Box(-counting) dimension.

A box is an interval in $\mathbb{R}$, a square in $\mathbb{R}^2$, a cube in $\mathbb{R}^3$, ...
How many boxes with side ϵ are needed to cover a bounded set X in $\mathbb{R}^k$?
Denote this number $N(\epsilon)$.

$X =$ unit interval	$X =$ unit square	$X =$ unit cube	In each case,
$N(\epsilon) \approx (1/\epsilon)^1$	$N(\epsilon) \approx (1/\epsilon)^2$	$N(\epsilon) \approx (1/\epsilon)^3$	$d \approx \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$.

Def. The box(-counting) dimension of a bounded set X in R^k is

$$\dim_B(X) = \lim_{\epsilon\to 0^+} \tfrac{\ln N(\epsilon)}{\ln(1/\epsilon)},$$

where $N(\epsilon)$ is the least number of boxed with side ϵ needed to cover X.

If the limit does not exist, we consider the upper and lower box dimensions, $\overline{\dim}_B(X)$ and $\underline{\dim}_B(X)$, defined as the corresponding lim inf and lim sup .

- If the limit along an exponentially decreasing sequence (ϵ_n) equals d, then $\dim_B(X) = d$

Example. $C =$ the Cantor set.

At step n, the set C_n consists of 2^n intervals of length $1/3^n$.

So we take $\epsilon_n = 1/3^n$. Then $N(\epsilon_n) = 2^n$ and $\frac{\ln N(\epsilon_n)}{\ln(1/\epsilon_n)} = \frac{\ln 2^n}{\ln 3^n} = \frac{\ln 2}{\ln 3}$.

Hence $\dim_B(C) = \frac{\ln 2}{\ln 3} = d_{sim}(C)$.

- For the examples we considered, it also holds that $\dim_B = d_{sim}$.
- Box dimension: pluses and minuses.
  (+) Relatively easy to define and to compute/estimate.
  (−) The limit does not necessarily exist, and we may only get liminf and limsup.
  (−) A countable set may have positive box dimension,
  and so adding countably many points to a set can change its box dimension.
- **Hausdorff dimension**, $\dim_H(X)$, does not have these drawbacks, but it is harder to define and compute or estimate.
  The construction involves covering X by balls of diameter *at most* ϵ.
- For any bounded set $X \in R^k$, $\dim_H(X) \le \underline{\dim}_B(x) \le \overline{\dim}_B(x) \le k$.
- For the sets that we considered, $\dim_H(X) = \dim_B(x) = \dim_{sim}(X)$.