Box dimension of countable sets.

Example 1. Let $X = \{0\} \cup \{\, \frac{1}{2^n} :\ n \in \mathbb{N}\,\} \subset \mathbb{R}$.

$\frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{2^n}$, so let $\epsilon_n = \frac{1}{2^n}$.

One box covers $[0, \frac{1}{2^n}]$, and each of $\frac{1}{2^{n-1}}, \ldots, \frac{1}{2}, 1$ requires its own box.

Thus $N(\epsilon_n) = n + 1$, and

$$\dim_B(X) = \lim_{n\to\infty} \frac{\ln N(\epsilon_n)}{\ln(1/\epsilon_n)} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(2^n)} = 0.$$

Example 2. Let $X = \{0\} \cup \{\, \frac{1}{n} :\ n \in \mathbb{N}\,\} \subset \mathbb{R}$.

$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{(n-1)(n)} > \frac{1}{n^2}$, so let $\epsilon_n = \frac{1}{n^2}$.

Each of $\frac{1}{n-1}, \ldots, \frac{1}{2}, 1$ requires its own box, so $N(\epsilon_n) \geq n$.

Also, n boxes will cover $[0, \frac{1}{n}]$, and so $N(\epsilon) \leq n + n = 2n$.

Then $\frac{1}{2} = \frac{\ln(n)}{\ln(n^2)} \leq \frac{\ln N(\epsilon_n)}{\ln(1/\epsilon_n)} \leq \frac{\ln(2n)}{\ln(n^2)} \to \frac{1}{2}$

and hence $\dim_B(X) = \frac{1}{2}$.