Semi-Riemannian Geometry

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Question 9: Lorentzian vector space

Let (V, g) be a linear space equipped with a Lorentzian inner-product.

Recall that a vector $v \in V$ is called, according to its causal type

- space-like if $g(v, v) > 0$ or $v = 0$,
- time-like if $g(v, v) < 0$,
- light-like if $g(v, v) = 0$.

The causal type of a subspace W of V is

- space-like if $g|_W$ is positive definite,
- time-like if $g|_W$ is non-degenerate of index 1,
- light-like if $g|_W$ is degenerate.

Previous of linear algebra

Proposition (1)

Let W be a subspace of V, then $dimW + dimW^{\perp} = dimV$.

Proof.

Let $\{e_1, \ldots, e_k\}$ be a orthonormal basis for W, and complete this basis to $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ for V.

A vector $v \in V$ can be written as $v = \sum_{i=1}^{n} v_i e_i$.

We observe that $v \in W^{\perp} \Leftrightarrow g(v, e_j) = 0$ for $j = 1, \ldots, k$. That is, writing $g_{ij} = g(e_i, e_j)$,

$$0 = g(v, e_j) = \sum_{i=1}^{n} v_i (e_i, e_j) = \sum_{i=1}^{n} g_{ij} v_i.$$

As the matrix (g_{ij}) of the metric is invertible, the system has exactly $n - k$ free variables, so the solution subspace of the system has dimension $n - k$. Thus, $dimW^{\perp} = n - k$. $\square$

Corollary (2)

Let W be a subspace of V. Then $(W^{\perp})^{\perp} = W$.

Proof.

By definition, if $u \in W$ such that $g(u, v) = 0, \forall v \in W^{\perp}$, then $u \in (W^{\perp})^{\perp}$. So, $W \subset (W^{\perp})^{\perp}$.

From Proposition (1), we have

$$dimW + dimW^{\perp} = dimV$$
$$dimW^{\perp} + dim(W^{\perp})^{\perp} = dimV.$$

Then $dimW = dim(W^{\perp})^{\perp}$. Therefore $W = (W^{\perp})^{\perp}$. $\square$

Previous of linear algebra

Proposition (3)

Let V be a inner product vector space, W a subspace of V. Then $g|_W$ is non-degenerate if and only if $V = W \oplus W^{\perp}$.

Proof.

From linear algebra, we know that

$$dim(W + W^{\perp}) + dim(W \cap W^{\perp}) = dimW + dimW^{\perp}.$$

From Proposition (1), we have $V = W \oplus W^{\perp}$ if and only if $dim(W \cap W^{\perp}) = 0$.

Now, we observe that $W \cap W^{\perp} = \{0\} \Leftrightarrow v \in W$ such that $g(v, w) = 0, \forall w \in W$ implies $v = 0 \Leftrightarrow g|_W$ is non-degenerate. $\square$

Previous of linear algebra

From Proposition (3) and Corollary (2), we have

Corollary (4)

Let W be a subspace of V. Then $g|_W$ is non-degenerate if and only if $g|_{W^\perp}$ is non-degenerate.

Exercise 1: (1) Show that

If $z \in V$ is time-like then $z^{\perp}$ is space-like and $V = \mathbb{R}z \oplus z^{\perp}$.

Proof.

As z is time-like, each vector of $\mathbb{R}z$ is of the form cz, for some $c \in \mathbb{R}$, so $g(cz, cz) = c^2 g(z, z)$. Thus, $\mathbb{R}z$ is time-like.

From Corollary (4), $g|_{z^{\perp}}$ is non-degenerate and it follows by Proposition (3) and Corollary (2) that $V = \mathbb{R}z \oplus z^{\perp}$.

Thus, $index(V) = index(\mathbb{R}z) + index(z^{\perp})$ which implies that $index(z^{\perp}) = 0$. Therefore $z^{\perp}$ is space-like. $\square$

Exercise 1: (2) Deduce that

(a) W is time-like if and only if $W^{\perp}$ is space-like.

Proof.

(a) Assume that $g|_W$ is non degenerate, then $g|_{W^{\perp}}$ is non degenerate. Then, $V = W \oplus W^{\perp}$ by Proposition (3), and thus $index(g) = index(g|_W) + index(g|_{W^{\perp}})$.

Now, if W is time-like, then $g|_W$ has index 1 and so $index(g|_{W^{\perp})} = 0$. Hence $W^{\perp}$ is space-like.

Conversely, if $W^{\perp}$ is space-like, then $g|_{W^{\perp}}$ is positive definite. So $index(g|_W) = 1$ and W is time-like. $\square$

Exercise 1: (2) Deduce that

(b) W is space-like if and only if $W^\perp$ is time-like.

Proof.
(b) It follows directly from (a) and the fact that $W = (W^\perp)^\perp$. $\square$

Exercise 1: (2) Deduce that

(c) W is light-like if and only if $W^{\perp}$ is light-like.

Proof.
(c) It follows from the above statements.

Suppose W light-like and, by contradiction, that

- $W^{\perp}$ is space-like, then W is time-like by (a). $\Rightarrow\!\!\times\!\!\Leftarrow$
- $W^{\perp}$ is time-like, then W is space-like by (b). $\Rightarrow\!\!\times\!\!\Leftarrow$.

$\square$

Exercise 1: (3) Show that

(a) W is time-like $\Leftrightarrow$ (b) W contains two linearly independent null vectors $\Leftrightarrow$ (c) W contains a time-like vector.

Proof.

(a) $\Rightarrow$ (b) As W is time-like, given an O.N.B $\{e_1, \ldots, e_k\}$ for W, we have that e_1 is time-like and e_k is space-like. It follows that

$$g(e_1 + e_k, e_1 + e_k) = -1 + 0 + 1 = 0$$
$$g(e_1 - e_k, e_1 - e_k) = 0,$$

then $e_1 + e_k$ and $e_1 - e_k$ are light-like vectors.

Now, for $a, b \in \mathbb{R}$, suppose that

$$0 = a(e_1 + e_k) + b(e_1 - e_k) = (a + b)e_1 + (a - b)e_k.$$

Since e_1, e_k are linearly independent, hence $a = b = 0$.

Then $e_1 + e_k$ and $e_1 - e_k$ are linearly independent null vectors. $\square$

Exercise 1: (3) Show that

(a) W is time-like $\Leftrightarrow$ (b) W contains two linearly independent null vectors $\Leftrightarrow$ (c) W contains a time-like vector.

Proof.
(b) $\Rightarrow$ (c) Let v, w be two light-like linearly independent vectors on W. Then

$$g(v+w, v+w) = 2g(v,w)$$
$$g(v-w, v-w) = -2g(v,w)$$

† We claim that $g(v,w) \neq 0$ if v, w are two light-like linearly independent vectors.

Therefore, either $v+w$ or $v-w$ is time-like. $\square$

†proof of the claim

If v, w are two light-like linearly independent vectors, then $g(v, w) \neq 0$.

Proof.

Suppose, on the contrary,that $g(v, w) = 0$ with v, w two light-like l.i. vectors on W.

Let u be a time-like vector in W such that $g(u, v) \neq 0$. Otherwise, $u \in W^{\perp}$ and it would imply that u is space-like.

Now, write $a = \frac{-g(u,w)}{g(u,v)} \in \mathbb{R}$, so that $g(u, w + av) = 0$, and then $w + av$ is space-like.

As $g(w + av, w + av) = g(w, w) + 2ag(w, v) + a^2 g(v, v) = 0$, then $w + av = 0$ which contradicts that w and v are l.i. $\Rightarrow\!\Leftarrow$ $\square$

Exercise 1: (3) Show that

(a) W is time-like $\Leftrightarrow$ (b) W contains two linearly independent null vectors $\Leftrightarrow$ (c) W contains a time-like vector.

Proof.

(c) $\Rightarrow$ (a) By assumption, W contains a time-like vector v, then $\mathbb{R}v$ is time-like and, we know that $z^{\perp}$ is space-like, with g inner product for all $z^{\perp}$ in $W^{\perp}$.

Thus $W^{\perp}$ is space-like, then W is time-like by (2)(a). $\square$

Exercise 1: (4) Show that

(a) W is light-like $\Leftrightarrow$ (b) W contains a null vector but no time-like vector $\Leftrightarrow$ (c) $W \cap \Lambda = L \setminus \{0\}$ where L is a one-dimensional subspace and Λ is the light-cone, that is the set of null vectors.

Proof.

(a) $\Rightarrow$ (b) We assume W light-like, so W contains a null vector but not a time-like vector by (3). Otherwise, W would be time-like. $\square$

Exercise 1: (4) Show that

(a) W is light-like $\Leftrightarrow$ (b) W contains a null vector but no time-like vector $\Leftrightarrow$ (c) $W \cap \Lambda = L \setminus \{0\}$ where L is a one-dimensional subspace and Λ is the light-cone, that is the set of null vectors.

Proof.

(b) $\Rightarrow$ (c) It is clear that $W \cap \Lambda \neq \emptyset$.

Moreover, $W \cap \Lambda$ cannot contain two l.i. null vectors because (3) would imply that W contains a time-like vector. $\square$

Exercise 1: (4) Show that

(a) W is light-like $\Leftrightarrow$ (b) W contains a null vector but no time-like vector $\Leftrightarrow$ (c) $W \cap \Lambda = L \setminus \{0\}$ where L is a one-dimensional subspace and Λ is the light-cone, that is the set of null vectors.

Proof.
(c) $\Rightarrow$ (a) W cannot be space-like since $w \in \Lambda$ implies $g(w, w) = 0$ with $w \neq 0$, while $w \in W$ space-like implies $w = 0$.

Furthermore, W cannot be time-like. Otherwise, W would contain two l.i. null vectors by (3). $\square$

Exercise 2

Let u be a time-like vector. Let

$$C(u) = \{v \in V,\ \text{time-like and } g(u,v) < 0\},$$

the time-cone containing u. Furthermore

$$C(-u) = -C(u) = \{v \in V,\ \text{time-like and } g(u,v) > 0\}.$$

Observe that $u^\perp$ is space-like.

Exercise 2: (1) Show that

The time-like vectors w and v are in the same time-cone if and only if $g(v, w) < 0$.

Proof.

We will show that if $v \in C(u)$ and w is time-like, then $w \in C(u)$ if and only if $g(v, w) < 0$.

Since $C\left(\frac{u}{|u|}\right) = C(u)$, we can assume u a time-like unit vector.

We write $v = au + \vec{x}$ and $w = bu + \vec{y}$, with $\vec{x}, \vec{y} \in u^{\perp}$ and $a, b \in \mathbb{R}$.

We observe that

- $0 > g(v, v) = a^2 g(u, u) + 2ag(u, \vec{x}) + g(\vec{x}, \vec{x}) = -a^2 + g(\vec{x}, \vec{x})$,
- $0 > g(w, w) = -b^2 + g(\vec{y}, \vec{y})$.

Then, $(ab)^2 > g(\vec{x}, \vec{x}) \cdot g(\vec{y}, \vec{y}) = |\vec{x}|^2 |\vec{y}|^2 \geq 0$.

Exercise 2: (1) Show that

The time-like vectors w and v are in the same time-cone if and only if $g(v, w) < 0$.

Proof.

As $\vec{x}, \vec{y}$ are space-like, by Cauchy-Schwarz inequality we have that

$$|g(\vec{x}, \vec{y})| \leq |\vec{x}||\vec{y}| < |ab|.$$

Now,

$$g(v, w) = -ab + g(\vec{x}, \vec{y}).$$

It follows from above that $\text{sign}(g(v, w)) = -\text{sign}(ab)$.

Since $v \in C(u)$, then $0 > g(u, v) = -a$, so $a > 0$ and thus

$$\text{sign}(g(v, w)) = -\text{sign}(b).$$

We deduce then that $w \in C(u)$ if and only if $b > 0$, because $0 > g(u, w) = -b$. Therefore $w \in C(u)$ if and only if $g(v, w) < 0$. $\square$

Exercise 2: (2) Show that

The time-cones $C(u)$ are convex sets.

Proof.

Let v, w be two time-like vectors in the same time-cone $C(u)$, so $g(v, w) < 0$. And $a, b \geq 0$ *(not both zero)*.

Then,

- $g(v, av + bw) = ag(v, v) + bg(v, w) < 0$,
- $g(av+bw, av+bw) = a^2 g(v, v) + 2ab g(v, w) + b^2 g(w, w) < 0$.

From the last inequality, we have that $av + bw$ is time-like, and from the first one, that $av + bw$ and v are in the same time-cone. $\square$

Exercise 2: (3) Show that

If v and w are time-like then $|g(v,w)| \geq |v||w|$.

Proof.

Suppose v, w time-like. Write $w = av + \vec{w}$, with $\vec{w} \in v^{\perp}$, $a \in \mathbb{R}$. Thus, $g(w,w) = a^2 g(v,v) + g(\vec{w},\vec{w})$.

Now $g(v,w) = ag(v,v)$, so $g(v,w)^2 = a^2 g(v,v)^2$. Replacing the above equality, and since $g(w,w) < 0$ and $g(\vec{w},\vec{w}) \geq 0$, we have

$$\begin{aligned} g(v,w)^2 &= (g(w,w) - g(\vec{w},\vec{w}))\, g(v,v) \\ &= g(w,w)g(v,v) - g(\vec{w},\vec{w})g(v,v) \\ &\geq g(w,w)g(v,v). \end{aligned}$$

Therefore $|g(v,w)|^2 \geq |w|^2|v|^2$.

Moreover, the equality holds if and only if $g(\vec{w},\vec{w}) = 0$. Since $\vec{w}$ is space-like, the equality holds if and only if $\vec{w} = 0$, what is, iff $w = av$. *(v and w are proportional or colinear)* $\square$

Exercise 2: (4) Show that

If the time-like vectors v and w are in the same $C(u)$ then there exists a real number $\phi \geq 0$ *(the hyperbolic angle)* such that $g(v, w) = -|v||w| \cosh \phi$.

Proof.

From reversed Cauchy-Schwarz ineq. (3), we have that $\frac{|g(v,w)|}{|v||w|} \geq 1$.

Since the hyperbolic cosine restriction to $[0, \infty)$ is bijective on $[1, \infty)$, there exists an unique real number $\phi \in [0, \infty)$ which satisfy $\cosh \phi = \frac{|g(v,w)|}{|v||w|}$.

As w and v are in the same time-cone, then $g(v, w) < 0$ and thus $|g(v, w)| = -g(v, w)$.

Therefore $g(v, w) = -|v||w| \cosh \phi$. $\square$

Exercise 2: (5) Show that

Deduce that if v and w are in the same time-cone then $|v| + |w| \leq |v + w|$.

Proof.

Since v and w are in the same time-cone $C(u)$, then $v + w \in C(u)$ by (2), and $g(v + w, v + w) < 0$ by (1).

Using reverse Cauchy-Schwarz inequality (3), we have that $|v||w| \leq |g(v, w)| = -g(v, w)$, because $g(v, w) < 0$ by (1). Thus,

$$\begin{aligned}(|v| + |w|)^2 &= |v|^2 + 2|v||w| + |w|^2 \\ &\leq -g(v, v) - 2g(v, w) - g(w, w) \\ &= -g(v + w, v + w) \\ &= |v + w|^2.\end{aligned}$$

Equality holds if and only if equality holds in the reverse Cauchy-Schwarz inequality. $\square$

Exercise 3:

We call a Lorentzian manifold (M, g) time-orientable if there exists a smooth *(global)* vector field V such that V_p is in the time-cone of T_pM.

Show that if there exists a nowhere zero vector field on M if and only if there exists a Lorentzian metric g such that (M, g) is time-orientable.

(Indication: If U is a unit vector field on a Riemannian manifold (M, h), then construct a Lorentzian metric on M, such that U is time-like.)

Show that if there exists a nowhere zero vector field on M if and only if there exists a Lorentzian metric g such that (M, g) is time-orientable.

Proof.

$\Rightarrow$ Let X be a nowhere vanishing vector field on M, so $U = \frac{X}{|X|}$ is a unit vector field on M.

Let h be a Riemannian metric on M such that $h(U, U) = 1$.

We define a new metric g by

$$g(X, Y) = h(X, Y) - 2h(U, X) \otimes h(U, Y), \qquad X, Y \in \mathfrak{X}(M)$$

Given $p \in M$, let $E_2|_p, \dots, E_n|_p \in T_pM$ be such that $\{E_1|_p, \dots, E_n|_p\}$ is an O.N.B for (T_pM, h_p), where $E_1|_p = U_p$.

Show that if there exists a nowhere zero vector field on M if and only if there exists a Lorentzian metric g such that (M, g) is time-orientable.

Proof.

$\Rightarrow$

Then $\{E_i|_p\}$ is an O.N.B for (T_pM, g_p), since

$$g(E_i, E_j) = h(E_i, E_j) = \delta_{i,j}, \quad \text{for } i, j = 2, \ldots, n.$$

$$g(U, E_j) = h(U, E_j) = 0, \quad \text{for } i, j = 2, \ldots, n.$$

$$g(U, U) = h(U, U) - 2h(U, U)^2 = -1.$$

So, g_p is a metric of index 1. Thus (M, g) is a Lorentzian manifold such that $g(U, U) < 0$, that is, U is time-like.

Then, assigning to each $p \in M$ the time-cone containing U_p gives a time-orientation.

Show that if there exists a nowhere zero vector field on M if and only if there exists a Lorentzian metric g such that (M, g) is time-orientable.

Proof.

$\Leftarrow$ We assume that M admits a time-orientable Lorentzian metric g.

Let $\{\Omega_\alpha, \varphi_\alpha\}_{\alpha \in I}$ be an atlas of M such that on each Ω_α there is a time-like vector field X_α, whose value at each $p \in \Omega_\alpha$ is in the time-cone of $T_p M$.

Let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity associated to the atlas such that $\text{supp}(\rho_\alpha) \subset \Omega_\alpha$, $0 \leq \rho_\alpha \leq 1$, $(\rho_\alpha)_{\alpha \in I}$ is locally finite, and $\sum_{\alpha \in I} \rho_\alpha(p) = 1$, for all $p \in M$.

Show that if there exists a nowhere zero vector field on M if and only if there exists a Lorentzian metric g such that (M, g) is time-orientable.

Proof.

$\Leftarrow$

Define $X = \sum_{\alpha \in I} \rho_\alpha X_\alpha$. Fixing $p \in M$, we have

$$X_p = \sum_{i=1}^{n} a_i X_i|_p,$$

where $a_i \in \mathbb{R}$, and $X_i|_p \in T_pM$ are time-like vectors.

As time-cones are convex sets, we have that X_p is a time-like vector in the same time-cone.

In particular, X is thus a nowhere vanishing time-like vector field. $\square$

Bibliography

O'NEILL, Barrett. *Semi-Riemannian geometry with applications to relativity*. Academic Press, 1983.