12

Introduction to the Laplace Transform

Assessment Problems

AP 12.1 [a] $\cosh \beta t = \dfrac{e^{\beta t} + e^{-\beta t}}{2}$

Therefore,

$$\mathcal{L}\{\cosh \beta t\} = \frac{1}{2}\int_{0^-}^{\infty} [e^{(s-\beta)t} + e^{-(s-\beta)t}]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-\beta)t}}{-(s-\beta)}\Big|_{0^-}^{\infty} + \frac{e^{-(s+\beta)t}}{-(s+\beta)}\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-\beta} + \frac{1}{s+\beta}\right) = \frac{s}{s^2-\beta^2}$$

[b] $\sinh \beta t = \dfrac{e^{\beta t} - e^{-\beta t}}{2}$

Therefore,

$$\mathcal{L}\{\sinh \beta t\} = \frac{1}{2}\int_{0^-}^{\infty} \left[e^{-(s-\beta)t} - e^{-(s+\beta)t}\right] dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-\beta)t}}{-(s-\beta)}\right]_{0^-}^{\infty} - \frac{1}{2}\left[\frac{e^{-(s+\beta)t}}{-(s+\beta)}\right]_{0^-}^{\infty}$$

$$= \frac{1}{2}\left(\frac{1}{s-\beta} - \frac{1}{s+\beta}\right) = \frac{\beta}{(s^2-\beta^2)}$$

AP 12.2 [a] Let $f(t) = te^{-at}$:

$$F(s) = \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

Now, $\quad \mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$

$$\text{So,}\quad \mathcal{L}\{t\cdot te^{-at}\} = -\frac{d}{ds}\left[\frac{1}{(s+a)^2}\right] = \frac{2}{(s+a)^3}$$

[b] Let $f(t) = e^{-at}\sinh\beta t$, then

$$\mathcal{L}\{f(t)\} = F(s) = \frac{\beta}{(s+a)^2-\beta^2}$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) = \frac{s(\beta)}{(s+a)^2-\beta^2} - 0 = \frac{\beta s}{(s+a)^2-\beta^2}$$

[c] Let $f(t) = \cos\omega t$. Then

$$F(s) = \frac{s}{(s^2+\omega^2)} \quad\text{and}\quad \frac{dF(s)}{ds} = \frac{-(s^2-\omega^2)}{(s^2+\omega^2)^2}$$

$$\text{Therefore}\quad \mathcal{L}\{t\cos\omega t\} = -\frac{dF(s)}{ds} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

AP 12.3

$$F(s) = \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = \frac{6-26+26}{(1)(2)} = 3; \qquad K_2 = \frac{24-52+26}{(-1)(1)} = 2$$

$$K_3 = \frac{54-78+26}{(-2)(-1)} = 1$$

Therefore $f(t) = [3e^{-t} + 2e^{-2t} + e^{-3t}]\,u(t)$

AP 12.4

$$F(s) = \frac{7s^2+63s+134}{(s+3)(s+4)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

$$K_1 = \frac{63-189-134}{1(2)} = 4; \qquad K_2 = \frac{112-252+134}{(-1)(1)} = 6$$

$$K_3 = \frac{175-315+134}{(-2)(-1)} = -3$$

$$f(t) = [4e^{-3t} + 6e^{-4t} - 3e^{-5t}]u(t)$$

AP 12.5

$$F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)}$$

$$s_{1,2} = -5 \pm \sqrt{25-169} = -5 \pm j12$$

$$F(s) = \frac{K_1}{s+5} + \frac{K_2}{s+5-j12} + \frac{K_2^*}{s+5+j12}$$

$$K_1 = \frac{10(25+119)}{25-50+169} = 10$$

$$K_2 = \frac{10[(-5+j12)^2+119]}{(j12)(j24)} = j4.17 = 4.17\underline{/90^\circ}$$

Therefore

$$\begin{aligned} f(t) &= [10e^{-5t} + 8.33e^{-5t}\cos(12t+90^\circ)]\,u(t) \\ &= [10e^{-5t} - 8.33e^{-5t}\sin 12t]\,u(t) \end{aligned}$$

AP 12.6

$$F(s) = \frac{4s^2+7s+1}{s(s+1)^2} = \frac{K_0}{s} + \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1}$$

$$K_0 = \frac{1}{(1)^2} = 1; \qquad K_1 = \frac{4-7+1}{-1} = 2$$

$$\begin{aligned} K_2 &= \frac{d}{ds}\left[\frac{4s^2+7s+1}{s}\right]_{s=-1} = \left.\frac{s(8s+7)-(4s^2+7s+1)}{s^2}\right|_{s=-1} \\ &= \frac{1+2}{1} = 3 \end{aligned}$$

Therefore $f(t) = [1 + 2te^{-t} + 3e^{-t}]\,u(t)$

AP 12.7

$$\begin{aligned} F(s) &= \frac{40}{(s^2+4s+5)^2} = \frac{40}{(s+2-j1)^2(s+2+j1)^2} \\ &= \frac{K_1}{(s+2-j1)^2} + \frac{K_2}{(s+2-j1)} + \frac{K_1^*}{(s+2+j1)^2} \\ &\quad + \frac{K_2^*}{(s+2+j1)} \end{aligned}$$

$$K_1 = \frac{40}{(j2)^2} = -10 = 10\underline{/180^\circ} \quad \text{and} \quad K_1^* = -10$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{s^3[4+(7/s)+(1/s)^2]}{s^3[1+(1/s)]^2}\right] = 4$$

$$\therefore\quad f(0^+) = 4$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{4s^2+7s+1}{(s+1)^2}\right] = 1$$

$$\therefore\quad f(\infty) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{40s}{s^4[1+(4/s)+(5/s^2)]^2}\right] = 0$$

$$\therefore\quad f(0^+) = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{40s}{(s^2+4s+5)^2}\right] = 0$$

$$\therefore\quad f(\infty) = 0$$

Problems

P 12.1 [a] $f(t) = 120 + 30t \qquad -4\,\text{s} \leq t \leq 0$

$$f(t) = 120 - 30t \qquad 0 \leq t \leq 8\,\text{s}$$

$$f(t) = -360 + 30t \qquad 8\,\text{s} \leq t \leq 12\,\text{s}$$

$$f(t) = 0 \qquad \text{elsewhere}$$

$$f(t) = (120 + 30t)[(u(t+4) - u(t)] + (120 - 30t)[u(t) - u(t-8)]$$
$$+(-360 + 30t)[u(t-8) - u(t-12)]$$

[b] $f(t) = 50\sin\frac{\pi}{2}t[u(t) - u(t-4)]$

$$= (50\sin\tfrac{\pi}{2}t)u(t) - (50\sin\tfrac{\pi}{2}t)u(t-4)$$

[c] $f(t) = (30 - 3t)t[u(t) - u(t-10)]$

P 12.2 [a] $(50 + 2.5t)[u(t+20) - u(t)] + (50 - 5t)[u(t) - u(t-10)]$

$$= (2.5t + 50)u(t+20) - 2.5tu(t) + (5t - 50)u(t-10)$$

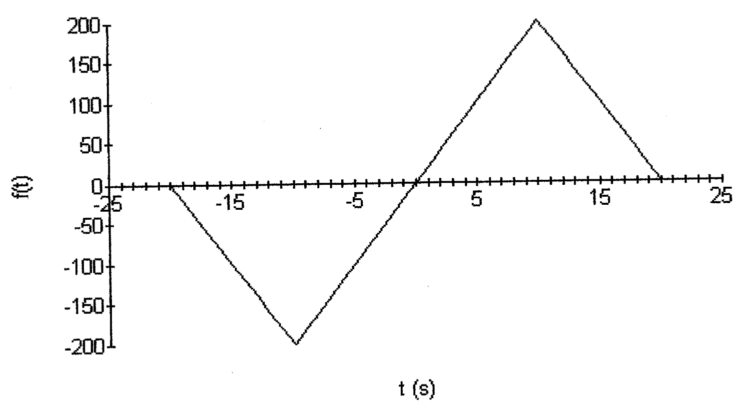
[b] $(5t + 45)[u(t+9) - u(t+6)] + 15[u(t+6) - u(t+3)] - 5t[u(t+3) - u(t-3)]$

$$-15[u(t-3) - u(t-6)] + (5t - 45)[u(t-6) - u(t-9)]$$

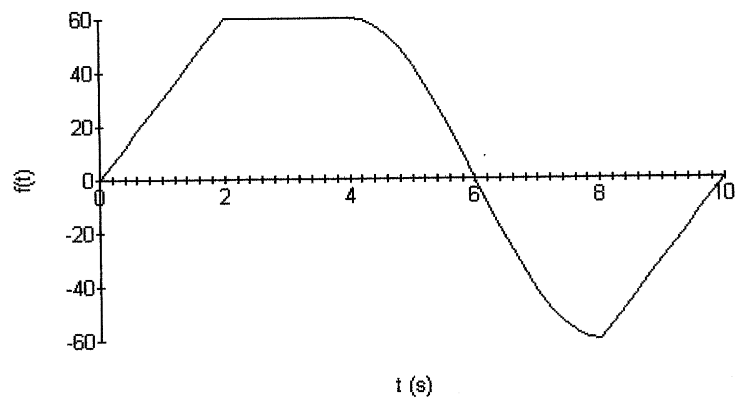
$$= 5(t+9)u(t+9) - 5(t+6)u(t+6) - 5(t+3)u(t+3) + 5(t-3)u(t-3)$$

$$+5(t-6)u(t-6) - 5(t-9)u(t-9)$$

P 12.3



P 12.4 [a]



[b] $f(t) = 30t[u(t) - u(t-2)] + 60[u(t-2) - u(t-4)]$
$+60\cos(\frac{\pi}{4}t - \pi)[u(t-4) - u(t-8)]$
$+(30t - 300)[u(t-8) - u(t-10)]$

P 12.5 [a] $A = \left(\frac{1}{2}\right) bh = \left(\frac{1}{2}\right)(2\varepsilon)\left(\frac{1}{\varepsilon}\right) = 1.0$

[b] 0; [c] ∞

P 12.6 [a] $I = \int_{-2}^{4} (t^3 + 4)\delta(t)\,dt + \int_{-2}^{4} 4(t^3 + 4)\delta(t-2)\,dt$

$= 4 + 4(8 + 4) = 52$

[b] $I = \int_{-3}^{4} t^2\delta(t)\,dt + \int_{-3}^{4} t^2\delta(t + 2.5)\,dt + 0$

$= 0^2 + (-2.5)^2 + 0 = 6.25$

P 12.7 $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(3 + j\omega)}{(4 + j\omega)} \cdot \pi\delta(\omega) \cdot e^{jt\omega}\,d\omega = \left(\frac{1}{2\pi}\right)\left(\frac{3 + j0}{4 + j0}\pi e^{-jt0}\right) = \frac{3}{8}$

P 12.8 As $\varepsilon \to 0$ the amplitude $\to \infty$; the duration $\to 0$; and the area is independent of ε, i.e.,

$$A = \int_{-\infty}^{\infty} \frac{\varepsilon}{\pi}\frac{1}{\varepsilon^2 + t^2}\,dt = 1$$

P 12.9 $F(s) = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}e^{-st}\,dt = \frac{e^{s\varepsilon} - e^{-s\varepsilon}}{2\varepsilon s}$

$$F(s) = \frac{1}{2s}\lim_{\varepsilon\to 0}\left[\frac{se^{s\varepsilon} + se^{-s\varepsilon}}{1}\right] = \frac{1}{2s}\cdot\frac{2s}{1} = 1$$

P 12.10 [a] Let $dv = \delta'(t-a)\,dt, \qquad v = \delta(t-a)$

$u = f(t), \qquad du = f'(t)\,dt$

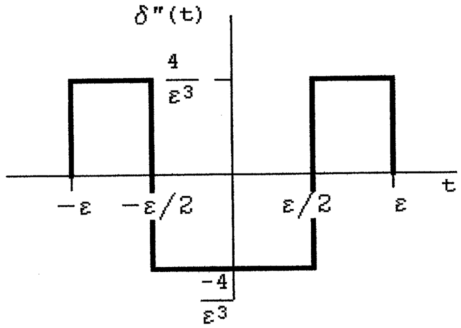
Therefore

$$\int_{-\infty}^{\infty} f(t)\delta'(t-a)\,dt = f(t)\delta(t-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t-a)f'(t)\,dt$$

$$= 0 - f'(a)$$

[b] $\mathcal{L}\{\delta'(t)\} = \int_{0^-}^{\infty} \delta'(t)e^{-st}\,dt = -\left[\frac{d(e^{-st})}{dt}\right]_{t=0} = -\left[-se^{-st}\right]_{t=0} = s$

P 12.11



$$F(s) = \int_{-\varepsilon}^{-\varepsilon/2} \frac{4}{\varepsilon^3}e^{-st}\,dt + \int_{-\varepsilon/2}^{\varepsilon/2} \left(\frac{-4}{\varepsilon^3}\right) e^{-st}\,dt + \int_{\varepsilon/2}^{\varepsilon} \frac{4}{\varepsilon^3}e^{-st}\,dt$$

Therefore $F(s) = \dfrac{4}{s\varepsilon^3}[e^{s\varepsilon} - 2e^{s\varepsilon/2} + 2e^{-s\varepsilon/2} - e^{-s\varepsilon}]$

$$\mathcal{L}\{\delta''(t)\} = \lim_{\varepsilon\to 0} F(s)$$

After applying L'Hopital's rule three times, we have

$$\lim_{\varepsilon\to 0} \frac{2s}{3}\left[se^{s\varepsilon} - \frac{s}{4}e^{s\varepsilon/2} - \frac{s}{4}e^{-s\varepsilon/2} + se^{-s\varepsilon}\right] = \frac{2s}{3}\left(\frac{3s}{2}\right)$$

Therefore $\mathcal{L}\{\delta''(t)\} = s^2$

P 12.12 $\mathcal{L}\left\{\dfrac{d^n f(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots,$

Therefore

$$\mathcal{L}\{\delta^n(t)\} = s^n(1) - s^{n-1}\delta(0^-) - s^{n-2}\delta'(0^-) - s^{n-3}\delta''(0^-) - \cdots = s^n$$

P 12.13 [a] $\mathcal{L}\{t\} = \dfrac{1}{s^2}; \qquad$ therefore $\mathcal{L}\{te^{-at}\} = \dfrac{1}{(s+a)^2}$

[b] $\sin \omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{j2}$

Therefore

$$\mathcal{L}\{\sin \omega t\} = \left(\frac{1}{j2}\right)\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \left(\frac{1}{j2}\right)\left(\frac{2j\omega}{s^2 + \omega^2}\right)$$

$$= \frac{\omega}{s^2 + \omega^2}$$

[c] $\sin(\omega t + \theta) = (\sin \omega t \cos \theta + \cos \omega t \sin \theta)$

Therefore

$$\mathcal{L}\{\sin(\omega t + \theta)\} = \cos \theta \mathcal{L}\{\sin \omega t\} + \sin \theta \mathcal{L}\{\cos \omega t\}$$

$$= \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}$$

[d] $\mathcal{L}\{t\} = \displaystyle\int_0^\infty te^{-st}\,dt = \frac{e^{-st}}{s^2}(-st - 1)\Big|_0^\infty = 0 - \frac{1}{s^2}(0 - 1) = \frac{1}{s^2}$

[e] $f(t) = \cosh t \cosh \theta + \sinh t \sinh \theta$

From Assessment Problem 12.1(a)

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2 - 1}$$

From Assessment Problem 12.1(b)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1}$$

$$\therefore \quad \mathcal{L}\{\cosh(t + \theta)\} = \cosh \theta \left[\frac{s}{(s^2 - 1)}\right] + \sinh \theta \left[\frac{1}{s^2 - 1}\right]$$

$$= \frac{\sinh \theta + s[\cosh \theta]}{(s^2 - 1)}$$

P 12.14 [a] $\mathcal{L}\{te^{-at}\} = \displaystyle\int_{0^-}^\infty te^{-(s+a)t}\,dt$

$$= \frac{e^{-(s+a)t}}{(s + a)^2}\Big[-(s + a)t - 1\Big]_{0^-}^\infty$$

$$= 0 + \frac{1}{(s + a)^2}$$

$$\therefore \quad \mathcal{L}\{te^{-at}\} = \frac{1}{(s + a)^2}$$

[b] $\mathcal{L}\left\{\dfrac{d}{dt}(te^{-at})u(t)\right\} = \dfrac{s}{(s + a)^2} - 0$

$$\mathcal{L}\left\{\frac{d}{dt}(te^{-at})u(t)\right\} = \frac{s}{(s + a)^2}$$

[c] $\frac{d}{dt}(te^{-at}) = -ate^{-at} + e^{-at}$

$$\mathcal{L}\{-ate^{-at} + e^{-at}\} = \frac{-a}{(s+a)^2} + \frac{1}{(s+a)} = \frac{-a}{(s+a)^2} + \frac{s+a}{(s+a)^2}$$

$$\therefore\ \mathcal{L}\left\{\frac{d}{dt}(te^{-at})\right\} = \frac{s}{(s+a)^2} \qquad \text{CHECKS}$$

P 12.15 [a] $\mathcal{L}\{f'(t)\} = \int_{-\varepsilon}^{\varepsilon} \frac{e^{-st}}{2\varepsilon}\,dt + \int_{\varepsilon}^{\infty} -ae^{-a(t-\varepsilon)}e^{-st}\,dt$

$$= \frac{1}{2s\varepsilon}(e^{s\varepsilon} - e^{-s\varepsilon}) - \left(\frac{a}{s+a}\right)e^{-s\varepsilon} = F(s)$$

$$\lim_{\varepsilon\to 0} F(s) = 1 - \frac{a}{s+a} = \frac{s}{s+a}$$

[b] $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

Therefore $\mathcal{L}\{f'(t)\} = sF(s) - f(0^-) = \frac{s}{s+a} - 0 = \frac{s}{s+a}$

P 12.16 $\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} [e^{-at}f(t)]e^{-st}\,dt = \int_{0^-}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

P 12.17 [a] $\mathcal{L}\left\{\int_{0^-}^{t} e^{-ax}\,dx\right\} = \frac{F(s)}{s} = \frac{1}{s(s+a)}$

[b] $\mathcal{L}\left\{\int_{0^-}^{t} y\,dy\right\} = \frac{1}{s}\left(\frac{1}{s^2}\right) = \frac{1}{s^3}$

[c] $\int_{0^-}^{t} e^{-ax}\,dx = \frac{1}{a} - \frac{e^{-at}}{a}$

$$\mathcal{L}\left\{\frac{1}{a} - \frac{e^{-at}}{a}\right\} = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] = \frac{1}{s(s+a)}$$

$$\int_{0^-}^{t} y\,dy = \frac{t^2}{2}; \qquad \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\cdot\frac{2}{s^3} = \frac{1}{s^3}$$

P 12.18 [a] $\mathcal{L}\left\{\frac{d\sin\omega t}{dt}u(t)\right\} = \frac{s\omega}{s^2+\omega^2} - \sin(0) = \frac{s\omega}{s^2+\omega^2}$

[b] $\mathcal{L}\left\{\frac{d\cos\omega t}{dt}u(t)\right\} = \frac{s^2}{s^2+\omega^2} - \cos(0) = \frac{s^2}{s^2+\omega^2} - 1 = \frac{-\omega^2}{s^2+\omega^2}$

[c] $\mathcal{L}\left\{\frac{d^3(t^2)}{dt^3}u(t)\right\} = s^3\left(\frac{2}{s^3}\right) - s^2(0) - s(0) - 2(0) = 2$

[d] $\dfrac{d\sin\omega t}{dt} = (\cos\omega t)\cdot\omega, \qquad \mathcal{L}\{\omega\cos\omega t\} = \dfrac{\omega s}{s^2+\omega^2}$

$$\frac{d\cos\omega t}{dt} = -\omega\sin\omega t$$

$$\mathcal{L}\{-\omega\sin\omega t\} = -\frac{\omega^2}{s^2+\omega^2}$$

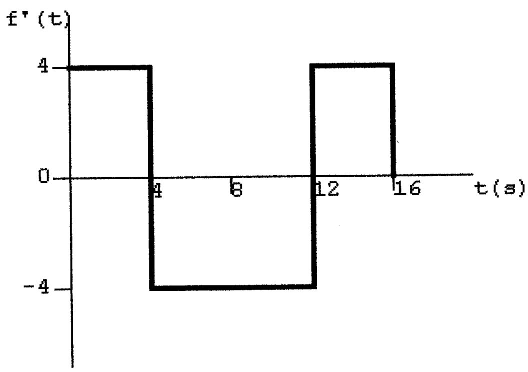
$$\frac{d^3(t^2)}{dt^3} = 2\delta(t); \qquad \mathcal{L}\{2\delta(t)\} = 2$$

P 12.19 [a] $f(t) = 4t[u(t) - u(t-4)]$

$$\begin{aligned} &+(32-4t)[u(t-4)-u(t-12)] \\ &+(4t-64)[u(t-12)-u(t-16)] \\ = {}& 4tu(t) - 8(t-4)u(t-4) \\ &+8(t-12)u(t-12) - 4(t-16)u(t-16) \end{aligned}$$

$$\therefore \quad F(s) = \frac{4[1-2e^{-4s}+2e^{-12s}-e^{-16s}]}{s^2}$$

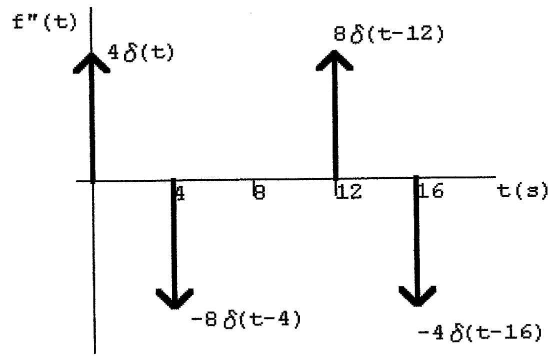
[b]



$$\begin{aligned} f'(t) &= 4[u(t)-u(t-4)] - 4[u(t-4)-u(t-12)] \\ &\quad +4[u(t-12)-u(t-16)] \\ &= 4u(t) - 8u(t-4) + 8u(t-12) - 4u(t-16) \end{aligned}$$

$$\mathcal{L}\{f'(t)\} = \frac{4[1-2e^{-4s}+2e^{-12s}-e^{-16s}]}{s}$$

[c]



$$f''(t) = 4\delta(t) - 8\delta(t-4) + 8\delta(t-12) - 4\delta(t-16)$$

$$\mathcal{L}\{f''(t)\} = 4[1 - 2e^{-4s} + 2e^{-12s} - e^{-16s}]$$

P 12.20 [a] $\int_{0^-}^{t} x\,dx = \frac{t^2}{2}$

$$\mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\int_{0^-}^{\infty} t^2 e^{-st}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-st}}{-s^3}(s^2t^2 + 2st + 2)\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2s^3}(2) = \frac{1}{s^3}$$

$$\therefore \quad \mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{1}{s^3}$$

[b] $\mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{\mathcal{L}\{t\}}{s} = \frac{1/s^2}{s} = \frac{1}{s^3}$

$$\therefore \quad \mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{1}{s^3} \qquad \text{CHECKS}$$

P 12.21 [a] $\mathcal{L}\{-20e^{-5(t-2)}u(t-2)\} = \frac{-20e^{-2s}}{(s+5)}$

[b] First rewrite $f(t)$ as

$$f(t) = (8t-8)u(t-1) + (24 - 8t - 8t + 8)u(t-2)$$
$$+(8t - 40 - 24 + 8t)u(t-4) - (8t-40)u(t-5)$$
$$= 8(t-1)u(t-1) - 16(t-2)u(t-2)$$
$$+16(t-4)u(t-4) - 8(t-5)u(t-5)$$

$$\therefore \quad F(s) = \frac{8[e^{-s} - 2e^{-2s} + 2e^{-4s} - e^{-5s}]}{s^2}$$

P 12.22 $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st}\,dt$

Let $u = at,\qquad du = a\,dt,\qquad u = 0^-\quad\text{when}\quad t = 0^-$

and $u = \infty\quad\text{when}\quad t = \infty$

Therefore $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(u)e^{-(u/a)s}\frac{du}{a} = \frac{1}{a}F(s/a)$

P 12.23 [a] $f_1(t) = e^{-at}\sin\omega t;\qquad F_1(s) = \frac{\omega}{(s+a)^2+\omega^2}$

$$F(s) = sF_1(s) - f_1(0^-) = \frac{s\omega}{(s+a)^2+\omega^2} - 0$$

[b] $f_1(t) = e^{-at}\cos\omega t;\qquad F_1(s) = \frac{s+a}{(s+a)^2+\omega^2}$

$$F(s) = \frac{F_1(s)}{s} = \frac{s+a}{s[(s+a)^2+\omega^2]}$$

[c] $\frac{d}{dt}[e^{-at}\sin\omega t] = \omega e^{-at}\cos\omega t - ae^{-at}\sin\omega t$

Therefore $F(s) = \frac{\omega(s+a)-\omega a}{(s+a)^2+\omega^2} = \frac{\omega s}{(s+a)^2+\omega^2}$

$$\int_{0^-}^{t} e^{-ax}\cos\omega x\,dx = \frac{-ae^{-at}\cos\omega t + \omega e^{-at}\sin\omega t + a}{a^2+\omega^2}$$

Therefore

$$F(s) = \frac{1}{a^2+\omega^2}\left[\frac{-a(s+a)}{(s+a)^2+\omega^2} + \frac{\omega^2}{(s+a)^2+\omega^2} + \frac{a}{s}\right]$$

$$= \frac{s+a}{s[(s+a)^2+\omega^2]}$$

P 12.24 [a] $\frac{dF(s)}{ds} = \frac{d}{ds}\left[\int_{0^-}^{\infty} f(t)e^{-st}\,dt\right] = -\int_{0^-}^{\infty} tf(t)e^{-st}\,dt$

Therefore $\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$

[b] $\frac{d^2F(s)}{ds^2} = \int_{0^-}^{\infty} t^2f(t)e^{-st}\,dt;\qquad \frac{d^3F(s)}{ds^3} = \int_{0^-}^{\infty} -t^3f(t)e^{-st}\,dt$

Therefore $\frac{d^nF(s)}{ds^n} = (-1)^n\int_{0^-}^{\infty} t^nf(t)e^{-st}\,dt = (-1)^n\mathcal{L}\{t^nf(t)\}$

[c] $\mathcal{L}\{t^5\} = \mathcal{L}\{t^4 t\} = (-1)^4 \dfrac{d^4}{ds^4}\left(\dfrac{1}{s^2}\right) = \dfrac{120}{s^6}$

$$\mathcal{L}\{t\sin\beta t\} = (-1)^1 \frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2\beta s}{(s^2+\beta^2)^2}$$

$\mathcal{L}\{te^{-t}\cosh t\}$:

From Assessment Problem 12.1(a),

$$F(s) = \mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}$$

$$\frac{dF}{ds} = \frac{(s^2-1)1 - s(2s)}{(s^2-1)^2} = -\frac{s^2+1}{(s^2-1)^2}$$

Therefore $\quad -\dfrac{dF}{ds} = \dfrac{s^2+1}{(s^2-1)^2}$

Thus

$$\mathcal{L}\{t\cosh t\} = \frac{s^2+1}{(s^2-1)^2}$$

$$\mathcal{L}\{e^{-t}t\cosh t\} = \frac{(s+1)^2+1}{[(s+1)^2-1]^2} = \frac{s^2+2s+2}{s^2(s+2)^2}$$

P 12.25 [a] $\displaystyle\int_s^\infty F(u)du = \int_s^\infty \left[\int_{0^-}^\infty f(t)e^{-ut}\,dt\right] du = \int_{0^-}^\infty \left[\int_s^\infty f(t)e^{-ut}\,du\right] dt$

$$= \int_{0^-}^\infty f(t)\int_s^\infty e^{-ut}\,du\,dt = \int_{0^-}^\infty f(t)\left[\frac{e^{-tu}}{-t}\Big|_s^\infty\right] dt$$

$$= \int_{0^-}^\infty f(t)\left[\frac{-e^{-st}}{-t}\right] dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

[b] $\mathcal{L}\{t\sin\beta t\} = \dfrac{2\beta s}{(s^2+\beta^2)^2}$

therefore $\quad \mathcal{L}\left\{\dfrac{t\sin\beta t}{t}\right\} = \displaystyle\int_s^\infty \left[\frac{2\beta u}{(u^2+\beta^2)^2}\right] du$

Let $\omega = u^2+\beta^2$, then $\omega = s^2+\beta^2$ when $u = s$, and $\omega = \infty$ when $u = \infty$; also $d\omega = 2u\,du$, thus

$$\mathcal{L}\left\{\frac{t\sin\beta t}{t}\right\} = \beta\int_{s^2+\beta^2}^\infty \left[\frac{d\omega}{\omega^2}\right] = \beta\left(\frac{-1}{\omega}\right)\Big|_{s^2+\beta^2}^\infty = \frac{\beta}{s^2+\beta^2}$$

P 12.26 $i_g(t) = 5\cos 10tu(t);$ so $I_g(s) = \dfrac{5s^2}{s^2+100}$

$$\frac{1}{RC} = 40; \qquad \frac{1}{LC} = 64; \qquad \frac{1}{C} = 40$$

Therefore $V = \dfrac{(40)(5)s^2}{(s^2+40s+64)(s^2+100)} = \dfrac{200s^2}{(s^2+40s+64)(s^2+100)}$

P 12.27 [a] $\dfrac{v_o - V_{\rm dc}}{R} + \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx + C\frac{dv_o}{dt} = 0$

$$\therefore \quad v_o + \frac{R}{L}\int_0^t v_o\,dx + RC\frac{dv_o}{dt} = V_{\rm dc}$$

[b] $V_o + \dfrac{R}{L}\dfrac{V_o}{s} + RCSV_o = \dfrac{V_{\rm dc}}{s}$

$$\therefore \quad sLV_o + RV_o + RCLs^2V_o = LV_{\rm dc}$$

$$\therefore \quad V_o(s) = \frac{(1/RC)V_{\rm dc}}{s^2 + (1/RC)s + (1/LC)}$$

[c] $i_o = \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx$

$$I_o(s) = \frac{V_o}{sL} = \frac{(1/RCL)V_{\rm dc}}{s[s^2 + (1/RC)s + (1/LC)]}$$

P 12.28 [a] $I_{\rm dc} = \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx + \frac{v_o}{R} + C\frac{dv_o}{dt}$

[b] $\dfrac{I_{\rm dc}}{s} = \dfrac{V_o(s)}{sL} + \dfrac{V_o(s)}{R} + sCV_o(s)$

$$\therefore \quad V_o(s) = \frac{I_{\rm dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

[c] $i_o = C\dfrac{dv_o}{dt}$

$$\therefore \quad I_o(s) = sCV_o(s) = \frac{sI_{\rm dc}}{s^2 + (1/RC)s + (1/LC)}$$

P 12.29 [a] $\dfrac{1}{L}\displaystyle\int_0^t v_1\,d\tau + \frac{v_1 - v_2}{R} = i_g$

and

$$C\frac{dv_2}{dt} + \frac{v_2}{R} - \frac{v_1}{R} = 0$$

[b] $$\frac{V_1}{sL} + \frac{V_1 - V_2}{R} = I_g$$

$$\frac{V_2 - V_1}{R} + sCV_2 = 0$$

or

$$(R + sL)V_1(s) - sLV_2(s) = RLsI_g(s)$$

$$-V_1(s) + (RCs + 1)V_2(s) = 0$$

Solving,

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

P 12.30 [a] For $t \geq 0^+$:

$$\frac{v_o}{R} + C\frac{dv_o}{dt} + i_o = 0$$

$$v_o = L\frac{di_o}{dt}; \qquad \frac{dv_o}{dt} = L\frac{d^2 i_o}{dt^2}$$

$$\therefore \quad \frac{L}{R}\frac{di_o}{dt} + LC\frac{d^2 i_o}{dt^2} + i_o = 0$$

$$\text{or} \quad \frac{d^2 i_o}{dt^2} + \frac{1}{RC}\frac{di_o}{dt} + \frac{1}{LC}i_o = 0$$

[b] $$s^2 I_o(s) - sI_{\text{dc}} - 0 + \frac{1}{RC}[sI_o(s) - I_{\text{dc}}] + \frac{1}{LC}I_o(s) = 0$$

$$I_o(s)\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right] = I_{\text{dc}}(s + 1/RC)$$

$$I_o(s) = \frac{I_{\text{dc}}[s + (1/RC)]}{[s^2 + (1/RC)s + (1/LC)]}$$

P 12.31 [a] For $t \geq 0^+$:

$$Ri_o + L\frac{di_o}{dt} + v_o = 0$$

$$i_o = C\frac{dv_o}{dt} \qquad \frac{di_o}{dt} = C\frac{d^2 v_o}{dt^2}$$

$$\therefore \quad RC\frac{dv_o}{dt} + LC\frac{d^2 v_o}{dt^2} + v_o = 0$$

or

$$\frac{d^2 v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC}v_o = 0$$

[b] $s^2V_o(s) - sV_{\rm dc} - 0 + \frac{R}{L}[sV_o(s) - V_{\rm dc}] + \frac{1}{LC}V_o(s) = 0$

$$V_o(s)\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right] = V_{\rm dc}(s + R/L)$$

$$V_o(s) = \frac{V_{\rm dc}[s + (R/L)]}{[s^2 + (R/L)s + (1/LC)]}$$

P 12.32 [a] $300 = 60i_1 + 25\frac{di_1}{dt} + 10\frac{d}{dt}(i_2 - i_1) + 5\frac{d}{dt}(i_1 - i_2) - 10\frac{di_1}{dt}$

$$0 = 5\frac{d}{dt}(i_2 - i_1) + 10\frac{di_1}{dt} + 40i_2$$

Simplifying the above equations gives:

$$300 = 60i_1 + 10\frac{di_1}{dt} + 5\frac{di_2}{dt}$$

$$0 = 40i_2 + 5\frac{di_1}{dt} + 5\frac{di_2}{dt}$$

[b] $\frac{300}{s} = (10s + 60)I_1(s) + 5sI_2(s)$

$$0 = 5sI_1(s) + (5s + 40)I_2(s)$$

[c] Solving the equations in (b),

$$I_1(s) = \frac{60(s + 8)}{s(s + 4)(s + 24)}$$

$$I_2(s) = \frac{-60}{(s + 4)(s + 24)}$$

P 12.33 $V(s) = \frac{200s^2}{(s^2 + 40s + 64)(s^2 + 100)}$

$s^2 + 40s + 64 = (s + 38.33)(s + 1.67);\qquad s^2 + 100 = (s - j10)(s + j10)$

Therefore

$$V(s) = \frac{200s^2}{(s + 38.33)(s + 1.67)(s - j10)(s + j10)}$$

$$= \frac{K_1}{s + 1.67} + \frac{K_2}{s + 38.33} + \frac{K_3}{s - j10} + \frac{K_3^*}{s + j10}$$

$$K_1 = \left.\frac{200s^2}{(s + 38.33)(s^2 + 100)}\right|_{s=-1.67} = 0.15$$

$$K_2 = \left.\frac{200s^2}{(s+1.67)(s^2+100)}\right|_{s=-38.33} = -5.11$$

$$K_3 = \left.\frac{200s^2}{(s+1.67)(s+38.33)(s+j10)}\right|_{s=j10} = 2.49\underline{/-5.14^\circ}$$

Therefore

$$v(t) = [4.98\cos(10t - 5.14^\circ) + 0.15e^{-1.67t} - 5.11e^{-38.33t}]u(t)\text{ V}$$

P 12.34 [a] $\dfrac{1}{LC} = \dfrac{10^9}{(0.8)(100)} = 1250 \times 10^4$

$$\frac{1}{RC} = \frac{10^6}{(10)(100)} = 1000$$

$$V_o(s) = \frac{70{,}000}{(s^2 + 1000s + 1250 \times 10^4)}$$

$$s_{1,2} = -500 \pm \sqrt{25 \times 10^4 - 1250 \times 10^4} = -500 \pm j3500\text{ rad/s}$$

$$V_o(s) = \frac{70{,}000}{(s+500-j3500)(s+500+j3500)}$$

$$= \frac{K}{s+500-j3500} + \frac{K^*}{s+500+j3500}$$

$$K = \frac{70{,}000}{(j7000)} = 10\underline{/-90^\circ}$$

$$\mathrm{V}_o(s) = \frac{10\underline{/-90^\circ}}{s+500-j3500} + \frac{10\underline{/90^\circ}}{s+500+j3500}$$

$$v_o(t) = [20e^{-500t}\cos(3500t - 90^\circ)]u(t)\text{ V} = [20e^{-500t}\sin 3500t]u(t)\text{ V}$$

[b] $I_o(s) = \dfrac{87{,}500}{s(s+500-j3500)(s+500+j3500)}$

$$= \frac{K_1}{s} + \frac{K_2}{s+500-j3500} + \frac{K_2^*}{s+500+j3500}$$

$$K_1 = \frac{87{,}500}{1250 \times 10^4} = 7\text{ mA}$$

$$K_2 = \frac{87{,}500}{(-500+j3500)(j7000)} = 3.5\underline{/171.87^\circ}\text{ mA}$$

$$i_o(t) = [7 + 7e^{-500t}\cos(3500t + 171.87^\circ)]u(t)\text{ mA}$$

P 12.35 [a] $$\frac{1}{RC} = \frac{10^9}{(4\times 10^3)(25)} = 10^4$$

$$\frac{1}{LC} = \frac{10^9}{(2.5)(25)} = 16\times 10^6$$

$$V_o(s) = \frac{40\times 10^6 I_{\text{dc}}}{s + 10{,}000s + 16\times 10^6}$$

$$= \frac{40\times 10^6 I_{\text{dc}}}{(s+2000)(s+8000)}$$

$$= \frac{120{,}000}{(s+2000)(s+8000)}$$

$$= \frac{K_1}{s+2000} + \frac{K_2}{s+8000}$$

$$K_1 = \frac{120{,}000}{6000} = 20; \qquad K_2 = \frac{120{,}000}{-6000} = -20$$

$$V_o(s) = \frac{20}{s+2000} - \frac{20}{s+8000}$$

$$v_o(t) = [20e^{-2000t} - 20e^{-8000t}]u(t)\ \text{V}$$

[b] $$I_o(s) = \frac{3\times 10^{-3}s}{(s+2000)(s+8000)}$$

$$= \frac{K_1}{s+2000} + \frac{K_2}{s+8000}$$

$$K_1 = \frac{-(3\times 10^{-3})(2000)}{6000} = -10^{-3}$$

$$K_2 = \frac{(3\times 10^{-3})(-8000)}{-6000} = 4\times 10^{-3}$$

$$I_o(s) = \frac{-10^{-3}}{s+2000} + \frac{4\times 10^{-3}}{s+8000}$$

$$i_o(t) = (4e^{-8000t} - e^{-2000t})u(t)\ \text{mA}$$

[c] $i_o(0) = 4 - 1 = 3\,\text{mA}$

Yes. The initial inductor current is zero by hypothesis, the initial resistor current is zero because the initial capacitor voltage is zero by hypothesis. Thus at $t = 0$ the source current appears in the capacitor.

P 12.36 $\frac{1}{C} = 2 \times 10^6; \qquad \frac{1}{LC} = 4 \times 10^6; \qquad \frac{R}{L} = 5000; \qquad I_g = \frac{0.015}{s}$

$$V_2(s) = \frac{30{,}000}{s^2 + 5000s + 4 \times 10^6}$$

$$s_1 = -1000; \qquad s_2 = -4000$$

$$V_2(s) = \frac{30{,}000}{(s+1000)(s+4000)}$$

$$= \frac{10}{s+1000} - \frac{10}{s+4000}$$

$$v_2(t) = [10e^{-1000t} - 10e^{-4000t}]u(t)\text{ V}$$

P 12.37 $\frac{1}{RC} = 10{,}000; \qquad \frac{1}{LC} = 16 \times 10^6$

$$I_o(s) = \frac{0.1(s+10{,}000)}{s^2 + 10{,}000s + 16 \times 10^6}$$

$$s_1 = -2000; \qquad s_2 = -8000$$

$$I_o(s) = \frac{0.1(s+10{,}000)}{(s+2000)(s+8000)} = \frac{K_1}{s+2000} + \frac{K_2}{s+8000}$$

$$K_1 = \frac{0.1(8000)}{6000} = 0.133$$

$$K_2 = \frac{0.1(2000)}{-6000} = -0.033$$

$$I_o(s) = \frac{0.133}{s+2000} - \frac{0.033}{s+8000}$$

$$i_o(t) = [133.33e^{-2000t} - 33.33e^{-8000t}]u(t)\text{ mA}$$

P 12.38 $\frac{R}{L} = 5000; \qquad \frac{1}{LC} = 4 \times 10^6$

$$V_o(s) = \frac{15(s+5000)}{s^2 + 5000s + 4 \times 10^6}$$

$$s_{1,2} = -2500 \pm \sqrt{6.25 \times 10^6 - 4 \times 10^6}$$

$s_1 = -1000$ rad/s; $\quad s_2 = -4000$ rad/s

$$V_o(s) = \frac{15(s+5000)}{(s+1000)(s+4000)} = \frac{K_1}{s+1000} + \frac{K_2}{s+4000}$$

$$K_1 = \frac{15(4000)}{3000} = 20\,\text{V}; \qquad K_2 = \frac{15(1000)}{-3000} = -5\,\text{V}$$

$$V_o(s) = \frac{20}{s+1000} - \frac{5}{s+4000}$$

$$v_o(t) = [20e^{-1000t} - 5e^{-4000t}]u(t)\,\text{V}$$

P 12.39 [a] $I_1(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+4} + \dfrac{K_3}{s+24}$

$$K_1 = \frac{(60)(8)}{(4)(24)} = 5; \qquad K_2 = \frac{(60)(4)}{(-4)(20)} = -3$$

$$K_3 = \frac{(60)(-16)}{(-24)(-20)} = -2$$

$$I_1(s) = \left(\frac{5}{s} - \frac{3}{s+4} - \frac{2}{s+24}\right)$$

$$i_1(t) = (5 - 3e^{-4t} - 2e^{-24t})u(t)\,\text{A}$$

$$I_2(s) = \frac{K_1}{s+4} + \frac{K_2}{s+24}$$

$$K_1 = \frac{-60}{20} = -3; \qquad K_2 = \frac{-60}{-20} = 3$$

$$I_2(s) = \left(\frac{-3}{s+4} + \frac{3}{s+24}\right)$$

$$i_2(t) = (3e^{-24t} - 3e^{-4t})u(t)\,\text{A}$$

[b] $i_1(\infty) = 5\,\text{A}; \qquad i_2(\infty) = 0\,\text{A}$

[c] Yes, at $t = \infty$

$$i_1 = \frac{300}{60} = 5\,\text{A}$$

Since i_1 is a dc current at $t = \infty$ there is no voltage induced in the 10 H inductor; hence, $i_2 = 0$. Also note that $i_1(0) = 0$ and $i_2(0) = 0$. Thus our solutions satisfy the condition of no initial energy stored in the circuit.

P 12.40 [a] $F(s) = \dfrac{K_1}{s+1} + \dfrac{K_2}{s+2} + \dfrac{K_3}{s+3}$

$$K_1 = \frac{18 - 66 + 54}{(1)(2)} = 3; \qquad K_2 = \frac{72 - 132 + 54}{(-1)(1)} = 6$$

$$K_3 = \frac{162 - 198 + 54}{(-2)(-1)} = 9$$

$$\therefore \quad f(t) = [3e^{-t} + 6e^{-2t} + 9e^{-3t}]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+2} + \dfrac{K_3}{s+3} + \dfrac{K_4}{s+5}$

$$K_1 = \left.\frac{8s^3 + 89s^2 + 311s + 300}{(s+2)(s+3)(s+5)}\right|_{s=0} = 10$$

$$K_2 = \left.\frac{8s^3 + 89s^2 + 311s + 300}{s(s+3)(s+5)}\right|_{s=-2} = 5$$

$$K_3 = \left.\frac{8s^3 + 89s^2 + 311s + 300}{s(s+2)(s+5)}\right|_{s=-3} = -8$$

$$K_4 = \left.\frac{8s^3 + 89s^2 + 311s + 300}{s(s+2)(s+3)}\right|_{s=-5} = 1$$

$$f(t) = [10 + 5e^{-2t} - 8e^{-3t} + e^{-5t}]u(t)$$

[c] $s_{1,2} = -6 \pm \sqrt{36 - 100} = -6 \pm j8$

$$F(s) = \frac{11s^2 + 172s + 700}{(s+2)(s+6-j8)(s+6+j8)}$$

$$= \frac{K_1}{s+2} + \frac{K_2}{s+6-j8} + \frac{K_2^*}{s+6+j8}$$

$$K_1 = \frac{44 - 344 + 700}{4 - 24 + 100} = 5$$

$$K_2 = \frac{11(-6+j8)^2 + 172(-6+j8) + 700}{(-4+j8)j16}$$

$$= 3 - j4 = 5\underline{/-53.13^\circ}$$

$$\therefore \quad f(t) = [5e^{-2t} + 10e^{-6t}\cos(8t - 53.13^\circ)]u(t)$$

[d] $s_{1,2} = -7 \pm \sqrt{49-625} = -7 \pm j24$

$$F(s) = \frac{56s^2 + 112s + 5000}{s(s+7-j24)(s+7+j24)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+7-j24} + \frac{K_2^*}{s+7+j24}$$

$$K_1 = \frac{5000}{625} = 8$$

$$K_2 = \frac{56(-7+j24)^2 + 112(-7+j24) + 5000}{(-7+j24)j48}$$

$$= 24 + j7 = 25\underline{/16.26^\circ}$$

$$\therefore \quad f(t) = [8 + 50e^{-7t}\cos(24t + 16.26^\circ)]u(t)$$

P 12.41 [a] $F(s) = \dfrac{K_1}{s^2} + \dfrac{K_2}{s} + \dfrac{K_3}{s+10}$

$$K_1 = \frac{8(s^2-5s+50)}{s+10}\bigg|_{s=0} = \frac{400}{10} = 40$$

$$K_2 = \frac{d}{ds}\left\{\frac{8(s^2-5s+50)}{s+10}\right\}\bigg|_{s=0}$$

$$= \frac{8(s+10)(2s-5) - 8(s^2-5s+50)(1)}{(s+10)^2}\bigg|_{s=0}$$

$$= \frac{10(-40) - 8(50)}{100} = -8$$

$$K_3 = \frac{8(s^2-5s+50)}{s^2}\bigg|_{s=-10} = \frac{8(100+50+50)}{100} = 16$$

$$F(s) = \frac{40}{s^2} - \frac{8}{s} + \frac{16}{s+10}$$

$$f(t) = [40t - 8 + 16e^{-10t}]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{(s+2)^2} + \dfrac{K_3}{s+2}$

$$K_1 = \frac{10(4)}{4} = 10; \qquad K_2 = \frac{10(12-8+4)}{-2} = -40$$

$$K_3 = \frac{d}{ds}\left\{\frac{10(3s^2+4s+4)}{s}\right\}\bigg|_{s=-2}$$

$$= \frac{10[(s)(6s+4) - (3s^2+4s+4)]}{s^2}\bigg|_{s=-2} = 20$$

$$F(s) = \frac{10}{s} - \frac{40}{(s+2)^2} + \frac{20}{s+2}$$

$$f(t) = [10 - 40te^{-2t} + 20e^{-2t}]u(t)$$

[c] $s_{1,2} = -2 \pm \sqrt{4-5} = -2 \pm j1$

$$F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+2-j1} + \frac{K_3^*}{s+2+j1}$$

$$K_1 = \frac{50}{5} = 10$$

$$K_2 = \frac{d}{ds}\left\{\frac{s^3 - 6s^2 + 15s + 50}{s^2 + 4s + 5}\right\}\Bigg|_{s=0}$$

$$= \frac{(s^2+4s+5)(3s^2-12s+15) - (s^3-6s^2+15s+50)(2s+4)}{(s^2+4s+5)^2}\Bigg|_{s=0}$$

$$= \frac{5(15) - 50(4)}{25} = -5$$

$$K_3 = \frac{s^3 - 6s^2 + 15s + 50}{s^2(s+2+j1)}\Bigg|_{s=-2+j1}$$

$$(-2+j1)^3 = -2 + j11; \qquad (-2+j1)^2 = 3 - j4$$

$$K_3 = \frac{-2 + j11 - 6(3-j4) + 15(-2+j1) + 50}{(3-j4)(j2)}$$

$$= 3 + j4 = 5\underline{/53.13^\circ}$$

$$F(s) = \frac{10}{s^2} - \frac{5}{s} + \frac{5\underline{/53.13^\circ}}{s+2-j1} + \frac{5\underline{/-53.13^\circ}}{s+2+j1}$$

$$f(t) = [10t - 5 + 10e^{-2t}\cos(t + 53.13^\circ)]u(t)$$

[d] $F(s) = \dfrac{K_1}{(s+2)^3} + \dfrac{K_2}{(s+2)^2} + \dfrac{K_3}{s+2}$

$$K_1 = s^2 + 6s + 5\Big|_{s=-2} = -3$$

$$K_2 = \frac{d}{ds}\{s^2 + 6s + 5\}\Big|_{s=-2} = 2s + 6\Big|_{s=-2} = 2$$

$$2K_3 = \frac{d}{ds}(2s+6)\Big|_{s=-2} = 2; \qquad K_3 = 1$$

$$F(s) = \frac{-3}{(s+2)^3} + \frac{2}{(s+2)^2} + \frac{1}{s+2}$$

$$f(t) = -\frac{3t^2e^{-2t}}{2} + 2te^{-2t} + e^{-2t} = [(2t - 1.5t^2 + 1)e^{-2t}]u(t)$$

[e] $s_{1,2} = -1 \pm \sqrt{1-5} = -1 \pm j2$

$$F(s) = \frac{K_1}{(s+1-j2)^2} + \frac{K_1^*}{(s+1+j2)^2} + \frac{K_2}{s+1-j2} + \frac{K_2^*}{(s+1+j2)}$$

$$K_1 = \frac{16s^3 + 72s^2 + 216s - 128}{(s+1+j2)^2}\bigg|_{s=-1+j2}$$

$$(-1+j2)^3 = 11 - j2; \qquad (-1+j2)^2 = -3 - j4$$

$$K_1 = \frac{176 - j32 - 216 - j288 - 216 + j432 - 128}{-16}$$

$$= 24 - j7 = 25\underline{/-16.26^\circ}$$

$$K_2 = \frac{d}{ds}\left\{\frac{16s^3 + 72s^2 + 216s - 128}{(s+1+j2)^2}\bigg|_{s=-1+j2}\right\}$$

$$= \frac{(s+1+j2)^2(48s^2 + 144s + 216)}{(s+1+j2)^4}\bigg|_{s=-1+j2}$$

$$- \frac{(16s^3 + 72s^2 + 216s - 128)2(s+1+j2)}{(s+1+j2)^4}\bigg|_{s=-1+j2}$$

$$= \frac{(j4)^2(-144 - j192 - 144 + j288 + 216) - (-384 + j112)(j8)}{(j4)^4}$$

$$= \frac{2048 + j1536}{256} = 8 + j6 = 10\underline{/36.87^\circ}$$

$$F(s) = \frac{25\underline{/-16.26^\circ}}{(s+1-j2)^2} + \frac{25\underline{/16.26^\circ}}{(s+1+j2)^2} + \frac{10\underline{/36.87^\circ}}{s+1-j2} + \frac{10\underline{/-36.87^\circ}}{s+1+j2}$$

$$f(t) = [50te^{-t}\cos(2t - 16.26^\circ) + 20e^{-t}\cos(2t + 36.87)]u(t)$$

P 12.42 [a]

$$F(s) = \begin{array}{r|l} & \quad\quad 10 \\ \hline s^2 + 6s + 5 & 10s^2 + 85s + 95 \\ & 10s^2 + 60s + 50 \\ \hline & \quad\quad 25s + 45 \end{array}$$

$$F(s) = 10 + \frac{25s + 45}{s^2 + 6s + 5} = 10 + \frac{K_1}{s+1} + \frac{K_2}{s+5}$$

$$K_1 = \frac{25s + 45}{s+5}\bigg|_{s=-1} = 5$$

$$K_2 = \frac{25s + 45}{s+1}\bigg|_{s=-5} = 20$$

$$F(s) = 10 + \frac{5}{s+1} + \frac{20}{s+5}$$

$$f(t) = 10\delta(t) + [5e^{-t} + 20e^{-5t}]u(t)$$

[b]

$$F(s) = \begin{array}{r|l} & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; 5 \\ \cline{2-2} s^2+4s+5 & 5s^2+40s+25 \\ & 5s^2+20s+25 \\ \cline{2-2} & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\; 20s \end{array}$$

$$F(s) = 5 + \frac{20s}{s^2+4s+5} = 5 + \frac{K_1}{s+2-j} + \frac{K_1^*}{s+2+j}$$

$$K_1 = \left.\frac{20s}{s+2+j}\right|_{s=-2+j} = 10 + j20 = 22.36\underline{/63.43^\circ}$$

$$F(s) = 5 + \frac{22.36\underline{/63.43^\circ}}{s+2-j} + \frac{22.36\underline{/-63.43^\circ}}{s+2+j}$$

$$f(t) = 5\delta(t) + 44.72e^{-2t}\cos(t + 63.43^\circ)u(t)$$

[c]

$$F(s) = \begin{array}{r|l} & \;\;\;\;\;\;\;\;\;\;\;\;\; s+5 \\ \cline{2-2} s+20 & s^2+25s+150 \\ & s^2+20s \\ \cline{2-2} & \;\;\;\;\;\; 5s+150 \\ & \;\;\;\;\;\; 5s+100 \\ \cline{2-2} & \;\;\;\;\;\;\;\;\;\;\;\;\; 50 \end{array}$$

$$F(s) = s + 5 + \frac{50}{(s+20)} = s + 5 + \frac{50}{s+20}$$

$$f(t) = \delta'(t) + 5\delta(t) + 50e^{-20t}u(t)$$

P 12.43 [a] $F(s) = \dfrac{K_1}{s^2} + \dfrac{K_2}{s} + \dfrac{K_3}{s+1-j2} + \dfrac{K_3^*}{s+1+j2}$

$$K_1 = \left.\frac{100(s+1)}{s^2+2s+5}\right|_{s=0} = 20$$

$$K_2 = \frac{d}{ds}\left[\frac{100(s+1)}{s^2+2s+5}\right] = \left[\frac{100}{s^2+2s+5} - \frac{100(s+1)(2s+2)}{(s^2+2s+5)^2}\right]_{s=0}$$

$$= 20 - 8 = 12$$

$$K_3 = \left.\frac{100(s+1)}{s^2(s+1+j2)}\right|_{s=-1+j2} = -6 + j8 = 10\underline{/126.87^\circ}$$

$$f(t) = [20t + 12 + 20e^{-t}\cos(2t + 126.87^\circ)]u(t)$$

[b] $F(s) = \dfrac{20s^2}{(s+1)^3} = \dfrac{K_1}{(s+1)^3} + \dfrac{K_2}{(s+1)^2} + \dfrac{K_3}{s+1}$

$\therefore \quad 20s^2 = K_1 + K_2(s+1) + K_3(s+1)^2$

$K_1 = 20s^2 \Big|_{s=-1} = 20$

After differentiating each side

$40s = 0 + K_2 + 2K_3(s+1); \qquad \therefore \quad K_2 = 40s \Big|_{s=-1} = -40$

After differentiating again

$40 = 0 + 2K_3; \qquad \therefore \quad K_3 = 20$

$\therefore \quad \dfrac{20s^2}{(s+1)^3} = \dfrac{20}{(s+1)^3} - \dfrac{40}{(s+1)^2} + \dfrac{20}{s+1}$

Test at $s = 0$:

$0 = 20 - 40 + 20 = 0 \qquad \text{OK}$

$f(t) = \dfrac{20t^2e^{-t}}{2!} - 40te^{-t} + 20e^{-t} = (10t^2 - 40t + 20)e^{-t}u(t)$

[c] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{(s+1)^3} + \dfrac{K_3}{(s+1)^2} + \dfrac{K_4}{s+1}$

$K_1 = \dfrac{40(s+2)}{(s+1)^3} \Big|_{s=0} = 80$

$K_2 = \dfrac{40(s+2)}{s} \Big|_{s=-1} = -40$

$K_3 = \dfrac{d}{ds}\left[\dfrac{40(s+2)}{s}\right] = \left[\dfrac{40}{s} - \dfrac{40(s+2)}{s^2}\right]_{s=-1} = -40 - 40 = -80$

$K_4 = \dfrac{1}{2}\dfrac{d}{ds}\left[\dfrac{40}{s} - \dfrac{40(s+2)}{s^2}\right]$

$= \dfrac{1}{2}\left[\dfrac{-40}{s^2} - \dfrac{40}{s^2} + \dfrac{80(s+2)}{s^3}\right]_{s=-1} = \dfrac{1}{2}(-40 - 40 - 80) = -80$

$f(t) = [80 - 20t^2e^{-t} - 80te^{-t} - 80e^{-t}]u(t)$

[d] $F(s) = \dfrac{5(s+2)^2}{s^4(s+1)} = \dfrac{K_1}{s+1} + \dfrac{K_2}{s^4} + \dfrac{K_3}{s^3} + \dfrac{K_4}{s^2} + \dfrac{K_5}{s}$

$K_1 = \dfrac{5(s+2)^2}{s^4} \Big|_{s=-1} = 5; \qquad K_2 = \dfrac{5(s+2)^2}{s+1} \Big|_{s=0} = 20$

$$\frac{5(s+2)^2}{s+1} = \frac{K_1 s^4}{s+1} + K_2 + K_3 s + K_4 s^2 + K_5 s^3$$

Differentiating each side gives

$$5\left[\frac{(s+1)2(s+2)-(s+2)^2}{(s+1)^2}\right] = \frac{K_1[4s^3(s+1)-s^4]}{(s+1)^2}$$

$$+0 + K_3 + 2K_4 s + 3K_5 s^2$$

$$\frac{5s(s+2)}{(s+1)^2} = \frac{K_1 s^3(3s+4)}{(s+1)^2} + K_3 + 2K_4 s + 3K_5 s^2$$

$$K_3 = \frac{5s(s+2)}{(s+1)^2}\bigg|_{s=0} = 0$$

Note that two more derivatives of the term involving K_1 will drop out at $s=0$. Hence,

$$2K_4 = 5\frac{d}{ds}\left[\frac{s(s+2)}{(s+1)^2}\right]_{s=0} - 6K_5 s\bigg|_{s=0}$$

$$2K_4 = 5\left\{\frac{(s+1)^2(2s+2)-s(s+2)2(s+1)}{(s+1)^4}\right\}\bigg|_{s=0}$$

$$= 5(s+1)\frac{2(s+1)^2-2s(s+2)}{(s+1)^4}\bigg|_{s=0}$$

$$= (5)\frac{2}{(s+1)^3}\bigg|_{s=0} = 10$$

$$\therefore\ K_4 = 5$$

Now differentiate once more to get

$$6K_5 = \frac{d}{ds}\left\{\frac{10}{(s+1)^3}\right\}\bigg|_{s=0}$$

$$= \frac{-30(s+1)^2}{(s+1)^6}\bigg|_{s=0}$$

$$= \frac{-30}{(s+1)^4}\bigg|_{s=0} = -30$$

$$\therefore\ K_5 = -5$$

$$\frac{5(s+2)^2}{s^4(s+1)} = \frac{5}{s+1} + \frac{20}{s^4} + \frac{0}{s^3} + \frac{5}{s^2} - \frac{5}{s}$$

$$= \frac{5}{s+1} + \frac{20}{s^4} + \frac{5}{s^2} - \frac{5}{s}$$

Test at $s = -2$:

$$0 = -5 + \frac{20}{16} + \frac{5}{4} + \frac{5}{2} = 0 \qquad \text{OK}$$

$$\therefore \quad F(s) = \frac{5}{s+1} + \frac{20}{s^4} + \frac{5}{s^2} - \frac{5}{s}$$

$$f(t) = 5e^{-t} + \frac{20t^3}{3!} + 5t - 5$$

$$= (5e^{5t} + \tfrac{10}{3}t^3 + 5t - 5)u(t)$$

P 12.44 $f(t) = \mathcal{L}^{-1}\left\{\dfrac{K}{s+\alpha-j\beta} + \dfrac{K^*}{s+\alpha+j\beta}\right\}$

$$= Ke^{-\alpha t}e^{j\beta t} + K^*e^{-\alpha t}e^{-j\beta t}$$

$$= |K|e^{-\alpha t}[e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}]$$

$$= |K|e^{-\alpha t}[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}]$$

$$= 2|K|e^{-\alpha t}\cos(\beta t + \theta)$$

P 12.45 [a] $\mathcal{L}\{t^n f(t)\} = (-1)^n \left[\dfrac{d^n F(s)}{ds^n}\right]$

Let $\quad f(t) = 1, \quad$ then $\quad F(s) = \dfrac{1}{s}, \quad$ thus $\quad \dfrac{d^n F(s)}{ds^n} = \dfrac{(-1)^n n!}{s^{(n+1)}}$

Therefore $\quad \mathcal{L}\{t^n\} = (-1)^n \left[\dfrac{(-1)^n n!}{s^{(n+1)}}\right] = \dfrac{n!}{s^{(n+1)}}$

It follows that $\quad \mathcal{L}\{t^{(r-1)}\} = \dfrac{(r-1)!}{s^r}$

and $\quad \mathcal{L}\{t^{(r-1)}e^{-at}\} = \dfrac{(r-1)!}{(s+a)^r}$

Therefore $\quad \dfrac{K}{(r-1)!}\mathcal{L}\{t^{r-1}e^{-at}\} = \dfrac{K}{(s+a)^r} = \mathcal{L}\left\{\dfrac{Kt^{r-1}e^{-at}}{(r-1)!}\right\}$

[b] $f(t) = \mathcal{L}^{-1}\left\{\dfrac{K}{(s+\alpha-j\beta)^r} + \dfrac{K^*}{(s+\alpha+j\beta)^r}\right\}$

Therefore

$$f(t) = \frac{Kt^{r-1}}{(r-1)!}e^{-(\alpha-j\beta)t} + \frac{K^*t^{r-1}}{(r-1)!}e^{-(\alpha+j\beta)t}$$

$$= \frac{|K|t^{r-1}e^{-\alpha t}}{(r-1)!}\left[e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}\right]$$

$$= \left[\frac{2|K|t^{r-1}e^{-\alpha t}}{(r-1)!}\right]\cos(\beta t + \theta)$$

P 12.46 [a] $\lim_{s\to\infty} sV(s) = \lim_{s\to\infty}\left[\frac{200s^3}{s^4[1+(40/s)+(64/s^2)][1+(100/s^2)]}\right] = 0$

Therefore $v(0^+) = 0$

[b] Yes, all of the poles of V are in the left-half of the complex plane. Therefore,

$$\lim_{s\to 0} sV(s) = \lim_{s\to 0}\left[\frac{200s^3}{(s^2+40s+64)(s^2+100)}\right] = 0$$

Therefore $v(\infty) = 0$

P 12.47 [a] $sF(s) = \frac{18s^3+66s^2+54s}{(s+1)(s+2)(s+3)}$

$\lim_{s\to 0} sF(s) = 0, \quad \therefore \quad f(\infty) = 0$

$\lim_{s\to\infty} sF(s) = 18, \quad \therefore \quad f(0^+) = 18$

[b] $sF(s) = \frac{8s^3+89s^2+311s+300}{(s+2)(s^2+8s+15)}$

$\lim_{s\to 0} sF(s) = 10; \quad \therefore \quad f(\infty) = 10$

$\lim_{s\to\infty} sF(s) = 8, \quad \therefore \quad f(0^+) = 8$

[c] $sF(s) = \frac{11s^3+172s^2+700s}{(s+2)(s^2+12s+100)}$

$\lim_{s\to 0} sF(s) = 0, \quad \therefore \quad f(\infty) = 0$

$\lim_{s\to\infty} sF(s) = 11, \quad \therefore \quad f(0^+) = 11$

[d] $sF(s) = \frac{56s^2+112s+5000}{(s^2+14s+625)}$

$\lim_{s\to 0} sF(s) = \frac{5000}{625} = 8, \quad \therefore \quad f(\infty) = 8$

$\lim_{s\to\infty} sF(s) = 56, \quad \therefore \quad f(0^+) = 56$

P 12.48 [a] $sF(s) = \dfrac{8(s^2 - 5s + 50)}{s(s + 10)}$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem.

$\lim_{s\to\infty} sF(s) = 8, \qquad \therefore\ f(0^+) = 8$

[b] $sF(s) = \dfrac{10(3s^2 + 4s + 4)}{(s + 2)^2}$

$\lim_{s\to 0} sF(s) = \dfrac{40}{4} = 10, \qquad \therefore\ f(\infty) = 10$

$\lim_{s\to\infty} sF(s) = 30, \qquad \therefore\ f(0^+) = 30$

[c] $sF(s) = \dfrac{s^3 - 6s^2 + 15s + 50}{s(s^2 + 4s + 5)}$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem.

$\lim_{s\to\infty} sF(s) = 1, \qquad \therefore\ f(0^+) = 1$

[d] $sF(s) = \dfrac{s^3 + 6s^2 + 5s}{(s + 2)^3}$

$\lim_{s\to 0} sF(s) = 0, \qquad \therefore\ f(\infty) = 0$

$\lim_{s\to\infty} sF(s) = 1, \qquad \therefore\ f(0^+) = 1$

[e] $sF(s) = \dfrac{16s^4 + 72s^3 + 216s^2 - 128s}{(s^2 + 2s + 5)^2}$

$\lim_{s\to 0} sF(s) = 0, \qquad \therefore\ f(\infty) = 0$

$\lim_{s\to\infty} sF(s) = 16, \qquad \therefore\ f(0^+) = 16$

P 12.49 All of the $F(s)$ functions referenced in this problem are improper rational functions, and thus the corresponding $f(t)$ functions contain impulses ($\delta(t)$). Thus, neither the initial value theorem nor the final value theorem may be applied to these $F(s)$ functions!

P 12.50 $sV_o(s) = \dfrac{sV_{\text{dc}}/RC}{s^2 + (1/RC)s + (1/LC)}$

$\lim_{s\to 0} sV_o(s) = 0, \qquad \therefore\ v_o(\infty) = 0$

$\lim_{s\to\infty} sV_o(s) = 0, \qquad \therefore\ v_o(0^+) = 0$

$$sI_o(s) = \frac{V_{dc}/RC)}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sI_o(s) = \frac{V_{dc}/RLC}{1/LC} = \frac{V_{dc}}{R}, \qquad \therefore \; i_o(\infty) = \frac{V_{dc}}{R}$$

$$\lim_{s\to\infty} sI_o(s) = 0, \qquad \therefore \; i_o(0^+) = 0$$

P 12.51 $$sV_o(s) = \frac{(I_{dc}/C)s}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sV_o(s) = 0, \qquad \therefore \; v_o(\infty) = 0$$

$$\lim_{s\to\infty} sV_o(s) = 0, \qquad \therefore \; v_o(0^+) = 0$$

$$sI_o(s) = \frac{s^2 I_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sI_o(s) = 0, \qquad \therefore \; i_o(\infty) = 0$$

$$\lim_{s\to\infty} sI_o(s) = I_{dc}, \qquad \therefore \; v_o(0^+) = I_{dc}$$

P 12.52 $$sI_o(s) = \frac{I_{dc}s[s + (1/RC)]}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sI_o(s) = 0, \qquad \therefore \; i_o(\infty) = 0$$

$$\lim_{s\to\infty} sI_o(s) = I_{dc}, \qquad \therefore \; i_o(0^+) = I_{dc}$$

P 12.53 [a] $$sF(s) = \frac{100(s+1)}{s(s^2 + 2s + 5)}$$

$F(s)$ has a second-order pole at the origin, so we cannot use the final value theorem here.

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \; f(0^+) = 0$$

[b] $$sF(s) = \frac{20s^3}{(s+1)^3}$$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \; f(\infty) = 0$$

$$\lim_{s\to\infty} sF(s) = 20, \qquad \therefore \; f(0^+) = 20$$

[c] $sF(s) = \dfrac{40(s+2)}{(s+1)^3}$

$$\lim_{s\to 0} sF(s) = 80, \qquad \therefore\ f(\infty) = 80$$

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore\ f(0^+) = 0$$

[d] $sF(s) = \dfrac{5s(s+2)^2}{s^4(s+1)} = \dfrac{5(s+2)^2}{s^3(s+1)}$

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore\ f(0^+) = 0$$

The final value theorem cannot be applied here, as $F(s)$ violates that requirement that all poles lie in the left-half plane, with the exception of a single pole at the origin. This $F(s)$ has four poles at the origin!