Types
CST Part IB Paper 8 & 9

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1 Simply-Typed λ-Calculus

Syntax

Types	T	$::=$	$1 \mid 0 \mid T_1 \times T_2 \mid T_1 + T_2 \mid T_1 \to T_2$
Terms	e	$::=$	$x \mid \langle\rangle \mid \langle e_1, e_2\rangle \mid \mathsf{fst}\ e \mid \mathsf{snd}\ e \mid \mathsf{L}\ e \mid \mathsf{R}\ e \mid \mathsf{case}(e, \mathsf{L}\ x \to e_1, \mathsf{R}\ y \to e_2)$
		$\mid$	$\lambda x : T.\ e \mid e_1\ e_2 \mid \mathsf{abort}$
Values	v	$::=$	$\langle\rangle \mid \langle v_1, v_2\rangle \mid \lambda x : T.\ e \mid \mathsf{L}\ v \mid \mathsf{R}\ v$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x : T$

Typing rules

(I: introduction rule, E: elimination rule, Hyp: hypothesis)

$$\frac{}{\Gamma \vdash \langle\rangle : 1}\ 1\mathrm{I} \qquad \frac{\Gamma \vdash e_1 : T_1 \qquad \Gamma \vdash e_2 : T_2}{\Gamma \vdash \langle e_1, e_2\rangle : 1}\ {\times}\mathrm{I} \qquad \frac{\Gamma \vdash e : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\ e : T_1}\ {\times}\mathrm{E}_1 \qquad \frac{\Gamma \vdash e : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\ e : T_2}\ {\times}\mathrm{E}_2$$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \textsc{Hyp} \qquad \frac{\Gamma, x : T \vdash e : T'}{\Gamma \vdash \lambda x : T.\ e : T \to T'}\ {\to}\mathrm{I} \qquad \frac{\Gamma \vdash e_1 : T \to T' \qquad \Gamma \vdash e_2 : T}{\Gamma \vdash e_1\ e_2 : T'}\ {\to}\mathrm{E}$$

$$\frac{\Gamma \vdash e : T_1}{\Gamma \vdash \mathsf{L}\ e : T_1 + T_2}\ {+}\mathrm{I}_1 \qquad \frac{\Gamma \vdash e : T_2}{\Gamma \vdash \mathsf{R}\ e : T_1 + T_2}\ {+}\mathrm{I}_2$$

$$\frac{\Gamma \vdash e : T_1 + T_2 \qquad \Gamma, x : X \vdash e_1 : T \qquad \Gamma, x : X \vdash e_2 : T}{\Gamma \vdash \mathsf{case}(e, \mathsf{L}\ x \to e_1, \mathsf{R}\ y \to e_2) : T}\ {+}\mathrm{E}$$

(No introduction for 0) $$\frac{\Gamma \vdash e : 0}{\Gamma \vdash \mathsf{abort}\ e : T}\ 0\mathrm{E}$$

Operational semantics

(No rule for unit) $$\frac{e_1 \rightsquigarrow e_1'}{\langle e_1, e_2\rangle \rightsquigarrow \langle e_1', e_2\rangle}\ \textsc{Pair1} \qquad \frac{e_2 \rightsquigarrow e_2'}{\langle v, e_2\rangle \rightsquigarrow \langle v, e_2'\rangle}\ \textsc{Pair2}$$

$$\frac{}{\mathsf{fst}\ \langle v_1, v_2\rangle \rightsquigarrow v_1}\ \textsc{Proj1} \qquad \frac{}{\mathsf{snd}\ \langle v_1, v_2\rangle \rightsquigarrow v_2}\ \textsc{Proj2} \qquad \frac{e \rightsquigarrow e'}{\mathsf{fst}\ e \rightsquigarrow \mathsf{fst}\ e'}\ \textsc{Proj3} \qquad \frac{e \rightsquigarrow e'}{\mathsf{snd}\ e \rightsquigarrow \mathsf{snd}\ e'}\ \textsc{Proj4}$$

$$\frac{e \rightsquigarrow e'}{\mathsf{L}\ e \rightsquigarrow \mathsf{L}\ e'}\ \textsc{Sum1} \qquad \frac{e \rightsquigarrow e'}{\mathsf{R}\ e \rightsquigarrow \mathsf{R}\ e'}\ \textsc{Sum2}$$

$$\frac{e \rightsquigarrow e'}{\mathsf{case}(e, \mathsf{L}\ x \to e_1, \mathsf{R}\ y \to e_2) \rightsquigarrow \mathsf{case}(e', \mathsf{L}\ x \to e_1, \mathsf{R}\ y \to e_2)}\ \textsc{Case1}$$

$$\frac{}{\mathsf{case}(\mathsf{L}\ v, \mathsf{L}\ x \to e_1, \mathsf{R}\ y \to e_2) \rightsquigarrow [v/x]e_1}\ \textsc{Case2} \qquad \frac{}{\mathsf{case}(\mathsf{R}\ v, \mathsf{L}\ x \to e_1, \mathsf{R}\ y \to e_2) \rightsquigarrow [v/y]e_2}\ \textsc{Case3}$$

$$\frac{e_1 \rightsquigarrow e_1'}{e_1\ e_2 \rightsquigarrow e_1'\ e_2}\ \textsc{App1} \qquad \frac{e_2 \rightsquigarrow e_2'}{v\ e_2 \rightsquigarrow v\ e_2'}\ \textsc{App2} \qquad \frac{}{(\lambda x : T.\ e)\ v \rightsquigarrow [v/x]e}\ \textsc{Fn}$$

$$\frac{e \rightsquigarrow e'}{\mathsf{abort}\ e \rightsquigarrow \mathsf{abort}\ e'}\ \textsc{Abort}$$

2 Polymorphic λ-Calculus

Syntax

Types	T	$::=$	$\tau \mid T_1 \to T_2 \mid \forall \alpha.\ T$
Terms	e	$::=$	$x \mid \lambda x : T.\ e \mid e_1\ e_2 \mid \Lambda \tau.\ e \mid e\ T$
Values	v	$::=$	$\lambda x : T.\ e \mid \Lambda \tau.\ e$
Type Contexts	Θ	$::=$	$\cdot \mid \Theta, \tau$
Term Contexts	Γ	$::=$	$\cdot \mid \Gamma, x : T$

Well-formedness of types

$$\frac{\tau \in \Theta}{\Theta \vdash \tau \text{ type}} \qquad \frac{\Theta \vdash T_1 \text{ type} \qquad \Theta \vdash T_2 \text{ type}}{\Theta \vdash T_1 \to T_2 \text{ type}} \qquad \frac{\Theta, \tau \vdash T \text{ type}}{\Theta \vdash \forall \tau.\ T \text{ type}}$$

Well-formedness of term contexts

$$\frac{}{\Theta \vdash \cdot \text{ ctx}} \qquad \frac{\Theta \vdash \Gamma \text{ ctx} \qquad \Theta \vdash T \text{ type}}{\Theta \vdash \Gamma, x : T \text{ ctx}}$$

Typing rules

$$\frac{x : T \in \Gamma}{\Theta; \Gamma \vdash x : T} \qquad \frac{\Theta \vdash T \text{ type} \qquad \Theta; \Gamma, x : T \vdash e : T'}{\Theta; \Gamma \vdash \lambda x : T.\ e : T \to T'} \qquad \frac{\Theta; \Gamma \vdash e_1 : T \to T' \qquad \Theta; \Gamma \vdash e_2 : T}{\Theta; \Gamma \vdash e_1\ e_2 : T'}$$

$$\frac{\Theta, \tau; \Gamma \vdash e : T'}{\Theta; \Gamma \vdash \Lambda \tau.\ e : \forall \tau.\ T'} \qquad \frac{\Theta; \Gamma \vdash e : \forall \tau.\ T' \qquad \Theta \vdash T \text{ type}}{\Theta; \Gamma \vdash e\ T : [T/\tau]T'}$$

Operational semantics