Universidade Federal do Ceará
Campus Sobral

Cálculo Diferencial e Integral II – 2021.2 (SBL0058)
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3a Avaliação Progressiva

Nome:____

1. Usando a Regra de L'Hôpital, encontre o valor dos limites:

(a) $\lim_{x\to 1^+}\Big(\dfrac{1}{\ln(x)} - \dfrac{1}{x-1}\Big)$;

$$\begin{aligned}
\lim_{x\to 1^+}\Big(\frac{1}{\ln(x)} - \frac{1}{x-1}\Big) &= \lim_{x\to 1^+} \frac{(x-1)-\ln(x)}{\ln(x)(x-1)} \\
&\overset{(*)}{=} \lim_{x\to 1^+} \frac{1-1/x}{(1/x)(x-1)+\ln(x)} \\
&= \lim_{x\to 1^+} \frac{x-1}{(x-1)+x\ln(x)} \\
&\overset{(**)}{=} \lim_{x\to 1^+} \frac{1}{1+\ln(x)+x(1/x)} \\
&= \lim_{x\to 1^+} \frac{1}{2+\ln(x)} \\
&= \frac{1}{2}
\end{aligned}$$

$$(*) \Leftarrow \begin{cases} \text{(i) } (x-1)-\ln(x) \text{ e } \ln(x)(x-1) \text{ são deriváveis em } (1,\infty) \\ \text{(ii) } (1/x)(x-1)+\ln(x) \neq 0 \text{ em } (1,\infty) \\ \text{(iii) } \lim_{x\to 1^+}[(x-1)-\ln(x)] = 0 \text{ e } \lim_{x\to 1^+} \ln(x)(x-1) = 0 \end{cases}$$

$$(**) \Leftarrow \begin{cases} \text{(i) } (x-1) \text{ e } (x-1)+x\ln(x) \text{ são deriváveis em } (1,\infty) \\ \text{(ii) } 1+\ln(x)+x(1/x) \neq 0 \text{ em } (1,\infty) \\ \text{(iii) } \lim_{x\to 1^+}(x-1) = 0 \text{ e } \lim_{x\to 1^+}[(x-1)+x\ln(x)] = 0 \end{cases}$$

(b) $\lim_{x\to\infty} \dfrac{x}{\ln(1+e^x)}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\ln(1+e^x)} &\overset{(*)}{=} \lim_{x\to\infty} \frac{1}{\frac{e^x}{1+e^x}} \\
&= \lim_{x\to\infty}(1+e^{-x}) \\
&= 1
\end{aligned}$$

$$(*) \Leftarrow \begin{cases} \text{(i) } x \text{ e } \ln(1+e^x) \text{ são deriváveis em } (0,\infty) \\ \text{(ii) } \dfrac{e^x}{1+e^x} \neq 0 \text{ em } (0,\infty) \\ \text{(iii) } \lim_{x\to\infty} x = \infty \text{ e } \lim_{x\to\infty} \ln(1+e^x) = \infty \end{cases}$$

2. Encontre o valor das integrais impróprias:

(a) $\displaystyle\int_1^{\infty} \frac{e^{1/x}}{x^2} dx;$

$$u = 1/x \Rightarrow du = -\frac{1}{x^2} dx$$

$$\begin{aligned}\int \frac{e^{1/x}}{x^2} dx &= \int e^u (-du)\\ &= -e^u + C\\ &= -e^{1/x} + C\end{aligned}$$

$$\begin{aligned}\int_1^{\infty} \frac{e^{1/x}}{x^2} dx &= \lim_{b\to\infty} \int_1^b \frac{e^{1/x}}{x^2} dx\\ &= \lim_{b\to\infty} \left[-e^{1/x}\right]_1^b\\ &= \lim_{b\to\infty} (-e^{1/x} + e)\\ &= e - 1\end{aligned}$$

(b) $\displaystyle\int_{-\infty}^{0} \frac{x}{x^4+16} dx.$

$$u = x^2 \Rightarrow du = 2x dx$$

$$\begin{aligned}\int \frac{x}{x^4+16} dx &= \int \frac{1}{u^2+16} \frac{du}{2}\\ &= \int \frac{1}{u^2+16} \frac{du}{2}\\ &= \frac{1}{2}\frac{1}{4} \operatorname{tg}^{-1}\left(\frac{u}{4}\right) + C\\ &= \frac{1}{8} \operatorname{tg}^{-1}\left(\frac{x^2}{4}\right) + C\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} \frac{x}{x^4+16} dx &= \lim_{a\to-\infty} \int_a^0 \frac{x}{x^4+16} dx\\ &= \lim_{a\to-\infty} \left[\frac{1}{8} \operatorname{tg}^{-1}\left(\frac{x^2}{4}\right)\right]_a^0\\ &= \lim_{a\to-\infty} \left[0 - \frac{1}{8} \operatorname{tg}^{-1}\left(\frac{a^2}{4}\right)\right]\\ &= \left[0 - \frac{1}{8}\frac{\pi}{2}\right] = -\frac{\pi}{16}\end{aligned}$$

3. Encontre a área da região interior ao círculo $r = \operatorname{sen}(\theta)$ e exterior à cardioide

$r = 1 - \cos(\theta)$.

$$\begin{aligned}(\operatorname{sen}\theta)^2 - (1-\cos\theta)^2 &= \operatorname{sen}^2\theta - 1 + 2\cos\theta - \cos^2\theta \\ &= \operatorname{sen}^2\theta - 1 + 2\cos\theta - (1-\operatorname{sen}^2\theta) \\ &= 2\operatorname{sen}^2\theta - 2 + 2\cos\theta \\ &= 1 - \cos 2\theta - 2 + 2\cos\theta \\ &= 2\cos\theta - \cos 2\theta - 1\end{aligned}$$

$$\begin{aligned}A &= \frac{1}{2}\int_0^{\pi/2} [(\operatorname{sen}\theta)^2 - (1-\cos\theta)^2]d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} (2\cos\theta - \cos 2\theta - 1)d\theta \\ &= \frac{1}{2}\Big[2\operatorname{sen}\theta - \frac{\operatorname{sen} 2\theta}{2} - \theta\Big]_0^{\pi/2} \\ &= \Big[\operatorname{sen}\theta - \frac{\operatorname{sen} 2\theta}{4} - \frac{\theta}{2}\Big]_0^{\pi/2} \\ &= \Big[1 - \frac{0}{4} - \frac{\pi}{4}\Big] - \Big[0 - \frac{0}{4} - \frac{0}{2}\Big] \\ &= 1 - \frac{\pi}{4}\end{aligned}$$

4. Calcule o volume do sólido gerado, pela rotação em torno do eixo $y = 2$, da região delimitada pelas curvas $y = \sqrt{x}$ e $y = x/2$.

$$\sqrt{x} = \frac{x}{2} \Rightarrow x = 0 \text{ ou } 4$$

$$\begin{aligned}V &= \pi\int_0^4 \Big[\Big(2 - \frac{x}{2}\Big)^2 - (2-\sqrt{x})^2\Big]dx \\ &= \pi\int_0^4 \Big[\Big(4 - 2x + \frac{x^2}{4}\Big) - (4 - 4x^{1/2} + x)\Big]dx \\ &= \pi\int_0^4 \Big(\frac{x^2}{4} + 4x^{1/2} - 3x\Big)dx \\ &= \pi\Big[\frac{1}{12}x^3 + \frac{8}{3}x^{3/2} - \frac{3}{2}x^2\Big]_0^4 \\ &= \pi\Big[\frac{1}{12}\cdot 4^3 + \frac{8}{3}\cdot 4^{3/2} - \frac{3}{2}\cdot 4^2\Big] \\ &= \frac{8}{3}\pi\end{aligned}$$

5. Calcule o volume do sólido gerado, pela rotação em torno do eixo $x = 0$, da região delimitada pela curva $x = 2\sqrt{x-1}$, e pela reta $y = x - 1$.

$$2\sqrt{x-1} = x - 1 \Rightarrow x = 1 \text{ ou } 5$$

$$
\begin{aligned}
A(x) &= 2\pi \cdot \text{raio} \cdot \text{altura} \\
&= 2\pi \cdot x \cdot [2\sqrt{x-1} - (x-1)]
\end{aligned}
$$

$$u = x - 1 \Rightarrow du = dx$$

$$
\begin{aligned}
V &= \int_1^5 A(x)dx \\
&= 2\pi \int_1^5 x \cdot [2\sqrt{x-1} - (x-1)]dx \\
&= 2\pi \int_0^4 (u+1)(2\sqrt{u} - u)du \\
&= 2\pi \int_0^4 (2u^{3/2} - u^2 + 2u^{1/2} - u)du \\
&= 2\pi \Big[\frac{4}{5}u^{5/2} - \frac{1}{3}u^3 + \frac{4}{3}u^{3/2} - \frac{1}{2}u^2\Big]_0^4 \\
&= 2\pi \Big[\frac{4}{5} \cdot 4^{5/2} - \frac{1}{3} \cdot 4^3 + \frac{4}{3} \cdot 4^{3/2} - \frac{1}{2} \cdot 4^2\Big] \\
&= \frac{208}{15}\pi
\end{aligned}
$$

6. Ache o comprimento de arco da curva $y = \frac{x^5}{10} + \frac{1}{6x^3}$ do ponto em que $x = 2$ ao ponto em que $x = 5$.

$$y = \frac{x^5}{10} + \frac{1}{6x^3} \Rightarrow y' = \frac{x^4}{2} - \frac{x^{-4}}{2}$$

$$
\begin{aligned}
\sqrt{1 + (y')^2} &= \sqrt{1 + \Big(\frac{x^4}{2} - \frac{x^{-4}}{2}\Big)^2} \\
&= \sqrt{1 + \frac{x^8}{4} - \frac{1}{2} + \frac{x^{-8}}{4}} \\
&= \sqrt{\frac{x^8}{4} + \frac{1}{2} + \frac{x^{-8}}{4}} \\
&= \sqrt{\Big(\frac{x^4}{2} + \frac{x^{-4}}{2}\Big)^2} \\
&= \frac{x^4}{2} + \frac{x^{-4}}{2}
\end{aligned}
$$

$$
\begin{aligned}
L &= \int_2^5 \sqrt{1 + (y')^2}dx \\
&= \int_2^5 \Big(\frac{x^4}{2} + \frac{x^{-4}}{2}\Big)dx \\
&= \Big[\frac{x^5}{10} - \frac{x^{-3}}{6}\Big]_2^5 \\
&= \frac{618639}{2000}
\end{aligned}
$$