbations; the use of redundancy and degeneracy as a buffer against perturbations; the equipping the system with a modular structure to enhance damage repair capacity and facilitate information sharing; and, the use of decoupling by which a module can be isolated from the rest of the system if its perturbations constitute a risk for the system.

In ecology, robustness is usually referred to as "resilience" [19], [26], [41]; but the main principle remains unchanged: robustness is the ability of an organization to persist in the presence of perturbations. The resilience of an ecosystem is measured by how fast the variables return to their equilibrium values following a perturbation.

B. Robustness in Engineering and Mathematics

By and large, the field of engineering regards the concept of robustness as being similar to the concept of reliability. Only in some recent literature, the word "reliability" is used when the system has hard constraints and uncertainty can break feasibility, while the word "robustness" is used when the problem is unconstrained [11].

In engineering, a system is considered reliable if it is robust against input and failure uncertainty. Consequently it has low reliability when a small amount of uncertainty brings about the possibility of failure [54]. A basic desideratum for robust control in practice is that the system is to remain stable in the face of perturbations. Since, instability may be equated with infinite loss, minimizing the worst case outcomes (when possible) guarantees stability [51]. More precisely, if the real dynamics of the process change, the performance of the system should not deteriorate beyond an unacceptable level [10].

In software engineering, programmers expect to design software which is robust outside the areas of specifications. One of the approaches to achieve robustness is to allow for redundancy, i.e., the software contains components with equivalent functionality, so that if one fails to perform properly, another one can provide the function that is needed. The challenge is to design the software system so that it can accommodate the additional components and capitalize on their redundant functionality [28].

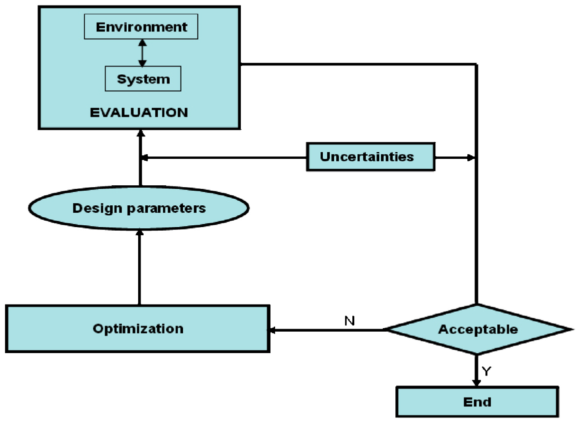


Fig. 2. Conceptual diagram of robustness in optimization.

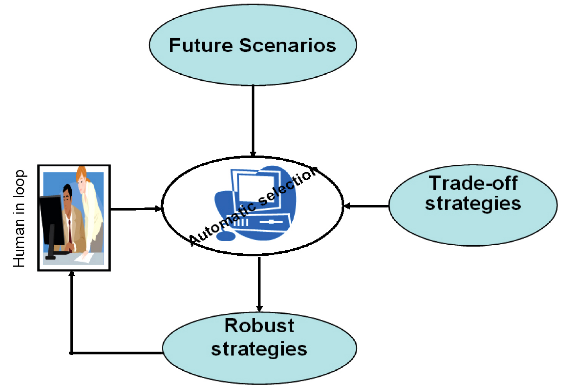


Fig. 3. Demonstration of robustness in decision-making.

In machine learning and statistics, robustness is considered the level of insensitivity of statistical models to small deviations from assumptions [53]. The assumptions may include the level of noise in the data or the parameters of the distribution of the latent variables.

"Robust statistics" is a particular branch of statistics. Nonparametric statistical methods allow to consider all possible distributions on a data set; parametric methods consider only "exact" distributions; while robust statistics take into account the full neighborhood of a parametric model. Other mechanisms to account for robustness are based on the use of ensembles [1], [29], [30], [50].

"*Robust optimization*" is dedicated to finding robust solutions subject to local perturbations in the *decision space* [5]. Robustness in optimization theory has been addressed in many problem domains under a variety of uncertainty types. This includes finding robust solutions in a noisy problem; tracking optima in a dynamic environment where a constraint or an objective changes; varying the values of the parameters; changing the mapping between the parameter and the design space; and alter one-to-many mappings when a solution has multiple different evaluations. A conceptual diagram of Robust Optimization is presented in Fig. 2.

C. Robustness in Decision Sciences

Robustness in decision-making is the capacity to deal with uncertainty. The difficulty of making decisions arises from existing uncertainty as it relates to decision parameters. In [36], a distinction is made between uncertainty that can be modeled probabilistically, in which case it can be quantified and a risk analysis approach can be adopted; and "deep uncertainty," which is difficult to quantify. Deep uncertainty occurs when analysts do not know or the stakeholders in a decision-making process cannot agree on: 1) the appropriate models to describe interactions among a system's variables; 2) the probability distributions to represent uncertainty about key parameters in the models; or 3) the evaluation of the desirability of alternative outcomes.

In economics, the main focus is on issues such as scales and levels at which economic models are applied. In modeling, the interaction among slow and fast-change variables is taken into account as well as the transfer of robustness among different levels of economies. Furthermore, economic models rely on some theoretical assumptions about uncertainties. Therefore economical definitions of robustness depend on the scope of such assumptions [23].

In organization science some research has focused on appropriately defining robust organizational architectures. In [4], the authors point out several parameters that affect the robustness of the organizations such as the skills of individuals and the size of the organization. Complexity and robustness prove useful categories to distinguish organizational structures [45]. Here, complexity is defined as the level of detail of information needed to correctly allocate agents within an organizational structure. Robustness, on the other hand, is the degree to which organizational performance is left unchanged under the effect of variation in the environment.

A key method of capturing the effect of uncertainty on the decision-making process is the use of scenarios [44]. In scenario planning, robust solutions are those that are as good as possible in simultaneously dealing with different plausible states of nature (scenarios) and adapting to different acceptable values of scenario parameters. Fig. 3 demonstrates this concept. The decision-maker is presented with a set of robust strategies to select from, while the automatic selection engine uses future scenarios, the trade-off strategies (i.e., robust Pareto strategies), and the DMAR to refine its selection of an appropriate strategy. In the remainder of this paper, we will discuss how to generate the set of robust Pareto solutions that the automatic selection engine uses as trade-off strategies.

IV. Robustness in Multiobjective Optimization

Over the last few years, the evolutionary multiobjective optimization (EMO) literature has paid more attention to the concept of robustness. Still, the number of papers on this topic in the single-objective domain is much larger than in the multiobjective domain. Overall, the concept of robustness in MOP is largely identical to the concept as developed for single-objective optimization problems—with the exception of moving from robust optimal solutions to a robust nondominated set of Pareto-optimal solutions. One of the major contributions to evolutionary multiobjective RO is made in

[14] where the authors define the following two types of robustness:

Definition 3 (Multiobjective Robust Solution of Type I): A solution $\vec{s}^*$ is called a multiobjective robust solution of type I, if it is the global feasible Pareto-optimal solution to the following multiobjective minimization problem (defined with respect to a δ-neighborhood $\mathcal{B}_\delta(\vec{s})$ of a solution $\vec{s}$ in the global Pareto optimal set S):

minimize $(f_1^e(\vec{s}), f_2^e(\vec{s}), ..., f_k^e(\vec{s}))$;

subject to $\vec{s} \in S$ where $f_j^e(\vec{s})$ is defined as follows:

$$f_j^e(\vec{s}) = \frac{1}{|\mathcal{B}_\delta(\vec{s})|} \int_{\vec{y} \in \mathcal{B}_\delta(\vec{s})} f_j(\vec{y}) dy. \quad (2)$$

Here, $|\mathcal{B}_\delta(\vec{s})|$ denotes the volume of the neighborhood.

Definition 4 (Multiobjective Robust Solution of Type II): A solution $\vec{s}^*$ is called a multiobjective robust solution of type II, if it is the global feasible Pareto-optimal solution to the following multiobjective minimization problem:

minimize $\vec{f}(\vec{s}) = (f_1(\vec{s}), f_2(\vec{s}), ..., f_k(\vec{s}))$;

subject to $\vec{s} \in S$ and

$$\frac{||\vec{f}^p(\vec{s}) - \vec{f}(\vec{s})||}{||\vec{f}(\vec{s})||} \leq \eta \quad (3)$$

where $\vec{f}^p(\vec{x})$ is the perturbed objective vector and it can be set, for example, as the worst function value in the neighborhood. η is a user-defined, small real parameter, and the operator $||.||$ is a suitable vector norm.

A similar approach is given in [20] where the authors analyze robustness against small variations in the decision variable space, but employ statistical methods to quantify robustness in terms of expectation and variance. The idea of worst case sensitivity as in Definition 4 is also used in [24]. There, the authors define robustness of a solution as the region of variation in decision space that still produces points in objective space that fall within a predefined range from the original point. Also, a robustness analysis can be found in [17] and [52]. Other work in the literature considers robustness as a response to noise in the objective functions with respect to fitness inheritance [7], probabilistic approximation [27], probabilistic dominance [42], and possibilistic archiving [21].

V. Methodology

A. New Types of Robustness in MultiObjective Optimization

What distinguishes multiobjective from single-objective decision-making is that the definition of optimality cannot be separated from the decision-maker's preference. A Pareto set is the set of optimal solutions before applying the decision-maker's preference that selects from the set the one solution to be adopted in the real world. As already emphasized, the decision-maker's preference strongly depends on his or her attitude to risk, i.e., DMAR. Therefore there are two components that contribute to the optimal decision: the Pareto optimal set and the DMAR. Concepts of robustness that are unique to MOP and not just derivatives of single-objective robustness types should involve both of these components.

We propose to measure robustness in relation to perturbations of the DMAR in two different ways. Firstly, we can consider the impact of DMAR variations on solutions in the search or decision space. DMAR can express itself in trends toward specific regions in decision space or toward particular relationships amongst decision variables. For instance, in the MOP example of finding the best mix of small and large trucks for a transportation company, the efficient solutions might range from configurations of many small trucks to those of a few large trucks with mixes of both small and large trucks in between. It is a general tendency in humans not to "put all eggs into one basket." In the fleet mix case, this means that many decision-makers are likely to prefer to pick heterogeneous solutions from the Pareto set (i.e., configurations that have a "balanced mix" of large and small trucks). More generically, decision-makers have DMAR related preferences to pick solutions that can be varied and, after variation, still provide efficiency in the sense of Pareto optimality.

In most practical cases, the degree to which such preferences impact upon decision-making is unknown. The analyst therefore is advised to assess the robustness of a solution with respect to other solutions in decision space. Note that the analyst's goal should be to identify such (robust) solutions that, after variation, remain in the Pareto optimal set.

We therefore define a solution to be DR if its perturbation in decision space does not move it away from the POF, i.e., drive it out of the set of Pareto optimal points (which is a set in objective space, see Section II). This is illustrated in the top half of Fig. 4. In other words, for a DR solution a perturbation in decision space does not alter its Pareto status, even though the functional value of a decision-maker's utility or preference is changed. So, we can formally define DR as follows.

Definition 5 (Dominance Robustness—DR): Dominance robustness of a Pareto-optimal solution $\vec{s}$ is defined as its ability to stay in the Pareto-optimal front when it is perturbed in decision space.

An application of this form of robustness is given in the example of an inter-disciplinary scientist who wants to win a competitive research grant. She can send her application to one and only one assessment panel of experts in a discipline covered by her research proposal. Her proposal might have some inherent flexibility in that some research components which relate to the different single disciplines can be traded off against each other. In order to maximize her chances of winning a grant, the researcher would want to send her application to that panel of experts who are unlikely to demand a redesign of her research project beyond the trade-offs that are acceptable to her.

Another example is a vehicle fleet mix determination problem, in which safety and speed objectives compete with each other. Here an analyst can determine all Pareto optimal solutions of the MOP. In configuration space, these solutions correspond to different mixes of vehicle types described, for example, by a certain number of fast cars with a low-safety rating and a certain number of slow cars with a high-safety rating. Without any inputs from users (i.e., decision-makers), the analyst cannot differentiate between any of the efficient solutions; they are "trade-off independent" [3]. However, if DR against the DMAR is taken into account then it is quite likely that extreme vehicle fleet configurations will be excluded in line with our above discussion about vehicle fleet mix example. Stated differently, efficient solutions with a high proportion of fast cars, are probably disliked by risk-averse users. The requirement for DR would exclude such configurations because any perturbation that would add even more cars that are fast would make the corresponding MOP solution dominated. Similarly, efficient solutions that correspond to very slow (but safe) fleets might be disfavored by risk takers. Fleet mixes that are balanced are more dominance robust (i.e., acceptable to both risk averse and risk taking users) because they are surrounded by configurations that correspond to efficient solutions.

Considering DR is very important in negotiations and when building alliances. In negotiation situations, one can evaluate a solution with respect to the kind of group preference characteristics that would maintain the solution at a level that is acceptable to the group. This evaluation aids in the identification of trade-offs and partners for alliances, and strengthens one's own negotiation position.

The second type of robustness—which we call PR—is based on the cost of transitions within the Pareto optimal set that are caused by the effect of perturbations in the decision-maker's preference or DMAR. In simple words, PR is related to the question what would happen and how costly would it be if the decision-maker changed his or her mind.

In most realistic examples, trade-offs are not being made with certainty. Often decision-makers have trade-off "bands" or intervals, i.e., are willing to vary one objective slightly for the benefits of others (or external factors that are not considered in the MOPs). Trade-off preferences can change over time, both because the decision-maker gets replaced by someone else or because the individuals themselves change, for instance, because economic circumstances change, new experiences result in modified value systems, the circle of influential friends alters, etc. Typically, variations in trade-off preferences are linked to risk attitudes. For instance, we might want to buy a new house based on price and size. When we find a nice candidate home, we might not only be interested in the price-size trade-off but also in the possibility of changing our mind later—e.g., when the birth of our third child will require the addition of a house extension. Here, we might find that the block of land restricts the potential of future variations, or that the price of the extension will be prohibitive according to our current financial position. If we are risk averse we would probably look into buying a different house; if we are risk takers we would possibly choose the home we found and deal with the need of extending the house at a later date.

In general, DMAR gives rise to potential variations in the trade-offs between the objectives used in the determination of the efficient set. At the time a MOP is analyzed, these potential variations are unknown and cause uncertainty. The analyst therefore should consider PR which is formally defined as follows.

Definition 6 (Preference Robustness—PR): Preference robustness of a Pareto-optimal solution $\vec{s}$ is defined as the minimum transition costs in decision space when $\vec{s}$ is perturbed in objective space.

For example, let us assume that a supply chain business needs to purchase a fleet of trucks that is to stay with the company for many years. Assume that the two objectives in this procurement problem are: 1) the acquisition cost of the fleet ("cost"), and 2) the percentage of projected future supply chain products that the company can offer with its newly acquired fleet ("efficacy"). Assume that these two objectives are in conflict such that a low-cost fleet can only offer the company's current supply chain solutions while a high-cost fleet is efficacious with respect to meeting the demand of the company's current clientele plus offering a diverse range of potential future supply chain products. If the company's chief executive officer (CEO) decides on the level of trade-off between cost and efficacy, his decision will be represented in the objective space by a particular solution. PR then measures by how much this solution in the decision space will change if the DMAR changes, e.g., when the current CEO is replaced by a new CEO whose preferences and DMAR are different.

To give a numerical example, assume that the company has decided to build its logistics fleet from two types of trucks: heavy and light. Assume that the option that the CEO chooses from the Pareto optimal set comprises ten heavy and 20 light trucks. If the DMAR changes slightly, a small perturbation along the Pareto front is incurred. a small change in the trade-off between cost and efficacy would mean that the original (10, 20) solution would need to be changed to a mix of eleven heavy and 21 light trucks in order to remain Pareto optimal. In a second scenario, however, the small change in trade-off might mean that the (10, 20) solution has to change to a (20, 0) solution. With respect to DMAR perturbations, the CEOs chosen (10, 20) solution thus is more preference-robust in the first scenario than in the second one. This is because a small movement away from the CEO's preference along the Pareto front results in a small alteration of decision variables in case of the first scenario but a huge change in case of the second scenario.

The key feature of both the DR and PR measures is that they are specifically tailored to MOP. DR describes robustness against perturbations in the decision space while the PR measure evaluates the effect of perturbations in the objective space. Both measures help the analyst to assess the impact of uncertainties that relate to unkown aspects of the DMAR. Fig. 4 illustrates the concept of both the DR and PR measures.

In summary, DR of a solution is its ability to stay close to the Pareto front, while PR of a solution is its closeness to other nondominated solutions in the decision space. Notice that DR is defined on the decision space and measured on the objective space, while PR is defined on the objective space and measured on the decision space.

B. Robustness Measures

In the previous section, we defined the concepts of DR and PR qualitatively. Now we propose measures that can quantify both concepts. Dominance robustness of a nondominated solution $\vec{s} \in S$ measures the *degree to which other nondominated solutions are generated* when $\vec{s}$ is perturbed within

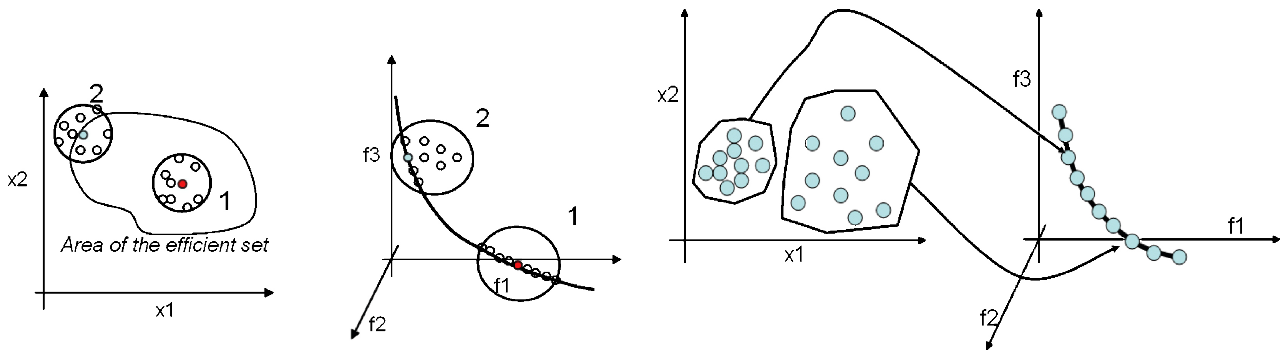


Fig. 4. Top half of the figure illustrates dominance robustness. On the left, the decision space contains two solutions 1 and 2. A perturbation of solution 1 in the decision space causes movements along the Pareto front (on the right) but not away from it because in the decision space, solution 1 is surrounded by nondominated solutions only. On the other hand, a decision-space perturbation to solution 2 would cause movements away from the Pareto front. Thus, solution 1 is DR, while solution 2 is not. In the bottom half of the figure, preference robustness is illustrated. On the left, there are two sets of solutions in decision space which map (via many-to-one mappings) to the Pareto front in objective space (on the right). In the smaller set, nondominated solutions are located very closely to each other, while in the set with the larger volume nondominated solutions lie further apart. Thus, a solution in the smaller set is more PR than a solution in the larger set because a change in the decision-maker's preferences causes smaller changes of the decision or search variables (x_1 and x_2).

its open δ neighborhood $\mathcal{B}_\delta(\vec{s})$ in the n-dimensional decision space. It thus is the percentage of dominated solutions within the neighborhood $\mathcal{B}_\delta$ of $\vec{s}$ and is calculated as follows:

$$f_d(\vec{s}) = \frac{1}{|\mathcal{B}_\delta(\vec{s})|} \int_{\vec{y}\in\mathcal{B}_\delta(\vec{s})} (\mathcal{G}(\vec{y}))\, dy^n \tag{4}$$

where $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}$ is a "dominance function" and $|\mathcal{B}_\delta(\vec{s})|$ is the volume of $\mathcal{B}_\delta(\vec{s})$ in the n-dimensional decision space. $\mathcal{G}(\vec{y}) = 1$ if $\vec{y}$ is a dominated solution, else $\mathcal{G}(\vec{y}) = 0$. Notice that it is computationally more efficient to look for dominated solutions than for nondominated ones.

If $\mathcal{B}_\delta(\vec{s})$ is a countable set, Eq. (4) can be rewritten as

$$f_d(\vec{s}) = \frac{\sum_{\vec{y}\in\mathcal{B}_\delta(\vec{s})} \mathcal{G}(\vec{y})}{|\mathcal{B}_\delta(\vec{s})|} \tag{5}$$

where we have used that, in countable sets, $|\mathcal{B}_\delta(\vec{s})| = \sum_{\vec{y}\in\mathcal{B}_\delta(\vec{s})}$ counts the solutions (or members of population P) that are within the open δ neighborhood $\mathcal{B}_\delta(\vec{s})$.

For PR, a measure is determined via an expected cost $c(\vec{f}(\vec{s})) : \vec{f}(S) \left(\subseteq \mathbb{R}^k\right) \to \mathbb{R}$. Here, $c(\vec{f}(\vec{s}))$ quantifies the cost incurred in decision space when $\vec{f}(\vec{s})$ is moved to a neighboring point in the k-dimensional objective space. If we have a continuous neighborhood $\mathcal{O}_\delta(\vec{f}(\vec{s}))$ in objective space, then the robustness indicator of a solution $\vec{s}$ is determined as follows:

$$f_c(\vec{s}) = \frac{1}{|\mathcal{O}_\delta(\vec{f}(\vec{s}))|} \int_{\vec{y}\in(\mathcal{O}_\delta(\vec{f}(\vec{s}))\bigcap POF)} \left(c(\vec{f}(\vec{y}))\right) df(\vec{y})^k \tag{6}$$

where $|\mathcal{O}_\delta(\vec{f}(\vec{s}))|$ is the volume of the neighborhood in objective space, and POF is the Pareto optimal front.

In many practical problems, we usually deal with a finite set of solutions. So, for each solution, given a neighborhood radius δ, there will be a finite number K of neighbors that surround a nondominated point (in the POF) and whose domain(s) is/are a finite number N of solutions in decision space. Then, f_c is approximated based on this finite set of N solutions. The transition cost varies depending on the problem domain; it could describe: the financial loss when changing from a solution to another; or, the extra cost for adding more resources needed to generate a new solution. In its simplest form, it is the average Euclidean distance between the solution and its neighbors. Then

$$f_c(\vec{s}) = \frac{1}{|\mathcal{O}_\delta(\vec{f}(\vec{s}))|} \sum_{\vec{f}(\vec{y})\in(\mathcal{O}_\delta(\vec{f}(\vec{s}))\bigcap POF)} D(\vec{s}, \vec{y}) \tag{7}$$

where $D(\vec{s}, \vec{y})$ is the Euclidean distance between $\vec{s}$ and its neighbor $\vec{y}$

$$D(\vec{s}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (s_i - y_i)^2}. \tag{8}$$

For both DR and PR, the smaller $f_d(\vec{s})$ or $f_c(\vec{s})$, the better the robustness $R(\vec{s})$ a solution $\vec{s}$ has. If $f_d(\vec{s})$ and $f_c(\vec{s})$ are normalized, the DR (or PR) robustness value of a solution $\vec{s}$ can be determined via $f_d(\vec{s})$ (similarly for $f_c(\vec{s})$) as follows:

$$R(\vec{s}) = 1 - f_d(\vec{s}). \tag{9}$$

Note, that for robust solutions of Types I and II as proposed in Deb and Gupta [14] (see Section IV), the set of robust solutions might be neither the original efficient set nor a local efficient set. However, for our new proposed types DR and PR, the robust set is a subset of the original efficient set. This is because we do not change the original objective but instead use the quantified robustness concept as an additional criterion to refine the original efficient set. This is advantageous because the user of the MOP output, i.e., the decision maker, will typically pick one and only one solution from the set of robust solution, given his or her preference (utility). Any distortion of the original Pareto optimal set caused by consideration of robustness against DMAR uncertainty would result in the creation of suboptimal solutions of the original MOP, which is undesirable.

The use of a neighborhood radius (δ) introduces a user-defined parameter to the MOP. As highlighted in the literature on robustness and robust MO, this is acceptable because δ can be considered a measure of the magnitude of correlations within the Pareto optimal set [14]. As such, in practice it is hard to think of robustness without this parameter.

C. Limitations of the DR Definition

From the definition in Eq. (4), it is obvious that in those continuous MOP for which the Pareto optimal set S has a dimension smaller than n or, more generally, forms a set of measure zero in $\mathbb{R}^n$, there would not be any dominance robust solution. This reflects the fact that if S is a set of measure zero then each solution $\vec{s}$ is densely surrounded by dominated solutions and any perturbation of $\vec{s}$ has probability 1 to result in $\vec{f}(\vec{s})$ being no longer a Pareto optimal point. Even though a few perturbations may still lead to nondominated solutions, sampling them has negligible (i.e., zero) probability. Stated differently, even the smallest change in decision parameters causes a solution to become suboptimal, and hence none of the nondominant solutions is DR.

Let n be the number of variables in a problem, m be the number of constraints, and k be the number of objectives. If this problem has continuous and differentiable objectives, then the Pareto front is part of a $k-1$ manifold under certain regularity conditions (see [25] for more details). Thus, if the problem's multiple objectives are continuous and differentiable and their number is less than the number of variables, no DR exists. In this case, we need to change the definition of the open balls in (4) as follows: the open ball $\mathcal{B}_\delta$ will be replaced by $\mathcal{B}\infty_\delta$—the intersection of $\mathcal{B}_\delta$ with a manifold that has the smallest possible (and potentially fractal) dimension to contain all nondominated solutions of $\mathcal{B}_\delta$.

Although the previous discussions overcome a theoretical problem associated with the DR definition in an MOP with continuous differentiable objectives, the new definition creates a computational problem; that is, how to estimate the dimension of the manifold. Luckily, we will see that we do not need to look at this computational challenge (at least not in this paper), mainly because of approximation errors that are inherent in some of the computational methods used to find the POS. We will use a function in the next section to demonstrate a case in which, theoretically, no DR solution exists according to the definition presented by Eq. (4) while, computationally, approximation errors make it possible to calculate a DR value. In these cases, the numerically obtained DR values can be used to approximate the open ball $\mathcal{B}\infty_\delta$.

It is important to say that the discussion above is limited to MOPs with continuous, differentiable objective functions and $k \leq n$ problems. If $k > n$, we will see—for instance, in the VIEN test problem (see Section VI-A)—that DR according to (4) exists. Moreover, even in the case where $k \leq n$, but where objective functions are discontinuous or nondifferentiable, DR to (4) may also exist. For example, assume $\vec{x} \in R^n$. Assume that a function f_1 has a value of 0 when $\vec{x}$ lies in the hypercube $1 \leq x_i \leq 3 \quad \forall\, i = 1 \cdots n$ and 1 otherwise. Assume that a function f_2 has a value of 0 when $5 \leq x_i \leq 7, \quad \forall\, i = 1 \cdots n$ and 1 otherwise. If we wish to minimize f_1 and f_2, clearly every solution in the two hypercubes will lie on the POS according to the dominance definition given in Section II. This is an example where potentially $n \gg k$, solutions of the nondominated set have a neighborhood of dimension n, DR exists, and the MOP is defined on a nondiscrete (compact) domain. In fact, we would argue that many real-world problems that we are dealing with are black-box optimizations, with the majority falling into this category.

D. Accounting for DR and PR in EMO

Despite that robust solutions according to both the DR and PR measures are a subset of the nondominated set, generating the nondominated set and filtering out the two subsets in a post-processing (posterior) step is not possible. The reason for this is that in many problems, there is a many-to-one mapping from the decision space to the objective space. In other words, there may exist two solutions in the decision space that map to the same objective values in the objective space. The most trivial example is when there is symmetry in the representation; thus two genotypes would map to the same phenotype and thus, map to the same values in objective space. If an EMO algorithm operates independently of the two DR and PR measures, any of the two solutions may appear in the nondominated set (they would not both appear because of the concept of dominance). However, one of them might be more robust than the other according to either DR or PR. This case is illustrated in the third test problem [Eq. (13) that is discussed later in this paper]. Hence, we must handle the DR and PR measures during the optimization process.

There are different ways to account for DR or PR in the EMO process. Three options are discussed below.

Option 1: Adding the robustness value as an objective. This may not be desirable in the DR and PR types of robustness since both types are defined on the nondominated set. As such, this additional objective would not introduce a solution that is not already in the original nondominated set. It would only spread the nondominated set into another dimension, making it more sparse. This may affect the evolutionary search negatively.

Option 2: Using the robustness measure during selection. Let $CV(\vec{x})$ be a constraint violation measure and $R(\vec{x})$ a robustness measure. A possible selection procedure is proposed as given on next page.

Option 3: Defining a penalty function to penalize solutions with low robustness. The new objective function $\vec{\bar{f}}(\vec{x}) = \left(\bar{f}_1, \ldots \bar{f}_k,\right)$ to be minimized can, for example, be defined as follows:

$$\bar{f}_i(\vec{x}) = f_i(\vec{x}) + \frac{1}{R(\vec{x}) + \epsilon} \qquad \forall i \in \{1, \ldots, k\} \tag{10}$$

with $\epsilon > 0$ being a small number to avoid potential singularities in the penalty term.[1] Here, robustness R is the complement of DR and PR calculated in (4) and (6).

It is Option 3 that we use in our case studies (see next section). When in these case studies, we refer to "nondominated set," "dominance," and the like, then we always mean the solutions of the original MOP.

[1]For all practical purposes we set $\epsilon = 1E-08$.

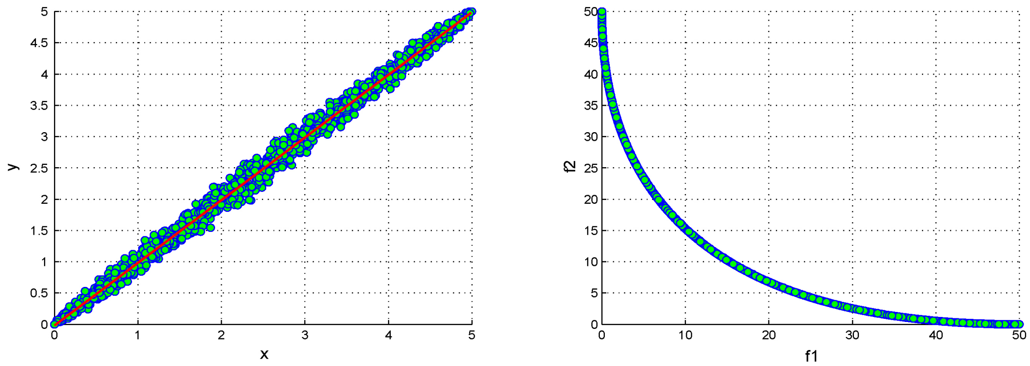


Fig. 5. Example of a nondominated set for BINH. The efficient set projected on both decision (left panel) and objective (right panel) spaces.

winner = **tournament**(a, b)
begin
if a is feasible and b is infeasible, *winner* = a
else if a is infeasible and b is feasible, *winner* = b
else if both a and b are infeasible
winner = **if** $(CV(a) < CV(b))$? a : b
else if both a and b are feasible
if a dominates b, *winner* = a
else if b dominates a, *winner* = b
else if $niching\_value(a) > niching\_value(b)$,
winner = a
else if $niching\_value(a) < niching\_value(b)$,
winner = b
else
{
–**if** $R(a) < R(b)$, *winner* = a
–**else if** $R(a) > R(b)$, *winner* = b
–**else** *winner* = random_choose(a, b)
}
end

VI. Case Studies

A. Problems for Testing Robustness

We modify two existing problems in the literature for the DR measure and introduce a new test problem for the PR measure.

For the DR measure, we use a problem introduced by Binh and Korn [6] with two objectives and a problem introduced by Viennet *et al.* [46] with three objectives. The shape of the Pareto curve is different in each. We call the former problem BINH and the latter VIEN. For BINH

$$\begin{aligned} f_1(x, y) &= x^2 + y^2 \\ f_2(x, y) &= (x-5)^2 + (y-5)^2 \end{aligned} \tag{11}$$

where $-5 \leq x, y \leq 10$. Both objectives are convex functions and there is a global efficient set lying on a straight line from (0, 0) to (5, 5). The POF is convex as depicted in the right panel of Fig. 5. However, for this problem, algorithms usually find an approximation of the efficient set in the shape of an ellipse being fat at the middle and sharpened to two extreme points (0, 0) and (5, 5) (see left panel of Fig. 5).

For VIEN

$$\begin{aligned} f_1(x, y) &= x^2 + (y-1)^2 \\ f_2(x, y) &= x^2 + (y+1)^2 + 1 \\ f_3(x, y) &= (x-1)^2 + y^2 + 2 \end{aligned} \tag{12}$$

where $-2 \leq x, y \leq 2$. VIEN is also a convex problem. The efficient set belongs to the area of the triangle with the three corner points (0, −1), (1, 0) and (0, 1). Fig. 6 depicts a nondominated set in the decision variable space (left panel) and the POF in the objective space (right panel).

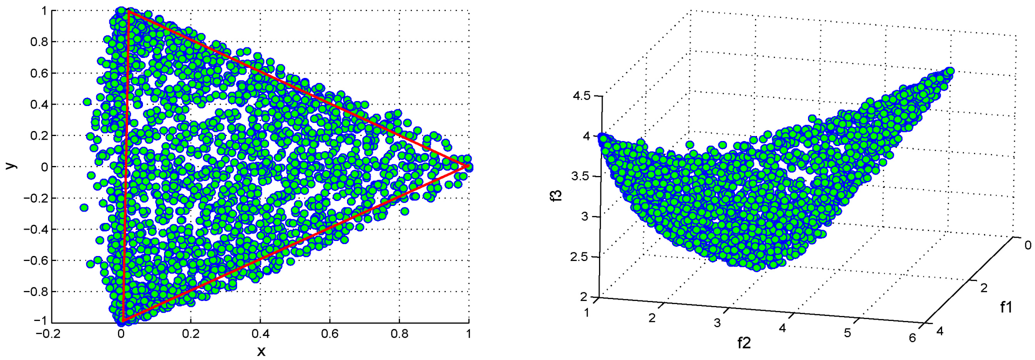
For PR, we introduce a test problem that we call preference robustness problem (PRP). In PRP the first objective, $f_1 : (x \in [x_{\min}, x_{\max}], y) \rightarrow \mathbb{R}$, is a function that maps intervals to points and effectively depends on one variable, x, only. The function's domain is divided into two equal parts. Each part uses a different resolution for forming intervals that are mapped to single points via function f_1. The second objective, f_2, is constructed from f_1 but depends (continuously) on y as well. f_1 and f_2 are in conflict.

The formal description of the PRP is as follows:

$$\begin{aligned} x_c &= \tfrac{1}{2}(x_{\max} + x_{\min}) \\ g(x) &= \begin{cases} x_{\min} + \text{floor}\left(\dfrac{x - x_{\min}}{r_1}\right) r_1, & \text{if } x < x_c \\ x_c + \text{floor}\left(\dfrac{x - x_c}{r_2}\right) r_2, & \text{else,} \end{cases} \\ f_1(x) &= 1 - 0.99\left(\exp\left(-\left(\frac{g(x) - \mu_1}{2\delta_1}\right)^2\right) + \right. \\ &\qquad \left. \exp\left(-\left(\frac{g(x) - \mu_2}{2\delta_2}\right)^2\right)\right) \\ f_2(x, y) &= 1 - \left(\frac{f_1}{1 + y^2}\right)^2 \end{aligned} \tag{13}$$

where x_c is the mid point of f_1's x-domain ($[x_{\min}, x_{\max}]$), r_1 and r_2 are resolutions for the two halfs of the domain (in our case study r_1 is set to 0.05 and r_2 is 0.2, i.e., the image of the interval ranging from $x_{\min}$ to x_c is four times denser than the image of the interval ranging from x_c to $x_{\max}$; see top panel of Fig. 7), $\mu_1 = -2.5$, $\delta_1 = 0.5$, $\mu_2 = 2.5$, and $\delta_2 = 0.5$; $x_{\min}$, and $x_{\max}$ are, respectively, the lower and upper domain bounds for x (in this paper: $x_{\min} = -20$ and $x_{\max} = 20$).

The PRP describes a many-to-one mapping from the decision space to the objective space. The efficient set is obtained



(a) (b)

Fig. 6. Example of a nondominated set for VIEN. The efficient set projected on both (a) decision and (b) objective spaces.

when $y = 0$ and $f_2 = 1 - f_1^2$ (see lower panel in Fig. 7). The interval $[x_{\min}, x_c]$ is mapped to floor $((x_{\max} - x_{\min})/(2r_1))$ points on the POF i.e., 400 points given the parameter set in our case study. The interval $[x_c, x_{\max}]$, on the other hand, is mapped to floor $((x_{\max} - x_{\min})/(2r_2)) = 100$ points on the POF. Therefore, the part of the functions' domain whose images have a higher resolution, i.e., $x \in [x_{\min}, x_c]$, will contribute more points to the POF than the other half of the domain. The PR measure therefore should prefer solutions with $x \in [x_{\min}, x_c]$.

It is also interesting to see how DR and PR are accounted for in general MOPs since it is usually difficult to know what characteristics problems have in practice. For this, we measured DR and PR on two classes of scalable problems (namely DTLZs) introduced in [16]: DTLZ3 and DTLZ6. They are described as follows:

—DTLZ3

$$\begin{aligned}
f_1(\vec{x}) &= (1 + g(x_M, \ldots, x_N)) \cos(x_1\pi/2) \cdots \\
&\quad \cdots \cos(x_2\pi/2) \cos(x_{M-2}\pi/2) \cdots \\
&\quad \cdots \cos(x_{M-1}\pi/2) \\
f_2(\vec{x}) &= (1 + g(x_M, \ldots, x_N)) \cos(x_1\pi/2) \cdots \\
&\quad \cdots \cos(x_2\pi/2) \cos(x_{M-2}\pi/2) \cdots \\
&\quad \cdots \sin(x_{M-1}\pi/2) \\
f_3(\vec{x}) &= (1 + g(x_M, \ldots, x_N)) \cos(x_1\pi/2) \cdots \\
&\quad \cdots \cos(x_2\pi/2) \sin(x_{M-2}\pi/2) \\
\cdots &= \vdots \\
f_{M-1}(\vec{x}) &= (1 + g(x_M, \ldots, x_N)) \cos(x_1\pi/2) \cdots \\
&\quad \cdots \sin(x_2\pi/2) \\
f_M(\vec{x}) &= (1 + g(x_M, \ldots, x_N)) \sin(x_1\pi/2).
\end{aligned}$$

Here M is the number of objectives, K is the number of active decision variables, $N = M + K - 1$, N is the total number of decision variables and $n = M + K - 1$, $K = |x_M|$, $0 \leq x_i \leq 1$, $i \in \{1, \ldots, N\}$, and $g(x_M, \ldots, x_N) = 100\left[K \sum_{x_{i=M} \in X_M}^{N} \left((x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))\right)\right]$ with X_M being the set of active decision variables. This problem has multiple (actually: $3^K - 1$) local POFs and a global one.

—DTLZ6

$$\begin{aligned}
f_1(\vec{x}) &= (1 + hg(x_M, \ldots, x_N)) \cos(\theta_1\pi/2) \cdots \\
&\quad \cdots \cos(\theta_2\pi/2) \cos(\theta_{M-2}\pi/2) \cdots \\
&\quad \cdots \cos(\theta_{M-1}\pi/2) \\
f_2(\vec{x}) &= (1 + hg(x_M, \ldots, x_N)) \cos(\theta_1\pi/2) \cdots \\
&\quad \cdots \cos(\theta_2\pi/2) \cos(\theta_{M-2}\pi/2) \cdots \\
&\quad \cdots \sin(\theta_{M-1}\pi/2) \\
f_3(\vec{x}) &= (1 + hg(x_M, \ldots, x_N)) \cos(\theta_1\pi/2) \cdots \\
&\quad \cdots \cos(\theta_2\pi/2) \cdots \\
&\quad \cdots \sin(\theta_{M-2}\pi/2) \\
\cdots &= \vdots \\
f_{M-1}(\vec{x}) &= (1 + hg(x_M, \ldots, x_N)) \cos(x_1\pi/2) \cdots \\
&\quad \cdots sin(\theta_2\pi/2) \\
f_M(\vec{x}) &= (1 + hg(x_M, \ldots, x_N)) \sin(x_1\pi/2).
\end{aligned}$$

Here $0 \leq x_i \leq 1$, $i \in \{1, \ldots, n\}$, $\theta_i = \frac{\pi}{4(1+hg(x_M,\ldots,x_N))}(1 + 2h(x_M, \ldots, x_N)gx_i)$, and $hg(x_M, \ldots, x_N) = \sum_{x_{i=M} \in X_M}^{N} x_i^{0.1}$.

B. Experimental Setup

We selected NSGA-II for all experiments in this paper. Any other EMO technique could be adopted as well. NSGA-II is an elitism-based multiobjective evolutionary algorithm [15]. Its main feature is an elitism-preserving operation. NSGA-II does not use an explicit archive; instead a population is used to store both elitist and nonelitist solutions for the next generation. However, for consistency, we can consider this population an archive by first setting the archive size equal to the initial population size. The current archive is then determined based on the combination of the current population and the previous archive. NSGA-II uses dominance ranking to classify the population into a number of layers, such that the first layer is the nondominated set in the population, the second layer is the nondominated set in the population with the first layer removed, and the third layer is the nondominated set in the population with the first as well as second layers removed and so on. The archive is created based on the order of ranking layers: the best rank being selected first. If the number of individuals in the archive is smaller than the population size, the next layer will be taken into account and so forth. If adding a layer makes the number of individuals in the archive exceed the initial population size, a truncation operator is applied to that layer using the so-called *crowding*

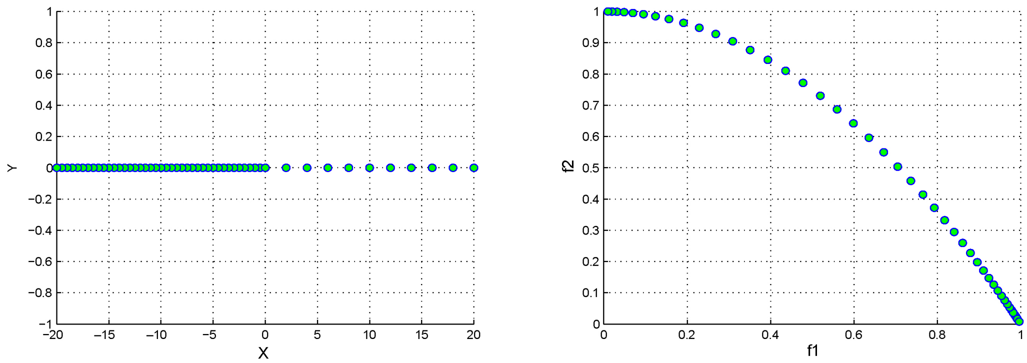


Fig. 7. Example of a nondominated set for PRP. Both the efficient set in decision space and its projection in objective space are shown.

distance D.[2] The truncation operator removes the individual with the smallest D. An offspring population of the same size as the initial population is then created from the archive using crowded tournament selection, crossover, and mutation operators. Crowded tournament selection is a traditional tournament selection method which uses the crowding distance to break the tie between two solutions of the same rank.

The population size is set to 100, the crossover rate is 0.9 and the mutation rate is 1/n. The distribution indices for crossover and mutation operators are, respectively, $\eta_c = 15$ and $\eta_m = 20$. For DR, each solution is being perturbed 500 times around its neighborhood. Furthermore, each experiment is run repeatedly for 30 times with different random seeds.

C. Results for DR

We report the DR measure [see (4), (5)] at the last generation for all 30 runs with a neighborhood radius $\delta = 0.3$. A scatter diagram of all solutions encountered in the 30 runs for each of the two test problems BINH and VIEN is presented in Figs. 8 and 9. Because some runs in NSGA-II failed to reach the true nondominated solutions, we see some dominated solutions included in the amalgamation of the 30 runs. Note that here we use the term "*NSGA-II*" alone to refer to the original NSGA-II without the robustness measure, while "*robust solution*" or "*DR solution*" or "NDR" is used for solutions found by NSGA-II using the robustness concept, Option 3 in Section V-D.

Fig. 8 shows the results for BINH. This is an example for which DR does not exist according to the original definition used in Eq. (4). However, we will see that because of computational errors, we can still calculate DR to approximate the modified definition in Section V-C. Practically, algorithms usually find an approximation of the set of nondominated solutions [which should lie on the straight line between (0, 0) and (5, 5)] in a shape of an ellipse covering this straight line [being fat in the middle and sharpening toward the two extreme points (0, 0) and (5, 5)]. The area occupied by the nondominated points obtained by NSGA-II is shown. From the

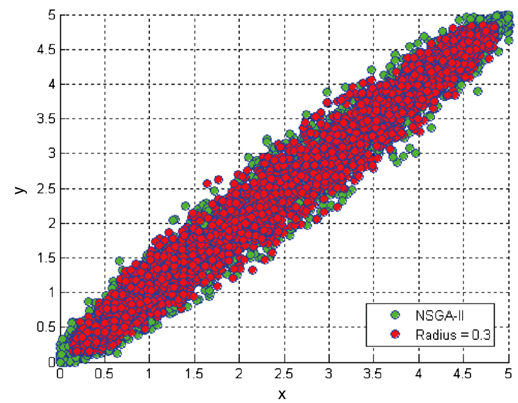


Fig. 8. Nondominated solutions found by NSGA-II (green) and the robust solutions (red, or dark color on the black and white version) for BINH.

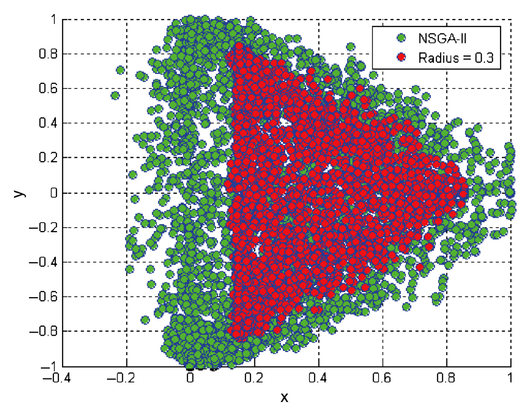


Fig. 9. Nondominated solutions found by NSGA-II (green) and the robust solutions (red) for VIEN.

figure, we can see that the shape formed by the DR solutions is slightly thinner than that of the approximated efficient set generated by NSGA-II. Further, the robust solutions are clearly separated from the two extreme points (0, 0) and (5, 5).

Results on the VIEN problem are depicted in Fig. 9. NSGA-II concentrates the solutions in the area in and around the triangle with corner points (−1, 0), (1, 0), and (0, 1). The figure shows that solutions that are close to the edges of the triangle are not robust since perturbations can easily generate solutions that are nondominated. The DR measure generates solutions well inside the triangle. It is very interesting

[2]The crowding distance D of a solution $\vec{x}$ is calculated as follows: the population is sorted according to each objective to find adjacent solutions to $\vec{x}$. Boundary solutions are assigned infinite D values. For nonboundary solutions, the average of the differences between the adjacent solutions in each objective is calculated and assigned to $\vec{x}$.

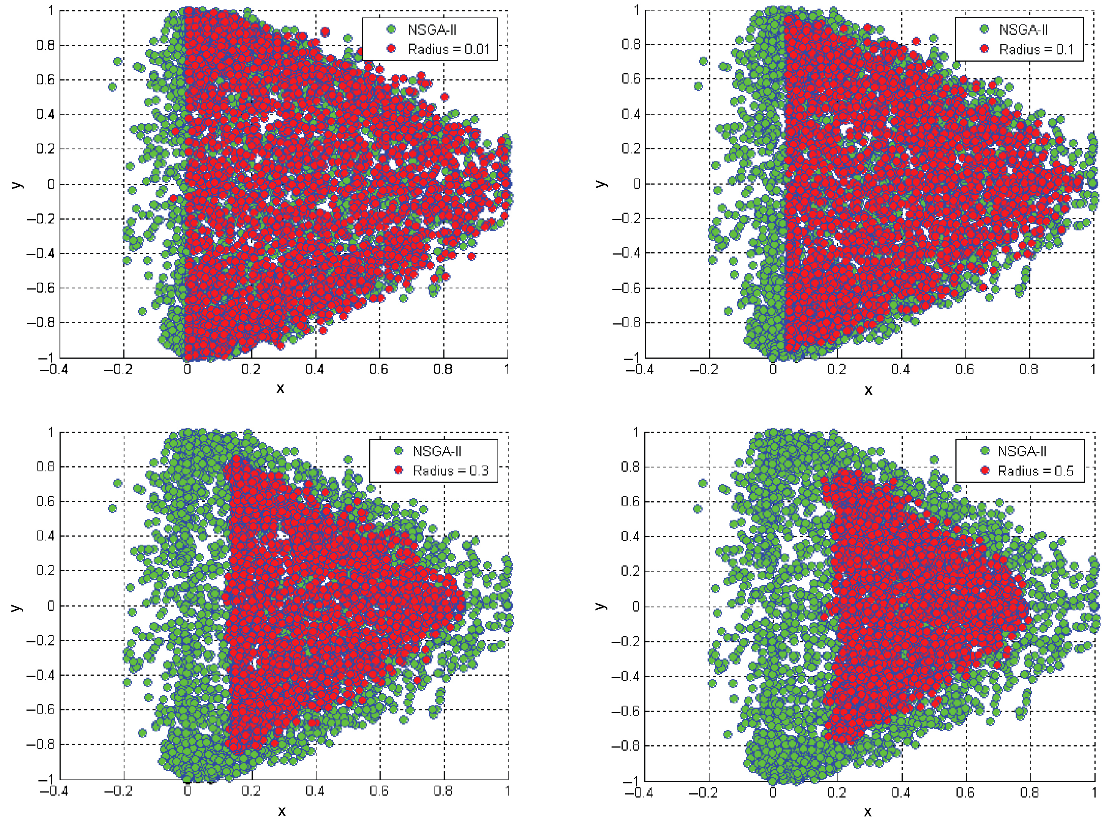


Fig. 10. Nondominated solutions found by NSGA-II (green) and the robust solutions (red) for VIEN with different radius for the neighborhood: (top left) 0.01; (top right) 0.1; (bottom left) 0.3; and (bottom right) 0.5.

to see that for both problems, the robust solutions tend to stay away from extreme points. This is in agreement with our intuition of how RO should account for uncertainty related to DMAR variations. As indicated earlier, decision-makers are unlikely to be willing to "put all eggs in one basket." Hence, "extreme solutions" would not be robust against a variation of DMAR, and a dominance-robustness based analysis is likely to provide a decision-maker with a smaller set of nondominated solutions to choose from. We find, as expected and not shown here, that the DR robust solution set of earlier (than the last) generations in each of the 30 runs is even smaller. This is due to the fact that for early generations NSGA-II has not evolved a good approximation of the nondominated set as yet. Thus, most identified solutions are surrounded by dominated solution and hence, are not dominance-robust.

Since, the measure of robustness is a function of the neighborhood radius δ used in the experiments, we varied δ and tested the following different values: 0.01, 0.1, 0.3, and 0.5. The effect of radius variation is illustrated in Fig. 10 for the VIEN problem. The figure shows that as the radius increases, the set of robust solutions shrinks.

We measured DR [as defined in Eq. (5)] for all solutions in the nondominated set found by NSGA-II without the DR penalty term and in the robust set evolved by NSGA-II with penalty term (NDR). Table I provides the minimum, maximum, mean, and standard deviation of DR taken over each of these two sets. The larger the value of the DR measure, the less robust a solution is.

It is expected that as the neighborhood radius δ increases, DR decreases as well. This is because the larger the neighborhood, the more likely it is that it contains dominated solutions. Adding DR as a penalty to the objective functions [as in (10)] generated a set of solutions that are much more dominance-robust than the original nondominated set. Actually, with the penalty term considered during optimization, the algorithm produces solutions that are on average one to two orders of magnitude more robust than those generated by NSGA-II without penalty term. Thus, adding the penalty term markedly improves the search for robust solutions.

The measured DR on the scalable problems of DTLZ3 and DTLZ6 are shown in Tables II and III. In the first table, we set the number of objectives to $M = 3$ and the δ radius to 0.1. The number of active decision variables K was changed from 5 to 20 in steps of 5, i.e., $K \in \{5, 10, 15, 20\}$. Thus, the number of decision variables $N = M + K - 1$ was $N \in \{7, 12, 17, 22\}$. Since, for these K and N values, both DTLZ3 and DTLZ6 are problems of dimensionality much greater than three, depicting the nondominant and robust sets is neither practical nor very illustrative.

It is clear from Table II that the robust set generated by NSGA-II with robustness NDR contains solutions with much better values for DR than does the nondominated set generated by NSGA-II without penalty term. For DTLZ3, the nondominated set obtained by NSGA-II had a mean DR value of around 1.0 while NDR produced a set of solutions with mean 0.3. For DTLZ6, NSGA-II obtained robustness values

TABLE I
MAXIMUM, MINIMUM, MEAN, AND STANDARD DEVIATION DR OF THE SET OF NONDOMINATED SOLUTIONS OVER ALL 30 RUNS FOR VIEN

Radius	Alg	Min	Max	Mean	Std
0.01	NSGA-II	0.000E+00	1.800E−02	2.014E−03	3.678E−03
	NDR	0.000E+00	1.400E−02	1.867E−05	3.684E−04
0.1	NSGA-II	0.000E+00	1.180E−01	1.855E−02	2.546E−02
	NDR	0.000E+00	6.000E−03	9.733E−05	5.098E−04
0.3	NSGA-II	0.000E+00	2.500E−01	4.990E−02	5.510E−02
	NDR	0.000E+00	8.000E−03	2.207E−04	7.734E−04
0.5	NSGA-II	0.000E+00	2.980E−01	7.486E−02	6.967E−02
	NDR	0.000E+00	1.200E−02	4.933E−04	1.061E−03

The smaller the DR value the more dominance-robust is a solution.

TABLE II
MAXIMUM, MINIMUM, MEAN, AND STANDARD DEVIATION DR OF THE SET OF NONDOMINATED SOLUTIONS OVER ALL 30 RUNS FOR DTLZ3 AND DTLZ6 WITH $M = 3$ AND DIFFERENT NUMBERS OF ACTIVE DECISION VARIABLES K

Probs	K	Alg	Min	Max	Mean	Std
DTLZ3	5	NSGA-II	9.439E−01	9.481E−01	9.458E−01	1.033E−03
		NDR	2.206E−01	4.450E−01	2.928E−01	4.621E−02
	10	NSGA-II	9.956E−01	9.967E−01	9.962E−01	2.514E−04
		NDR	2.434E−01	4.228E−01	3.160E−01	5.163E−02
	15	NSGA-II	9.934E−01	9.996E−01	9.976E−01	1.576E−03
		NDR	2.390E−01	4.265E−01	3.449E−01	4.692E−02
	20	NSGA-II	9.732E−01	9.946E−01	9.861E−01	5.390E−03
		NDR	2.786E−01	4.700E−01	3.593E−01	5.516E−02
DTLZ6	5	NSGA-II	6.574E−01	6.888E−01	6.733E−01	7.635E−03
		NDR	5.404E−02	1.283E−01	1.177E−01	1.867E−02
	10	NSGA-II	8.627E−01	8.861E−01	8.734E−01	7.185E−03
		NDR	1.224E−01	1.287E−01	1.258E−01	1.784E−03
	15	NSGA-II	9.242E−01	9.498E−01	9.402E−01	5.895E−03
		NDR	1.181E−01	1.338E−01	1.244E−01	3.832E−03
	20	NSGA-II	9.385E−01	9.601E−01	9.498E−01	6.076E−03
		NDR	1.147E−01	1.282E−01	1.222E−01	3.466E−03

that, in terms of DR, were about 8 times worse than those generated by NDR. Clearly, the solutions found by NDR were, on average, more robust than the ones obtained by NSGA-II. A change in the dimension (number of decision variables) of the DTLZ MOP did not result in any different generic outcomes: NDR obtained sets of solutions that, on average were much more robust than solutions in the nondominated set evolved by NSGA-II.

We also tested the robustness of solution sets generated by NSGA-II and NDR in DTLZ problems of different scale. In order to do so, we set the number of objectives to $M \in \{3, 4, 5, 6\}$ and the number of active decision variables to $K = 10$. As can be seen from Table III, with increasing number of objectives, both algorithms, NSGA-II and NDR, obtained greater values for DR. In all cases, NDR evolved a set of solutions that, on average, were more robust than the nondominated solutions found by NSGA-II.

D. Results for PR

In this section, we report on the PRP. Fig. 11 depicts the nondominated solutions found by NSGA-II in all 30 runs for a neighborhood radius δ of 0.3. Unlike for the DR measure, here δ is defined in the objective space. The global nondominated set lies on the x-axis ($y = 0$) and forms two line segments of differing resolution (the denser segment spans from −20 to 0, and the sparser one from 0 to 20). Because some runs in NSGA-II failed to reach the true nondominated solutions, we see some dominated solutions included in the amalgamation of the 30 runs.

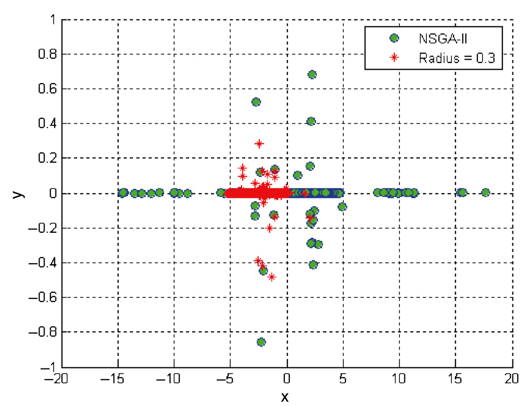


Fig. 11. NonDominated solutions found by NSGA-II (green) and the robust solutions (red star) for PRP.

The NSGA-II technique with the modified penalty term identified successfully solutions that lie inside the nondominated set. Moreover, the PR measure [see (6), (7)] prefers solutions that are close to each other in the decision space.

TABLE III

DR VALUES OF THE SET OF NONDOMINATED SOLUTIONS OVER ALL 30 RUNS FOR DTLZ3 AND DTLZ6 WITH DIFFERENT NUMBERS OF OBJECTIVES, M, AND $K = 10$

Probs	M	Alg	Min	Max	Mean	Std
8*DTLZ3	3	NSGA-II	9.956E−01	9.967E−01	9.962E−01	2.514E−04
		NDR	2.434E−01	4.228E−01	3.160E−01	5.163E−02
	4	NSGA-II	7.939E−01	9.591E−01	8.801E−01	3.262E−02
		NDR	1.212E−01	2.491E−01	1.616E−01	2.666E−02
	5	NSGA-II	4.925E−01	6.290E−01	5.592E−01	3.804E−02
		NDR	6.491E−02	1.425E−01	1.028E−01	2.040E−02
	6	NSGA-II	3.061E−01	4.269E−01	3.672E−01	3.270E−02
		NDR	5.427E−02	9.479E−02	7.206E−02	8.788E−03
DTLZ6	3	NSGA-II	8.627E−01	8.861E−01	8.734E−01	7.185E−03
		NDR	1.224E−01	1.287E−01	1.258E−01	1.784E−03
	4	NSGA-II	3.787E−01	4.477E−01	4.127E−01	1.584E−02
		NDR	7.686E−02	8.918E−02	8.298E−02	3.177E−03
	5	NSGA-II	1.411E−01	2.372E−01	1.912E−01	2.551E−02
		NDR	4.334E−02	5.260E−02	4.722E−02	2.310E−03
	6	NSGA-II	5.560E−02	1.153E−01	7.896E−02	1.713E−02
		NDR	2.496E−02	3.042E−02	2.823E−02	1.249E−03

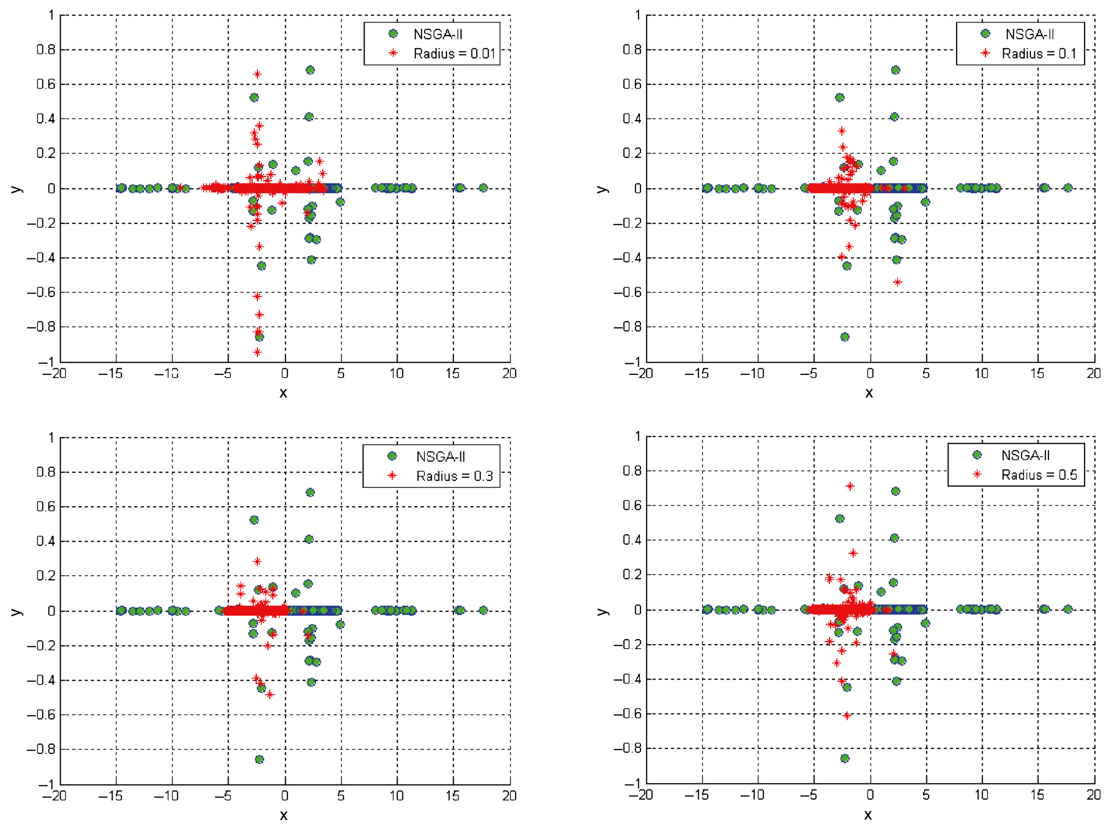


Fig. 12. Nondominated solutions found by NSGA-II (green) and the robust solutions (red star) for PRP with different radius for the neighborhood: 0.01 (top), 0.1, 0.3, and 0.5 (bottom).

TABLE IV

VALUES OF THE SET OF NONDOMINATED SOLUTIONS OVER ALL 30 RUNS FOR PRP

Radius	Alg	Min	Max	Mean	Std
0.01	NSGA-II	2.000E−15	4.000E+01	8.570E+00	1.481E+01
	NPR	2.796E−12	4.000E+01	8.068E−01	5.320E+00
0.1	NSGA-II	1.622E−01	4.000E+01	3.528E+00	1.991E+00
	NPR	2.523E−02	5.014E+00	1.706E−01	2.253E−01
0.3	NSGA-II	1.434E+00	1.710E+01	3.289E+00	1.314E+00
	NPR	1.238E−01	3.209E+00	2.771E−01	1.421E−01
0.5	NSGA-II	1.903E+00	1.706E+01	3.247E+00	1.290E+00
	NPR	2.122E−01	3.158E+00	3.770E−01	1.489E−01

TABLE V

THE PR VALUES OF THE SET OF NONDOMINATED SOLUTIONS OVER ALL 30 RUNS FOR DTLZ3 AND DTLZ6

Probs	K	Alg	Min	Max	Mean	Std
DTLZ3	5	NSGA-II	1.291E−01	2.449E−01	1.868E−01	3.344E−02
		NPR	8.878E−02	2.193E−01	1.455E−01	3.142E−02
	10	NSGA-II	1.599E−01	2.775E−01	2.045E−01	3.040E−02
		NPR	7.355E−02	2.359E−01	1.338E−01	3.855E−02
	15	NSGA-II	2.088E−01	9.403E−01	5.357E−01	2.078E−01
		NPR	1.165E−01	7.036E−01	2.980E−01	1.546E−01
	20	NSGA-II	8.650E−01	1.001E+00	9.635E−01	3.187E−02
		NPR	8.115E−01	1.000E+00	9.608E−01	4.640E−02
DTLZ6	5	NSGA-II	9.283E−02	4.320E−01	2.889E−01	9.545E−02
		NPR	3.610E−02	1.138E−01	6.129E−02	2.250E−02
	10	NSGA-II	1.910E−01	4.702E−01	3.412E−01	6.149E−02
		NPR	3.798E−02	2.572E−01	9.842E−02	5.140E−02
	15	NSGA-II	1.886E−01	3.831E−01	2.656E−01	4.413E−02
		NPR	5.688E−02	2.428E−01	1.381E−01	4.577E−02
	20	NSGA-II	1.301E−01	2.983E−01	2.286E−01	4.479E−02
		NPR	1.008E−01	2.260E−01	1.548E−01	2.925E−02

TABLE VI

THE PR VALUES OF THE SET OF NONDOMINATED SOLUTIONS OVER ALL 30 RUNS FOR DTLZ3 AND DTLZ6

Probs	M	Alg	Min	Max	Mean	Std
DTLZ3	3	NSGA-II	1.599E−01	2.775E−01	2.045E−01	3.040E−02
		NPR	7.355E−02	2.359E−01	1.338E−01	3.855E−02
	4	NSGA-II	9.605E−01	1.000E+00	9.955E−01	9.764E−03
		NPR	9.592E−01	1.000E+00	9.840E−01	1.178E−02
	5	NSGA-II	9.802E−01	1.000E+00	9.974E−01	5.747E−03
		NPR	9.700E−01	1.000E+00	9.891E−01	1.045E−02
	6	NSGA-II	9.803E−01	1.000E+00	9.990E−01	3.984E−03
		NPR	9.704E−01	1.000E+00	9.863E−01	1.038E−02
DTLZ6	3	NSGA-II	1.910E−01	4.702E−01	3.412E−01	6.149E−02
		NPR	3.798E−02	2.572E−01	9.842E−02	5.140E−02
	4	NSGA-II	9.348E−01	1.002E+00	9.736E−01	1.707E−02
		NPR	7.994E−01	9.798E−01	9.111E−01	4.202E−02
	5	NSGA-II	9.771E−01	1.010E+00	9.991E−01	8.137E−03
		NPR	9.443E−01	1.000E+00	9.810E−01	1.588E−02
	6	NSGA-II	9.890E−01	1.025E+00	1.007E+00	7.833E−03
		NPR	9.735E−01	1.000E+00	9.930E−01	8.739E−03

As such, it prefers solutions on the left side of the x-axis, i.e., solutions with negative x values, where the resolution of the discretization is high.

In Fig. 12, we show different experiments with different neighborhood radii to study the impact of the neighborhood on the set of PR solutions. With a neighborhood radius of 0.01, solutions spread widely, even into the low-resolution area ($x > 0$). As the neighborhood size increases, a higher pressure on the PR set is created and solutions that fall in the low-resolution area ($x > 0$) start to disappear. It is also important to note that for $\delta = 0.01$ a PR solution was found close to $x = -10$. As the neighborhood size increases, PR solutions cluster more between -5 and 0. This is mainly a result of the performance of NSGA-II on this problem. The nondominated set generated by NSGA-II shows discontinuities, which is an artifact of NSGA-II. This artifact causes high-PR values for some solutions (especially the ones on the left of -5). As such, no robust solution appears in that area when δ is sufficiently large.

We measured PR for the nondominated set found by NSGA-II with and without the PR penalty value, similarly to the DR measurement described in the previous section.

The results for the PRP test problem are reported in Table IV for different neighborhood sizes. Here, "NPR" denotes the EC algorithm in which the penalty term PR was added to the objectives, as in (10). A value close to zero for the PR indicates that solutions that are close to each other in objective space are also very close to each other in decision space. As can be seen from Table IV, NPR generated a set with a large percentage of PR nondominated solutions.

We also tested the new robustness measure PR on the two scalable problems of DTLZ3 and DTLZ6, and the corresponding values are shown in Table V. As before, NPR obtained sets of nondominated solutions that contain a larger proportion of PR solutions than in the nondominated sets evolved by NSGA-II. NPR behaved stably in all cases reported here.

The scaling behavior of PR with increasing number of objectives is shown in Table VI. NPR still did well and obtained sets of solutions that, on average, are more PR than are solutions in the nondominated sets generated by NSGA-II. Unlike for DR, PR values increase with increasing number of objectives. This indicates that because of the many-to-one mapping nature of the two DTLZ problems and because the number of nondominated solutions increases with a growing number of objectives, there are more solutions that are far separated in decision space but close together in objective space. Thus, both NPR and NSGA-II obtained sets in which the average PR of solutions decreases as the number of objectives increases.

VII. Conclusion

The DMAR is a key factor in decision sciences. It contributes to the preference values or utilities of each objective. Uncertainties that arise from variations in the DMAR can result in significant perturbations to the decisions made in MOP. Thus, it is important for the MOP community to develop RO methods that aid in the study of DMAR effects on the efficient set of MOP solutions.

In this paper, the consideration of uncertainties arising from the DMAR led us to introduce two new types of robustness measures, namely DR and PR. Neither of them has any analogue in single-objective RO problems. Instead, they both make use of the representations of the efficient set of solutions in the different MOP spaces, notably the decision space and the objective space. DR measures the impact of a perturbation in the decision space on the decision-maker's preference while PR measures the impact of a perturbation in the decision-maker's preference on the decision space.

In our mathematical definitions of DR and PR, we abstracted the concepts and did not make explicit the dependence on the DMAR. Thus, DR and PR have a wider applicability than just describing robustness against perturbations caused by DMAR variations. The formal definitions have two noteworthy characteristics. Firstly, robustness is added as a new criterion to the original objectives of the MOP. Thus, the sets of multiobjective DR and PR solutions are subsets of the true efficient set, i.e., the RO with DR and PR results in a reduced set of nondominated solutions to the original MOP for decision-makers to consider. Secondly, neighborhood radii δ feature in the DR and PR definitions. These radii can be considered "robustness correlation lengths"—i.e., the distances in decision space (for DR) and in objective space (for PR) over which perturbative variations of a solution are "felt" by other solutions in the efficient set. It is these parameters δ through which the effect of DMAR variations on the MOP solutions can be studied quantitatively.

There are various options for implementing DR and PR in EMO. We had argued that the best results can be expected for implementations in which robustness-related penalty terms are added to the original objective functions of the MOP. We used this method in three case studies. These studies: 1) have confirmed the intuitive notions of the novel robustness measures, in particular the dependence of the robust efficient sets on the neighborhood radii δ; and 2) have demonstrated that the DR and PR efficient sets are subsets of the true efficient set.

The novel measures DR and PR are true MOP-specific robustness measures. They are likely to prove particularly useful when the mapping of the efficient set in search space to its image in objective space (i.e., the POF) is not bijective but many-to-one. For instance, in systems engineering design problems, DR and PR are probably useful measures that can help quantify anticipated improvements (or not) of design principles based on redundancy, degeneracy and the like. Another area of research in which DR and PR can be utilized to great benefit is optimization in the context of group decision-making and negotiations. Using DR and PR to study the mappings of clusters in the efficient set (which can be considered the preferred solutions of decision-makers in groups) is a very interesting problem and is the focus of some of our current research efforts.

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Lam Thu Bui received the Ph.D. degree in computer science from the University of New South Wales (UNSW), Canberra Campus, Australia, in 2007.

After graduation, he was a Post-Doctoral Research Fellow with UNSW until 2009. He has been involved in academics including teaching and research since 1998. He is currently the Head of the Department of Software Engineering, Le Quy Don University, Hanoi, Vietnam. He is the co-editor of the book *MultiObjective Optimization in Computational Intelligence: Theory and Practice* (Hershey, PA: IGI Global, Information Science Reference Series, 2008). His current research interests include evolutionary computation, with an emphasis on evolutionary multiobjective optimization.

Dr. Bui has been a member of the program committees of several conferences and workshops in the field of evolutionary computing, such as the IEEE Congress on Evolutionary Computation and the Genetic and Evolutionary Computation Conference.

Hussein Abbass (SM'05) is a Professor and the Chair of Information Technology, University of New South Wales, Canberra Campus, Australia, at the Australian Defense Force Academy, Canberra, ACT, Australia. He has 190+ refereed papers. His current research interest lies in applying computational intelligence, risk assessment, and human decision making to transportation and capability planning projects.

Prof. Abbass is a Fellow of the Australian Computer Society, an Associate Fellow of the Australian Institute of Management, Hindmarsh, SA, Australia, the Chair of the ACS National Committee on Complex Systems, Chair of the IEEE Task Force on Complex Adaptive Systems and Artificial Life, and a member of a number of national and international committees including the IEEE Technical Committee on Data Mining and IEEE Working Group on Soft Computing in the SMC society. He is an Associate Editor of the IEEE TRANSACTIONS ON EVOLUTIONARY COMPUTATION, IEEE COMPUTATIONAL INTELLIGENCE MAGAZINE, and *International Journal of Intelligent Computing and Cybernetics*.

Michael Barlow received the Ph.D. degree in computer science from the University of New South Wales (UNSW), Kensington, NSW, Australia, in 1992.

From 1991 to 1994, he was with the Department of Electrical Engineering, University of Queensland, Brisbane, QLD, Australia, before taking up a 2-year post-doctoral position with Human Interface Laboratories, Nippon Telegraph and Telephone, Japan. For the past decade, he has been with the Australian Defense Force Academy, Canberra, ACT, Australia. He is currently the Director of the Virtual Environments and Simulation Laboratory and the Deputy Director of the Defense and Security Applications Research Center, UNSW. His current research interests include multiagent systems, simulation, games technology, immersion, virtual environments, human decision-making and behavioral models, and automatic speech processing—all with a complex systems perspective.

Axel Bender received the Ph.D. degree in theoretical particle physics from Tübingen University, Tübingen, Germany, in 1994.

He is currently a Senior Research Scientist with the Land Operations Division, Defense, Science and Technology Organization (DSTO), Edinburgh, SA, Australia. From 1994 to 1996, he undertook research in nonperturbative quantum field theories with Argonne National Laboratory, Argonne, IL. From 1996 to 1998, he was a Post-Doctoral Research Fellow with the Special Research Center for the Subatomic Structure of Matter, University of Adelaide, SA. From 1998 to 2003, he was a Statistics Research Officer with the National Center for Vocational Education Research, and joined the DSTO in 2003. Since 2003, he has been working on supply chain management problems, complex systems, military experimentation, and science and technology support to defense capability development projects.