Zoids ACM-ICPC Notebook 2018 (C++)

Contents

1 Geometry

1.1 Convex Hull Algorithm

```
bool compare(PT a,PT b){    return a.y<b.y || (a.y==b.y && a.x<b.x); }
double cross(PT o,PT a, PT b)
{
        return (a.x-o.x)*(b.y-o.y)-(a.y-o.y)*(b.x-o.x);
}

vector<PT> ConvexHull(vector<PT> p) {    int n=p.size();    int k=0;
    vector<PT> h(2*n);
    sort(p.begin(),p.end(),compare);
        //build lower hull
        for(int i=0;i<n;++i)
        {
                while(k>=2 && cross(h[k-2],h[k-1],p[i])<=0) k--;
                h[k++]=p[i];
        }
        //build top hull
        for(int i=n-2,t=k+1;i>=0;--i)
        {
                while(k>=t && cross(h[k-2],h[k-1],p[i])<=0) k--;
                h[k++]=p[i];
        }
        h.resize(k);
        return h;
}
```

1.2 Delaunay Triangulation

```
/*
Stanford notebook
----------------------------------------------------
Delaunay Algorithm Does not handle degenerate cases
Running time: O(n^4)
INPUT: x[] = x-coordinates
            y[] = y-coordinates
OUTPUT: triples = a vector containing m triples
(indices corresponding to triangle vertices)
----------------------------------------------------
*/

typedef double T;

struct triple {
        int i, j, k;
        triple() {}
        triple(int i, int j, int k) : i(i), j(j), k(k) {}
};

vector<triple> delaunayTriangulation(vector<T>& x, vector<T>& y)
{
        int n = x.size();
        vector<T> z(n);
        vector<triple> ret;

    for (int i = 0; i < n; i++)
        z[i] = x[i] * x[i] + y[i] * y[i];
    for (int i = 0; i < n-2; i++)
    {
        for (int j = i+1; j < n; j++)
        {
                for (int k = i+1; k < n; k++)
                {
                        if (j == k) continue;
                        double xn = (y[j]-y[i])*(z[k]-z[i]) - (y[k]-y[i])*(z[j]-z[i]);
                double yn = (x[k]-x[i])*(z[j]-z[i]) - (x[j]-x[i])*(z[k]-z[i]);
                double zn = (x[j]-x[i])*(y[k]-y[i]) - (x[k]-x[i])*(y[j]-y[i]);
                bool flag = zn < 0;
                for (int m = 0; flag && m < n; m++)
                        flag = flag && ((x[m]-x[i])*xn + (y[m]-y[i])*yn + (z[m]-z[i])*zn <= 0);
                if (flag) ret.push_back(triple(i, j, k));
            }
        }
    }
    return ret;
}

int main() {
        T xs[]={0, 0, 1, 0.9};
        T ys[]={0, 1, 0, 0.9};
        vector<T> x(&xs[0], &xs[4]), y(&ys[0], &ys[4]);
        vector<triple> tri = delaunayTriangulation(x, y);
        //expected: 0 1 3
        //          0 3 2
        int i;
        for(i = 0; i < tri.size(); i++)
                printf("%d %d %d\n", tri[i].i, tri[i].j, tri[i].k);
        return 0;
}
```

1.3 Various Geometry Functions

```
// Stanford Notebook
double INF = 1e100;
double EPS = 1e-12;


struct PT {
        double x, y;
        PT() {}
        PT(double x, double y) : x(x), y(y) {}
        PT(const PT &p) : x(p.x), y(p.y)    {}
        PT operator + (const PT &p)  const { return PT(x+p.x, y+p.y); }
        PT operator - (const PT &p)  const { return PT(x-p.x, y-p.y); }
        PT operator * (double c)     const { return PT(x*c,   y*c  ); }
        PT operator / (double c)     const { return PT(x/c,   y/c  ); }
};

double dot(PT p, PT q)     { return p.x*q.x+p.y*q.y; }
double dist2(PT p, PT q)   { return dot(p-q,p-q); }
double cross(PT p, PT q)   { return p.x*q.y-p.y*q.x; }
```

```
// rotate a point CCW or CW around the origin
PT RotateCCW90(PT p)   { return PT(-p.y,p.x); }
PT RotateCW90(PT p)    { return PT(p.y,-p.x); }
PT RotateCCW(PT p, double t) {     return PT(p.x*cos(t)-p.y*sin(t), p.x*sin(t)+p.y*cos(t)); }

// project point c onto line through a and b // assuming a != b
PT ProjectPointLine(PT a, PT b, PT c) { return a + (b-a)*dot(c-a, b-a)/dot(b-a, b-a); }
// project point c onto line segment through a and b
PT ProjectPointSegment(PT a, PT b, PT c)
{
        double r = dot(b-a,b-a);
        if (fabs(r) < EPS) return a;
        r = dot(c-a, b-a)/r;
        if (r < 0) return a;
        if (r > 1) return b;
        return a + (b-a)*r;
}

// compute distance from c to segment between a and b
double DistancePointSegment(PT a, PT b, PT c)
{
        return sqrt(dist2(c, ProjectPointSegment(a, b, c)));
}

// compute distance between point (x,y,z) and plane ax+by+cz=d
double DistancePointPlane(double x, double y, double z, double a, double b, double c, double d)
{
    return fabs(a*x+b*y+c*z-d)/sqrt(a*a+b*b+c*c);
}

// determine if lines from a to b and c to d are parallel or collinear
bool LinesParallel(PT a, PT b, PT c, PT d) {  return fabs(cross(b-a, c-d)) < EPS; }

bool LinesCollinear(PT a, PT b, PT c, PT d)
{
        return LinesParallel(a, b, c, d)
        && fabs(cross(a-b, a-c)) < EPS
        && fabs(cross(c-d, c-a)) < EPS;
}


// determine if line segment from a to b intersects with
// line segment from c to d
bool SegmentsIntersect(PT a, PT b, PT c, PT d)
{
        if (LinesCollinear(a, b, c, d))
        {
                if (dist2(a, c) < EPS || dist2(a, d) < EPS ||
                dist2(b, c) < EPS || dist2(b, d) < EPS) return true;
                if (dot(c-a, c-b) > 0 && dot(d-a, d-b) > 0 && dot(c-b, d-b) > 0)
                        return false;
                return true;
        }
        if (cross(d-a, b-a) * cross(c-a, b-a) > 0)
                return false;
        if (cross(a-c, d-c) * cross(b-c, d-c) > 0)
                return false;
        return true;
}


// compute intersection of line passing through a and b
// with line passing through c and d, assuming that unique
// intersection exists; for segment intersection, check if
// segments intersect first
PT ComputeLineIntersection(PT a, PT b, PT c, PT d)
{
        b=b-a; d=c-d; c=c-a;
        assert(dot(b, b) > EPS && dot(d, d) > EPS);
        return a + b*cross(c, d)/cross(b, d);
}

// compute center of circle given three points
PT ComputeCircleCenter(PT a, PT b, PT c)
{
        b=(a+b)/2;
        c=(a+c)/2;
        return ComputeLineIntersection(b, b+RotateCW90(a-b), c, c+RotateCW90(a-c));
}

// determine if point is in a possibly non-convex polygon (by William
// Randolph Franklin); returns 1 for strictly interior points, 0 for
// strictly exterior points, and 0 or 1 for the remaining points.
// Note that it is possible to convert this into an *exact* test using
// integer arithmetic by taking care of the division appropriately
// (making sure to deal with signs properly) and then by writing exact
// tests for checking point on polygon boundary
bool PointInPolygon(const vector<PT> &p, PT q)
{
        bool c = 0;
        for (int i = 0; i < p.size(); i++)
        {
                int j = (i+1)%p.size();
                if ((p[i].y <= q.y && q.y < p[j].y ||
                p[j].y <= q.y && q.y < p[i].y) &&
                q.x < p[i].x + (p[j].x - p[i].x) * (q.y - p[i].y) / (p[j].y - p[i].y))
                c = !c;
        }
        return c;
}


// determine if point is on the boundary of a polygon
bool PointOnPolygon(const vector<PT> &p, PT q)
{
        for (int i = 0; i < p.size(); i++)
                if (dist2(ProjectPointSegment(p[i], p[(i+1)%p.size()], q), q) < EPS)
                        return true;
        return false;
}

// compute intersection of line through points a and b with
 // circle centered at c with radius r > 0
 vector<PT> CircleLineIntersection(PT a, PT b, PT c, double r)
 {
        vector<PT> ret;
        b = b-a; a = a-c;
        double A = dot(b, b);
        double B = dot(a, b);
        double C = dot(a, a) - r*r;
        double D = B*B - A*C;
        if (D < -EPS) return ret;
        ret.push_back(c+a+b*(-B+sqrt(D+EPS))/A);
        if (D > EPS)
                ret.push_back(c+a+b*(-B-sqrt(D))/A);
        return ret;
}


// compute intersection of circle centered at a with radius r
// with circle centered at b with radius R
vector<PT> CircleCircleIntersection(PT a, PT b, double r, double R)
{
        vector<PT> ret;
        double d = sqrt(dist2(a, b));
        if (d > r+R || d+min(r, R) < max(r, R)) return ret;
        double x = (d*d-R*R+r*r)/(2*d);
        double y = sqrt(r*r-x*x);
        PT v = (b-a)/d;
        ret.push_back(a+v*x + RotateCCW90(v)*y);
        if (y > 0) ret.push_back(a+v*x - RotateCCW90(v)*y);
        return ret;
}

// This code computes the area or centroid of a (possibly nonconvex)
// polygon, assuming that the coordinates are listed in a clockwise or
// counterclockwise fashion.  Note that the centroid is often known as
// the "center of gravity" or "center of mass".
double ComputeSignedArea(const vector<PT> &p)
{
        double area = 0;
        for(int i = 0; i < p.size(); i++)
        {
                int j = (i+1) % p.size();
                area += p[i].x*p[j].y - p[j].x*p[i].y;
        }
        return area / 2.0;
}

double ComputeArea(const vector<PT> &p) {    return fabs(ComputeSignedArea(p)); }

PT ComputeCentroid(const vector<PT> &p)
{
        PT c(0,0);
        double scale = 6.0 * ComputeSignedArea(p);
        for (int i = 0; i < p.size(); i++)
        {
                int j = (i+1) % p.size();
                c = c + (p[i]+p[j])*(p[i].x*p[j].y - p[j].x*p[i].y);
        }
        return c / scale;
}

// tests whether or not a given polygon (in CW or CCW order) is simple
bool IsSimple(const vector<PT> &p)
{
        for (int i = 0; i < p.size(); i++)
        {
                for (int k = i+1; k < p.size(); k++)
                {
                        int j = (i+1) % p.size();
                        int l = (k+1) % p.size();
                        if (i == l || j == k) continue;
```

```
				if (SegmentsIntersect(p[i], p[j], p[k], p[l]))
					return false;
				}
			}
		return true;
	}
```

2 Graphs

2.1 Dijkstra's Algorithm

```
//Dijkstra Algorithm
int t,n,m,s,e;
vector<ii> edges[N];//pair<NodeEnd,dist>
int distances[N]; // =INF=0x3f3f3f3f
int parent[N]; // =-1

int Dijkstra()
{
	vector<ii> :: iterator it;
	priority_queue< ii, vector<ii>, greater<ii> > pq;
	distances[s]=0;
	pq.push(ii(distances[s],s));
	while(!pq.empty())
	{
		ii p = pq.top();
		pq.pop();
		int d=p.first;
		int a=p.second;
		for(it=edges[a].begin();it!=edges[a].end();++it)
		{
			if(distances[it->first]>distances[a]+it->second)
			{
			    distances[it->first]=distances[a]+it->second;
			    parent[it->first]=a;
			    pq.push(ii(distances[it->first],it->first));
			}
		}
	}
	return distances[e];
}
```

2.2 Max Flow (Dinic's Algorithm)

```
// Stanford Notebook
typedef long long LL;

struct Edge
{
	int from, to, cap, flow, index;
	Edge(int from, int to, int cap, int flow, int index) :
	from(from), to(to), cap(cap), flow(flow), index(index) {}
	LL rcap() { return cap - flow; }
};

struct Dinic
{
	int N;
	vector<vector<Edge> > G;
	vector<vector<Edge *> > Lf;
	vector<int> layer;
	vector<int> Q;
	Dinic(int N) : N(N), G(N), Q(N) {}
	void AddEdge(int from, int to, int cap)
	{
		if (from == to) return;
		G[from].push_back(Edge(from, to, cap, 0, G[to].size()));
		G[to].push_back(Edge(to, from, 0, 0, G[from].size() - 1));
	}

	LL BlockingFlow(int s, int t)
	{
		layer.clear();
		layer.resize(N, -1);
		layer[s] = 0;
		Lf.clear(); Lf.resize(N);

	int head = 0, tail = 0;
		Q[tail++] = s;
	while (head < tail)
		{
	int x = Q[head++];
			for (int i = 0; i < G[x].size(); i++)
			{
				Edge &e = G[x][i]; if (e.rcap() <= 0) continue;
				if (layer[e.to] == -1)
				{
					layer[e.to] = layer[e.from] + 1;
					Q[tail++] = e.to;
				}
				if (layer[e.to] > layer[e.from])
				{
					Lf[e.from].push_back(&e);
				}
			}
		}
	if (layer[t] == -1) return 0;
	LL totflow = 0;
		vector<Edge *> P;
		while (!Lf[s].empty())
		{
			int curr = P.empty() ? s : P.back()->to;
			if (curr == t)
			{
				// Augment
				LL amt = P.front()->rcap();
				for (int i = 0; i < P.size(); ++i)
				{
					amt = min(amt, P[i]->rcap());
				}
				totflow += amt;
				for (int i = P.size() - 1; i >= 0; --i)
				{
					P[i]->flow += amt;
					G[P[i]->to][P[i]->index].flow -= amt;
					if (P[i]->rcap() <= 0)
					{
						Lf[P[i]->from].pop_back();
						P.resize(i);
					}
				}
			}
			else if (Lf[curr].empty())
			{
				// Retreat
				P.pop_back();
				for (int i = 0; i < N; ++i)
					for (int j = 0; j < Lf[i].size(); ++j)
						if (Lf[i][j]->to == curr)
							Lf[i].erase(Lf[i].begin() + j);
			}
			else
			{
				// Advance
				P.push_back(Lf[curr].back());
			}
		}
		return totflow;
	}

	LL GetMaxFlow(int s, int t)
	{
		LL totflow = 0;
		while (LL flow = BlockingFlow(s, t))
			totflow += flow;
		return totflow;
	}
};
```

2.3 Max Flow (Edmonds-Karp Algorithm)

```
/*
Stanford Notebook
MinCostMaxFlow (adjacency matrix, Edmonds and Karp 1972)
This implementation keeps track of forward and reverse edges separately
(so you can set cap[i][j] != cap[j][i]). For a regular max flow, set all edge costs to 0.
Running time, O(|V|^2) cost per augmentation
	max flow:           O(|V|^3) augmentations
	min cost max flow:  O(|V|^4 * MAX_EDGE_COST) augmentations
INPUT: - graph, constructed using AddEdge()
	   - source
	   - sink
OUTPUT: - (maximum flow value, minimum cost value)
		- To obtain the actual flow, look at positive values only.
*/

typedef vector<int> VI;
```

```
typedef vector<VI> VVI;
typedef long long L;
typedef vector<L> VL;
typedef vector<VL> VVL;
typedef pair<int, int> PII;
typedef vector<PII> VPII;

const L INF = numeric_limits<L>::max() / 4;

struct MinCostMaxFlow {
        int N;
        VVL cap, flow, cost;
        VI found;
        VL dist, pi, width;
        VPII dad;

    MinCostMaxFlow(int N) :
    N(N), cap(N, VL(N)), flow(N, VL(N)), cost(N, VL(N)),
    found(N), dist(N), pi(N), width(N), dad(N) {}

    void AddEdge(int from, int to, L cap, L cost)
    {
        this->cap[from][to] = cap;
        this->cost[from][to] = cost;
    }

    void Relax(int s, int k, L cap, L cost, int dir)
    {
        L val = dist[s] + pi[s] - pi[k] + cost;
        if (cap && val < dist[k])
        {
                dist[k] = val;
                dad[k] = make_pair(s, dir);
                width[k] = min(cap, width[s]);
        }
    }

    L Dijkstra(int s, int t)
    {
        fill(found.begin(), found.end(), false);
        fill(dist.begin(), dist.end(), INF);
        fill(width.begin(), width.end(), 0);
        dist[s] = 0;
        width[s] = INF;

        while (s != -1)
        {
                int best = -1;
                found[s] = true;
                for (int k = 0; k < N; k++)
                {
                        if (found[k]) continue;
                        Relax(s, k, cap[s][k] - flow[s][k], cost[s][k], 1);
                        Relax(s, k, flow[k][s], -cost[k][s], -1);
                        if (best == -1 || dist[k] < dist[best]) best = k;
                }
            s = best;
        }

        for (int k = 0; k < N; k++)
                pi[k] = min(pi[k] + dist[k], INF);
        return width[t];
    }

    pair<L, L> GetMaxFlow(int s, int t)
    {
        L totflow = 0, totcost = 0;
        while (L amt = Dijkstra(s, t))
        {
                totflow += amt;
                for (int x = t; x != s; x = dad[x].first)
                {
                        if (dad[x].second == 1)
                        {
                                flow[dad[x].first][x] += amt;
                                totcost += amt * cost[dad[x].first][x];
                        }
                        else
                        {
                                flow[x][dad[x].first] -= amt;
                                totcost -= amt * cost[x][dad[x].first];
                        }
                }
        }
        return make_pair(totflow, totcost);
    }
};
```

2.4 Eulerian Path

```
struct Edge;
typedef list<Edge>::iterator iter;
struct Edge
{
        int next_vertex;
        iter reverse_edge;
    Edge(int next_vertex) :next_vertex(next_vertex) { }
};

const int max_vertices = ;
int num_vertices;
list<Edge> adj[max_vertices]; // adjacency list
vector<int> path;

void find_path(int v)
{
        while(adj[v].size() > 0)
        {
                int vn = adj[v].front().next_vertex;
        adj[vn].erase(adj[v].front().reverse_edge);
        adj[v].pop_front();
        find_path(vn);
    }
    path.push_back(v);
}

void add_edge(int a, int b)
{
        adj[a].push_front(Edge(b));
        iter ita = adj[a].begin();
        adj[b].push_front(Edge(a));
        iter itb = adj[b].begin();
        ita->reverse_edge = itb;
        itb->reverse_edge = ita;
}
```

2.5 Hopcroft-Karp Algorithm

```
#include <vector>

vector<int> g[N];
int r[N], l[N], n, m, e, a, b;

// generate g;

bool dfs(int v)
{
        if(vis[v]) return false;
        vis[v] = true;
        for(int u=0; u<g[v].size(); ++u)
        {
                if(!r[g[v][u]])
                {
                        l[v]=g[v][u];
                        r[g[v][u]]=v;
                        return true;
                }
        }
        for(int u=0; u<g[v].size(); ++u)
        {
                if(dfs(r[g[v][u]]))
                {
                        l[v]=g[v][u];
                        r[g[v][u]]=v;
                        return true;
                }
        }
        return false;
}

void hopcroft_karp()
{
        bool change = true;
        while(change)
        {
                change = false;
                fill(vis, vis+n+1, false);
                for(int i=1; i<=n; ++i)
                        if(!l[i])
                                change|= dfs(i);
        }
}
```

2.6 Lowest Common Ancestor

```
const int max_nodes, log_max_nodes;

int num_nodes, log_num_nodes, root;
vector<int> children[max_nodes];
// children[i] contains the children of node i int A[max_nodes][log_max_nodes+1];      // A[i][j] is
      the 2^j-th ancestor of node i, or -1 if that ancestor does not exist int L[max_nodes];
                              // L[i] is the distance between node i and the root
// floor of the binary logarithm of n
int lb(unsigned int n)
{
        if(n==0)  return -1;
        int p = 0;
        if (n >= 1<<16) { n >>= 16; p += 16; }
        if (n >= 1<< 8) { n >>=  8; p +=  8; }
        if (n >= 1<< 4) { n >>=  4; p +=  4; }
        if (n >= 1<< 2) { n >>=  2; p +=  2; }
        if (n >= 1<< 1) {           p +=  1; }
        return p;
}

void DFS(int i, int l)
{
        L[i] = l;
        for(int j = 0; j < children[i].size(); j++)
                DFS(children[i][j], l+1);
}

int LCA(int p, int q) {
        // ensure node p is at least as deep as node q
        if(L[p] < L[q])  swap(p, q);

        // "binary search" for the ancestor of node p situated on the same level as q
        for(int i = log_num_nodes; i >= 0; i--)
                if(L[p] - (1<<i) >= L[q])
        p = A[p][i];
        if(p == q)   return p;

         // "binary search" for the LCA
        for(int i = log_num_nodes; i >= 0; i--)
        {
                if(A[p][i] != -1 && A[p][i] != A[q][i])
                {
                        p = A[p][i];
                        q = A[q][i];
                }
         }
    return A[p][0];
}

int main(int argc,char* argv[])
{
        // read num_nodes, the total number of nodes
        log_num_nodes=lb(num_nodes);
        for(int i = 0; i < num_nodes; i++)
        {
                int p;
                // read p, the parent of node i or -1 if node i is the root
        A[i][0] = p;
        if(p != -1)  children[p].push_back(i);
        else  root = i;
    }
    // precompute A using dynamic programming
    for(int j = 1; j <= log_num_nodes; j++)
        for(int i = 0; i < num_nodes; i++)
                if(A[i][j-1] != -1)  A[i][j] = A[A[i][j-1]][j-1];
                else  A[i][j] = -1;
    // precompute L
        DFS(root, 0);
        return 0;
  }
```

2.7 Strongly Connected Components

```
#include <memory.h>

struct edge
{
        int e, nxt;
};

int V, E;
edge e[MAXE], er[MAXE];
int sp[MAXV], spr[MAXV];
int group_cnt, group_num[MAXV];
bool v[MAXV];
int stk[MAXV];

void fill_forward(int x)
{
        int i;
        v[x]=true;
        for(i=sp[x];i;i=e[i].nxt)
                if(!v[e[i].e]) fill_forward(e[i].e);
        stk[++stk[0]]=x;
}

void fill_backward(int x)
{
        int i;
        v[x]=false;
        group_num[x]=group_cnt;
        for(i=spr[x];i;i=er[i].nxt)
                if(v[er[i].e])
                        fill_backward(er[i].e);
}

void add_edge(int v1, int v2) //add edge v1->v2
{
        e[++E].e=v2;
        e[E].nxt=sp [v1];
        sp[v1]=E;
        er[E].e=v1;
        er[E].nxt=spr[v2];
        spr[v2]=E;
}

void SCC()
{
        int i;
        stk[0]=0;
        memset(v, false, sizeof(v));
        for(i=1;i<=V;i++)
                if(!v[i])
                        fill_forward(i);
        group_cnt=0;
        for(i=stk[0];i>=1;i--)
                if(v[stk[i]]){group_cnt++; fill_backward(stk[i]);}
}
```

2.8 Union-Find Set

```
struct Edge // MST
{
        int a,b,d;
        bool operator < (const Edge &E) const{
        return this->d <E.d;
        }
};
int ranks[M]; int c[N];

int Find(int x)
{
        int y=x;
        while(y!=c[y])
                y=c[y];
        while(x!=c[x])
        {
                int aux=c[x];
                c[x]=y;
                x=aux;
        }
        return y;
}

void Union(int x,int y)
{
        if(ranks[x]>ranks[y])
                c[x]=y;
        else
                c[y]=x;
        if(ranks[x]==ranks[y])
                ranks[y]++;
}
```

3 Tree

3.1 Cartesian Tree

```cpp
struct Tr
{
        Tr *l,*r;
        int key,pr,cnt,val,rev;
        long long sum;
        Tr(int new_key,int new_pr,int new_val)
        {
                rev=0;
                key=new_key;
                cnt=1;
                l=r=NULL;
                pr=new_pr;
                val=new_val;
                sum=new_val;
        }
};


#define T Tr*
T R=NULL;

int cnt(T t)
{
        if(!t) return 0;
        return t->cnt;
}

void upd_cnt(T &t)
{
        if(t) t->cnt=cnt(t->l)+cnt(t->r)+1;
}
long long sum(T t)
{
        if(!t) return 0;
        return t->sum;
}
void upd_sum(T &t)
{
    if(t) t->sum=sum(t->l)+sum(t->r)+t->val;
}
void push(T &t)
{
        if(t && t->rev)
        {
                t->rev=0;
                swap(t->l,t->r);
                upd_sum(t);
                if(t->l) t->l->rev^=1;
                if(t->r) t->r->rev^=1;
        }
}


void split(T t,T &l,T &r,int key,int add)
{
        if(!t)
                return void(l=r=NULL);
        push(t);
        upd_cnt(t);
        int current_key=add+cnt(t->l)+1;
        if(key<=current_key)
                split(t->l,l,t->l,key,add),r=t;
    else
                split(t->r,t->r,r,key,current_key),l=t;
        upd_cnt(t);
        upd_sum(t);
}


void merge(T &t,T l,T r)
{
        push(l);
        push(r);
        if(!l || !r)
                t=l?l:r;
        else if(l->pr>r->pr)
                merge(l->r,l->r,r), t=l;
        else
                merge(r->l,l,r->l), t=r;
        upd_cnt(t);
        upd_sum(t);
}


void insert(T &t,T it,int add)
{
        push(t);
        if(!t)
        {
                t=it;
                upd_cnt(t);
                return;
        }
        upd_sum(t);
        if(it->pr > t->pr)
                split(t,it->l,it->r,it->key,add),t=it;
        else if(it->key>add+cnt(t->l)+1)
                insert(t->r,it,add+cnt(t->l)+1);
        else
                insert(t->l,it,add);
        upd_sum(t);
        upd_cnt(t);
}

void print(T t)
{
        if(!t) return;
        print(t->l);
        cout<<t->val<<" ";
        print(t->r);
}

void reverse(int left,int right)
{
        Tr *t1,*t2,*t3;
        t1=t2=t3=NULL;
        split(R,t1,t2,left,0);
        split(t2,t2,t3,right-left+2,0);
        t2->rev^=1;
        merge(R,t1,t2);
        merge(R,R,t3);
}

void get_sum(int left,int right)
{
        Tr *t1,*t2,*t3;
        t1=t2=t3=NULL;
        split(R,t1,t2,left,0);
        split(t2,t2,t3,right-left+2,0);
        cout<<t2->sum<<"\n";
        merge(R,t1,t2);
        merge(R,R,t3);
}


int n,m, q,a,b;
void example()
{
        ios_base::sync_with_stdio(0);
        cin.tie(0);
        freopen("reverse.in","r",stdin);
        freopen("reverse.out","w",stdout);
        srand(time(0));
        cin>>n>>m;
        for(int i=1;i<=n;++i)
        {
                cin>>a;
                T it=new Tr(i,rand()+1,a);
                insert(R,it,0);
        }
        for(int i=0;i<m;++i)
        {
                cin>>q>>a>>b;
                if(q) reverse(a,b);
                else  get_sum(a,b);
        }
        return 0;
}
```

3.2 Segment Tree

```cpp
//Segment Tree
#include <iostream>
#define N (1<<18)

using namespace std;
//int find(vector <int>& C, int x){return (C[x]==x) ? x : C[x]=find(C, C[x]);} C++
//int find(int x){return (C[x]==x)?x:C[x]=find(C[x]);}
```

```
typedef pair<int,int> ii;
ii arb[N]={{0,0}};
int n,m,a,b,v;
char type;

void update(int node,int l,int r,int a,int b,int val,int p)
{
        if(a<=l && r<=b)
        {
                arb[node].first=val;
                arb[node].second=p;
        }
        else
        {
                int mid=(l+r)/2;
                if(a<=mid)
                        update(node*2,l,mid,a,b,val,p);
                if(b>mid)
                        update(2*node+1,mid+1,r,a,b,val,p);
        }
}

pair<int,int> search(int node,int l,int r,int a)
{
        if(a==l && a==r)
        {
                return arb[node];
        }
        else
        {
                int mid=(l+r)/2;
                ii cur;
                if(a<=mid)
                        cur=search(2*node,l,mid,a);
                else
                        cur=search(2*node+1,mid+1,r,a);
                if(cur.second<arb[node].second)
                        return arb[node];
                else
                        return cur;
        }
}
```

4 Math

4.1 Extended Euclid's Algorithm

```
// tested on https://uva.onlinejudge.org/index.php?option=com_onlinejudge&Itemid=8&category=24&page=
      show_problem&problem=1045

struct EuclidReturn{
    Long u , v , d;
    EuclidReturn( Long u , Long v, Long d ) : u( u ) , v( v ) , d( d ) {}
};

EuclidReturn Extended_Euclid( Long a , Long b){
    if( b == 0 ) return EuclidReturn( 1 , 0 , a );
    EuclidReturn aux = Extended_Euclid( b , a%b );
    Long v = aux.u - (a/b)*aux.v;
    return EuclidReturn( aux.v , v , aux.d );
}
```

4.2 PollardRho + MillerRabin

```
// tested on https://uva.onlinejudge.org/index.php?option=onlinejudge&Itemid=99999999&category=791&
      page=show_problem&problem=2471
typedef unsigned long long ull;
typedef vector<ull> vull;

struct Pollard_Rho
{
        ull q;
        vull v;
        Pollard_Rho(){}
        Pollard_Rho( ull x ) {
                q = x;
        }
        ull gcd(ull a, ull b){
            if(b == 0) return a;
            return gcd( b , a % b );
        }
        ull mul(ull a,ull b,ull c){
            ull x = 0, y = a % c;
            while(b > 0){
                if(b%2 == 1){
                    x = (x+y)%c;
                }
                y = (y*2)%c;
                b /= 2;
            }
            return x%c;
        }
        ull modd(ull a,ull b,ull c){
            ull x=1,y=a;
            while(b > 0){
                if(b%2 == 1){
                    x=mul(x,y,c);
                }
                y = mul(y,y,c);
                b /= 2;
            }
            return x%c;
        }
        bool Miller(ull p,int iteration){ // isPrime? O(iteration * (log(n)) ^ 3 )
            if(p<2){
                return false;
            }
            if(p!=2 && p%2==0){
                return false;
            }
            ull s=p-1;
            while(s%2==0){
                s/=2;
            }
            for(int i=0;i<iteration;i++){
                ull a=rand()%(p-1)+1,temp=s;
                ull mod=modd(a,temp,p);
                while(temp!=p-1 && mod!=1 && mod!=p-1){
                    mod=mul(mod,mod,p);
                    temp *= 2;
                }
                if(mod!=p-1 && temp%2==0){
                    return false;
                }
            }
            return true;
        }
        ull rho(ull n){
            if( n % 2 == 0 ) return 2;
            ull x = 2 , y = 2 , d = 1;
            int c = rand() % n + 1;
            while( d == 1 ){
                x = (mul( x , x , n ) + c)%n;
                y = (mul( y , y , n ) + c)%n;
                y = (mul( y , y , n ) + c)%n;
                if( x - y >= 0 ) d = gcd( x - y , n );
                else d = gcd( y - x , n );
            }
            return d;
        }
        void factor(ull n){
            if (n == 1) return;
            if( Miller(n , 10) ){ // 10 is good enough for most cases
                if(q != n) v.push_back(n);
                return;
            }
            ull divisor = rho(n);
            factor(divisor);
            factor(n/divisor);
        }
        vull primefact( ull num ) // O(num ^ (1/4))
        {
                v.clear();
                q = num;
                factor( num );
                sort( ALL(v) );
                if( v.empty() ) // primos o 1
                        v.push_back( num );
                return v;
        }
    map<ull, int> primeFactorsDescomposition(ull num) { // returns pairs of {prime, exponent}
        vull pf = primefact(num);
        map<ull, int> pd; // prime descomposition
        for (int i = 0; i < (int)pf.size(); i++) {
            pd[pf[i]]++;
        }
        return pd;
    }
};
```

4.3 Sieve

```
const int MAXN = (int)1e5;

bool prime[MAXN+1];

void sieve() {// O(nlglgn)
        memset(prime, true, sizeof(prime));

        prime[0] = false;
        prime[1] = false;

        for(int i=2; i*i<=MAXN; i++)
                if(prime[i])
                        for(int j=i*i; j<=MAXN; j+=i)
                                prime[j]=false;
}

const int MAXN = (int)3e8;

bitset <MAXN+1> notprime;

void sieve() { // careful as pair numbers are not marked as notprime
    for(int i=3; i*i<=MAXN; i+=2)
        if(!notprime[i])
                for(int j=i*i; j<=MAXN; j+=(i<<1))
                        notprime[j] = true;
}
```

4.4 Fermat's Little Theorem

```
if P is prime then:
a ^ p = a mod p

And if a is not divisible by p then:
a ^ (p - 1) = 1 mod p
```

4.5 Euler's Theorem

```
a ^ phi(n) = 1 mod n  iff (if and only if)  n and a are coprimes

Bonus:

let n = p1 ^ a1 * p2 ^ a2 ...

phi(n) = (p1 - 1) * p1 ^ (a1 - 1) * (p2 - 1) * p2 ^ (a2 - 1) ...

phi(n) = n * (for each distinct prime 'p' that divides n: the product of (1 - 1 / p))
```

4.6 Chinese Remainder Theorem

```
Dados k enteros positivos {ni}, tales que ni y nj son coprimos (i!=j).
Para cualquier {ai}, existe x tal que:

x % ni = ai

Todas las soluciones son congruentes modulo N = n1*n2*...*nk

r*ni + s*N/ni = 1 -> ei = s*N/ni   -> ei % nj = 0
                     r*ni + ei = 1 -> ei % ni = 1

x = a1*e1 + a2*e2 + ... + ak*ek

// ax = 1(mod n)
Long modular_inverse(Long a, Long n){
    EuclidReturn aux = Extended_Euclid(a,n);
    return ((aux.u/aux.d)%n+n)%n;
}

// rem y mod tienen el mismo numero de elementos
long long chinese_remainder(vector<Long> rem, vector<Long> mod){
    long long ans = rem[0],m = mod[0];
    int n = rem.size();

    for(int i=1;i<n;++i){
        int a = modular_inverse(m,mod[i]);
        int b = modular_inverse(mod[i],m);
        ans = (ans*b*mod[i]+rem[i]*a*m)%(m*mod[i]);
        m *= mod[i];
    }

    return ans;
}


Chinese Remainder Theorem: Strong Form
======================================
(thanks to https://forthright48.com/2017/11/chinese-remainder-theorem-part-2-non-coprime-moduli.html)

Given two sequences of numbers A=[a1,a2,   ,an] and M=[m1,m2,   ,mn], a solution to x exists for the
      following n congrunce equations:

 x  a 1 (mod m1)
 x  a 2 (mod m2)

 x  a n (mod mn)


if,  a i  a j  (mod GCD(mi,mj)) and the solution will be unique modulo L=LCM(m1,m2,   ,mn)

Implementation O(n * log(L)):

// tested on https://open.kattis.com/problems/generalchineseremainder

/**
    A CRT solver which works even when moduli are not pairwise coprime
    1. Add equations using addEquation() method
    2. Call solve() to get {x, N} pair, where x is the unique solution modulo N. (returns -1, -1 if no
         solution)
    Assumptions:
        1. LCM of all mods will fit into long long.
*/
class ChineseRemainderTheorem {
    typedef long long vlong;
    typedef pair<vlong,vlong> pll;
    typedef __int128 overflowtype;
    //typedef long long overflowtype;

    /** CRT Equations stored as pairs of vector. See addEquation()*/
    vector<pll> equations;

public:
    void clear() {
        equations.clear();
    }

    /** Add equation of the form x = r (mod m)*/
    void addEquation( vlong r, vlong m ) {
        equations.push_back({r, m});
    }
    pll solve() {
        if (equations.size() == 0) return {-1,-1}; /// No equations to solve

        vlong a1 = equations[0].first;
        vlong m1 = equations[0].second;
        a1 %= m1;
        /** Initially x = a_0 (mod m_0)*/

        /** Merge the solution with remaining equations */
        for ( int i = 1; i < equations.size(); i++ ) {
            vlong a2 = equations[i].first;
            vlong m2 = equations[i].second;

            EuclidReturn euclidReturn1 = Extended_Euclid(m1, m2);
            vlong g = euclidReturn1.d;
            if ( a1 % g != a2 % g ) return {-1,-1}; /// Conflict in equations

            /** Merge the two equations*/
            vlong p, q;
            EuclidReturn euclidReturn = Extended_Euclid(m1/g, m2/g);
            p = euclidReturn.u;
            q = euclidReturn.v;

            vlong mod = m1 / g * m2;
            vlong x = ( (overflowtype)a1 * (m2/g) % mod *q % mod + (overflowtype)a2 * (m1/g) % mod * p
                 % mod ) % mod;

            /** Merged equation*/
            a1 = x;
            if ( a1 < 0 ) a1 += mod;
            m1 = mod;
        }
        return {a1, m1};
    }
};
```

5 Strings

5.1 Aho-Corasick Algorithm

```
/*
Implementation - Benoit Chabod
------------------------------
Aho Corasick algorithm
------------------------------
*/

struct node
{
        int f;
        map<char, int> g;
        vector<short> out;
        node(int fail = -1): f(fail) {}
};

vector<node> nodes;

void add_str(const string & s, int num)
{
        int cur = 0;
        int n = s.size();
        for(int i = 0; i < n; i++)
        {
                auto it = nodes[cur].g.find(s[i]);
                if(it == nodes[cur].g.end())
                {
                        nodes[cur].g[s[i]] = nodes.size();
                        cur = nodes.size();
                        nodes.push_back(node());
                }
                else
                {
                        cur = it->second;
                }
        }
        nodes[cur].out.push_back(num);
}

void init_fail()
{
        int cur = 0;
        queue<int> q;
        q.push(cur);

    while( !q.empty() )
    {
        cur = q.front();
        map<char, int>::iterator it;
        for(it = nodes[cur].g.begin(); it != nodes[cur].g.end(); it++)
        {
                int child = it->second;
                int pfail = nodes[cur].f;
                char ch = it->first;
                map<char, int>::iterator f;
                while( pfail != -1 && ((f = nodes[pfail].g.find(ch)) == nodes[pfail].g.end()) )
                {
                        pfail = nodes[pfail].f;
                }
                nodes[child].f = (pfail == -1)? 0 : f->second;
                pfail = nodes[child].f;
                nodes[child].out.insert(nodes[child].out.end(),nodes[pfail].out.begin(),nodes[pfail].
                      out.end());
                q.push(child);
        }
        q.pop();
    }
}

// Usage
void usage()
{
        nodes.push_back(node());
        for [each word] add_str(word,i)
                init_fail();
        for [each letter]
        {
                map<char, int>::iterator f;
                while( cur != -1 && ((f = nodes[cur].g.find(letter)) == nodes[cur].g.end()) )
                        cur = nodes[cur].f;
                if( cur == -1 )
                {
                        cur = 0;
                        continue;
                }
                cur = f->second;
                for(auto v : nodes[cur].out)
                {
                        // Word v was found
                }
        }
}
```

5.2 Knuth-Morris-Pratt Algorithm

```
/*
---------------
KMP/Pi function
Note : cin>>(s+1) (the operations in the pi-function start at 1)
---------------
*/
void preKmp()
{
        int k;
        k=kmpNext[1]=0;
        for(int i=2;i<=n;++i)
        {
                while(k && p[k+1]!=p[i]) k=kmpNext[k];
                if(p[k+1]==p[i])
                        k++;
                kmpNext[i]=k;
        }
}

void KMP()
{
        preKmp();
        int k=0;

        for(int i=1;i<=m;++i)
        {
                while(k && p[k+1]!=s[i])
                        k=kmpNext[k];
                if(p[k+1]==s[i])
                        k++;
                if(k==n)
                {
                        // here we have a match
                        k=kmpNext[k];
                }
        }
}
```

5.3 Suffix Array

```
//Suffix Array

struct SuffixArray
{
        const int L;
        string s;
        vector<vector<int> > P;
        vector<pair<pair<int,int>,int> > M;
        SuffixArray(const string &s) : L(s.length()), s(s), P(1, vector<int>(L, 0)), M(L)
        {
                for (int i = 0; i < L; i++)
                        P[0][i] = int(s[i]);
                for (int skip = 1, level = 1; skip < L; skip *= 2, level++)
                {
                        P.push_back(vector<int>(L, 0));
                        for (int i = 0; i < L; i++)
                                M[i] = make_pair(make_pair(P[level-1][i], i + skip < L ? P[level-1][i
                                      + skip] : -1000), i);
                        sort(M.begin(), M.end());
                        for (int i = 0; i < L; i++)
                                P[level][M[i].second] = (i > 0 && M[i].first == M[i-1].first) ? P[
                                      level] [M[i-1].second] : i;
                }
        }

    vector<int> GetSuffixArray()
    {
        return P.back();
        }
```

```
	// returns the length of the longest common prefix of s[i...L-1] and s[j...L-1]
	int LongestCommonPrefix(int i, int j)
	{
		int len = 0;
		if (i == j) return L - i;
		for (int k = P.size() - 1; k >= 0 && i < L && j < L; k--)
		{
			if (P[k][i] == P[k][j])
			{
			    i += 1 << k;
			    j += 1 << k;
			    len += 1 << k;
			}
		}
		return len;
	}
};


int main()
{
	// bobocel is the 0'th suffix
	//  obocel is the 5'th suffix
	//   bocel is the 1'st suffix
	//    ocel is the 6'th suffix
	//     cel is the 2'nd suffix
	//      el is the 3'rd suffix
	//       l is the 4'th suffix
	SuffixArray suffix("bobocel");
	vector<int> v = suffix.GetSuffixArray();
	// Expected output: 0 5 1 6 2 3 4
	//                  2
	for (int i = 0; i < v.size(); i++)
		cout << v[i] << " ";
	cout << endl;
	cout << suffix.LongestCommonPrefix(0, 2) << endl;
}
```

6 Techniques

6.1 Various algorithm techniques

```
Recursion
Divide and conquer
	Finding interesting points in N log N
Greedy algorithm
	Scheduling
	Max contigous subvector sum
	Invariants
	Huffman encoding
Graph theory
	Dynamic graphs (extra book-keeping)
	Breadth first search
	Depth first search
	* Normal trees / DFS trees
	Dijkstra's algoritm
	MST: Prim's algoritm
	Bellman-Ford
	Konig's theorem and vertex cover
	Min-cost max flow
	Lovasz toggle
	Matrix tree theorem
	Maximal matching, general graphs
	Hopcroft-Karp
	Hall's marriage theorem
	Graphical sequences
	Floyd-Warshall
	Eulercykler
	Flow networks
	* Augumenting paths
	* Edmonds-Karp
	Bipartite matching
	Min. path cover
	Topological sorting
	Strongly connected components
	2-SAT
	Cutvertices, cutedges och biconnected components
	Edge coloring
	* Trees
	Vertex coloring
	* Bipartite graphs (=> trees)
	* 3^n (special case of set cover)
	Diameter and centroid
	K'th shortest path
	Shortest cycle
Dynamic programmering
	Knapsack
	Coin change
	Longest common subsequence
	Longest increasing subsequence
	Number of paths in a dag
	Shortest path in a dag
	Dynprog over intervals
	Dynprog over subsets
	Dynprog over probabilities
	Dynprog over trees
	3^n set cover
	Divide and conquer
	Knuth optimization
	Convex hull optimizations
	RMQ (sparse table a.k.a 2^k-jumps)
	Bitonic cycle
	Log partitioning (loop over most restricted)
Combinatorics
	Computation of binomial coefficients
	Pigeon-hole principle
	Inclusion/exclusion
	Catalan number
	Pick's theorem
Number theory
	Integer parts
	Divisibility
	Euklidean algorithm
	Modular arithmetic
	* Modular multiplication
	* Modular inverses
	* Modular exponentiation by squaring
	Chinese remainder theorem
	Fermat's small theorem
	Euler's theorem
	Phi function
	Frobenius number
	Quadratic reciprocity
	Pollard-Rho
	Miller-Rabin
	Hensel lifting
	Vieta root jumping
Game theory
	Combinatorial games
	Game trees
	Mini-max
	Nim
	Games on graphs
	Games on graphs with loops
	Grundy numbers
	Bipartite games without repetition
	General games without repetition
	Alpha-beta pruning
Probability theory
Optimization
	Binary search
	Ternary search
	Unimodality and convex functions
	Binary search on derivative
Numerical methods
	Numeric integration
	Newton's method
	Root-finding with binary/ternary search
	Golden section search
Matrices
	Gaussian elimination
	Exponentiation by squaring
Sorting
	Radix sort
Geometry
	Coordinates and vectors
	* Cross product
	* Scalar product
	Convex hull
	Polygon cut
	Closest pair
	Coordinate-compression
	Quadtrees
	KD-trees
	All segment-segment intersection
Sweeping
	Discretization (convert to events and sweep)
	Angle sweeping
	Line sweeping
	Discrete second derivatives
Strings
	Longest common substring
	Palindrome subsequences
	Knuth-Morris-Pratt
	Tries
	Rolling polynom hashes
	Suffix array
```

```
        Suffix tree
        Aho-Corasick
        Manacher's algorithm
        Letter position lists
Combinatorial search
        Meet in the middle
        Brute-force with pruning
        Best-first (A*)
        Bidirectional search
        Iterative deepening DFS / A*
Data structures
        LCA (2^k-jumps in trees in general)
        Pull/push-technique on trees
        Heavy-light decomposition
        Centroid decomposition
        Lazy propagation
        Self-balancing trees
        Convex hull trick (wcipeg.com/wiki/Convex_hull_trick)
        Monotone queues / monotone stacks / sliding queues
        Sliding queue using 2 stacks
        Persistent segment tree
```

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\displaystyle\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\displaystyle\left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[{n \atop n}\right] = 1$, **17.** $\displaystyle\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\displaystyle\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\displaystyle\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ **26.** $\displaystyle\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \sum_{k=0}^{n} \begin{bmatrix} k \\ m \end{bmatrix} n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix},$ **39.** $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\left\{ {n \atop m} \right\} = \sum_k \binom{n}{k} \left\{ {k+1 \atop m+1} \right\} (-1)^{n-k},$ **41.** $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k},$

42. $\left\{ {m+n+1 \atop m} \right\} = \sum_{k=0}^{m} k \left\{ {n+k \atop k} \right\},$ **43.** $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m} k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix},$

44. $\binom{n}{m} = \sum_k \left\{ {n+1 \atop k+1} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$ **45.** $(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ {k \atop m} \right\} (-1)^{m-k}, \quad \text{for } n \geq m,$

46. $\left\{ {n \atop n-m} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix},$ **47.** $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ {m+k \atop k} \right\},$

48. $\left\{ {n \atop \ell+m} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ {k \atop \ell} \right\} \left\{ {n-k \atop m} \right\} \binom{n}{k},$ **49.** $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.$

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \leq 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \quad \vdots \quad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n \left(\frac{c^m - 1}{c - 1} \right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$$

We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\ &= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ \ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

$$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040,\ 40320,\ 362880,\ \ldots$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x) \Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

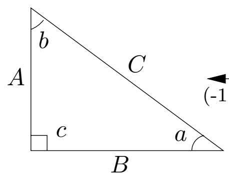
$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$

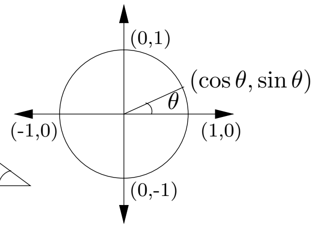
$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$

$$\csc a = C/A, \quad \sec a = C/B,$$

$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$

$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$

$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$

$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$

$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$

$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$

$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$

$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$

$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$

$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$

$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

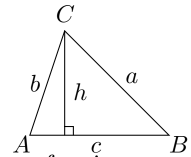
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$

$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

. . . in mathematics you don't understand things, you just get used to them.
– J. von Neumann



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$

$$= \tfrac{1}{2}ab \sin C,$$

$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$

$$s = \tfrac{1}{2}(a + b + c),$$

$$s_a = s - a,$$

$$s_b = s - b,$$

$$s_c = s - c.$$

More identities:

$$\sin \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

$$\cos \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

$$\tan \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{1 - \cos x}{\sin x},$$

$$= \frac{\sin x}{1 + \cos x},$$

$$\cot \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$

$$= \frac{1 + \cos x}{\sin x},$$

$$= \frac{\sin x}{1 - \cos x},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$\sin x = \frac{\sinh ix}{i},$$

$$\cos x = \cosh ix,$$

$$\tan x = \frac{\tanh ix}{i}.$$

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i-1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k,\ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

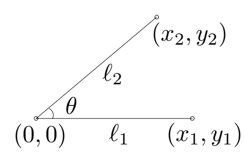
Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\left[\lvert x_1 - x_0\rvert^p + \lvert y_1 - y_0\rvert^p\right]^{1/p},$$
$$\lim_{p \to \infty} \left[\lvert x_1 - x_0\rvert^p + \lvert y_1 - y_0\rvert^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:

$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:

$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Derivatives:

1. $\frac{d(cu)}{dx} = c\frac{du}{dx}$, **2.** $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$, **3.** $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$,

4. $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$, **5.** $\frac{d(u/v)}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\frac{d(e^{cu})}{dx} = ce^{cu}\frac{du}{dx}$,

7. $\frac{d(c^u)}{dx} = (\ln c)c^u\frac{du}{dx}$, **8.** $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$,

9. $\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$, **10.** $\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$,

11. $\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$, **12.** $\frac{d(\cot u)}{dx} = \csc^2 u\frac{du}{dx}$,

13. $\frac{d(\sec u)}{dx} = \tan u \sec u\frac{du}{dx}$, **14.** $\frac{d(\csc u)}{dx} = -\cot u \csc u\frac{du}{dx}$,

15. $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, **16.** $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$,

17. $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$, **18.** $\frac{d(\text{arccot}\, u)}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$,

19. $\frac{d(\text{arcsec}\, u)}{dx} = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$, **20.** $\frac{d(\text{arccsc}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$,

21. $\frac{d(\sinh u)}{dx} = \cosh u\frac{du}{dx}$, **22.** $\frac{d(\cosh u)}{dx} = \sinh u\frac{du}{dx}$,

23. $\frac{d(\tanh u)}{dx} = \text{sech}^2 u\frac{du}{dx}$, **24.** $\frac{d(\coth u)}{dx} = -\text{csch}^2 u\frac{du}{dx}$,

25. $\frac{d(\text{sech}\, u)}{dx} = -\text{sech}\, u \tanh u\frac{du}{dx}$, **26.** $\frac{d(\text{csch}\, u)}{dx} = -\text{csch}\, u \coth u\frac{du}{dx}$,

27. $\frac{d(\text{arcsinh}\, u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$, **28.** $\frac{d(\text{arccosh}\, u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$,

29. $\frac{d(\text{arctanh}\, u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$, **30.** $\frac{d(\text{arccoth}\, u)}{dx} = \frac{1}{u^2-1}\frac{du}{dx}$,

31. $\frac{d(\text{arcsech}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$, **32.** $\frac{d(\text{arccsch}\, u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$.

Integrals:

1. $\int cu\,dx = c\int u\,dx$, **2.** $\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\int \frac{1}{x}dx = \ln x$, **5.** $\int e^x\,dx = e^x$,

6. $\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\int \sin x\,dx = -\cos x$, **9.** $\int \cos x\,dx = \sin x$,

10. $\int \tan x\,dx = -\ln|\cos x|$, **11.** $\int \cot x\,dx = \ln|\cos x|$,

12. $\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\int \arcsin\frac{x}{a}dx = \arcsin\frac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0$,

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2-x^2}, \quad a>0,$

16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2}\ln(a^2+x^2), \quad a>0,$

17. $\int \sin^2(ax)dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax)dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\, dx = \tan x,$

20. $\int \csc^2 x\, dx = -\cot x,$

21. $\int \sin^n x\, dx = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2} x\, dx,$

22. $\int \cos^n x\, dx = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2} x\, dx,$

23. $\int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \quad n \neq 1,$

24. $\int \cot^n x\, dx = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \quad n \neq 1,$

25. $\int \sec^n x\, dx = \dfrac{\tan x \sec^{n-1} x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\, dx, \quad n \neq 1,$

26. $\int \csc^n x\, dx = -\dfrac{\cot x \csc^{n-1} x}{n-1} + \dfrac{n-2}{n-1}\int \csc^{n-2} x\, dx, \quad n \neq 1,$

27. $\int \sinh x\, dx = \cosh x,$

28. $\int \cosh x\, dx = \sinh x,$

29. $\int \tanh x\, dx = \ln|\cosh x|,$

30. $\int \coth x\, dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\, dx = \arctan \sinh x,$

32. $\int \operatorname{csch} x\, dx = \ln\left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x\, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2+a^2}, \quad a>0,$

37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2}\ln|a^2-x^2|,$

38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a>0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a>0, \end{cases}$

39. $\int \dfrac{dx}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{a^2+x^2}\right), \quad a>0,$

40. $\int \dfrac{dx}{a^2+x^2} = \frac{1}{a}\arctan \frac{x}{a}, \quad a>0,$

41. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin \frac{x}{a}, \quad a>0,$

42. $\int (a^2-x^2)^{3/2} dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin \frac{x}{a}, \quad a>0,$

43. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a}, \quad a>0,$

44. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right|,$

45. $\int \dfrac{dx}{(a^2-x^2)^{3/2}} = \dfrac{x}{a^2\sqrt{a^2-x^2}},$

46. $\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{a^2 \pm x^2}\right|,$

47. $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right|, \quad a>0,$

48. $\int \dfrac{dx}{ax^2+bx} = \dfrac{1}{a}\ln\left|\dfrac{x}{a+bx}\right|,$

49. $\int x\sqrt{a+bx}\, dx = \dfrac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$

50. $\int \dfrac{\sqrt{a+bx}}{x}\, dx = 2\sqrt{a+bx} + a\int \dfrac{1}{x\sqrt{a+bx}} dx,$

51. $\int \dfrac{x}{\sqrt{a+bx}}\, dx = \dfrac{1}{\sqrt{2}}\ln\left|\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|, \quad a>0,$

52. $\int \dfrac{\sqrt{a^2-x^2}}{x}\, dx = \sqrt{a^2-x^2} - a\ln\left|\dfrac{a+\sqrt{a^2-x^2}}{x}\right|,$

53. $\int x\sqrt{a^2-x^2}\, dx = -\frac{1}{3}(a^2-x^2)^{3/2},$

54. $\int x^2\sqrt{a^2-x^2}\, dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin \frac{x}{a}, \quad a>0,$

55. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\dfrac{a+\sqrt{a^2-x^2}}{x}\right|,$

56. $\int \dfrac{x\, dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2},$

57. $\int \dfrac{x^2\, dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin \frac{x}{a,}, \quad a>0,$

58. $\int \dfrac{\sqrt{a^2+x^2}}{x}\, dx = \sqrt{a^2+x^2} - a\ln\left|\dfrac{a+\sqrt{a^2+x^2}}{x}\right|,$

59. $\int \dfrac{\sqrt{x^2-a^2}}{x}\, dx = \sqrt{x^2-a^2} - a\arccos \frac{a}{|x|}, \quad a>0,$

60. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

61. $\int \dfrac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\dfrac{x}{a+\sqrt{a^2+x^2}}\right|,$

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \tfrac{1}{a}\arccos\tfrac{a}{|x|}, \quad a>0,$ **63.** $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$ **65.** $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\ \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\ \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$$

68. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$$

71. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$

72. $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax)+\tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax)-\tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\displaystyle\int x^n e^{ax}\,dx = \frac{x^ne^{ax}}{a}-\tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1}-\frac{1}{(n+1)^2}\right),$

76. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m-\frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$$\begin{array}{lcccc}
x^1 = & x^{\underline{1}} & = & x^{\overline{1}} \\
x^2 = & x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{2}}-x^{\overline{1}} \\
x^3 = & x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}} \\
x^4 = & x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}} \\
x^5 = & x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} & = & x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{array}$$

$$\begin{array}{lclc}
x^{\overline{1}} = & x^1 & x^{\underline{1}} = & x^1 \\
x^{\overline{2}} = & x^2+x^1 & x^{\underline{2}} = & x^2-x^1 \\
x^{\overline{3}} = & x^3+3x^2+2x^1 & x^{\underline{3}} = & x^3-3x^2+2x^1 \\
x^{\overline{4}} = & x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} = & x^4-6x^3+11x^2-6x^1 \\
x^{\overline{5}} = & x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} = & x^5-10x^4+35x^3-50x^2+24x^1
\end{array}$$

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\, f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u+\Delta v,$

$\Delta(uv) = u\Delta v+\mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$

$\Delta(H_x) = x^{\underline{-1}}, \qquad \Delta(2^x) = 2^x,$

$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x+\sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv-\sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$
$= 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$
$= 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^{n}\left\{ n \atop k \right\}x^{\underline{k}} = \sum_{k=1}^{n}\left\{ n \atop k \right\}(-1)^{n-k}x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^{n}\left[n \atop k \right](-1)^{n-k}x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^{n}\left[n \atop k \right]x^k.$$

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} ix^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i}\right]x^i,$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n}\right]\frac{n!x^i}{i!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^i i!^2}{(i+1)(2i+1)!}x^{2i}.$$

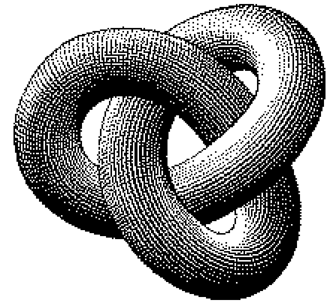
$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{{i \atop n}\right\}x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\left\{{i \atop n}\right\}\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1\\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2\\ \vdots \qquad \vdots \qquad &\vdots\\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\ne 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Fibonacci Numbers

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \frac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1}\\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1\\ 1 & 1 \end{pmatrix}^n.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)