Rotation number - continued.

Prop. Rotation number is an invariant of orientation-preserving topological conjugacy, i.e., if f and h are orient.-pres. homeomorphisms of S^1, then $\tau(h \circ f \circ h^{-1}) = \tau(f)$.

Pf Let $F, H: \mathbb{R} \to \mathbb{R}$ be lifts of f and h, resp.
Then H^{-1} is a lift of h^{-1} and $H \circ F \circ H^{-1}$ is a lift of $h \circ f \circ h^{-1}$ (Check!)

Let $x \in \mathbb{R}$. We have: $\frac{1}{n}((HFH^{-1})^n(x) - x) = \frac{1}{n}((HF^nH^{-1})(x) - x) =$

$= \frac{1}{n}\left(H(F^nH^{-1}(x)) - F^nH^{-1}(x)\right) + \frac{1}{n}\left(F^n(H^{-1}(x)) - H^{-1}(x)\right) + \frac{1}{n}\left(H^{-1}(x) - x\right)$

$|H(F^nH^{-1}(x)) - F^nH^{-1}(x)| = |H(z_n) - z_n| \leq \max_{z \in [0,1]} |H(z) - z|$. (holds since h is orientation-pres.)

So the first and the third terms $\to 0$ as $n \to \infty$.

Denoting $H^{-1}(x)$ by y, we obtain

$$\tau(HFH^{-1}) = \lim_{n\to\infty} \frac{1}{n}((HFH^{-1})^n(x) - x) = \lim_{n\to\infty} \frac{1}{n}(F^n(y) - y) = \tau(F),$$

and so $\tau(hfh^{-1}) = \tau(f)$. □

Note The result does not hold for an orientation-reversing conjugacy.
For example, for $\alpha \neq \frac{1}{2} \bmod 1$, R_α and $R_{-\alpha}$ have different rotation numbers, however, $h(x) = -x$ is a top. conjugacy between them (check!)

Rational rotation number.

Assumption: f is an orientation-pres. homeomorphism of S^1.

Prop. f has a fixed point $\Longleftrightarrow \tau(f) = 0$.

Pf ($\Rightarrow$) Suppose $f(\bar{x}) = \bar{x}$ for some $\bar{x} \in S^1$. Let $x \in \mathbb{R}$ be s.t. $\pi(x) = \bar{x}$. Then for any lift F of f, $F(x) = x + k$, $k \in \mathbb{Z}$, and there is F with $F(x) = x$.
For this F, $\tau(F) = \lim_{n\to\infty} \frac{1}{n}(F^n(x) - x) = 0$. So $\tau(f) = 0$.

($\Leftarrow$) (by contraposition). Suppose f has no fixed pt.
Let $F: \mathbb{R} \to \mathbb{R}$ be a lift of f with $F(0) \in [0,1)$.
Then for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, $F(x) \neq x + k$ (otherwise $\pi(x)$ is fixed).
Since $F(0) = F(0) - 0 \in (0,1)$, $0 < F(x) - x < 1$ for all x.
Since $F(x) - x$ is cont. on $[0,1]$, there is $\delta > 0$ s.t. $\delta \leq F(x) - x \leq 1 - \delta$ on $[0,1]$, and hence on $\mathbb{R}$ by periodicity. Adding $\delta \leq F^{i+1}(0) - F^i(0) \leq 1 - \delta$ for $i = 0, \dots, n-1$, we obtain $n\delta \leq F^n(0) \leq (1-\delta)n$ and hence $\delta \leq \frac{F^n(0) - 0}{n} \leq 1 - \delta$.
Hence $\tau(F) \in [\delta, 1-\delta]$, and so $\tau(f) \neq 0$. □

Lemma: For any $m \in \mathbb{N}$, $\tau(f^m) = m \cdot \tau(f)$

Pf (Ex).

Prop. The rotation number of f is rational $\iff$ f has a periodic pt.

Pf ($\Rightarrow$) Suppose $\tau(f) = \frac{p}{q} \in \mathbb{Q}$, where $q \in \mathbb{N}$ (Choose τ in $[0,1)$)

Then $\tau(f^q) = q \cdot \tau(f) = p = 0 \mod 1$. Thus f^q has a fixed pt.

This pt is periodic with period q for f.

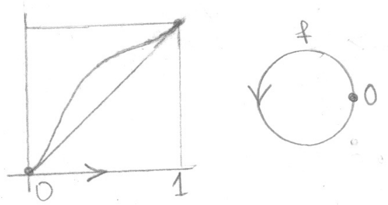
($\Leftarrow$) If f has a periodic pt, then f^q has a fixed pt, then $\tau(f^q) = 0$, and hence $\tau(f) \in \mathbb{Q}$ by the Lemma. $\square$

Remark Let $\tau(f) = \frac{p}{q}$, where $p, q \in \mathbb{N}$.

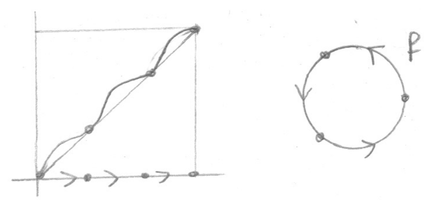
- all periodic pts for f have the same prime period, q. (Prop. 11.1.5)
- For every periodic pt $\bar{x} = f^q(\bar{x}) \in S^1$, the ordering of $\{\bar{x}, f(\bar{x}), f^2(x), \dots, f^{q-1}(x)\}$ on S^1 is the same as the ordering of $\{0, \frac{p}{q}, \frac{2p}{q}, \dots, \frac{(q-1)p}{q}\}$ ($\leftarrow$ the orbit of 0 under $R_{p/q}$) (Prop. 11.2.1)

However In general, f is not top. conjugate to $R_{p/q}$, moreover, none is a factor of the other. Smoothness of f does not help.

Ex



OR



Since f has a fixed pt, $\tau(f) = 0$. However, f is not top. conjugate to $R_0 = Id$, and none is a factor of the other (Explain why)

Suppose $\tau(f) = \frac{p}{q}$. Then $\tau(f^q) = 0$.

If f is conjugate/semiconjugate to $R_{p/q}$, then
f^q —— " —— to $(R_{p/q})^q = Id$,
which is not necessarily the case.