Solutions of massless Vlasov equation on a fixed FLRW background with $\mathbb{R}^3$ spatial topology

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Outline of the talk

General Relativity

General relativity is a geometric theory of gravitation whose main object of study are the Lorentzian manifolds $(\mathcal{M}^{1+3}, g)$ satisfying the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the energy momentum tensor of matter and κ is the Einstein gravitational constant.

General Relativity

The energy momentum tensor $T_{\mu\nu}$ takes the form

$$T^{\mu\nu}(t,x) = \int_{\mathcal{P}} f(t,x,p) p^\mu p^\nu \frac{\sqrt{|\det g|}}{-p^0} dp^1 dp^2 dp^3, \qquad (1)$$

where indices are raised and lowered with respect to metric g (so that, for example, $p_0 = g_{0\mu} p^\mu$).

† Here t is also denoted x^0.
Greek indices, such as μ, ν. range over $0, 1, 2, 3$.
Latin indices, such as i, j, k range over $1, 2, 3$.

General Relativity

Several metrics exist based on the exact solutions of the Einstein field equations

1. The simplest solution in Minkowski space-time $(\mathbb{R}^4, g)$ is

$$g = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

2. The most general metric for an expanding, homogeneous, and isotropic universe is the Friedman-Lemaître-Robertson-Walker metric. The FLRW spacetime is described on

$$\mathcal{M} = I \times \Sigma, \qquad g = -dt^2 + a(t)^2 g_\Sigma,$$

where $I \subset \mathbb{R}$, (Σ, g_Σ) is a constant curvature manifold and $a : I \to (0, \infty)$ is an appropriate *scale factor*.

Collisionless many-particle systems in GR

We introduce a density distribution function $f\colon \mathcal{P} \to [0,\infty)$ defined on [1]

$$\mathcal{P} = \left\{(t,x,p) \in T\mathcal{M} : g_{(t,x)}(p,p) = -m^2\right\} \subset T\mathcal{M},$$

where p is future-directed. $\mathcal{P}$ is a submanifold called the **mass-shell**.

The two cases that concern us

1. massive particles: $m > 0$ (WLOG $m = 1$),
2. mass-less particles: $m = 0$.

[1] f represents a collection of particles at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^3$ with momentum $p \in \mathbb{R}^3$.

The Vlasov equation

We introduce the Vlasov equation by

$$p^{\mu}\partial_{x^{\mu}} f - \Gamma^{i}_{\mu\nu} p^{\mu} p^{\nu} \partial_{p^{i}} f = 0. \tag{2}$$

In particular, this equation on Minkowski space in cartesian coordinates becomes

$$p^{0}\partial_{t} f + p^{i}\partial_{x^{i}} f = 0.$$

Free transport equations

1. The classical Vlasov equation *(non-relativistic)*:

$$f(t, x, p) = f^0(x - tp, p), \qquad \text{that solves} \quad \partial_t f + p^i \partial_{x^i} f = 0.$$

2. The relativistic Vlasov equation:

$$f(t, x, p) = f^0(x - t\frac{p}{p^0}, p), \qquad \text{that solves} \quad p^0 \partial_t f + p^i \partial_{x^i} f = 0.$$

† Where, abusing notation, we write p for p^i

Classical estimates

For the classical case, we have

$$\int_{\mathbb{R}^3} f(t,x,p)dp = \int_{\mathbb{R}^3} f^0(x-pt,p)dp$$
$$\leq \sup_{w\in\mathbb{R}^3} \int_{\mathbb{R}^3} f^0(x-pt,w)dp.$$

For $t>0$, we can apply the change of variable $y = x - pt$ and the appropriate scaling to get the decay estimates

$$\left|\int_{\mathbb{R}^3} f(t,x,p)dp\right| \leq \frac{1}{t^3} \sup_{w\in\mathbb{R}^3} \int_{\mathbb{R}^3} \left|f^0(y,w)\right| dy.$$

massive case: $m = 1$

For the relativistic case,

$$\int_{\mathbb{R}^3} f(t,x,p)dp \leq \int_{\mathbb{R}^3} \sup_{w\in\mathbb{R}^3} f^0\left(x - \frac{p}{p^0}t, w\right)dp.$$

Applying the change of variable $y = x - \frac{p}{p^0}t$, assuming that f has compact support[2], so that, we can bound the Jacobian by a constant C_v to get

$$\int_{\mathbb{R}^3} |f(t,x,p)|dp \leq \frac{C_v}{t^3} \int_{\mathbb{R}^3} \sup_{w\in\mathbb{R}^3} |f^0(y,w)|dy,$$

[2] Considering $\frac{p}{p^0} \leq \frac{R}{\sqrt{1+R^2}} < 1$, we have that if $|p| \leq R$ and $|x| \geq R + \frac{R}{\sqrt{1+R^2}}$ then $|x - t\frac{p}{p^0}| \geq R$.

$$V = \sup\left\{ p \in \mathbb{R}^3 : \left|\frac{p}{p^0}\right| < 1;\ \exists x \in \mathbb{R}^3 : |x - t\frac{p}{p^0}| \geq R > 0 \text{ for all } t > 0, \quad \text{such that} \quad f(t,x,p) \neq 0 \right\}.$$

massless case: $m = 0$

Using polar coordinates $r = |p|, p = r\omega$ we may write

$$\int_{\mathbb{R}^3} f(t,x,p)dp = \int_{\mathbb{R}^3} f^0(x - \frac{p}{|p|}t, p)dp$$
$$= \int_0^\infty \int_{\mathbb{S}^2} f^0(x - t\omega, r\omega)d\omega dr,$$

then, on the sphere *(rather than the whole space)* applying the change of variables $\omega \to t\omega = \gamma$, where $|J| = t^2$. Then, $d\gamma = t^2 d\omega$, so that

$$\int_{\mathbb{R}^3} f(t,x,p)dp = \frac{1}{t^2}\int_0^\infty \int_{\mathbb{S}^2} f^0(x - \gamma, rt^{-1}\gamma)d\gamma dr$$
$$\leq \frac{\tilde{C}_v}{t^2} \sup_{x,v\in\mathbb{R}^3} \int_0^\infty \int_{\mathbb{S}^2} f^0(x - \gamma, v)d\gamma dr.$$

Spatially homogeneous FLRW solutions

For any smooth sufficiently decaying function $\mu : [0,\infty) \to [0,\infty)$, $\mu \neq 0$, the metric $g_\circ$ and $f_\circ$ defined by [3]

$$g_\circ = -dt^2 + a(t)^2 \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right),$$

$$f_\circ(t,x,p) = \mu\left(a(t)^4 |p|^2\right),$$

where

$$a(t) = t^{\frac{1}{2}} \left(\frac{4\varrho}{3}\right)^{\frac{1}{4}}, \qquad \varrho = \int |p| \mu(|p|^2) dp,$$

define a solution of (1)-(2) on $\mathcal{M}_\circ = (0,\infty) \times \mathbb{R}^3$.

[3] Constant curvature manifold $(\Sigma, g_\Sigma) = (\mathbb{R}^3, g_{eucl})$

The FLRW metrics in double null gauge

A double null gauge consists of functions $u, v : \mathcal{Q} \to \mathbb{R}$ that foliate $\mathcal{Q}$ in outgoing (ingoing) null lines, where $\mathcal{Q} := \mathcal{M}/_{SO(3)}$ to introduce spherical symmetry assumption.

It can be complemented with coordinates (θ^1, θ^2) on $\mathbb{S}^2$ to local coordinates $(u, v, \theta^1, \theta^2)$ for $\mathcal{M}$.

The metric g can be written in double null form

$$g = -\Omega^2 dudv + R^2\gamma,$$

where γ is the unit round metric on $\mathbb{S}^2$, Ω is a function on $\mathcal{Q}$ and $R : \mathcal{Q} \to \mathbb{R}$ is the area radius function.

The FLRW metrics in double null gauge

For $t \in (0, \infty)$, define double null coordinates

$$u = t^{\frac{1}{2}} - \frac{r}{2}, \qquad v = t^{\frac{1}{2}} + \frac{r}{2},$$

where

$$r = \left(\frac{4\varrho}{3}\right)^{\frac{1}{4}} \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)^{\frac{1}{2}}.$$

Since $t \geq 0$ and $r \geq 0$ in $\mathcal{M}_\circ$, we have

$$v \geq 0, \qquad v \geq u, \qquad v \geq -u.$$

The FLRW metrics in double null gauge

The FLRW metric $g_\circ$ in the above double null gauge takes the form

$$g_\circ = -4t\,du\,dv + tr^2\gamma, \qquad t = \frac{1}{4}(v+u)^2, \qquad r = v - u, \qquad (3)$$

defined on the quotient manifold

$$\mathcal{Q}_\circ = \{(u,v) \in \mathbb{R}^2 : v \geq 0, v \geq u, v \geq -u\}.$$

Thus, $g_\circ$ can be written as

$$g_\circ = -\Omega_\circ^2\,du\,dv + R_\circ^2\gamma, \qquad \text{where } \Omega_\circ^2 = 4t, \qquad R_\circ = t^{\frac{1}{2}} r,$$

with $\sqrt{-\det g_\circ} = 2t^2 r^2 \sqrt{\det\gamma}$.

The FLRW metrics in double null gauge

A given spherically symmetric double null gauge $(u, v, \theta^1, \theta^2)$ for $(\mathcal{M}, g)$, induces a coordinate system[4] $(u, v, \theta^1, \theta^2, p^v, p^1, p^2)$ on the mass shell $\mathcal{P}$.

Moreover, in a given double null gauge, f can be written as

$$f(x, p) = f(u, v, p^v, L),$$

where A and B range over 1 and 2, $L = (R^4 \gamma_{AB} p^A p^B)^{\frac{1}{2}}$ is the angular momentum on $\mathcal{P}$.

[4] p^μ is defined by the mass shell relation $g_{\mu\nu} p^\mu p^\nu = 0$

The FLRW metrics in double null gauge

Thus, the components of the energy-momentum tensor T on $\mathcal{M}$ takes the form

$$T_{uu}(u,v) = \frac{\Omega^4}{R^2}\frac{\pi}{2}\int_0^\infty\int_0^\infty f(u,v,p^v,L)p^v L dL dp^v, \quad (4)$$

$$T_{uv}(u,v) = \frac{\Omega^4}{R^2}\frac{\pi}{2}\int_0^\infty\int_0^\infty f(u,v,p^v,L)p^u L dL dp^v, \quad (5)$$

$$T_{vv}(u,v) = \frac{\Omega^4}{R^2}\frac{\pi}{2}\int_0^\infty\int_0^\infty f(u,v,p^v,L)\frac{(p^u)^2}{p^v} L dL dp^v, \quad (6)$$

with $p^u(u,v,p^v,L)$ defined by

$$p^u = \frac{R^2\gamma_{AB}p^A p^B}{\Omega^2 p^v} = \frac{L^2}{\Omega^2 R^2 p^v}.$$

The massless Vlasov equation on an FLRW background

Theorem

Let f be a solution of the massless Vlasov equation (2) on $(\mathcal{M}_\circ, g_\circ)$, where $g_\circ$ is the FLRW metric (3), such that $f_1 = f|_{t=1}$ is compactly supported. The components of the energy-momentum tensor satisfy

$$T_{uu} \lesssim \frac{\|f_1\|_{L^\infty}}{t^2}, \qquad T_{uv} \lesssim \frac{\|f_1\|_{L^\infty}}{t^3}, \qquad T_{vv} \lesssim \frac{\|f_1\|_{L^\infty}}{t^4}$$

for $t \geq 1$.

The massless Vlasov equation on an FLRW background

Sketch of proof:

- Consider $cr \leq t^{\frac{1}{2}} \leq Cr$ in supp f,
- then, from the conservation of angular momentum, the mass relation and properties of null geodesics in FLRW, it is possible to have the following bounds

$$(\gamma_{ab}p^a p^b)^{\frac{1}{2}} \leq \frac{L_0}{t^2}, \qquad 0 \leq p^u \leq \frac{Cp^v}{t},$$
$$0 \leq p^v \leq \frac{C}{t}, \qquad ct^2 \leq R_\circ^2 \leq Ct^2.$$

The massless Vlasov equation on an FLRW background

Hence, in view of expression (4), we can get

$$T_{uu} = \frac{\Omega_\circ^4}{R_\circ^2}\frac{\pi}{2}\int_0^{L_0}\int_0^{\frac{C}{t}} f(u,v,p^v,L)p^v L dL dp^v \lesssim \frac{\|f_1\|_{L^\infty}}{t^2}.$$

Similarly, in view of (5) and (6), we can get

$$T_{uv} = \frac{\Omega_\circ^4}{R_\circ^2}\frac{\pi}{2}\int_0^{L_0}\int_0^{\frac{C}{t}} f(u,v,p^v,L)p^u dp^v L dL \lesssim \frac{\|f_1\|_{L^\infty}}{t^3},$$

$$T_{vv} = \frac{\Omega_\circ^4}{R_\circ^2}\frac{\pi}{2}\int_0^{L_0}\int_0^{\frac{C}{t}} f(u,v,p^v,L)\frac{(p^u)^2}{p^v} dp^v L dL \lesssim \frac{\|f_1\|_{L^\infty}}{t^4}.$$

References

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