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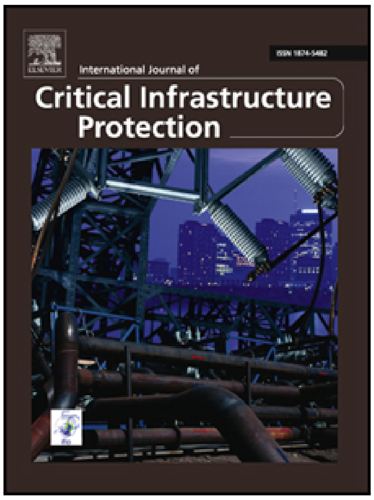

Dynamics of Interdependent Critical Infrastructures – a mathematical model with unexpected results

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Dynamics of Interdependent Critical Infrastructures – a mathematical model with unexpected results

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Abstract

The article develops a formal mathematical model which describes the dynamic interactions of interdependent critical infrastructures in discrete time steps, i.e. the direct and indirect consequences (cascading effects) of a disrupted infrastructure to depending infrastructures. The main result here is that a disturbed infrastructure with only a reduced performance level draws (on the long run) all other infrastructures to the same reduced performance level. Also the development of all infrastructures after a full repair of the disturbances is investigated. The related result is that even after a full repair the infrastructures cannot return to full performance without external support. Both results are proved as mathematical theorems. Visualized examples are shown to demonstrate the results.

Keywords

Interdependence, cascading effects, mathematical model, operability, intrinsic capacity

1 Introduction

Critical Infrastructures have been a focal point of civil security research in the US and Europe for many years, for example in the EU FP7 security program or in EU H2020. An important feature of critical infrastructures is interdependence, meaning that the operability of one critical infrastructure depends on the operability of one or more other critical infrastructures and vice versa. Typical examples of interdependent Critical Infrastructures are railways, electrical grids, power plants and gas grids (on a national level). It is obvious that e.g. the railway functioning depends on the power supply, the power supply depends on the gas grid etc.

Critical Infrastructures and their direct and indirect interdependencies (the latter ones also called cascading effects) have been investigated from many different points of view. A comprehensive analysis regarding the basic understanding of interdependencies including examples of interacting Critical Infrastructures in the reality is found e.g. [15]. Among the different approaches to analyze the mutual dependencies of Critical Infrastructures are System Dynamics based methods for which [3] (with usage of the well-known Vensim Software tool [19]) is a recent example and network based approaches, e.g. figured out in [8]. A complete overview of the topic including classification is given in [13]. One of the most prominent approaches is an adaption of the economic input-output based model which goes back to Wassily Leontief 1973. His input-output model is a quantitative economic technique that describes the interdependencies between different branches of a national economy, see [10] for a complete description. Based on Leontief's technical coefficient matrix Haimes and Jiang ([4]) developed an input-output inoperability model (IIM) for Critical Infrastructures. It introduced the basic concept of inoperability of a Critical Infrastructure and was further elaborated in [5] and [7] and in [16] with focus on demand reduction. In [11] the IMM is augmented by a time discrete dynamic model for interdependent Critical Infrastructures (DIIM) to describe the development of their inoperabilities over time; this model is extended in [1].

Using the DIIM concept, Oliva, Panzieri and Setola ([12] and [14], p.70) describe explicitly the inoperability of a given infrastructure at time k+1 as a function of all inoperabilities and of external causes (i.e. disruptions) at time k; a similar function is given in [6]. In [17] and [18] stability criteria for the long-term development of the inoperabilities of Critical Infrastructures are formulated, i.e. for $X(k+1) = X(k)$.

The present research will take up the concepts of DIIM including the disruption concept but proposes a new formal mathematical model. We develop a revised recursive definition of operabilities especially regarding the coefficient matrix and the incorporation of timely disruptions. This enables us not only to show alternative steady-state solutions for operabilities but also the way how they approximate these solutions. As in the previous approaches cited above, the price for the formalization is a high level of abstraction and certain simplifications regarding the structure of outputs, time delays, material stocks, working at maximum level etc. Due to such simplifications a direct application to real CIs is potentially difficult. However, even though the results are based on abstractions, we believe that they allow us to better understand the mechanisms by which different CIs interact. In section 6 some future paths towards a more realistic model are outlined.

The results refer to the long term operabilities of all involved infrastructures

a. after a long lasting partial or total disruption of one or more infrastructures and
b. after a time-limited disruption of one or more infrastructures and its subsequent "repair" without any disruption in the following time steps.

The first main result is that one disrupted infrastructure with a given reduced operability will on the long run "draw" all other infrastructures to the same operability level. The second result is that even after all disruptions are repaired the involved infrastructures will not return to their full operability but instead converge to a common reduced level of operability.

These results will be proven in section 4. Sections 2 and 3 introduce the formal model and show two examples. Section 5 demonstrates the proven results with an example from section 3 as well. Section 6 draws conclusions with open questions and shows paths for future research.

2 The Model

We consider a set $K = \{K_1, K_2, \dots, K_n\}$ of critical infrastructures, from now on also denoted as infrastructure or as CI[1]. The operability $X^{(t)}_i$ of a CI K_i at time t ($t \geq 0$ a natural number) is a real value between 0 and 1 denoting to which extent K_i can "do its work" at time t, compared to a standard under optimal circumstances[2]. $X^{(t)}_i = 1$ means this standard, a full scale work at time t, $X^{(t)}_i = 0$ means no operation at all. We also use the term X_I for the operability of K_i if it does not refer to a specific time step t. Also we denote by $X^{(t)}$ the column vector of all operabilities at time t: $X^{(t)} := (X^{(t)}_1, \dots, X^{(t)}_n)^T$ and call it the operability of K.

As announced above, a main interest in our model will be the interdependency. Informally spoken, interdependency of several CIs means that each CI "influences" each other CI in a direct or indirect way: E.g. a reduced output of the first CI will mean a reduced input into at least one other CI and hence lead to a reduced output of the latter one. We combine this input-output analysis with a discrete time series analysis as done e.g. in [17]: We want to derive each $X^{(t+1)}_i$ from all of the values $X^{(t)}_I$, i.e. from all inputs delivered to K_i in the previous time step. At first, a formal definition of full interdependent CIs is needed.

Definition 1:

A system ***K*** of full interdependent CIs consists of the following:

1. a set $K = \{K_1, K_2, \dots, K_n\}$ of infrastructures;
2. an n-tuple $e = (1, e_2, \dots, e_n)$, describing for each infrastructure K_i its relative economic strength[3], compared to K_1 in the normal full operability state; e is also called the strength vector;
3. an input-output nxn-matrix $O = (o_{ij})$. O refers to K and o_{ij} denotes the (relative) output of K_j for K_i with the following properties:
    a. $0 \leq o_{ij} \leq 1$ for $1 \leq i, j \leq n$;

[1] We do not explain here the term CI but take it over e.g. from 0 and many other sources.
[2] This notion is preferred to the frequently used notion of inoperability (= 1-operability) as it will simplify the proof of one of the main results.
[3] A common measure for the economic strength can be the Net Sales amount for all products and services that are delivered to the own and the other CIs (deliveries to instances outside the CIs are ignored). It is not discussed here how this amount can be assessed or calculated. In either case we mean the amount in a given time period.

b. $0 < o_{ii} < 1$ for all i;

c. $\sum_{i=1}^{n} o_{ij} = 1$ for $1 \le j \le n$, meaning that each entry in a column denotes the fraction of the whole output of the respective K_j;

d. In each subset $K' \subsetneq K$ there is at least one K_j with at least one output $o_{ij} > 0$ such that K_i is not an element of K'.

Explanation of the properties of matrix O:

Properties 3.a and 3.c mean that each column O_j describes the output (percentage) distribution from infrastructure K_j to all infrastructures (including itself, but excluding the output to external instances). Hence O maps correctly the output relations between the CIs. Observe that the rows of O (indicating the inputs to a K_i) do not sum up to 1.

Property 3.b is the self-providing condition. It means that each infrastructure always works "for itself" to a certain positive extent but never to a full extent. This is fully plausible, e.g. a power plant uses a small part of the generated power for its own operation, a railway undertaking providing material transports for other CIs also will transport material for its own network sustaining. On the other hand a CI working completely for itself does not make any sense.

Property 3.d is the full interdependency condition. Each K_j "influences" each other K_i either directly (in this case $o_{ij} > 0$) or indirectly via a sequence of outputs (e.g. $o_{kj} > 0$ and $o_{ik} > 0$). This assumption is central and makes sense: A missing interdependency of K_i would mean that either a disruption of K_i has no consequences for the other CIs or K_i is not affected by any other CI disruption. Such a constellation is not typical for real systems of CIs.

Direct consequences of the above properties are

I. $o_{ij} < 1$ for all $\le i, j \le n$,

II. $o_{ij} > 0$ for at least one i in each column O_j with $i \neq j$.

Vice versa, for a given i, all the values o_{ij}, $1 \le j \le n$, describe all inputs for infrastructure K_i. We are interested in determining the relative contributions of these inputs. These contributions are not correctly described by the elements in row i as the economic strengths e_j do not occur. Therefore we define the "weighted" matrix O'[4] = (o'_{ij}) with

$o'_{ij} := o_{ij} * e_j$ and

$O'_i := \sum_{j=1}^{n} o_{ij} * e_j$ as row sum.

O'_i is denoted as dependency level[5] of the i-th CI; it will be used for the recursive definition of $X^{(t)}_i$ below. It is easy to check that matrix O' also satisfies properties 3.a, 3.b, and 3.d of definition 1. The elements of the i-th row of O' can be regarded as the influences or dependencies that K_i is exposed to. A big value of o'_{ij} results from a large relative output from K_j to K_i and/or a large economic strength of K_j. Note that neither the columns nor the rows necessarily sum up to 1.

Additionally to the "external" influences described above there is a second relevant kind of influence for each CI: the intrinsic capacity of K_i, which describes how far technical, personnel related and other prerequisites are given "inside the CI" such that K_i can work fully or only partially or not at all. Note that this capacity does not depend on how strong the supporting K_j work.

We denote the intrinsic capacity of K_i at time t by $T^{(t)}_i$ and define it (similar to X_i) as a relative value between 0 and 1. An intrinsic capacity smaller than 1 is also denoted by disruption or disturbance. In reality, such disruptions will be induced typically by catastrophic events like criminal or terrorist attacks, natural hazards or big technical failures. Once a disruption is there it will take a number of time steps until it is partly or fully repaired. $T^{(t)}$ denotes the n-vector ($T^{(t)}_1$, …, $T^{(t)}_n$) and T denotes the infinite sequence $T^{(1)}$, $T^{(2)}_I$, …

Given this intrinsic capacity for each K_i it is plausible to assume that the operability $X^{(t)}_i$ cannot exceed $T^{(t)}_i$. Apart from this, the subsequent recursive definition follows the mechanism which is used in [4], [14], [17] and other publications: An operability is derived from the (dynamic previous) operabilities of the supporting K_j, potential disruptions, and (constant) distribution shares for K_i. Constant o'_{ij} means that they do not change when the CI operability goes below

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[4] O' is similar to the *Leontief Coefficients Matrix* but not the same (the consumers are lacking); therefore we avoid this term here.

[5] In 0/0 the term dependency index δ_i is used similarly, however in a dynamic way, depending on the disruption time.

1, each K_i follows it's predefined "distribution program". This leads in a first step to the preliminary

Definition 2:

$$X^{(0)} := T^{(0)} \text{ and } X^{(t+1)}_i := \min (T^{(t+1)}_i, \sum_{j=1}^{n} o'_{ij}*X^{(t)}_j) \text{ for } 1 \le i \le n \text{ and } t \ge 0.$$

However there are several problems with this definition. First the condition $X^{(t)}_i \le 1$ cannot be guaranteed as a dependency level O'_i may exceed 1. Second a dependency level above 1 for K_i would imply that a constellation exists where K_i has full operability though some or all supporting infrastructures deliver only reduced input to it. Third a dependency index smaller than 1 for K_i implies that K_i comes to a 0 operability though the supporting infrastructures still provide input. The latter points are not regarded plausible. Therefore we interpret the entries o'_{ij} , $1 \le j \le n$ of the i-th row as the relative contributions of all K_j to K_i which all together describe exactly the complete dependency. This allows a normalization of o'_{ij} with respect to the dependency index: $r_{ij} := o'_{ij}/O'_i$ as normalized input of K_j for K_i. The associated dependency matrix is $R := (r_{ij})$. By definition, $\sum_{j=1}^{n} r_{ij} = 1$ and the properties 3.a, 3.b, and 3.c in definition 1 hold for R as well. One may say that r_{ij} is the percentage to which the performance of K_i depends on the performance of K_j. The above definition 2 is therefore obsolete. It is replaced by

Definition 3:

$$X^{(0)} := T^{(0)} \text{ and } X^{(t+1)}_i := \min (T^{(t+1)}_i, \sum_{j=1}^{n} r_{ij}*X^{(t)}_j) \text{ for } 1 \le i \le n \text{ and } t \ge 0$$

or, in the notation of vector multiplication,

$$X^{(t+1)}_i := \min (T^{(t+1)}_i, R_i*X^{(t)}) \text{ where } R_i \text{ is the i-th row of matrix R.}$$

Definition 3 guarantees that all operabilities have values between 0 and 1 at each time. Moreover it indicates that the derived dependency matrix R will play a central role for the results to prove. The derivation of R from the output distributions of O is clearly a suitable method but in principle not a necessary condition. In [17] and [18] a method is described and performed to estimate values for dependency matrices from empirical questionnaires data. [3] uses empirical data stemming from a survey of CI experts which is described in detail in [9]. Therefore, if R respects the conditions stated above it is not relevant where they are derived from. This is discussed further in section 3.

With definition 3 our model of full interdependent Critical Infrastructures is complete. It will be illustrated by the following

3 Examples with 5 and 11 CIs

This section contains two examples. The first one is simple and fictional and demonstrates how the dependency matrix R is derived from the output matrix O. The components of our example system ***K*** are

$K = \{K_1, K_2, \dots , K_5\}$;

$e = (1 \ \ 0.8 \ \ 0.7 \ \ 0.7 \ \ 0.5)$ describes the relative strengths of the last 4 CIs relative to the first one;

is the output distribution. E.g. Column $O_1 = (0.1 \quad 0.2 \ \ 0.4 \ \ 0.1 \ \ 0.2)^T$ means that K_1 takes 10% of its output for its own operation, 20% is delivered to K_2, 40% to K_3, 10% to K_4, and 20% to K_5. As stated above the part delivered to the consumers remains out of consideration here. The other columns have the respective meanings. According to the definitions in section 2 we derive

and through division of each element by its row sum we get the derived dependency matrix

It is easy to check that R satisfies all required properties. This means that e.g. the first row describes the percentage distribution of all inputs for CI K_1. r_{14} = 0.156 means that K_1 depends at 15.6% on input from K_4, r_{44} = 0.187 shows that K_4 depends to a higher degree on its own input than all other CIs do.

The second example uses empirical data which have been in collected in [9]. They describe dependencies between 11 CIs in nine countries in Europe and North America which are based on a survey among CI providers. The resulting dependency matrix is not derived from an output distribution matrix O (as figured out in the previous section) but can be used for our purposes. The original matrix from [9][6] is

describing the dependencies in a non-normalized form and

is the resulting normalized dependency matrix with row sums equal 1. Note that the matrix does not fulfill our condition $r_{ii} > 0$ as this was excluded by the authors for methodological reasons. Nevertheless it will be useful to come back to this system in section 5 and see that it behaves in the same way as our standard systems do.

4 Results

We derive two relevant results about how the operabilities of interdependent infrastructures develop over time in case of an intrinsic disruption in one CI and what happens when this disruption is “healed” after a certain number of time steps. For the proofs we use the fact that the behavior of the system $\boldsymbol{K}$ does not depend on the numbering of the K_i. E.g. to interchange two infrastructures K_i and K_j in the system first interchange the i-th and the j-th column of O and then the i-th and the j-th row. Additionally all intrinsic capacity vectors $T^{(t)}$ for $t \geq 0$ and the strength vector e have to be modified accordingly (interchange the i-th and the j-th component). The same holds for any permutation of the n-tuple $(K_1, K_2, \dots, K_n)$ as it can be constructed as a finite series of interchanges.

This property of any output matrix O clearly holds in the same way for the input matrix R and will be used in the first theorem.

[6] The matrix had to be transposed as the influencing Cis of a given CI were not in a row but in a column.

Theorem 1:

Let $\boldsymbol{K}$ = (K, e, O) be a system of CIs and R the associated input matrix[7]. Assume as intrinsic capacity vector $T^{(t)} = (1, \ldots, c, \ldots, 1)$ for all $t \geq 0$, Then, with growing t, all operabilities X_j, converge to c from above; formally

(a) $\lim_{t\to\infty} X^{(t)}_i = c$ for $1 \leq i \leq n$,

(b) $X^{(t)}_k = c$ for $t \geq 0$, where K_k is the CI with disturbance c,

(c) $X^{(t)}_i > c$ for all other K_i and all $t \geq 0$.

Theorem 1 means that one infrastructure with a constant intrinsic disturbance will (asymptotically) "draw" all other infrastructures to its own reduced operability level. E.g. if K_1 has intrinsic capacity c = .4 then it will work with at most 40% of its normal power and each other K_i will on the long run work as well with only 40% of K_i's normal power.

Via renumbering we can assume that $T^{(t)} = (c, 1, \ldots, 1)$ for all $t \geq 0$, i.e. K_1 is restricted to c and all other K_i have full intrinsic capacity.

The proof of theorem 1 requires a preliminary statement about the behavior the operabilities.

Lemma 1.1:

Let $C = (c_1, \ldots, c_n)$ be a vector of intrinsic capacities, $0 \leq c_i \leq 1$ and $T^{(t)} = C$ for $t \geq 0$. Then $X^{(t)}_i \leq X^{(t-1)}_i$ for all $t \geq 1$ and $1 \leq i \leq n$.

Proof (by induction over t):

<u>t = 1:</u>

$X^{(0)}_i = 1$ by assumption and $X^{(1)}_i \leq 1$ by definition for $1 \leq i \leq n$.

<u>t -> t+1:</u>

For $1 \leq i \leq n$ assume that $X^{(t)}_i \leq X^{(t-1)}_i$. We have to show that $X^{(t+1)}_i \leq X^{(t)}_i$.

For each i we have according to definition 2:

[7] As figured out in section 2, R need not necessarily be derived from O, if the required properties are fulfilled.

$X^{(t+1)}_i$ = min (c_i, r_{i1}*$X^{(t)}_1$ + r_{i2}*$X^{(t)}_2$ + … + r_{in}* $X^{(t)}_n$)

≤ min (c_i, r_{i1}* $X^{(t-1)}_1$ + r_{i2}*$X^{(t-1)}_2$ + … + r_{in}*$X^{(t-1)}_n$) by induction assumption

= $X^{(t)}_i$ for i > 1.

Proof of theorem 1 for c = 0:

Renumber the infrastructures K_i, i > 1, such that the n_1 infrastructures to which K_1 delivers (o_{i1} > 0, hence r_{i1} > 0) have the first n_1 numbers. We know

$X^{(1)}_1$ = 0 and

$X^{(1)}_i = r_{i1}*0 + r_{i2}*X^{(0)}_2 + \ldots + r_{in}*X^{(0)}_n \leq r_{i1}*0 + r_{i2} + \ldots + r_{in} = 1 - r_{i1}$ for $i \leq n_1$. (1)

From lemma 1.1 it follows that the same relation is true for any t > 1, on the whole:

$X^{(t)}_i \leq 1 - r_{i1}$ for $i \leq n_1$ and $t \geq 1$. (2)

Property (d) in the definition of matrix O implies that at least one K_j, $1 < j \leq n_1$, must deliver to the next n_2 infrastructures (after renumbering), $n_2 > 0$, i.e. there is at least one pair (K_i,K_j), $n_1 < i \leq n_2$, with $r_{ij} > 0$. For each such K_j we have

$X^{(2)}_i = 0*0 + r_{i2}*X^{(1)}_2 + \ldots r_{ij}*X^{(1)}_j + \ldots + r_{in}*X^{(1)}_n$

$\leq 0 + r_{i2} + \ldots r_{ij}*(1- r_{i1}) + \ldots + r_{in} = 1 - r_{i1}*r_{ij}$ and $r_{i1}, r_{ij} > 0$.

Again, due to lemma 1.1 this inequation remains valid for all t > 2, on the whole

$X^{(t)}_i \leq 1 - r_{i1}*r_{ij}$ for $n_1 \leq i \leq n_2$ and $t \geq 2$.

The above operation can be repeated m times until K_n is reached. This means that

$X^{(m)}_i \leq 1 - r_{i1}$ for $i \leq n_1$,

$X^{(m)}_i \leq 1 - r_{i1}* r_{ij}$ for appropriate indexes $n_1 \leq i \leq n_2$, (3)

and so forth. As shown above all these boundaries are smaller than 1. Let r be the biggest of them. Then we have

$0 < X^{(m)}_i \leq r$ for $1 < i \leq n$, (4) the first part of the inequation resulting from the fact that no entry r_{ij} equals 1.

Perform the time steps m+1, m+2, … in the same way as above but starting with value r instead of 1 for the $X^{(m)}_I$, i > 1. Then inequation (1) becomes

$X^{(m+1)}_i = r_{i1}*0 + r_{i2}*X^{(m)}_2 + … + r_{in}*X^{(m)}_n \leq 0*r_{i1} + r_{i2}*r + … + r_{in}*r = (1 - r_{i1})*r$ for $i \leq n_1$, and (2) becomes

$X^{(t)}_i \leq (1 - r_{i1})*r$ for $i \leq n_1$ and $t \geq m+1$.

Inequation (3) will be adjusted accordingly.

This shows: after 2m steps all operabilities $X^{(2m)}_I$ are smaller than or equal r^2. After z*m steps we have $X^{(zm)}_i < r^z$. As r < 1, all $X^{(t)}_I$ must converge to 0 for t -> ∞, which shows part (a) of the theorem. Part (b) follows from lemma 1.1 and part (c) follows from inequation (4).

Consider now the general case c > 0:

We define a second intrinsic capacity vector U for ***K***:

$U^{(t)}_I$:= (0, 1-c, 1-c, …, 1-c) for all t ≥ 0. The operabilities under U are denoted by $Y^{(t)}_I$.

Obviously we have $Y^{(0)}_1 = 0$ and $Y^{(0)}_i = 1 – c$ for i > 1. Lemma 1.1 applies with intrinsic capacity vector $U^{(t)}_I$. Consequently, we can follow the first part of the proof, and this means all $Y^{(t)}_i$ converge to 0 for growing t.

On the other hand (for the original intrinsic capacity vector T) $X^{(0)}_i = Y^{(0)}_i + c$. Hence

$X^{(1)}_I = r_{i1}*(Y^{(0)}_1 + c) + … + r_{in}*(Y^{(0)}_n + c) = r_{i1}*Y^{(0)}_1 + … + r_{in}*Y^{(0)}_n + c = Y^{(1)}_i + c$.

By induction it follows

$X^{(t)}_i = Y^{(t)}_i + c$ for all t ≥ 0. Thus $X^{(t)}_i$ must converge to c.

Corollary 1.2:

If several infrastructures K_1, …, K_r have reduced intrinsic capacities c_1, …, c_r, then all operabilities will converge to the smallest c_i.

Proof:

The proof is a slight modification of the proof for theorem 1.

Theorem 2:

Let ***K*** = (K, e, O) be a system of critical infrastructures and R be the corresponding dependency matrix. If the intrinsic capacity of one infrastructure K_i is reduced to a value c < 1 for z time periods and returns to 1 from period z+1 on, then all operabilities $X^{(t)}_i$, $1 \le i \le n$ converge to a common value X with c < X < X_{max}, where X_{max} is largest of the operabilities $X^{(z+1)}_i$.

The meaning of theorem 2 is that, once a CI is disturbed for at least one period (and hence affects the other CIs according to theorem 1) the operabilities of <u>all</u> CIs cannot return to 1 even if the disturbance is "repaired".

Proof:

After the end of the intrinsic disturbance at time z definition 2 reduces to

$X^{(t+z+1)}_I = \sum_{j=1}^{n} r_{ij} * X^{(t+z)}_j$; Without loss of generality we can assume z = 0; in the notation of matrix multiplication we then write

$X^{(t+1)} = R * X^{(t)}$,

According to the rules of matrix multiplication (associativity) this means

$X^{(t)} = R^t * X^{(0)}$, where R^t is the t-th power of matrix R.

How does R^t develop with growing t? To answer this question we refer to the theory of Markov Chains. The reader who is familiar with basics of probability theory and/or Markov Chains may have noticed that our matrix R is a stochastic matrix; all entries of a row R_i are between 0 and 1 and sum up to 1. It is therefore a transition matrix for a Markov Chain. We refer to [2] for the following definitions. Moreover, property (d) of the definition of matrix R means that, in the language of Markov, R is irreducible, see also definition 1 in [2]. The third property (b) (r_{ii} >0) ensures that, again in the language of Markov, R is aperiodic, see definition 2 in . R is, on the whole, a stochastic, irreducible, and aperiodic matrix. Therefore theorem 7.4 in [2] applies which states that with growing t, R^t converges to a stochastic matrix R* in which all rows are equal; formally

$\lim_{t\to\infty} R^t = R^*$, and for any i and j we have $R^*_i = R^*_j$.

Consequently

$\lim_{t\to\infty} R^t * X^{(0)} = (\lim_{t\to\infty} R^t) * X^{(0)} = R^* X^{(0)} = (R^*_1 * X^{(0)}, \ldots, R^*_1 * X^{(0)})^T = (X, \ldots, X)^T$ for some constant $X \le 1$. This proves the first part of the theorem.

It remains to show $c < X < X_{max}$,. Recall the general notation

$$X^{(t+1)}_i = r_{i1} * X^{(t)}_1 + r_{i2} * X^{(t)}_2 + \ldots + r_{i,n-1} * X^{(t)}_{n-1} + r_{i,n} * X^{(t)}_n, \qquad (5)$$

which is valid if the intrinsic capacities all equal 1. According to (5) $X^{(t+1)}_i$ is a weighted average of all the $X^{(t)}_i$, therefore

$X_{max} \ge X^{(t+1)}_i \ge c$ for all i and all t > z.

Clearly R* consists only of strictly positive elements r^*_{ij}, see also theorem 7.4 (i) in [2]. Therefore a number t_0 must exist such that for all $t > t_0$, R^t consists of strictly positive elements $r_{ij}^{(t)}$ as well. For such a t equation (5) becomes

$$X^{(t+1)}_i = r_{i1}^{(t)} * X^{(0)}_1 + r_{i2}^{(t)} * X^{(0)}_2 + \ldots + r_{i,n-1}^{(t)} * X^{(0)}_{n-1} + r_{in}^{(t)} * X^{(0)}_n,$$

and due to the positive coefficients and equation (5) it follows for such a t

$X_{max} > X^{(t+1)}_i > c$ for all i. This completes the proof of theorem 2.

Corollary 2.1:

If several infrastructures $K_1, \ldots, K_r$ have reduced intrinsic capacities $c_1, \ldots, c_r$, then after the end of all disturbances at time z, all operabilities will converge to a value c with $X_{max} > c > c_{min}$ where X_{max} is the greatest operability at time z and c_{min} is the smallest of the disturbances c_i.

Proof:

The proof is a slight modification of the proof for theorem 2.

5 Example with 11 CIs, continued

We come back to the second example system of 11 CIs from section 3 with the original dependency matrix R' and the normalized matrix R. We visualize the behavior of the 11 CIs under the conditions of theorems 1 and 2. As said in section 3 the matrix R does not fulfil the condition $r_{ii} > 0$. Nevertheless the subsequent calculations of the $X^{(t+1)}_i$ according to definition 3 show the same behavior than those proven in the theorems. They can be done easily with rather simple excel operations. We use these calculations and their visualizations as well.

Table 6 shows the assumed intrinsic disturbances: Water is supposed to work with an intrinsic capacity of 30%, Order Safety works with an intrinsic capacity of 50%, all other CIs have no disturbance.

Figure 1 shows how the operabilities of all CIs converge to the reduced capacity 0.3 of CI TLC wireless.

For a demonstration of theorem 2 we choose for CIs Transport, ICT and Chemical a different disturbance over different time periods. For the details see *Table 7*. From time step 9 on all intrinsic capacities return to 1.

Figure 2 shows how under these assumptions the operabilities converge to a value below 1 and above 0.3, in this case around 0.44.

6 Conclusion and avenues for future research

This paper has demonstrated with a formal mathematical model the dynamic behavior of interdependent critical infrastructures when one or several infrastructures are disturbed for one or several time periods. Two properties were proven: First, when the intrinsic capacities of one or more infrastructures are reduced to a certain minimum level $c<1$, the operabilities of all infrastructures in the system converge over time to c from above. Second, after all intrinsic capacities of the damaged infrastructures are restored, the operabilities of all infrastructures don't go back to 1 but instead converge to a common value X where $c < X < 1$.

It was stated in section 1 that our model is simplified in different aspects and hence the results may not be applied directly. Nevertheless they indicate practical implications for practitioners operating interdependent critical infrastructures and especially dealing with disturbances in them. On the one hand, they imply a need to quickly restore damaged critical infrastructures in order to avoid other critical infrastructures (without intrinsic disturbance) being drawn to the same reduced operability. On the other hand, they suggest that, to restore the operability of the whole system of interdependent critical infrastructures, it is not enough to simply repair the disturbed infrastructure(s) and then doing business as usual.

Two potential fields of future research result from this observation. First, preliminary tests suggest that a path to full restoration of all CIs ($X^{(t)} = 1$) could be to prioritize the outputs of the infrastructures. Roughly spoken, instead of continuing to distribute the outputs according to the matrix O (its columns for each CI), managers of the CIs might decide to give a larger share of its (reduced) outputs to some preferred infrastructures in the system, while giving less to others. It is beyond the scope of this paper to decide about the CIs to prefer or to prove how this behavior affects the operability of the system. Future research directed towards exploring the dynamics of interdependent critical infrastructures, therefore, should include the possibility that managers of damaged critical infrastructures change the way their output is distributed to other infrastructures.

A second avenue for future research concerns a model design that is nearer to the reality of interdependent CIs than the present one. Some examples: What is the influence of disturbed CIs who do not work at their maximum performance during the standard operation? Or, are critical infrastructure operabilities and interdependencies more deterministic (as in our model) or more stochastic in nature? In reality, a stochastic behavior may be true: Instead of being fully determined by a combination of past operabilities and fixed dependencies, the output of a critical infrastructure may fluctuate due to elements of chance. Similarly, relations between the infrastructures – what infrastructures deliver what part of their output to what other infrastructures – may vary slightly over time periods. This begs the question whether the behavior of such a stochastic system of interdependent critical infrastructures conforms to the behavior of a deterministic one. Again, first simulations suggest that there are differences. For example, Figure 3 shows a (fictive) system of interdependent critical infrastructures in which the output vector $X^{(t)}$ contains a small stochastic element. As in the deterministic case, the operabilities of the five critical infrastructures converge to a common level. In contrast to the deterministic case, this level does not stay constant over time.

Another example is the future usage of inputs that have been delivered to a CI K_i but cannot be used due to an actual disturbance in K_i. In our model such inputs are lost and ignored for the later steps. This might be realistic in some cases; e.g. if a CI K_i cannot use the delivered electrical power than this power will probably be lost for K_i as storage of electrical power is very difficult. However in other cases it is not realistic. It could therefore be relevant to specify the model with more variables such that unused resources in a CI can be stored and used in later steps.

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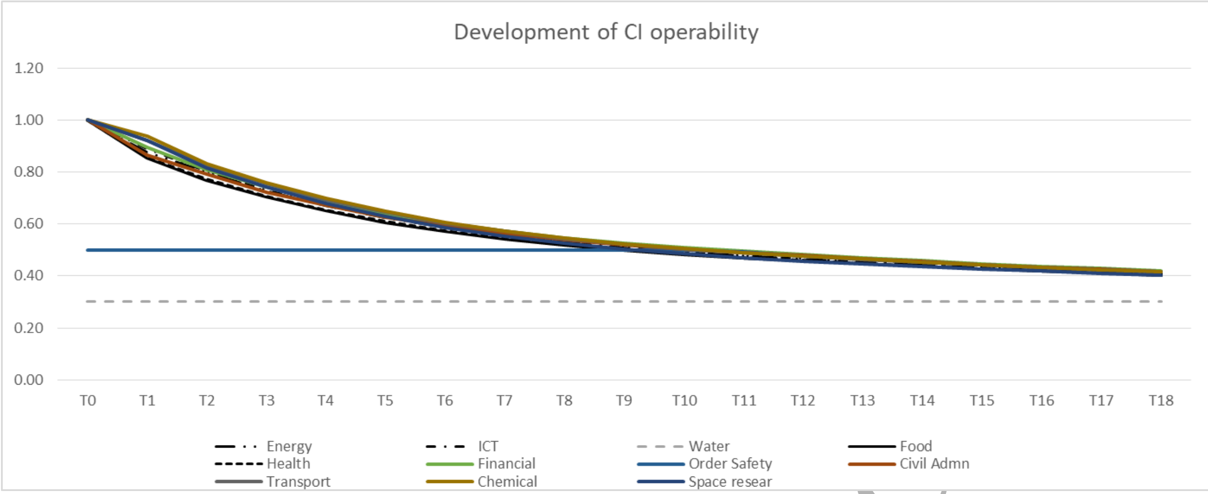

Figure 1: Development of operabilities with intrinsic disturbances as defined in table 6

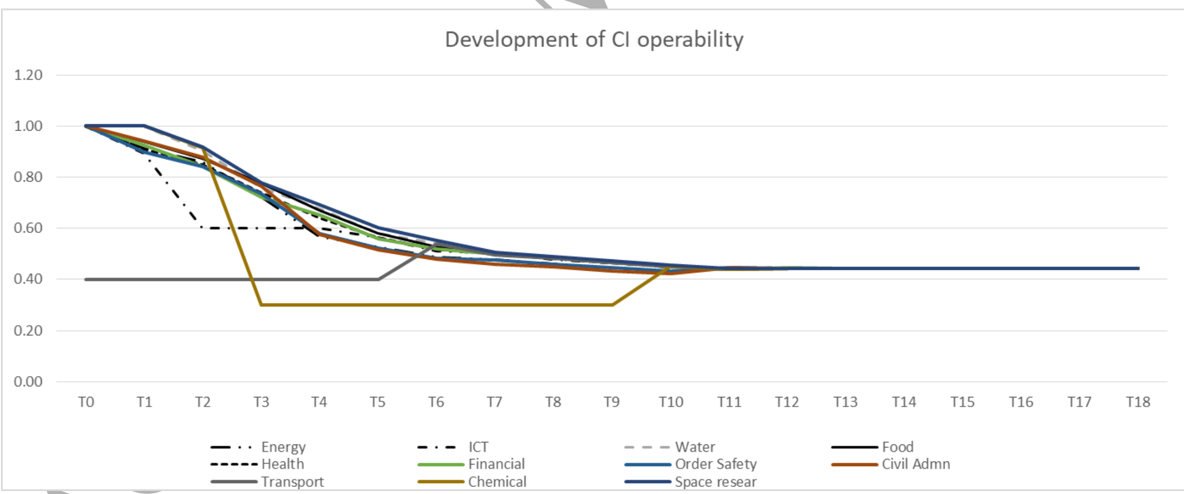

Figure 2: Development of operabilities during and after intrinsic disturbances as defined in table 7

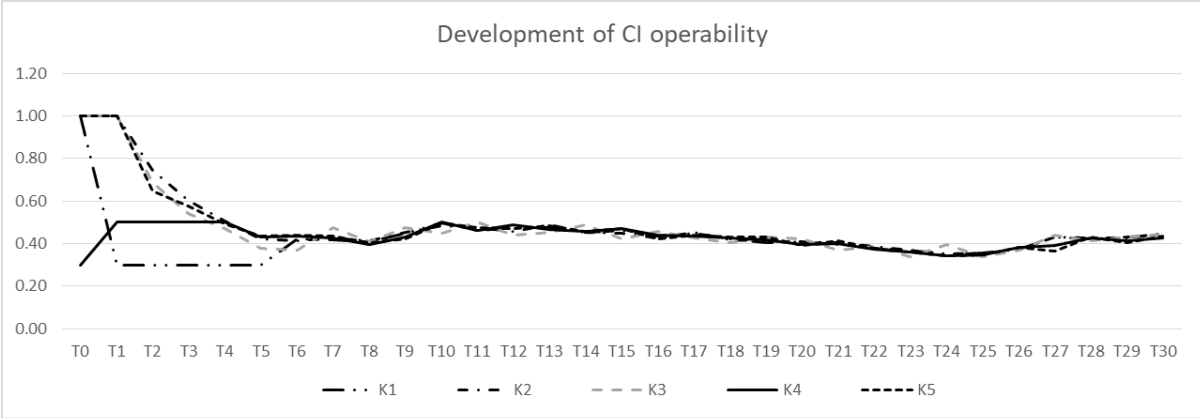

Figure 3: Example for a development of operabilities with a stochastic element in the CI outputs

Table 1: Example output distribution for 5 CIs

	K1	K2	K3	K4	K5	
O =	0.1	0.2	0.3	0.1	0.2	K1
	0.2	0.05	0.3	0.2	0.25	K2
	0.4	0.1	0.05	0.05	0.4	K3
	0.1	0.4	0.2	0.2	0.1	K4
	0.2	0.25	0.15	0.45	0.05	K5
	1	1	1	1	1	

Table 2: The resulting matrix O'

	K1	K2	K3	K4	K5	Row sums
O' =	0.1	0.16	0.21	0.07	0.1	0.64
	0.2	0.04	0.21	0.14	0.125	0.715
	0.4	0.08	0.035	0.035	0.2	0.75
	0.1	0.32	0.14	0.14	0.05	0.75
	0.2	0.2	0.105	0.315	0.025	0.845

Table 3: The derived dependency matrix R

	0.156	0.25	0.328	0.109	0.156	1
	0.28	0.056	0.294	0.196	0.175	1
R =	0.533	0.107	0.047	0.047	0.267	1
	0.133	0.427	0.187	0.187	0.067	1
	0.237	0.237	0.124	0.373	0.03	1

Table 4: Non-normalized dependency matrix with empirical data from Italian CIs

0	Energy	ICT	Water	Food	Health	Financial	Order Safety	Civil Admn	Transport	Chemical	Space resear
Energy	0.00	2.67	0.83	0.00	0.50	0.17	0.83	0.33	1.17	1.50	0.17
ICT	0.86	0.00	0.57	0.14	0.14	0.71	0.43	0.86	1.00	0.29	0.57
Water	1.33	1.00	0.00	0.00	0.00	0.00	0.33	0.00	0.00	0.00	0.00
Food	2.89	1.67	1.56	0.00	0.78	1.22	1.00	0.38	1.11	0.22	0.00
Health	1.40	2.20	1.20	0.60	0.00	0.20	1.00	1.00	1.40	0.40	0.00
Financial	2.67	2.33	0.00	0.00	0.00	0.00	1.67	0.33	1.00	0.00	0.00
Order Safe-ty	1.67	2.67	0.33	0.33	1.67	0.33	0.00	1.00	2.00	2.00	0.00
Civil Admn	0.40	1.40	0.20	0.20	0.60	0.00	1.40	0.00	0.60	1.40	0.00
Transport	2.40	2.40	0.20	0.00	0.00	0.60	0.80	0.20	0.00	0.20	0.20
Chemical	4.67	2.67	0.33	0.33	0.33	1.33	1.00	1.00	0.00	0.00	0.00
Space re-sear	1.33	1.00	0.33	0.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 5: Normalized dependency matrix

	Energy	ICT	Water	Food	Health	Financial	Order Safety	Civil Admn	Transport	Chemical	Space resear
Energy	0.00	0.33	0.10	0.00	0.06	0.02	0.10	0.04	0.14	0.18	0.02
ICT	0.15	0.00	0.10	0.03	0.03	0.13	0.08	0.15	0.18	0.05	0.10
Water	0.50	0.38	0.00	0.00	0.00	0.00	0.12	0.00	0.00	0.00	0.00

	Food	0.27	0.15	0.14	0.00	0.07	0.11	0.09	0.04	0.10	0.02	0.00
R =	Health	0.15	0.23	0.13	0.06	0.00	0.02	0.11	0.11	0.15	0.04	0.00
	Financial	0.33	0.29	0.00	0.00	0.00	0.00	0.21	0.04	0.13	0.00	0.00
	Order Safety	0.14	0.22	0.03	0.03	0.14	0.03	0.00	0.08	0.17	0.17	0.00
	Civil Admn	0.06	0.23	0.03	0.03	0.10	0.00	0.23	0.00	0.10	0.23	0.00
	Transport	0.34	0.34	0.03	0.00	0.00	0.09	0.11	0.03	0.00	0.03	0.03
	Chemical	0.40	0.23	0.03	0.03	0.03	0.11	0.09	0.09	0.00	0.00	0.00
	Space resear	0.44	0.33	0.11	0.11	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 6: Intrinsic capacities for the 11 CIs in the first 18 time steps, according to the assumption of theorem 1 and corollary 1.1

	Intrinsic Capacity at time T0 - T18																		
Energy	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
ICT	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Water	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
Food	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Health	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Financial	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Order Safety	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
Civil Admn	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Transport	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Chemical	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Space resear	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 7: Intrinsic capcities for the 11 CIs in the first 9 time steps, according to the assumption of theorem 2

	Intrinsic Capacity at time T0 - T18																		
Energy	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
ICT	1.00	1.00	0.60	0.60	0.60	0.60	0.60	0.60	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Water	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Food	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Health	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Financial	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Order Safety	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Civil Admn	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Transport	0.40	0.40	0.40	0.40	0.40	0.40	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Chemical	1.00	1.00	1.00	0.30	0.30	0.30	0.30	0.30	0.30	0.30	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Space resear	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00