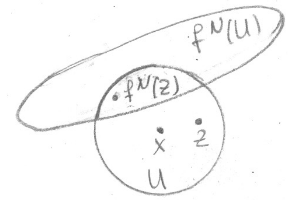
Recurrence properties (continued)

Nonwandering points

Recall that the set of (pos./neg.) recurrent points is not necessarily closed. The set of nonwandering points, $NW(f)$, is closed and invariant, it contains all recurrent pts, and all ω- and α-limit sets.

As before, we are considering $f: X \to X$, where X is a compact metric space and f is a continuous map.

Def a point $x \in X$ is nonwandering if for any open set U containing x there is $N \in \mathbb{N}$ s.t. $f^N(U) \cap U \neq \emptyset$.



Comments

- x is nonwandering $\iff$ for every $\varepsilon > 0$ there is a point z in $B_\varepsilon(x)$ and $N \in \mathbb{N}$ such that $f^N(z) \in B_\varepsilon(x)$.
- a nonwandering pt. is not necessarily positively recurrent.
- x is nonwandering $\iff$ for any open set U containg x there is an arbitrarily large $N \in \mathbb{N}$ s.t. $f^N(U) \cap U \neq \emptyset$

  Pf ($\Rightarrow$) Suppose $f^n(U) \cap U = \emptyset$ for all $n > n_0$. Then x is not periodic, and hence there exists an open $V \ni x$ s.t. $V \cap f^i(V) = \emptyset$ for $i = 1, \dots, n_0$. Then for $W = U \cap V$ there is no $N \in \mathbb{N}$ s.t. $f^N(W) \cap W \neq \emptyset$.
- If f is invertible and x is nonwandering, then for any open $U \ni x$ there is an arbitrarily large $N \in \mathbb{N}$ s.t. $f^{-N}(U) \cap U \neq \emptyset$. (Explain)

The set of all nonwandering pts for f is denoted $NW(f)$.

Properties of $NW(f)$

- $NW(f)$ is closed

  Pf Its complement is open. Indeed, if $x \notin NW(f)$, then there is an open $U \ni x$ s.t. $f^n(U) \cap U = \emptyset$ for all $n \in \mathbb{N}$. Then $y \in U \Rightarrow y \notin NW(f)$. □
- $NW(f)$ is f-invariant.

  Pf let $x \in NW(f)$. let V be an open set containg $f(x)$, and let $U = f^{-1}(V)$. Then there is $N \in \mathbb{N}$ s.t. $f^N(U) \cap U \neq \emptyset$, and hence $f^N(V) \cap V \neq \emptyset$.
- $NW(f)$ contains all (positively) recurrent pts (Clear)
- $NW(f) \neq \emptyset$ since there is a pos. rec. pt. (for compact X)

- If f is invertible, $NW(f)$ contains all negatively recurrent pts. (Explain)

- For every $x \in X$, $NW(f)$ contains the ω-limit set for x.

  Pf Let $y \in \omega(x)$. Then there is a strictly increasing sequence (n_k) s.t. $f^{n_k}(x) \to y$. Let U be an open nbhd of y. Then $f^{n_k}(x) \in U$ for all $k \geq k_0$.

  Hence for $N = n_{k_0+1} - n_{k_0}$, $f^N(U) \cap U \neq \emptyset$.

- If f is invertible, $NW(f)$ contains $\alpha(x)$ for every x. (Why?)

Note another way to explain $NW(f) \neq \emptyset$:
Since $\omega(x) \neq \emptyset$ and $\omega(x) \subseteq NW(f)$, $NW(f) \neq \emptyset$.

- Thus for an invertible f we have (omit for non-invertible)

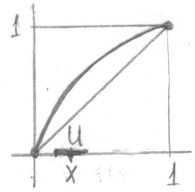
$$\underbrace{\{\text{Periodic pts}\}}_{\text{may be } \emptyset} \subseteq \{\text{recurrent pts}\} \subseteq \left(\{\text{positively rec. pts}\} \cup \{\text{negatively rec. pts}\}\right) \subseteq \underbrace{NW(f)}_{\text{closed}}$$

We can also consider the closures of

Examples

- $NW(R_\alpha) = S^1$ and $NW(E_m) = S^1$ since recurrent pts are dense

- For a hyperbolic automorphism f of $\mathbb{T}^2$, $NW(f) = \mathbb{T}^2$ since periodic pts are dense.

- 

For this interval map f, $NW(f) = \{0, 1\}$.
Indeed, for any $x \in (0,1)$, there is an open interval $U \ni x$ s.t. $f^N(U) \cap U = \emptyset$ for all sufficiently large N.
In fact, there is U s.t. $f^N(U) \cap U = \emptyset$ for all $n \in \mathbb{N}$.
(Explain)

(?) Does topological transitivity of f imply $NW(f) = X$?
(Yes / No / Yes under an additional assumption)

(?) Does topological mixing imply $NW(f) = X$?