Feedback Linearization for a Nonlinear Quarter-Car Model

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Contents

1 Abstract

A quarter car model is a common simplified model used to study the dynamics of a car suspension. The system is modelled as two masses on top of each other with a spring and damper attached between them and a spring and damper attached to the ground. The lower mass represents the tire and the upper mass represents one-quarter the mass of the car, hence the name Quarter-Car Model. The springs and dampers are used to represent the tire and suspension's characteristics with respect to vertical displacement and velocity.

The purpose of this project is to minimize the vertical displacement of the car using a controlled force actuator implemented in the suspension setup. Knowing that the springs and dampers represent a tire and a complex setup assembly, the spring will be modeled as a nonlinear element. Then using feedback linearization technique learned in class, we design a nonlinear controller. The tracking performance of the controller in the presence known road disturbance will be illustrated using Matlab simulation.

2 Introduction

In the project, feedback linearization is applied to design to design a nonlinear controller for a quarter-car suspension system. The system is well known and ample research can be found. Research on this topic is done or a quarter, half, or full car model with minor adjustments. This system has been studied as a linear system and controlled using a PID controller. Traditional linear methods for optimization have been performed on the system.

Moreover, modern techniques and nonlinear control techniques have been used to control this system. Feedback linearization have been done of the viscous damper dynamics (1). In this project, the main reference will be a paper on feedback linearization for a half car suspension system (2) which presents a technique that would help in tracking a desired trajectory. In addition to the method learned in class (3), this technique will be used to simulate trajectory tracking of a quarter-car model with the help of feedback linearization.

3 Model

3.1 Nomenclature

M	Quarter mass of the vehicle's body
m	Mass of one wheel
F_s	Elastic force of suspension
F_d	Damping force of suspension
F_t	Elastic force of tire
F_b	Damping force of tire
w	Road input
z_1	Un-sprung mass displacement
z_2	Vertical displacement of the body
u	Controller

3.2 Kinetics

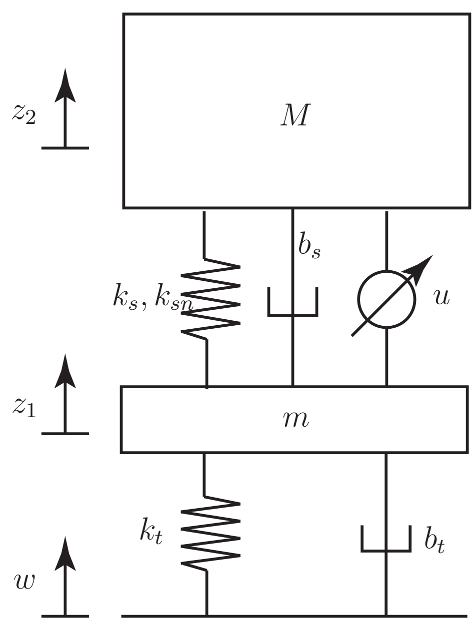


Figure 1: Quarter-Car Model

$$m\ddot{z}_1 = F_s + F_d - F_t - F_b - u \tag{1}$$
$$M\ddot{z}_2 = -F_s - F_d + u \tag{2}$$

The forces produces by the springs and dampers are the following:

$$F_s = k_s(z_2 - z_1) + k_{sn}(z_2 - z_1)^3 \tag{3}$$
$$F_d = b_s(\dot{z}_2 - \dot{z}_1) \tag{4}$$
$$F_t = k_t(z_1 - w) \tag{5}$$
$$F_b = b_t(\dot{z}_1 - \dot{w}) \tag{6}$$

where k_s and k_{sn} are respectively the linear and nonlinear stiffness coefficients of the suspension; b_s is the damping coefficient of the suspension. k_t is the stiffness of the tire, and b_t is its damping coefficient. Define the road input

$$w = \begin{cases} 1 - cos(8\pi t) & ; 1 \leq t \leq 1.25 \\ 0 & ; otherwise \end{cases}$$

Define state variables

$$x_1 = z_2,\ x_2 = \dot{z}_2,\ x_3 = z_1,\ x_4 = \dot{z}_1$$

The system can be rewritten in the following state-space form

$$\dot{x}_1 = x_2 \tag{7}$$
$$\dot{x}_2 = \frac{1}{M}(-F_s - F_d + u) \tag{8}$$
$$\dot{x}_3 = x_4 \tag{9}$$
$$\dot{x}_4 = \frac{1}{m}(F_s + F_d - F_t - F_b - u) \tag{10}$$
$$y = x_1 \tag{11}$$

where

$$F_s = k_s(x_1 - x_3) + k_{sn}(x_1 - x_3)^3 \tag{12}$$
$$F_d = b_s(x_2 - x_4) \tag{13}$$
$$F_t = k_t(x_3 - w) \tag{14}$$
$$F_b = b_t(x_4 - \dot{w}) \tag{15}$$

The system parameters used throughout this project are utilized utilized in paper (4) are given in Table 1

Parameters	Values	Parameters	Values
M	600 kg	m	100 kg
k_s	15000 N/m	b_s	1500 Ns/m
k_t	200000 N/m	b_t	2000 Ns/m
k_{sn}	1000 N/m		

Table 1: System parameters

4 Nonlinear Controller Design

In this part, feedback linearization is used to find the controller. The controller should satisfy the following requirements:

1. Asymptotic stability
2. Good command tracking

4.1 System

The system is represented in the following form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{16}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$ is the state vector,u is the control input, and $y = x_1$ is the output. $f(x)$, $g(x)$, and $h(x)$ are given as follows

$$f(x) = \begin{bmatrix} x_2 \\ \frac{1}{M}(-F_s - F_d) \\ x_4 \\ \frac{1}{m}(F_s + F_d - F_t - F_b) \end{bmatrix}; \; g(x) = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{m} \end{bmatrix}; \; h(x) = [x_1] \tag{17}$$

4.2 Feedback Linearization

4.2.1 Coordinate Change

Define the coordinate system

$$y = x_1 = y_1 \tag{18}$$

$$\dot{y} = \dot{y}_1 = \dot{x}_1 = x_2 = y_2 \tag{19}$$

$$\ddot{y} = \dot{y}_2 = \dot{x}_2 = \frac{1}{M}(-F_s - F_d) + \frac{1}{M}u \tag{20}$$

$$L_f h = \frac{\partial h}{\partial x} \times f(x) = x_2 \tag{21}$$
$$L_g h = \frac{\partial h}{\partial x} \times g(x) = 0 \tag{22}$$
$$L_f^2 h = \frac{1}{M}(-F_s - F_d) \tag{23}$$
$$L_g L_f h = \frac{1}{M} \tag{24}$$

since $L_g h = 0$ and $L_g L_f h \neq 0 \quad \forall x \in R^4$, then the system relative degree is 2. Also, $n - r = 4 - 2 = 2$

Then, the following coordinate change is found

$$\begin{bmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f(x) \\ \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 + w \\ Mx_2 + mx_4 - mw \end{bmatrix} \tag{25}$$

ϕ_1 and ϕ_2 are selected such that $L_g\phi = 0$, and ϕ is invertible. For the latter, ϕ is defined everywhere, thus it is invertible. For the former:

$$L_g\phi = \begin{bmatrix} L_g\phi_1 \\ L_g\phi_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1}g_1 + \frac{\partial\phi_1}{\partial x_2}g_2 + \frac{\partial\phi_1}{\partial x_3}g_3 + \frac{\partial\phi_1}{\partial x_4}g_4 \\ \frac{\partial\phi_2}{\partial x_1}g_1 + \frac{\partial\phi_2}{\partial x_2}g_2 + \frac{\partial\phi_2}{\partial x_3}g_3 + \frac{\partial\phi_2}{\partial x_4}g_4 \end{bmatrix} = 0 \tag{26}$$

but $g_1 = g_3 = 0$, and $g_2 = \frac{1}{M}$, $g_2 = -\frac{1}{m}$, then equation 26 yields to

$$\begin{aligned} \frac{1}{M}\frac{\partial\phi_1}{\partial x_2} - \frac{1}{m}\frac{\partial\phi_1}{\partial x_4} &= 0 \\ \frac{1}{M}\frac{\partial\phi_2}{\partial x_2} - \frac{1}{m}\frac{\partial\phi_2}{\partial x_4} &= 0 \end{aligned} \tag{27}$$

Knowing that $x_4 = \frac{z_2 - Mx_2 + m\dot{w}}{m} = \frac{z_2 - My_2 + m\dot{w}}{m}$ and using equations 17 and 26, equation 27 is verified as follows

$$\dot{\phi}_1 = -\dot{x}_3 + \dot{w} = -x_4 + \dot{w} = -\frac{z_2 - My_2 + m\dot{w}}{m} + \dot{w} = -\frac{z_2 - My_2}{m} \tag{28}$$

$$\begin{aligned} \dot{\phi}_2 &= M\dot{x}_2 + m\dot{x}_4 = -F_s - F_d + u + F_s + F_d - F_t - F_b - u = -F_t - F_b \\ &= -k_t(x_3 - w) - b_t(x_4 - \dot{w}) = -k_t(w - z_1 - w) - b_t(\frac{z_2 - My_2 + m\dot{w}}{m} - \dot{w}) \\ &= k_t z_1 - b_t\frac{z_2 - My_2}{m} \end{aligned} \tag{29}$$

It is clear that $\dot{\phi}$ is not a function of u, and $L_g\phi = 0$

4.2.2 Controller

The controller is designed such it cancels the effect of the nonlinear part.

$$u = \frac{1}{L_g L_f h(x)}[v - L_f^2 h(x)] = M[v + k_s(x_1 - x_3) + k_{sn}(x_1 - x_3)^3 + b_s(x_2 - x_4)] \quad (30)$$

4.2.3 Closed Loop Dynamics

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= v = -\gamma_1 y_1 - \gamma_2 y_2 \end{aligned} \quad (31)$$

and

$$A = \begin{bmatrix} 0 & 1 \\ -\gamma_1 & -\gamma_2 \end{bmatrix}$$

Choose $-\gamma_1$ and $-\gamma_2$ such that $Real(\lambda_1, \lambda_2) < 0$ for the controller to be stable

4.2.4 Zero Dynamics

Check for zero dynamics

$$\dot{z} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -\frac{z_2 - My_2}{m} \\ k_t z_1 - b_t \frac{z_2 - My_2}{m} \end{bmatrix} \quad (32)$$

check that $z = 0$ is asymptotically stable when $y_1 = y_2 = 0$

$$\dot{z} = f_z(z, 0, 0) = \begin{bmatrix} -\frac{1}{m} z_2 \\ k_t z_1 - \frac{b_t}{m} z_2 \end{bmatrix} \quad (33)$$

taking $V(z) = \frac{m}{2} z_1^2 + \frac{1}{2k} z_2^2$, $\dot{V} < 0 \ \forall x \in R - \{0\}$. This means that $z = 0$ g.a.s

5 Simulation

For the simulation, we use a different method (2) to find the control that insures command tracking. Using input-output linearization, u will have the same form as equation 30 but v will take a different form than equation 31. y is required to track a smooth signal y_q

$$u = \frac{1}{L_g L_f h(x)}[v - L_f^2 h(x)] = M[v + k_s(x_1 - x_3) + k_{sn}(x_1 - x_3)^3 + b_s(x_2 - x_4)] \quad (34)$$

where

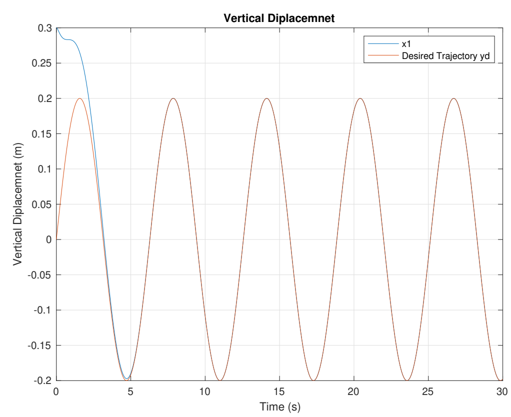
$$\begin{aligned} v &= \ddot{y}_d - K_1 e - K_2 \dot{e} \\ e &= y - y_d \end{aligned} \quad (35)$$

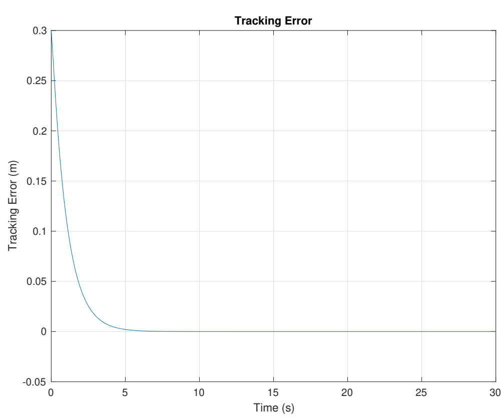
This will lead to

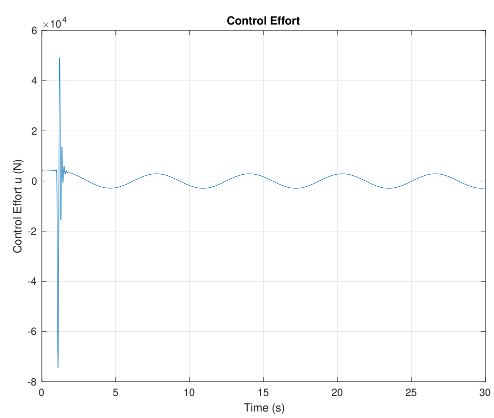
$$\begin{aligned}\ddot{y} &= \ddot{y}_d - K_1 e - K_2 \dot{e} \\ \ddot{e} + K_2 \dot{e} + K_1 e &= 0\end{aligned} \tag{36}$$

when $K_1 > 0$ and $K_2 > 0$, the error will decay to zero. This means that the output will follow the desired trajectory.

In the following simulation, the desired trajectory was chosen to be $y_d = 0.2sin(t)$.
The initial conditions of the system were assigned as follow: $x = [0.3 \quad 0 \quad 0 \quad 0]^T$
The controller parameters were chosen to have a fast response and to minimise overshoot. After trying several configurations, they were chosen as $K_1 = 10$ and $K_2 = 11$
Below are the simulation results. The Matlab code can be accessed using the Github Repository Link.







It is clear that the tracking is accurate and effective; thus, the behavior is asymptotically stable. Also the control effort is smooth unless a major stiff disturbance occurs. This disturbance can be visualized at $t = 1$ s where a lot of control effort is needed.

References

[1] J. Pedro and O. Dahunsi, "Neural network based feedback linearization control of a servo-hydraulic vehicle suspension system," International Journal of Applied Mathematics and Computer Science, vol. 21, no. 1, pp. 137–147, 2011.

[2] S. Singh, R. Negi, and B. Pratap, "Nonlinear feedback linearization controller design for half car suspension system," in 2013 Annual IEEE India Conference (INDICON), pp. 1–6, IEEE, 2013.

[3] H. K. Khalil and J. W. Grizzle, Nonlinear systems, vol. 3. Prentice hall Upper Saddle River, NJ, 2002.

[4] W. Sun, Z. Zhao, and H. Gao, "Saturated adaptive robust control for active suspension systems," IEEE Transactions on industrial electronics, vol. 60, no. 9, pp. 3889–3896, 2012.