

Discrete Mathematics

CST Part IA Paper 2

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1 Proof

1. Some mathematical jargon:

- **Statement:** A sentence that is either true or false — but not both.
- **Predicate:** A statement whose truth depends on the value of one or more variables.
- **Theorem:** A very important true statement.
- **Proposition:** A less important but nonetheless interesting true statement.
- **Lemma:** A true statement used in proving other true statements.
- **Corollary:** A true statement that is a simple deduction from a theorem or proposition.
- **Conjecture:** A statement believed to be true, but for which we have no proof.
- **Proof:** Logical explanation of why a statement is true; a method for establishing truth.
- **Logic:** The study of methods and principles used to distinguish good (correct) from bad (incorrect) reasoning.
- **Axiom:** A basic assumption about a mathematical situation. Axioms can be considered facts that do not need to be proved (just to get us going in a subject) or they can be used in definitions.
- **Definition:** An explanation of the mathematical meaning of a word (or phrase). The word (or phrase) is generally defined in terms of properties.
- A statement is *simple* (or *atomic*) when it cannot be broken into other statements, and it is *composite* when it is built by using several (simple or composite statements) connected by logical expressions

2. Contraposition:

The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

3. Modus Ponens: If P and $P \Rightarrow Q$ holds then so does Q .

$$\frac{P \quad P \Rightarrow Q}{Q}$$

4. Some notations:

- Implication: \Rightarrow
- Bi-implication: \iff
- Universal quantification: $\forall x.P(x)$
- Existential quantification: $\exists x.P(x)$
- Unique existence: $\exists!x.P(x)$

$$\exists!x.P(x) \iff \exists x.P(x) \wedge (\forall y.\forall z.(P(y) \wedge P(z)) \Rightarrow y = z)$$

- Conjunction: \wedge
- Disjunction: \vee
- Negation: \neg

5. Equality axioms:

- Every individual is equal to itself.

$$\forall x.x = x$$

- (Leibniz equality) For any pair of equal individuals, if a property holds for one of them, then it also holds for the other one.

$$\forall x. \forall y. x = y \Rightarrow (P(x) \Rightarrow P(y))$$

2 Numbers

1. Definitions of real numbers. A real number is:

- **rational** if it is of the form $\frac{m}{n}$ for a pair of integers m and n ; otherwise it is **irrational**;
- **positive** if it is greater than 0, and **negative** if it is smaller than 0;
- **nonnegative** if it is greater than or equal to 0, and **nonpositive** if it is smaller than or equal to 0;
- **natural** if it is a nonnegative integer.

2. Additive structure $(\mathbb{N}, 0, +)$ of natural numbers with zero and addition is a commutative monoid (a *monoid* is a semigroup with an identity element; a *semigroup* preserves closure and associativity):

- Monoid laws:

$$0 + n = n + 0 = n \quad (\text{identity})$$

$$(l + m) + n = l + (m + n) \quad (\text{associativity})$$

- Commutativity law:

$$m + n = n + m$$

3. Multiplicative structure $(\mathbb{N}, 1, \cdot)$ of natural numbers with one and multiplication is a commutative monoid):

- Monoid laws:

$$1 \cdot n = n \cdot 1 = n$$

$$(l \cdot m) \cdot n = l \cdot (m \cdot n)$$

- Commutativity law:

$$m \cdot n = n \cdot m$$

4. The overall structure $(\mathbb{N}, 0, +, 1, \cdot)$ is a commutative semiring:

- $(\mathbb{N}, 0, +)$ is a commutative monoid;
- $(\mathbb{N}, 1, \cdot)$ is a monoid;
- Multiplication is distributive over addition:

$$l \cdot (m + n) = l \cdot m + l \cdot n$$

- Multiplication by 0 annihilates \mathbb{N} :

$$0 \cdot n = n \cdot 0 = 0$$

5. Cancellation:

- Additive cancellation: for all natural numbers k, m, n ,

$$k + m = k + n \Rightarrow m = n$$

- Multiplicative cancellation: for all natural numbers k, m, n ,

$$\text{if } k \neq 0 \text{ then } k \cdot m = k \cdot n \Rightarrow m = n$$

6. Inverses:

- A number x is said to admit an **additive inverse** whenever there exists a number y such that $x + y = 0$;
- A number x is said to admit an **multiplicative inverse** whenever there exists a number y such that $x \cdot y = 1$.

7. The integers \mathbb{Z} form a commutative ring, and the rationals \mathbb{Q} form a field:

- A *group* is a monoid in which every element has an inverse;
- A *ring* is a semiring $(0, +), (1, \cdot)$ where $(0, +)$ is a commutative group. It is commutative if $(1, \cdot)$ is also commutative;
- A *field* is a ring where every non-zero element has a multiplicative inverse.

8. Divisibility and congruence:

- Let d and n be integers. We say that d divides n , and write $d \mid n$, whenever there exists an integer k such that $n = k \cdot d$;
- Fix a positive integer m . For integers a and b , we say that a is congruent to b modulo m , and write $a \equiv b \pmod{m}$, whenever $m \mid (a - b)$.

9. For all prime numbers p and integers $0 \leq m \leq p$, either $\binom{p}{m} \equiv 0 \pmod{p}$ or $\binom{p}{m} \equiv 1 \pmod{p}$.
For $0 < m < p$, $p \mid \binom{p}{m}$ and $(p - m) \mid \binom{p-1}{m}$.

10. The Freshman's Dream: For all natural numbers m , n and primes p ,

$$(m + n)^p \equiv m^p + n^p \pmod{p}$$

11. The Dropout Lemma: For all natural numbers m and primes p ,

$$(m + 1)^p \equiv m^p + 1 \pmod{p}$$

12. The Many Dropout Lemma: For all natural numbers m and i , and primes p ,

$$(m + i)^p \equiv m^p + i \pmod{p}$$

13. Fermat's Little Theorem: For all natural numbers i and primes p ,

- (1) $i^p \equiv i \pmod{p}$, and
- (2) $i^{p-1} \equiv 1 \pmod{p}$ whenever i is not a multiple of p .

14. The Division Theorem: For every natural number m and positive natural number n , there exists a unique pair of integers q and r such that $q \geq 0$, $0 \leq r < n$, and $m = q \cdot n + r$.

15. Modular arithmetic: For all natural numbers $m > 1$, the modular-arithmetic structure

$$(\mathbb{Z}_m, 0, +_m, 1, \cdot_m)$$

is a commutative ring.

For prime p , \mathbb{Z}_p is a field.

16. Greatest Common Divisor: For all positive integers m and n ,

$$\gcd(m, n) = \begin{cases} n & , \text{if } n \mid m \\ \gcd(n, \text{rem}(m, n)) & , \text{otherwise} \end{cases}$$

17. Some fundamental properties of gcds:

- Commutativity: $\gcd(m, n) = \gcd(n, m)$,
- Associativity: $\gcd(l, \gcd(m, n)) = \gcd(\gcd(l, m), n)$,
- Distributivity: $\gcd(l \cdot m, l \cdot n) = l \cdot \gcd(m, n)$.

18. Theorem: For positive integers k , m , and n , if $k \mid (m \cdot n)$ and $\gcd(k, m) = 1$ then $k \mid n$.

Corollary (Euclid's Theorem): For positive integers m , n , and prime p , if $p \mid (m \cdot n)$ then $p \mid m$ or $p \mid n$.

19. For all positive integers m and n ,

- (1) $n \cdot \text{lc}_2(m, n) \equiv \gcd(m, n) \pmod{m}$, and
- (2) whenever $\gcd(m, n) = 1$,
 $[\text{lc}_2(m, n)]_m$ is the multiplicative inverse of $[n]_m$ in \mathbb{Z}_m .

20. Principle of Induction:

Let $P(m)$ be a statement for m ranging over the natural numbers greater than or equal to a fixed natural number l . If

- $P(l)$ holds, and
- $\forall n \geq l$ in $\mathbb{N}.$ $(P(n) \Rightarrow P(n+1))$ also holds,

then

- $\forall m \geq l$ in $\mathbb{N}.$ $P(m)$ holds.

21. Principle of Strong Induction:

Let $P(m)$ be a statement for m ranging over the natural numbers greater than or equal to a fixed natural number l . If

- $P(l)$ holds, and
- $\forall n \geq l$ in $\mathbb{N}.$ $(\forall k \in [l..n]. P(k)) \Rightarrow P(n+1)$ also holds,

then

- $\forall m \geq l$ in $\mathbb{N}.$ $P(m)$ holds.

22. Well-Founded Induction:

Definition: a *well-founded relation* is a binary relation \prec on a set A such that there are no infinite descending chains $\cdots \prec a_i \prec \cdots \prec a_1 \prec a_0$. When $a \prec b$ we say a is a *predecessor* of b .

Principle of Well-Founded Induction: Let \prec be a well-founded relation on a set A . if

- $\forall a \in A. (\forall b \prec a. P(b)) \Rightarrow P(a)$ holds,

then

- $\forall a \in A. P(a)$ holds.

23. Fundamental Theorem of Arithmetic: For every positive integer n there is a unique finite ordered sequence of primes $(p_1 \leq \cdots \leq p_l)$ with $l \in \mathbb{N}$ such that

$$n = \times_{i=1}^l p_i.$$

3 Sets

1. Axioms:

- Extensionality axiom: Two sets are equal if they have the same elements.

$$\forall \text{ sets } A, B . A = B \iff (\forall x. x \in A \iff x \in B)$$

- Powerset axiom: For any set, there is a set consisting of all its subsets.
- Pairing axiom: For every a and b , there is a set with a and b as its only elements.
- Union axiom: Every collection of sets has a union.
- Infinity axiom: There is an infinite set, containing \emptyset and closed under successor.
($\text{Succ}(x) =_{\text{def}} x \cup \{x\}$)
- Axiom of choice: Every surjection has a section (right inverse).
- Replacement axiom: The direct image of every definable functional property on a set is a set.

2. Cardinality:

- \forall finite set $U . \# \mathcal{P}(U) = 2^{\#U}$
- \forall sets $A, B . \#(A \times B) = \#A \times \#B$
- \forall sets $A, B . \#(A \uplus B) = \#A + \#B$

3. Subsets:

$$A \subseteq B \iff (\forall x. x \in A \Rightarrow x \in B)$$

$$A \subset B \iff (A \subseteq B \wedge A \neq B)$$

Reflexivity: \forall set $A . A \subseteq A$

Transitivity: \forall set $A, B, C . (A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$

Antisymmetry: \forall set $A, B . (A \subseteq B \wedge B \subseteq A) \Rightarrow A = B$

4. Separation principle: For any set A and any definable property P , there is a set containing precisely those elements of A for which the property P holds.

$$\{x \in A \mid P(x)\}$$

5. The powerset Boolean algebra: $(\mathcal{P}(U), \emptyset, U, \cup, \cap, (\cdot)^c)$

- For all $A, B \in \mathcal{P}(U)$,

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\} \in \mathcal{P}(U)$$

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\} \in \mathcal{P}(U)$$

$$A^c = \{x \in U \mid \neg(x \in A)\} \in \mathcal{P}(U)$$

- The union operation \cup and the intersection operation \cap are associative, commutative, and idempotent:

$$(A \cup B) \cup C = A \cup (B \cup C), A \cup B = B \cup A, A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C), A \cap B = B \cap A, A \cap A = A$$

- The *empty set* \emptyset is a neutral element for \cup and the *universal set* U is a neutral element for \cap :

$$\emptyset \cup A = U \cap A = A$$

- The empty set \emptyset is an annihilator for \cap and the universal set U is an annihilator for \cup :

$$\emptyset \cap A = \emptyset$$

$$U \cup A = U$$

- With respect to each other, the union operation \cup and the intersection operation \cap are distributive and absorptive:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A \cap (A \cup B) = A$$

- The complement operation $(\cdot)^c$ satisfies complementation laws:

$$A \cup A^c = U, A \cap A^c = \emptyset$$

6. Ordered pair: $\langle a, b \rangle =_{\text{def}} \{\{a\}, \{a, b\}\}$

Fundamental property or ordered pairing:

$$\forall a, b, x, y. \langle a, b \rangle = \langle x, y \rangle \iff (a = x \wedge b = y)$$

7. Big Unions: Let U be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$ (i.e. $\mathcal{F} \subseteq \mathcal{P}(U)$),

$$\bigcup \mathcal{F} =_{\text{def}} \{x \in U \mid \exists A \in \mathcal{F}. x \in A\} \in \mathcal{P}(U)$$

Idea:

$$\bigcup \{A_1, A_2, \dots\} = (A_1 \cup A_2 \cup \dots) \subseteq U$$

8. Big Intersections: Let U be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$ (i.e. $\mathcal{F} \subseteq \mathcal{P}(U)$),

$$\bigcap \mathcal{F} =_{\text{def}} \{x \in U \mid \forall A \in \mathcal{F}. x \in A\} \in \mathcal{P}(U)$$

Idea:

$$\bigcap \{A_1, A_2, \dots\} = (A_1 \cap A_2 \cap \dots) \subseteq U$$

9. Tagging: $\{l\} \times A$

10. Disjoint Unions: $A \uplus B =_{\text{def}} (\{1\} \times A) \cup (\{2\} \times B)$

$$\forall x. x \in (A \uplus B) \iff (\exists a \in A. x = (1, a)) \vee (\exists b \in B. x = (2, b))$$

4 Relations

1. Some notations and definitions:

- Relation: \rightarrow

For all finite sets A and B , $\#Rel(A, B) = 2^{\#A \cdot \#B}$

- Partial function: \rightarrow

Set of partial functions: \Rightarrow

Every partial function $f : A \rightarrow B$ satisfies that: for each element a of A there is at most one element b of B such that $a f b$.

$$\forall f \in Rel(A, B). f \in (A \Rightarrow B) \iff \forall a \in A. \forall b_1, b_2 \in B. a f b_1 \wedge a f b_2 \Rightarrow b_1 = b_2$$

For all finite sets A and B , $\#(A \Rightarrow B) = (\#B + 1)^{\#A}$

- Mapping: \mapsto

- Function: \rightarrow

Set of functions: \Rightarrow

A partial function is total if its domain of definition coincides with its source.

$$\forall f \in (A \Rightarrow B). f \in (A \rightarrow B) \iff \forall a \in A. \exists b \in B. a f b$$

$$\forall f \in Rel(A, B). f \in (A \rightarrow B) \iff \forall a \in A. \exists! b \in B. a f b$$

For all finite sets A and B , $\#(A \rightarrow B) = \#B^{\#A}$

- Injection: \hookrightarrow

A function $f : A \rightarrow B$ is injective whenever

$$\forall a_1, a_2 \in A. f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

- Surjection: \twoheadrightarrow

A function $f : A \rightarrow B$ is surjective whenever

$$\forall b \in B. \exists a \in A. f(a) = b$$

For all finite sets A and B , $\#Sur(A, B) =$

- Bijection: A function $f : A \rightarrow B$ is bijective whenever there exists a (necessarily unique) function $g : B \rightarrow A$ (referred to as the inverse of f) such that

$$g \circ f = \text{id}_A \quad \text{and} \quad f \circ g = \text{id}_B$$

For all finite sets A and B ,

$$\#Bij(A, B) = \begin{cases} 0 & , \text{if } \#A \neq \#B \\ n! & , \text{if } \#A = \#B = n \end{cases}$$

2. Composition:

Composition of two relations $R : A \rightarrow B$ and $S : B \rightarrow C$:

$$S \circ R : A \rightarrow C$$

Relational composition is associative and has the identity relation as neutral element:

$$\forall R : A \rightarrow B, S : B \rightarrow C, T : C \rightarrow D. (T \circ S) \circ R = T \circ (S \circ R)$$

$$\forall R : A \rightarrow B. R \circ \text{id}_A = \text{id}_B \circ R = R$$

$R^{\circ n}$: R composed with itself n times.

$$R^{\circ*} = \bigcup_{n \in \mathbb{N}} R^{\circ n}$$

3. Preorders:

A preorder (P, \sqsubseteq) consists of a set P and a relation \sqsubseteq on P satisfying the following two axioms:

- Reflexivity: $\forall x \in P. x \sqsubseteq x$
- Transitivity: $\forall x, y, z \in P. (x \sqsubseteq y \wedge y \sqsubseteq z) \Rightarrow x \sqsubseteq z$

$R^{\circ*}$ is the reflexive-transitive closure of R

$R^{\circ*}$ is the least preorder containing R

$R^{\circ*}$ is the preorder freely generated by R

4. Isomorphism: \cong

Two sets A and B are isomorphic (and have the same cardinality) whenever there is a bijection between them,

5. Equivalence relations:

A relation E on a set A is an equivalence relation whenever it is:

- (1) Reflexive: $\forall x \in A. x E x$
- (2) Symmetric: $\forall x, y \in A. x E y \Rightarrow y E x$
- (3) Transitive: $\forall x, y, z \in A. (x E y \wedge y E z) \Rightarrow x E z$

6. Set partitions:

A partition P of a set A is a set of non-empty subsets of A (that is, $P \subseteq \mathcal{P}(A)$ and $\emptyset \notin P$), whose elements are typically referred to as blocks, such that

- (1) The union of all blocks yields A : $\bigcup P = A$, and
- (2) All blocks are pairwise disjoint: $\forall B_1, B_2 \in P. B_1 \neq B_2 \Rightarrow B_1 \cap B_2 = \emptyset$

For every set A : $\text{EqRel}(A) \cong \text{Part}(A)$

7. Enumerability:

A set A is enumerable whenever there exists a surjection $(\mathbb{N} \twoheadrightarrow A)$, or a injection $(A \hookrightarrow \mathbb{N})$, referred to as an enumeration.

A countable set is one that is either empty or enumerable.

8. Relational images and functional images:

Let $R : A \rightarrow B$ be a relation.

- The direct image of $X \subseteq A$ under R is the set $\vec{R}(X) \subseteq B$:

$$\vec{R}(X) = \{b \in B \mid \exists x \in X. x R b\}$$

This construction yields a function $\vec{R} : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$.

- The inverse image of $Y \subseteq B$ under R is the set $\overleftarrow{R}(Y) \subseteq A$:

$$\overleftarrow{R}(Y) = \{a \in A \mid \forall b \in B. a R b \Rightarrow b \in Y\}$$

This construction yields a function $\overleftarrow{R}(Y) : \mathcal{P}(B) \rightarrow \mathcal{P}(A)$.

Let $f : A \rightarrow B$ be a function.

- For all $X \subseteq A$, $\vec{f}(X) = \{b \in B \mid \exists a \in X. f(a) = b\};$
- For all $Y \subseteq B$, $\overleftarrow{f}(Y) = \{a \in A \mid f(a) \in Y\}.$