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Homework 4

Question 1:

Define a random variable X by the following procedure. Draw a card from a standard deck of playing cards. If the card is knave, queen, or king, then $X = 11$. If the card is an ace, then $X = 1$; otherwise, X is the number of the card (i.e. two through ten). Now define a second random variable Y by the following procedure. When you evaluate X, you look at the color of the card. If the card is red, then $Y = X - 1$; otherwise, $Y = X + 1$.

(a) What is $P(\{X \leq 2\})$?

$$P(\{X \leq 2\}) = \frac{cards\ that\ result\ in\ X \leq 2}{total\ number\ of\ cards}$$

X must either be an Ace or a 2. Since there are four aces and four 2s in the deck:

$$P(\{X \leq 2\}) = \frac{4+4}{52} = \frac{8}{52} = \frac{2}{13}$$

(b) What is $P(\{X \geq 10\})$?

$$P(\{X \geq 10\}) = \frac{cards\ that\ result\ in\ X \geq 10}{total\ number\ of\ cards}$$

X must be a 10 or a jack, queen, or ace. Since there is four of each in a deck:

$$P(\{X \geq 10\}) = \frac{4 * 4}{52} = \frac{16}{52} = \frac{4}{13}$$

(c) What is $P(\{X \geq Y\})$?

X is greater than Y if and only if the card is red. Since half of the deck is red:

$$P(\{X \geq Y\}) = \frac{1}{2}$$

(d) What is the probability distribution of $Y - X$?

Since Y is defined as either one greater than X or one less than X, $Y - X$ will either result in 1 or -1. Since both outcomes are equally as likey because

$$P(\{ Y = X - 1\}) = P(\{card\ being\ red\}) = \frac{1}{2}$$

$$P(\{ Y = X + 1\}) = P(\{card\ being\ black\}) = \frac{1}{2}$$

Therefore:

$$P(\{ Y - X = 1\}) = \frac{1}{2}$$

$$P(\{ Y - X = -1\}) = \frac{1}{2}$$

(e) What is $P(\{Y \geq 12\})$?

The max value of Y is 12 and it is achieved when X = 11 and the color of the card is black.

$$P(\{Y \geq 12\}) = \frac{X = 11\ and\ color\ is\ black}{52}$$

X = 11 when the card is Jack, Queen, or King and there are 2 suits that make the card black. Therefore:

$$P(\{Y \geq 12\}) = \frac{2+2+2}{52} = \frac{6}{52} = \frac{3}{26}$$

Question 2:

Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We consider a game with two players. Each player has a deck of 40 cards. Each player shuffles their deck, then deals seven cards, called their hand. The rest of each player's deck is called their library. Assume that player one has 10 land cards in their deck and player two has 20. Write L_1 for the number of lands in player one's hand and L_2 for the number of lands in player two's hand. Write L_t for the number of lands in the top 10 cards of player one's library.

(a) Write $S = L_1 + L_2$. What is $P(\{S = 0\})$?

$$P(\{S = 0\}) = \frac{player\ 1\ all\ non\ land\ cards}{player\ 1\ all\ possible\ hands} \times \frac{player\ 2\ all\ non\ land\ cards}{player\ 2\ all\ possible\ hands}$$

$$\frac{player\ 1\ all\ non\ land\ cards}{player\ 1\ all\ possible\ hands} = \frac{(30\ choose\ 7)}{(40\ choose\ 7)}$$

$$\frac{player\ 2\ all\ non\ land\ cards}{player\ 2\ all\ possible\ hands} = \frac{(20\ choose\ 7)}{(40\ choose\ 7)}$$

$$P(\{S = 0\}) = \frac{30\ choose\ 7}{40\ choose\ 7} \times \frac{20\ choose\ 7}{40\ choose\ 7}$$

(b) Write $D = L_1 - L_2$. What is $P(\{D = 0\})$?

$P(\{D = 0\}) =$ the sum of all cases where $L_1 = L_2$

$$P(L_1 = i) = \frac{(10\ choose\ i) \times (30\ choose\ (7-i))}{40\ choose\ 7}$$

$$P(L_2 = i) = \frac{(20\ choose\ i) \times (20\ choose\ (7-i))}{40\ choose\ 7}$$

Therefore:

$$P(\{D = 0\}) = \sum_{i=0}^{7}\left[\frac{(10\ choose\ i) \times (30\ choose\ (7-i))}{40\ choose\ 7} \times \frac{(20\ choose\ i) \times (20\ choose\ (7-i))}{40\ choose\ 7}\right]$$

(c) What is the probability distribution for L_1?

$L_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$P(L_1 = i) = \frac{(10\ choose\ i) \times (30\ choose\ (7-i))}{40\ choose\ 7}$

Therefore:

$P(L_1 = 0) = \frac{30\ choose\ 7}{40\ choose\ 7}$

$P(L_1 = 1) = \frac{(10\ choose\ 1) \times (30\ choose\ 6)}{40\ choose\ 7}$

$P(L_1 = 2) = \frac{(10\ choose\ 2)\times (30\ choose\ 5)}{40\ choose\ 7}$

$P(L_1 = 3) = \frac{(10\ choose\ 3) \times (30\ choose\ 4)}{40\ choose\ 7}$

$P(L_1 = 4) = \frac{(10\ choose\ 4) \times (30\ choose\ 3)}{40\ choose\ 7}$

$P(L_1 = 5) = \frac{(10\ choose\ 5) \times (30\ choose\ 2)}{40\ choose\ 7}$

$P(L_1 = 6) = \frac{(10\ choose\ 6) \times(30\ choose\ 1)}{40\ choose\ 7}$

$P(L_1 = 7) = \frac{(10\ choose\ 7)}{40\ choose\ 7}$

(d) Write out the probability distribution for $P(\{L_1|L_t = 10\})$?

If the first 10 cards of player one's deck are all land cards then player one will draw only land cards. Therefore:

$P(L_1 = 7) = 1$

And for i = 0,1,2,3,4,5,6

$P(L_1 = i) = 0$

(e) Write out the probability distribution for $P(\{L_1|L_t = 5\})$?

If we know there are 5 land cards in the top 10 cards of player ones deck we can rewrite the probability calculation as

For i = 0,1,6,7

$P(\{L_1 = i|L_t = 5\}) = 0$

This is because there are not enough land cards in the top ten to have 6 or 7 land cards in your hand and there are too few non-land card to have only 0 or 1 land cards in your hand.

For i = 2,3,4,5

$$P(\{L_1 = i | L_t = 5\}) = \frac{(5\ choose\ i) \times (5\ choose\ (7-i))}{10\ choose\ 7}$$

Therefore:

$$P(L_1 = 0 | L_t = 5) = 0$$

$$P(L_1 = 1 | L_t = 5) = 0$$

$$P(L_1 = 2 | L_t = 5) = \frac{(5\ choose\ 2) \times (5\ choose\ 5)}{10\ choose\ 7}$$

$$P(L_1 = 3 | L_t = 5) = \frac{(5\ choose\ 3) \times (5\ choose\ 4)}{10\ choose\ 7}$$

$$P(L_1 = 4 | L_t = 5) = \frac{(5\ choose\ 4) \times (5\ choose\ 3)}{10\ choose\ 7}$$

$$P(L_1 = 5 | L_t = 5) = \frac{(5\ choose\ 5) \times (50\ choose\ 2)}{10\ choose\ 7}$$

$$P(L_1 = 6 | L_t = 5) = 0$$

$$P(L_1 = 7 | L_t = 5) = 0$$

Question 3:

Determine $E[X]$,$E[Y]$,$E[(3X - 2Y)^2]$

To calculate E[X] we can use

$E[X] = \sum[i \times P(\{X = i\})]$

Where i is all of the possible values for x so 1,2, 3, 5

To calculate $P(\{x\})$ we need to sum all of the values that x can equal.

So,

$P(\{X = 1\}) = 0.05 + 0.05 + 0.1 = 0.2$
$P(\{X = 2\}) = 0.1 + 0.09 + 0.21 = 0.4$
$P(\{X = 3\}) = 0.05 + 0.01 + 0.04 = 0.1$
$P(\{X = 5\}) = 0.1 + 0.05 + 0.15 = 0.3$

So,

$E[X] = 1 * 0.2 + 2 * 0.4 + 3 * 0.1 + 5 * 0.1 = 1.8$

The same equation can be used for $E[Y]$

$P(\{Y = 1\}) = 0.05 + 0.1 + 0.05 + 0.1 = 0.3$
$P(\{Y = 2\}) = 0.05 + 0.09 + 0.01 + 0.05 = 0.2$
$P(\{Y = 4\}) = 0.1 + 0.21 + 0.04 + 0.15 = 0.5$
$E[Y] = 1 * 0.3 + 2 * 0.2 + 3 * 0.5 = 2.2$

$E[(3X - 2Y)^2] = E[9X^2] - E[12XY] + E[4Y^2]$
$= 9 \times E[X^2] + (- 12 \times E[XY]) + 4 * E[Y^2]$

To calculate $E[N^2]$ where N is X or Y we can use:

$E[N] = \sum\left[i^2 \times P(\{N = i\}\right]$

So,

$E[X^2] = 1 * 0.2 + 2^2 * 0.4 + 3^2 * 0.1 + 5^2 * 0.5 = 15.2$
$E[Y^2] = 1 * 0.3 + 2^2 * 0.2 + 3^2 * 0.5 = 5.6$

For $E[XY]$ we can use

$$E[XY] = \sum_i \sum_j [i * j \times P(\{X = i, Y = j\}]$$

Where i is all the possible values for X and j is the possible values for Y.

So,

$$\begin{aligned} E[XY] = \ & 1 * 1 * 0.05 + 1 * 2 * 0.05 + 1 * 4 * 0.1 \\ & + 2 * 1 * 0.1 + 2 * 2 * 0.09 + 2 * 4 * 0.21 \\ & + 3 * 1 * 0.05 + 3 * 2 + 0.01 + 3 * 4 * 0.04 \\ & + 4 * 1 * 0.1 + 5 * 2 * 0.05 + 5 * 4 * 0.15 = 7.93 \end{aligned}$$

Now we plug those values into

$$E[(3X - 2Y)^2] = 9 \times E[X^2] + (-12 \times E[XY]) + 4 * E[Y^2]$$

$$E[(3X - 2Y)^2] = 9 * 15.2 + (-12 * 5.6) + 4 * 7.93 = 101.32$$

Question 4:

a) Calculate the possible values of W and the corresponding probabilities.

Projects success	Net Return	Probability
None	-300	$(1 - 0.2)(1 - 0.3)(1 - 0.6) = 0.224$
P_1 only	700	$(0.2)(1 - 0.3)(1 - 0.6) = 0.056$
P_2 only	200	$(1 - 0.2)(0.3)(1 - 0.6) = 0.096$
P_3 only	-200	$(1 - 0.2)(1 - 0.3)(0.6) = 0.336$
P_1 and P_2	1200	$(0.2)(0.3)(1 - 0.6) = 0.024$
P_1 and P_3	800	$(0.2)(1 - 0.3)(0.6) = 0.084$
P_2 and P_3	300	$(1 - 0.2)(0.3)(0.6) = 0.144$
P_1,P_2,P_3	1300	$(0.2)(0.3)(0.6) = 0.036$

b) What is the expected value of the winnings, E[W]?

$$E[W] = \Sigma\left[W_i * P(W_i)\right]$$

$$= (-300\times0.224) + (700\times0.056) + (200\times0.096) + (-200\times0.336)$$
$$+ (1200\times0.024) + (800\times0.084) + (300\times0.144) + (1300\times0.036)$$
$$= -67.2 + 39.2 + 19.2 - 67.2 + 28.8 + 67.2 + 43.2 + 46.8$$
$$= 109.2$$

Question 5:

Program:

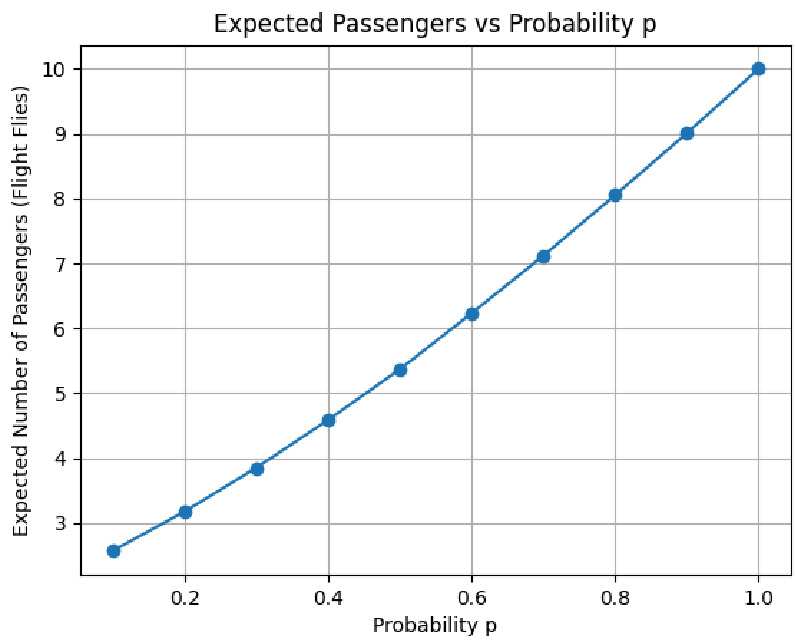
```
import numpy as np
import matplotlib.pyplot as plt

def tickets_to_sell(p):
    return np.ceil(10 / p)

def simulate_flight(p, N=10**5):
    seats = 10
    expected_passengers = []

    for _ in range(N):
        passengers = np.random.binomial(1, p, size=seats)
        women = np.random.binomial(1, 0.5, size=seats) * passengers

        if np.sum(women) >= 2:
            expected_passengers.append(np.sum(passengers))

    return np.mean(expected_passengers)

p_values = np.arange(0.1, 1.1, 0.1)
expected_passengers_flying = []

for p in p_values:
    print("tickets to sell if p = ",p ,": ",tickets_to_sell(p))

print()

for p in p_values:
    x = simulate_flight(p)
    print("Expected passengersif p = ",p ,": ",x)
    expected_passengers_flying.append(x)

plt.plot(p_values, expected_passengers_flying, marker='o')
plt.xlabel('Probability p')
plt.ylabel('Expected Number of Passengers (Flight Flies)')
plt.title('Expected Passengers vs Probability p')
plt.grid(True)
```

```
plt.show()
```

a) Tickets to sell for greater than 10 passengers.

```
tickets to sell if p =  0.1 :  100.0
tickets to sell if p =  0.2 :  50.0
tickets to sell if p =  0.30000000000000004 :  34.0
tickets to sell if p =  0.4 :  25.0
tickets to sell if p =  0.5 :  20.0
tickets to sell if p =  0.6 :  17.0
tickets to sell if p =  0.7000000000000001 :  15.0
tickets to sell if p =  0.8 :  13.0
tickets to sell if p =  0.9 :  12.0
tickets to sell if p =  1.0 :  10.0
```

b) Expected number of passengers.




```
Expected passengersif p =  0.1 :  2.5553757225433524
Expected passengersif p =  0.2 :  3.1768655867066276
Expected passengersif p =  0.30000000000000004 :  3.8431346723740534
Expected passengersif p =  0.4 :  4.583219524735295
Expected passengersif p =  0.5 :  5.377152690449401
Expected passengersif p =  0.6 :  6.219962770093546
Expected passengersif p =  0.7000000000000001 :  7.12262922307465
Expected passengersif p =  0.8 :  8.0458951387869
Expected passengersif p =  0.9 :  9.01562436049605
Expected passengersif p =  1.0 :  10.0
```