

NST Part IA Mathematics

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1. Parallel and perpendicular components of a vector:

$$\underline{a}_{\parallel} = (\underline{a} \cdot \underline{\hat{n}})\underline{\hat{n}}$$
$$\underline{a}_{\perp} = \underline{a} - (\underline{a} \cdot \underline{\hat{n}})\underline{\hat{n}}$$

2. Vector triple product:

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c})$$
$$[\underline{a}, \underline{b}, \underline{c}] = [\underline{b}, \underline{c}, \underline{a}] = [\underline{c}, \underline{a}, \underline{b}]$$
$$[\underline{a}, \underline{b}, \underline{c}] = -[\underline{a}, \underline{c}, \underline{b}]$$

3. $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

4. Plane equation:

$$\underline{r} \cdot \underline{\hat{n}} = \underline{a} \cdot \underline{\hat{n}} = d$$

$|d|$ is perpendicular distance of plane from origin for unit normal $\underline{\hat{n}}$

5. Polar coordinates:

$$\underline{\hat{r}} = \cos \phi \underline{i} + \sin \phi \underline{j}$$
$$\underline{\hat{\phi}} = -\sin \phi \underline{i} + \cos \phi \underline{j}$$
$$dS = r dr d\phi$$

6. Cylindrical coordinates:

$$x = r \cos \phi$$
$$y = r \sin \phi$$
$$z = z$$
$$dV = r dr d\phi dz$$

7. Spherical coordinates:

$$x = r \sin \theta \cos \phi$$
$$y = r \sin \theta \sin \phi$$
$$z = r \cos \theta$$
$$\underline{\hat{r}} = \sin \theta \cos \phi \underline{i} + \sin \theta \sin \phi \underline{j} + \cos \theta \underline{k}$$
$$\underline{\hat{\theta}} = \cos \theta \cos \phi \underline{i} + \cos \theta \sin \phi \underline{j} - \sin \theta \underline{k}$$
$$\underline{\hat{\phi}} = -\sin \phi \underline{i} + \cos \phi \underline{j}$$
$$dV = r^2 \sin \theta dr d\theta d\phi$$
$$dS = r^2 \sin \theta d\theta d\phi$$

8. Leibnitz's formula:

$$\frac{d^n(fg)}{dx^n} = \sum_{i=0}^n \binom{n}{i} f^{(n-i)} g^{(i)}$$

9. Limits:

$$\lim_{x \rightarrow a} f(x) = K \text{ means that } \forall \epsilon > 0. \exists \delta > 0. (0 < |x - a| < \delta) \implies (|f(x) - K| < \epsilon)$$

$$\lim_{x \rightarrow a+} f(x) = K \text{ means that } \forall \epsilon > 0. \exists \delta > 0. (0 < x - a < \delta) \implies (|f(x) - K| < \epsilon)$$

$$\lim_{x \rightarrow a-} f(x) = K \text{ means that } \forall \epsilon > 0. \exists \delta > 0. (0 < a - x < \delta) \implies (|f(x) - K| < \epsilon)$$

$$\lim_{x \rightarrow \infty} f(x) = K \text{ means that } \forall \epsilon > 0. \exists X > 0. (x > X) \implies (|f(x) - K| < \epsilon)$$

10. Continuity: $f(x)$ is continuous at a if:

- 1) $f(a)$ exists;
- 2) $\lim_{x \rightarrow a} f(x)$ exists and equals $f(a)$

11. Taylor's series:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$$

12. Hyperbolic functions:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

13. Differentiation of integrals wrt parameters:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + \frac{db}{dt} f(b, t) - \frac{da}{dt} f(a, t)$$

14. Schwarz's inequality:

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \left(\int_a^b f^2(x)dx \right) \left(\int_a^b g^2(x)dx \right)$$

15. Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

16. Bayes' theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

17. Poisson distribution:

$$P(X=r) = e^{-\lambda} \frac{\lambda^r}{r!}$$

$$\text{mean} = \text{variance} = \lambda$$

18. Lifetime distribution:

$$f(t) = \lambda e^{-\lambda t}$$

$$\text{mean} = \frac{1}{\lambda}, \text{variance} = \frac{1}{\lambda^2}$$

19. Gaussian (normal) distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

mean = μ , variance = σ^2

20. Linear 1st-order ODE:

$$\frac{dy}{dx} + p(x)y = f(x):$$

$$y = \frac{1}{\mu(x)} \int \mu(x)f(x)dx, \text{ where } \mu(x) = e^{\int p(x)dx}$$

21. Linear 2nd-order ODE (constant coefficients):

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x):$$

Complementary function: consider the auxiliary equation $\lambda^2 + a\lambda + b = 0$:

- Case of real, distinct roots λ_1, λ_2 : $y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
- Case of equal roots α : $y_c = (A + Bx)e^{\alpha x}$
- Case of complex roots $\alpha \pm \beta i$: $y_c = e^{\alpha x}(A \sin \beta x + B \cos \beta x)$

Particular integral:

- Case $f(x)$ is polynomial: try polynomial with same degree (or higher if needed)
- Case $f(x) = Ce^{kx}$: try $y_p = De^{kx}$ (or Dxe^{kx} , or Dx^2e^{kx})
- Case $f(x) = C_1 \sin kx + C_2 \cos kx$: try $y_p = D_1 \sin kx + D_2 \cos kx$ (or $D_1 x \sin kx + D_2 x \cos kx$)

22. Partial differentiation:

- 1) Differential relations:

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

- 2) Chain rule:

$$\begin{aligned} \left(\frac{\partial f}{\partial u}\right)_v &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v \\ \left(\frac{\partial f}{\partial v}\right)_u &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u \end{aligned}$$

- 3) Reciprocity:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1$$

- 4) Cyclic relations:

$$\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x = -1$$

- 5) Exact differential:

$$P(x, y)dx + Q(x, y)dy \text{ is an exact differential iff } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

- 6) Taylor series:

$$\begin{aligned} f(x, y) &= f(x_0, y_0) \\ &+ f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &+ \frac{1}{2!}(f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2) + \dots \end{aligned}$$

23. Stationary points of multi-variable functions:

f has a stationary point if $f_x = f_y = 0$:

- Local minimum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} > 0$ and $f_{yy} > 0$;
- Local maximum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} < 0$ and $f_{yy} < 0$;
- Saddle point: $f_{xx}f_{yy} < f_{xy}^2$

24. Lagrange multipliers and Lagrangian function:

To find the stationary points of $f(x, y)$ subject to the constraint $g(x, y) = 0$:
define the Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

and solve $L_x = L_y = L_\lambda = 0$.

25. Gradient of a scalar field:

$$\nabla \Phi = (\Phi_x, \Phi_y, \Phi_z)$$

$$d\Phi = (\nabla \Phi) \cdot d\mathbf{x}$$

26. Divergence of a vector field:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

27. Curl of a vector field:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (F_x, F_y, F_z) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$\nabla \times (\nabla \Phi) = \mathbf{0}$$

28. Normal to a surface:

$$\mathbf{n} = \frac{\nabla \Phi}{|\nabla \Phi|}$$

29. Line integral of a scalar field:

$$\int_{\Gamma} \Phi ds = \int_{s_1}^{s_2} \Phi(\mathbf{x}(s)) ds = \int_{t_1}^{t_2} \Phi(\mathbf{x}(t)) \left| \frac{d\mathbf{x}}{dt} \right| dt$$

30. Line integral of a vector field:

$$\int_{\Gamma} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt} dt$$

31. Conservative vector fields: $\mathbf{F} = \nabla \Phi$

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{x} = \int_{\Gamma} (\nabla \Phi) \cdot d\mathbf{x} = \Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2)$$

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{x} = \oint_{\Gamma} (\nabla \Phi) \cdot d\mathbf{x} = 0$$

32. Surface integral (flux):

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} dS$$

33. Gauss's theorem (divergence theorem):

$$\int_V (\nabla \cdot \mathbf{F}) dV = \int_S \mathbf{F} \cdot d\mathbf{S}, \text{ where } S \text{ is the bounding surface of } V$$

34. Stoke's theorem (curl theorem):

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{x}, \text{ where } C \text{ is the boundary of } S \text{ (also called } \partial S)$$

35. Decomposing matrix \mathbf{M} as sum of a symmetric matrix \mathbf{S} and an anti-symmetric matrix \mathbf{A} :
 $\mathbf{S} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)$, $\mathbf{A} = \frac{1}{2}(\mathbf{M} - \mathbf{M}^T)$

For an anti-symmetric matrix \mathbf{A} , $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$ for any column vector \mathbf{x} .

36. Hermitian conjugation:

If $\mathbf{A} = (a_{ij})$, then the Hermitian conjugate is $\mathbf{A}^\dagger = (\mathbf{A}^T)^* = (\mathbf{A}^*)^T = (a_{ji}^*)$

Hermitian matrix: $\mathbf{A}^\dagger = \mathbf{A}$

37. Trace: For an $n \times n$ matrix \mathbf{A} , $\text{trace}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$

The trace of the product of a symmetric and an antisymmetric matrix is 0.

$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$.

The results can be generalised and holds for any cyclic permutation of the order of multiplication.

38. Minors and cofactors:

For an $n \times n$ matrix $\mathbf{A} = (a_{ij})$, let \mathbf{M}_{ij} be an $(n-1) \times (n-1)$ submatrix:

- Minor of the element a_{ij} of \mathbf{A} : $|\mathbf{M}_{ij}|$
- Cofactor of a_{ij} : $A_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|$

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Classical adjoint: $(\text{adj} \mathbf{A})_{ij} = A_{ji}$

$$\begin{pmatrix} A_{11} & A_{21} & \dots & A_{j1} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{j2} & \dots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1i} & A_{2i} & \dots & A_{ji} & \dots & A_{ni} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{jn} & \dots & A_{nn} \end{pmatrix}$$

39. Determinants:

$$|\mathbf{A}| = \sum_{j=1}^n a_{ij} A_{ij} \text{ for any fixed } i; \text{ or}$$

$$|\mathbf{A}| = \sum_{i=1}^n a_{ij} A_{ij} \text{ for any fixed } j; \text{ or}$$

product of the elements on the diagonal if the matrix is triangular.

- $\mathbf{A}(\text{adj} \mathbf{A}) = (\det \mathbf{A}) \mathbf{I}$
- Interchanging any two rows or columns of \mathbf{A} changes the sign of $\det \mathbf{A}$;
- $\det \mathbf{A} = 0$ if any two rows or columns are the same;
- Multiplying all the elements of any one row or column of \mathbf{A} by λ multiplies $\det \mathbf{A}$ by λ ;
- Adding a multiple of row (column) on another row (column) leaves $\det \mathbf{A}$ unchanged;
- $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$
- $\det \mathbf{A} = \det \mathbf{A}^T$

40. Eigenvalues:

$$- \det \mathbf{A} = \prod_{i=1}^n \lambda_i$$

$$- \text{trace}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$$

41. Diagonalisation of real symmetric matrices: if \mathbf{A} is real symmetric, then:

- \mathbf{A} has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$;
- \mathbf{A} has n linearly independent eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ that form an orthonormal basis;
- \mathbf{A} can be diagonalised by setting $\mathbf{X} = (\mathbf{e}_1 \ \mathbf{e}_2 \ \dots \ \mathbf{e}_n)$, then

$$\mathbf{A}' = \mathbf{X}^T \mathbf{A} \mathbf{X} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$