An Exact Method for the Vehicle Routing Problem with Backhauls

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We consider the problem in which a fleet of vehicles located at a central depot is to be optimally used to serve a set of customers partitioned into two subsets of linehaul and backhaul customers. Each route starts and ends at the depot and the backhaul customers must be visited after the linehaul customers. A new (0–1) integer programming formulation of this problem is presented. We describe a procedure that computes a valid lower bound to the optimal solution cost by combining different heuristic methods for solving the dual of the LP-relaxation of the exact formulation. An algorithm for the exact solution of the problem is presented. Computational tests on problems proposed in the literature show the effectiveness of the proposed algorithms in solving problems up to 100 customers.

The purpose of this article is to present an exact algorithm for the Vehicle Routing Problem with Backhauls (VRPB) defined as follows. Let $G = (V, A)$ be a directed graph such that $V = \{0\} \cup L \cup B$, where $L = \{1, \ldots, n\}$ corresponds to n linehaul customers, $B = \{n + 1, \ldots, n + m\}$ corresponds to m backhaul customers and the vertex 0 represents the depot. A non-negative cost d_{ij} is associated with each arc $(i, j) \in A$ and a non-negative integer quantity q_i is associated with each customer $i \in L \cup B$. A fleet of M identical vehicles of capacity Q is located at the depot and must be used to supply the linehaul customers and make collections from the backhaul customers. It is required that every route performed by a vehicle starts and ends at the depot and that the load of all linehauls and backhauls does not exceed, separately, the vehicle capacity. Furthermore, in any feasible route, all linehaul customers must precede all backhaul customers. The cost of a route corresponds to the sum of the cost of the arcs forming such route. The problem we consider is of designing M routes, one for each vehicle, so that each customer is visited exactly once and the sum of the route costs is minimized. Fig. 1 shows an example of a VRPB solution. It is easy to see that if $L = \emptyset$ or $B = \emptyset$ then the VRPB reduces to the Vehicle Routing Problem (VRP), proving that the problem is NP-hard (see GAREY and JOHNSON, 1979).

Heuristic methods based on the savings algorithm of CLARKE and WRIGHT (1964) for the VRP have been proposed by DEIF and BODIN (1984), CASCO et al. (1988), and GOETSCHALCKX and JACOBS-BLECHA (1989, 1993). TOTH and VIGO (1996) describe a new cluster-first–route-second heuristic based on a Lagrangean relaxation of a new formulation of the VRPB. Computational results show that the algorithm of Toth and Vigo outperforms the two heuristics of Goetschalckx and Jacobs-Blecha (1989, 1993). ANILY (1996) describes a heuristic method for the VRPB that converges to the optimal solution, under mild probabilistic conditions and when there are no restrictions on the order in which the linehaul and backhaul customers are visited. A 1.5 approximation algorithm for the single vehicle version of the VRPB is given by GENDREAU, HERTZ, and LAPORTE (1997).

An exact algorithm was proposed by YANO et al. (1987) for the special case in which the number of customers of each type in a route is not greater than 4. TOTH and VIGO (1997) describe an exact branch and bound method that uses a Lagrangian lower bound strengthened by adding valid inequalities in a cutting plane fashion.

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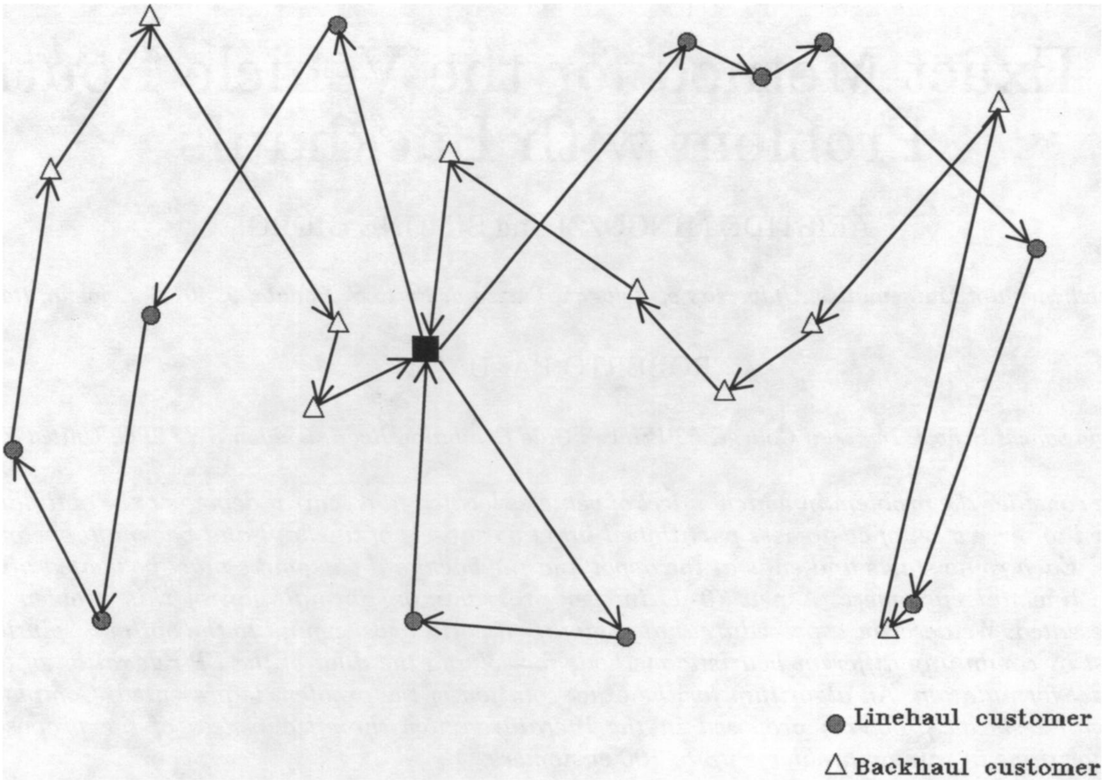

Fig. 1. Example of a VRPB solution.

In this paper, we describe a new (0–1) integer programming formulation of the VRPB based upon a set-partitioning approach. We use a heuristic procedure to solve the dual problem, called D, of the LP-relaxation of the integer formulation to obtain a valid lower bound to the VRPB. This procedure, called HDS, combines two different heuristic algorithms, each one finding a feasible solution to D without requiring the entire set of the dual constraints. The dual solution thus obtained and a valid upper bound to the VRPB are then used to reduce the number of routes (e.g., the variables of the integer formulation) which may form an optimal solution.

However, the size of the reduced integer problem might still be too large for solving it by a branch and bound method. In this case, we propose a procedure, called EHP, that consists of reducing the number of variables of the integer program so that the resulting problem can be solved by an integer programming solver. This method may terminate without having found an optimal solution. However, procedure EHP provides a means to estimate the maximum deviation from optimality of the VRPB solution obtained. Computational tests on two classes of test problems from the literature show the effectiveness of the proposed method in solving problems up to about 100 customers.

The paper is organized as follows. Section 1 describes the notation used throughout the paper. Section 2 gives the new integer programming model for VRPB and presents different pricing procedures for reducing the variables of the ILP model. Section 3 presents the bounding procedure HDS. Algorithm EHP is described in Section 4, and computational results are given in Section 5.

1. NOTATION

IN THIS SECTION, we describe additional notation used in the paper. To ensure feasibility, we assume that $M \geqslant \text{Max}[M_L, M_B]$, where M_L (resp. M_B) is the minimum number of vehicles needed to visit all the linehaul (resp. backhaul) customers. The values M_L and M_B can be computed by solving the Bin Packing Problem (see MARTELLO and TOTH, 1990) associated with the linehaul and backhaul customers. Following Toth and Vigo (1997), we assume that routes containing only backhaul customers are not allowed. We must point out that the method we are going to describe for solving the VRPB can be easily extended to deal with the case where $M_L < M_B$.

Let us denote by $G_L = (L_0, A_L)$ and $G_B = (B_0, A_B)$

the two subgraphs of G induced by the linehaul and backhaul customers, respectively, where

$$\begin{aligned} &L_0 = L \cup \{0\} \\ \text{and}\quad &A_{\mathrm{L}} = \{(i, j)\colon (i, j) \in A \text{ s.t. } i, j \in L_0\} \\ &B_0 = B \cup \{0\} \\ \text{and}\quad &A_{\mathrm{B}} = \{(i, j)\colon (i, j) \in A \text{ s.t. } i, j \in B_0\} \end{aligned} \tag{1}$$

Let us define $A_0 = \{(i, j)\colon (i, j) \in A$ s.t. $i \in L, j \in B_0\}$.

An elementary path P in G_{L} starting at vertex 0 (resp. in G_{B} ending at vertex 0) is called a feasible path if its load satisfies the inequalities

$$Q^{\mathrm{L}}_{\min} \leqslant \sum_{i \in P} q_i \leqslant Q \quad \left(\text{resp. } Q^{\mathrm{B}}_{\min} \leqslant \sum_{i \in P} q_i \leqslant Q\right), \tag{2}$$

where $Q^{\mathrm{L}}_{\min}$ (resp. $Q^{\mathrm{B}}_{\min}$) represents the minimum load of linehaul customers (resp. backhaul customers) of any feasible path in G_{L} (resp. G_{B}).

The values $Q^{\mathrm{L}}_{\min}$ and $Q^{\mathrm{B}}_{\min}$ are computed as follows:

$$Q^{\mathrm{L}}_{\min} = \mathrm{Max}\left[0, \left(\sum_{i \in L} q_i\right) - (M - 1)Q\right]$$

and

$$Q^{\mathrm{B}}_{\min} = \mathrm{Max}\left[0, \left(\sum_{i \in B} q_i\right) - (M - 1)Q\right].$$

We will use $t(P)$ to indicate both the terminal vertex of a feasible path P in G_{L} and the starting vertex of a feasible path P in G_{B}. Note that any pair of feasible paths P in G_{L} and P' in G_{B} and the arc $(t(P), t(P')) \in A_0$ form a feasible route that is obtained by appending to the end of P the arc $(t(P), t(P'))$ and the path P'. Furthermore, any feasible path P in G_{L} leads to a feasible route involving linehaul customers only by appending to P the arc $(t(P), 0) \in A_0$. Since we assume $M_{\mathrm{L}} \geqslant M_{\mathrm{B}}$, no feasible route exists involving backhaul customers only. Finally, we must observe that the M routes of any feasible VRPB solution consist of M feasible paths in G_{L}, at least M_{B} feasible paths in G_{B} and M arcs of the subset A_0.

2. MATHEMATICAL FORMULATION

LET $\mathscr{L}$ BE THE INDEX set of all feasible paths in G_{L}. We denote with $\mathscr{L}_i \subseteq \mathscr{L}$ (resp. $\mathscr{L}^{\mathrm{E}}_i \subseteq \mathscr{L}$) the index set of all paths passing through (resp. ending at) customer $i \in L$. Let $\mathscr{B}$ be the index set of all feasible paths in G_{B}. We denote by $\mathscr{B}_i \subseteq \mathscr{B}$ (resp. $\mathscr{B}^{\mathrm{S}}_i \subseteq \mathscr{B}$) the index set of all paths passing through (resp. starting at) customer $i \in B$. We indicate with c_ℓ the cost of path $\ell \in \mathscr{L} \cup \mathscr{B}$. In the following, we will use $t(P_\ell)$ and/or t_ℓ, to denote the terminal vertex of path P_ℓ, if $\ell \in \mathscr{L}$, or the starting vertex of path P_ℓ, if $\ell \in \mathscr{B}$.

Let us define the following binary variables: x_ℓ, $\ell \in \mathscr{L}$, y_ℓ, $\ell \in \mathscr{B}$ and ξ_{ij}, $(i, j) \in A_0$. We have $x_\ell = 1$, $y_{\ell'} = 1$ and $\xi_{ij} = 1$ if and only if the paths $\ell \in \mathscr{L}$, $\ell' \in \mathscr{B}$ and the arc $(i, j) \in A_0$ are in the optimal VRPB solution.

An integer programming formulation of the VRPB is as follows.

(IP)

$$z(\mathrm{IP}) = \mathrm{Min} \sum_{\ell \in \mathscr{L}} c_\ell x_\ell + \sum_{\ell \in \mathscr{B}} c_\ell y_\ell + \sum_{(i,j) \in A_0} d_{ij}\xi_{ij} \tag{3}$$

$$\text{subject to} \quad \sum_{\ell \in \mathscr{L}_i} x_\ell = 1, \quad i \in L \tag{4}$$

$$\sum_{\ell \in \mathscr{B}_j} y_\ell = 1, \quad j \in B \tag{5}$$

$$\sum_{\ell \in \mathscr{L}^{\mathrm{E}}_i} x_\ell - \sum_{j \in \mathrm{B}_0} \xi_{ij} = 0, \quad i \in L \tag{6}$$

$$\sum_{\ell \in \mathscr{B}^{\mathrm{S}}_j} y_\ell - \sum_{i \in \mathrm{L}} \xi_{ij} = 0, \quad j \in B \tag{7}$$

$$\sum_{(i,j) \in A_0} \xi_{ij} = M \tag{8}$$

$$x_\ell \in \{0, 1\}, \quad \ell \in \mathscr{L}, \quad y_\ell \in \{0, 1\}, \quad \ell \in \mathscr{B}, \quad \xi_{ij} \in \{0, 1\}, \quad (i, j) \in A_0. \tag{9}$$

Equations 4 and 5 require that each vertex $i \in L$ and $j \in B$ be visited by one route. Equation 6 forces the solution to contain an arc of A_0 starting at vertex $i \in L$ whenever such solution contains a feasible path in G_{L} ending at vertex $i \in L$. Equation 7 requires in the solution an arc (i, j) with $i \in L$ and $j \in B$ if such a solution contains a feasible path in G_{B} starting at vertex $j \in B$. Equation 8 forces the solution to contain M routes by requiring that M arcs of the set A_0 are in solution. Since the set A_0 contains all arcs $(i, 0)$, $\forall i \in L$, routes containing only linehaul customers are allowed.

Problem IP cannot be solved directly, even for problems of moderate size because the number of variables may be too large. In this paper, we describe a heuristic procedure that finds a feasible solution of the dual problem D of the linear relaxation of IP, thus providing a valid lower bound to the VRPB. This procedure does not require the explicit generation of the path set $\mathscr{L}$ and $\mathscr{B}$. Moreover, this dual solution is used by the exact procedure that solves the VRPB to reduce drastically the sets $\mathscr{L}$ and

$\mathcal{B}$, removing those paths that cannot belong to any optimal VRPB solution.

Let u_i, $i \in L$ and v_j, $j \in B$, be the dual variables associated with constraints 4 and 5, respectively. Indicate by α_i, $i \in L$, and β_j, $j \in B$, the dual variables associated with constraints 6 and 7, respectively. Finally, associate with constraint 8 the dual variable w.

The dual of the LP-relaxation of IP is

$$\text{(D)} \qquad z(\mathrm{D}) = \operatorname{Max} \sum_{i \in \mathrm{L}} u_i + \sum_{j \in \mathrm{B}} v_j + Mw \tag{10}$$

subject to

$$\sum_{k \in P_\ell} u_k + \alpha_i \leq c_\ell, \quad \ell \in \mathcal{L}_i^{\mathrm{E}}, \quad i \in L \tag{11}$$

$$\sum_{k \in P_\ell} v_k + \beta_j \leq c_\ell, \quad \ell \in \mathcal{B}_j^{\mathrm{S}}, \quad j \in B \tag{12}$$

$$-\alpha_i - \beta_j + w \leq d_{ij}, \quad (i, j) \in A_0 \tag{13}$$

$$\left.\begin{array}{lll} u_i, \alpha_i & \text{unrestricted,} & i \in L \\ v_j, \beta_j & \text{unrestricted,} & j \in B \\ w & \text{unrestricted.} & \end{array}\right\} \tag{14}$$

Note that we assume $\beta_0 = 0$ in the dual constraints 13, $\forall (i, 0) \in A_0$.

2.1 Variable Reduction of Problem IP

Let $(\mathbf{u}', \mathbf{v}', \boldsymbol{\alpha}', \boldsymbol{\beta}', w')$ be a feasible solution of D of cost $z'(\mathrm{D})$ and let $(\mathbf{x}', \mathbf{y}', \xi')$ be a feasible solution of IP of cost $z'(\mathrm{IP})$. We denote by c'_ℓ and d'_{ij} the reduced costs according to $(\mathbf{u}', \mathbf{v}', \boldsymbol{\alpha}', \boldsymbol{\beta}', w')$ of each path $\ell \in \mathcal{L} \cup \mathcal{B}$ and each arc $(i, j) \in A_0$, respectively, that is

$$\begin{aligned} c'_\ell &= c_\ell - \sum_{k \in P_\ell} u'_k - \alpha'_i, \quad \ell \in \mathcal{L}_i^{\mathrm{E}}, \quad i \in L, \\ c'_\ell &= c_\ell - \sum_{k \in P_\ell} v'_k - \beta'_j, \quad \ell \in \mathcal{B}_j^{\mathrm{S}}, \quad j \in B, \\ d'_{ij} &= d_{ij} + \alpha'_i + \beta'_j - w', \quad (i, j) \in A_0. \end{aligned} \tag{15}$$

From linear programming duality we have $z'(\mathrm{D}) \leq z'(\mathrm{IP})$. Furthermore, we can use these two solutions to reduce the variables of IP as it is established in the following theorem.

THEOREM 1. *Let $X = \{\ell\colon \ell \in \mathcal{L}, \text{s.t. } x'_\ell = 1\}$, $Y = \{\ell\colon \ell \in \mathcal{B}, \text{s.t. } y'_\ell = 1\}$ and $H = \{(i, j)\colon (i, j) \in A_0, \text{s.t. } \xi'_{ij} = 1\}$. The following relationship holds:*

$$z'(\mathrm{IP}) = z'(\mathrm{D}) + \sum_{\ell \in X} c'_\ell + \sum_{\ell \in Y} c'_\ell + \sum_{(i,j) \in H} d'_{ij}. \tag{16}$$

Proof. Follows directly from linear programming duality.

COROLLARY 1. *Let $z(\mathrm{UB})$ be the cost of a feasible VRPB solution and $(\mathbf{u}', \mathbf{v}', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathrm{w}')$ be a feasible solution of D of cost $z'(\mathrm{D})$. Any optimal solution of IP of cost less than $z(\mathrm{UB})$ cannot contain any path $\ell \in \mathcal{L} \cup \mathcal{B}$ or any arc $(i, j) \in A_0$ whose reduced cost is greater or equal to $z(\mathrm{UB}) - z'(\mathrm{D})$.*

Proof. Follows directly from Theorem 1.

Corollary 1 states that an optimal VRPB solution can be obtained by replacing in problem IP the sets $\mathcal{L}$, $\mathcal{B}$, and A_0 with the subsets $\mathcal{L}' \subseteq \mathcal{L}$, $\mathcal{B}' \subseteq \mathcal{B}$ and $A'_0 \subseteq A_0$, defined as

$$\begin{aligned} \mathcal{L}' &= \{\ell\colon \ell \in \mathcal{L}, \text{ s.t. } c'_\ell < z(\mathrm{UB}) - z'(\mathrm{D})\}, \\ \mathcal{B}' &= \{\ell\colon \ell \in \mathcal{B}, \text{ s.t. } c'_\ell < z(\mathrm{UB}) - z'(\mathrm{D})\}, \\ A'_0 &= \{(i, j)\colon (i, j) \in A_0, \text{ s.t. } d'_{ij} < z(\mathrm{UB}) - z'(\mathrm{D})\}. \end{aligned} \tag{17}$$

Note that expressions 17 require the computation of the reduced costs of the paths of $\mathcal{L}$ and $\mathcal{B}$ and of the arcs of A_0. The effectiveness of expressions 17 increases if the gap between the upper bound $z(\mathrm{UB})$ and the lower bound $z'(\mathrm{D})$ is small.

2.1.1 A Better Variable Reduction

A further reduction of the sets $\mathcal{L}'$, $\mathcal{B}'$, and A'_0 can be achieved by means of the following observations.

Reduction of $\mathcal{L}'$. Consider a VRPB solution containing a given path $\ell \in \mathcal{L}'$. If this solution is feasible, it must also contain a path from $t_\ell \in L$ to the depot to complete the route emerging from ℓ. We have two cases:

(i) the path completing ℓ consists of the arc $(t_\ell, 0)$. In this case, from Theorem 1, the resulting VRPB solution cannot be smaller than

$$z'(\mathrm{D}) + c'_\ell + d'_{t_\ell 0}; \tag{18}$$

(ii) the path completing ℓ starts at t_ℓ, goes directly to some backhaul customer (say j) and returns to the depot passing through backhaul customers. This VRPB solution has a cost not smaller than

$$z'(\mathrm{D}) + c'_\ell + \operatorname*{Min}_{j \in \mathrm{B}} \left[d'_{t_\ell j} + \operatorname*{Min}_{r \in \mathcal{B}_j'^{\mathrm{S}}} [c'_r] \right], \tag{19}$$

where $\mathcal{B}_j'^{\mathrm{S}} = \mathcal{B}_j^{\mathrm{S}} \cap \mathcal{B}'$, $j \in B$.

COROLLARY 2. *An optimal VRPB solution cannot contain any path $\ell \in \mathcal{L}'$ such that*

$$c'_\ell + \operatorname*{Min}_{j \in \mathrm{B}_0} \left[d'_{t_\ell j} + \operatorname*{Min}_{r \in \mathcal{B}_j'^{\mathrm{S}}} [c'_r] \right] \geq z(\mathrm{UB}) - z'(\mathrm{D}), \tag{20}$$

where we assume

$$\operatorname*{Min}_{r \in \mathcal{B}_0'^{\mathrm{S}}} [c'_r] = 0.$$

Reduction of $\mathscr{B}'$. Observations similar to those made to establish Corollary 2 lead to the following corollary.

COROLLARY 3. *An optimal VRPB solution cannot contain any path $\ell \in \mathscr{B}'$ such that*

$$c'_\ell + \underset{i \in L}{\text{Min}} \left[\underset{r \in \mathscr{L}'^{\text{E}}_i}{\text{Min}} [c'_r] + d'_{i t_\ell} \right] \geq z(\text{UB}) - z'(\text{D}), \tag{21}$$

where $\mathscr{L}'^{\text{E}}_i = \mathscr{L}^{\text{E}}_i \cap \mathscr{L}'$, $i \in L$.

Reduction of A'_0. Any VRPB solution containing the arc $(i, j) \in A'_0$ must contain either a path of the set $\mathscr{L}'^{\text{E}}_i$ and a path of $\mathscr{B}'^{\text{S}}_j$, if $j \neq 0$, or the arc $(i, 0)$, if $j = 0$. Hence, from Theorem 1, we have the following corollary.

COROLLARY 4. *An optimal VRPB solution cannot contain the arc $(i, j) \in A_0$ if*

$$\underset{\ell \in \mathscr{L}'^{\text{E}}_i}{\text{Min}} [c'_\ell] + d'_{ij} + \underset{r \in \mathscr{B}'^{\text{S}}_j}{\text{Min}} [c'_r] \geq z(\text{UB}) - z'(\text{D}). \tag{22}$$

Note that the tests described might be insufficient for reducing the size of the sets $\mathscr{L}'$ and $\mathscr{B}'$ so that problem IP can become solvable. This might happen even if the gap $z(\text{UB}) - z'(\text{D})$ is small. In Section 4, we describe a procedure to reduce $\mathscr{L}'$ and $\mathscr{B}'$ so that the resulting problem IP can be solved but without any guarantee that the solution obtained is an optimal VRPB solution. However, the reduction of the sets $\mathscr{L}'$ and $\mathscr{B}'$ is such that it is possible to estimate the maximum distance from optimality of the solution obtained.

3. HEURISTIC PROCEDURE FOR SOLVING PROBLEM D

IT IS WELL KNOWN from linear programming duality, that the cost of any feasible solution to D is a lower bound to the optimal solution cost of IP. In this section, we describe a heuristic procedure (called HDS) for finding a feasible solution to D that is based on the following general idea. A feasible solution $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2 + \cdots + \mathbf{w}^k$ of the linear program,

$$\text{(LP)} \qquad \text{Max } \mathbf{wb}$$
$$\text{subject to} \quad \mathbf{wA} \leq \mathbf{c}$$
$$\mathbf{w} \text{ unrestricted},$$

can be obtained by successively solving a sequence of k linear programs $\text{LP}^1, \text{LP}^2, \ldots, \text{LP}^k$ by means of k different heuristic procedures $\text{H}^1, \text{H}^2, \ldots, \text{H}^k$. Procedure H^r finds a feasible solution $\mathbf{w}^r$ of the linear program LP^r defined as

$$(\text{LP}^r) \qquad \text{Max } \mathbf{w}^r\mathbf{b}$$
$$\text{subject to} \quad \mathbf{w}^r\mathbf{A} \leq \mathbf{c}^r$$
$$\mathbf{w}^r \text{ unrestricted},$$

where $\mathbf{c}^r = \mathbf{c} - (\mathbf{w}^0 + \mathbf{w}^1 + \cdots + \mathbf{w}^{r-1})\mathbf{A}$ and $\mathbf{w}^0 = \mathbf{0}$. Note that linear program LP^1 coincides with LP.

This general method has been applied by MINGOZZI, CHRISTOFIDES, and HADJICONSTANTINOU (1994) for solving the Vehicle Routing Problem, by BIANCO, MINGOZZI, and RICCIARDELLI (1994) for the Multiple Depot Vehicle Scheduling, and by MINGOZZI et al. (1995) for the Crew Scheduling Problem.

The procedure, HDS, that we propose for solving problem D involves two heuristic procedures H^1 and H^2 used in sequence. Procedure H^1 finds a feasible solution $(\mathbf{u}^1, \mathbf{v}^1, \alpha^1, \beta^1, w^1)$ of problem D^1 ($\equiv$ D) without requiring the generation of the sets $\mathscr{L}$ and $\mathscr{B}$. The second procedure, H^2, solves problem D^2 that is obtained from D by replacing the path costs c_ℓ, $\ell \in \mathscr{L} \cup \mathscr{B}$ and the arc costs d_{ij}, $(i, j) \in A_0$ with the reduced costs c'_ℓ and d'_{ij} computed according to the solution $(\mathbf{u}^1, \mathbf{v}^1, \alpha^1, \beta^1, w^1)$ obtained by procedure H^1. Procedure H^2 requires the generation of limited subsets of the sets $\mathscr{L}$ and $\mathscr{B}$.

3.1 Procedure H^1

This procedure is based on the observation that any route of a feasible VRPB solution contains an arc of the set A_0. A lower bound to the VRPB can be obtained as follows. Associate to each arc $(i, j) \in A_0$ a cost representing a lower bound on the cost of the least cost route passing through it. Therefore, the sum of the costs of the M vertex–disjoint arcs of minimum cost of A_0 is a valid lower bound to the VRPB. This problem, called $\text{PR}(\boldsymbol{\lambda}, \boldsymbol{\mu})$, is defined as follows. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$ be two vectors of unrestricted real numbers associated with the linehaul and backhaul customers, respectively.

Let us associate with each arc $(i, j) \in A$ a cost $\bar{d}_{ij}$ as follows:

$$\bar{d}_{ij} = \begin{cases} d_{ij} - \lambda_j & \text{if } j \in L \\ d_{ij} - \mu_j & \text{if } j \in B. \end{cases} \tag{23}$$

Denote by φ^{L}_i (resp. φ^{B}_j) a lower bound to the cost of the least cost feasible path ending at vertex i of G_{L} (resp. starting at vertex j of G_{B}) using the arc costs $\{\bar{d}_{ij}\}$ defined by expressions 23. Therefore, the values φ^{L}_i, $i \in L$, and φ^{B}_j, $j \in B$, satisfy the inequalities

$$\begin{aligned} \varphi^{\text{L}}_i &\leq c_\ell - \sum_{k \in P_\ell} \lambda_k, \quad \ell \in \mathscr{L}^{\text{E}}_i, \quad i \in L, \\ \varphi^{\text{B}}_j &\leq c_\ell - \sum_{k \in P_\ell} \mu_k, \quad \ell \in \mathscr{B}^{\text{S}}_j, \quad j \in B. \end{aligned} \tag{24}$$

A valid lower bound b_{ij} to the cost of the least cost route passing through arc $(i, j) \in A_0$ can be computed as

$$b_{ij} = \varphi_i^{\mathrm{L}} + d_{ij} + \varphi_j^{\mathrm{B}}, \quad \forall (i, j) \in A_0, \tag{25}$$

where we assume $\varphi_0^{\mathrm{B}} = 0$. In subsection 3.1.2, we describe a method for computing φ_i^{L} and φ_j^{B} that does not require the enumeration of the path sets $\mathcal{L}$ and $\mathcal{B}$.

The mathematical formulation of PR($\boldsymbol{\lambda}, \boldsymbol{\mu}$) is as follows.

PR($\boldsymbol{\lambda}, \boldsymbol{\mu}$)

$$z(\mathrm{PR}(\boldsymbol{\lambda}, \boldsymbol{\mu})) = \mathrm{Min} \sum_{(i,j)\in A_0} b_{ij}\xi_{ij} + \sum_{i\in L} \lambda_i + \sum_{j\in B} \mu_j \tag{26}$$

$$\text{subject to} \quad \sum_{j\in B_0} \xi_{ij} \leqslant 1, \quad i \in L \tag{27}$$

$$\sum_{i\in L} \xi_{ij} \leqslant 1, \quad j \in B \tag{28}$$

$$\sum_{(i,j)\in A_0} \xi_{ij} = M \tag{29}$$

$$\xi_{ij} \in \{0, 1\}, \quad (i,j) \in A_0. \tag{30}$$

Constraints 27 and 28 force the arcs of a feasible PR($\boldsymbol{\lambda}, \boldsymbol{\mu}$) solution to be vertex–disjoint, and constraint 29 requires that exactly M arcs are in the solution.

Let $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$ and $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_m)$ be two vectors of dual variables associated with constraints 27 and 28, respectively, and let δ be the dual variable associated with constraint 29. The dual of the LP-relaxation of PR($\boldsymbol{\lambda}, \boldsymbol{\mu}$), called DPR($\boldsymbol{\lambda}, \boldsymbol{\mu}$), is then

DPR($\boldsymbol{\lambda}, \boldsymbol{\mu}$)

$$z(\mathrm{DPR}(\boldsymbol{\lambda}, \boldsymbol{\mu})) = \mathrm{Max} \sum_{i\in L} \eta_i + \sum_{j\in B} \nu_j + M\delta + \sum_{i\in L} \lambda_i + \sum_{j\in B} \mu_j \tag{31}$$

$$\text{subject to} \quad \eta_i + \nu_j + \delta \leqslant b_{ij}, \quad (i, j) \in A_0 \tag{32}$$

$$\left.\begin{array}{ll} \eta_i \leqslant 0, & i \in L \\ \nu_j \leqslant 0, & j \in B \\ \delta \text{ unrestricted.} & \end{array}\right\} \tag{33}$$

Note that we assume $\nu_0 = 0$ in inequality 32. In the following theorem, we prove that a feasible solution of D can be obtained from any feasible solution of DPR($\boldsymbol{\lambda}, \boldsymbol{\mu}$) and the predetermined vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, showing that z(PR($\boldsymbol{\lambda}, \boldsymbol{\mu}$)), for any $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, is a valid lower bound to the VRPB.

THEOREM 2. *Let* ($\boldsymbol{\eta}, \boldsymbol{\nu}, \delta$) *be a feasible solution of* DPR($\boldsymbol{\lambda}, \boldsymbol{\mu}$) *of cost* z(DPR($\boldsymbol{\lambda}, \boldsymbol{\mu}$)) *for a given pair of vectors* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$. *A feasible solution of* D *of cost* $z(\mathrm{D}) = z(\mathrm{DPR}(\boldsymbol{\lambda}, \boldsymbol{\mu}))$ *is given by*

$$\begin{aligned} &u_i = \lambda_i + \eta_i, \qquad \alpha_i = \varphi_i^{\mathrm{L}} - \eta_i, \quad i \in L \\ &v_j = \mu_j + \nu_j, \qquad \beta_j = \varphi_j^{\mathrm{B}} - \nu_j, \quad j \in B \\ &w = \delta, \end{aligned} \tag{34}$$

hence, z(DPR($\boldsymbol{\lambda}, \boldsymbol{\mu}$)) *is a valid lower bound for IP for any* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$.

Proof. It is easy to see that the values of the dual variables of D computed according to Eqs. 34 satisfy constraints 14.

Consider a path $\ell \in \mathcal{L}_i^{\mathrm{E}}$. Substituting into the left-hand side of inequalities 11 the values of $\mathbf{u}$ and α_i, given by Eqs. 34, yields

$$\sum_{k\in P_\ell} u_k + \alpha_i = \sum_{k\in P_\ell} \lambda_k + \sum_{k\in P_\ell} \eta_k + \varphi_i^L - \eta_i. \tag{35}$$

Note that

$$\sum_{k\in P_\ell} \eta_k - \eta_i = \sum_{k\in P_\ell\setminus\{i\}} \eta_k$$

and, since $\eta_k \leqslant 0$, $k \in L$, we have

$$\sum_{k\in P_\ell\setminus\{i\}} \eta_k \leqslant 0.$$

Therefore, from Eq. 35, we obtain

$$\sum_{k\in P_\ell} u_k + \alpha_i \leqslant \sum_{k\in P_\ell} \lambda_k + \varphi_i^{\mathrm{L}}. \tag{36}$$

From inequalities 24 and 36, we have

$$\sum_{k\in P_\ell} u_k + \alpha_i \leqslant \sum_{k\in P_\ell} \lambda_k + c_\ell - \sum_{k\in P_\ell} \lambda_k (= c_\ell). \tag{37}$$

Inequality 37 shows that the values of $\mathbf{u}$ and $\boldsymbol{\alpha}$, given by Eqs. 34, satisfy constraints 11. In a similar way, it is easy to show that the values of $\mathbf{v}$ and $\boldsymbol{\beta}$, given by Eqs. 34 satisfy inequalities 12. Finally, let us prove that the values of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and w, given by Eqs. 34, satisfy inequalities 13. In fact, for each arc $(i, j) \in A_0$ we have

$$-\alpha_i - \beta_j + w = -\varphi_i^{\mathrm{L}} + \eta_i - \varphi_j^{\mathrm{B}} + \nu_j + \delta. \tag{38}$$

From inequalities 32 and the definition of b_{ij}, we obtain

$$\eta_i + \nu_j + \delta \leqslant \varphi_i^{\mathrm{L}} + \varphi_j^{\mathrm{B}} + d_{ij} \tag{39}$$

or

$$-\varphi_i^{\mathrm{L}} + \eta_i - \varphi_j^{\mathrm{B}} + \nu_j + \delta \leqslant d_{ij}. \tag{40}$$

From Eqs. 38 and inequalities 40, we obtain inequalities 13. □

In MINGOZZI et al. (1996), it is shown that an optimal solution ($\boldsymbol{\eta}^*$, $\boldsymbol{\nu}^*$, δ^*) of DPR($\boldsymbol{\lambda}$, $\boldsymbol{\mu}$), for given vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, can be efficiently computed in $O[(n + m)^3]$ time by transforming PR($\boldsymbol{\lambda}$, $\boldsymbol{\mu}$) into a transportation problem.

3.1.1 Improving the Value z(PR($\boldsymbol{\lambda}$, $\boldsymbol{\mu}$))

Algorithm H^1 is an iterative procedure that finds a feasible solution of the problem D^1 by finding a feasible solution of the problem

$$\operatorname*{Max}_{\lambda, \mu} [z(\mathrm{PR}(\boldsymbol{\lambda}, \boldsymbol{\mu}))]. \tag{41}$$

An iteration of H^1 consists of computing new vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ and of finding a new solution of the resulting problem PR($\boldsymbol{\lambda}$, $\boldsymbol{\mu}$). The method used for changing $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, at each iteration, is as follows.

Let ξ^* be an optimal PR($\boldsymbol{\lambda}$, $\boldsymbol{\mu}$) solution for given $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ and $H^* = \{(i, j): (i, j) \in A_0, \xi^*_{ij} = 1\}$. Denote with L^* (resp. B^*) the set of linehaul (resp. backhaul) terminal vertices of the arcs of H^* (i.e., $L^* = \{i: (i, j) \in H^*\}$ and $B^* = \{j: (i, j) \in H^*\}$). We indicate with P^L_i, $i \in L$, and P^B_j, $j \in B$, the paths of cost φ^{L}_i and φ^{B}_j, respectively. Let h_i be the number of times that vertex $i \in L \cup B$ appears in the paths P^{L}_k, $k \in L^*$ and P^{B}_k, $k \in B^*$. It is obvious that, in any feasible VRPB solution, we have $h_i = 1$, $i \in L \cup B$; hence, a subgradient optimization method can be used to change $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as follows.

$$\lambda_i = \lambda_i - \varepsilon \frac{z(\mathrm{UB}) - z(\mathrm{PR}(\boldsymbol{\lambda}, \boldsymbol{\mu}))}{\sum_{j\in L} (h_j - 1)^2 + \sum_{j\in B} (h_j - 1)^2} (h_i - 1), \quad i \in L \cup B. \tag{42}$$

The solution of D^1 is given by Eqs. 34 using the values of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ that produce the best approximate solution of problem 41 and the values of $\boldsymbol{\eta}$, $\boldsymbol{\nu}$, and $\boldsymbol{\delta}$ of an optimal solution of the corresponding problem DPR($\boldsymbol{\lambda}$, $\boldsymbol{\mu}$).

3.1.2 Computing φ^{L}_i, $i \in L$ *and* φ^{B}_j, $j \in B$

We describe a method for computing φ^{L}_i, $i \in L$, an equivalent method can be used for computing φ^{B}_j, $j \in B$. A path (not necessarily elementary) $\boldsymbol{\Phi} = (0, i_1, i_2, \ldots, i_k)$ in the graph G_{L} and such that $\Sigma_{i\in\Phi}\, q_i = q$ is called a q-path (see CHRISTOFIDES, MINGOZZI, and TOTH (1981a)). Let $f_i(q)$ be the cost of the least cost q-path in G_{L} from depot 0 to vertex $i \in L$, using arc costs $\{\bar{d}_{ij}\}$ defined by expression (23). Christofides et al. (1981a) describe a dynamic programming algorithm of complexity $O(Qn^2)$ for computing the value $f_i(q)$, for each $i \in L$, and $q = q_i, q_i + 1, \ldots, Q$ with the restriction that the q-path corresponding to $f_i(q)$ should not contain loops formed by three consecutive vertices. Using function $f_i(q)$, the lower bound φ^{L}_i, $i \in L$, can be computed as

$$\varphi^{\mathrm{L}}_i = \operatorname*{Min}_{q_i \le q \le Q} [f_i(q)]. \tag{43}$$

3.2 Procedure H^2

Let ($\mathbf{u}^1$, $\mathbf{v}^1$, $\boldsymbol{\alpha}^1$, $\boldsymbol{\beta}^1$, w^1) be a feasible solution of D^1 of cost $z(\mathrm{D}^1)$ produced by procedure H^1. The reduced costs of the variables of problem IP are given by

$$\begin{aligned} c^2_\ell &= c_\ell - \sum_{k\in P_\ell} u^1_k - \alpha^1_{t_\ell}, \quad \ell \in \mathscr{L}, \\ c^2_\ell &= c_\ell - \sum_{k\in P_\ell} v^1_k - \beta^1_{t_\ell}, \quad \ell \in \mathscr{B}, \\ d^2_{ij} &= d_{ij} + \alpha^1_i + \beta^1_j - w^1, \quad (i, j) \in A_0. \end{aligned} \tag{44}$$

We denote by D^2 the problem obtained from D by replacing $\{c_\ell\}$ with $\{c^2_\ell\}$ and $\{d_{ij}\}$ with $\{d^2_{ij}\}$.

Problem D^2 cannot be solved directly because the number of constraints may be too large. We note here that it is not convenient to use H^1 for solving D^2. In fact, it can be easily shown that this would be equivalent to double the number of subgradient iterations performed by H^1 in solving D^1. Thus the value of $z(\mathrm{D}^2)$, obtained by H^1 in solving D^2, is going to be small (eventually zero) provided that the maximum number of subgradient iterations allowed to H^1 for solving D^1 is sufficiently large.

In this section, we describe a heuristic procedure, called H^2, for reducing the number of constraints of D^2 so that the resulting problem, called $\bar{\mathrm{D}}^2$, can be solved directly and any solution of $\bar{\mathrm{D}}^2$ is a feasible D^2 solution. Problem $\bar{\mathrm{D}}^2$ is obtained from D^2 as follows:

(i) reduce the number of constraints 11 and 12 by replacing $\mathscr{L}$ and $\mathscr{B}$ with subsets $\bar{\mathscr{L}} \subseteq \mathscr{L}$ and $\bar{\mathscr{B}} \subseteq \mathscr{B}$ of limited size;
(ii) add constraints to force any $\bar{\mathrm{D}}^2$ solution to satisfy constraints 11 for any $\ell \in \mathscr{L}\backslash\bar{\mathscr{L}}$ and constraints 12 for any $\ell \in \mathscr{B}\backslash\bar{\mathscr{B}}$.

Reduced problem $\bar{D}^2$. Let $\bar{\mathscr{L}} \subseteq \mathscr{L}$ and $\bar{\mathscr{B}} \subseteq \mathscr{B}$ be the subsets of paths satisfying the conditions

$$\begin{aligned} &\operatorname*{Max}_{\ell\in\bar{\mathscr{L}}} [c^2_\ell] \le \operatorname*{Min}_{\ell\in\mathscr{L}\backslash\bar{\mathscr{L}}} [c^2_\ell] && \text{(a)} \\ &\operatorname*{Max}_{\ell\in\bar{\mathscr{B}}} [c^2_\ell] \le \operatorname*{Min}_{\ell\in\mathscr{B}\backslash\bar{\mathscr{B}}} [c^2_\ell] && \text{(b)} \\ &c^2_\ell < z(\mathrm{UB}) - z(\mathrm{D}^1), \quad \ell \in \bar{\mathscr{L}} \cup \bar{\mathscr{B}} && \text{(c)} \end{aligned} \tag{45}$$

We set $\bar{\mathscr{L}}^{\mathrm{E}}_i = \bar{\mathscr{L}} \cap \mathscr{L}^{\mathrm{E}}_i$, $i \in L$ and $\bar{\mathscr{B}}^{\mathrm{S}}_j = \bar{\mathscr{B}} \cap \mathscr{B}^{\mathrm{S}}_j$, $j \in B$.

For generating the two sets $\bar{\mathscr{L}}$ and $\bar{\mathscr{B}}$, we used a dynamic programming procedure, called GEN$\mathscr{P}$, described in Mingozzi et al. (1996). GEN$\mathscr{P}$ generates the sets $\bar{\mathscr{L}}$ and $\bar{\mathscr{B}}$ satisfying conditions 45 with the additional restrictions that $|\bar{\mathscr{L}}| \leqslant$ Maxsize and $|\bar{\mathscr{B}}| \leqslant$ Maxsize, where Maxsize is a predefined positive integer to ensure that no memory overflow occurs. Note that real-world VRPB constraints can, at this stage, be easily considered by removing from $\bar{\mathscr{L}}$ and $\bar{\mathscr{B}}$ any infeasible path. The reduced problem $\bar{\mathrm{D}}^2$ is as follows.

$$(\bar{D}^2) \qquad z(\bar{\mathrm{D}}^2) = \mathrm{Max} \sum_{i \in L} u_i + \sum_{j \in B} v_j + Mw \tag{46}$$

subject to
$$\sum_{k \in P_\ell} u_k + \alpha_i \leqslant c_\ell^2, \quad \ell \in \bar{\mathscr{L}}_i^{\mathrm{E}},\ i \in L \tag{47}$$

$$\sum_{k \in P_\ell} v_k + \beta_j \leqslant c_\ell^2, \quad \ell \in \bar{\mathscr{B}}_j^{\mathrm{S}},\ j \in B \tag{48}$$

$$-\alpha_i - \beta_j + w \leqslant d_{ij}^2, \quad (i, j) \in A_0 \tag{49}$$

$$u_i + \delta_i \leqslant U_i, \quad i \in L \tag{50}$$

$$\alpha_i - \delta_i \leqslant 0, \quad i \in L \tag{51}$$

$$v_j + \theta_j \leqslant V_j, \quad j \in B \tag{52}$$

$$\beta_j - \theta_j \leqslant 0, \quad j \in B \tag{53}$$

$$\begin{array}{lll} u_i,\ \alpha_i \text{ unrestricted}, & \delta_i \geqslant 0 & i \in L \\ v_j,\ \beta_j \text{ unrestricted}, & \theta_j \geqslant 0 & j \in B \\ w \quad \text{unrestricted}. & & \end{array} \tag{54}$$

Constraints 50, 51, 52, and 53 ensure that any solution of $\bar{\mathrm{D}}^2$ is a feasible D^2 solution if the upper bounds U_i, $i \in L$ and V_j, $j \in B$ are chosen such that

$$\begin{array}{ll} \sum_{i \in P_\ell} U_i \leqslant c_\ell^2, \quad \ell \in \mathscr{L} \backslash \bar{\mathscr{L}} & \text{(a)} \\ \sum_{j \in P_\ell} V_j \leqslant c_\ell^2, \quad \ell \in \mathscr{B} \backslash \bar{\mathscr{B}}. & \text{(b)} \end{array} \tag{55}$$

THEOREM 3. *Any feasible solution to* $\bar{\mathrm{D}}^2$ *is also a feasible solution of* D^2 *with the same objective function value.*

Proof. Let us consider the dual constraint 11 of path $\ell \in \mathscr{L}_i^{\mathrm{E}} \backslash \bar{\mathscr{L}}_i^{\mathrm{E}}$ for a given $i \in L$. From inequalities 50 and 51, we have

$$\sum_{k \in P_\ell} u_k + \alpha_i \leqslant \sum_{k \in P_\ell} U_k - \sum_{k \in P_\ell} \delta_k + \delta_i,$$

and from inequalities 55.a, since $\delta_i \geqslant 0$, $\forall i \in L$, we have

$$\sum_{k \in P_\ell} u_k + \alpha_i \leqslant c_\ell^2 - \sum_{k \in P_\ell} \delta_k + \delta_i \leqslant c_\ell^2.$$

Hence, any solution of $\bar{\mathrm{D}}^2$ satisfies the dual constraint 11 for any $\ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$. In a similar way, we can show that a feasible $\bar{\mathrm{D}}^2$ solution satisfies constraints 12 for any $\ell \in \mathscr{B} \backslash \bar{\mathscr{B}}$. □

3.2.1 Computation of U_i, $i \in L$ and V_j, $j \in B$

In computing U_i, $\forall i \in L$, we must consider two cases.

A. $c_\ell^2 \geqslant z(\mathrm{UB}) - z(\mathrm{D}^1)$, $\forall \ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$. From Corollary 1, no path $\ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$ can belong to an optimal VRPB solution, hence, we can set $U_i = \infty$, $\forall i \in L$.

B. $c_\ell^2 < z(\mathrm{UB}) - z(\mathrm{D}^1)$, for some $\ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$. In this case, every optimal VRPB solution might contain some path $\ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$. We can set

$$U_i = q_i \hat{c}^{\mathrm{L}} / Q, \quad \forall i \in L, \tag{56}$$

where

$$\hat{c}^{\mathrm{L}} = \mathrm{Max}_{\ell \in \bar{\mathscr{L}}} [c_\ell^2]. \tag{57}$$

It is easy to show that the $\{U_i\}$ defined according to expression 56 and 57 satisfy inequalities 55a. For any $\ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$ we have

$$\sum_{i \in P_\ell} U_i = \sum_{i \in P_\ell} q_i \hat{c}^{\mathrm{L}} / Q. \tag{58}$$

Since, from 57, we know that $\hat{c}^{\mathrm{L}} \leqslant c_\ell^2$, $\ell \in \mathscr{L} \backslash \bar{\mathscr{L}}$, Eq. 58 then becomes

$$\sum_{i \in P_\ell} U_i \leqslant c_\ell^2 \sum_{i \in P_\ell} q_i / Q \leqslant c_\ell^2.$$

Also, for the computation of V_j, $\forall j \in B$, we must consider two cases.

C. $c_\ell^2 \geqslant z(\mathrm{UB}) - z(\mathrm{D}^1)$, $\forall \ell \in \mathscr{B} \backslash \bar{\mathscr{B}}$. This case is analogous to case A above. We can set $V_j = \infty$, $\forall j \in B$.

D. $c_\ell^2 < z(\mathrm{UB}) - z(\mathrm{D}^1)$, for some $\ell \in \mathscr{B} \backslash \bar{\mathscr{B}}$. We can set

$$V_j = q_j \hat{c}^{\mathrm{B}} / Q, \quad \forall j \in B, \tag{59}$$

where

$$\hat{c}^{\mathrm{B}} = \mathrm{Max}_{\ell \in \bar{\mathscr{B}}} [c_\ell^2]. \tag{60}$$

The proof that the $\{V_j\}$ computed according to expressions 59 and 60 satisfies inequalities 55b is similar to the one given in case B above. Problem $\bar{\mathrm{D}}^2$ can be considered the dual of the following problem $\overline{\mathrm{IP}}^2$.

TABLE I
Problems of Class A: Lower Bounds

Problem	Problem Data n	m	M	M_B	$z(IP)$	HDS $z(D^1)$	t_{H^1}	$z'(D)$	t_{HDS}	$\%E_{HDS}$	TV $\%E_{TV}$
A1	20	5	8	2	229886	215233	2.1	227079	4.0	98.8	98.3
A2	20	5	5	1	180119	170474	1.2	177869	3.4	98.8	98.1
A3	20	5	4	1	163405	154512	5.8	163405	9.1	100.0	100.0
A4	20	5	3	1	155796	148452	5.2	155796	11.6	100.0	100.0
B1	20	10	7	4	239080	233869	10.6	233869	13.2	97.8	96.0
B2	20	10	5	3	198048	193176	29.9	193880	39.1	97.9	97.4
B3	20	10	3	2	169372	169372	3.9	169372	3.9	100.0	100.0
C1	20	20	7	6	249448	236825	9.4	244857	13.6	98.2	95.7
C2	20	20	5	4	215020	207305	10.8	208495	13.9	97.0	96.5
C3	20	20	5	3	199346	197522	18.8	199346	24.5	100.0	99.8
C4	20	20	4	3	195366	193542	18.7	195367	24.4	100.0	100.0
D1	30	8	12	3	322530	306565	3.7	318671	6.2	98.8	97.0
D2	30	8	11	3	316709	292534	3.6	310929	8.5	98.2	94.5
D3	30	8	7	2	239479	224657	4.5	231931	17.6	96.8	95.9
D4	30	8	5	2	205832	194225	21.9	198301	49.4	96.3	95.4
E1	30	15	7	3	238880	229614	5.7	238880	11.2	100.0	95.1
E2	30	15	4	2	212263	206362	21.3	212263	40.9	100.0	97.9
E3	30	15	4	2	206659	199031	32.8	204360	62.2	98.9	98.2
F1	30	30	6	6	263173	248195	6.4	256287	66.4	97.4	96.6
F2	30	30	7	6	265213	254285	6.5	262342	27.6	98.9	98.3
F3	30	30	5	4	241120	229452	9.1	238221	74.8	98.8	98.0
F4	30	30	4	3	233861	221136	11.2	227576	91.3	97.3	97.3
G1	45	12	10	3	306305	292859	14.2	299522	43.3	97.8	91.1
G2	45	12	6	2	245441	237618	24.3	242423	63.6	98.8	93.3
G3	45	12	5	2	229507	221566	30.7	223205	80.7	97.3	96.2
G4	45	12	6	2	232521	223271	30.9	226712	71.6	97.5	96.5
G5	45	12	5	1	221730	213131	38.2	217204	81.7	98.0	97.9
G6	45	12	4	1	213457	204187	49.4	207116	102.8	97.0	96.6
H1	45	23	6	3	268933	262397	98.7	264609	130.5	98.4	96.6
H2	45	23	5	3	253365	249237	67.4	251972	143.7	99.5	99.4
H3	45	23	4	2	247449	242391	79.6	245860	171.3	99.4	99.2
H4	45	23	5	2	250221	244114	32.1	249239	176.4	99.6	99.7
H5	45	23	4	2	246121	239537	94.8	244450	263.4	99.3	99.3
H6	45	23	5	2	249135	243664	74.1	247832	169.4	99.5	99.4
I1	45	45	10	9	353021	338580	56.5	342376	193.2	97.0	n.a.
I2	45	45	7	7	309943	301904	80.2	305923	198.3	98.7	n.a.
I3	45	45	5	5	294833	281061	122.7	285158	274.0	96.7	n.a.
I4	45	45	6	5	295988	286849	121.1	289314	299.6	97.7	n.a.
I5	45	45	7	5	301226	293773	120.6	295935	304.6	98.2	n.a.
J1	75	19	10	3	335006	323922	94.6	329466	150.5	98.3	n.a.
J2	75	19	8	2	315644	295532	123.4	299069	217.2	94.7	n.a.
J3	75	19	6	2	282447	268495	185.2	271767	362.8	96.2	n.a.
J4	75	19	7	2	300548	281414	147.2	285203	259.8	94.9	n.a.
K1	75	38	10	5	394637	379113	104.9	385215	187.2	97.6	n.a.
K2	75	38	8	4	362360	351581	117.7	357327	223.3	98.6	n.a.
K3	75	38	9	4	365693	354651	115.7	360365	219.4	98.5	n.a.
K4	75	38	7	3	358308	336260	142.3	340958	264.1	95.2	n.a.
Average %dev										98.2	97.4
Minimum %dev										94.7	91.1

TABLE II

Problems of Class A: Exact Method EHP

	Problem Data						EHP				TV
Problem	n	m	M	M_B	z(UB)	z(IP)	LS	$\|\mathscr{L}'\|$	$\|\mathscr{B}'\|$	t_{EHP}	t_{TV}
A1	20	5	8	2	229886	229886	∞	125	7	5	902
A2	20	5	5	1	180119	180119	∞	242	13	4	209
A3	20	5	4	1	163405	163405[1]	∞	—	—	10	3
A4	20	5	3	1	155796	155796[1]	∞	—	—	12	3
B1	20	10	7	4	239080	239080	∞	307	69	14	148
B2	20	10	5	3	198048	198048	∞	386	126	40	151
B3	20	10	3	2	169372	169372[1]	∞	—	—	4	1
C1	20	20	7	6	253318	249448	∞	945	574	17	227
C2	20	20	5	4	215020	215020	∞	1144	772	18	322
C3	20	20	5	3	199346	199346[1]	∞	—	—	25	84
C4	20	20	4	3	195367	195366[1]	∞	—	—	25	5
D1	30	8	12	3	322705	322530	∞	339	32	9	289
D2	30	8	11	3	318476	316709	∞	1158	47	13	491
D3	30	8	7	2	239479	239479	∞	4132	160	51	*
D4	30	8	5	2	205832	205832	∞	14696	191	161	*
E1	30	15	7	3	238880	238880[1]	∞	—	—	12	476
E2	30	15	4	2	212263	212263[1]	∞	—	—	41	788
E3	30	15	4	2	206659	206659	∞	996	288	64	482
F1	30	30	6	6	263929	263173	268630	7201	12019	2049	756
F2	30	30	7	6	265214	265213	∞	805	978	44	891
F3	30	30	5	4	241121	241120	246458	1115	1981	76	468
F4	30	30	4	3	233862	233861	234671	22708	33442	173	3523
G1	45	12	10	3	306959	306305	308396	24678	271	3556	*
G2	45	12	6	2	245441	245441	247176	13705	105	229	*
G3	45	12	5	2	230170	229507[2]	227049	38180	351	967	4225
G4	45	12	6	2	232647	232521[2]	230648	21336	115	89	*
G5	45	12	5	1	221899	221730[2]	220508	17556	434	157	3433
G6	45	12	4	1	213457	213457[3]	209922	18946	763	103	840
H1	45	23	6	3	270719	268933[2]	265930	2202	374	454	1344
H2	45	23	5	3	253365	253365	256154	6654	534	221	5020
H3	45	23	4	2	247536	247449	249200	5987	1724	177	1465
H4	45	23	5	2	250221	250221	253120	2194	872	179	1287
H5	45	23	4	2	246121	246121	247526	13356	2156	277	406
H6	45	23	5	2	249135	249135	250351	3462	1086	173	416
I1	45	45	10	9	354410	353021[2]	349787	55702	57332	20225	n.a.
I2	45	45	7	7	315184	309943	310965	16854	16678	6395	n.a.
I3	45	45	5	5	298367	294833[2]	285787	37767	19714	18045	n.a.
I4	45	45	6	5	295988	295988[2]	293375	46873	40119	20055	n.a.
I5	45	45	7	5	302709	301226[2]	300060	48245	40870	8642	n.a.
J1	75	19	10	3	343476	335006[2]	331204	1298	9769	1640	n.a.
J2	75	19	8	2	315644	315644[3]	300485	1318	29849	218	n.a.
J3	75	19	6	2	282447	282447[3]	272889	827	26266	363	n.a.
J4	75	19	7	2	300548	300548[3]	286404	504	25603	260	n.a.
K1	75	38	10	5	408303	394637[2]	387804	3713	58698	*	n.a.
K2	75	38	8	4	372423	362360[2]	359157	2693	54446	2618	n.a.
K3	75	38	9	4	374417	365693[2]	362516	4556	52029	3812	n.a.
K4	75	38	7	3	358308	358308[3]	342184	1166	47759	265	n.a.

[1]Optimal solution obtained by procedure HDS.

[2]z(IP) is the cost of the best VRPB solution found by procedure EHP.

[3]No solution found by algorithm EHP of cost smaller than z(UB).

TABLE III
Problems of Class B: Lower Bounds

Problem	Problem Data n	m	M	M_B	$z(\mathrm{IP})$	HDS $z(\mathrm{D}^1)$	t_{H^1}	$z'(\mathrm{D})$	t_{HDS}	%E_{HDS}	TV %E_{TV}
eil2250	11	10	3	2	371	369	2.8	371	5.1	100.0	100.0
eil2266	14	7	3	1	366	366	1.0	366	3.0	100.0	100.0
eil2280	17	4	3	1	375	366	3.5	372	5.8	99.2	98.9
eil2350	11	11	2	1	682	682	0.4	682	0.4	100.0	100.0
eil2366	15	7	2	1	649	604	4.5	645	7.6	99.4	98.8
eil2380	18	4	2	2	623	610	5.5	615	8.9	98.7	98.1
eil3050	15	14	2	2	501	473	7.7	501	7.7	100.0	100.0
eil3066	20	9	3	1	537	492	6.4	524	14.1	97.6	98.5
eil3080	24	5	3	1	514	488	7.7	503	24.7	97.9	100.0
eil3350	16	16	3	2	738	737	21.8	738	45.4	100.0	98.4
eil3366	22	10	3	1	750	746	15.3	750	26.6	100.0	94.8
eil3380	26	6	3	1	736	727	18.0	731	42.2	99.3	93.9
eil5150	25	25	3	3	559	550	38.7	557	65.2	99.6	99.3
eil5166	34	16	4	2	548	541	40.2	544	60.6	99.3	97.8
eil5180	40	10	4	1	565	552	47.5	554	104.0	98.1	98.0
eilA7650	37	38	6	5	739	730	67.0	733	110.0	99.2	98.2
eilA7666	50	25	7	4	768	756	75.0	760	135.0	99.0	95.4
eilA7680	60	15	8	2	781	758	89.3	763	195.0	97.7	90.5
eilB7650	37	38	8	7	801	794	45.0	795	62.5	99.3	97.6
eilB7666	50	25	10	5	873	860	54.8	864	97.7	99.0	91.2
eilB7680	60	15	12	3	919	908	65.6	914	115.0	99.5	85.2
eilC7650	37	38	5	4	713	699	88.5	705	186.0	98.9	98.9
eilC7666	50	25	6	3	734	725	100.0	728	196.0	99.2	97.6
eilC7680	60	15	7	2	733	713	62.3	717	131.0	97.8	93.7
eilD7650	37	38	4	3	690	684	109.3	688	182.0	99.7	99.7
eilD7666	50	25	5	2	715	704	119.0	705	236.0	98.6	98.5
eilD7680	60	15	6	2	694	683	140.0	687	310.0	99.0	95.6
eilA10150	50	50	4	4	843	800	167.5	812	363.2	96.3	96.3
eilA10166	67	33	6	3	846	841	201.0	843	373.0	99.6	99.2
eilA10180	80	20	6	2	908	830	222.9	833	430.9	91.7	89.5
eilB10150	50	50	7	7	933	888	96.0	892	210.0	95.6	n.a.
eilB10166	67	33	9	5	1056	937	118.8	941	292.6	89.1	n.a.
eilB10180	80	20	11	3	1022	992	132.5	993	306.9	97.2	n.a.
Average %dev										98.3	96.8
Minimum %dev										89.1	85.2

($\overline{\mathrm{IP}}^2$)

$$z(\overline{\mathrm{IP}}^2) = \mathrm{Min} \sum_{\ell \in \bar{\mathcal{L}}} c_\ell^2 x_\ell + \sum_{\ell \in \bar{\mathcal{B}}} c_\ell^2 y_\ell + \sum_{(i,j) \in A_0} d_{ij}^2 \xi_{ij} + \sum_{i \in L} U_i x_i^\alpha + \sum_{j \in B} V_j y_j^\beta \quad (61)$$

subject to

$$\sum_{\ell \in \bar{\mathcal{L}}_i} x_\ell + x_i^\alpha = 1, \quad i \in L \quad (62)$$

$$\sum_{\ell \in \bar{\mathcal{B}}_j} y_\ell + y_j^\beta = 1, \quad j \in B \quad (63)$$

$$\sum_{\ell \in \bar{\mathcal{L}}_i^{\mathrm{E}}} x_\ell - \sum_{j \in B_0} \xi_{ij} + \xi_i^\alpha = 0, \quad i \in L \quad (64)$$

$$\sum_{\ell \in \bar{\mathcal{B}}_j^{\mathrm{S}}} y_\ell - \sum_{i \in L} \xi_{ij} + \xi_j^\beta = 0, \quad j \in B \quad (65)$$

$$\sum_{(i,j) \in A_0} \xi_{ij} = M \quad (66)$$

$$x_i^\alpha - \xi_i^\alpha \geqslant 0, \quad i \in L \quad (67)$$

$$y_j^\beta - \xi_j^\beta \geqslant 0, \quad j \in B \quad (68)$$

$$\mathbf{x}, \mathbf{y}, \mathbf{x}^\alpha, \mathbf{y}^\beta, \boldsymbol{\xi}, \boldsymbol{\xi}^\alpha, \boldsymbol{\xi}^\beta \geqslant 0. \quad (69)$$

Procedure H^2 consists of finding an optimal solution $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{x}^{\alpha*}, \mathbf{y}^{\beta*}, \boldsymbol{\xi}^*, \boldsymbol{\xi}^{\alpha*}, \boldsymbol{\xi}^{\beta*})$ of $\overline{\mathrm{IP}}^2$ of cost $z(\overline{\mathrm{IP}}^2)$ and the corresponding optimal dual variables $(\mathbf{u}^*, \mathbf{v}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, w^*)$. Hence, we have $z(\mathrm{D}^2) = z(\bar{\mathrm{D}}^2)$ and $\mathbf{u}^2 = \mathbf{u}^*$, $\mathbf{v}^2 = \mathbf{v}^*$, $\boldsymbol{\alpha}^2 = \boldsymbol{\alpha}^*$, $\boldsymbol{\beta}^2 = \boldsymbol{\beta}^*$, $w^2 = w^*$. Pro-

TABLE IV
Problems of Class B: Exact Method EHP

	Problem Data						EHP				TV
Problem	n	m	M	M_B	z(UB)	z(IP)	LS	$\lvert\mathcal{L}'\rvert$	$\lvert\mathcal{B}'\rvert$	t_{EHP}	t_{TV}
eil2250	11	10	3	2	389	371[1]	∞	—	—	6	3
eil2266	14	7	3	1	366	366[1]	∞	—	—	3	6
eil2280	17	4	3	1	375	375	∞	196	6	6	55
eil2350	11	11	2	1	682	682[1]	∞	—	—	1	2
eil2366	15	7	2	1	649	649	∞	157	33	7	65
eil2380	18	4	2	2	625	623	∞	657	4	9	36
eil3050	15	14	2	2	501	501[1]	∞	—	—	8	3
eil3066	20	9	3	1	542	537	∞	2442	136	17	119
eil3080	24	5	3	1	519	514	∞	7948	12	31	13
eil3350	16	16	3	2	764	738[1]	763	—	—	46	292
eil3366	22	10	3	1	763	750	∞	—	—	27	1338
eil3380	26	6	3	1	761	736	748	6745	93	44	1655
eil5150	25	25	3	3	561	559	∞	420	925	66	441
eil5166	34	16	4	2	551	548	553	4366	463	68	2754
eil5180	40	10	4	1	584	565	566	32322	236	691	4436
eilA7650	37	38	6	5	756	739	743	5127	14206	884	15931
eilA7666	50	25	7	4	776	768	770	35299	5635	1205	13464
eilA7680	60	15	8	2	839	781[2]	772	44942	718	596	*
eilB7650	37	38	8	7	836	801	∞	2669	9183	124	16345
eilB7666	50	25	10	5	897	873	∞	20246	1879	2918	12990
eilB7680	60	15	12	3	951	919	927	28181	349	821	10413
eilC7650	37	38	5	4	714	713	715	12061	27537	16659	10344
eilC7666	50	25	6	3	748	734	736	44074	2100	952	*
eilC7680	60	15	7	2	757	733[2]	724	42230	1314	*	*
eilD7650	37	38	4	3	704	690	695	4753	9050	197	401
eilD7666	50	25	5	2	730	715[2]	711	37745	16371	5023	*
eilD7680	60	15	6	2	715	694[2]	691	41252	827	20148	*
eilA10150	50	50	4	4	849	843[3]	816	4402	8835	364	*
eilA10166	67	33	6	3	879	846	847	20587	15371	434	10913
eilA10180	80	20	6	2	908	908[3]	835	1159	55640	431	*
eilB10150	50	50	7	7	954	933[2]	900	22571	16152	*	n.a.
eilB10166	67	33	9	5	1056	1056[3]	946	4331	28666	293	n.a.
eilB10180	80	20	11	3	1076	1022[2]	996	4939	64198	20199	n.a.

[1]Optimal solution obtained by procedure HDS.
[2]z(IP) is the cost of the best VRPB solution found by procedure EHP.
[3]No solution found by algorithm EHP of cost smaller than z(UB).

cedure HDS finds a solution $(\mathbf{u}', \mathbf{v}', \boldsymbol{\alpha}', \boldsymbol{\beta}', w')$ of D of cost $z'(\mathrm{D}) = z(\mathrm{D}^1) + z(\mathrm{D}^2)$ by setting $\mathbf{u}' = \mathbf{u}^1 + \mathbf{u}^2$, $\mathbf{v}' = \mathbf{v}^1 + \mathbf{v}^2$, $\boldsymbol{\alpha}' = \boldsymbol{\alpha}^1 + \boldsymbol{\alpha}^2$, $\boldsymbol{\beta}' = \boldsymbol{\beta}^1 + \boldsymbol{\beta}^2$, $w' = w^1 + w^2$.

An optimal VRPB solution. We can observe that an optimal $\overline{\mathrm{IP}}^2$ solution, under some circumstances, can correspond to an optimal VRPB solution. In fact, we have the following cases:

A. $\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\xi}^*$ integer and $\mathbf{x}^{\alpha *} = \boldsymbol{\xi}^{\alpha *} = \mathbf{0}$, $\mathbf{y}^{\beta *} = \boldsymbol{\xi}^{\beta *} = \mathbf{0}$.
This solution is an optimal VRPB solution of cost $z(\mathrm{IP}) = z'(\mathrm{D})$.

B. $\mathbf{x}^*$ or $\mathbf{y}^*$ or $\boldsymbol{\xi}^*$ not integer and $U_i = \infty$, $i \in L$, and $V_j = \infty$, $j \in \mathrm{B}$.
In this case, all paths of any optimal VRPB solution are contained in the two sets $\bar{\mathcal{L}}$ and $\bar{\mathcal{B}}$ and an optimal VRPB solution can be obtained by solving problem IP after having replaced the sets $\mathcal{L}$ and $\mathcal{B}$ with $\bar{\mathcal{L}}$ and $\bar{\mathcal{B}}$.

C. $\mathbf{x}^{\alpha *} \neq \mathbf{0}$ or $\mathbf{y}^{\beta *} \neq \mathbf{0}$.
This $\overline{\mathrm{IP}}^2$ solution is not feasible for the VRPB. Furthermore, the sets $\bar{\mathcal{L}}$ and $\bar{\mathcal{B}}$ might not contain any feasible and/or optimal VRPB solution.

4. AN EXACT METHOD FOR SOLVING THE VRPB

IN THIS SECTION, we describe an exact method for the VRPB, called EHP, that consists of reducing the number of variables of the integer program IP so that the resulting problem can be solved by an integer programming solver (CPLEX, 1993). This method may terminate, under certain circumstances, without having found an optimal solution.

Let ($\mathbf{u}'$, $\mathbf{v}'$, $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$, w') be the solution of D of cost $z'(\mathrm{D})$ obtained by procedure HDS and let c'_ℓ, $\ell \in \mathcal{L} \cup \mathcal{B}$, and d'_{ij}, $(i, j) \in A_0$ be the reduced costs corresponding to this dual solution. We could attempt to solve IP as indicated by Corollary 1, that is, we might generate $\mathcal{L}'$, $\mathcal{B}'$, and A'_0 according to expressions 17 and then solve IP using $\mathcal{L}'$, $\mathcal{B}'$, and A'_0 instead of $\mathcal{L}$, $\mathcal{B}$, and A_0. However, the size of $\mathcal{L}'$ and/or $\mathcal{B}'$ may be too large; hence we propose generating $\mathcal{L}'$ and $\mathcal{B}'$ so that their size is limited and the resulting problem IP′ becomes solvable. By means of the same algorithm GEN$\mathcal{P}$ used by procedure H^2 (see Section 3.2), we generate $\mathcal{L}'$, $\mathcal{B}'$ satisfying conditions 55 where the reduced costs $\{c^2_\ell\}$ are replaced with $\{c'_\ell\}$, and $z(\mathrm{D}^1)$ is substituted with $z'(\mathrm{D})$. Note that the size of each set $\mathcal{L}'$ and $\mathcal{B}'$ is limited by the value of Maxsize used in algorithm GEN$\mathcal{P}$. Moreover, the sets $\mathcal{L}'$, $\mathcal{B}'$, and A'_0 can be further reduced by applying Corollaries 2, 3, and 4 of Section 2.1.1.

Let $\mathbf{x}^*$, $\mathbf{y}^*$, $\boldsymbol{\xi}^*$ be an optimal solution of IP′ of cost $z(\mathrm{IP}')$ (we assume $z(\mathrm{IP}') = \infty$ if the sets $\mathcal{L}'$ and $\mathcal{B}'$ do not contain any optimal VRPB solution). If $z(\mathrm{IP}') < \infty$ then solution $\mathbf{x}^*$, $\mathbf{y}^*$, $\boldsymbol{\xi}^*$ is a feasible VRPB solution and it may be also an optimal one. Let

$$\Delta = \mathrm{Min}\left\{ \underset{\ell \in \mathcal{L}'}{\mathrm{Max}}\,[c'_\ell],\ \underset{\ell \in \mathcal{B}'}{\mathrm{Max}}\,[c'_\ell] \right\}.$$

We have the following cases:

1. $z(\mathrm{IP}') \leqslant z'(\mathrm{D}) + \Delta$. In this case, the optimal solution of IP′ is also an optimal VRPB solution. This derives from Theorem 1 as any VRPB solution involving at least one path of $\mathcal{L}\backslash\mathcal{L}'$ or $\mathcal{B}\backslash\mathcal{B}'$ has a cost greater or equal to $z'(\mathrm{D}) + \Delta$.
2. $z(\mathrm{IP}') > z'(\mathrm{D}) + \Delta$. The optimal solution of IP′ might not be an optimal VRPB solution. However, it is easy to note that, in this case, $z'(\mathrm{D}) + \Delta$ is a valid lower bound to any optimal VRPB solution.

The optimal solution of IP′ is obtained by means of the integer programming code CPLEX 3.0.

5. COMPUTATIONAL RESULTS

THE ALGORITHM EHP described in Section 4 has been coded in Fortran 77 and run on a Silicon Graphics Indy (MIPS R4400/200 MHz processor) on two classes of test problems. We have used CPLEX 3.0 as the LP-solver in procedure H^2 and as the integer programming solver in EHP.

The problems of class A correspond to a subset of the VRPB instances proposed by Goetschalckx and Jacobs-Blecha (1989). The problems of class B have been generated by Toth and Vigo (1996) from VRP problems known in the literature. For each VRP problem, three VRPB instances have been generated, each one corresponding to a linehaul customer percentage of 50, 66, and 80%, respectively. Problem input data of class B have been kindly provided by Toth and Vigo. All test problems are also available via E-mail from the OR-library (see BEASLEY, 1990).

To our knowledge, the only exact method presented in the literature for solving these problems has been proposed by Toth and Vigo (1997). The tables show the following columns.

- $z(\mathrm{IP})$ = cost of the optimal VRPB solution (or cost of the best known solution).
- $z(\mathrm{UB})$ = cost of the VRPB solution found by the heuristic algorithm of Toth and Vigo (1996).
- $z(\mathrm{D}^1)$ = lower bound produced by procedure H^1 after 200 subgradient iterations.
- t_{H^1} = computing time spent by the bounding procedure H^1.
- $z'(\mathrm{D})$ = final lower bound produced by procedure HDS.
- t_{HDS} = total computing time of procedure HDS.
- $\%E_{HDS}$ = percentage error of the lower bound $z'(\mathrm{D})$ computed by procedure HDS.
- $|\mathcal{L}'|$ = number of linehaul paths generated in EHP.
- $|\mathcal{B}'|$ = number of backhaul paths generated in EHP.
- LS = $z'(\mathrm{D}) + \Delta$, where Δ is the value defined in Section 4, and it is used by EHP to show the optimality of $z(\mathrm{IP})$ [we set $LS = \infty$ if $c'_\ell > z(\mathrm{UB}) - z'(\mathrm{D})$, $\forall \ell \in (\mathcal{L} \cup \mathcal{B})\backslash(\mathcal{L}' \cup \mathcal{B}')$].
- t_{EHP} = total computing time of procedure EHP including t_{HDS}. We impose a time limit of 25,000 CPU seconds. If the time limit is reached, the instance is marked with an asterisk.
- $\%E_{TV}$ = percentage error of the lower bound produced by Toth and Vigo (1997).
- t_{TV} = computing time of the exact method TV proposed by Toth and Vigo (seconds of a Pentium 60 MHz personal computer). If an imposed time limit of 6000 CPU seconds has been reached, the instance is marked with an asterisk. Instances not attempted by Toth and Vigo are marked with n.a.

The percentage errors $\%\mathrm{E}_{\mathrm{HDS}}$ and $\%\mathrm{E}_{\mathrm{TV}}$ are computed as the ratio of the lower bound divided by $z(\mathrm{IP})$ and multiplied by 100. The parameter Maxsize, used in GEN$\mathcal{P}$, has been set to 70,000 in both procedures H^2 and EHP.

Tables I and III show the quality of the lower bounds produced by procedure HDS and by Toth and Vigo for the two classes of problems. Columns $\%E_{HDS}$ and $\%E_{TV}$ of both tables show that the lower bound obtained by HDS is greater than the lower bound produced by Toth and Vigo, the average values being $\%E_{HDS} = 98.2$ and $\%E_{TV} = 97.4$ for problems of class A and $\%E_{HDS} = 98.3$ and $\%E_{TV} = 96.8$ for problems of class B. In fact, out of 64 cases where comparison is possible, only in 3 of these did the procedure of Toth and Vigo give a superior lower bound. Tables II and IV report the results obtained by the exact method EHP and the exact algorithm of Toth and Vigo. Note that it is difficult to compare directly the computing times required by the two methods because they are relative to different machines. In our experience, the Pentium 60 MHz used by Toth and Vigo is about four times slower than the Silicon Graphics Indy we used. Tables II and IV indicate that EHP is capable of solving problems up to 90 customers of class A and up to 100 customers of class B. For some problems, EHP cannot prove the optimality of the solution produced (this happens when $LS < z(\mathrm{IP})$), however, the distance between $z(\mathrm{IP})$ and LS is small. The computing time required by CPLEX in procedure EHP to solve problem IP′ is given by $t_{\mathrm{EHP}} - t_{\mathrm{HDS}}$. We can observe that the CPU time consumed by CPLEX becomes the main component of the total time required by EHP to solve some problems of both classes A and B. We note here that, for algorithm EHP, it is better to have only a few customers per path (say, an average of 15 customers/path) and that the problem should be tight (i.e., the ratios $(\Sigma_{i\in L}\, q_i)/(M_L Q)$ and $(\Sigma_{i\in B}\, q_i)/(M_B Q)$ should be greater than, say, 0.9). In this case, the sizes of $\mathcal{L}'$ and $\mathcal{B}'$ are then small and EHP can be a potentially useful tool for solving practical VRPB problems.

6. CONCLUSION

IN THIS PAPER, we have described an exact algorithm for the basic Vehicle Routing Problem with Backhauls (VRPB) based on a new (0–1) integer programming formulation. We have presented a method for computing the lower bound by finding a feasible solution of the dual of the LP-relaxation of its integer program. Such a dual solution is obtained by combining two different bounding procedures where the structure of the last bound is such that additional constraints found in real-world VRPBs can be considered. The exact method uses the dual solution and a method for limiting the variables of the integer program so that the resulting problem can be solved by CPLEX. The overall bounding procedure proved to be effective, being able to produce a lower bound whose value, on average, was at least 98.2% of the optimum. Computational results show that the proposed method is able to solve exactly VRPBs of size up to 100 customers within the imposed time limit of 25,000 seconds.

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