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Times-m map of S^1 (continued)

a point with dense orbit for E_3:

$z = 0.\underbrace{0}\underbrace{1}\underbrace{2}\underbrace{00}\underbrace{01}\underbrace{02}\underbrace{10}\underbrace{11}\underbrace{12}\underbrace{20}\underbrace{21}\underbrace{22}\underbrace{000}\ldots\underbrace{222}\ldots$ in base 3

$O(z)$ is dense since it visits every interval $[\frac{k}{3^n}, \frac{k+1}{3^n}]$, $n \in \mathbb{N}$, $0 \le k < 3^n$, and hence every open interval.

another explanation: If $x = 0.a_1 \ldots a_n ** \ldots_3$ and $y = 0.a_1 \ldots a_n ** \ldots_3$, then $d(x,y) \le \frac{1}{3^n}$. So if a base 3 expansion of z contains every finite sequence of $0, 1, 2$, then $O(z)$ is dense.

Given any $x \in S^1$ and $\varepsilon > 0$, we can find y s.t. $d(x,y) < \varepsilon$ and $O(y)$ is dense:

let $x = 0.a_1 a_2 \ldots_3$. Take n s.t. $\frac{1}{3^n} < \varepsilon$ and $y = 0.a_1 \ldots a_n \underbrace{0}\underbrace{1}\underbrace{2}\underbrace{00}\underbrace{01}\ldots_3$.

Similarly for E_m.

- Thus for $E_m : S^1 \to S^1$
  - periodic points are dense in S^1, points with dense orbits are dense,
  - for any sufficiently close x, y, $d(E_m(x), E_m(y)) = 3d(x,y)$
  - for any $x \neq y$, there is $n \in \mathbb{N} \cup \{0\}$ s.t. $d(E_m^n(x), E_m^n(y)) \ge \frac{1}{2m}$.

Hence the orbits of nearby points do not remain close, and an arbitrarily small change in x (an error) may yield an orbit with a different behavior.

So $E_m : S^1 \to S^1$ differs dramatically from contractions and circle rotations.

a definition of chaos

Let $f : X \to X$ be a continuous map of a compact metric space.

We say that f is <u>chaotic</u> if periodic points are dense in X and f is topologically transitive, i.e. there is a point with dense orbit.

<u>Note</u> If X does not have isolated points, then top. transitivity implies that points with dense orbits are dense in X (Explain why).

Closed invariant sets for E_m.

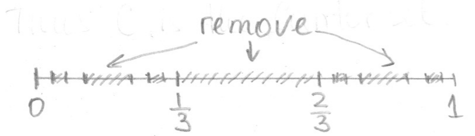
The orbit of any periodic point is a closed invariant set.

An infinite closed invariant set for E_3:

Let $C = \{x \in [0,1] : x$ can be written in base 3 without using 1$\}$

In S^1, we identify 0 and 1, i.e. $0.00\ldots_3$ and $0.222\ldots_3$.

The set C is invariant under E_3. To obtain C, we need to remove the numbers that require 1 in their base 3 expansion, i.e. we remove the open middle thirds of the intervals. Thus C is the Cantor set.



The Cantor set is closed, uncountable, and has Lebesgue measure 0.

? Does the restriction $E_3|_C$ have periodic points? Yes, any periodic sequence of 0's and 2's yields a periodic pts in C.
For example, $x = 0.020202\ldots_3 = \frac{2}{3^2} + \frac{2}{3^4} + \cdots = 2 \sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1}{4}$.

? Is the restriction $E_3|_C$ transitive?
Yes, $z = 0.\underbrace{0}\underbrace{2}\underbrace{00}\underbrace{02}\underbrace{20}\underbrace{22}\underbrace{000}\ldots\underbrace{222}\ldots_3 \in C$ and $O(z)$ is dense in C.

Similarly we can obtain closed invariant sets for E_m.

Shifts on sequence spaces and semiconjugacy

We used the shift on sequence of m symbols to study E_m.
Let Ω_m^R be the set of all one-sided sequences of $0, 1, \ldots, m-1$, i.e.

$$\Omega_m^R = \{ \omega = (\omega_0, \omega_1, \omega_2, \ldots) : \omega_i \in \{0, 1, \ldots, m-1\}, i = 0, 1, 2, \ldots \}$$

Let us consider the following distance on Ω_m^R:

for $\omega, \omega' \in \Omega_m^R$, $d(\omega, \omega') = 2^{-\min\{i : \omega_i \neq \omega'_i\}}$.

It is a distance and (Ω_m^R, d) is a compact metric space.
The left shift σ on Ω_m^R is given by $\sigma((\omega_0, \omega_1, \omega_2, \ldots)) = (\omega_1, \omega_2, \ldots)$

Let h be the map from Ω_m^R to S^1 given by

$h((\omega_0 \omega_1 \omega_2 \ldots)) = 0.\omega_0 \omega_1 \omega_2 \ldots$ in base m.

Then h is continuous since $d(\omega, \omega') < \frac{1}{2^n} \Rightarrow (\omega_0, \ldots, \omega_n) = (\omega'_0, \ldots, \omega'_n) \Rightarrow$
$\Rightarrow d(h(\omega), h(\omega')) \leq \frac{1}{m^{n+1}}$,

and h is surjective, but not injective.
Note that the following diagram is commutative:

$$\begin{tikzcd} \Omega_m^R \arrow[r, "\sigma"] \arrow[d, "h"'] & \Omega_m^R \arrow[d, "h"] \\ S^1 \arrow[r, "E_m"] & S^1 \end{tikzcd}$$

that is, $h \circ \sigma = E_m \circ h$.

$E_m : S^1 \to S^1$ is a factor of $\sigma : \Omega_m^R \to \Omega_m^R$,
and h is a semiconjugacy.