Introduction to Probability
CST Part IA Paper 1

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1 Prerequisites and Introduction

1. Combinatorics:

Counting tasks on n objects			
Permutations (sort objects)		Combinations (choose r objects)	
Distinct	Indistinct	Distinct 1 group	Distinct k groups
$n!$	$\frac{n!}{n_1!n_2!\cdots n_r!}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$

Pascal's identity: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad (1 \le r \le n)$

Binomial theorem: $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

2. Probability axioms:

Axiom 1: For any event E, $0 \le \mathbb{P}[E] \le 1$
Axiom 2: Probability of the sample space S is $\mathbb{P}[S] = 1$
Axiom 3: If E and F are mutually exclusive (i.e., $E \cap F = \varnothing$), then $\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F]$
In general, for all mutually exclusive events $E_1, E_2, \cdots,$

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} E_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[E_i]$$

3. General inclusion-exclusion principle: $\mathbb{P}\left[\bigcup_{i=1}^{n} E_i\right] = \sum_{r=1}^{n} (-1)^{r+1} \left(\sum_{i_1 < \cdots < i_r}^{n} \mathbb{P}[E_{i_1} \cap \cdots \cap E_{i_r}] \right)$

Case $n = 2$: $\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F]$

4. Union bound (Boole's inequality): For any events $E_1, E_2, \cdots, E_n$,

$$\mathbb{P}\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \mathbb{P}[E_i]$$

5. Conditional probability (original and conditioning on event G):

Chain rule:

$$\mathbb{P}[EF] = \mathbb{P}[E|F]\mathbb{P}[F] \qquad \mathbb{P}[EF|G] = \mathbb{P}[E|FG]\mathbb{P}[F|G]$$

Multiplication rule:

$$\mathbb{P}[E_1 E_2 \cdots E_n] = \mathbb{P}[E_1]\mathbb{P}[E_2|E_1] \cdots [E_n|E_1 \cdots E_{n-1}]$$
$$\mathbb{P}[E_1 E_2 \cdots E_n|G] = \mathbb{P}[E_1|G]\mathbb{P}[E_2|E_1 G] \cdots [E_n|E_1 \cdots E_{n-1} G]$$

Independence of E and F:

$$\mathbb{P}[EF] = \mathbb{P}[E]\mathbb{P}[F] \qquad \mathbb{P}[EF|G] = \mathbb{P}[E|G]\mathbb{P}[F|G]$$
$$\mathbb{P}[E|F] = \mathbb{P}[E] \qquad \mathbb{P}[E|FG] = \mathbb{P}[E|G]$$

Law of total probability:

$$\mathbb{P}[E] = \mathbb{P}[EF] + \mathbb{P}[EF^\complement] = \mathbb{P}[E|F]\mathbb{P}[F] + \mathbb{P}[E|F^\complement]\mathbb{P}[F^\complement]$$
$$\mathbb{P}[E|G] = \mathbb{P}[EF|G] + \mathbb{P}[EF^\complement|G] = \mathbb{P}[E|FG]\mathbb{P}[F|G] + \mathbb{P}[E|F^\complement G]\mathbb{P}[F^\complement|G]$$

In general, for disjoint events $F_1, F_2, \cdots, F_n$ such that $F_1 \cup \cdots \cup F_n = S$,

$$\mathbb{P}[E] = \sum_{i=1}^{n} \mathbb{P}[E|F_i]\mathbb{P}[F_i] \qquad \mathbb{P}[E|G] = \sum_{i=1}^{n} \mathbb{P}[E|F_i G]\mathbb{P}[F_i|G]$$

Bayes' theorem:

$$\mathbb{P}[F|E] = \frac{\mathbb{P}[E|F]\mathbb{P}[F]}{\mathbb{P}[E]} \qquad \mathbb{P}[F|EG] = \frac{\mathbb{P}[E|FG]\mathbb{P}[F|G]}{\mathbb{P}[E|G]}$$

6. Confusion matrix:

<table>
<tr><td colspan="2"></td><td colspan="2">Actual condition</td></tr>
<tr><td></td><td>Total population</td><td>Positive F</td><td>Negative $F^\complement$</td></tr>
<tr><td rowspan="2">Predicted condition</td><td>Positive E</td><td>True positive
$\mathbb{P}[E|F]$</td><td>False positive
$\mathbb{P}[E|F^\complement]$</td></tr>
<tr><td>Negative $E^\complement$</td><td>False negative
$\mathbb{P}[E^\complement|F]$</td><td>True negative
$\mathbb{P}[E^\complement|F^\complement]$</td></tr>
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2 Random Variables

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3 Moments and Limit Theorems

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4 Applications and Statistics

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