1 Definitions:

A single finite **clothoid** curve is parametrically defined by arc length parameter s, where:

- $s = 0$ corresponds to the start of the clothoid
- $s = s_{max}$ corresponds to the end of the clothoid, which is also the total arc length
- s can also take on negative values if the clothoid is evaluated backward along the path
- Technically, there is no "end" to the mathematical clothoid, i.e. s_{max} is only needed when constructing finite paths using finite-length clothoid segments

The **tangent angle**, in radians, at point s on the clothoid: $\phi(s) = \phi_0 + \kappa_0 s + \frac{1}{2}\kappa_p s^2 \ [rad]$

where: ϕ_0 is the initial tangent angle [rad], at $s = 0$ [m]
κ_0 is the initial curvature $\left[\frac{rad}{m}\right]$ at $s = 0$
κ_p is the (assumed) constant curvature rate $\left[\frac{rad}{m^2}\right]$ throughout the entire clothoid

The **curvature** at point s on the clothoid: $\kappa(s) = \frac{d\phi(s)}{ds} = \kappa_0 + \kappa_p s = \frac{1}{R(s)} \left[\frac{rad}{m}\right]$

where: $R(s)$ is the instantaneous radius [m] of curvature at s

$\kappa_p = \left(\frac{d\kappa}{ds}\right) \left[\frac{rad}{m^2}\right]$ is the constant curvature rate

Note that for a ground vehicle:

- The tangent angle $\phi(s)$ is the same as the vehicle heading angle $\psi(s)$, but only if the clothoid path is measured at the center of the rear axle and not at any other point
- The front wheel angle $\delta(s)$ is monotonically related to the curvature $\kappa(s)$
- The steering wheel angle $\theta(s)$ is monotonically related to the front wheel angle $\delta(s)$
- Therefore, the steering wheel angle is also monotonically related to the curvature
- This monotonic relationship is often accurately modeled as linear

2 Mathematical definition of a clothoid, "Euler spiral", "Cornu spiral":

$$x(s) = x_0 + \int_0^s cos\big(\phi(\eta)\big)d\eta$$

$$y(s) = y_0 + \int_0^s sin\big(\phi(\eta)\big)d\eta$$

where: x_0 is the initial x-direction coordinate [m] of the clothoid at $s = 0$
y_0 is the initial y-direction coordinate [m] of the clothoid at $s = 0$

These foundational definitions are too complicated to work with directly, so a great deal of effort has gone into "simplifying" the result.

This simplification usually leads to one of several definitions for Fresnel integrals (see Matlab FRESNELC and FRESNELS). The most common definition is given by:

$$C(s) = \int_0^s cos\left(\frac{\pi}{2}\eta^2\right) d\eta$$
$$S(s) = \int_0^s sin\left(\frac{\pi}{2}\eta^2\right) d\eta$$

Using Fresnel integral definitions, the general solution for a parametric clothoid is given by:

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + sign(\kappa_p)\sqrt{\frac{\pi}{|\kappa_p|}} e^{i\left(\phi_0 - \frac{{\kappa_0}^2}{2\kappa_p}\right)} \left[C\left(\frac{\kappa_0 + \kappa_p s}{\sqrt{\pi|\kappa_p|}}\right) - C\left(\frac{\kappa_0}{\sqrt{\pi|\kappa_p|}}\right) \right] + i\sqrt{\frac{\pi}{|\kappa_p|}} e^{i\left(\phi_0 - \frac{{\kappa_0}^2}{2\kappa_p}\right)} \left[S\left(\frac{\kappa_0 + \kappa_p s}{\sqrt{\pi|\kappa_p|}}\right) - S\left(\frac{\kappa_0}{\sqrt{\pi|\kappa_p|}}\right) \right]$$

Note that this result is only valid for $\kappa_p \neq 0$; however, this is not really a simplified result!

A further "simplification" of the Fresnel integrals is also given by:

$$C(s) = \frac{1}{2} + f(s)sin\left(\frac{\pi}{2}s^2\right) - g(s)cos\left(\frac{\pi}{2}s^2\right)$$
$$S(s) = \frac{1}{2} - f(s)cos\left(\frac{\pi}{2}s^2\right) - g(s)sin\left(\frac{\pi}{2}s^2\right)$$

where the auxiliary functions $f(s)$ and $g(s)$ are defined as (which are not directly used!):

$$f(s) = \left(\frac{1}{2} - S(s)\right) cos\left(\frac{\pi}{2}s^2\right) - \left(\frac{1}{2} - C(s)\right) sin\left(\frac{\pi}{2}s^2\right)$$
$$g(s) = \left(\frac{1}{2} - C(s)\right) cos\left(\frac{\pi}{2}s^2\right) + \left(\frac{1}{2} - S(s)\right) sin\left(\frac{\pi}{2}s^2\right)$$

These auxiliary functions are well behaved and are related to each other through derivatives (which are also not directly used!):

$$\frac{df(s)}{ds} = -\pi s g(s)$$
$$\frac{dg(s)}{ds} = \pi s f(s) - 1$$

The auxiliary functions are accurately approximated with polynomial ratios [Wilde, 2009]:

$$f(s) \approx \frac{1 + 0.926s}{2 + 1.792s + 3.104s^2}$$

$$g(s) \approx \frac{1}{2 + 4.142s + 3.492s^2 + 6.67s^3}$$

Other low-order polynomial approximations to the Fresnel integrals exist, such as [McCrae & Singh, 2009].

3 Numerical evaluation of parametric clothoid curve:

An alternative method for numerically evaluating the foundational clothoid definition is to use a trapezoidal rule to approximate the integral.

Given: $x_0, y_0, \phi_0, \kappa_0, \kappa_p, s_{max}$

Find: $x(s), y(s) \quad for \quad s \in [0, s_{max}]$

Note that this numerical approach assumes that the clothoid length is known apriori.

To solve this problem, we can discretize the parametric interval into a finite number N of equally spaced (not required, but still a good idea) arc-length intervals:

$$s_n = n\,\Delta s \qquad where \qquad \Delta s \equiv \frac{s_{max}}{N-1} \qquad n \in \{0,1,\ldots,N-1\}$$

Care should be taken when choosing N to insure that Δs is sufficiently small. For example, for a ground vehicle path:

$$\Delta s \leq 1\,m$$

Next, the foundational clothoid integrals can be converted to summations using the Trapezoidal rule for approximating integrals:

$$x(s_n) \approx x_0 + \left(\frac{\Delta s}{2}\right) \sum_{m=1}^{n} \Big(cos\big(\phi(s_m)\big) + cos\big(\phi(s_{m-1})\big)\Big)$$

$$y(s_n) \approx y_0 + \left(\frac{\Delta s}{2}\right) \sum_{m=1}^{n} \Big(sin\big(\phi(s_m)\big) + sin\big(\phi(s_{m-1})\big)\Big)$$

This approximation is more useful in the following recursive form:

$$x(s_n) = x(s_{n-1}) + \left(\frac{\Delta s}{2}\right)\Big(cos\big(\phi(s_m)\big) + cos\big(\phi(s_{m-1})\big)\Big)$$

$$y(s_n) = y(s_{n-1}) + \left(\frac{\Delta s}{2}\right)\Big(sin\big(\phi(s_m)\big) + sin\big(\phi(s_{m-1})\big)\Big)$$

The evaluation of parametric clothoid path coordinates $x(s_n)$ and $y(s_n)$ is then straightforward.

There is no need for complicated transformations with the trapezoidal integration approach; however, it is recognized that evaluation of trig functions can be expensive in real-time.

One potential advantage of some "simplified" methods is that they enable direct computation of the clothoid path coordinate at the end of the path s_{max} without the need for computing any intermediate path coordinates.

In contrast, the proposed trapezoidal method requires computation of multiple intermediate path coordinates to accumulate the integral. The proposed trapezoidal method will never result in imaginary results, and it has no restrictions like the "simplified" method does, such as $\kappa_p \neq 0$.

4 Alternative numerical evaluation of parametric clothoid curve

For autonomous vehicle applications, we need to be able to determine the optimal curvature rate κ_p such that the clothoid reaches a specified target point (x_T, y_T) within some tolerance ε.

Given: $\phi_0, \kappa_0, \kappa_p, x_0, y_0, x_T, y_T, \varepsilon$
Find: s_{max}, κ_p

If the desired path from (x_0, y_0) to (x_T, y_T) is a straight line, then s_{max} is simply the length of that straight line and $\kappa_p = 0$. We are only interested in the more general case where the path is not a straight line.

There is no known closed-form solution to this problem; however, we can divide the problem into two parts, each of which do have numerical solutions. First, we modify the requirements above to the following:

Given: $\phi_0, \kappa_0, \kappa_p, x_0, y_0, \kappa_p, R_T, \varepsilon$
Find: s_{max}

where $R_T = \sqrt{(x_T - x_0)^2 + (y_T - y_0)^2}$ is the radial distance between the initial and target positions. In this problem, we only seek to find the clothoid length s_{max} for a given curvature rate that reaches the desired radial distance within some tolerance. This is a much easier problem to solve. Once we have the solution to this problem, a second algorithm can be used to determine the optimal value of curvature rate to hit the target.

This alternative numerical approach is similar to the previous numerical approach with one key difference. Instead of a pre-computed grid of path points s_n, we will step along the clothoid with equally spaced pre-defined steps. When the path gets close to the end, we will need to take smaller steps to reach the target radius.

In order to reduce the memory requirements, the individual computational steps from above can be converted to recursive form. The recursive update for the path position is:

$$s_n = s_{n-1} + \tilde{s}_n, \quad and \quad s_0 = 0, \quad \tilde{s}_n = \Delta s$$

For the path tangent angle:

$$\phi_n \equiv \phi(s_n) = \phi_0 + \kappa_0 s_n + \tfrac{1}{2}\kappa_p s_n^2$$

$$\phi_{n-1} \equiv \phi(s_{n-1}) = \phi_0 + \kappa_0 s_{n-1} + \tfrac{1}{2}\kappa_p s_{n-1}^2$$

Subtracting these two equations, we have:

$$\phi_n - \phi_{n-1} = \kappa_0(s_n - s_{n-1}) + \tfrac{1}{2}\kappa_p(s_n^2 - s_{n-1}^2) = \kappa_0\tilde{s}_n + \tfrac{1}{2}\kappa_p(s_n^2 - s_{n-1}^2)$$

Expanding the last term, we have:

$$(s_n^2 - s_{n-1}^2) = (s_n - s_{n-1})(s_n + s_{n-1}) = \tilde{s}_n(s_n + s_{n-1})$$

$$(s_n^2 - s_{n-1}^2) = \tilde{s}_n(s_{n-1} + \tilde{s}_n + s_{n-1}) = \tilde{s}_n(2s_{n-1} + \tilde{s}_n)$$

Therefore, the recursive tangent angle update equation is:

$$\phi_n = \phi_{n-1} + \kappa_0\tilde{s}_n + \tfrac{1}{2}\kappa_p\tilde{s}_n(2s_{n-1} + \tilde{s}_n)$$

Or alternatively:

$$\phi_n = \phi_{n-1} + \tilde{s}_n\left(\kappa_0 + \kappa_p\left(s_{n-1} + \tfrac{\tilde{s}_n}{2}\right)\right)$$

The recursive update for trapezoidal integration of the northing and easting coordinates is the same as before:

$$x_n = x_{n-1} + \left(cos(\phi_n) + cos(\phi_{n-1})\right)\left(\frac{\tilde{s}_n}{2}\right)$$
$$y_n = y_n + \left(sin(\phi_n) + sin(\phi_{n-1})\right)\left(\frac{\tilde{s}_n}{2}\right)$$

After each incremental step $\tilde{s}_n$ along the clothoid path, we first need to evaluate the radial distance [m] to the current point:

$$D_n = \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}$$

If the current radial distance is far enough away from the target radial distance, i.e. if

$$D_n < (R_T - \tilde{s}_n - \varepsilon)$$

then continue stepping along the path with the nominal increment $\tilde{s}_n = \Delta s$. Note that this is a conservative bound since the threshold radius $(R_T - \tilde{s}_n)$ assumes that the clothoid path approaches the circle of radius R_T perpendicularly. Under the assumption that the clothoid path actually approaches the circle of radius R_T perpendicularly, the path should never exceed the target radius in the next step because it can only step with path length $\tilde{s}_n$.

The only other alternative is for the current radial distance D_n to be close to the target radial distance, i.e.

$$(R_T - \tilde{s}_n - \varepsilon) \leq D_n \leq (R_T + \varepsilon)$$

In this case, the next clothoid path length $\tilde{s}_n$ will need to be scaled based on the current and previous radial distance:

$$\tilde{s}_n = \tilde{s}_{n-1}\left(\frac{R_T - D_n}{R_T - D_{n-1}}\right)$$

This choice of path length will move the clothoid endpoint closer to the target radius; however, the number of steps will vary depending on the angle of approach. Approaching the target radius from a perpendicular to the circle will result in the least number of steps. Approaching the target radius tangentially to the circle will result in a larger number of steps. In practice, the algorithm can continue stepping until the target distance is within the desired bound:

$$|D_n - R_T| \leq \varepsilon$$

5 Boundary Condition Matching for Multiple Connected Clothoids

Assumption: A continuous drivable road section traverses from point A to point B. This road section was constructed with two continuous clothoid segments. The curvature and heading angle are also continuous at all points in the road section.

Objective: Knowing only the information at A and B, how can we determine the clothoids?

Clothoid Segment 1 begins at point A and ends at point M:

Known:	$\{x_A, y_A, \phi_A, \kappa_A\}$	
Unknown:	$\{x_M, y_M, \phi_M, \kappa_M, \kappa_1', s_{1M}\}$	
Definitions:		
Curvature	$\kappa_1(s_{1M}) = \kappa_M$	$\kappa_1(s_1) = \kappa_A + \kappa_1'\, s_1$
Heading angle	$\phi_1(s_{1M}) = \phi_M$	$\phi_1(s_1) = \phi_A + \kappa_A s_1 + \frac{1}{2}\kappa_1' s_1^2$
X position	$x_1(s_{1M}) = x_M$	$x_1(s_1) = x_A + \int_0^{s_1} cos\big(\phi_1(\tau)\big)d\tau$
Y position	$y_1(s_{1M}) = y_M$	$y_1(s_1) = y_A + \int_0^{s_1} sin\big(\phi_1(\tau)\big)d\tau$
	$0 \leq s_1 \leq s_{1M}$	

Clothoid Segment 2 begins at point B and ends at point M:

Known: $\{x_B, y_B, \phi_B, \kappa_B\}$
Unknown: $\{x_M, y_M, \phi_M, \kappa_M, \kappa_2', s_{2M}\}$
Definitions:

Curvature	$\kappa_2(s_{2M}) = \kappa_M$	$\kappa_2(s_2) = \kappa_B + \kappa_2'\, s_2$
Heading angle	$\phi_2(s_{2M}) = \phi_M$	$\phi_2(s_2) = \phi_B + \kappa_B s_2 + \frac{1}{2}\kappa_2' s_2^2$
X position	$x_2(s_{2M}) = x_M$	$x_2(s_2) = x_B + \int_0^{s_2} cos\big(\phi_2(\tau)\big)d\tau$
Y position	$y_2(s_{2M}) = y_M$	$y_2(s_2) = y_B + \int_0^{s_2} sin\big(\phi_2(\tau)\big)d\tau$
	$0 \le s_2 \le s_{2M}$	

Clothoid Matching Conditions at point M:

Condition 1: Curvature Continuity

$$\kappa_M = \kappa_1(s_{1M}) = \kappa_2(s_{2M}) \quad \rightarrow \quad \kappa_A + \kappa_1'\, s_{1M} = \kappa_B + \kappa_2'\, s_{2M}$$

Condition 2: Heading Continuity

$$\phi_M = \phi_1(s_{1M}) = \phi_2(s_{2M}) \quad \rightarrow \quad \phi_A + \kappa_A\, s_{1M} + \tfrac{1}{2}\kappa_1' s_{1M}^2 = \phi_B + \kappa_B\, s_{2M} + \tfrac{1}{2}\kappa_2' s_{2M}^2$$

Condition 3: X position Continuity

$$x_M = x_1(s_{1M}) = x_2(s_{2M}) \quad \rightarrow \quad x_A + \int_0^{s_{1M}} cos\big(\phi_1(\tau)\big)d\tau = x_B + \int_0^{s_{2M}} cos\big(\phi_2(\tau)\big)d\tau$$

Condition 4: Y position Continuity

$$y_M = y_1(s_{1M}) = y_2(s_{2M}) \quad \rightarrow \quad y_A + \int_0^{s_{1M}} sin\big(\phi_1(\tau)\big)d\tau = y_B + \int_0^{s_{2M}} sin\big(\phi_2(\tau)\big)d\tau$$

Thus, we have 4 equations (Continuity Conditions 1-4), and 4 unknowns $\{\kappa_1', s_{1M}, \kappa_2', s_{2M}\}$.

Note, the quantities $\{x_M, y_M, \phi_M, \kappa_M\}$ were also listed as unknowns above; however, these are "derived" quantities in the sense that once we have solutions for the actual unknowns (shaded red) above, these four quantities are easily computed, therefore they are not independent unknown quantities, and may not need to be part of an explicit solution.

Linear Matching Equations:

Conditions 1 and 2 are linear when we assume that s_{1M} and s_{2M} are known:

$$\kappa_1'\, s_{1M} - \kappa_2'\, s_{2M} = \kappa_B - \kappa_A$$
$$\tfrac{1}{2}\kappa_1' s_{1M}^2 - \tfrac{1}{2}\kappa_2' s_{2M}^2 = \phi_B + \kappa_B\, s_{2M} - \phi_A - \kappa_A\, s_{1M}$$

In linear algebra form:

$$\begin{bmatrix} s_{1M} & -s_{2M} \\ s_{1M}^2 & -s_{2M}^2 \end{bmatrix} \begin{bmatrix} \kappa_1' \\ \kappa_2' \end{bmatrix} = \begin{bmatrix} \kappa_B - \kappa_A \\ 2(\phi_B - \phi_A + \kappa_B\, s_{2M} - \kappa_A\, s_{1M}) \end{bmatrix}$$

Note that for a unique solution to exist, we must have:

$$s_{2M}s_{1M}^2 - s_{1M}s_{2M}^2 \ne 0 \quad \rightarrow \quad s_{1M} \ne s_{2M}$$

Fortunately, this should never be a problem, i.e. this linear system of equations will always be non-singular. We will always choose:

$$s_{1M} > 0 \quad and \quad s_{2M} < 0$$

because the clothoid starting at B must be defined in the negative s direction to connect to the end of the clothoid starting at A.