Projected Entangled Pair States (PEPS)

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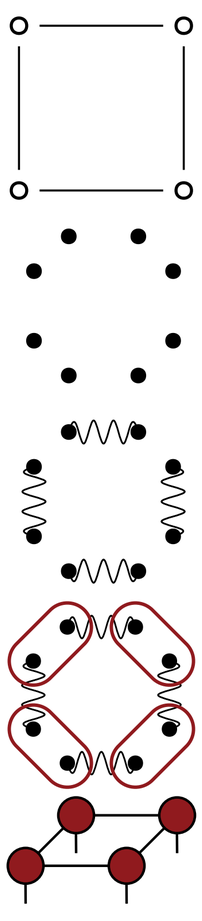
I. Definition

Projected Entangled Pair States (PEPS) are a class of tensor network states that efficiently parametrise quantum states with finite entanglement. They are a generalization of Matrix Product States (MPS).



- Given a physical sytem composed of local sites with site-Hilbertspaces $\mathcal{H}_i \equiv \mathbb{C}^{d_i}$ situated on a graph with edges $E = \{e_{i,j}\}$ and vertices $V = \{v_i\}$.
- For each vertex v_i, and for each edge $e_{i,j}$ connected to v_i (and v_j), associate an ancilla degree of freedom $a_{i,j} \in \mathcal{H}_{i,j} \equiv \mathbb{C}^{d_{i,j}}$.
- Maximally entangle ancillae $a_{i,j}$ and $a_{j,i}$ associated with each edge $e_{i,j}$:

$$|\Phi_{i,j}) = \sum_{l=1}^{d_{i,j}} |l\rangle \otimes |l\rangle \tag{1}$$

- Apply a linear map $\mathcal{P}_i : \bigotimes_{j_i} \mathcal{H}_{a_{i,j}} \to \mathcal{H}_i$ to the ancillae of site i, to obtain the projected entangled pair states (PEPS) $|\Psi\rangle = \bigotimes_{e \in E} \mathcal{P}_e \, |\Phi_e)$.
- The final PEPS is a tensor network state that has the same connectivity as the original graph and that lives in the total Hilbert space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$.

[Cirac et. al. 2021]

II. Entanglement Area Law

Entanglement Area Law
The entanglement entropy of a region $\mathcal{A}$ of quantum state with finite (local) entanglement scales as $\partial\mathcal{A}$, the boundary of $\mathcal{A}$.

$$S_{\mathcal{A}} \sim \partial\mathcal{A} \tag{2}$$

This is in constrast with the volume law most states follow.
Given the Schmidt decomposition of a state $|\Psi\rangle$ across a bipartition of the system into the "In" system $\mathcal{A}$ and the "Out" system $\mathcal{B}$ (where $\sum_i \lambda_i^2 = 1$):

$$|\Psi\rangle = \sum_i \lambda_i \, |I_i\rangle \otimes |O_i\rangle \tag{3}$$

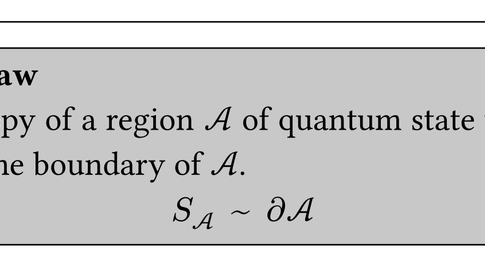
The entanglement entropy of the region $\mathcal{A}$ is given by:

$$S_{\mathcal{A}} = -\sum_i \lambda_i^2 \log(\lambda_i^2) \tag{4}$$

The maximal value this can take is when all the $\lambda_i = \frac{1}{N_S}$, with N_S the Schmidt rank.
For finitely entangled systems, $S_{\mathcal{A}}$ is thus bounded by the finite Schmidt rank N_S.

$$S_{\mathcal{A}} \leq \log(N_S) \tag{5}$$

It turns out that PEPS satisfy an **area law** for their entanglement entropy **by construction**. This property makes PEPS an efficient representation of quantum states with finite (local) entanglement.



The entanglement entropy of the shaded area $\mathcal{A}$ is the sum of the entanglement entropy across the cut virtual bonds, which by construction all have a finite bond dimension (and Schmidt rank) χ.
The entanglement entropy of a single cut is bounded by $\log(\chi)$. The total entanglement entropy of the region $\mathcal{A}$ is thus proportional to the number of cut virtual bonds times $\log(\chi)$.

$$S_{\mathcal{A}} \sim \log(\chi) \cdot \partial\mathcal{A} \tag{6}$$

III. Parent Hamiltonians

Parent Hamiltonians for PEPS
Every injective PEPS is the unique ground state of a *local*, *frustration free* parent Hamiltonian.

Consider the same shaded region $\mathcal{A}$ as before. The support space $\mathcal{S}_{\mathcal{A}}$ of the reduced density matrix $\rho_{\mathcal{A}}$ grows as $\partial\mathcal{A}$ while the Hilbert space $\mathcal{H}_{\mathcal{A}}$ grows as $\mathcal{A}$.
Therefore we can always find a sufficiently large region such that $\rho_{\mathcal{A}}$ has a non-trivial kernel $\ker(\rho_{\mathcal{A}}) = \mathcal{H}_{\mathcal{A}} \setminus \mathcal{S}_{\mathcal{A}}$.
It is thus the groundstate of every Hamiltonian h with:

- $h \geq 0$
- $\ker(h) = \mathcal{S}_{\mathcal{A}}$

since by construction $\mathrm{tr}(\rho_{\mathcal{A}} h) = 0$.
The Hamiltonian can be extended to the entire lattice by assigning a region $\mathcal{R}_v$ to every vertex v of the lattice and defining

$$H = \sum_v h_v \otimes \mathbb{1} \tag{7}$$

The Hamiltonian is *local* since it is the sum of local terms h_v acting on the region $\mathcal{R}_v$, and it is *frustration free* since every term h_v is minimized by the PEPS.
[**Perez-Garcia et. al. 2007**]

IV. Computation of local expectation values

COMPUTATION OF EXPECTATION VALUES????????

V. Computational complexity of PEPS contractions

How Complex are these peps really? It can't be that NP-Hard.