Last time: Suspension flow, or flow under a function.

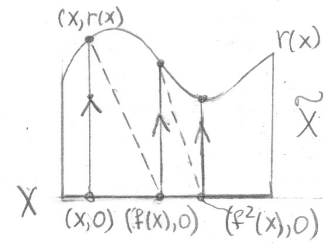
X - a compact metric space, $f: X \to X$ a homeomorphism

$r: X \to (0,\infty)$ a continuous function, called the roof (or ceiling) function.

Note: r is bounded away from 0. We consider the quotient space

$\tilde{X} = X_{r,f} = \{(x,s) : x \in X,\ 0 \le s \le r(x)\} / (x, r(x)) \sim (f(x), 0)$.

Under the suspension flow $\Phi = \{\varphi^t\}$ on $\tilde{X}$, points move along vertical segments with unit speed until they reach the "roof".



Some properties of $\Phi = \{\varphi^t\}$.

- Φ has no fixed points. If $x \in X$ is a fixed point for f, then $(x,0)$ is a periodic point for Φ with minimal period $r(x)$.
- $x \in X$ is periodic for $f \iff (x,0) \in \tilde{X}$ is periodic for $\Phi \iff$ for any $s \in [0, r(x))$, (x,s) is periodic for Φ.
  If n is the min. period of x, then the min. period of $(x,0)$ is $\sum_{i=0}^{n-1} r(f^i(x))$.
- The orbit of $x \in X$ under f is dense in $X \iff$ the orbit of $(x,0) \in \tilde{X}$ under Φ is dense in $\tilde{X}$.
  So Φ is top. transitive $\iff f$ is top. transitive.
- Top. mixing
  If $r \equiv 1$ (or is a constant function), then Φ is <u>not</u> top. mixing, even if f is top. mixing. Indeed, let $U = V = X \times (0, \frac{1}{4}) \subset \tilde{X}$.
  Then there is no T s.t. $\varphi^t(U) \cap V \neq \emptyset$ for all $t \ge T$.
  In general, top. mixing for Φ depends on $f: X \to X$ and on r.

Equivalence for flows.

Def Two flows $\{\varphi^t\}$ on M and $\{\psi^t\}$ on N are top. conjugate, or C^0 flow equivalent if there exists a homeomorphism $h: M \to N$ such that $\varphi^t = h^{-1} \circ \psi^t \circ h$ for <u>all</u> $t \in \mathbb{R}$.

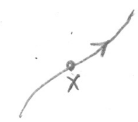
$$\begin{array}{ccc} M & \xrightarrow{\varphi^t} & M \\ h \downarrow & & \downarrow h \\ N & \xrightarrow{\psi^t} & N \end{array}$$

Note Two C^n flows $\{\varphi^t\}$ and $\{\psi^t\}$ are C^m, $m \le n$, flow equivalent if there exists a C^m diffeomorphism $h: M \to N$ s.t. $\varphi^t = h^{-1} \circ \psi^t \circ h$ for all t.

Ex The suspension flow $\{\varphi^t\}$ over $R_\alpha : S^1 \to S^1$ with $r=1$ and the linear flow $\{T^t_{(\alpha,1)}\}$ on $\mathbb{T}^2$ are top. conjugate.
In fact, they are C^∞ flow equivalent

Time change

Def. Let $\{\varphi^t\}$ and $\{\psi^t\}$ be two flows on M. We say that $\{\varphi^t\}$ is a time change of $\{\psi^t\}$ if for each $x \in M$ the orbits $O^\varphi(x) = \{\varphi^t(x)\}_{t\in\mathbb{R}}$ and $O^\psi(x) = \{\psi^t(x)\}_{t\in\mathbb{R}}$ coincide and are traced in the same direction as t increases.



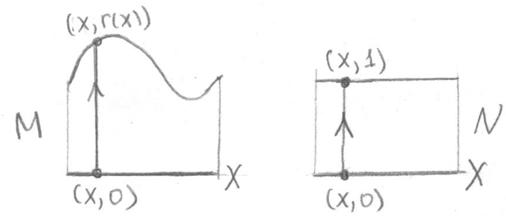
Note
- Let $\{\varphi^t\}$ be a flow on M, and let $x \in M$. Then there are only 3 possibilities: (1) x is fixed by $\{\varphi^t\}$, (2) $\varphi^t(x) \neq x$ for all $t \in \mathbb{R}$, (3) x is periodic with a min. period $T>0$.
- If $\{\psi^t\}$ is a time change of $\{\varphi^t\}$, then their fixed points are the same, and their non-fixed periodic pts are the same, but periods may be different.

Orbit equivalence.

Def Two flows $\{\varphi^t\}$ on M and $\{\psi^t\}$ on N are C^0 orbit equivalent if there exists a homeomorphism $h: M \to N$ s.t. the flow $\{\tilde{\varphi}^t\}$ on M given by $\tilde{\varphi}^t = h^{-1} \circ \psi^t \circ h$ is a time change of $\{\varphi^t\}$.

Note
- h maps the orbits of $\{\varphi^t\}$ to the orbits of $\{\psi^t\}$, preserving the direction.
- C^0 flow equivalence (top. conjugacy) $\underset{\not\Leftarrow}{\Rightarrow}$ C^0 orbit equivalence.

Ex Two suspension flows over $f: X \to X$
$\{\varphi^t\}$ with roof function $r(x)$, and
$\{\psi^t\}$ with roof function 1.



The map $h(x,s) = (x, \frac{s}{r(x)})$ from M to N gives C^0 orbit equivalence, but not top. conjugacy.

Note. $\{\varphi^t\}$ and $\{\psi^t\}$ may or may not be top. conjugate.
a certain relation between $r(x)$ and 1 ensures top. conjugacy.