Feedback Linearization for a Nonlinear Quarter-Car Model

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ECH 267
Final Project

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Overview

1. Model
2. Controller Design
3. Simulation

Model

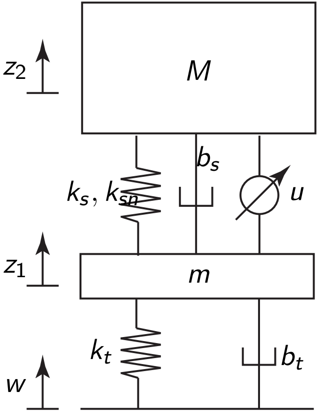


Figure: Quarter-Car Model

$$m\ddot{z}_1 = F_s + F_d - F_t - F_b - u \tag{1}$$

$$M\ddot{z}_2 = -F_s - F_d + u \tag{2}$$

The forces produces by the springs and dampers are the following:

$$F_s = k_s(z_2 - z_1) + k_{sn}(z_2 - z_1)^3 \tag{3}$$

$$F_d = b_s(\dot{z}_2 - \dot{z}_1) \tag{4}$$

$$F_t = k_t(z_1 - w) \tag{5}$$

$$F_b = b_t(\dot{z}_1 - \dot{w}) \tag{6}$$

State Space

Define state variables

$$x_1 = z_2,\ x_2 = \dot{z}_2,\ x_3 = z_1,\ x_4 = \dot{z}_1$$

$$\dot{x}_1 = x_2 \tag{7}$$

$$\dot{x}_2 = \frac{1}{M}(-F_s - F_d + u) \tag{8}$$

$$\dot{x}_3 = x_4 \tag{9}$$

$$\dot{x}_4 = \frac{1}{m}(F_s + F_d - F_t - F_b - u) \tag{10}$$

$$y = x_1 \tag{11}$$

where

$$F_s = k_s(x_1 - x_3) + k_{sn}(x_1 - x_3)^3 \tag{12}$$

$$F_d = b_s(x_2 - x_4) \tag{13}$$

$$F_t = k_t(x_3 - w) \tag{14}$$

$$F_b = b_t(x_4 - \dot{w}) \tag{15}$$

System

The system is represented in the following form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{16}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$ is the state vector, u is the control input, and $y = x_1$ is the output. $f(x)$, $g(x)$, and $h(x)$ are given as follows

$$f(x) = \begin{bmatrix} x_2 \\ \frac{1}{M}(-F_s - F_d) \\ x_4 \\ \frac{1}{m}(F_s + F_d - F_t - F_b) \end{bmatrix} ; \ g(x) = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{m} \end{bmatrix} ; \ h(x) = [x_1] \tag{17}$$

Lie Derivatives and Coordinate Change

$$y = x_1 = y_1 \tag{18}$$

$$\dot{y} = \dot{y}_1 = \dot{x}_1 = x_2 = y_2 \tag{19}$$

$$\ddot{y} = \dot{y}_2 = \dot{x}_2 = \frac{1}{M}(-F_s - F_d) + \frac{1}{M}u \tag{20}$$

the system relative degree is 2. Also, $n - r = 4 - 2 = 2$

$$L_f h = \frac{\partial h}{\partial x} \times f(x) = x_2 \tag{21}$$

$$L_g h = \frac{\partial h}{\partial x} \times g(x) = 0 \tag{22}$$

$$L_f^2 h = \frac{1}{M}(-F_s - F_d) \tag{23}$$

$$L_g L_f h = \frac{1}{M} \tag{24}$$

Coordinate Change and Zero Dynamics

$$\begin{bmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f(x) \\ \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 + w \\ Mx_2 + mx_4 - mw \end{bmatrix} \tag{25}$$

ϕ_1 and ϕ_2 are selected such that $L_g\phi = 0$, and T is a diffeomorphism

Check for zero dynamics

$$\dot{z} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -\frac{z_2 - My_2}{m} \\ k_t z_1 - b_t \frac{z_2 - My_2}{m} \end{bmatrix} \tag{26}$$

$$\dot{z} = f_z(z, 0, 0) = \begin{bmatrix} -\frac{1}{m} z_2 \\ k_t z_1 - \frac{b_t}{m} z_2 \end{bmatrix} \tag{27}$$

is Horowitz or can be proven a.s using a Lyapunov function

Controller and Closed Loop Dynamics

$$u = \frac{1}{L_g L_f h(x)}[v - L_f^2 h(x)] = M[v + k_s(x_1 - x_3) + k_{sn}(x_1 - x_3)^3 + b_s(x_2 - x_4)] \tag{28}$$

close loop dynamics:

$$A = \begin{bmatrix} 0 & 1 \\ -\gamma_1 & -\gamma_2 \end{bmatrix}$$

Choose $-\gamma_1$ and $-\gamma_2$ such that $Real(\lambda_1, \lambda_2) < 0$ for the controller to be stable

Controller

y is required to track a smooth signal y_q

$$u = \frac{1}{L_g L_f h(x)}[v - L_f^2 h(x)] = M[v + k_s(x_1 - x_3) + k_{sn}(x_1 - x_3)^3 + b_s(x_2 - x_4)] \tag{29}$$

where

$$\begin{aligned} v &= \ddot{y}_d - K_1 e - K_2 \dot{e} \\ e &= y - y_d \end{aligned} \tag{30}$$

This will lead to

$$\begin{aligned} \ddot{y} &= \ddot{y}_d - K_1 e - K_2 \dot{e} \\ \ddot{e} + K_2 \dot{e} + K_1 e &= 0 \end{aligned} \tag{31}$$

when $K_1 > 0$ and $K_2 > 0$, the error will decay to zero. This means that the output will follow the desired trajectory.

Parameters

Parameters	Values
M	600 kg
m	100 kg
k_s	15000 N/m
b_s	1500 Ns/m
k_t	200000 N/m
b_t	2000 Ns/m
k_{sn}	1000 N/m

Table: System parameters

$$w = \begin{cases} 1 - cos(8\pi t) & ; 1 \leq t \leq 1.5 \\ 0 & ; otherwise \end{cases}$$

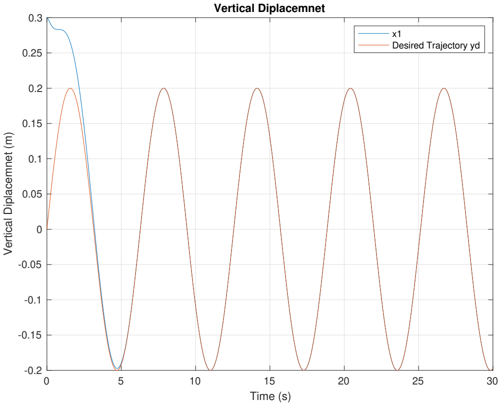
$$y_d = 0.2sin(t)$$

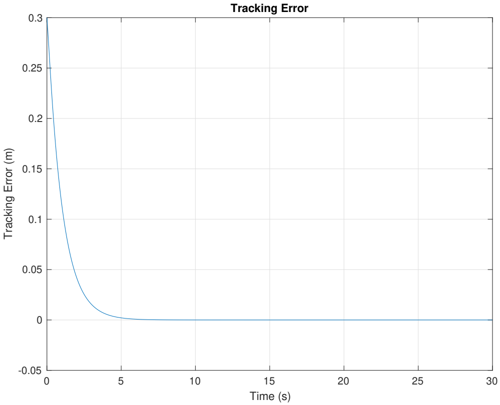
$$x = [0.3 \quad 0 \quad 0 \quad 0]^T$$

$$K_1 = 10$$

$$K_2 = 11$$

Results


Vertical Diplacemnet
x1
Desired Trajectory yd
Vertical Diplacemnet (m)
Time (s)
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
5
10
15
20
25
30


Tracking Error
Tracking Error (m)
Time (s)
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
5
10
15
20
25
30

Results Cont.