NST Part IA Mathematics

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1. Parallel and perpendicular components of a vector:
   $\mathbf{a}_{\parallel} = (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$
   $\mathbf{a}_{\perp} = \mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$

2. Vector triple product:
   $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
   $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}]$
   $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$

3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

4. Plane equation:
   $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = d$
   $|d|$ is perpendicular distance of plane from origin for unit normal $\hat{\mathbf{n}}$

5. Polar coordinates:
   $\hat{\mathbf{r}} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$
   $\hat{\boldsymbol{\phi}} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$
   $dS = r\,dr\,d\phi$

6. Cylindrical coordinates:
   $x = r\cos\phi$
   $y = r\sin\phi$
   $z = z$
   $dV = r\,dr\,d\phi\,dz$

7. Spherical coordinates:
   $x = r\sin\theta\cos\phi$
   $y = r\sin\theta\sin\phi$
   $z = r\cos\theta$

   $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$
   $\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$
   $\hat{\boldsymbol{\phi}} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$

   $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$
   $dS = r^2\sin\theta\,d\theta\,d\phi$

8. Leibnitz's formula:

   $$\frac{d^n(fg)}{dx^n} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)}$$

9. Limits:

   $\lim_{x \to a} f(x) = K$ means that $\forall \epsilon > 0. \exists \delta > 0.\ (0 < |x - a| < \delta) \implies (|f(x) - K| < \epsilon)$

   $\lim_{x \to a+} f(x) = K$ means that $\forall \epsilon > 0. \exists \delta > 0.\ (0 < x - a < \delta) \implies (|f(x) - K| < \epsilon)$

   $\lim_{x \to a-} f(x) = K$ means that $\forall \epsilon > 0. \exists \delta > 0.\ (0 < a - x < \delta) \implies (|f(x) - K| < \epsilon)$

   $\lim_{x \to \infty} f(x) = K$ means that $\forall \epsilon > 0. \exists X > 0.\ (x > X) \implies (|f(x) - K| < \epsilon)$

10. Continuity: $f(x)$ is continuous at a if:

    - $f(a)$ exists;
    - $\lim_{x \to a} f(x)$ exists and equals $f(a)$.

11. Taylor's series:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$$

12. Hyperbolic functions:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

13. Differentiation of integrals wrt parameters:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t)$$

14. Schwarz's inequality:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)$$

15. Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

16. Bayes' theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

17. Poisson distribution:

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}$$

mean = variance = λ

18. Lifetime distribution:

$$f(t) = \lambda e^{-\lambda t}$$

mean = $\frac{1}{\lambda}$, variance = $\frac{1}{\lambda^2}$

19. Gaussian (normal) distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

mean $= \mu$, variance $= \sigma^2$

20. Linear 1st-order ODE:

$$\frac{dy}{dx} + p(x)y = f(x):$$

$$y = \frac{1}{\mu(x)} \int \mu(x)f(x)dx, \text{ where } \mu(x) = e^{\int p(x)dx}$$

21. Linear 2nd-order ODE (constant coefficients):

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x):$$

Complementary function: consider the auxiliary equation $\lambda^2 + a\lambda + b = 0$:

- Case of real, distinct roots λ_1, λ_2: $y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
- Case of equal roots α: $y_c = (A + Bx)e^{\alpha x}$
- Case of complex roots $\alpha \pm \beta i$: $y_c = e^{\alpha x}(A \sin\beta x + B \cos\beta x)$

Particular integral:

- Case $f(x)$ is polynomial: try polynomial with same degree (or higher if needed)
- Case $f(x) = Ce^{kx}$: try $y_p = De^{kx}$ (or Dxe^{kx}, or $Dx^2 e^{kx}$)
- Case $f(x) = C_1 \sin kx + C_2 \cos kx$: try $y_p = D_1 \sin kx + D_2 \cos kx$ (or $D_1 x \sin kx + D_2 x \cos kx$)

22. Partial differentiation:

- Differential relations:

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

- Chain rule:

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

$$\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

- Reciprocity:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1$$

- Cyclic relations:

$$\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x = -1$$

- Exact differential:

$P(x, y)dx + Q(x, y)dy$ is an exact differential iff $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

- Taylor series:

$$\begin{aligned} f(x, y) = {} & f(x_0, y_0) \\ & + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ & + \frac{1}{2!}\left(f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2\right) + \cdots \end{aligned}$$

23. Stationary points of multi-variable functions: f has a stationary point if $f_x = f_y = 0$

- Local minimum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} > 0$ and $f_{yy} > 0$
- Local maximum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} < 0$ and $f_{yy} < 0$
- Saddle point: $f_{xx}f_{yy} < f_{xy}^2$

24. Lagrange multipliers and Lagrangian function:
    To find the stationary points of $f(x, y)$ subject to the constraint $g(x, y) = 0$:
    define the Lagrangian function

    $$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

    and solve $L_x = L_y = L_\lambda = 0$.

25. Gradient of a scalar field:
    $$\nabla \Phi = (\Phi_x, \Phi_y, \Phi_z)$$
    $$d\Phi = (\nabla \Phi) \cdot d\mathbf{x}$$

26. Divergence of a vector field:

    $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

    $$\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

27. Curl of a vector field:

    $$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (F_x, F_y, F_z) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

    $$\nabla \times (\nabla \Phi) = \mathbf{0}$$

28. Normal to a surface:

    $$\mathbf{n} = \frac{\nabla \Phi}{|\nabla \Phi|}$$

29. Line integral of a scalar field:

    $$\int_\Gamma \Phi ds = \int_{s_1}^{s_2} \Phi(\mathbf{x}(s)) ds = \int_{t_1}^{t_2} \Phi(\mathbf{x}(t)) \left| \frac{d\mathbf{x}}{dt} \right| dt$$

30. Line integral of a vector field:

    $$\int_\Gamma \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt} dt$$

31. Conservative vector fields: $\mathbf{F} = \nabla \Phi$

    $$\int_\Gamma \mathbf{F} \cdot d\mathbf{x} = \int_\Gamma (\nabla \Phi) \cdot d\mathbf{x} = \Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2)$$

    $$\oint_\Gamma \mathbf{F} \cdot d\mathbf{x} = \oint_\Gamma (\nabla \Phi) \cdot d\mathbf{x} = 0$$

32. Surface integral (flux):

    $$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} dS$$

33. Gauss's theorem (divergence theorem):

    $$\int_V (\nabla \cdot \mathbf{F}) dV = \int_S \mathbf{F} \cdot d\mathbf{S}, \text{ where } S \text{ is the bounding surface of } V$$

34. Stoke's theorem (curl theorem):

    $$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{x}, \text{ where } C \text{ is the boundary of } S \text{ (also called } \partial S)$$

35. Decomposing matrix $\mathbf{M}$ as sum of a symmetric matrix $\mathbf{S}$ and an anti-symmetric matrix $\mathbf{A}$:
$\mathbf{S} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)$, $\mathbf{A} = \frac{1}{2}(\mathbf{M} - \mathbf{M}^T)$
For an anti-symmetric matrix $\mathbf{A}$, $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ for any column vector $\mathbf{x}$.

36. Hermitian conjugation:
If $\mathbf{A} = (a_{ij})$, then the Hermitian conjugate is $\mathbf{A}^\dagger = (\mathbf{A}^T)^* = (\mathbf{A}^*)^T = (a_{ji}^*)$
Hermitian matrix: $\mathbf{A}^\dagger = \mathbf{A}$

37. Trace: For an $n \times n$ matrix $\mathbf{A}$, $\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$

    The trace of the product of a symmetric and an antisymmetric matrix is 0.
    $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$.
    The results can be generalised and holds for any cyclic permutation of the order of multiplication.

38. Minors and cofactors:
    For an $n \times n$ matrix $\mathbf{A} = (a_{ij})$, let $\mathbf{M}_{ij}$ be an $(n-1) \times (n-1)$ sumathbfatrix:

    - Minor of the element a_{ij} of $\mathbf{A}$: $|\mathbf{M}_{ij}|$
    - Cofactor of a_{ij}: $A_{ij} = (-1)^{i+j}|\mathbf{M}_{ij}|$

    $$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

    - Classical adjoint: $(\text{adj}\mathbf{A})_{ij} = A_{ji}$

    $$\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1i} & A_{2i} & \cdots & A_{ji} & \cdots & A_{ni} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{jn} & \cdots & A_{nn} \end{pmatrix}$$

39. Determinants:

    $|\mathbf{A}| = \sum_{j=1}^{n} a_{ij}A_{ij}$ for any fixed i; or

    $|\mathbf{A}| = \sum_{i=1}^{n} a_{ij}A_{ij}$ for any fixed j; or

    product of the elements on the diagonal if the matrix is triangular.

    - $\mathbf{A}(\text{adj}\mathbf{A}) = (\det\mathbf{A})\mathbf{I}$
    - Interchanging any two rows or columns of $\mathbf{A}$ changes the sign of $\det\mathbf{A}$
    - $\det\mathbf{A} = 0$ if any two rows or columns are the same
    - Multiplying all the elements of any one row or column of $\mathbf{A}$ by λ multiplies $\det\mathbf{A}$ by λ
    - Adding a multiple of row (column) on another row (column) leaves $\det\mathbf{A}$ unchanged
    - $\det\mathbf{AB} = (\det\mathbf{A})(\det\mathbf{B})$
    - $\det\mathbf{A} = \det\mathbf{A}^T$

40. Eigenvalues:

    - $\det\mathbf{A} = \prod_{i=1}^{n} \lambda_i$

    - $\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$

41. Diagonalisation of real symmetric matrices: if $\mathbf{A}$ is real symmetric, then:

- $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$;
- $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ that form an orthonormal basis;
- $\mathbf{A}$ can be diagonalised by setting $\mathbf{X} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{pmatrix}$, then

$$\mathbf{A}' = \mathbf{X}^T \mathbf{A} \mathbf{X} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$