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GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Advanced Computational Physics Lab

Programming Project 1: Statistical Physics of Disordered Systems

Introductory lectures:
Wednesday, April 10th 2:00 pm & Friday, April 12th 10:00 am

Tutor:
Rafael Díaz: `rafael.diaz@theorie.physik.uni-goettingen.de`

Organization

Most discussions will be held in a Rocket Chat group on `https://chat.gwdg.de` to facilitate communication between the participants and enable timely responses to your questions. Please contact me so that I can add you to the group. I will be around weekly for Q/A session on *Thursdays, 11:00 - 12:30*. You can find me at my office (A.02.106), or we can meet via Zoom if you email beforehand.

Introduction

This project provides an introduction to disordered systems and some techniques for analysing them. You will learn to implement the Cavity Method (also known as Belief Propagation algorithm), which allows you to make inference on networks and ultimately retrieve the relevant average properties of a physical model. The first exercise is on the one–dimensional Ising chain, which acts as an illustrative example to familiarize yourself with the algorithm. The second exercise explores the connection between Random Matrix Theory and graphical models. You will derive the famous Wigner semicircle law by computing the spectral density of a matrix that represents weighted Gaussian interactions in a fully connected graph. The third exercise is on the Anderson model, a simplified model for conduction by electrons in random lattices. The last problem concerns the Bouchaud trap model, which aims to describe the dynamics of glassy systems in a coarse–grained picture. Whereas the first two problems are mandatory, you should choose one among the last two problems to work on. Both involve the same amount of work in terms of computational effort.

Exercise 1: Belief propagation algorithm (Ising 1D chain) (10 points + 5 points for code)

Consider the ferromagnetic 1–dimensional Ising model with coupling constant $J = 1$ in the presence of an external magnetic field h. The system is described by a vector $\vec{\sigma} = (\sigma_1, \cdots, \sigma_N)$ with $\sigma_i \in \{+1, -1\}$. The Hamiltonian of the system is given by

$$H(\vec{\sigma}) = -\sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i \tag{1}$$

and the associated probability distribution in the presence of a thermal bath with inverse temperature $\beta = 1/k_B T$ (canonical ensemble) is

$$P(\vec{\sigma}) = \frac{1}{\mathcal{Z}} \mathrm{e}^{-\beta H(\vec{\sigma})} \tag{2}$$

with the partition sum

$$\mathcal{Z} = \sum_{\sigma_1 = \pm 1, \cdots, \sigma_N = \pm 1} \mathrm{e}^{-\beta H(\vec{\sigma})} \,. \tag{3}$$

Belief Propagation is an algorithm to perform inference on graphs (networks) that aims to estimate the marginal distributions $P(\sigma_i)$ for any node i in the graph. This is relevant in our case, because we are interested in computing statistical properties like the mean magnetization, which is given by

$$\langle \sigma_i \rangle = P(\sigma_i = +1) - P(\sigma_i = -1) \tag{4}$$

In the thermodynamic limit we expect the spins to be indistinguishable and therefore $\langle \sigma_i \rangle$ for any node i in the bulk coincides with the mean magnetization $\langle M \rangle = \frac{1}{N} \sum_{j=1}^{N} \langle \sigma_j \rangle$.

In order to estimate (4) let us start by writing an expression for the marginal $P(\sigma_i)$. If we sum the expression (2) over the spins at all the sites or "nodes" except node i, we arrive at the relation

$$P(\sigma_i) \propto \mathrm{e}^{\beta h \sigma_i} \sum_{\sigma_{i-1} = \pm 1, \sigma_{i+1} = \pm 1} \mathrm{e}^{\beta(\sigma_{i-1}\sigma_i + \sigma_i \sigma_{i+1})} P^{(i)}(\sigma_{i-1}, \sigma_{i+1}) \tag{5}$$

Here $P^{(i)}(\sigma_{i-1}, \sigma_{i+1})$ denotes a joint probability distribution of two spins in a system where spin i has been removed ("cavity") and the proportionality is up to a normalization factor. Due to the structure of our problem (1-D chain) it is plausible to assume that the joint distribution of the neighbours of i factorises, i.e.

$$P^{(i)}(\sigma_{i-1}, \sigma_{i+1}) = P^{(i)}(\sigma_{i-1}) P^{(i)}(\sigma_{i+1}) \tag{6}$$

Substitution of this into eq. (5) leads to

$$P(\sigma_i) \propto \mathrm{e}^{\beta h \sigma_i} \prod_{k \in \partial i} \sum_{\sigma_k = \pm 1} \mathrm{e}^{\beta \sigma_i \sigma_k} P^{(i)}(\sigma_k) \tag{7}$$

with ∂i denoting the neighbouring nodes of i. The distribution $P^{(i)}(\sigma_k)$ is again a cavity distribution, now for a single spin; it corresponds to a probability distribution on a

modified graph where the edge from i to k has been removed. This is also called a *message*, since it carries the information needed to compute the marginal at node i along the edge $k \to i$.

We next express the cavity distributions iteratively in terms of other cavity distributions. For a cavity created at i one expresses the left message (from $i-1$ to i) as

$$P^{(i)}(\sigma_{i-1}) = \mathrm{e}^{\beta h \sigma_{i-1}} \sum_{\sigma_{i-2}=\pm 1} \mathrm{e}^{\beta \sigma_{i-1}\sigma_{i-2}} P^{(i-1)}(\sigma_{i-2}) \tag{8}$$

and similarly for the right message. In our current setting, the distributions are defined over binary variables (the spins) and thus can be parameterized by a single real number. Let us choose for convenience the following representation

$$P^{(j)}(\sigma_k) = \frac{1}{2}(1 + \sigma_k \tanh(\beta \omega_k^{(j)})) \tag{9}$$

which implies that $P^{(j)}(\sigma_k) \propto \mathrm{e}^{\beta \omega_k^{(j)} \sigma_k}$. The cavity parameter $\omega_{i-1}^{(i)}$ is then a rescaled log–likelihood ratio given by

$$\omega_k^{(j)} = \frac{1}{2\beta} \log \left(\frac{P^{(j)}(\sigma_k = +1)}{P^{(j)}(\sigma_k = -1)} \right) \tag{10}$$

Equation (8) with the parametrization (9) becomes an equation for $\omega_{i-1}^{(i)}$. In this way one obtains a recursive relation for the cavity parameters on the whole chain (both left and right oriented).

1. Obtain the equation for the cavity parameter $\omega_{i-1}^{(i)}$ in terms of $\omega_{i-2}^{(i-1)}$.

2. Consider a chain with 50 nodes. Assume a uniform distribution for $P^{(2)}(\sigma_1 = \pm 1) = 1/2$. Fix the numerical values of the parameters (e.g. $\beta = 1/2, h = 1/3$) and solve numerically for all the $\{\omega_{i-1}^{(i)}\}$. Make a plot of them as a function of the cavity position.

3. Consider a different initial cavity distribution, say $P^{(2)}(\sigma_1 = +1) = 3/4$ and repeat the previous exercise with the same parameters. Discuss the role of the initialization in the algorithm.

4. In the thermodynamic limit $N \to \infty$, one expects that the recursive relation found in problem (1) converges to a fixed point, $\omega^*(\beta, h)$, regardless of the initialization at the boundary. Fix $h = 1$, find numerically the fixed point and with this estimate the mean magnetization (Eq. (4)) for different values of β. Interpret your result and compare it with the analytical prediction. **Hint:** you need to normalize expression (7).

Exercise 2: Random matrix theory (15 points + 10 points for code)

Consider an ensemble of random matrices $\mathcal{M}$ of dimension $N \times N$ with elements of the form

$$M_{ij} = A_{ij} J_{ij} = M_{ji} \tag{11}$$

Here $\mathbf{A}$ is the adjacency matrix of a Random Regular Graph (RRG), with mean connectivity c, and $\mathbf{J}$ is a symmetric random matrix with entries sampled from a Gaussian distribution with mean zero and variance $1/c$, i.e. $J_{ij} \sim \mathcal{N}(0, 1/c)$. The J_{ij} can be interpreted as the edge weights among nodes i and j.

We want to investigate the eigenvalue distribution of matrices from this ensemble. For a single realization $\mathbf{M}$ this spectral density is

$$\rho^{\mathbf{M}}(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i^{\mathbf{M}}) \tag{12}$$

In the limit $N \to \infty$ we expect that this density is self-averaging and hence tends to an $\mathbf{M}$–independent function, i.e.

$$\rho(\lambda) = \langle \rho^{\mathbf{M}}(\lambda) \rangle_{\mathcal{M}} = \rho^{\mathbf{M}}(\lambda) \tag{13}$$

Part 1: Direct Diagonalization

1. Generate 10 different random matrices from $\mathcal{M}$ with sizes $N = 2^{10}$ and $c = 3$. Estimate the mean spectral density using Direct Diagonalization and a histogram to display the distribution. Discuss the role of the binning on the estimation and the error bars.
   Note: To generate Random Regular Graphs the libraries **NetworkX** in Python or **LightGraphs** in Julia are recommended; you can export the associated adjacency matrix from there. For more information about algorithms for generating RRGs, see Steger and Wormald's reference in the Literature section.

2. Generate 10 different random matrices from $\mathcal{M}$ with sizes $N = 2^{10}$ for a fully connected network, i.e. $c = N - 1$. Estimate the mean spectral density using Direct Diagonalization and a histogram to display the distribution.

3. For the fully connected network in the limit $N \to \infty$, the spectral density is given by the Wigner semicircle law, namely

   $$\rho(\lambda) = \frac{\sqrt{4 - \lambda^2}}{2\pi} \tag{14}$$

   Verify the property of self–averaging by generating a relatively large graph, $N = 2^{12}$ and computing the distribution of its eigenvalues. Show that this is consistent with the semicircle law above. Discuss the possible sources of error in the estimation.

4. Wigner's semicircle law is ubiquitous in physics and emerged originally in an attempt to describe the energy spectrum of heavy nuclear systems and to understand

their level spacing distribution. The fact that regardless of the exact matrix elements of the Hamiltonian its spectral properties can be understood with random matrices led to the development of Random Matrix Theory and the idea of universality classes.

Discuss to which universality class the fully connected network belongs. Show numerically that for c finite but sufficiently large, the associated matrices also fall into the same universality class by looking at the corresponding spectral density.

Part 2: Cavity method

Following the lecture, we write the spectral density as proportional to the (imaginary part of) trace of the resolvent matrix $\mathbf{G}$:

$$\rho(\lambda) = \lim_{\epsilon\to 0} \frac{1}{\pi N} \sum_{j=1}^{N} \mathrm{Im} G_{jj}(\lambda - i\epsilon), \qquad \mathbf{G}(z) = (z\mathbf{1} - \mathbf{M})^{-1} \tag{15}$$

We consider this as the covariance matrix of a (complex) Gaussian distribution with zero mean:

$$P(\mathbf{x}) \propto \mathrm{e}^{-\frac{i}{2}\mathbf{x}^T\mathbf{G}^{-1}\mathbf{x}} = \mathrm{e}^{-\frac{i}{2}\sum_{k,j} x_k x_j((\lambda - i\epsilon)\delta_{k,j} - M_{kj})} \tag{16}$$

Within this construction, the diagonal resolvent elements are essentially the marginals of $P(\mathbf{x})$:

$$G_{jj} = i\langle x_j^2\rangle = i\int dx_1 \cdots dx_N \, x_j^2 P(\mathbf{x}) = i\int dx_j \, x_j^2 P(x_j) \tag{17}$$

with $P(x_j)$ the marginal distribution at node j. In the cavity approach, one writes the marginals in terms of cavity distributions as

$$P(x_j) \propto \mathrm{e}^{-\frac{i}{2}(\lambda - i\epsilon)x_j^2} \int d\mathbf{x}_{\partial j} \mathrm{e}^{ix_j \sum_{k\in\partial j} x_k M_{kj}} P^{(j)}(\mathbf{x}_{\partial j}) \tag{18}$$

where $P^{(j)}(\mathbf{x}_{\partial j})$ denotes the probability distribution of the variables $\{x_k\}$ on the nodes that are neighbours of j on the RRG. In the tree–like approximation, namely $P^{(j)}(\mathbf{x}_{\partial j}) = \prod_{k\in\partial j} P^{(j)}(x_k)$ one gets a simplified equation for $P(x_j)$ in terms of individual cavity distributions. Those in turn satisfy the following recursive equation

$$P^{(j)}(x_k) \propto \mathrm{e}^{-\frac{i}{2}(\lambda - i\epsilon)x_k^2} \prod_{l\in\partial k\setminus j} \int dx_l \mathrm{e}^{ix_l x_k M_{kl}} P^{(k)}(x_l) \tag{19}$$

5. Consider the following Gaussian ansatz for the cavity and marginal distributions:

$$P^{(k)}(x_l) \propto \mathrm{e}^{-\frac{1}{2}\omega_l^{(k)} x_l^2}, \qquad P(x_j) \propto \mathrm{e}^{-\frac{1}{2}\omega_j x_j^2} \tag{20}$$

Derive a set of equations for the cavity precisions $\{\omega_k^{(j)}\}$ and for the marginal precisions ω_j in terms of the cavity precisions.

6. Consider a sparse matrix of size $N = 2^{11}$ with $c = 3$. Solve the associated cavity equations and estimate the spectral density using equation (15). Discuss the role of ϵ by playing with different values. Compare your results with Direct Diagonalization for the same matrix.

7. Justify why in the mean–field limit ($A_{ij} = 1 \forall\, i \neq j$ and $N \to \infty$), the cavity precisions may be approximated by the marginals, i.e. $\omega_k^{(j)} \approx \omega_k$

8. In terms of the Gaussian ansatz (20), the diagonal resolvent entries are

$$G_{jj} = i\omega_j^{-1} \tag{21}$$

Let us introduce the following variable for the mean resolvent

$$g = \frac{1}{N}\sum_{j=1}^{N} G_{jj} = \frac{i}{N}\sum_{j=1}^{N}\frac{1}{\omega_j} \tag{22}$$

Derive an equation for g in the mean–field limit by using the assumption discussed in the previous question. Solve this equation and evaluate the spectral density (15). Compare with the analytical result.

Exercise 3: Anderson model on an RRG [1] (15 points + 25 points for code)

The Anderson model paved the way for the modern point of view of metal–insulator transitions. It models a gas of noninteracting electrons in a crystalline array of potential wells. If all the wells have the same depth an electron could move freely along the array and hence the system as a whole is a conductor. On the other hand, if the well depths are randomly chosen within a range W, there exists a critical value W_c (dimension–dependent) such that all the states become localized and the system as a whole is an insulator.

In terms of fermionic creation and and annihilation operators, the description of non-interacting spinless electrons in a disordered potential on a generic network is given by the following Hamiltonian

$$H = -\sum_{\langle i,j\rangle}(c_i^\dagger c_j + c_j^\dagger c_i) + \sum_{i=1}^{N} E_i c_i^\dagger c_i \,, \tag{23}$$

where N is the total number of nodes, the sum in the first term is over pairs of neighbours and $\{E_i\}$ are the on–site energies , which are taken as independent random variables each uniformly distributed in the interval $[-W/2, W/2]$:

$$\rho(E) = \frac{1}{W}\Theta\left(\frac{W}{2} - |E|\right) \tag{24}$$

An important network in the history of Anderson Localization phenomena is the Cayley tree. The model (23) set on a Cayley tree can be analytically solved and led to the confirmation of Anderson's conjecture that loops in high dimensions are not crucial for localizing the electrons.

As a matrix, H can be seen as the sum of the (negative) adjacency matrix and a diagonal matrix with entries $\{E_1, \cdots, E_N\}$, i.e.

$$H_{ij} = -A_{ij} + E_i\delta_{ij} \tag{25}$$

It is our aim to study this model on a Random Regular Graph (RRG) with connectivity c, which mimics better than a Cayley tree the bulk of an infinite dimensional system because all sites of the RRG are equivalent.

Following the steps sketched in Exercise 2, one can show that the cavity method leads to the following system of equations for the cavity and marginal precisions:

$$\omega_j = i(\lambda - i\epsilon - E_j) + \sum_{k\in\partial j}\frac{1}{\omega_k^{(j)}}\,, \quad \omega_k^{(j)} = i(\lambda - i\epsilon - E_k) + \sum_{l\in\partial k\setminus j}\frac{1}{\omega_l^{(k)}}\,, \tag{26}$$

the diagonal resolvent entries being $G_{jj} = i\omega_j^{-1}$.

In the thermodynamic limit $N \to \infty$, we can convert the equation for the cavity precisions into a self-consistent equation for their probability distribution $P(\omega)$. This is

$$P(\omega) = \int dE\rho(E)\prod_{l=1}^{c-1} dP(\omega_l)\,\delta\left(\omega - i(\lambda - i\epsilon - E) - \sum_{l=1}^{c-1}\frac{1}{\omega_l}\right) \tag{27}$$

[1] Choose one among this or exercise 4

The analogous equation for the distribution of the marginals $P(\Omega)$ in terms of the cavity distribution $P(\omega)$ reads

$$P(\Omega) = \int dE \rho(E) \prod_{l=1}^{c} dP(\omega_l)\, \delta\left(\Omega - i(\lambda - i\epsilon - E) - \sum_{l=1}^{c} \frac{1}{\omega_l}\right) \tag{28}$$

The numerical solution of equation (27) is performed via Population Dynamics as discussed in the lecture.

1. Discuss the physical meaning of the eigenvalues of H. What physical information can be extracted from the associated spectral density?

2. Consider an RRG with fixed connectivity $c = 3$ and size $N = 2^{10}$. Estimate the spectral density both via Direct Diagonalization and via the single instance Cavity Method (with $\epsilon = 10^{-3}$) for a low value of the disorder $W = 0.3$.

3. For the same $W = 0.3$, estimate the spectral density with the Population Dynamics algorithm. Consider a population size $N_p = 10^3$. Before evaluating equation (28) show that the population of the cavity ω's has reached equilibrium.

4. Instead of the distribution of cavity precisions, one could consider the distribution of the cavity variances $g = i/\omega$. Anderson and Thouless proposed that the distribution of its imaginary part $P(\mathrm{Im}g)$ in the limit $\epsilon \to 0$ characterizes the localization properties of the system for a fixed value of the disorder (and for a given λ). If the typical value of $\mathrm{Im}g$ tends to zero in this limit the associated states are considered localized, otherwise they are considered extended.

   Fix $\lambda = 0$, use populations of size $N_p = 10^3, 5 \times 10^3, 10^4$ and study how the typical value of the distribution $g^{\mathrm{typ}} = \mathrm{e}^{\langle \ln \mathrm{Im}g \rangle}$ changes from $O(1)$ to $O(\epsilon)$ as W increases across the range $W = \{10, 11, \cdots, 20\}$. (**Hint:** Use a very small ϵ, like 10^{-300} to easily distinguish among both cases.) Suggest a way to estimate W_c. Compare your results with the analytical value $W_c \approx 18.2$. Give an interpretation of the localization transition in terms of the physics of the model.

5. One can extend Anderson and Thouless' argument to the typical value of the *marginal* variances $G = i/\Omega$. For the same $\lambda = 0$, $N_p = 10^4$ and a moderately small $\epsilon = 10^{-6}$, compare the whole distribution of the marginals $P(\mathrm{Im}G)$ in the localized and extended domains (choose suitable W values). Explain why in the case of localized states the spectral density (proportional to the mean of the distribution) is $O(1)$ even though the typical value is $O(\epsilon)$.

6. A common method for studying the localization properties of a vector $\mathbf{v}$ is via the scaling of the inverse participation ratio (IPR). This is defined as

$$I_2(\mathbf{v}) = \frac{\sum_{i=1}^{N} v_i^4}{\left(\sum_{i=1}^{N} v_i^2\right)^2} \tag{29}$$

   and is a measure of how (un-)evenly the squared norm of a vector is distributed across its entries. A vector is considered localized if its entries are concentrated on

a few nodes in the large N limit and then $I_2 \sim O(1)$. On the other hand, a vector is considered extended if its entries are roughly equally distributed and therefore $I_2 \sim 1/N$ in the same large N limit.

Consider a small window of eigenvectors around $\lambda = 0$ and study how the mean value of NI_2 (for at least ten instances) behaves for different matrix sizes ($N = 2^{10}, 2^{11}, 2^{12}$) as the disorder increases from $W = 5$ to $W = 10$. Discuss your results and explain why it is hard to estimate the critical value W_c using this criterion.

Exercise 4: Bouchaud trap model on an RRG [2] (15 points + 25 points for code)

The Bouchaud trap model is a simplified model of glassy dynamics. It considers the configuration space of a glassy system as a collection of nodes (and therefore as a network) with associated random energy depths E_i. On this a Markov jump dynamics takes place. The hopping rates (from trap i to j) are taken as being of Arrhenius form,

$$w_{ji} = \frac{1}{c} \mathrm{e}^{-\beta E_i} \tag{30}$$

with c the mean connectivity of the network and $\beta = 1/T$ the inverse temperature (the Boltzmann constant is set to one). The random trap depths are selected from an exponential distribution with mean 1, i.e.

$$\rho(E) = \mathrm{e}^{-E} \tag{31}$$

Overall, the model is encoded in terms of the following master equation for the probability vector $\mathbf{p}$:

$$\frac{d\mathbf{p}(t)}{dt} = \mathbf{M}\mathbf{p}(t) \tag{32}$$

with $M_{ij} = A_{ij} w_{ij}$ the transition rates, $\mathbf{A}$ the adjacency matrix of the network and $M_{ii} = -\sum_{j \neq i} M_{ji}$. The model becomes more strongly disordered for low T, as the variation in trap depths translates into a broader distribution of transition rates there.

The transition rates (30) obey detailed balance with respect to the Boltzmann equilibrium distribution

$$\lim_{t \to \infty} \mathbf{p}(t) = \mathbf{p}_{\mathrm{eq}}(E) = \frac{1}{\mathcal{Z}} \left(\mathrm{e}^{\beta E_1}, \cdots, \mathrm{e}^{\beta E_N} \right) \tag{33}$$

with $\mathcal{Z}$ a normalization constant. In order to apply the cavity method for the analysis of the statistical properties of the master operator $\mathbf{M}$ we need to symmetrize it using the equilibrium distribution:

$$M^s_{ij} = \mathrm{e}^{-\beta E_i/2} M_{ij} \mathrm{e}^{\beta E_j/2} \tag{34}$$

This is a similarity transformation and thus $\mathbf{M}^s$ and $\mathbf{M}$ have the same spectrum.

As discussed in the lecture, the Gaussian ansatz in the cavity equations leads to the following system of equations for the marginal and cavity precisions:

$$\omega_j = ic((\lambda - i\epsilon)\tau_j + 1) + \sum_{k \in \partial j} \frac{1}{\omega_k^{(j)}}, \quad \omega_k^{(j)} = ic((\lambda - i\epsilon)\tau_k + 1) + \sum_{l \in \partial k \setminus j} \frac{1}{\omega_l^{(k)}} \tag{35}$$

with $\tau_j = \mathrm{e}^{\beta E_j}$. The diagonal resolvent entries G_{jj} are then given by

$$G_{jj} = \frac{i\tau_j c}{\omega_j} \tag{36}$$

[2] Choose one among this or exercise 3

In the thermodynamic limit $N \to \infty$ we can convert the second equation in (35) (for the cavity precisions) into a self consistent equation for the probability distribution of the cavity precisions, $P(\omega)$:

$$P(\omega) = \int dE \rho(E) \prod_{l=1}^{c-1} dP(\omega_l)\, \delta \left(\omega - ic((\lambda - i\epsilon)\mathrm{e}^{\beta E} + 1) - \sum_{l=1}^{c-1} \frac{1}{\omega_l} \right) \tag{37}$$

In the same limit, the equation for the distribution of the marginal precisions $P(\Omega)$ in terms of the distribution of cavity precisions $P(\omega)$ reads

$$P(\Omega) = \int dE \rho(E) \prod_{l=1}^{c} dP(\omega_l)\, \delta \left(\Omega - ic((\lambda - i\epsilon)\mathrm{e}^{\beta E} + 1) - \sum_{l=1}^{c} \frac{1}{\omega_l} \right) \tag{38}$$

The numerical solution of equation (37) is performed via Population Dynamics as discussed in the lecture.

1. Discuss the physical meaning of the eigenvalues of $\mathbf{M}$. What physical information can be extracted from the associated spectral density?

2. Consider an RRG with fixed connectivity $c = 3$ and size $N = 2^{10}$. Estimate the spectral density via Direct Diagonalization and the single instance Cavity Method (with $\epsilon = 10^{-3}$) for a relative high value of disorder $T = 1/2$.

3. For the same $T = 1/2$, estimate the spectral density with the Population Dynamics algorithm. Consider a population size $N_p = 10^3$. Before evaluating equation (28) show that the population of cavity ω's has reached equilibrium. Compare your results with the analytical mean–field ($c \to \infty$) form

$$\rho(\lambda) \simeq |\lambda|^{T-1}, \quad \lambda \to 0 \tag{39}$$

   Hint: Use a logarithmic λ-grid for better visualization.

4. Instead of the distribution of cavity precisions, one could consider the distribution of cavity variances, $g = \frac{i\tau c}{\omega}$. Anderson and Thouless proposed that the distribution of its imaginary part $P(\mathrm{Im}g)$ in the limit $\epsilon \to 0$ characterizes the localization properties of the system for a fixed value of the disorder (and for a given λ). If the typical value of $\mathrm{Im}g$ tends to zero in this limit the associated states are considered localized, otherwise they are considered extended.

   Fix $T = 1.0$, use populations of size $N_p = 10^3, 5 \times 10^3, 10^4$ and study how the typical value of the distribution $g^{\mathrm{typ}} = \mathrm{e}^{\langle \ln \mathrm{Im}g \rangle}$ changes from $O(1)$ to $O(\epsilon)$ as λ goes to zero $\lambda = -10^{-5}, -10^{-4.5}, \cdots, -10^{-1}$. (**Hint:** Use a very small ϵ, like 10^{-300} to easily distinguish among both cases.) Suggest a way to estimate λ_c. Give an interpretation to the existence of a *mobility edge* (λ_c) separating localized from extended states in terms of the physics of the model.

5. One can extend Anderson and Thouless' argument to the typical value of the *marginal* variances, $G = i\tau c/\Omega$. For the same $T = 1.0$, $N_p = 10^4$ and a moderately small $\epsilon = 10^{-6}$, compare the whole distribution of the marginals $P(\mathrm{Im}G)$ in the localized and extended domains (choose suitable λ values). Explain why in the case of localized states the spectral density (proportional to the mean of the distribution) is $O(1)$ even though the typical value is $O(\epsilon)$.

6. A common method for studying the localization properties of a vector $\mathbf{v}$ is via the scaling of the inverse participation ratio (IPR). This is defined as

$$I_2(\mathbf{v}) = \frac{\sum_{i=1}^{N} v_i^4}{\left(\sum_{i=1}^{N} v_i^2\right)^2} \tag{40}$$

and it is a measure of how (un-)evenly the squared norm of a vector is distributed across its entries. A vector is considered localized if its entries are concentrated on a few nodes in the large N limit and then $I_2 \sim O(1)$. On the other hand, a vector is considered extended if its entries are roughly equally distributed and therefore $I_2 \sim 1/N$ in the same large N limit.

Consider a small window of eigenvectors around $\lambda = -10^{-3}, -10^{-2.7}, \cdots, -10^{-0.6}$; and study how the mean value of NI_2 (for at least ten instances) behaves for different matrix sizes ($N = 2^{10}, 2^{11}, 2^{12}$) and for a fixed disorder parameter $T = 1.0$. Discuss your results and explain why it is hard to estimate the critical value λ_c using this criterion.

To facilitate the supervision, please set up a git repository at gitlab.gwdg.de and give reading rights to the tutor.

Criteria for grading:

- **(40 points)** Upload the codes that you use to solve the exercises to the repository. Include a `makefile` or `README` file, as well as instructions on how to run your program. The code needs to be executable and the tutor needs to be able to verify that your code produced the data presented in your report. In particular, make sure that all your data is reproducible with the parameter sets you provide. Make sure that the code is well documented. Also, please include all additional routines (e.g. jupyter-notebooks) you used to process the data.
- Prepare a written report (e.g. using LaTeX, maximum 10 pages, plus figures) in which you explain the project and your results. Hand in your report via e-mail to your tutor. This is freeze time: the tutor will pull the code from the repository and no further changes will be accepted at this point.
- Discuss the numerical accuracy and possible numerical errors.
- **(20 points)** Formal aspects: Proper use of units, preparation of figures and figure captions, suitable choice of figures (and number of figures). Each figure should be discussed in the text of your report.
- **(40 points)** Correctness and quality of the answers to the topic-specific questions.

References

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