Box(-counting) dimension.

A box is an interval in $\mathbb{R}$, a square in $\mathbb{R}^2$, a cube in $\mathbb{R}^3$, ...
How many boxes with side ϵ are needed to cover a bounded set X in $\mathbb{R}^k$?
Denote this number $N(\epsilon)$.

X = unit interval	X = unit square	X = unit cube	In each case,
$N(\epsilon) \approx (1/\epsilon)^1$	$N(\epsilon) \approx (1/\epsilon)^2$	$N(\epsilon) \approx (1/\epsilon)^3$	$d \approx \dfrac{\ln N(\epsilon)}{\ln(1/\epsilon)}$.

Def. The **box(-counting) dimension** of a bounded set X in $\mathbb{R}^k$ is

$$\dim_B(X) = \lim_{\epsilon \to 0^+} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}, \quad \textit{if the limit exists,}$$

where $N(\epsilon)$ is the least number of boxed with side ϵ needed to cover X.

If the limit does not exist, we consider the upper and lower box dimensions, $\overline{\dim}_B(X)$ and $\underline{\dim}_B(X)$, defined as the corresponding $\liminf$ and $\limsup$.

- If the limit along a decreasing sequence $\epsilon_n \to 0$ with $\epsilon_{n+1} \geq c\epsilon_n$ equals d, then $\dim_B(X) = d$.

Example. C = the Cantor set.
At step n, the set C_n consists of 2^n intervals of length $1/3^n$.
So we take $\epsilon_n = 1/3^n$. Then $N(\epsilon_n) = 2^n$ and $\dfrac{\ln N(\epsilon_n)}{\ln(1/\epsilon_n)} = \dfrac{\ln 2^n}{\ln 3^n} = \dfrac{\ln 2}{\ln 3}$.
Hence $\dim_B(C) = \dfrac{\ln 2}{\ln 3} = d_{sim}(C)$.

- For the examples we considered, it also holds that $\dim_B = d_{sim}$.
  Find their similarity and box dimensions.

- Box dimension: pluses and minuses.
  (+) Relatively easy to define and to compute/estimate.
  (−) The limit does not necessarily exist, and we may only get liminf and limsup.
  (−) A countable set may have positive box dimension *(see next slide)*,
  and so adding countably many points to a set can change its box dimension.

- **Hausdorff dimension**, $\dim_H(X)$, does not have these drawbacks, but it is harder to define and compute/estimate.
  The construction involves covering X by balls of diameter *at most* ϵ.

- For any bounded set $X \subset \mathbb{R}^k$, $\dim_H(X) \leq \underline{\dim}_B(x) \leq \overline{\dim}_B(x) \leq k$.

- For the sets that we considered, $\dim_H(X) = \dim_B(X) = \dim_{sim}(X)$.