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Symbolic dynamical systems

Full shift on m symbols.

Let $m \geq 2$ be an integer.
The set $\mathcal{A}_m = \{0, 1, \ldots, m-1\}$ is an alphabet, its elements – symbols.
A finite sequence of symbols is a word.

$\Omega_m^R = \mathcal{A}_m^{\mathbb{N}_0}$ = the set of one-sided sequences of elements of $\mathcal{A}_m$ =
$= \{\omega = (\omega_i)_{i \in \mathbb{N}_0} : \omega_i \in \mathcal{A}_m \text{ for } i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$

$\Omega_m = \mathcal{A}_m^{\mathbb{Z}}$ = the set of two-sided sequences of elements of $\mathcal{A}_m$ =
$= \{\omega = (\omega_i)_{i \in \mathbb{Z}} : \omega_i \in \mathcal{A}_m \text{ for } i \in \mathbb{Z}\}$.

A cylinder in Ω_m or Ω_m^R is a set of the form

$$C_{\alpha_1, \ldots, \alpha_k}^{n_1, \ldots, n_k} = \{\omega \in \Omega_m^{(R)} : \omega_{n_i} = \alpha_i \text{ for } i = 1, \ldots, k\},$$

where $n_1 < \ldots < n_k$ and $\alpha_1, \ldots, \alpha_k \in \mathcal{A}_m$. (It is the set of ω with given $\omega_{n_1}, \ldots, \omega_{n_k}$)
The number k of fixed digits is called the rank of the cylinder.

Topology on $\Omega_m^{(R)}$

Consider $\mathcal{A}_m$ with discrete topology, and the product topology on Ω_m.
Recall that a basis of the product topology consists of the sets of the form
$\prod_{i \in \mathbb{Z}} U_i$, where each U_i is open in $\mathcal{A}_m$, and $U_i = \mathcal{A}_m$ for all but finitely many i.
Each such set is a finite union of cylinders. So the cylinders also give a basis of the product topology. General open sets are finite or countably infinite unions of cylinders.
Since $\mathcal{A}_m$ is compact, $\Omega_m^{(R)}$ with the product topology is also compact.
Note that the cylinders are also closed as their complements are open (Why?)
We can use this to show that $\Omega_m^{(R)}$ is totally disconnected (Explain).

Metrics on Ω_m and Ω_m^R

We will consider the following metrics:

On $\Omega_m^{(R)}$, $\quad d(\omega, \omega') = 2^{-\min\{i : \omega_i \neq \omega_i'\}}$

On Ω_m, $\quad d(\omega, \omega') = 2^{-\min\{|i| : \omega_i \neq \omega_i'\}}$

- d is a distance on $\Omega_m^{(R)}$
  $d(\omega, \omega') \leq d(\omega, \omega'') + d(\omega'', \omega')$ holds since $\omega_i \neq \omega_i' \Rightarrow \omega_i \neq \omega_i''$ or $\omega_i' \neq \omega_i''$.

- For any $\omega, \omega' \in \Omega_m^{(R)}$, $d(\omega, \omega') \leq 1$.

On Ω_m^R: • $d(\omega,\omega') = 2^{-k} \iff \omega_i = \omega_i'$ for $i=0,..,k-1$, and $\omega_k \neq \omega_k'$

• The open ball $B(\omega, 2^{-k}) = \{\omega' \in \Omega_m^R : d(\omega,\omega') < 2^{-k}\} =$
$= \{\omega' \in \Omega_m^R : \omega_i = \omega_i'$ for $i = 0,..,k\} =$
$= C^{0,\dots,k}_{\omega_0,\dots,\omega_k}$, a cylinder of rank $k+1$.

• For any $2^{-k-1} < r \le 2^{-k}$, $B(\omega, r) = B(\omega, 2^{-k})$

On Ω_m: • $d(\omega,\omega') = 2^{-k} \iff \omega_i = \omega_i'$ for $i = -(k-1), \dots, 0, \dots, k-1$
and ($\omega_k \neq \omega_k'$ or $\omega_{-k} \neq \omega_{-k}'$)

• $B(\omega, 2^{-k}) = \{\omega' \in \Omega_m : \omega_i = \omega_i'$ for $i = -k, \dots, 0, \dots, k\} =$
$= C^{-k,\dots,k}_{\omega_{-k},\dots,\omega_k}$, a cylinder.

Other metrics on Ω_m and similarly on Ω_m^R.

$$d_\lambda(\omega,\omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega_n'|}{\lambda^{|n|}} \quad \text{and} \quad \tilde{d}_\lambda(\omega,\omega') = \sum_{n=-\infty}^{\infty} \frac{\delta(\omega_n,\omega_n')}{\lambda^{|n|}},$$

$\delta(\omega_n, \omega_n') =$
1, if $\omega_n \neq \omega_n'$
0 if $\omega_n = \omega_n'$

where λ is chosen large enough so that symmetric cylinders $C^{-k,\dots,0,\dots,k}_{\alpha_{-k},\dots,\alpha_k}$ are open balls. all these metrics generate the same topology.

The shift map

On Ω_m^R, $\sigma^R((\omega_0, \omega_1, \omega_2, \dots)) = (\omega_1, \omega_2, \dots)$
On Ω_m, $\sigma((\omega_i)_{i\in\mathbb{Z}}) = (\omega_{i+1})_{i\in\mathbb{Z}}$, i.e. $(\dots, \omega_{-1}, \underset{\uparrow}{\omega_0}, \omega_1, \dots) \mapsto (\dots, \omega_{-1}, \omega_0, \underset{\uparrow}{\omega_1}, \dots)$

Properties:

• $\sigma: \Omega_m \to \Omega_m$ is invertible
• $\sigma_R: \Omega_m^R \to \Omega_m^R$ is surjective, but not injective, it is m-to-1
• Both σ and σ_R are continuous since the pre-image of a cylinder is a cylinder (Check)
• $\sigma: \Omega_m \to \Omega_m$ is a homeomorphism.
• $\sigma_R: \Omega_m^R \to \Omega_m^R$ is expanding since for any ω, ω' with $d(\omega,\omega') \le \frac{1}{2}$,
$d(\sigma_R(\omega), \sigma_R(\omega')) = 2\, d(\omega,\omega')$ (why?)

Def. $f: X \to X$ is expanding if there is a constant $\lambda > 1$ such that for any sufficiently close $x, y \in X$, $d(f(x), f(y)) \ge \lambda\, d(x,y)$.

• $E_m: S^1 \to S^1$ is also expanding, $\lambda = m$.
• expanding $\Rightarrow$ expansive.