COCYCLES AND LOCAL RIGIDITY FOR PARTIALLY HYPERBOLIC SYSTEMS

Lecture notes

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1. Partially hyperbolic automorphisms ans diffeomorphisms

1.1. Partially hyperbolic toral automorphisms.

Let $L \in SL(d, \mathbb{Z})$. The action of a matrix L on $\mathbb{R}^d$ induces an automorphism of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, which we denote by the same letter.

L is called *hyperbolic* or *Anosov* if the matrix has no eigenvalues of modulus 1.

L is called *partially hyperbolic* if some but not all eigenvalues of L have modulus 1. Then L has eigenvalues of modulus > 1 and of modulus < 1.

An example: (a hyperbolic automorphism of $\mathbb{T}^{d_1}$) $\times$ (identity on T^{d_2}), such as $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $\mathbb{T}^2$ $\times$ (Id on $\mathbb{T}$). Such products are reducible.

An automorphism L is *irreducible* if the map $L : \mathbb{R}^n \to \mathbb{R}^n$ has no nontrivial rational invariant subspace, equivalently, if its characteristic polynomial $p(L)$ is irreducible over $\mathbb{Q}$. The eigenvalues of an irreducible L are always simple. However, distinct eigenvalues may have the same absolute value.

For an irreducible PH automorphism L, we need $n \geq 4$.

An example (Pugh-Shub '69): $L = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$.

An automorphism L is ergodic with respect to the Lebesgue measure m on $\mathbb{T}^d \iff$ L is mixing w.r. to $m \iff$ no eigenvalue of L is a root of unity.

1.2. Partially hyperbolic and Anosov diffeomorphisms.

Let $\mathcal{M}$ be a compact connected smooth manifold.
A diffeomorphism f of $\mathcal{M}$ is *partially hyperbolic* if there exist a Df-invariant splitting of the tangent bundle $T\mathcal{M} = E^s \oplus E^c \oplus E^u$ with non-trivial E^s and E^u such that for some Riemannian metric on $\mathcal{M}$ and continuous functions $\nu < 1$, $\hat{\nu} < 1$, γ, $\hat{\gamma}$,

$$\text{(PH)} \qquad \|Df(\mathbf{v}^s)\| < \nu(x) < \gamma(x) < \|Df(\mathbf{v}^c)\| < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \|Df(\mathbf{v}^u)\|.$$

for all unit vectors $\mathbf{v}^s \in E^s(x)$, $\mathbf{v}^c \in E^c(x)$, and $\mathbf{v}^u \in E^u(x)$.

If E^c is trivial, f is called *hyperbolic* or *Anosov*.

The sub-bundles E^s, E^u, and E^c are called, *stable, unstable,* and *center.* They depend Hölder continuously on the base point. E^s and E^u are tangent to the stable and unstable foliations W^s and W^u, respectively. The leaves of W^s and W^u are as smooth as f.

The sub-bundles E^c, $E^{cu} = E^u \oplus E^c$, and $E^{cs} = E^s \oplus E^c$ are not necessarily integrable, i.e. tangent to foliations.

If f is C^1 close to a partially hyperbolic automorphism L, then f is *dynamically coherent* by [HPS77], that is, the bundles E^c, E^{cu}, and E^{cs} are tangent to foliations W^c, W^{cu}, and W^{cs} with C^r leaves, where $r > 1$ satisfies $\nu < \gamma^r$ and $\hat{\nu} < \hat{\gamma}^r$.

An su-path in $\mathcal{M}$ is a concatenation of finitely many (C^1) subpaths each lying entirely in a single leaf of W^s or W^u. A partially hyperbolic diffeomorphism f is called *accessible* if any two points in $\mathcal{M}$ can be connected by an su-path.

For nearby x and y, this path is not necessarily short. However, there exist constants k and ℓ depending on $\mathcal{M}$ and f such that for any x and y there exists an su-path consisting of at most k subpaths, each of length at most ℓ [W13].

A partially hyperbolic toral automorphism L is not accessible since E^s and E^u are jointly integrable, i.e., $E^s \oplus E^u$ is tangent to a foliation W^{su}, and hence any su-path starting at x remains in the leaf $W^{su}(x)$.

We say that f is volume-preserving if it preserves a probability measure μ in the measure class of the volume induced by a Riemannian metric. It was proven in [BW10] that if f is C^2, accessible, and center bunched, then f is ergodic with respect to such μ. The diffeomorphism f is called *center bunched* if the functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ can be chosen so that $\nu < \gamma\hat{\gamma}$ and $\hat{\nu} < \gamma\hat{\gamma}$. It follows that

$$\|Df|_{E^c}\| \cdot \|(Df|_{E^c})^{-1}\| \cdot \nu < \hat{\gamma}^{-1}\gamma^{-1}\nu < 1 \quad \text{and} \quad \|Df|_{E^c}\| \cdot \|(Df|_{E^c})^{-1}\| \cdot \hat{\nu} < 1,$$

which means that non-conformality of f on E^c is dominated by the contraction on E^s and expansion on E^u. In particular, f is center bunched if E^c is one dimensional or if Df is conformal on E^c.

Hyperbolic diffeomorphisms are trivially center bunched, and accessible by the local product structure of stable and unstable manifolds.

The *local stable manifold* $W^s_{\mathrm{loc}}(x)$ is a ball of a fixed small radius ρ around x in the intrinsic metric of W^s. Local unstable manifolds are defined similarly. We choose ρ sufficiently small so that $\|D_y f\| < \nu(x)$ for all $W^s_{\mathrm{loc}}(x)$ and so that $W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(z)$ consists of a single point for any sufficiently close $x, z \in \mathcal{M}$. The second property gives the *local product structure* of the stable and unstable manifolds.

We will consider

• A C^2 PH diffeomorphism f that is volume-preserving, accessible and center bunched, or

• a C^2 or $C^{1+\text{Hölder}}$ hyperbolic (Anosov) diffeomorphism f.

2. Linear cocycles over partially hyperbolic diffeomorphisms

2.1. Linear cocycles.

Let f be a PH or hyperbolic diffeomorphism of a compact connected manifold $\mathcal{M}$.

Let $P : \mathcal{E} \to \mathcal{M}$ be a finite dimensional vector bundle over $\mathcal{M}$, for example, $T\mathcal{M}$, E^s, E^u, E^c. A *linear cocycle* over f is an automorphism of $\mathcal{E}$ that projects to f, that is, a homeomorphism $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ such that $P \circ \mathcal{A} = f \circ P$ and for each $x \in \mathcal{M}$ the map $\mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{fx}$ between the fibers is a linear isomorphism.

We use the following notations for the iterates of $\mathcal{A}$: $\mathcal{A}_x^0 = \text{Id}$, and for $n \in \mathbb{N}$,
$\mathcal{A}_x^n = \mathcal{A}_{f^{n-1}x} \circ \cdots \circ \mathcal{A}_{fx} \circ \mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{f^n x}$ and $\mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1} : \mathcal{E}_x \to \mathcal{E}_{f^{-n}x}$.
Clearly, $\mathcal{A}$ satisfies the *cocycle equation* $\mathcal{A}_x^{n+k} = \mathcal{A}_{f^k x}^n \circ \mathcal{A}_x^k$.

In the case of a trivial vector bundle $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, any linear cocycle $\mathcal{A}$ can be identified with a matrix-valued function $A : \mathcal{M} \to GL(d, \mathbb{R})$ given by $A(x) = \mathcal{A}_x$. We call such $\mathcal{A}$ a $GL(d, \mathbb{R})$-valued cocycle.

Definition 2.1. *Let $A : \mathcal{M} \to GL(d, \mathbb{R})$ be continuous. The $GL(d, \mathbb{R})$-valued cocycle over f generated by A is the map $\mathcal{A} : \mathcal{M} \times \mathbb{Z} \to GL(d, \mathbb{R})$ defined by*
$\mathcal{A}(x, 0) = \mathcal{A}_x^0 = Id$, and for $n \in \mathbb{N}$,

$$\mathcal{A}(x, n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) \quad \textit{and} \quad \mathcal{A}(x, -n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

More generally, instead of $GL(d, \mathbb{R})$ we can consider cocycles with values in any topological group G with a complete metric.

For us, the primary example of a linear cocycle is the differential Df viewed as an automorphism of the tangent bundle $T\mathcal{M}$ or its restriction to a Df-invariant sub-bundle $\mathcal{E}' \subset T\mathcal{M}$, such as the stable or unstable sub-bundle. In these examples,

$$\mathcal{A}_x = D_x f \;\text{ or }\; \mathcal{A}_x = Df|_{\mathcal{E}'(x)} \quad \text{and} \quad \mathcal{A}_x^n = D_x f^n \;\text{ or }\; \mathcal{A}_x^n = Df^n|_{\mathcal{E}'(x)}.$$

Since the stable and unstable sub-bundles are Hölder continuous but usually not more regular, we will assume that the vector bundle $\mathcal{E}$ is Hölder continuous and we will consider linear cocycles in the Hölder category. Also, Hölder continuity is needed to obtain results, even for scalar cocycles.

We fix a Hölder continuous Riemannian metric on $\mathcal{E}$. A linear cocycle $\mathcal{A}$ is called *Hölder continuous* if the fiber maps $\mathcal{A}_x$ depend Hölder continuously on x, i.e. there exist K, $\beta > 0$ such that for all nearby $x, y \in \mathcal{M}$

$$d(\mathcal{A}_x, \mathcal{A}_y) = \|\mathcal{A}_x - \mathcal{A}_y\| + \|(\mathcal{A}_x)^{-1} - (\mathcal{A}_y)^{-1}\| \leq K \cdot \text{dist}(x, y)^\beta,$$

where $\mathcal{A}_x$ and $\mathcal{A}_y$ are viewed as matrices using local coordinates and $\|.\|$ is the operator norm.

2.2. **Fiber bunching and holonomies.**

In the study of cocycles, comparing their iterates along exponentially converging orbits, such as $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$, where $y \in W^s(x)$ plays an important role. For linear cocycles, the *fiber bunching* condition, ensures their convergence.

Definition 2.2. *A cocycle $\mathcal{A}$ cocycle is* conformal *if $\|\mathcal{A}_x\| \cdot \|(\mathcal{A}_x)^{-1}\| = 1$ for all x, i.e., $\|\mathcal{A}_x u\| = \|\mathcal{A}_x v\| = \alpha(x)$ for all unit vectors $u, v \in \mathcal{E}_x$.*
The cocycle is uniformly quasiconformal *if there exists a constant K such that*

$$\|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| = \frac{\max\{\, \| \mathcal{A}_x^n(v) \|: \|v\|=1 \,\}}{\min\{\, \| \mathcal{A}_x^n(v) \|: \|v\|=1 \,\}} \le K \quad \textit{for all } x \in X \textit{ and } n \in \mathbb{Z}.$$

$\mathcal{A}_x^n$ maps the unit sphere to an ellipsoid, and $\|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\|$ is the ratio of its largest and smallest semi-axes.

Definition 2.3. *A β-Hölder linear cocycle $\mathcal{A}$ over a PH diffeomorphism f is* s fiber bunched *is there exists $\theta < 1$ such that*

$$\text{(FB1)} \qquad \|\mathcal{A}_x\| \cdot \|(\mathcal{A}_x)^{-1}\| \cdot \nu(x)^\beta < \theta \qquad \textit{for all } x \in \mathcal{M},$$

or more generally if there exist $\theta < 1$ and L such that

$$\text{(FB2)} \qquad \|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < L\,\theta^n \textit{ for all } x \in \mathcal{M} \textit{ and } n \in \mathbb{N},$$

where ν is such that $\|Df(\mathbf{v}^s)\| < \nu(x)$ and $\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x)$.
u fiber bunching is defined similarly using $\hat{\nu}$: $\|\mathcal{A}_x^{-n}\| \cdot \|\mathcal{A}_{f^{-n}x}^n\| \cdot (\hat{\nu}_x^{-n})^\beta < L\,\theta^n$.
$\mathcal{A}$ is fiber bunched *if it is both s and u fiber bunched.*

Clearly, conformal and uniformly quasiconformal cocycles are fiber bunched, and so are cocycles that are sufficiently close to conformal.

An important role in the study of cocycles is played by holonomies. In the context of linear cocycles, this notion was introduced in [BV04, V08] and further studied in [AV10, ASV13, KS13, S15, KS16]. Existence of holonomies was proved in [AV10, ASV13] under the stronger assumption (FB1) and under (FB2) in [S15], bundle setting was considered in [KS13], and uniqueness/non-uniqueness of holonomies (as maps satisfying properties H1-H4 below) was considered in [KS16]. For a fiber bunched linear cocycle $\mathcal{A}$, a holonomy can be obtained as the limit of the products $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$. Estimates for such products have been studied for various types of group-valued cocycles whose growth is slower than the contraction/expansion in the base (see e.g. [NT, PW01, LW10]).

Proposition 2.4. (Existence and properties of holonomies)
Let $\mathcal{A}$ be a β-Hölder linear cocycle over f. If $\mathcal{A}$ is s fiber bunched, then for every $x \in \mathcal{M}$ and $y \in W^s(x)$ the limit

$$\text{(Hol)} \qquad H_{x,y} = H_{x,y}^{\mathcal{A},s} = \lim_{n\to\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n,$$

called a stable holonomy of $\mathcal{A}$, *exists and satisfies*

(H1) *$H_{x,y}$ is an invertible linear map from $\mathcal{E}_x$ to $\mathcal{E}_y$;*

(H2) *$H_{x,x} = Id$ and $H_{y,z} \circ H_{x,y} = H_{x,z}$, which implies $(H_{x,y})^{-1} = H_{y,x}$;*

(H3) $H_{x,y} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y} \circ \mathcal{A}_x^n$ *for all* $n \in \mathbb{N}$*;*

(H4) $\|H_{x,y} - Id\| \le c\, dist(x,y)^\beta$, *where* c *is independent of* x *and* $y \in W^s_{loc}(x)$.

The map $H^{\mathcal{A},s} : (x,y) \mapsto H^{\mathcal{A},s}_{x,y}$, *where* $x \in \mathcal{M}$ *and* $y \in W^s_{loc}(x)$, *is continuous.*

The unstable holonomy $H^{\mathcal{A},u}$ for u fiber bunched $\mathcal{A}$ is defined similarly:

$$H^{\mathcal{A},u}_{x,y} = \lim_{n\to\infty} (\mathcal{A}_y^{-n})^{-1} \circ \mathcal{A}_x^{-n} \quad \text{for } y \in W^u(x).$$

Proof. We outline a proof under the "one-step" fiber bunching assumption (FB1).

Let $x \in X$ and $y \in W^s_{\text{loc}}(x)$. The key step is to show that the sequence $((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n)$ is Cauchy. Denoting $x_i = f^i(x)$ and $y_i = f^i(y)$, we obtain

$$\begin{aligned}
&(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = (\mathcal{A}_y^{n-1})^{-1} \circ \left((\mathcal{A}_{y_{n-1}})^{-1} \circ \mathcal{A}_{x_{n-1}}\right) \circ \mathcal{A}_x^{n-1} = \\
&= (\mathcal{A}_y^{n-1})^{-1} \circ (\mathrm{Id} + r_{n-1}) \circ \mathcal{A}_x^{n-1} = (\mathcal{A}_y^{n-1})^{-1} \circ \mathcal{A}_x^{n-1} + (\mathcal{A}_y^{n-1})^{-1} \circ r_{n-1} \circ \mathcal{A}_x^{n-1} = \\
&= \cdots = \mathrm{Id} + \sum_{i=0}^{n-1} (\mathcal{A}_y^i)^{-1} \circ r_i \circ \mathcal{A}_x^i, \quad \text{where } r_i = (\mathcal{A}_{y_i})^{-1} \circ \mathcal{A}_{x_i} - \mathrm{Id}.
\end{aligned}$$

Since $\mathcal{A}$ is β-Hölder continuous and $y \in W^s_{\text{loc}}(x)$, for each $i \in \mathbb{N}$ we have

$$\|r_i\| \le \|(\mathcal{A}_{y_i})^{-1}\| \cdot \|\mathcal{A}_{x_i} - \mathcal{A}_{y_i}\| \le c_1 \operatorname{dist}(x_i, y_i)^\beta \le c_1 (\operatorname{dist}(x,y)\, \nu_y^i)^\beta$$

Using (FB1) and exponential convergence of $\operatorname{dist}(x_i, y_i)$ to 0, we show that for all $i \ge 0$,

$$\|(\mathcal{A}_y^i)^{-1}\| \cdot \|\mathcal{A}_x^i\| \le c_2\, \theta^i (\nu_y^i)^{-\beta}.$$

and so

$$\|(\mathcal{A}_y^i)^{-1} \circ r_i \circ \mathcal{A}_x^i\| \le c_3 \operatorname{dist}(x,y)^\beta\, \theta^i.$$

Therefore, for every $n \in \mathbb{N}$,

$$\|\mathrm{Id} - (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n\| \le \sum_{i=0}^{n-1} \|(\mathcal{A}_y^i)^{-1} \circ r_i \circ \mathcal{A}_x^i\| \le c_3 \operatorname{dist}(x,y)^\beta \sum_{i=0}^{n-1} \theta^i \le c \operatorname{dist}(x,y)^\beta.$$

Also, since

$$\|(\mathcal{A}_y^{n+1})^{-1} \circ \mathcal{A}_x^{n+1} - (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n\| = \|(\mathcal{A}_y^n)^{-1} \circ r_n \circ \mathcal{A}_x^n\| \le c_3 \operatorname{dist}(x,y)^\beta\, \theta^n,$$

the sequence $\{(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n\}$ is Cauchy and hence it has a limit $H^{\mathcal{A},s}_{x,y} \in GL(d,\mathbb{R})$. Since the convergence is uniform on the set of pairs (x,y) where $y \in W^s_{\text{loc}}(x)$, the map $H^{\mathcal{A},s}$ is continuous. It is easy to see that the map $H^{\mathcal{A},s}_{x,y}$ satisfies (H2, H3, H4). It can be extended to any $y \in W^s(x)$ using (H3). $\square$

3. Cohomology of $GL(d,\mathbb{R})$-valued cocycles

A survey of results and techniques for linear cocycles over hyperbolic systems is given in [S].

Assumptions.

(AH) f is a C^2 (or $C^{1+\text{Hölder}}$) hyperbolic diffeomorphism, and μ is an ergodic f-invariant measure μ with full support and local product structure, for example the measure of maximal entropy, or the invariant volume if it exists.

A measure μ has *local product structure* if it is locally equivalent to the product of its conditional measures on $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(x)$.

(APH) f is an accessible and center bunched partially hyperbolic C^2 diffeomorphism preserving a volume μ.

- $\mathcal{A}$ and $\mathcal{B}$ are β-Hölder continuous $GL(d,\mathbb{R})$-valued cocycles over f as in (AH) or (APH).

Cohomology is a natural notion of equivalence for cocycles.

Definition 3.1. *Cocycles $\mathcal{A}$ and $\mathcal{B}$ over f are (measurably, continuously)* cohomologous *if there exists a (measurable, continuous) function $C : X \to GL(d,\mathbb{R})$ such that*
$$\mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1} \quad \textit{(almost everywhere / for all } x \in X.)$$
The function C is called a conjugacy *between $\mathcal{A}$ and $\mathcal{B}$, or a* transfer map.

The map $C(x)$ can be viewed as a coordinate change in the fiber $\mathcal{E}_x$.

Similarly, linear cocycles are cohomologous if for (every, a.e.) $x \in \mathcal{M}$ there is an invertible linear map $C_x : \mathcal{E}_x \to \mathcal{E}_x$ such that $\mathcal{A}_x = C_{fx} \circ \mathcal{B}_x \circ C_x^{-1}$.

Question. *Is every measurable conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ continuous, that is, does it coincide with a continuous conjugacy on a set of full measure?*

For cocycles over hyperbolic systems, this question has been extensively studied starting with the work of Livšic [Li71, Li72].

- The answer is Yes for *scalar* cocycles over hyperbolic f as in (AH), moreover, C is β-Hölder [Li71].
- Also, Yes for for *scalar* cocycles over PH f as in (APH), moreover, C is β'-Hölder for some β' [W13]. This is a rare example when Hölder continuity is proven in PH case.

Unlike $\mathbb{R}$, the group $GL(d,\mathbb{R})$ with $d > 1$ is not abelian. This case is more complicated, and the answer is not always positive.

- In [PW01] Pollicott and Walkden gave an example of $GL(2,\mathbb{R})$-valued cocycles over a hyperbolic automorphism f that are measurably but not continuously cohomologous. Note that $f(0) = 0$. Let $\epsilon > 0$, $A(x) = \begin{bmatrix} \alpha(x) & \epsilon \\ 0 & 1 \end{bmatrix}$ and $B(x) = \begin{bmatrix} \alpha(x) & 0 \\ 0 & 1 \end{bmatrix}$, where $\alpha(0) = 1$ and $1 - \epsilon < \alpha(x) < 1$ for $x \neq 0$. Since the matrices $A(x_0)$ and $B(x_0)$ are not conjugate, the cocycles $\mathcal{A}$ and $\mathcal{B}$ are not continuously cohomologous. A measurable conjugacy is constructed in the form $C(x) = \begin{bmatrix} 1 & c(x) \\ 0 & 1 \end{bmatrix}$. Both A and B can be made arbitrarily close to the identity, and hence the cocycles are fiber bunched.

Thus fiber bunching of both cocycles does not guarantee continuity of a measurable conjugacy between them. Continuity holds under a stronger assumption that one of the cocycles is uniformly quasiconformal.

Theorem 3.2. (H) [S15] *Let f and μ be as in (AH). If $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ is uniformly quasiconformal, then any μ-measurable conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ is Hölder continuous.*

The fiber bunching assumption on $\mathcal{A}$ can be removed in the hyperbolic case: Butler deduced it from the other assumptions of Theorem 3.2 in [Bt18].

Theorem 3.3. (PH) [KS16] *Let f and μ be as in (APH). If $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ is uniformly quasiconformal, then any μ-measurable conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ is continuous.*

Outline of the proof of Theorems 3.2 and 3.3. Since $\mathcal{A}$ and $\mathcal{B}$ are fiber bunched, they have holonomies $H^{\mathcal{A},s}_{x,y} = \lim_{n\to\infty} (\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x$, $H^{\mathcal{A},u}$, $H^{\mathcal{B},s}$, and $H^{\mathcal{B},u}$.

The key step is to show that C *intertwines the holonomies* of $\mathcal{A}$ and $\mathcal{B}$ on a set of full measure, that is, there exists a set $Y \subset \mathcal{M}$ with $\mu(Y) = 1$ such that

$$H^{\mathcal{A},s}_{x,y} = C(y) \circ H^{\mathcal{B},s}_{x,y} \circ C(x)^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^s(x), \tag{3.1}$$

and similarly for the unstable holonomies.

To prove (3.1) we consider $x \in \mathcal{M}$ and $y \in W^s(x)$. Iterating forward, we reduce to the case of $y \in W^s_{\text{loc}}(x)$. Since $\mathcal{A}(x) = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}$, we have

$$\begin{aligned} (\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x \;&=\; C(y) \circ (\mathcal{B}^n_y)^{-1} \circ C(f^n y)^{-1} \circ C(f^n x) \circ \mathcal{B}^n_x \circ C(x)^{-1} = \\ &= C(y) \circ (\mathcal{B}^n_y)^{-1} \circ (\text{Id} + r_n) \circ \mathcal{B}^n_x \circ C(x)^{-1} = \\ &= C(y) \circ (\mathcal{B}^n_y)^{-1} \circ \mathcal{B}^n_x \circ C(x)^{-1} + C(y) \circ (\mathcal{B}^n_y)^{-1} \circ r_n \circ \mathcal{B}^n_x \circ C(x)^{-1}. \end{aligned} \tag{3.2}$$

where

$$\|r_n\| = \|C(f^n y)^{-1} \circ C(f^n x) - \text{Id}\| \le \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\|.$$

Since C is μ-measurable, by Lusin's theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ such that C is uniformly continuous on S and hence $\|C\|$ and $\|C^{-1}\|$ are bounded on S. Let Y be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem $\mu(Y) = 1$. If x and y are in Y, there exists a sequence $\{n_i\}$ such that $f^{n_i}x$ and $f^{n_i}y$ are in S for all i. It follows that $\|r_{n_i}\| \to 0$ as $i \to \infty$.

The product

$$\|(\mathcal{B}^n_y)^{-1}\| \cdot \|\mathcal{B}^n_x\| = \|(\mathcal{B}^n_y)^{-1}\| \cdot \|H^{\mathcal{B},s}_{f^n y, f_n x} \circ \mathcal{B}^n_y \circ H^{\mathcal{B},s}_{x,y}\| \le \|(\mathcal{B}^n_y)^{-1}\| \cdot \|\mathcal{B}^n_y\| \cdot \|H^{\mathcal{B},s}_{f^n y, f_n x}\| \cdot \|H^{\mathcal{B},s}_{x,y}\|$$

is uniformly bounded since the cocycle $\mathcal{B}$ is uniformly quasiconformal and the norms of the holonomies are uniformly bounded over x and $y \in W^s_{\text{loc}}(x)$.

Thus for every x and y in Y such that $y \in W^s_{\text{loc}}(x)$, the last term in (3.2) tends to 0 along a subsequence, and (3.1) follows.

In the hyperbolic case the argument is completed as follows. Equation (3.1) gives $C(y) = H^{\mathcal{A},s}_{x,y} \circ C(x) \circ H^{\mathcal{B},s}_{x,y}$ and then Hölder continuity of the holonomies along W^s yields essential Hölder continuity of C along W^s_{loc}, that is,

$$d(C(y), C(x)) \le k\, \text{dist}(x,y)^\beta \quad \text{for all } x,y \in Y \text{ such that } y \in W^s_{\text{loc}}(x), \tag{3.3}$$

where constant k depends on $C(x)$. Similarly, we obtain essential Hölder continuity of C along W^u_{loc}. We consider a small open set

$$U = W^s_{\text{loc}}(x_0) \times W^u_{\text{loc}}(x_0) \overset{\text{def}}{=} \{W^u_{\text{loc}}(x) \cap W^s_{\text{loc}}(y) \mid x \in W^s_{\text{loc}}(x_0),\ y \in W^u_{\text{loc}}(x_0)\}.$$

Since μ has local product structure and full support, for almost all local stable leaves in U, the set of points of Y on the leaf has full conditional measure, and the conditional measures on almost all leaves have full support. Denote by Y_U the set of points in $U \cap Y$ on such leaves. Then by the local product structure of μ, for any two points x and z in Y_U there exists a point $y \in W^s_{\text{loc}}(x) \cap Y_U$ such that $W^u_{\text{loc}}(y) \cap W^s_{\text{loc}}(z)$ is also in Y_U. Then the inequality (3.3) can be used to show that C is uniformly bounded in $GL(d,\mathbb{R})$ on Y_U, and then, by covering $\mathcal{M}$ by such neighborhoods U, on a set of full measure $\tilde{Y} \subseteq Y$. Thus the constant k in (3.3) can be chosen uniform on $\tilde{Y}$. Now using again (3.3) and the local product structure of the stable and unstable manifolds we obtain that

$$d(C(x), C(z)) \le c\, \text{dist}(x,z)^\beta \text{ for all } x, z \text{ in a set of full measure } \tilde{Y}.$$

The set $\bar{Y} = \bigcap_{n=-\infty}^{\infty} f^n(\tilde{Y})$ is f-invariant and is dense in $\mathcal{M}$ since μ has full support and $\mu(\bar{Y}) = 1$. So we can extend C from $\bar{Y}$ and obtain a Hölder continuous conjugacy on $\mathcal{M}$.

In the partially hyperbolic case the local product structure is replaced by accessibility. In this case, continuity of C on $\mathcal{M}$ follows from the intertwining of holonomies using [ASV13, Theorem D]. □

4. Continuity of a measurable conjugacy to a constant cocycle

In our recent work [KSW] we establish Hölder continuity of a measurable conjugacy for perturbations of constant cocycles, which we then use in our local rigidity results.

Theorem 4.1. (H) [KSW]
Let f and μ be as in (AH) and let $\mathcal{A}$ is be a constant $GL(d,\mathbb{R})$-valued cocycle over f. Then for any β-Hölder continuous $GL(d,\mathbb{R})$-valued cocycle $\mathcal{B}$ sufficiently C^0 close to $\mathcal{A}$, any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is β'-Hölder for some $\beta' > 0$.

Ideas of the proof. Let A and B be the generators of $\mathcal{A}$ and $\mathcal{B}$.
Let $\rho_1 < \dots < \rho_\ell$ be the distinct moduli of the eigenvalues of A and let

$$\mathbb{R}^d = E^1 \oplus \dots \oplus E^\ell \tag{4.1}$$

be the corresponding A-invariant splitting, where E^i is the sum of generalized eigenspaces of A corresponding to eigenvalues of modulus ρ_i. Using an adapted norm on $\mathbb{R}^d$ we have

$$(\rho_i - \epsilon)^n \le \|\mathcal{A}_i^n u\| \le (\rho_i + \epsilon)^n \quad \text{for any unit vector } u \in E^i, \ \text{ where } \mathcal{A}_i = \mathcal{A}|_{E^i}.$$

If B is sufficiently C^0 close to A, then $\mathcal{B}$ has a Hölder continuous invariant splitting $\mathbb{R}^d = \mathcal{E}^1_x \oplus \cdots \oplus \mathcal{E}^\ell_x$ close to (4.1) so that the restrictions $\mathcal{B}_i = \mathcal{B}|_{\mathcal{E}^i}$ satisfy

$$(\rho_i - 2\epsilon)^n \le \|\mathcal{B}_i^n u\| \le (\rho_i + 2\epsilon)^n \quad \text{for any unit vector } u \in \mathcal{E}^i.$$

Hence all restrictions $\mathcal{B}_i$ are fiber bunched if ϵ is sufficiently small.

Let C be a measurable conjugacy satisfying $\mathcal{B}_x = C(fx)\,\mathcal{A}_x\,C(x)^{-1}$.

First we show that $C_x(E^i) = \mathcal{E}^i_x$ for μ a.e. x, and hence $C_i = C|_{E^i}$ is a measurable conjugacy between fiber bunched cocycles $\mathcal{A}_i$ and $\mathcal{B}_i$. The main step is to show that each C_i is Hölder for all $i = 1, \ldots, \ell$, and hence so is C.

If $\mathcal{A}$ is *diagonalizable over* $\mathbb{C}$, then each $\mathcal{A}_i$ is uniformly quasiconformal (conjugate to a conformal matrix). So in this case continuity of C_i follows from Theorem 3.2.

In the general case one has to deal with non-conformality due to Jordan blocks, and the proof becomes much more complicated. We use a more general result for cocycles with one Lyapunov exponent, which we discuss later. □

4.1. **An application to local rigidity of hyperbolic systems.**

You may see [KSW22] *for a brief survey of local rigidity results.*

By the *structural stability* of hyperbolic systems, if L is a hyperbolic automorphism of $\mathbb{T}^d$ (or more generally a hyperbolic diffeomorphism of $\mathcal{M}$) and f is a diffeomorphism sufficiently C^1 close to L, then f is hyperbolic and topologically conjugate to f. The latter means that there exists a homeomorphism h of $\mathbb{T}^d$, called a *conjugacy*, such that

$$L \circ h = h \circ f, \qquad \text{that is,} \qquad f = h^{-1} \circ L \circ h.$$

Such h is unique in a C^0 neighborhood of the identity. A conjugacy h is always bi-Hölder, but it is usually not even C^1. The problem of establishing smoothness of the conjugacy h from some weaker assumptions is often referred to as *local rigidity,* in the sense that weak equivalence of L and f implies strong equivalence. In the next theorem we show that weak differentiability of h implies its smoothness.

Theorem 4.2. (H) [KSW] *Let L be a hyperbolic automorphism of $\mathbb{T}^d$ and let f be a $C^{1+Hölder}$ diffeomorphism C^1 close to L. Suppose that the conjugacy h between f and L is Lipschitz, or more generally in the Sobolev space $W^{1,q}(\mathbb{T}^d)$ of L^q functions with L^q weak partial derivatives of first order, with $q > d$. Then h is a $C^{1+Hölder}$ diffeomorphism.*

Outline of the proof. Formally differentiating $L \circ h = h \circ f$ we obtain

$$L \circ Dh = Dh \circ Df.$$

We use the assumption $h \in W^{1,q}$ with $q > d$ to show that its Jacoby matrix of partial derivatives is invertible and gives the differential of h for Lebesgue almost every point of $\mathbb{T}^N$, and that f preserves a volume. Then the equation above yields that $C = Dh$ is a measurable conjugacy between L and Df, as linear cocycles over f. Then Dh is Hölder continuous by Theorem 4.1, and so h is a $C^{1+\text{Hölder}}$ diffeomorphism.

Remark. We also obtain estimates of $\|h - \mathrm{Id}\|_{C^{1+\beta}}$ using the estimate for the conjugacy between cocycles in Theorem 4.1. This plays an important role in establishing higher regularity of h. Using different techniques, we then show that if L is irreducible (or more generally weakly irreducible), f is C^∞ and $C^{r(L)}$ close to L, and h is as in Theorem 4.2, then h is C^∞. While Theorem 4.2 holds for *any* L, higher regularity of h does not hold in general for reducible L [dlL92].

5. Conformal structures

5.1. **Basics.** A conformal structure on $\mathbb{R}^d$, $d \geq 2$, is a class of proportional inner products. The space $\mathcal{C}^d$ of conformal structures on $\mathbb{R}^d$ can be identified with the space of real symmetric positive definite $d \times d$ matrices with determinant 1.

For a $GL(d, \mathbb{R})$-valued cocycle $\mathcal{A}$ over $f : \mathcal{M} \to \mathcal{M}$ an *invariant conformal structure* is a function $\tau : \mathcal{M} \to \mathcal{C}^d$ such that

$$\mathcal{A}_x(\tau(x)) = \tau(fx) \quad \text{for all } x \in \mathcal{M}.$$

This means that the linear map $\mathcal{A}_x$ maps the family of homothetic ellipsoids in E_x corresponding to $\tau(x)$ to the family of ellipsoids in $\mathcal{E}_{fx}$ corresponding to $\tau(fx)$. Invariant conformal structures are defined similarly for linear cocycles on a vector bundle $\mathcal{E}$. A cocycle $\mathcal{A}$ is conformal with respect to a Riemannian metric on $\mathcal{E}$ if and only if it preserves the conformal structure associated with this metric.

5.2. **Continuity of a measurable invariant conformal structure.**

Theorem 5.1. (H) [KS13] *Let f and μ be as in (AH), and let a $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder fiber bunched linear cocycle over f. Then any $\mathcal{A}$-invariant μ-measurable conformal structure τ on $\mathcal{E}$ is β-Hölder continuous, i.e., coincides with an $\mathcal{A}$-invariant β-Hölder continuous conformal structure on a set of full measure.*

Theorem 5.2. (PH) [KS13] *Let f and μ be as in (APH), and let a $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder fiber bunched linear cocycle over f. Then any $\mathcal{A}$-invariant μ-measurable conformal structure τ on $\mathcal{E}$ is continuous.*

Recall that fiber bunched cocycles have holonomies, and that $H^s_{x,y}$ is a linear map from $\mathcal{E}_x$ to $\mathcal{E}_y$, where $y \in W^s(x)$.

The key step in the proofs of Theorems 5.1 and 5.2 is showing that a μ-measurable $\mathcal{A}$-invariant conformal structure τ is essentially invariant under the stable and unstable holonomies of $\mathcal{A}$. That is, there is a set $Y \subseteq \mathcal{M}$ of full measure such that

$$\tau(y) = H^s_{x,y}(\tau(x)) \;\text{ for all } x, y \in Y \;\text{ such that } y \in W^s_{loc}(x),$$

and similarly for the unstable holonomies.
The proof shares many features with that of Theorem 3.2.

6. Cocycles with one Lyapunov exponent

6.1. **The Lyapunov exponents of a cocycle.**

The following theorem applies to continuous $GL(d, \mathbb{R})$-valued cocycles over f as in (AH) or (APH).

Theorem 6.1 (Oseledets Multiplicative Ergodic Theorem (MET))**.** [O68] *Let f be an invertible ergodic measure-preserving transformation of a probability space (X, μ). Let $\mathcal{A}$ be a measurable $GL(d, \mathbb{R})$-valued cocycle over f such that $\log \|\mathcal{A}_x\|$ and $\log \|\mathcal{A}_x^{-1}\|$ are in $L^1(X, \mu)$.*

Then there exist numbers $\lambda_1 < \cdots < \lambda_\ell$, an f-invariant set Λ with $\mu(\Lambda) = 1$, and an $\mathcal{A}$-invariant decomposition

$$\mathcal{E}_x = \mathbb{R}^d = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^\ell \quad \textit{for } x \in \Lambda$$

such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\mathcal{A}_x^n v\| = \lambda_i \quad \textit{for any } i = 1, \ldots, \ell \textit{ and any } 0 \neq v \in \mathcal{E}_x^i.$$

The numbers $\lambda_1, \ldots, \lambda_\ell$ are called the *Lyapunov exponents* of $\mathcal{A}$ with respect to μ.

The exponents and the decomposition depend on the choice of μ. We also note that

$$\lambda_\ell(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}_x^n\| \;\text{ and }\; \lambda_1(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|(\mathcal{A}_x^n)^{-1}\|^{-1} \;\text{ for } \mu\text{-a.e. } x \in \mathcal{M}.$$

These limits exist and are constant μ a.e. by the Subadditive Ergodic Theorem.

Now we consider linear cocycles with *one Lyapunov exponent* with respect to μ, i.e., with $\ell = 1$.

Clearly, a uniformly quasiconformal cocycle has one exponent with respect to any measure. Having one exponent with respect to every measure does not imply uniform quasiconformality, consider for example a constant cocycle $A(x) \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Fiber bunched cocycles do not necessarily have one exponent: consider a constant cocycle $A(x) \equiv \text{diag}(1, 1 + \epsilon)$ with a sufficiently small $\epsilon > 0$.

Cocycles over with one exponent with respect to the volume (or just one invariant measure) are not necessarily fiber bunched. For example, let $A(x) = \text{diag}(a_1(x), a_2(x))$ over a hyperbolic L, where we take $a_1(0) = 1$ and $a_2(0) = N$ large enough to violate the fiber bunching condition at fixed point 0. For any measure $\mu \neq \delta_0$ we can choose functions a_1 and a_2 so that $\int_{\mathbb{T}^d} \log a_1(x) d\mu = \int_{\mathbb{T}^d} \log a_2(x) d\mu$, and hence $\mathcal{A}$ has one exponent with respect to μ.

However, cocycles over H or PH f with one exponent with respect to *each* invariant measure are fiber bunched, as in this case $\|\mathcal{A}_x^n\| \|(\mathcal{A}_x^n)^{-1}\|$ grows sub-exponentially [KS13].

When $\mathcal{A}$ has more than one Lyapunov exponent, the invariant sub-bundles $\mathcal{E}^i$ given the Oseledets MET are measurable but not necessarily continuous. The next two theorems establish continuity of measurable invariant sub-bundles for fiber bunched cocycles with one exponent.

Theorem 6.2. (H) ([KS13] *applying* [AV10]) *Let f and μ be as in (AH) and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder fiber bunched linear cocycle over f. If $\mathcal{A}$ has one exponent with respect to μ, then any μ-measurable $\mathcal{A}$-invariant sub-bundle of $\mathcal{E}$ is β-Hölder.*

Theorem 6.3. (PH) ([KS13] *applying* [ASV13]) *Let f and μ be as in (APH) and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder fiber bunched linear cocycle over f. If $\mathcal{A}$ has one exponent with respect to μ, then any μ-measurable $\mathcal{A}$-invariant sub-bundle of $\mathcal{E}$ is continuous.*

Again, continuity follows from invariance under holonomies. We work with the Grassman manifold $\mathcal{G}_x$ of all d'-dimensional subspaces in $\mathcal{E}_x$, the corresponding fiber bundle $\mathcal{G}$ over $\mathcal{M}$, the cocycle induced by $\mathcal{A}$ on $\mathcal{G}$ and its holonomies.

Using the results on continuity of measurable conformal structures and subbundles, we obtain a structure theorem for cocycles with one exponent.

Theorem 6.4. (PH, H)[KS13] *Let f and μ be as in (APH) or (AH), and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a fiber bunched β-Hölder continuous linear cocycle over f with one exponent with respect to μ. Then there exists a finite cover $\tilde{\mathcal{A}} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of $\mathcal{A}$ and $N \in \mathbb{N}$ such that $\tilde{\mathcal{A}}^N$ satisfies the following property. There exist a flag of continuous $\tilde{\mathcal{A}}^N$-invariant sub-bundles*

$$\{0\} = \tilde{\mathcal{E}}^0 \subset \tilde{\mathcal{E}}^1 \subset \cdots \subset \tilde{\mathcal{E}}^{k-1} \subset \tilde{\mathcal{E}}^k = \tilde{\mathcal{E}}, \tag{6.1}$$

a continuous conformal structure on $\tilde{\mathcal{E}}^1$ invariant under $\tilde{\mathcal{A}}^N$, and continuous conformal structures on the factor bundles $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$, $i = 2, \dots, k$, invariant under the factor-cocycles induced by $\tilde{\mathcal{A}}^N$.

(H) *In the hyperbolic case, for f and μ are as in (AH), the bundles and structures are β-Hölder continuous, moreover, if $\mathcal{A}$ has one exponent with respect to* each *ergodic f-invariant measure, then passing to a finite cover and power of $\mathcal{A}$ is not needed.*

Outline of the proof of Theorem 6.4 for the basic hyperbolic case. We trivialize the bundle $\mathcal{E}$ on a set of full measure, that is, we measurably identify $\mathcal{E}$ with $\mathcal{M} \times \mathbb{R}^d$ and view $\mathcal{A}$ as a $GL(d, \mathbb{R})$-valued cocycle. Zimmer's Amenable Reduction gives that $\mathcal{A}$ is measurably cohomologous to a cocycle $\mathcal{B}$ with values in an amenable subgroup G of $GL(d, \mathbb{R})$. Such G has a finite index subgroup contained in a conjugate of a group of block-triangular matrices of the form

$$\begin{bmatrix} B_1 & * & \dots & * \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & B_k \end{bmatrix}$$

where each diagonal block B_i is a scalar multiple of a $d_i \times d_i$ orthogonal matrix.

We give an argument for the simple case when G itself is contained in one of these subgroups. Then the sub-bundle V^i spanned by the first $d_1 + \dots + d_i$ coordinate vectors in $\mathbb{R}^d$ is $\mathcal{B}$-invariant for $i = 1, \dots, k$. Let C be a measurable conjugacy satisfying $\mathcal{B}(x) = C^{-1}(fx)\, \mathcal{A}(x)\, C(x)$. Denoting $\mathcal{E}^i_x = C(x)V^i$ we obtain the corresponding flag of measurable $\mathcal{A}$-invariant sub-bundles

$$\mathcal{E}^1 \subset \mathcal{E}^2 \subset \cdots \subset \mathcal{E}^k = \mathcal{E} \quad \text{with} \quad \dim \mathcal{E}^i = d_1 + \cdots + d_i.$$

Theorem 6.3 gives β-Hölder continuity of the sub-bundles $\mathcal{E}^i$. Since $B_1(x)$ is a conformal matrix for μ-a.e. x, the push forward by C of the standard conformal structure on V^1 is invariant under the restriction of $\mathcal{A}$ to $\mathcal{E}^1$ and hence β-Hölder continuous by Theorem 5.2.

Similarly, we consider the factor-bundles $\mathcal{E}^i/\mathcal{E}^{i-1}$ over $\mathcal{M}$ with the natural induced cocycle $\mathcal{A}^{(i)}$. Since the matrix of the map induced by B on $V^i/V^{i-1} = \mathbb{R}^{d_i}$ is B_i, it preserves the standard conformal structure on $\mathbb{R}^{d_i}$. Pushing it forward by C we obtain a measurable conformal structure τ_i on $\mathcal{E}^i/\mathcal{E}^{i-1}$ invariant under $\mathcal{A}^{(i)}$. The holonomies $H^{\mathcal{A},s}$ and $H^{\mathcal{A},u}$ induce holonomies for $\mathcal{A}^{(i)}$ on $\mathcal{E}^i/\mathcal{E}^{i-1}$. Then τ_i is essentially invariant under these holonomies and hence is also β-Hölder continuous on $\mathcal{M}$. $\square$

Recall that the example from [PW01] shows that two fiber bunched cocycles can be measurably but not continuously cohomologous. Note that the cocycles in the example have two exponents: 0 and $\int_{\mathcal{M}} \log \alpha(x) d\mu < 0$ with respect to the volume. The following result extends Theorem 3.2 to the optimal generality when cocycle $\mathcal{A}$ has one Lyapunov exponent instead of being uniformly quasiconformal. This result also completes the proof of Theorem 4.1 in the general (non-diagonalizable) case, as the theorem applies to any constant cocycle $\mathcal{A}$ with all eigenvalues of the same modulus.

Theorem 6.5. (H) [KSW] *Let f and μ be as in (AH) and let $\mathcal{A}$ and $\mathcal{B}$ be β-Hölder linear cocycles over f. Suppose that $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has one Lyapunov exponent for each ergodic f-invariant measure. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is β-Hölder continuous.*

We use Theorem 6.4 in the proof of Theorem 6.5.

7. Local rigidity of Lyapunov spectrum for toral automorphisms

7.1. Hyperbolic case.

Recall that if L is hyperbolic automorphism of $\mathbb{T}^d$ and a diffeomorphism f is sufficiently C^1 close to L, then f is hyperbolic and topologically conjugate to f. A conjugacy h is always bi-Hölder, but it is usually not even C^1 (see Section 4.1). Various conditions that ensure C^1 conjugacy have been studied. One of the weakest is the assumption that f and L have the same Lyapunov exponents with respect to the volume, which is clearly necessary. Somewhat surprisingly, it is sufficient for a broad class of L. We note that this rigidity result relies on the fact that volume is the measure of maximal entropy for L, and it does not hold in general when L is replaced by a non-linear system.

Theorem 7.1. (H) [GKS20] *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus and there are no pairs of eigenvalues $\lambda, -\lambda$ or $i\lambda, -i\lambda$, where λ is real. Let f be a volume-preserving C^2 diffeomorphism of $\mathbb{T}^d$ sufficiently C^1-close to L. If the Lyapunov exponents of f with respect to the volume are the same as the Lyapunov exponents of L, then f is $C^{1+\text{Hölder}}$ conjugate to L.*

The irreducibility assumption is needed since for reducible L the conjugacy may not be smooth [dlL92]. The assumptions on the eigenvalues are used in the proof, but may not be necessary. The theorem extends the earlier result by Saghin and Yang [SaY19] for the case of simple Lyapunov exponents. Allowing double Lyapunov exponents we obtain the result for a much broader class of L. In fact, toral automorphisms satisfying the assumptions of Theorem 7.1 are generic in the following sense. Consider the set of matrices in $SL(d, \mathbb{Z})$ of norm at most T. Then the proportion of matrices corresponding to automorphisms that satisfy our assumptions tends to 1 as $T \to \infty$.

Outline of the proof of Theorem 7.1.

We denote by E^u the unstable sub-bundle of L. Let $1 < \rho_1 < \cdots < \rho_\ell$ be the distinct moduli of the unstable eigenvalues of L, and let $E^u = E_1 \oplus \cdots \oplus E_\ell$ be the corresponding splitting. We denote the corresponding linear foliations by W_i.

Since f is C^1 close to L, f is also Anosov, and its unstable subbundle $\mathcal{E}^u$ splits into a direct sum of ℓ invariant Hölder continuous subbundles close to the corresponding subbundles for L: $\mathcal{E}^u = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell$. It can be shown that each $\mathcal{E}^i$ is tangent to a foliation $\mathcal{W}_i$ with $C^{1+\text{Hölder}}$ leaves.

Let h be the topological conjugacy between f and L close to the identity. There are two main arguments in the proof:
(1) showing that $h(\mathcal{W}_i) = W_i$ (irreducibility is used in this argument);
(2) then proving that h is $C^{1+\text{Hölder}}$ along $\mathcal{W}_i$.

Once we have this for each $\mathcal{W}_i$, Journe's lemma gives that h is $C^{1+\text{Hölder}}$ along $\mathcal{W}^u$. Then regularity of h along $\mathcal{W}^s$ is obtained similarly, and $C^{1+\text{Hölder}}$ regularity on $\mathbb{T}^d$ follows. This approach does not yield higher regularity of h since the leaves of $\mathcal{W}_i$ are no better than $C^{1+\text{Hölder}}$ in general.

We outline the argument for (2). Let $\mathcal{A}_i = L|_{E_i}$ and $\mathcal{B}_i = Df|_{\mathcal{E}_i}$.

Since no three eigenvalues of L have the same modulus, each E_i has dimension one or two. Once we know that $h(\mathcal{W}_i) = W_i$, establishing that its regularity along $\mathcal{W}_i$ relies on conformality of $\mathcal{B}_i$, which is trivial for one-dimensional $\mathcal{E}_i$. To prove conformality for two-dimensional $\mathcal{E}_i$ we use the following corollary of Theorem 6.4.

Corollary 7.2. *Let f and μ be as in (AH), and let $\mathcal{B}$ be a fiber bunched β-Hölder continuous linear cocycle over f with one exponent. Suppose that the fibers of $\mathcal{E}$ are two-dimensional. Then at least one of the following holds:*

(i) $\mathcal{B}$ *is conformal with respect to a β-Hölder continuous Riemannian metric on $\mathcal{E}$;*
(ii) $\mathcal{B}$ *preserves a β-Hölder continuous one dimensional sub-bundle;*
(iii) $\mathcal{B}$ *preserves a β-Hölder continuous field of two transverse lines.*

Since L has no pairs of eigenvalues of the form $\lambda, -\lambda$ or $i\lambda, -i\lambda$ where λ is real by the assumption (and λ, λ by irreducibility), L has no invariant lines or pairs of lines. The same property holds for the derivative at a fixed point for any sufficiently small perturbation f. It follows that options (ii) and (iii) are impossible for $\mathcal{B}_i$ and hence it is conformal.

We use conformality of $\mathcal{B}_i$ to show that h is Lipschitz along the leaves of $\mathcal{W}_i$. Then we differentiate the conjugacy equation $h \circ f = L \circ l$ a.e. along the leaves of $\mathcal{W}_i$ and show that $D(h|_{\mathcal{W}_i})$ is a measurable conjugacy between $\mathcal{A}_i$ and $\mathcal{B}_i$. Hence it is Hölder by Theorem 3.2, and so H is $C^{1+\text{Hölder}}$ along $\mathcal{W}_i$. $\square$

7.2. **Partially hyperbolic case.**

Now we consider a partially hyperbolic toral automorphism L. In contrast to the hyperbolic case, a C^1 small perturbation f of L is not necessarily topologically conjugate to it. Existence of a topological conjugacy was established for a certain class of partially hyperbolic L and their perturbations described below.

We say that a toral automorphism L *totally irreducible* if L^n is irreducible for every $n \in \mathbb{N}$. Let L be a totally irreducible automorphism of $\mathbb{T}^d$ with two-dimensional center bundle, i.e., with exactly two eigenvalues on the unit circle. By the results in [RH05] and [AV10] the following holds:

If f is a volume-preserving diffeomorphism sufficiently C^N close to L such that $Df|_{\mathcal{E}^c}$ has one exponent, then f is not accessible, f is bi-Hölder conjugate to L, and the conjugacy is $C^{1+\text{Hölder}}$ along the leaves of the center foliation.
It suffices to take $N = 5$ if $d > 4$, and $N = 22$ if $d = 4$.

The following theorem establishes Lyapunov spectrum rigidity for such toral automorphisms with simple real stable and unstable eigenvalues.

Theorem 7.3. (PH) [GKS20] *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a totally irreducible automorphism with exactly two eigenvalues of modulus one and simple real eigenvalues away from the unit circle. Let f be a volume-preserving C^N-small perturbation of L such that the Lyapunov exponents of f with respect to the volume are the same as the Lyapunov exponents of L. Then f is $C^{1+\text{Hölder}}$ conjugate to L.*

Since L and f are not accessible, we cannot apply the results for linear cocycles discussed above, and so we cannot handle subbundles $\mathcal{E}_i$ of dimension greater than one. Smoothness of h along the one-dimensional foliations $\mathcal{W}_i$ is obtained as in [SaY19] by showing that h maps the absolutely continuous conditional measures on the leaves of $\mathcal{W}^i$ to those on the leaves of W_i. This yields $C^{1+\text{Hölder}}$ regularity along $\mathcal{W}^u$ and $\mathcal{W}^s$, and the regularity along $\mathcal{W}^c$ is given by [AV10].

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