# MATH 507 Dynamical Systems.

## Introduction

A disrete-time dynamical system is a set $X$ and a map $f: X \to X$.
Usually, the following settings are considered:

- $X$ - a (compact) topological or metric space, $f$ - a continuous map
- $X$ - a (compact) smooth manifold, $f$ - a differentiable map.
- $X$ - a (probability) measure space, $f$ - a measure-preserving map.

We are interested in the long-term behavior of the system.

Iterates of $f$: $f^0 = Id$, $f^n = \underbrace{f \circ \ldots \circ f}_{n}$, $n \in \mathbb{N}$,
for an invertible $f$ we can also consider $f^{-n} = f^{-1} \circ \ldots \circ f^{-1}$.

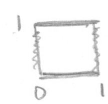
For an invertible $f$, the orbit of a point $x \in X$ is $\{f^n(x) : x \in \mathbb{Z}\}$
the positive semi-orbit is $\{f^n(x) : n \geq 0\}$
the negative semi-orbit is $\{f^n(x) : n \leq 0\}$

For a non-invertible $f$, the orbit of $x$ is $\{f^n(x) : n \geq 0\}$.

$x \in X$ is a fixed point for $f$ if $f(x) = x$

$x \in X$ is a periodic point for $f$ if $f^n(x) = x$ for some $n \geq 1$.
Such $n$ is called a period of $x$, the smallest such $n$ is the prime period of p.

Some examples of dynamical systems (will be discussed in the course)

- Contractions on metric spaces
- Circle rotations: $S^1 = \mathbb{R}/\mathbb{Z}$ 
  $R_\alpha(x) = x + \alpha \mod 1$
- Translations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ 

  $f_{\alpha,\beta}(x,y) = (x+\alpha, y+\beta) \mod 1$
- Times-m map of the cirle, $m \in \{2, 3, \ldots\}$
  $E_m(x) = mx \mod 1$
- Symbolic dynamical systems
  Ex Full one-sided shift on two symbols
  $X = \{(x_n)_{n \geq 0} : x_n \in \{0, 1\}\}$ $\quad f((x_n)) = (x_{n+1})$ left shift
- Hyperbolic automorphisms of a torus
  Ex Arnold's cat map. $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\det A = 1$, e.values $0 < \lambda_1 < 1 < \lambda_2$.
  $A: \mathbb{R}^2 \to \mathbb{R}^2$ projects to an invertible map of $\mathbb{T}^2$.

## Some questions

- are there fixed points? are they attracting, repelling, or neither?
- are there periodic points?
  How does the number of periodic points of period n grow with n?
- Is there a point whose orbit is dense in X?
- For an open set $U \subset X$, what can we say about $f^n(U)$?
- are there any (closed) sets invariant under $f$?
- How is the orbit of $x \in X$ distributed?
  Does x return close to itself?
- For nearby $x, y \in X$, do their orbits remain close?
  Can an arbitrarily small change in x (an error) produce a very different orbit? Is the system chaotic?
- How complex is the orbit structure of the system?
  and how to measure the complexity?
- What does it mean for two systems to be "qualitatively the same"?
  (topologically conjugate)? Which systems are/are not top. conjugate?
  What are invariants of top. conjugacy?
  Is a system top. conjugate to a "model"?
- Is a small perturbation of a system top. conjugate to it?

<u>a continuous-time dynamical system</u> (a flow)

is a set X and a one-parameter family $\{f^t : t \in \mathbb{R}\}$ of maps $f^t : X \to X$ such that $f^0 = Id$ and $f^{s+t} = f^s \circ f^t$ for all $s, t \in \mathbb{R}$

<u>Ex</u> Linear flow on the torus $\mathbb{T}^2$

$$f^t(x,y) = (x+\alpha t,\ y+\beta t) \mod 1$$