# CLASSICAL RESULTS IN COMPARISON GEOMETRY

*by*

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***Abstract.*** — This study explores the influence of curvature, particularly its sign, on the topology of manifolds. Focusing on three significant results in geometry: the Bonnet-Myers theorem, Synge's theorem, and the Cartan-Hadamard theorem, showing how curvature constrains and shapes the topological properties of manifolds.

## Contents



## 1. Introduction

We have essentially studied the interaction that exists between local and global properties of a Riemannian manifold. By a local property, we mean a property that depends on the behavior of the manifold in the neighborhood of a point, and by global property we mean a property that depends on the behavior of the manifold as a whole.

We discuss two applications of the formula for the second variation of energy of a geodesic. The first (Bonnet-Myers theorem) states that a complete manifold whose curvature is positive is then compact, and its diameter can be estimated in terms of the bounds of the curvature. The second (Synge theorem) asserts the simple connectivity of a compact, orientable, even-dimensional manifold whose sectional curvature is positive. Finally, we discuss the Cartan-Hadamard theorem, in which a local condition (non-positive sectional curvature) together with weak global restrictions (complete and simply connected) implies a strong global restriction (diffeomorphic to $\mathbb{R}^n$).

This report begins with a concise introduction to Riemannian geometry, providing the necessary foundational concepts and tools. Following this, I will delve into explanations and proofs of the three chosen theorems: the Bonnet-Myers theorem, Synge's theorem, and the Cartan-Hadamard theorem, in spirit of [**dC92**].

***Key words.*** — Riemannian geometry, variation formula, Jacobi fields. *Supervised by Dr. Juan Souto.*

## 2. Preliminaries

In this section, we will introduce the main notions for a *(smooth)* Riemannian Manifold such as the Levi-Civita connection, curvature, geodesics, and Jacobi fields. More details and proofs can be found in [**dC92**]. [**Lee18**] and [**GHL04**].

**Notation:** By $M$ we always denote a connected, smooth manifold of dimension $n$ and the manifold topology should be hausdorff and satisfy the second countable axiom. For $p \in M$, the tangent space at $p$ is denoted by $T_pM$, and $TM$ denotes the tangent bundle. If $N$ is another manifold and $f : M \to N$ a smooth (i.e. $C^\infty$) map, its differential at some point $p \in M$ is always denoted by $df_p : T_pM \to T_{f(p)}N$. If $c : I \to M$ is a (smooth) curve, we denote its tangent vector by $\dot{c}(t) = \frac{d}{dt}c(t) \in T_{c(t)}M$.

A Riemannian metric $g$ on $M$ will be denoted by an inner product $g_p(\cdot,\cdot) = \langle \cdot,\cdot \rangle_p$ on $T_pM$ for each $p \in M$ , such that the assignment depends smoothly on $p$. We define the norm of a vector by $|v| = \sqrt{\langle v, v\rangle}$, and the length of a curve $c : I \to M$ by $\ell(c) = \int_I |\dot{c}(t)|dt$. We denote the distance between two points $x, y \in M$ by $d(x, y) = inf\{\ell(c)$ where $c : [a, b] \to M$ with $c(a) = x$ and $c(b) = y\}$. If $\ell(c) = d(x, y)$ for some curve $c$, then we say that $c$ is shortest.

Until the end, if it is not specified, all the considered manifolds will be given with a Riemannian metric. If the metric is not explicitly defined, we will call it $g$ and use the notations as above.

A vector field on a smooth manifold $M$ is a section $X : M \to TM$ of the tangent bundle, that we denote by $X \in \Gamma(TM)$.

### 2.1. Connection and parallel transport.

— Now we list the properties that the derivative of vector fields should have, as follows.

***Definition 2.1*** **(Affine Connection)**. — An affine connection on a Riemannian manifold $(M, g)$ is a map $\nabla : \Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM)$ given by $\nabla(X, Y) = \nabla_X Y$ such that, for each $X, Y, Z \in \Gamma(TM)$, $a, b \in \mathbb{R}$ and $f, h \in C^\infty(M)$ verify the following properties:

1. $\nabla_X Y$ is $\mathbb{R}$-linear on $Y$: $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z$,
2. $\nabla_X Y$ is $C^\infty(M)$-linear on $X$: $\nabla_{fX+hY} Z = f\nabla_X Z + h\nabla_Y Z$,
3. $\nabla_X fY = (Xf)Y + f\nabla_X Y$,

We will call $\nabla_X Y$ the covariant derivative of $Y$ in the direction of the vector field $X$.

***Proposition 2.2***. — *Let $M$ be a smooth manifold with an affine connection $\nabla$. There exists a unique correspondence which associates to a vector field $V$ along the smooth curve $c : I \to M$ another vector field $\frac{\nabla V}{dt}$ along $c$, called the covariant derivative of $V$ along $c$, such that*

1. $\frac{\nabla}{dt}(V + W) = \frac{\nabla}{dt}V + \frac{\nabla}{dt}W$*, where $W$ is a vector field along $c$ and $f$ is a smooth function on $I$.*
2. $\frac{\nabla}{dt}(fV) = \frac{df}{dt}V + f\frac{\nabla}{dt}V$.
3. *If $V$ is induced by a vector field $Y \in \Gamma(TM)$, i.e., $V(t) = Y(c(t))$, then $\frac{\nabla}{dt}V = \nabla_{\dot{c}(t)}Y$.*

Suppose $M$ and $N$ are smooth manifolds and $\varphi : M \to N$ is a diffeomorphism. For a smooth vector field $X \in \Gamma(TM)$ recall that the pushforward of $X$ is the unique vector field $\varphi_* X \in \Gamma(TN)$ that satisfies $d\varphi_p(X_p) = (\varphi_* X)_{\varphi(p)}$ for all $p \in M$.

We define the pullback connection $(\varphi^*\tilde{\nabla})_X Y$, which verifies to be linear over $\mathbb{R}$ in $Y$, to be linear over $C^\infty(M)$ in $X$, and the product rule in $Y$, that is, it is a connection.

***Lemma 2.3* (Pullback Connection).** — *Suppose $M$ and $N$ are smooth manifolds. If $\tilde{\nabla}$ is a connection in $TN$ and $\varphi : M \to N$ is a diffeomorphism, then the map $\varphi^*\tilde{\nabla} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by*

$$(\varphi^*\tilde{\nabla})_X Y = (\varphi^{-1})_*\big(\tilde{\nabla}_{\varphi_* X}(\varphi_* Y)\big)$$

*is a connection in $TM$, called the pullback of $\tilde{\nabla}$ by $\varphi$.*

***Definition 2.4* (Levi-Civita Connection).** — An affine connection $\nabla$ on a Riemannian manifold $(M, g)$ such that, for each $X, Y, Z \in \Gamma(TM)$, verifies the following properties:

1. $\nabla$ is symmetric or torsion-free: $\nabla_X Y - \nabla_Y X = [X, Y]$,
2. $\nabla$ is metric compatible: $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

is called the Levi-Civita connection.

Consider $2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle$, known as Koszul's formula. Note that the right-hand side is determined by the metric. And that the $\nabla_X Y$ defined by the above formula satisfies all conditions of Levi-Civita connections. So, it provides the uniqueness of a torsion-free and metric-compatible condition. This allows to state

***Theorem 2.5* (The fundamental theorem of Riemannian geometry)**
*On any Riemannian manifold $(M, g)$, there is a unique Levi-Civita connection $\nabla$.*

***Proposition 2.6*.** — *Let $M$ be a Riemannian manifold. A connection $\nabla$ on $M$ is compatible with a metric if and only if for any vector fields $V$ and $W$ along the differentiable curve $c : I \to M$ we have*

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{\nabla}{dt}V, W\rangle + \langle V, \frac{\nabla}{dt}W\rangle,\ t \in I.$$

***Proposition 2.7* (Naturality of the Levi-Civita Connection).** — *Suppose $(M, g)$ and $(N, h)$ are Riemannian manifold, and let $\nabla$ denote the Levi-Civita connection of $g$ and $\tilde{\nabla}$ that of $h$. If $\varphi : M \to N$ is an isometry, then $\varphi^*\tilde{\nabla} = \nabla$.*

*Proof.* — By uniqueness of the Levi-Civita connection, it suffices to show that the pullback connection $\varphi^*\tilde{\nabla}$ is symmetric and compatible with $g$. The fact that $\varphi$ is an isometry means that for any $X, Y \in \Gamma(TM)$ and $p \in M$

$$\langle Y, Z\rangle_p = \langle \varphi_* Y, \varphi_* Z\rangle_{\varphi(p)}$$

Therefore

$$\begin{aligned}
X\langle Y, Z\rangle &= X(\langle \varphi_* Y, \varphi_* Z\rangle \circ \varphi) \\
&= \big((\varphi_* X)\langle \varphi_* Y, \varphi_* Z\rangle\big) \circ \varphi \\
&= \big(\langle \tilde{\nabla}_{\varphi_* X}(\varphi_* Y), \varphi_* Z\rangle + \langle (\varphi_* Y), \tilde{\nabla}_{\varphi_* X}(\varphi_* Z)\rangle\big) \circ \varphi \\
&= \langle (\varphi^{-1})_* \tilde{\nabla}_{\varphi_* X}(\varphi_* Y), Z\rangle + \langle Y, (\varphi^{-1})_* \tilde{\nabla}_{\varphi_* X}(\varphi_* Z)\rangle \\
&= \langle (\varphi^*\tilde{\nabla})_X Y, Z\rangle + \langle Y, (\varphi^*\tilde{\nabla})_X Z\rangle
\end{aligned}$$

which shows that the pullback connection is compatible with $g$. Symmetry is proved as follows

$$\begin{aligned}
(\varphi^*\tilde{\nabla})_X Y - (\varphi^*\tilde{\nabla})_Y X &= (\varphi^{-1})_*\big(\tilde{\nabla}_{\varphi_* X}(\varphi_* Y) - \nabla_{\varphi_* Y}(\varphi_* X)\big) \\
&= (\varphi^{-1})_*[\varphi_* X, \varphi_* Y] \\
&= [X, Y]
\end{aligned}$$

□

***Remark 2.8.*** — Let $\phi : M \to N$ is an inmersion, and $X, Y \in \Gamma(TM)$ and $\tilde{X}, \tilde{Y} \in \Gamma(TN)$ verifying $\tilde{X}_{\phi(p)} = d\phi_p X$ and $\tilde{Y}_{\phi(p)} = d\phi_p Y$, for all $p \in M$. Then $[\tilde{X}, \tilde{Y}]_{\phi(p)} = d\phi_p[X, Y]$. It follows that, if $[X, Y] = 0$, then $[\tilde{X}, \tilde{Y}]_{\phi(p)} = 0$.

In the following, $\nabla$ will always denote a Levi-Civita connection.

***Remark 2.9.*** — If $M$ is smooth manifold with $\nabla$ torsion-free, and $\gamma : (-\epsilon, \epsilon) \times I \to M$ a curve with $s \in (-\epsilon, \epsilon)$, $t \in I$. Then

$$\frac{\nabla}{ds}\frac{\partial\gamma}{\partial t} - \frac{\nabla}{dt}\frac{\partial\gamma}{\partial s} = \left[\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right].$$

Notice that for $f \in C^\infty(M)$, we have

$$\begin{aligned}\left[\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right] f &= \frac{\partial\gamma}{\partial s}\left(\frac{\partial\gamma}{\partial t} f\right) - \frac{\partial\gamma}{\partial t}\left(\frac{\partial\gamma}{\partial s} f\right) \\ &= \frac{\partial}{\partial s}\frac{\partial}{\partial t}(f\circ\gamma) - \frac{\partial}{\partial t}\frac{\partial}{\partial s}(f\circ\gamma) \\ &= \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right](f\circ\gamma) \\ &= 0,\end{aligned}$$

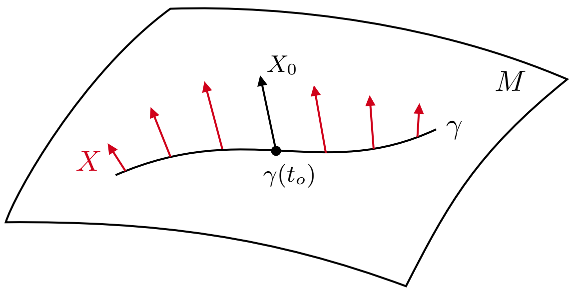
then we have

$$\frac{\nabla}{ds}\frac{\partial\gamma}{\partial t} = \frac{\nabla}{dt}\frac{\partial\gamma}{\partial s}.$$

***Definition 2.10.*** — Let $M$ be a differentiable manifold with $\nabla$ a connection. We say a vector field $X$ is parallel along $\gamma : I \to M$ if $\frac{\nabla}{dt}X = 0$, for all $t \in I$.

Due to the classical existence and uniqueness theorem in ODE, we have the following theorem

***Theorem 2.11.*** — *For any curve $\gamma : [a, b] \to M$ , any $t_0 \in [a, b]$ and any $X_0 \in T_{\gamma(t_0)}M$, there exists a unique vector field $X$ along $\gamma$ which is parallel, such that $X(\gamma(t_0)) = X_0$.*




***Definition 2.12.*** — We will call the map

$$\begin{array}{rccc} P_{\gamma,t_0,t} : & T_{\gamma(t_0)}M & \longrightarrow & T_{\gamma(t)}M \\ & X_0 = X(\gamma(t_0)) & \mapsto & X(\gamma(t)) \end{array}$$

the parallel transport from $\gamma(t_0)$ to $\gamma(t)$ along $\gamma$, where $X$ is the parallel vector field along $\gamma$ such that $X(\gamma(t_0)) = X_0$.

From the fact that the solution of a homogeneous linear ODE system that depends linearly on initial data is linear and, that $P_{\gamma,t_0,t}$ is invertible because

$$P_{\gamma,t_0,t} \circ P_{\gamma,t,t_0} = Id.$$

Thus, we have the following statement

***Lemma 2.13.*** — *Any parallel transport $P_{\gamma,t_0,t}$ is a linear isomorphism.*

For a Riemannian manifold, each tangent space $T_pM$ is not only a linear space but a linear space with an inner product.

***Proposition 2.14.*** — *The parallel transport $P_{\gamma,t_0,t}$ with respect to a metric compatible affine connection $\nabla$ is an isometry between $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$.*

*Proof.* — Let $\nabla$ be a metric-compatible affine connection and $\gamma : I \to M$ a curve. Let $V_0, W_0 \in T_{\gamma(t_0)}M$ and $V, W$ the corresponding parallel vector field along $\gamma$. Then

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{\nabla}{dt}V, W\rangle + \langle V, \frac{\nabla}{dt}W\rangle = 0.$$

Therefore $\langle V, W\rangle$ is constant, hence

$$\begin{aligned}\big\langle P_{\gamma,t_0,t}(V_0), P_{\gamma,t_0,t}(W_0)\big\rangle &= \big\langle V\big(\gamma(t)\big), W\big(\gamma(t)\big)\big\rangle \\ &= \big\langle V\big(\gamma(t_0)\big), W\big(\gamma(t_0)\big)\big\rangle \\ &= \big\langle V_0, W_0\big\rangle.\end{aligned}$$

This proves that parallel transport is an isometry. □

***Lemma 2.15.*** — *Given any smooth curve $\gamma$ in $M$, every orthonormal basis at a point of $\gamma$ can be extended to a parallel orthonormal frame along $\gamma$.*

*Proof.* — Suppose $\gamma : I \to M$ is a smooth curve and $\{e_i\}$ is an orthonormal basis for $T_{\gamma(t_0)}M$, for some $t_0 \in I$. We can extend each $e_i$ by parallel transport to obtain a smooth parallel vector field $E_i$ along $\gamma$, and since that parallel transport is a linear isometry, then the resulting $n-$tuple $\{E_i\}$ is an orthonormal frame at all points of $\gamma$. □

Let us recall the definition of orientation in differentiable manifolds

***Definition 2.16.*** — Let $M$ be a smooth manifold of dimension $n$.

1. Two charts $(U_\alpha, \phi_\alpha)$ and $(U_\beta, \phi_\beta)$ are orientation compatible if the transition map $\phi_\beta \circ \phi_\alpha^{-1}$ satisfies
$$det\big(d(\phi_\beta \circ \phi_\alpha^{-1})\big)_x > 0, \quad \text{for all } x \in \phi_\alpha(U_\alpha \cap U_\beta).$$
2. An orientation of $M$ is an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ whose charts are pairwise orientation compatible. We say $M$ is orientable if it has an orientation.

In the opposite case, we say that $M$ is non-orientable.

***Remark 2.17.*** — If $M$ is orientable and connected there exists exactly two distinct orientations on $M$. Two differentiable structures that are orientable, determine *the same orientation* if their union again satisfies the definition.

Now, let $M$ and $N$ be n-dimensional connected manifolds and $f : M \to N$ a diffeomorphism. Suppose both of these manifolds are equipped with orientations, $f$ induces and orientation on $N$ which may or may not coincide with the initial orientation of $N$. In the first case, we say that $f$ preserves the orientation and in the second case, that $f$ reverses the orientation.

***Proposition 2.18.*** — *The parallel transport on an oriented Riemannian manifold $M$ is orientation-preserving.*

*Proof.* — Let $e_1, \cdots, e_n$ be an orientable basis of $T_{\gamma(t_0)}M$ and let $e_i(s) = P_{\gamma,t_0,s}(e_i)$, $s \in [t_0, t]$, be the parallel transport of $e_i$, from $t_0$ to $s$, along $\gamma$.
Since in any oriented local coordinates $(U, x^i)$, we can write $e_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x^j}$, then the orientation of $e_i(s)$ is determined by the sign of $det(a_{ij}(s))$, which is continuous with respecto to $s$. Thus, since $det(a_{ij}(t_0)) > 0$, we have that $det(a_{ij}(s)) > 0$, for all $s \in [t_0, t]$. So $P$ preserves the orientation. □

**2.2. Curvature. —** Now, we come to one of the central concepts of Riemannian geometry

***Definition 2.19* (Riemannian curvature).** — The curvature of a Riemannian manifold $M$ is a correspondence that to each pair of vector fields $X, Y \in \Gamma(TM)$ associates the map $R(X,Y) : \Gamma(TM) \to \Gamma(TM)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{1}$$

for all $Z \in \Gamma(TM)$, where $\nabla$ is the Levi-Civita connection.

***Remark 2.20.*** — Various definitions of the Riemannian curvature tensor may difier by a sign in the literature. Be careful with such detail!

***Remark 2.21.*** — As $R(X,Y)Z = 0$ if $M = \mathbb{R}^n$, we can say that $R(X,Y)Z$ measures how much $M$ differs from being flat. Also, considering a coordinate basis $(x^i)$ around a point $p \in M$. Using $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$, the last term in (1) drops out and the curvature $R$ can be interpreted as measuring the non-commutativity of the covariant derivative.

Now, we introduce some important properties of the curvature $R$

***Proposition 2.22.*** — *The Riemannian curvature $R$ of a Riemannian Manifold $M$ verifies the following properties:*

1. *Bilinearity in $\Gamma(TM) \times \Gamma(TM)$: Let $f, g \in C^\infty(M)$. $X_1, X_2, Y_1, Y_2 \in \Gamma(TM)$*
   (a) $R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$,
   (b) $R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2)$.

2. *Linearity $R(X,Y) : \Gamma(TM) \to \Gamma(TM)$: Let $X, Y, Z, W \in \Gamma(TM)$, $f, g \in C^\infty(M)$*
   (a) $R(X,Y)(Z + W) = R(X,Y)Z + R(X,Y)W$,
   (b) $R(X,Y)(fZ) = fR(X,Y)Z$.

***Remark 2.23.*** — Note $R$ is linear in all variables, so, it is a tensor.

***Proposition 2.24.*** — *The Riemann curvature satisfies, for $X, Y, Z, T \in \Gamma(TM)$,*

1. $\langle R(X,Y)Z, T\rangle = -\langle R(Y,X)Z, T\rangle$
2. $\langle R(X,Y)Z, T\rangle = -\langle R(X,Y)T, Z\rangle$
3. $\langle R(X,Y)Z, T\rangle = \langle R(Z,T)X, Y\rangle$

***Remark 2.25.*** — Local isometries must preserve the curvature tensor in the following sense, as we have seen with the Levi-Civita connection. If $f : M \to N$ is a local isometry between two Riemannian manifolds, then

$$R_{f(p)}\big(df_p(X_p), df_p(Y_p)\big)df_p(Z_p) = R_p(X_p, Y_p)Z_p$$

for every $p \in M$ and every $X, Y, Z \in \Gamma(TM)$.

***Definition 2.26* (Sectional curvature).** — Let $M$ be a Riemannian manifold and $p \in M$. If $\sigma$ is a 2−dimensional subspace of the tangent space $T_pM$, then we define the sectional curvature of $\sigma$ at $p$ to be

$$K(\sigma) = \big\langle R(e_1, e_2)e_1, e_2\big\rangle$$

for any orthonormal basis $e_1, e_2$ of $\sigma$.

Consider that, given a vector space $X$, we denote by $|x \wedge y|$ the expression $\sqrt{|x|^2|y|^2 - \langle x, y\rangle^2}$, which represents the area of the two-dimensional parallelogram determined by the vectors $x, y \in X$. Thus, if $x, y$ is an arbitrary basis for the 2−plane $\sigma$, the sectional curvature of $\sigma$ is given by

$$K(x, y) = K(\sigma) = \frac{\big\langle R(x, y)x, y\big\rangle}{|x \wedge y|^2}.$$

***Definition 2.27* (Ricci curvature).** — Let $x = e_n$ be a unit vector in $T_pM$, we take an orthonormal basis $\{e_1, e_2, \ldots, e_{n-1}\}$ of the hyperplane in $T_pM$ orthogonal to $x$. The following average

$$Ric_p(x) = \frac{1}{n-1}\sum_{i=1}^{n-1}\big\langle R(x, e_i)x, e_i\big\rangle, \qquad i = 1, 2, \ldots, n-1.$$

is called the Ricci curvature in the direction $x$.

***Remark 2.28.*** — Given a unit-length tangent vector $x \in T_pM$, we obtain the Ricci curvature of $X$ at $p$ by extending $x = e_n$ to an orthonormal basis $e_1, \ldots, e_n$ and then $Ric_p(x)$ is given by the above definition. On the other hand, the sectional curvature $K(x, e_i)$ for $i < n$, (remember the $e_i$ are orthonormal) is given by $K(x, e_i) = \langle R(x, e_i)x, e_i\rangle$. Thus, if $K(x, e_i) \geq C > 0$, we have

$$Ric_p(x) = \sum_{i=1}^{n-1}\big\langle R(x, e_i)x, e_i\big\rangle \geq C > 0.$$

So you also get a lower bound on the Ricci curvature.

Now, we are going to introduce a relation that will be useful for the formulas of variation.

***Lemma 2.29.*** — *Let $V = V(s, t)$ be a vector field along a smooth curve $f : (-\epsilon, \epsilon) \times I \subset \mathbb{R}^2 \to M$ with $s \in (-\epsilon, \epsilon)$, $t \in I$. Then*

$$\frac{\nabla}{dt}\frac{\nabla}{ds}V - \frac{\nabla}{ds}\frac{\nabla}{dt}V = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})V.$$

*Proof.* — Let $(\mathcal{U}, x)$ be a coordinate system based at $p \in M$. Thus $V = \sum_i v^i X_i$, where $v^i = v^i(s, t)$ and $X_i = \frac{\partial}{\partial x^i}$. Then

$$\frac{\nabla}{ds}V = \frac{\nabla}{ds}\Big(\sum_i v^i X_i\Big) = \sum_i\Big(v^i\frac{\nabla}{ds}X_i + \frac{\partial v^i}{\partial s}X_i\Big),$$

and so

$$\frac{\nabla}{dt}\frac{\nabla}{ds}V = \sum_i\Big(v^i\frac{\nabla}{dt}\frac{\nabla}{ds}X_i + \frac{\partial v^i}{\partial t}\frac{\nabla}{ds}X_i + \frac{\partial v^i}{\partial s}\frac{\nabla}{dt}X_i + \frac{\partial^2 v^i}{\partial t\partial s}X_i\Big).$$

Then, interchanging the values of $s$ and $t$, and subtracting, we get

$$\frac{\nabla}{dt}\frac{\nabla}{ds}V - \frac{\nabla}{ds}\frac{\nabla}{dt}V = \sum_i v^i\Big(\frac{\nabla}{dt}\frac{\nabla}{ds}X_i - \frac{\nabla}{ds}\frac{\nabla}{dt}X_i\Big)$$

Setting $f(s, t) = \big(x^1(s, t), \ldots, x^n(s, t)\big)$. Then $\frac{\partial f}{\partial s} = \sum_j \frac{\partial x^j}{\partial s}X_j$ and $\frac{\partial f}{\partial t} = \sum_k \frac{\partial x^k}{\partial t}X_k$. Thus, by Proposition 2.2-3. we consider that

$$\nabla_{\frac{\partial f}{\partial s}}X_i = \nabla_{\sum_j \frac{\partial x^j}{\partial s}X_j}X_i = \sum_j \frac{\partial x^j}{\partial s}\nabla_{X_j}X_i.$$

We have

$$\begin{aligned}\frac{\nabla}{dt}\frac{\nabla}{ds}X_i &= \frac{\nabla}{dt}\bigg(\sum_j \frac{\partial x^j}{\partial s}\nabla_{X_j}X_i\bigg)\\ &= \sum_j\left(\frac{\partial^2 x^j}{\partial t\partial s}\nabla_{X_j}X_i + \frac{\partial x^j}{\partial s}\left(\nabla_{\frac{\partial f}{\partial t}}\nabla_{X_j}X_i\right)\right)\\ &= \sum_j \frac{\partial^2 x^j}{\partial t\partial s}\nabla_{X_j}X_i + \sum_{j,k}\frac{\partial x^j}{\partial s}\frac{\partial x^k}{\partial t}\nabla_{X_k}\nabla_{X_j}X_i.\end{aligned}$$

Therefore

$$\frac{\nabla}{dt}\frac{\nabla}{ds}X_i - \frac{\nabla}{ds}\frac{\nabla}{dt}X_i = \sum_{j,k}\frac{\partial x^j}{\partial s}\frac{\partial x^k}{\partial t}\Big(\nabla_{X_k}\nabla_{X_j} - \nabla_{X_j}\nabla_{X_k}\Big)X_i.$$

Joining everything together and using (1), we get

$$\begin{aligned}\frac{\nabla}{dt}\frac{\nabla}{ds}V - \frac{\nabla}{ds}\frac{\nabla}{dt}V &= \sum_{i,j,k} v^i\frac{\partial x^j}{\partial s}\frac{\partial x^k}{\partial t}R(X_j, X_k)X_i\\ &= R\Big(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big)V.\end{aligned}$$

□

**2.3. Geodesics, the exponential map & Hopf-Rinow theorem.** — In what follows, $M$ will be a Riemannian manifold, together with its Levi-Civita connection.

***Definition 2.30.*** — A parametrized curve $\gamma : I \to M$ is a geodesic at $t_0 \in I$ if $\frac{\nabla}{dt}\dot\gamma = 0$ at the point $t_0$. If $\gamma$ is a geodesic at $t$, for all $t \in I$, we say that $\gamma$ is a geodesic. If $[a, b] \subset I$ and $\gamma : I \to M$ is a geodesic, the restriction of $\gamma$ to $[a, b]$ is called a geodesic segment joining $\gamma(a)$ to $\gamma(b)$.

***Remark 2.31.*** — At times, by abuse of language, we refer to the image $\gamma(I)$ of a geodesic $\gamma$, as a geodesic.

Let $(M, g)$ be a Riemannian manifold. Recall that for any $p \in M$ and any $v \in T_pM$, there exists a unique geodesic $\gamma(t) = \gamma(t, p, v)$ such that $\gamma(0) = p$, $\dot\gamma(0) = v$. Moreover, the geodesic $\gamma(t)$ depends smoothly on $p$ and $v$.

***Remark 2.32.*** — If the geodesic $\gamma(t, p, v)$ is defined on the interval $(-\epsilon, \epsilon)$, then the geodesic $\gamma(t, p, av)$ with $0 < a \in \mathbb{R}$, is defined on the interval $(-\frac{\epsilon}{a}, \frac{\epsilon}{a})$ and

$$\gamma(t, p, av) = \gamma(at, p, v).$$

The existence and uniqueness theorem for ODE can be used to prove the following proposition:

***Proposition 2.33.*** — *Given $p \in M$, there exists a neighborhood $V$ of $p$ in $M$, a number $\epsilon > 0$ and a $C^\infty$ mapping $\gamma : (-2, 2) \times \mathcal{U} \to M$ where $\mathcal{U} = \{(q, w) \in TM, q \in V, w \in T_qM, |w| < \epsilon\}$ such that $t \to \gamma(t, q, w)$, $t \in (-2, 2)$, is the unique geodesic of $M$ which, at the instant $t = 0$, passes through $q$ with velocity $w$, for every $q \in V$ and for every $w \in T_qM$, with $|w| < \epsilon$.*

***Definition 2.34.*** — The map $\exp : \mathcal{U} \to M$ defined by

$$\exp(p, v) = \gamma(1, p, v) = \gamma(|v|, p, \frac{v}{|v|}), \qquad (p, v) \in \mathcal{U},$$

is called the exponential map on $\mathcal{U}$.

***Remark 2.35.*** — According to the smooth dependence in ODE theory, the exponential map is smooth. In most of the applications, we shall utilize the restriction of exp to an open subset of the tangent space $T_pM$, that is, we define $\exp_p : B_\epsilon(0) \subset T_pM \to M$ by $\exp_p(v) = \exp(p, v)$.

Geometrically, $\exp_p(v)$ is a point of $M$ obtained by going out the length equal to $|v|$, starting from $p$, along a geodesic which passes through $p$ with velocity equal to $\frac{v}{|v|}$.

Here, and in what follows, we denote by $B_\epsilon(0)$ an open ball with center at the origin 0 of $T_pM$ and of radius $\epsilon$. By definition, we have that $\exp_p$ is differentiable and that $\exp_p(0) = p$.

The following lemma will be very useful

***Proposition 2.36.*** — *Given $p \in M$, there exists an $\epsilon > 0$ such that $\exp_p : B_\epsilon(0) \subset T_pM \to M$ is a diffeomorphism of $B_\epsilon(0)$ onto an open subset of $M$.*

*Proof.* — For any $p \in M$ , if we identify $T_0(T_pM) \cong T_pM$ , then for any $v \in T_0(T_pM)$,

$$(d\exp_p)_0(v) = \frac{d}{dt}\bigg|_0 \exp_p(tv) = \frac{d}{dt}\bigg|_0 \gamma(1, p, tv) = \frac{d}{dt}\bigg|_0 \gamma(t, p, v) = v.$$

Thus, $(d\exp_p)_0 = Id\big|_{T_pM} : T_pM \to T_pM$, and we get the statement as a consequence of the inverse function theorem. □

***Definition 2.37.*** — A Riemannian manifold $M$ is geodesically complete if for all $p \in M$, the exponential map, $\exp_p$, is defined for all $v \in T_pM$, i.e., if any geodesic $\gamma(t)$ starting from $p$ is defined for all values of the parameter $t \in \mathbb{R}$.

***Remark 2.38.*** — Observe that if there exists a minimizing geodesic $\gamma$ joining $p$ to $q$ *(which is not always true)* then $d(p, q) = \ell(\gamma)$.

We introduce Hopf-Rinow's theorem which characterizes the completeness of a Riemannian manifold in which the geodesics have nice properties of being globally defined and minimizing the curves between two points.

***Theorem 2.39* (Hopf-Rinow).** — *Let $M$ be a Riemannian manifold and let $p \in M$. The following assertions are equivalent:*

1. *$\exp_p$ is defined on all of $T_pM$.*
2. *The closed and bounded sets of $M$ are compact.*
3. *$M$ is complete as a metric space.*
4. *$M$ is geodesically complete.*
5. *Assume $M$ is non-compact. There exists a sequence of compact subsets $K_n \subset M$, $K_n \subset int(K_{n+1})$ and $\bigcup_n K_n = M$, such that if $q_n \notin K_n$ then $d(p, q_n) \to \infty$.*

*In addition, any of the statements above implies that*

6. *For any $q \in M$, there exists a geodesic $\gamma$ joining $p$ to $q$ with $\ell(\gamma) = d(p, q)$.*

***Corollary 2.40.*** — *If $M$ is compact then $M$ is geodesically complete.*

In the proposition that follows, we use some facts on the fundamental group and covering spaces [See section 5]

***Proposition 2.41.*** — *Let $M$ be a connected Riemannian manifold and $\pi : \tilde{M} \to M$ the universal covering of $M$ with the covering metric $\tilde{g} = \pi^*g$. To each vector field $X \in \Gamma(TM)$ we associate a vector field $\pi^*(X) \in \Gamma(T\tilde{M})$, defined by $\pi^*(X)_{\tilde{p}} = (d\pi_{\tilde{p}})^{-1}(X_{\pi(\tilde{p})})$. Then we have*

1. *$\tilde{\nabla}_{\pi^*X}(\pi^*Y) = \nabla_X Y$ for all $X, Y \in \Gamma(TM)$.*

2. *If $\gamma$ is a geodesic in $M$ , then each lift of $\gamma$ over $\pi$ is a geodesic in $\tilde{M}$.*
3. $R_{\tilde{g}}(\pi^*X, \pi^*Y)\pi^*Z = R_g(X,Y)Z$ *for all* $X, Y, Z \in \Gamma(TM)$.
4. *If $M$ is complete, then $\tilde{M}$ is complete.*

*Proof.* — From 1. to 3. are general properties of local isometries.

To show 4., let $\tilde{\gamma}_0$ be a geodesic in $\tilde{M}$, starting at $\tilde{p}$. By completeness of $M$, the geodesic $\gamma_0 = \pi \circ \tilde{\gamma}_0$ can be extended to a geodesic $\gamma$, which is defined for all $t$. Its lift starting at $\tilde{p}$ is an extension of $\tilde{\gamma}_0$ which is defined for all $t$. □

**2.4. Energy variation formulas and Jacobi fields.** — Now we specify the idea of "neighboring curves" to a given curve. But first, recall that for any curve $c : [a, 0] \to M$, we have the arc length of $c$ given by

$$\ell(c) := \int_0^a \left|\frac{dc}{dt}\right| dt,$$

and the energy of $c$

$$E(c) := \int_0^a \left|\frac{dc}{dt}\right|^2 dt.$$

Setting $f \equiv 1$ and $g = \left|\frac{dc}{dt}\right|$ in Schwarz inequality, we have

$$\left(\int_0^a fg\ dt\right)^2 \leq \int_0^a f^2 dt \cdot \int_0^a g^2 dt$$

we obtain

$$\ell(c)^2 \leq aE(c),$$

and equality occurs if and only if $g$ is constant, that is, if and only if $t$ is proportional to arc length. From these considerations, it follows that

***Lemma 2.42.*** — *Let $p, q \in M$, and let $\gamma : [0, a] \to M$ be a minimizing geodesic joining $p$ and $q$. Then, for all curves $c : [0, a] \to M$ joining $p$ to $q$,*

$$aE(\gamma) = \ell(\gamma)^2 \leq \ell(c)^2 \leq aE(c)$$

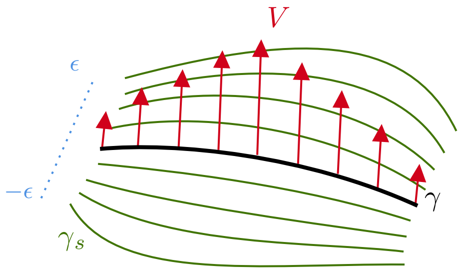
*with the equality holding if and only if $c$ is a minimizing geodesic.*

***Definition 2.43.*** — Let $\gamma : [a, b] \to M$ be a smooth curve in $M$ and $\epsilon > 0$. A variation of $\gamma$ is a continuous map $f : (-\epsilon, \epsilon) \times [a, b] \to M$ such that $f(0, t) = \gamma(t)$ for $t \in [a, b]$.

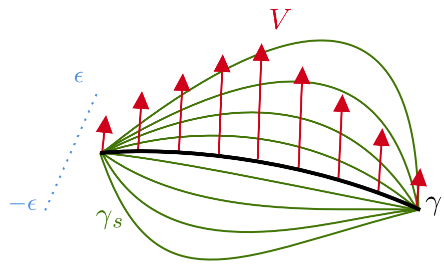
A variation is said to be proper if $f(s, a) = \gamma(a)$ and $f(s, b) = \gamma(b)$, for all $s \in (-\epsilon, \epsilon)$.

The variational field of a variation $f$ is a smooth vector field $V$ along $\gamma$ defined by $V(t) = \frac{\partial f}{\partial s}(0, t)$.

In what follows, we will also denote $\gamma_s(t) = f(s, t)$ and $\frac{\partial f}{\partial t}(s, t) = \dot{\gamma}_s(t)$.




$\gamma_s$ is a variation of $\gamma$




$\gamma_s$ is a proper variation of $\gamma$

***Theorem 2.44*** **(Formula of the first variation of the energy of a curve)**
*Let $\gamma_s(t)$ be a smooth variation of a curve $\gamma$. If $E : (-\epsilon, \epsilon) \to \mathbb{R}$ is the energy of $\gamma_s(t)$, then*

$$\frac{1}{2}\frac{d}{ds}E(\gamma_s(t))\Big|_{s=0} = \langle V(t), \dot\gamma(t)\rangle\Big|_a^b - \int_a^b \left\langle V(t), \frac{\nabla}{dt}\dot\gamma(t)\right\rangle dt \tag{2}$$

*Proof.* — Differentiating the energy of $\gamma$ and using properties of the connection and Remark 2.9, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}E(\gamma_s(t)) &= \frac{1}{2}\int_a^b \frac{d}{ds}\langle \dot\gamma_s(t), \dot\gamma_s(t)\rangle dt\\ &= \int_a^b \left\langle \frac{\nabla}{ds}\dot\gamma_s(t), \dot\gamma_s(t)\right\rangle dt\\ &= \int_a^b \left\langle \frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}, \dot\gamma_s(t)\right\rangle dt\\ &= \int_a^b \frac{d}{dt}\left\langle \frac{\partial\gamma_s(t)}{\partial s}, \dot\gamma_s(t)\right\rangle - \left\langle \frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{dt}\dot\gamma_s(t)\right\rangle dt\end{aligned}$$

Evaluating $s = 0$, we obtain (2). □

***Remark 2.45.*** — In particular, if $\gamma_s(t)$ is a proper variation of $\gamma$, then

$$\frac{1}{2}\frac{d}{ds}E(\gamma_s(t))\Big|_{s=0} = -\int_a^b \left\langle V(t), \frac{\nabla}{dt}\dot\gamma(t)\right\rangle dt$$

We refer the reader to **[dC92]** for the more general treatment of piecewise differentiable curves and their variations.

***Proposition 2.46.*** — *A smooth curve $\gamma : [a, b] \to M$ is a geodesic if and only if, for every proper variation $\gamma_s(t)$ of $\gamma$, we have $\frac{d}{ds}E(\gamma_s(t))\big|_{s=0} = 0$.*

*Proof.* — Let $V$ be the associated proper variational vector field, so $V(a) = V(b) = 0$. From the first variation formula Remark 2.45, it follows that, if $\gamma$ is a geodesic, then $\frac{\nabla}{dt}\dot\gamma(t) = 0$. Therefore $\frac{d}{ds}E(\gamma_s(t))\big|_{s=0} = 0$.

Conversely, suppose $\frac{d}{ds}E(\gamma_s(t))\big|_{s=0} = 0$, for every proper variation. Let $f : [a, b] \to \mathbb{R}$ be a smooth function such that $f(t) > 0$ for $t \in (a, b)$ with $f(a) = f(b) = 0$, and set $V(t) = f(t)\frac{\nabla}{dt}\dot\gamma(t)$. Thus, $V$ is a smooth vector field along $\gamma$ with $V(a) = V(b) = 0$, which implies

$$\int_a^b f\left|\frac{\nabla}{dt}\dot\gamma(t)\right|^2 dt = 0.$$

Then we have that $\gamma$ is a geodesic. □

***Theorem 2.47*** **(Formula of the second variation of the energy of a curve)**
*Let $\gamma : [a, b] \to M$ be a geodesic. Assume $\gamma_s(t)$ a variation of $\gamma$ and $V = V(t)$ the variation vector field along $\gamma$. Let $E : (-\epsilon, \epsilon) \to \mathbb{R}$ be the energy function associated to $\gamma_s(t)$. Then*

$$\frac{1}{2}\frac{d^2}{dt^2}E(\gamma_s(t))\Big|_{s=0} = -\int_a^b \left\langle V, \frac{\nabla}{dt}\frac{\nabla}{dt}V + R\big(\dot\gamma(t), V\big)\dot\gamma(t)\right\rangle dt + \left\langle V, \frac{\nabla}{dt}V\right\rangle\Big|_a^b + \left\langle \frac{\nabla}{ds}V, \dot\gamma(t)\right\rangle\Big|_a^b \tag{3}$$

*Proof.* — From the proof of the first variation formula, we derive and using properties of the connection and Lemma 2.29, we have

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}E(\gamma_s(t)) &= \int_a^b \frac{d}{ds}\left\langle \frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right\rangle dt \\
&= \int_a^b \left\langle \frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right\rangle + \left\langle \frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{ds}\dot{\gamma}_s(t)\right\rangle dt \\
&= \int_a^b \left\langle \frac{\nabla}{dt}\frac{\nabla}{ds}\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right\rangle - \left\langle R\left(\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right)\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right\rangle + \left\langle \frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{ds}\dot{\gamma}_s(t)\right\rangle dt \\
&= \int_a^b \frac{d}{dt}\left\langle \frac{\nabla}{ds}\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right\rangle - \left\langle \frac{\nabla}{ds}\frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{dt}\dot{\gamma}_s(t)\right\rangle - \left\langle R\left(\dot{\gamma}_s(t), \frac{\partial\gamma_s(t)}{\partial s}\right)\dot{\gamma}_s(t), \frac{\partial\gamma_s(t)}{\partial s}\right\rangle \\
&\quad + \frac{d}{dt}\left\langle \frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{ds}\dot{\gamma}_s(t)\right\rangle - \left\langle \frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{dt}\frac{\nabla}{ds}\dot{\gamma}_s(t)\right\rangle dt \\
&= \left\langle \frac{\nabla}{ds}\frac{\partial\gamma_s(t)}{\partial s}, \dot{\gamma}_s(t)\right\rangle\Big|_a^b + \left\langle \frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{ds}\dot{\gamma}_s(t)\right\rangle\Big|_a^b - \int_a^b \left\langle \frac{\nabla}{ds}\frac{\partial\gamma_s(t)}{\partial s}, \frac{\nabla}{dt}\dot{\gamma}_s(t)\right\rangle \\
&\quad - \left\langle R\left(\dot{\gamma}_s(t), \frac{\partial\gamma_s(t)}{\partial s}\right)\dot{\gamma}_s(t), \frac{\partial\gamma_s(t)}{\partial s}\right\rangle - \left\langle \frac{\nabla}{dt}\frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}, \frac{\partial\gamma_s(t)}{\partial s}\right\rangle dt.
\end{aligned}$$

Taking $s = 0$, and since $\gamma(t)$ is geodesic, the third term vanishes, then we obtain (3). □

***Remark 2.48.*** — In particular, if the variation $\gamma_s(t)$ is proper, then $V(a) = V(b) = 0$ and we have

$$\frac{1}{2}\frac{d^2}{dt^2}E(\gamma_s(t))\Big|_{s=0} = -\int_a^b \left\langle V(t), \frac{\nabla}{dt}\frac{\nabla}{dt}V(t) + R\big(\dot{\gamma}(t), V(t)\big)\dot{\gamma}(t)\right\rangle dt$$

***Remark 2.49.*** — If the sectional curvature of $M$ is nonpositive, then the second variation is nonnegative for all proper normal variational fields $V$ and all normal nontrivial geodesics $\gamma$, that is, in the case that the variational field is normal to the curve (i.e. $V(t) \perp \dot{\gamma}(t)$ for all $t \in [a, b]$).

Now, suppose that $\gamma : [a, b] \to M$ is a geodesic and $f$ is a geodesic variation of $\gamma$, i.e., each $\gamma_s(\cdot) = f(\cdot, s)$ is a geodesic.

Let $J$ be its variation field. Then $\frac{\nabla}{dt}\big(\frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s}\big) = \frac{\nabla}{dt}\frac{\nabla}{dt}J$. Using Lemma 2.29, we have

$$\frac{\nabla}{dt}\frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial s} = \frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial\gamma_s(t)}{\partial t} + R\left(\frac{\partial\gamma_s(t)}{\partial s}, \frac{\partial\gamma_s(t)}{\partial t}\right)\frac{\partial\gamma_s(t)}{\partial t}.$$

Evaluating $s = 0$, we get

$$\frac{\nabla}{dt}\frac{\nabla}{dt}J(t) = \frac{\nabla}{ds}\frac{\nabla}{dt}\dot{\gamma}(t) + R\big(J(t), \dot{\gamma}(t)\big)\dot{\gamma}(t)$$

hence

$$\frac{\nabla}{dt}\frac{\nabla}{dt}J(t) + R\big(\dot{\gamma}(t), J(t)\big)\dot{\gamma}(t) = 0 \tag{4}$$

***Definition 2.50.*** — A vector field along a geodesic $\gamma$ is said to be a Jacobi field if it satisfies the so-called Jacobi equation (4) for all $t \in [a, b]$.

***Remark 2.51.*** — The above computation shows that the variational vector field of a geodesic variation is a Jacobi field.

A Jacobi field is determined by its initial conditions $J(0)$ and $\frac{\nabla}{dt}J(0)$. Indeed, let $e_1(t),\dots,e_n(t)$ be parallel, orthonormal fields along $\gamma$. We write $J(t)=\sum_i f_i(t)e_i(t)$, $i=1,\dots,n=dimM$. Then

$$\frac{\nabla}{dt}\frac{\nabla}{dt}J(t)=\sum_i \ddot{f}_i(t)e_i(t)$$

and,

$$\begin{aligned} R\big(\dot\gamma(t),J(t)\big)\dot\gamma &= \sum_j \big\langle R\big(\dot\gamma,J(t)\big)\dot\gamma,e_j\big\rangle e_j \\ &= \sum_{i,j} f_i(t)\big\langle R(\dot\gamma,e_i)\dot\gamma,e_j\big\rangle e_j \\ &= \sum_{i,j} f_i(t)a_{ij}e_j. \end{aligned}$$

Therefore, the Jacobi equation (4) is equivalent to the system

$$\ddot{f}_j(t)+\sum_i a_{ij}(t)f_i(t), \qquad j=1,\dots,n,$$

which is a linear system of second order. Hence, given the initial conditions $J(0)$ and $\frac{\nabla}{dt}J(0)$, there exists a $C^\infty$ solution of the system, defined on $[0,a]$. There exists, therefore, $2n$ linearly independent Jacobi fields along $\gamma$.

***Proposition 2.52.*** — *Let $\gamma:[0,a]\to M$ be a geodesic, and put $\gamma(0)=p$. Given $u,v\in T_{\gamma(0)}M$, there exists a unique Jacobi field $J$ such that $J(0)=u$ and $\frac{\nabla}{dt}J(0)=v$.*

***Proposition 2.53.*** — *Let $\gamma:[0,a]\to M$ be a geodesic and let $J$ be a Jacobi field along $\gamma$ with $J(0)=0$. Put $\frac{\nabla}{dt}J(0)=w$ and $\dot\gamma(0)=v$. Consider $w\in T_{av}(T_{\gamma(0)}M)$ and construct a curve $v(s)\in T_{\gamma(0)}M$ with $v(0)=av$ and $\dot v(0)=aw$. Put $f(t,s)=exp_p\big(\frac{t}{a}v(s)\big)$ with $p=\gamma(0)$ and define $\bar J$ by $\bar J(t)=\frac{\partial f(t,0)}{\partial s}$. Then $\bar J=J$ in $[0,a]$.*

*Proof.* — Notice that

$$\bar J(t)=\frac{\partial f(s,t)}{\partial t}\bigg|_{s=0}=(\mathrm{dexp}_p)_{tv}(tw),$$

then

$$\bar J(0)=(\mathrm{dexp}_p)_0(0)=0=J(0).$$

Furthermore,

$$\begin{aligned} \frac{\nabla}{dt}\frac{\partial f(s,t)}{\partial s}\bigg|_{s=0} &= \frac{\nabla}{dt}\big(d(\exp_p)_{tv}(tw)\big) \\ &= \frac{\nabla}{dt}\big(t(\exp_p)_{tv}(w)\big) \\ &= (\mathrm{dexp}_p)_{tv}(w)+t\frac{\nabla}{dt}\big((\mathrm{dexp}_p)_{tv}(w)\big) \end{aligned}$$

So, we get

$$\frac{\nabla}{dt}\bar J(0)=(\mathrm{dexp}_p)_0(w)=w=\frac{\nabla}{dt}J(0).$$

Since $J(0)=\bar J(0)=0$ and $\frac{\nabla}{dt}J(0)=\frac{\nabla}{dt}\bar J(0)=w$. Hence, from the uniqueness theorem, we conclude that $\bar J=J$. $\square$

***Corollary 2.54.*** — *Let $\gamma : [0,a] \to M$ be a geodesic. Then, a Jacobi field $J$ along $\gamma$ with $J(0) = 0$ is given by $J(t) = (dexp_p)_{t\dot\gamma(0)}\big(t\frac{\nabla}{dt}J(0)\big)$, $t \in [0,a]$.*

***Definition 2.55.*** — Let $\gamma : [0,a] \to M$ be a geodesic. The point $\gamma(t_0)$ is said to be conjugate to $\gamma(0)$ along $\gamma$, with $t_0 \in (0,a]$, if there exists a Jacobi field $J$ along $\gamma$, not identically zero, with $J(0) = 0 = J(t_0)$. The maximun number of such linearly independent fields is called the multiplicity of the conjugate point $\gamma(t_0)$.

***Remark 2.56.*** — If $dimM = n$, there exists exactly $n$ linearly independent Jacobi fields along the geodesic $\gamma$, which are zero at $\gamma(0)$.

***Proposition 2.57.*** — *Let $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p$. The point $q = \gamma(t_0), t_0 \in (0,a]$, is conjugate to $p$ along $\gamma$ if and only if $v_0 = t_0\dot\gamma(0)$ is a critical point of $exp_p$.*

*In addition, the multiplicity of $q$ as a conjugate point of $p$ is equal to the dimension of the kernel of the linear map $(dexp_p)_{v_0}$.*

*Proof.* — The point $q = \gamma(t_0)$ is a conjugate point of $p$ along $\gamma$ if and only if there exists a non-zero Jacobi field $J$ along $\gamma$ with $J(0) = 0 = J(t_0)$.

Let $v = \dot\gamma(0)$ and $w = \frac{\nabla}{dt}J(0)$.

From Corollary 2.54 we have that

$$J(t) = (\mathrm{dexp}_p)_{tv}(tw), t \in [0,a].$$

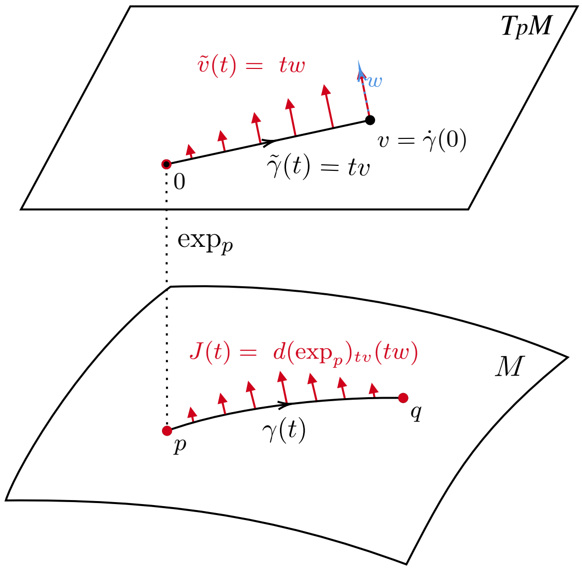
Note that $J(t) \neq 0 \iff w \neq 0$.

Therefore $q$ is conjugate to $p$ if and only if

$$0 = J(t_0) = (\mathrm{dexp}_p)_{t_0 v}(t_0 w), w \neq 0,$$

that is, if and only if $t_0 v$ is a critical point of $\exp_p$.

□




## 3. Bonnet-Myers theorem & Synge theorem

We now go into some applications of the formula for the second variation of energy. First, we consider manifolds with positive curvature. One of the most important facts about such manifolds is the following theorem, which was first proved in 1941 by Sumner B. Myers, building on the earlier work of Ossian Bonnet, Heinz Hopf, and John L. Synge.

Recall that every Riemannian manifold $M$ is assumed to be $n-$dimensional, connected and smooth.

***Theorem 3.1*** **(Bonnet-Myers).** — *Let $(M,g)$ be a complete Riemannian manifold with $Ric_p(v) \geq \frac{1}{r^2} > 0$, for all $p \in M$ and for all $v \in T_pM$, $|v| = 1$. Then $M$ is compact and $diamM \leq \pi r$.*

*Proof.* — $\boxed{\text{Strategy}}$ Assume the hypotheses. Let $p$ and $q$ any two points in $M$. Since $M$ is complete, there exists $\gamma : [0,1] \to M$ a minimizing geodesics joining $p$ and $q$, by Theorem 2.39 *(Hopf-Rinow)*. Let us parameterize it by arclength, so that $\gamma : [0,1] \to M$ and $|\dot\gamma(t)| = 1$. Notice that it is enough to show that $\ell(\gamma) \leq \pi r$, because $M$ would be bounded and complete, therefore compact. In addition, since $d(p,q) \leq \pi r$, for all $p,q \in M$, it follows that $\mathrm{diam}M = \sup_{p,q\in M} d(p,q) \leq \pi r$.

Since $\gamma$ is minimizing, we have a nonnegative second variation for all proper fields $V$ along $\gamma$. We will show that if $\ell(\gamma) > \pi r$, then the second variation above inequality would not hold.

Suppose then, on the contrary, that $\ell(\gamma) > \pi r$.

$\boxed{\text{Step 1}}$ Construction of vector fields along $\gamma$.
Let $\{e_i(t)\}$ be an orthonormal basis of $T_{\gamma(t)}M$, with each $e_i(t)$ parallel along $\gamma$, for all $t \in [0,1]$ *(orthonormal complement of* $\gamma$*)*, and so that $e_n(t) = \frac{\dot\gamma(t)}{\ell(\gamma)}$. Let $V_j$ be a vector field along $\gamma$ given by $V_j(t) = \sin(\pi t)e_j(t)$ with $j = 1, \ldots, n-1$. and $t \in [0,1]$. Thus, $V_j(0) = 0 = V_j(1)$. Then $V_j$ generates a proper variation of $\gamma$, whose energy we denote by $E_j$.

$\boxed{\text{Step 2}}$ Using the formula of the second variation of the energy.
We will use the second variation formula for the proper case *(Remark 2.48)*. So, considering that

$$\begin{aligned}\frac{\nabla}{dt}\sin(\pi t)e_j(t) &= \frac{\partial \sin(\pi t)}{\partial t}e_j(t) + \sin(\pi t)\nabla_{\frac{\partial\gamma}{\partial t}}e_j(t)\\ &= \pi\cos(\pi t)e_j(t),\end{aligned}$$

we have that

$$\begin{aligned}\frac{\nabla}{dt}\frac{\nabla}{dt}\sin(\pi t)e_j(t) &= \pi\frac{\partial \cos(\pi t)}{\partial t}e_j(t) + \pi\cos(\pi t)\nabla_{\frac{\partial\gamma}{\partial t}}e_j(t)\\ &= -\pi^2\sin(\pi t)e_j(t).\end{aligned}$$

And, since

$$R(\dot\gamma(t), V_j)\dot\gamma(t) = \sin(\pi t)\ell(\gamma)^2 R\big(e_n(t), e_j(t)\big)e_n(t)$$

hence

$$\big\langle R(\dot\gamma(t), V_j)\dot\gamma(t), V_j\big\rangle = \sin^2(\pi t)\ell(\gamma)^2 R\big(e_n(t), e_j(t), e_n(t), e_j(t)\big).$$

Putting everything together in the formula, we have

$$\frac{d^2}{dt^2}E_j(\gamma_s(t))\bigg|_{s=0} = \int_0^1 \sin^2(\pi t)\Big(\pi^2 - \ell(\gamma)^2 R\big(e_n(t), e_j(t), e_n(t), e_j(t)\big)\Big)dt$$

Summing on j, we get

$$\sum_{j=1}^{n-1}\frac{d^2}{dt^2}E_j(\gamma_s(t))\bigg|_{s=0} = \int_0^1 (n-1)\sin^2(\pi t)\Big(\pi^2 - \ell(\gamma)^2 Ric_\gamma\big(e_n(t)\big)\Big)dt.$$

Since $Ric_\gamma(e_n(t)) \geq \frac{1}{r^2}$ and $\ell(\gamma) > \pi r$, we have

$$\ell(\gamma)^2 Ric_\gamma(e_n(t)) > \pi^2,$$

then

$$\sum_{j=1}^{n-1}\frac{d^2}{dt^2}E_j(\gamma_s(t))\bigg|_{s=0} < \int_0^1 (n-1)\sin^2(\pi t)\left(\pi^2 - \pi^2\right)dt = 0.$$

$\boxed{\text{Step 3}}$ Conclusion: finding a contradiction.
Therefore, there exists a variation of $\gamma$ for which $\frac{d^2}{dt^2}E_j(\gamma_s(t))\big|_{s=0} < 0$. However, since $\gamma$ is a minimal geodesic, its length is smaller than or equal to that of any curve joining $p$ to $q$. Thus, for every variation of $\gamma$ we should have $\frac{d^2}{dt^2}E(\gamma_s(t))\big|_{s=0} \geq 0$. We obtain therefore a contradiction. Hence $\ell(\gamma) \leq \pi r$, which verifies the statement. $\square$

In the corollary that follows, we use some facts on the fundamental group and covering spaces [See **Appendix A: Covering spaces**].

***Corollary 3.2.*** — *Let $M$ be a complete Riemannian manifold with $Ric_p(v) \geq \epsilon > 0$, for all $p \in M$ and for all $v \in T_pM$. Then the fundamental group $\pi_1(M)$ is finite.*

*Proof.* — Let $\pi : \tilde{M} \to M$ be the universal covering of $M$, endowed with the pullback metric $\tilde{g} = \pi^* g$. Then $(\tilde{M}, \tilde{g})$ is also a complete Riemannian manifold whose Ricci curvature is bounded below by a positive number. By Theorem 3.1 *(Bonnet-Myers)*, we have $\tilde{M}$ compact. As a consequence, $\pi : \tilde{M} \to M$ has to be a finite covering (Lemma .9). So $\pi_1(M)$ is finite. □

The following application of the formula for the second variation was proved in 1936 by John L. Synge. To prove it, we need an important result, due essentially to A. Weinstein. But first, we need the following linear algebra fact [see Chap.9 sec.3 [**dC92**] ].

***Lemma 3.3.*** — *Let $A$ be an orthogonal linear transformation of $\mathbb{R}^{n-1}$ and suppose $det(A) = (-1)^n$. Then $A$ leaves invariant some non-zero vector of $\mathbb{R}^{n-1}$.*

*Proof.* — We have that $A$ is an orthogonal matrix of a $(n-1)$ dimensional vector space.

1. If $n$ is even, then $det(A) = 1$ and the dimension is odd.
   Thus, $A$ has $(n-1)$ eigenvalues of the form $-1$ or $1$, because $A$ is orthogonal. Since the product of complex eigenvalues of $A$ is non-negative, we have that $A$ must have an even number of eigenvalues $-1$ and an odd number of eigenvalues $1$, that is, at least one eigenvalue $1$. So $Av = v$ for some non-zero vector $v \in \mathbb{R}^{n-1}$.
2. If $n$ is odd, then $Det(A) = -1$.
   In this case, $A$ has an odd number of eigenvalues $-1$ and an odd number of eigenvalues $1$, that is, at least one eigenvalue $1$. So $Av = v$ for some non-zero vector $v \in \mathbb{R}^{n-1}$.

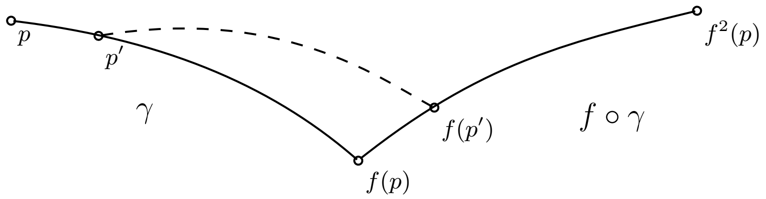
□

***Theorem 3.4* (Weinstein).** — *Let $f$ be an isometry of a compact oriented Riemannian manifold $M$ of dimension $n$. Suppose that $M$ has positive sectional curvature and that $f$ preserves the orientation of $M$ if $n$ is even, and reverses it if $n$ is odd. Then $f$ has a fixed point, i.e., there exists $p \in M$ with $f(p) = p$.*

*Proof.* — $\boxed{\text{Strategy}}$ Assume the hypotheses. Suppose, on the contrary, that $f$ has no fixed point.

Let $p \in M$ such that $d(p, f(p))$ attains a minimum in $M$, because $M$ is compact. Since $M$ is complete, then there exists a normalized minimizing geodesic $\gamma : [0, \ell] \to M$ joining $p$ to $f(p)$.
The idea is to build a variation $h$ of $\gamma$ and then use the second variation formula to obtain a contradiction to the fact that $\gamma$ minimizes the energy. A complication arises because we cannot build a proper variation, so our variation has to be carefully chosen so that, in particular, the boundary terms in the second variation formula vanish. We see that it will be useful to find a geodesic $\beta$ starting at p which is initially orthogonal to $\gamma$. This will then give us the direction in which to vary $\gamma$.

$\boxed{\text{Step 1}}$ Consider the geodesic $f \circ \gamma$ which joins $f(p)$ to $f^2(p)$. We want to show that $(f \dot{\circ} \gamma)(0) = \dot{\gamma}(\ell)$.
Let $p' = \gamma(t')$, $0 \neq t' \neq \ell$, and $f(p') = f \circ \gamma(t')$. Since $f$ is an isometry, then we have $d(p, p') = d\big(f(p), f(p')\big)$.




By triangle inequality,

$$\begin{aligned} d\big(p', f(p')\big) &\leq d\big(p', f(p)\big) + d\big(f(p), f(p')\big) \\ &= d\big(p', f(p)\big) + d(p, p') \\ &= d\big(p, f(p)\big), \end{aligned}$$

which is a minimum. So $d\big(p', f(p')\big) = d\big(p', f(p)\big) + d\big(f(p), f(p')\big)$. Therefore, the curve formed by $\gamma$ and $f \circ \gamma$ is a geodesic, because it is a minimizing curve. Hence $\dot{\gamma}(\ell) = (f \dot{\circ} \gamma)(0)$.

Step 2 Let $\tilde{A} = P_{\gamma,\gamma(\ell),\gamma(0)} \circ df_p : T_pM \to T_pM$ be an isometry, obtained as the composition of the differential of $f$ with $P$ the parallel backward transport along $\gamma$. We want to show that $\tilde{A}$ fixes $\dot{\gamma}(0)$.
Since $\dot{\gamma}$ is parallel along $\gamma$, and by the previous step we have

$$\tilde{A}(\dot{\gamma}(0)) = (P_{\gamma,\gamma(\ell),\gamma(0)} \circ df_p)(\dot{\gamma}(0)) = P_{\gamma,\gamma(\ell),\gamma(0)}\big(d(f \circ \gamma)(0)\big) = P_{\gamma,\gamma(\ell),\gamma(0)}(\dot{\gamma}(\ell)) = \dot{\gamma}(0).$$

Step 3 The restriction of $\tilde{A}$ to the orthogonal complement of $\dot{\gamma}(0)$ fixes a non-zero vector.
Consider the restriction of the isometry $\tilde{A}$ to the orthogonal complement of $\dot{\gamma}(0)$ by

$$A : \tilde{A}\big|_{\left(\dot{\gamma}(0)\right)^{\perp}} : \big(\dot{\gamma}(0)\big)^{\perp} \to \big(\dot{\gamma}(0)\big)^{\perp}.$$

Then $A$ is an orthogonal transformation on $\mathbb{R}^{n-1}$. We just need to make sure the determinant condition for Lemma 3.3 is satisfied.

Since $\tilde{A}(\dot{\gamma}(0)) = \dot{\gamma}(0)$, then $\dot{\gamma}(0)$ is an eigenvector of $\tilde{A}$ with eigenvalue 1, which implies that $det(A) = det(\tilde{A})$. Since $P$ is an orientation-preserving isometry by Proposition 2.18, then $det(P) = 1$, and since by hypothesis $f$ preserves or reverses the orientation according to the parity of $n$, we have $det(df_p) = (-1)^n$. Thus

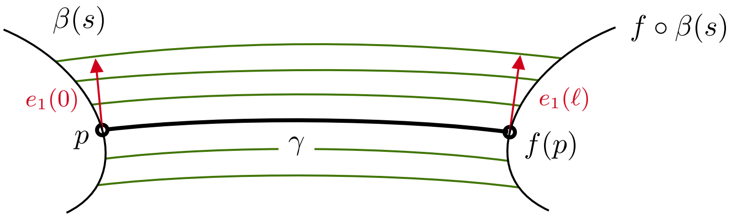
$$det(A) = det(\tilde{A}) = det(P) \cdot det(df_p) = (-1)^n.$$

From Lemma 3.3, we have that $A$ leaves a non-zero vector invariant.

Step 4 Let $e_1(t)$ be a unit parallel vector field along $\gamma$ such that $e_1(t)$ belongs to the orthogonal complement of $\dot{\gamma}(t)$, for each $t \in [0, \ell]$, and $e_1(0)$ is invariant by $A$.

Let $\beta(s)$ with $s \in (-\epsilon, \epsilon)$ be a geodesic with $\beta(0) = p$ and $\dot{\beta}(0) = e_1(0)$. Since $A(e_1(0)) = e_1(0)$, we have $df_p(e_1(0)) = e_1(\ell)$. So the geodesic $f \circ \beta$ is such that $f \circ \beta(0) = f(p)$ and $(f \dot{\circ} \beta)(0) = e_1(\ell)$.

Let $h$ be a variation of $\gamma$ given by $h(s, t) = \exp_{\gamma(t)}(se_1(t))$, $s \in (-\epsilon, \epsilon)$, $t \in [0, \ell]$. Note that, since $h(s, 0) = \beta(s)$, then $h(s, \ell) = \exp_{f(p)}(se_1(\ell)) = f \circ \beta(s)$.




Therefore

$$V(t) = \frac{\partial}{\partial s}\exp_{\gamma(t)}(se_1(t))\bigg|_{s=0} = e_1(t),$$

hence we have that $\frac{\nabla}{dt}V(t) = 0$ because $e_1(t)$ is parallel along $\gamma$. Now, applying the Second variation formula (3) we see that all of the boundary terms vanish, as $\beta$ and $f\circ\beta$ are geodesics and $\frac{\nabla}{dt}V(t) = 0$. In addition, the term involving $\frac{\nabla}{dt}\frac{\nabla}{dt}V(t) = 0$ also vanishes, and so we get

$$\begin{aligned}\frac{d^2}{dt^2}E(h)\bigg|_{s=0} &= -\int_0^\ell \langle V(t), R\big(\dot\gamma(t), V(t)\big)\dot\gamma(t)\rangle dt \\ &= -\int_0^\ell K\big(\dot\gamma(t), e_1(t)\big)dt\end{aligned}$$

Since $K > 0$, then $\frac{d^2}{dt^2}E(h)\Big|_{s=0} < 0$.

$\boxed{\text{Step 5}}$ Conclusion: finding a contradiction.

This shows that there exists a curve $c$ in the variation $h$, such that

$$\ell(c)^2 \leq \ell E(c) \leq \ell E(\gamma) \leq \ell(\gamma)^2.$$

Since the curves in the variation join some point $q = c(0)$ to $f(q) = c(\ell)$ so that $d\big(q, f(q)\big) < d\big(p, f(p)\big)$, therefore we obtain then a contradiction to the fact that $d(p, f(p))$ is a minimum. $\square$

In the next theorem, we use some facts of covering spaces [ See **Appendix: Covering spaces**].

***Theorem 3.5* (Synge).** — *Let $M$ be a compact Riemannian manifold with positive sectional curvature.*

1. *If $M$ is orientable and $n$ is even, then $M$ is simply connected.*
2. *If $n$ is odd, then $M$ is orientable.*

*Proof.* — $\boxed{\text{Strategy}}$ We will consider $\pi : \tilde M \to M$, the universal cover of $M$ for the first case, and the orientable double-cover for the second case, with a deck transformation $f \in \mathrm{Deck}(\pi)$. We want to use the Weinstein theorem to conclude that the deck transformation group is trivial. To be able to use it, we need to verify that $\tilde M$ is compact with positive sectional curvature and that $f$ is an isometry that preserves or reverses the orientation, respectively.

1. Suppose that $M$ is orientable and $n$ is even.

   $\boxed{\text{Step 1}}$ Claim: the manifold $\tilde M$ verifies the hypothesis of Weinstein's theorem.

   Let $\pi : \tilde M \to M$ be the universal covering of $M$ with $\pi^* g$ the induced metric for $\tilde M$ and the induced orientation in $\tilde M$. Because $M$ is compact and has positive sectional curvature $K > 0$, so the sectional curvature attains a minimum on the compact $M$, so $K \geq \epsilon > 0$. From the fact that $\pi$ is local isometry, the same curvature condition holds in $\tilde M$, that is, $\tilde K \geq \epsilon > 0$. Since $M$ is compact, then $M$ is complete as a metric space, and thus it is geodesically complete by Theorem 2.39 *(Hopf-Rinow)*.

Since $\tilde{M}$ is complete and by Remark 2.28, we can use Theorem 3.1 *(Bonnet-Myers)* to conclude that $\tilde{M}$ is compact.

$\boxed{\text{Step 2}}$ Claim: the deck transformation $f$ verifies the hypothesis of Weinstein's theorem.
Let $f : \tilde{M} \to \tilde{M}$ be a deck transformation of $\tilde{M}$, that is $\pi \circ f = f$.
We see that $f$ is an isometry of $\tilde{M}$. Indeed, since $\pi$ is a local isometry for any $q \in \tilde{M}$, we have

$$\begin{aligned}\langle u, v\rangle_q &= \langle d\pi_q u, d\pi_q v\rangle_{\pi(q)} \\ &= \left\langle d\pi_{f(q)}(df_q(u)), d\pi_{f(q)}(df_q(v))\right\rangle_{\pi(f(q))} \\ &= \left\langle df_q(u), df_q(v)\right\rangle_{f(q)}.\end{aligned}$$

Notice that, from the way that we oriented $\tilde{M}$, $f$ preserves orientation because locally the deck transformation has the form $(\pi|_{\tilde{U}_2})^{-1} \circ (\pi|_{\tilde{U}_1})$ for some open subsets $\tilde{U}_1, \tilde{U}_2 \subset \tilde{M}$ of an appropriate neighborhood $U \subset M$, and $\pi$ is itself orientation preserving because we used it to pullback the oriented atlas of $M$ to give $\tilde{M}$ its atlas and therefore its orientation.

$\boxed{\text{Step 3}}$ Conclusion.
Because $n$ is even, we can apply Theorem 3.4 *(Weinstein)* to conclude that $f$ has a fixed point, but a covering transformation that has a fixed point is the identity. So the deck transformation group of $\tilde{M}$, which is isomorphic to the fundamental group of $M$, is trivial. Therefore $M$ is simply connected.

2. Suppose $n$ is odd, and by way of contradiction, that $M$ is not orientable. Consider the orientable double-cover $\pi : \tilde{M} \to M$, and the induced metric $\pi^* g$.
   Let $f : \tilde{M} \to \tilde{M}$ be a nontrivial deck transformation which reverses orientation of $\tilde{M}$, which exists because $M$ is not orientable and $M \cong {}^{\tilde{M}}/_{deck(\pi)}$. Therefore, by Theorem 3.4 $f$ has a fixed point, that is, $f$ is the identity, which contradicts the fact that reverses orientation. We conclude then that $M$ is orientable.

□

## 4. Cartan-Hadamard theorem

Our major local-to-global theorem provides a complete topological characterization of complete, simply connected manifolds of nonpositive sectional curvature. This was proved in 1928 by Élie Cartan, generalizing earlier proofs for surfaces by Hans Carl Friedrich von Mangoldt and Jacques Hadamard.

Before proving it, we will introduce two lemmas. The first one, says that any non-positive curvature manifold has no conjugate points.

***Lemma 4.1***. — *Let $(M, g)$ be a complete Riemannian manifold with non-positive sectional curvature $K \leq 0$, then there are no conjugate points in any geodesics in $M$.*

*Proof.* — Let $p \in M$ and $J(t)$ be a non-zero Jacobi field along a geodesic $\gamma : [0, a] \to M$ with $\gamma(0) = p$, $J(0) = 0$ and $\frac{\nabla}{dt} J(0) \neq 0$. Note that

$$\frac{d}{dt}\langle J(t), J(t)\rangle = 2\langle \frac{\nabla}{dt} J(t), J(t)\rangle$$

then, from Jacobi's equation (4) and Definition 2.26 we have

$$\begin{aligned}\frac{d^2}{dt^2}\langle J(t), J(t)\rangle &= 2\langle \frac{\nabla}{dt}J(t), \frac{\nabla}{dt}J(t)\rangle + 2\langle \frac{\nabla}{dt}\frac{\nabla}{dt}J(t), J(t)\rangle \\ &= 2\left|\frac{\nabla}{dt}J(t)\right|^2 - 2\langle R(\dot\gamma(t), J(t))\dot\gamma(t), J(t)\rangle \\ &= 2\left|\frac{\nabla}{dt}J(t)\right|^2 \underbrace{-2K\big(\dot\gamma(t), J(t)\big)}_{\geq 0}|\dot\gamma(t)\wedge J(t)|^2,\end{aligned}$$

so $\frac{d^2}{dt^2}\langle J(t), J(t)\rangle \geq 0$, hence $\frac{d}{dt}\langle J(t), J(t)\rangle$ is increasing. Since $\frac{d}{dt}\langle J(0), J(0)\rangle = 0$, then we have $\frac{d}{dt}\langle J(t), J(t)\rangle \geq 0$ for all $t \in [0, a]$, and thus we deduce that $\langle J(t), J(t)\rangle$ is increasing. Now, since $J(0) = 0$ and $\frac{\nabla}{dt}J(0) \neq 0$, then $J(t) \neq 0$ for small $t > 0$. Otherwise, we can see that if exists $t_0 \in (0, a]$ such that $J(t_0) = 0$ it would imply that $J(t) = 0$ for all $t$, which contradicts the fact that $J(t)$ is non-zero. Therefore $J(t)$ has no zero along $\gamma$, and we may conclude that $p$ has no conjugate points along $\gamma$. □

We prove now the second useful lemma, which states that a geometric property of $f$ *(local isometry)*, implies a topological property of $f$ *(covering map)*.

***Lemma 4.2.*** — *Let $(\tilde M, \tilde g)$ and $(M, g)$ be two Riemannian manifolds. Assume $\tilde M$ complete and $f : \tilde M \to M$ a local isometry. Then $f$ is a covering map.*

*Proof.* — We will prove this in four steps.

$\boxed{\text{Step 1}}$ Claim: lift geodesics in $M$ to geodesics in $\tilde M$.

We say that $f$ has the path-lifting property for geodesics if for any $p \in M$, $\tilde p \in f^{-1}(p)$ and $\gamma$ be a geodesic with $\gamma(0) = p$, we have that exists a unique geodesic $\tilde\gamma$ starting at $\tilde p$ such that $f \circ \tilde\gamma = \gamma$.

$$\begin{tikzcd} & \tilde M \arrow[d, "f"] \\ I \arrow[ur, dotted, "\tilde\gamma"] \arrow[r, "\gamma"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & M \end{tikzcd}$$

We want to prove that $f$ has the path-lifting property. Given $p \in M, \tilde p \in f^{-1}(p)$ and a geodesic $\gamma$ with $\gamma(0) = p$. Consider $\dot\gamma(0) = v \in T_pM$. Let $\tilde v \in T_{\tilde p}\tilde M$ such that $v = df_{\tilde p}(\tilde v)$, which is well defined since $df_{\tilde p} : T_{\tilde p}\tilde M \to T_pM$ is a linear isomorphism.

Let $\tilde\gamma$ a geodesic in $\tilde M$ with $\tilde\gamma(0) = \tilde p$ and $\dot{\tilde\gamma}(0) = \tilde v$. Since $f$ is a local isometry, then $f \circ \tilde\gamma$ is a geodesic.

To prove that $f \circ \tilde\gamma = \gamma$ it suffices to check that verifies for initial points and velocity. Observe that

$$f(\tilde\gamma(0)) = f(\tilde p) = p = \gamma(0)$$

and

$$df_{\tilde p}(\dot{\tilde\gamma}(0)) = df_{\tilde p}(\tilde v) = v = \dot\gamma(0).$$

Thus, we have $f \circ \tilde\gamma = \gamma$.

$\boxed{\text{Step 2}}$ Claim: $\tilde M$ complete implies $M$ complete.

Since $\tilde M$ is complete, $\tilde\gamma$ is defined on all of $\mathbb{R}$. Since $f$ is a local isometry, maps geodesics to geodesics. So, we have that $f \circ \tilde\gamma = \gamma$ is defined on all of $\mathbb{R}$, therefore $M$ is complete.

$\boxed{\text{Step 3}}$ Claim: $f$ is surjective.

Let $\tilde{p} \in \tilde{M}$ and write $f(\tilde{p}) = p$. Let $q \in M$ be arbitrary. Since $M$ is complete, there exists a geodesic $\gamma : [0, b] \to M$ joining $p$ to $q$ by Theorem 2.39 *(Hopf-Rinow)*. Let $\tilde{\gamma}$ the lift of $\gamma$ starting at $\tilde{p}$. Thus,

$$f(\tilde{\gamma}(b)) = \gamma(b) = q.$$

So, there exist $\tilde{q} = \tilde{\gamma}(b) \in \tilde{M}$ such that $f(\tilde{q}) = q$.

$\boxed{\text{Step 4}}$ Claim: $f$ verifies covering properties. i.e., that any $p \in M$ has an open neighborhood $B_\epsilon(p)$ such that $f^{-1}(B_\epsilon(p))$ is a disjoint union of open subsets $B_\epsilon(\tilde{p}) \subset \tilde{M}$ which are homeomorphic to $B_\epsilon(p)$.

- Let any $p \in M$ and $\epsilon > 0$ such that $\exp_p : B_\epsilon(0_{T_pM}) \to B_\epsilon(p)$ is a diffeomorphism. We want to show first that for any $\tilde{p} \in f^{-1}(p)$, the following diagram commutes

$$\begin{tikzcd}
B_\epsilon(0_{T_{\tilde{p}}\tilde{M}}) \subset T_{\tilde{p}}\tilde{M} \arrow[r, "df_{\tilde{p}}"] \arrow[d, "exp_{\tilde{p}}"'] \arrow[dr, phantom, "\circlearrowright"] & T_pM \supset B_\epsilon(0_{T_pM}) \arrow[d, "exp_p"] \\
B_\epsilon(\tilde{p}) \subset \tilde{M} \arrow[r, "f"'] & M \supset B_\epsilon(p)
\end{tikzcd}$$

  Given $\tilde{v} \in B_\epsilon(0_{T_{\tilde{p}}\tilde{M}})$ and $\tilde{\gamma}$ a geodesic starting at $\tilde{p}$ with velocity $\tilde{v}$. We have $\exp_{\tilde{p}}(\tilde{v}) = \tilde{\gamma}(1)$ and then $f(\exp_{\tilde{p}}(\tilde{v})) = f(\tilde{\gamma}(1))$.

  Let $v = df_{\tilde{p}}(\tilde{v})$ and $\gamma$ be a geodesic starting at $p$ with velocity $v$. We thus have that $\exp_p(df_{\tilde{p}}(\tilde{v})) = \exp_p(v) = \gamma(1)$. But since $f$ is a local isometry, $\gamma = f \circ \tilde{\gamma}$ is geodesic and then we have

$$\exp_p(df_{\tilde{p}}(\tilde{v})) = \exp_p(v) = \gamma(1) = f(\tilde{\gamma}(1)) = f(\exp_{\tilde{p}}(\tilde{v})).$$

  Since $\tilde{v}$ was arbitrary, therefore

$$\exp_p \circ df_{\tilde{p}} = f \circ \exp_{\tilde{p}}$$

  We want to deduce now, that for any $\tilde{p} \in f^{-1}(p)$, $f : B_\epsilon(\tilde{p}) \to B_\epsilon(p)$ is a diffeomorphism. Since it is a local isometry, we already know that it is a local diffeomorphism and we only need to show that $f$ is injective and surjective.

  From the above diagram, we know that the right and top arrows are diffeomorphisms. This implies that the composition of the bottom and left arrow is a diffeomorphism, so we have that if $\exp_{\tilde{p}} : B_\epsilon(0_{T_{\tilde{p}}\tilde{M}}) \to B_\epsilon(\tilde{p})$ is surjective, then $f$ is injective. So, we want to show that for all $\tilde{q} \in B_\epsilon(\tilde{p})$, there exists $\tilde{v} \in B_\epsilon(0_{T_{\tilde{p}}\tilde{M}})$ such that $\exp_{\tilde{p}}(\tilde{v}) = \tilde{q}$.

  Let $\tilde{\gamma} : [0, 1] \to \tilde{M}$ be a minimizing geodesic with $\tilde{\gamma}(0) = \tilde{p}$ and $\gamma(1) = \tilde{q}$. Consider $\tilde{v} = \dot{\tilde{\gamma}}(0)$ so that we need to verify that $|\tilde{v}| < \epsilon$.

  Let's consider the assignment for length $t \mapsto |\dot{\tilde{\gamma}}(t)|$ and energy, $t \mapsto |\dot{\tilde{\gamma}}(t)|^2$, so that

$$\frac{d}{dt}\langle \dot{\tilde{\gamma}}(t), \dot{\tilde{\gamma}}(t) \rangle = 2\langle \frac{\nabla}{dt}\dot{\tilde{\gamma}}(t), \dot{\tilde{\gamma}}(t) \rangle = 0,$$

  and the fact that $\tilde{\gamma}$ is a geodesic from $\tilde{p}$ to $\tilde{q} \in B_\epsilon(\tilde{p})$, imply that

$$\epsilon > \ell\left(\dot{\tilde{\gamma}}(t)\Big|_{[0,1]}\right) = \int_0^1 |\dot{\tilde{\gamma}}(t)| dt = \int_0^1 |\dot{\tilde{\gamma}}(0)| dt = |\dot{\tilde{\gamma}}(0)| = |\tilde{v}|.$$

  We can conclude that $\exp_{\tilde{p}}$ is surjective, then $f$ is injective. Consequently, as $f$ is already surjective and a local diffeomorphism, this implies that it is a diffeomorphism.

- •• Claim: for any $p \in M$, $\{\tilde{p}_\alpha\}_{\alpha\in A} \in f^{-1}(p)$ we have $f^{-1}(B_\epsilon(p)) = \bigcup_{\alpha\in A} B_\epsilon(\tilde{p}_\alpha)$.

  Since we proved that $f : B_\epsilon(\tilde{p}) \to B_\epsilon(p)$ is a diffeomorphism. It suffices to show that $f^{-1}(B_\epsilon(p)) \subset \bigcup_{\alpha\in A} B_\epsilon(\tilde{p}_\alpha)$.

  Consider $\tilde{q} \in f^{-1}(B_\epsilon(p))$ and write $q = f(\tilde{q})$. This means $q \in B_\epsilon(p)$. Because $M$ is complete, Theorem 2.39 *(Hopf-Rinow)* tells us that there is a minimizing geodesic $\gamma$ joining $q$ and $p$, with $l = d_g(p, q) < \epsilon$. Let $\tilde{\gamma}$ be the lift of $\gamma$ staring at $\tilde{q}$. Thus, $f(\tilde{\gamma}(l) = \gamma(l) = p$. So exists an $\alpha \in A$ such that $\tilde{\gamma}(l) = \tilde{p}_\alpha$. Since $f$ is a local isometry, $\ell(\gamma) = \ell(\tilde{\gamma}) < \epsilon$. So $\tilde{q} \in B_\epsilon(\tilde{p}_\alpha)$. Then we have $f^{-1}(B_\epsilon(p)) \subset \bigcup_{\alpha\in A} B_\epsilon(\tilde{p}_\alpha)$.

- • • • Claim: for any $p \in M$, let $\tilde{p}_\alpha$ and $\tilde{p}_\beta$ be two distinct points in $f^{-1}(p)$. We have that $B_\epsilon(\tilde{p}_\alpha)$ and $B_\epsilon(\tilde{p}_\beta)$ are disjoints.

  Suppose, on the contrary, that there exists $\tilde{q} \in B_\epsilon(\tilde{p}_\alpha) \cap B_\epsilon(\tilde{p}_\beta)$ with $\tilde{p}_\alpha \neq \tilde{p}_\beta$. Let $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}_\beta$ be minimal geodesics starting from $\tilde{p}_\alpha$ and $\tilde{p}_\beta$ to $\tilde{q}$, respectively. It follows that $f \circ \tilde{\gamma}_\alpha$ and $f \circ \tilde{\gamma}_\beta$ are minimal geodesics in $M$, both starting from $p$ to $f(\tilde{q})$. Since $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}_\beta$ are lifts of $f \circ \tilde{\gamma}_\alpha$ and $f \circ \tilde{\gamma}_\beta$ from the same initial point $p$, we then deduce that $\tilde{p}_\alpha = \tilde{p}_\beta$ by uniqueness of the lift. This contradicts the fact that $\tilde{p}_\alpha \neq \tilde{p}_\beta$.

We have just shown that $f^{-1}(B_\epsilon(p))$ is equal to the disjoint union of $B_\epsilon(\tilde{p})$ for $\tilde{p} \in f^{-1}(p)$, which are all diffeomorphic to $B_\epsilon(p)$. Therefore $f$ is a covering map. □

***Remark 4.3.*** — We can see that this lemma fails if we don't assume $(\tilde{M}, \tilde{g})$ to be complete.

Now, we are ready to prove

***Theorem 4.4* (Cartan-Hadamard).** — *Let $(M, g)$ be a connected, complete $n-$dimensional Riemannian manifold with non-positive sectional curvature $K \leq 0$. Then $exp_p : T_pM \to M$ is a covering map. In particular, if $M$ is simply connected, then $M$ is diffeomorphic to $\mathbb{R}^n$.*

*Proof.* — We will prove this in four steps. Note first that for a given $p \in M$. Since $M$ is complete, then $\exp_p$ is defined on all of $T_pM$.

Step 1 Claim: $\exp_p : T_pM \to M$ is a local diffeomorphism.
By Lemma 4.1, there are no conjugate points in any geodesics in $M$. Then, $d\exp_p$ has trivial kernel and is, therefore, an isomorphism. Therefore $\exp_p$ is a local diffeomorphism.

Step 2 Claim: $\exp_p : T_pM \to M$ is a local isometry.
Define a Riemannian metric $\tilde{g}$ on $T_pM$ given by $\tilde{g} = (\exp_p)^* g$. This turns $\exp_p : (T_pM, \tilde{g}) \to (M, g)$ into a local isometry, because we have for $v, w \in T_0(T_pM)$, identifying $T_0(T_pM) \cong T_pM$, that

$$\langle v, w\rangle_{\tilde{g}} = \langle d\exp_p(v), d\exp_p(w)\rangle_g$$

where we can see that $\tilde{g}$ is bilinear and symmetric. The non-degeneracy then comes from the fact that $d\exp_p$ is injective.

Step 3 Claim: $(T_pM, \tilde{g})$ is complete.
Since $\exp_p$ is a local isometry, the geodesics through the origin of $T_pM$ are precisely the straight lines going through zero, indeed, by definition of the exponential map, they are mapped to the geodesics $\gamma(t) = \exp_p(tv)$ under $\exp_p$, and preimages of geodesics under local isometries are again geodesics. Hence $(T_pM, \tilde{g})$ is geodesically complete at 0. By Theorem 2.39 *(Hopf-Rinow)* we have that $(T_pM, \tilde{g})$ is then complete.

$\boxed{\text{Step 4}}$ Conclusion.
We use Lemma 4.2 and we are done! Moreover, if $M$ is simply connected with $\exp_p : T_pM \to M$ a cover, then $\exp_p$ is a global homeomorphism. Since $\exp_p$ is a local diffeomorphism, it must be a global diffeomorphism. Thus, since $T_pM \cong \mathbb{R}^n$ we have that $M$ is diffeomorphic to $\mathbb{R}^n$. $\square$

Because of this theorem, a complete, simply connected Riemannian manifold with nonpositive sectional curvature is called a Cartan–Hadamard manifold. The basic examples are Euclidean and hyperbolic spaces.

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**Appendix: Covering spaces** [For further information and proofs, please refer to [**Mas77**]]

***Definition .1.*** — Suppose $\tilde{X}$ and $X$ are connected smooth manifolds. A map $\pi : \tilde{X} \to X$ is called a smooth covering map if

- $\pi$ is continuous and surjective, and
- each point $p \in X$ has a neighborhood $U$ that is evenly covered by $\pi$, meaning that $U$ is connected and each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto $U$ by $\pi$.

$X$ is called the base of the covering, and $\tilde{X}$ is called a covering space of $X$.

***Definition .2.*** — Let $(M, g)$ and $(\tilde{M}, \tilde{g})$ be two Riemannian manifolds. A map $\pi : \tilde{M} \to M$ is a Riemannian covering if it is a smooth covering map and a local isometry.

***Definition .3.*** — Let $\tilde{M}$ and $M$ be smooth manifolds, and let $\pi : \tilde{M} \to M$ be a covering map. A covering transformation *(or deck transformation)* of $\pi$ is a homeomorphism $\varphi : \tilde{M} \to \tilde{M}$ such that $\pi \circ \varphi = \pi$. i.e., the following diagram commutes

$$\begin{tikzcd} \tilde{M} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{M} \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

***Definition .4.*** — The set $A(\tilde{M}, p)$ of all covering transformation, called the covering group of $\pi$, is a group under the composition, acting on $\tilde{M}$ on the left.

***Lemma .5.*** — *Let $\varphi_0$ and $\varphi_1$ be homomorphisms of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$. If there exists any point $x \in \tilde{X}_1$ such that $\varphi_0(x) = \varphi_1(x)$, then $\varphi_0 = \varphi_1$.*

***Corollary .6.*** — *The group $A(\tilde{X}, p)$ operates without fixed points on the space $\tilde{X}$. i.e., if $\varphi \in A(\tilde{X}, p)$ and $\varphi \neq 1$, then $\varphi$ has no fixed points.*

***Definition .7.*** — A covering $\pi : \tilde{X} \to X$ is called a universal covering if $\tilde{X}$ is simply connected.

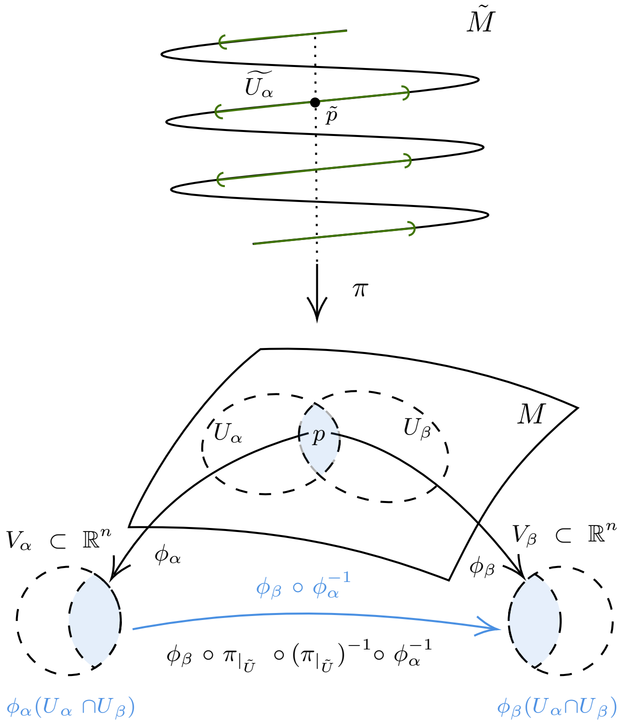
***Corollary .8.*** — *Let $(\tilde{X}, p)$ be a universal covering space of $X$. Then, $A(\tilde{X}, p)$ is isomorphic to the fundamental group of $X$, denoted by $\pi_1(X)$, and the order of the group $\pi_1(X)$ is equal to the number of sheets of the covering space $(\tilde{X}, p)$.*

Notice that if $\tilde{X}$ is compact, because points of X are closed, then $\pi^{-1}(x)$ is closed, and so compact (and discrete!), so finite. Because $|\pi^{-1}(x)| = |\pi_1(X)|$, we state

***Lemma .9.*** — *Let $\pi : \tilde{X} \to X$ the universal covering. If $\tilde{X}$ is compact, then $\pi_1(X)$ is finite.*

***Lemma .10***. — *Let $\pi : \tilde{M} \to M$ be the universal covering of a connected, oriented Riemannian manifold $M$. Then $\pi$ is orientation preserving.*

*Proof.* — Since $M$ is orientable, following the Definition 2.16 we have that $det(\phi_\beta \circ \phi_\alpha^{-1}) > 0$. Then, it follows that $det(\phi_\beta \circ \pi|_{\tilde{U}} \circ (\pi|_{\tilde{U}})^{-1} \circ \phi_\alpha^{-1}) > 0$.

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