Introduction to Probability
CST Part IA Paper 1

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1 Prerequisites and Introduction

1. Combinatorics:

Counting tasks on n objects			
Permutations (sort objects)		Combinations (choose r objects)	
Distinct	Indistinct	Distinct 1 group	Distinct k groups
$n!$	$\dfrac{n!}{n_1!n_2!\cdots n_r!}$	$\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$	$\dbinom{n}{n_1, n_2, \cdots, n_k} = \dfrac{n!}{n_1!n_2!\cdots n_k!}$

Pascal's identity: $\dbinom{n}{r} = \dbinom{n-1}{r-1} + \dbinom{n-1}{r} \qquad (1 \leq r \leq n)$

Binomial theorem: $(x+y)^n = \displaystyle\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

2. Frequentist definition of probability: $\mathbb{P}[E] = \displaystyle\lim_{n\to\infty} \frac{\text{\# Trials where } E \text{ occurs}}{\text{\# Total trials } (n)}$

3. Probability axioms:

Axiom 1: For any event E, $0 \leq \mathbb{P}[E] \leq 1$
Axiom 2: Probability of the sample space S is $\mathbb{P}[S] = 1$
Axiom 3: If E and F are mutually exclusive (i.e., $E \cap F = \varnothing$), then $\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F]$
In general, for all mutually exclusive events $E_1, E_2, \cdots$,

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} E_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[E_i]$$

4. General inclusion-exclusion principle: $\mathbb{P}\left[\displaystyle\bigcup_{i=1}^{n} E_i\right] = \displaystyle\sum_{r=1}^{n} (-1)^{r+1} \left(\sum_{i_1 < \cdots < i_r}^{n} \mathbb{P}[E_{i_1} \cap \cdots \cap E_{i_r}] \right)$

Case $n=2$: $\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F]$

5. Union bound (Boole's inequality): For any events $E_1, E_2, \cdots, E_n$,

$$\mathbb{P}\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} \mathbb{P}[E_i]$$

2 Random Variables

1.

3 Moments and Limit Theorems

1.

4 Applications and Statistics

1.