Universidade Federal do Ceará
Campus Sobral

Cálculo Diferencial e Integral II – 2021.1 (SBL0058)
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3a Avaliação Progressiva – 2a Chamada

Nome:____

1. Usando a Regra de L'Hôpital, encontre o valor dos limites:

(a) $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{\cos x - 1}$;

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} &\overset{(*)}{=} \lim_{x\to 0} \frac{e^x - 1}{-\operatorname{sen} x} \\
&\overset{(**)}{=} \lim_{x\to 0} \frac{e^x}{-\cos x} \\
&= \frac{1}{-1} = -1
\end{aligned}$$

$$(*) \Leftarrow \begin{cases} \text{(i) } e^x - x - 1 \text{ e } \cos x - 1 \text{ são deriváveis em } (-1,1) \\ \text{(ii) } -\operatorname{sen} x \neq 0 \text{ em } (-1,1)\setminus\{0\} \\ \text{(iii) } \lim\limits_{x\to 0}(e^x - x - 1) = 0 \text{ e } \lim\limits_{x\to 0}(\cos x - 1) = 0 \end{cases}$$

$$(**) \Leftarrow \begin{cases} \text{(i) } e^x - 1 \text{ e } -\operatorname{sen} x \text{ são deriváveis em } (-1,1) \\ \text{(ii) } -\cos x \neq 0 \text{ em } (-1,1) \\ \text{(iii) } \lim\limits_{x\to 0}(e^x - 1) = 0 \text{ e } \lim\limits_{x\to 0}(-\operatorname{sen} x) = 0 \end{cases}$$

(b) $\lim\limits_{x\to\infty} x^{1/\sqrt{x}}$.

$$\begin{aligned}
\lim_{x\to\infty} x^{1/\sqrt{x}} &= \lim_{x\to\infty} \exp(\ln(x^{1/\sqrt{x}})) \\
&= \exp\Big(\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}\Big) \\
&\overset{(*)}{=} \exp\Big(\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}\frac{1}{\sqrt{x}}}\Big) \\
&= \exp\Big(\lim_{x\to\infty} \frac{2}{\sqrt{x}}\Big) \\
&= \exp(0) = e
\end{aligned}$$

$$(*) \Leftarrow \begin{cases} \text{(i) } \ln x \text{ e } \sqrt{x} \text{ são deriváveis em } (0,\infty) \\ \text{(ii) } \dfrac{1}{2}\dfrac{1}{\sqrt{x}} \neq 0 \text{ em } (0,\infty) \\ \text{(iii) } \lim\limits_{x\to\infty} \ln x = \infty \text{ e } \lim\limits_{x\to\infty} \sqrt{x} = \infty \end{cases}$$

2. Encontre o valor das integrais impróprias:

(a) $\displaystyle\int_{-\infty}^{0} \frac{1}{1-x^2}dx;$

$$\begin{aligned}
\int \frac{1}{1-x^2}dx &= \int \frac{1}{2}\Big(\frac{1}{x+1} - \frac{1}{x-1}\Big)dx \\
&= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C \\
&= \frac{1}{2}\ln\Big|\frac{x+1}{x-1}\Big| + C
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{1-x^2}dx &= \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{\sqrt{1-x}}dx \\
&= \lim_{a\to-\infty}\Big[\frac{1}{2}\ln\Big|\frac{x+1}{x-1}\Big|\Big]_{a}^{0} \\
&= \lim_{a\to-\infty}\Big[\frac{1}{2}\ln\Big|\frac{0+1}{0-1}\Big| - \frac{1}{2}\ln\Big|\frac{a+1}{a-1}\Big|\Big] \\
&= -\frac{1}{2}\ln\Big|\frac{1+(\lim_{a\to-\infty}\frac{1}{a})}{1-(\lim_{a\to-\infty}\frac{1}{a})}\Big| \\
&= 0
\end{aligned}$$

(b) $\displaystyle\int_{1}^{\infty} \frac{\ln(x)}{x^2}dx.$

$$u = \ln(x) \Rightarrow du = \frac{1}{x}dx$$

$$dv = \frac{1}{x^2}dx \Rightarrow v = -\frac{1}{x}$$

$$\begin{aligned}
\int \frac{\ln(x)}{x^2}dx &= uv - \int vdu \\
&= -\frac{\ln(x)}{x} - \int\Big(-\frac{1}{x}\Big)\Big(\frac{1}{x}dx\Big) \\
&= -\frac{\ln(x)}{x} - \frac{1}{x} + C
\end{aligned}$$

$$\begin{aligned}
\int_{1}^{\infty} \frac{\ln(x)}{x^2}dx &= \lim_{b\to\infty}\int_{1}^{b} \frac{\ln(x)}{x^2}dx \\
&= \lim_{b\to\infty}\Big[-\frac{\ln x}{x} - \frac{1}{x}\Big]_{1}^{b} \\
&= \lim_{b\to\infty}\Big[-\frac{\ln b}{b} - \frac{1}{b} + \frac{\ln 1}{1} + \frac{1}{1}\Big] \\
&= 1
\end{aligned}$$

3. Calcule a área da região delimitada pela curva $r = 3 + 2\,\text{sen}(\theta)$.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} [3 + 2\,\text{sen}(\theta)]^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} [9 + 12\,\text{sen}(\theta) + 4\,\text{sen}^2(\theta)] d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \Big(9 + 12\,\text{sen}(\theta) + 4\frac{1-\cos(2\theta)}{2}\Big) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} [11 + 12\,\text{sen}(\theta) - 2\cos(2\theta)] d\theta \\
&= \frac{1}{2}\Big[11\theta + 12\cos\theta - \text{sen}(2\theta)\Big]_0^{2\pi} \\
&= 11\pi
\end{aligned}$$

4. Calcule o volume do sólido gerado, pela rotação em torno do eixo $y = -2$, da região delimitada pelas curvas $y = \sqrt{x}$ e $y = x/3$.

$$\sqrt{x} = \frac{x}{3} \Rightarrow x = 0 \text{ ou } 9$$

$$\begin{aligned}
V &= \pi\int_0^9 \Big[(2+\sqrt{x})^2 - \Big(2 + \frac{x}{3}\Big)^2\Big] dx \\
&= \pi\int_0^9 \Big[(4 + 4\sqrt{x} + x) - \Big(4 + \frac{4}{3}x + \frac{x^2}{9}\Big)\Big] dx \\
&= \pi\int_0^9 \Big(4x^{1/2} - \frac{1}{3}x - \frac{x^2}{9}\Big) dx \\
&= \pi\Big[\frac{8}{3}x^{3/2} - \frac{1}{6}x^2 - \frac{x^3}{27}\Big]_0^9 \\
&= \frac{63}{2}\pi
\end{aligned}$$

5. Calcule o volume do sólido gerado, pela rotação em torno do eixo $y = -1$, da região delimitada pela curva $x = (y-2)^2$, e pela reta $y = x$.

$$y = (y-2)^2 \Rightarrow y = 1 \text{ ou } 4$$

$$\begin{aligned}
A(y) &= 2\pi \cdot \text{raio} \cdot \text{altura} \\
&= 2\pi(y+1)(y - (y-2)^2) \\
&= 2\pi(-y^3 + 4y^2 + y - 4)
\end{aligned}$$

$$\begin{aligned}
V &= \int_1^4 A(y) dy \\
&= 2\pi\int_1^4 (-y^3 + 4y^2 + y - 4) dy \\
&= 2\pi\Big[-\frac{y^4}{4} + \frac{4}{3}y^3 + \frac{1}{2}y^2 - 4y\Big]_1^4 \\
&= 2\pi \cdot \frac{63}{4} = \frac{63}{2}\pi
\end{aligned}$$

6. Ache o comprimento de arco da curva $y = \frac{1}{4}(e^{2x} + e^{-2x})$ do ponto em que $x = 0$ ao ponto em que $x = 1$.

$$y = \frac{1}{4}(e^{2x} + e^{-2x}) \Rightarrow y' = \frac{1}{2}(e^{2x} - e^{-2x})$$

$$\begin{aligned}
L &= \int_0^1 \sqrt{1 + [\frac{1}{2}(e^{2x} - e^{-2x})]^2} dx \\
&= \int_0^1 \sqrt{1 + \frac{1}{4}(e^{4x} - 2 + e^{-4x})} dx \\
&= \int_0^1 \sqrt{\frac{1}{4}(e^{4x} + 2 + e^{-4x})} dx \\
&= \int_0^1 \sqrt{[\frac{1}{2}(e^{2x} + e^{-2x})]^2} dx \\
&= \int_0^1 \frac{1}{2}(e^{2x} + e^{-2x}) dx \\
&= \left[\frac{1}{4}(e^{2x} - e^{-2x})\right]_0^1 \\
&= \frac{1}{4}(e^2 - e^{-2})
\end{aligned}$$