Universidade Federal do Ceará
Campus Sobral

Cálculo Diferencial e Integral II – 2020.2 (SBL0058)
Prof. Rui F. Vigelis

3a Avaliação Progressiva

Nome:____

1. Usando a Regra de L'Hôpital, encontre o valor dos limites:

(a) $\lim_{x\to 0}(1-x)^{1/x}$;

$$\begin{aligned}
\lim_{x\to 0}(1-x)^{1/x} &= \lim_{x\to 0}\exp(\ln((1-x)^{1/x})) \\
&= \exp\left(\lim_{x\to 0}\frac{\ln(1-x)}{x}\right) \\
&\overset{(*)}{=} \exp\left(\lim_{x\to 0}\frac{\frac{-1}{1-x}}{1}\right) \\
&= \exp(-1) = 1/e
\end{aligned}$$

$$(*) \Leftarrow \begin{cases} \text{(i) } \ln(1-x) \text{ e } x \text{ são deriváveis em } (-\infty,1) \\ \text{(ii) } 1 \neq 0 \text{ em } (-\infty,1) \\ \text{(iii) } \lim_{x\to 0}\ln(1-x) = 0 \text{ e } \lim_{x\to 0} x = 0 \end{cases}$$

(b) $\lim_{x\to\infty}\frac{\ln x}{x^2}$.

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln x}{x^2} &\overset{(*)}{=} \lim_{x\to\infty}\frac{1/x}{2x} \\
&= \frac{1}{2}\lim_{x\to\infty}\frac{1}{x^2} \\
&= \frac{1}{2}\cdot 0 = 0
\end{aligned}$$

$$(*) \Leftarrow \begin{cases} \text{(i) } \ln x \text{ e } x^2 \text{ são deriváveis em } (0,\infty) \\ \text{(ii) } 2x \neq 0 \text{ em } (0,\infty) \\ \text{(iii) } \lim_{x\to\infty}\ln x = \infty \text{ e } \lim_{x\to\infty} x^2 = \infty \end{cases}$$

2. Encontre o valor das integrais impróprias:

(a) $\int_0^\infty \frac{1}{e^x+1}dx$;

$$u = e^x + 1 \Rightarrow du = e^x dx \Rightarrow \frac{du}{u-1} = dx$$

$$
\begin{aligned}
\int \frac{1}{e^x+1}dx &= \int \frac{1}{u}\frac{du}{u-1} \\
&= \int \Big(\frac{1}{u-1} - \frac{1}{u}\Big)du \\
&= \ln|u-1| + \ln|u| + C \\
&= \ln|e^x| - \ln|e^x+1| + C \\
&= \ln\Big(\frac{e^x}{e^x+1}\Big) + C \\
&= \ln\Big(\frac{1}{1+e^{-x}}\Big) + C
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty \frac{1}{e^x+1}dx &= \lim_{b\to\infty}\int_0^b \frac{1}{e^x+1}dx \\
&= \lim_{b\to\infty}\Big[\ln\Big(\frac{1}{1+e^{-x}}\Big)\Big]_0^b \\
&= \lim_{b\to\infty}\Big[\ln\Big(\frac{1}{1+e^{-b}}\Big) - \ln\Big(\frac{1}{1+e^{0}}\Big)\Big] \\
&= \ln\Big(\frac{1}{1+\lim_{b\to\infty} e^{-b}}\Big) - \ln\Big(\frac{1}{1+1}\Big) \\
&= \ln\Big(\frac{1}{1+0}\Big) - \ln\Big(\frac{1}{2}\Big) \\
&= \ln(2)
\end{aligned}
$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+9}dx.$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^2+9}dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{x^2+9}dx + \lim_{b\to\infty}\int_0^\infty \frac{1}{x^2+9}dx \\
&= \lim_{a\to-\infty}\Big[\frac{1}{3}\operatorname{tg}^{-1}\Big(\frac{x}{3}\Big)\Big]_a^0 + \lim_{b\to\infty}\Big[\frac{1}{3}\operatorname{tg}^{-1}\Big(\frac{x}{3}\Big)\Big]_0^b \\
&= \lim_{a\to-\infty}\Big[\frac{1}{3}\operatorname{tg}^{-1}(0) - \frac{1}{3}\operatorname{tg}^{-1}\Big(\frac{a}{3}\Big)\Big] + \lim_{b\to\infty}\Big[\frac{1}{3}\operatorname{tg}^{-1}\Big(\frac{b}{3}\Big) - \frac{1}{3}\operatorname{tg}^{-1}(0)\Big] \\
&= \Big[0 + \frac{1}{3}\frac{\pi}{2}\Big] + \Big[\frac{1}{3}\frac{\pi}{2} - 0\Big] = \frac{\pi}{3}
\end{aligned}
$$

3. Calcule a área da região que se situa dentro de $r = 3 + 2\operatorname{sen}\theta$ e fora de $r = 2$.

$$
3 + 2\operatorname{sen}\theta = 2 \Rightarrow \operatorname{sen}\theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}, \frac{7\pi}{6}
$$

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}[(3+2\operatorname{sen}\theta)^2-4]d\theta\\
&= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}[(5+12\operatorname{sen}\theta+4\operatorname{sen}^2\theta]d\theta\\
&= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}[(5+12\operatorname{sen}\theta+4\frac{1}{2}(1-\cos 2\theta)]d\theta\\
&= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}[(7+12\operatorname{sen}\theta-2\cos 2\theta]d\theta\\
&= \frac{1}{2}\Big[7\theta-12\cos\theta-\operatorname{sen}2\theta\Big]_{-\pi/6}^{7\pi/6}\\
&= \frac{11\sqrt{3}}{2}+\frac{14\pi}{3}
\end{aligned}
$$

4. Calcule o volume do sólido gerado, pela rotação em torno do eixo $y=-1$, da região delimitada pelas curvas $y=\sqrt[3]{x}$ e $y=x/4$, que se situa no primeiro quadrante do plano cartesiano.

$$\sqrt[3]{x}=\frac{x}{4}\Rightarrow x=0 \text{ ou } 8$$

$$
\begin{aligned}
V &= \pi\int_0^8[(1+\sqrt[3]{x})^2-(1+\frac{x}{4})^2]dx\\
&= \pi\int_0^8[(1+2x^{1/3}+x^{2/3})-(1+\frac{x}{2}+\frac{x}{16})]dy\\
&= \pi\int_0^8(2x^{1/3}+x^{2/3}-\frac{x}{2}-\frac{x^2}{16})dy\\
&= \pi\Big[\frac{3}{2}x^{4/3}+\frac{3}{5}x^{5/3}-\frac{x^2}{4}-\frac{x^3}{48}\Big]_0^8\\
&= \pi\Big[\frac{3}{2}\cdot 8^{4/3}+\frac{3}{5}\cdot 8^{5/3}-\frac{8^2}{4}-\frac{8^3}{48}\Big]\\
&= \frac{248}{15}\pi
\end{aligned}
$$

5. Calcule o volume do sólido gerado, pela rotação em torno do eixo $x=6$, da região delimitada pela curva $y=2\sqrt{x-1}$, e pela reta $y=x-1$.

$$2\sqrt{x-1}=(x-1)\Rightarrow\sqrt{x-1}(\sqrt{x-1}-2)=0\Rightarrow x=1 \text{ ou } x=5$$

$$
\begin{aligned}
A(x) &= 2\pi\cdot\text{raio}\cdot\text{altura}\\
&= 2\pi(6-x)(2\sqrt{x-1}-(x-1))\\
&= 2\pi(x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1})
\end{aligned}
$$

$$u=x-1\Rightarrow du=dx$$

$$
\begin{aligned}
\int 2x\sqrt{x-1}dx &= 2\int (u+1)u^{1/2}du \\
&= 2\int (u^{3/2}+u^{1/2})du \\
&= 2\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C \\
&= \frac{4}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}+C
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_1^5 A(x)dx \\
&= \int_1^5 2\pi(x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1})dx \\
&= 2\pi\Big[\frac{1}{3}x^3-\frac{7}{2}x^2+6x+8(x-1)^{3/2}-\frac{4}{3}(x-1)^{3/2}-\frac{4}{5}(x-1)^{5/2}\Big]_1^5 \\
&= 2\pi\cdot\frac{136}{15} \\
&= \frac{272}{15}\pi
\end{aligned}
$$

6. Ache o comprimento de arco da curva $y=\frac{2}{3}x^{3/2}$ do ponto em que $x=0$ ao ponto em que $x=3$.

$$
y=\frac{2}{3}x^{3/2}\Rightarrow y'=\frac{2}{3}\frac{3}{2}x^{\frac{3}{2}-1}=x^{1/2}
$$

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+(y')^2}dx \\
&= \int_0^3 \sqrt{1+(x^{1/2})^2}dx \\
&= \int_0^3 \sqrt{1+x}dx \\
&= \Big[\frac{2}{3}(1+x)^{3/2}\Big]_0^3 \\
&= \frac{2}{3}(1+3)^{3/2}-\frac{2}{3}(1+0)^{3/2} \\
&= \frac{2}{3}\cdot 8-\frac{2}{3}=\frac{14}{3}
\end{aligned}
$$