NST Mathematics

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1. Parallel and perpendicular components of a vector:
   $\underline{a}_{\parallel} = (\underline{a} \cdot \hat{\underline{n}})\hat{\underline{n}}$
   $\underline{a}_{\perp} = \underline{a} - (\underline{a} \cdot \hat{\underline{n}})\hat{\underline{n}}$

2. Vector triple product:
   $[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c})$
   $[\underline{a}, \underline{b}, \underline{c}] = [\underline{b}, \underline{c}, \underline{a}] = [\underline{c}, \underline{a}, \underline{b}]$
   $[\underline{a}, \underline{b}, \underline{c}] = -[\underline{a}, \underline{c}, \underline{b}]$

3. $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

4. Plane equation:
   $\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}} = d$
   $|d|$ is perpendicular distance of plane from origin for unit normal $\hat{\underline{n}}$

5. Polar coordinates:
   $\hat{\underline{r}} = \cos\phi \underline{i} + \sin\phi \underline{j}$
   $\hat{\underline{\phi}} = -\sin\phi \underline{i} + \cos\phi \underline{j}$
   $dS = r dr d\phi$

6. Cylindrical coordinates:
   $x = r\cos\phi$
   $y = r\sin\phi$
   $z = z$
   $dV = r dr d\phi dz$

7. Spherical coordinates:
   $x = r\sin\theta\cos\phi$
   $y = r\sin\theta\sin\phi$
   $z = r\cos\theta$

   $\hat{\underline{r}} = \sin\theta\cos\phi \underline{i} + \sin\theta\sin\phi \underline{j} + \cos\theta \underline{k}$
   $\hat{\underline{\theta}} = \cos\theta\cos\phi \underline{i} + \cos\theta\sin\phi \underline{j} - \sin\theta \underline{k}$
   $\hat{\underline{\phi}} = -\sin\phi \underline{i} + \cos\phi \underline{j}$

   $dV = r^2 \sin\theta dr d\theta d\phi$
   $dS = r^2 \sin\theta d\theta d\phi$

8. Leibnitz's formula:
   $$\frac{d^n(fg)}{dx^n} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)}$$

9. Limits:
   $$\lim_{x\to a} f(x) = K \text{ means that } \forall\epsilon > 0.\exists\delta > 0.\ (0 < |x-a| < \delta) \implies (\ |f(x) - K| < \epsilon)$$
   $$\lim_{x\to a+} f(x) = K \text{ means that } \forall\epsilon > 0.\exists\delta > 0.\ (0 < x-a < \delta) \implies (\ |f(x) - K| < \epsilon)$$
   $$\lim_{x\to a-} f(x) = K \text{ means that } \forall\epsilon > 0.\exists\delta > 0.\ (0 < a-x < \delta) \implies (\ |f(x) - K| < \epsilon)$$
   $$\lim_{x\to\infty} f(x) = K \text{ means that } \forall\epsilon > 0.\exists X > 0.\ (x > X) \implies (\ |f(x) - K| < \epsilon)$$

10. Continuity: $f(x)$ is continuous at a if:
    1) $f(a)$ exists;
    2) $\lim_{x\to a} f(x)$ exists and equals $f(a)$

11. Taylor's series:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$$

12. Hyperbolic functions:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

13. Differentiation of integrals wrt parameters:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t)$$

14. Schwarz's inequality:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)$$

15. Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

16. Bayes' theorem:
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

17. Poisson distribution:
$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}$$
mean = variance = λ

18. Lifetime distribution:
$f(t) = \lambda e^{-\lambda t}$
mean = $\frac{1}{\lambda}$, variance = $\frac{1}{\lambda^2}$

19. Gaussian (normal) distribution:
$f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
mean $= \mu$, variance $= \sigma^2$

20. Linear 1st-order ODE:
$\dfrac{dy}{dx} + p(x)y = f(x)$:

$y = \dfrac{1}{\mu(x)} \displaystyle\int \mu(x) f(x) dx$, where $\mu(x) = e^{\int p(x)dx}$

21. Linear 2nd-order ODE (constant coefficients):

$\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = f(x)$:

Complementary function: consider the auxiliary equation $\lambda^2 + a\lambda + b = 0$:
   - Case of real, distinct roots λ_1, λ_2: $y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
   - Case of equal roots α: $y_c = (A + Bx)e^{\alpha x}$
   - Case of complex roots $\alpha \pm \beta i$: $y_c = e^{\alpha x}(A \sin \beta x + B \cos \beta x)$

Particular integral:
   - Case $f(x)$ is polynomial: try polynomial with same degree (or higher if needed)
   - Case $f(x) = Ce^{kx}$: try $y_p = De^{kx}$ (or Dxe^{kx}, or Dx^2e^{kx})
   - Case $f(x) = C_1 \sin kx + C_2 \cos kx$: try $y_p = D_1 \sin kx + D_2 \cos kx$ (or $D_1 x \sin kx + D_2 x \cos kx$)

22. Partial differentiation:

   1) Differential relations:
   $$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$
   2) Chain rule:
   $$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$
   $$\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$
   3) Reciprocity:
   $$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1$$
   4) Cyclic relations:
   $$\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x = -1$$
   5) Exact differential:
   $P(x, y)dx + Q(x, y)dy$ is an exact differential iff $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
   6) Taylor series:
   $$\begin{aligned} f(x, y) = {} & f(x_0, y_0) \\ & + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ & + \frac{1}{2!}\left(f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2\right) + \cdots \end{aligned}$$

23. Stationary points of multi-variable functions:
f has a stationary point if $f_x = f_y = 0$:
   - Local minimum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} > 0$ and $f_{yy} > 0$;
   - Local maximum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} < 0$ and $f_{yy} < 0$;
   - Saddle point: $f_{xx}f_{yy} < f_{xy}^2$

24. Lagrange multipliers and Lagrangian function:
To find the stationary points of $f(x, y)$ subject to the constraint $g(x, y) = 0$:
define the Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

and solve $L_x = L_y = L_\lambda = 0$.

25. Gradient of a scalar field:
$\boldsymbol{\nabla}\Phi = (\Phi_x, \Phi_y, \Phi_z)$
$d\Phi = (\boldsymbol{\nabla}\Phi) \cdot d\boldsymbol{x}$

26. Divergence of a vector field:

$$\text{div } \boldsymbol{F} = \boldsymbol{\nabla} \cdot \boldsymbol{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\Phi) = \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

27. Curl of a vector field:

$$\text{curl } \boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Phi) = \boldsymbol{0}$$

28. Normal to a surface:

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla}\Phi}{|\boldsymbol{\nabla}\Phi|}$$

29. Line integral of a scalar field:

$$\int_\Gamma \Phi ds = \int_{s_1}^{s_2} \Phi(\boldsymbol{x}(s)) ds = \int_{t_1}^{t_2} \Phi(\boldsymbol{x}(t)) \left|\frac{d\boldsymbol{x}}{dt}\right| dt$$

30. Line integral of a vector field:

$$\int_\Gamma \boldsymbol{F}(\boldsymbol{x}) \cdot d\boldsymbol{x} = \int_{t_1}^{t_2} \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \frac{d\boldsymbol{x}}{dt} dt$$

31. Conservative vector fields: $\boldsymbol{F} = \boldsymbol{\nabla}\Phi$

$$\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{x} = \int_\Gamma (\boldsymbol{\nabla}\Phi) \cdot d\boldsymbol{x} = \Phi(\boldsymbol{x}_1) - \Phi(\boldsymbol{x}_2)$$

$$\oint_\Gamma \boldsymbol{F} \cdot d\boldsymbol{x} = \oint_\Gamma (\boldsymbol{\nabla}\Phi) \cdot d\boldsymbol{x} = 0$$

32. Surface integral (flux):

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$$

33. Gauss's theorem (divergence theorem):

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{F}) dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}, \text{ where } S \text{ is the bounding surface of } V$$

34. Stoke's theorem (curl theorem):

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_C \boldsymbol{F} \cdot d\boldsymbol{x}, \text{ where } C \text{ is the boundary of } S \text{ (also called } \partial S)$$

35. Decomposing matrix $\boldsymbol{M}$ as sum of a symmetric matrix $\boldsymbol{S}$ and an anti-symmetric matrix $\boldsymbol{A}$:
$\boldsymbol{S} = \frac{1}{2}(\boldsymbol{M} + \boldsymbol{M}^T)$, $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{M} - \boldsymbol{M}^T)$
For an anti-symmetric matrix $\boldsymbol{A}$, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$ for any column vector $\boldsymbol{x}$.

36. Hermitian conjugation:
If $\boldsymbol{A} = (a_{ij})$, then the Hermitian conjugate is $\boldsymbol{A}^\dagger = (\boldsymbol{A}^T)^* = (\boldsymbol{A}^*)^T = (a_{ji}^*)$
Hermitian matrix: $\boldsymbol{A}^\dagger = \boldsymbol{A}$

37. Trace: For an $n \times n$ matrix $\boldsymbol{A}$, trace($\boldsymbol{A}$)=$\displaystyle\sum_{i=1}^{n} a_{ii}$
The trace of the product of a symmetric and an antisymmetric matrix is 0.
trace($\boldsymbol{AB}$)=trace($\boldsymbol{BA}$).
The results can be generalised and holds for any cyclic permutation of the order of multiplication.

38. Minors and cofactors:
For an $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$, let $\boldsymbol{M}_{ij}$ be an $(n-1) \times (n-1)$ submatrix:
   - Minor of the element a_{ij} of $\boldsymbol{A}$: $|\boldsymbol{M}_{ij}|$
   - Cofactor of a_{ij}: $A_{ij} = (-1)^{i+j}|\boldsymbol{M}_{ij}|$

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

   - Classical adjoint: $(\text{adj}\boldsymbol{A})_{ij} = A_{ji}$

$$\begin{pmatrix} A_{11} & A_{21} & \dots & A_{j1} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{j2} & \dots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1i} & A_{2i} & \dots & A_{ji} & \dots & A_{ni} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{jn} & \dots & A_{nn} \end{pmatrix}$$

39. Determinants:
$|\boldsymbol{A}| = \displaystyle\sum_{j=1}^{n} a_{ij}A_{ij}$ for any fixed i; or
$|\boldsymbol{A}| = \displaystyle\sum_{i=1}^{n} a_{ij}A_{ij}$ for any fixed j; or
product of the elements on the diagonal if the matrix is triangular.
   - $\boldsymbol{A}(\text{adj}\boldsymbol{A}) = (\det\boldsymbol{A})\boldsymbol{I}$
   - Interchanging any two rows or columns of $\boldsymbol{A}$ changes the sign of $\det\boldsymbol{A}$;
   - $\det\boldsymbol{A} = 0$ if any two rows or columns are the same;
   - Multiplying all the elements of any one row or column of $\boldsymbol{A}$ by λ multiplies $\det\boldsymbol{A}$ by λ;
   - Adding a multiple of row (column) on another row (column) leaves $\det\boldsymbol{A}$ unchanged;
   - $\det\boldsymbol{AB} = (\det\boldsymbol{A})(\det\boldsymbol{B})$
   - $\det\boldsymbol{A} = \det\boldsymbol{A}^T$

40. Eigenvalues:
   - $\det\boldsymbol{A} = \displaystyle\prod_{i=1}^{n} \lambda_i$
   - $\text{trace}(\boldsymbol{A}) = \displaystyle\sum_{i=1}^{n} \lambda_i$

41. Diagonalisation of real symmetric matrices: if $\boldsymbol{A}$ is real symmetric, then:

   - $\boldsymbol{A}$ has n distinct eigenvalues λ_1, λ_2, ..., λ_n;
   - $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, ..., $\boldsymbol{e}_n$ that form an orthonormal basis;
   - $\boldsymbol{A}$ can be diagonalised by setting $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \dots & \boldsymbol{e}_n \end{pmatrix}$, then

$$\boldsymbol{A}' = \boldsymbol{X}^T \boldsymbol{A} \boldsymbol{X} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$