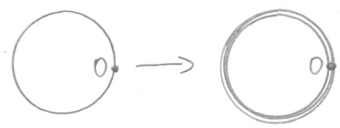
Times-m map of the circle

Let $m \in \mathbb{N}$, $m \geq 2$. $E_m : S^1 \to S^1$ is given by $E_m(x) = mx \mod 1$.

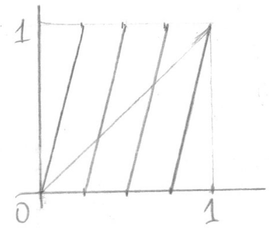
E_m is well-defined since $x - y \in \mathbb{Z} \Rightarrow mx - my \in \mathbb{Z}$.

"E_m stretches the circle by a factor of m and wraps it around itself m times."



On the interval $[0,1)$
$x \mapsto \{mx\}$ (fractional part)

The graph for $m=4$:

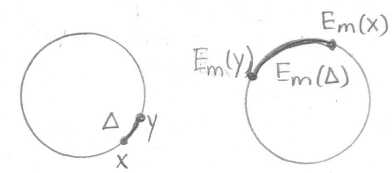


The map E_m is m-to-1, i.e. each $x \in S^1$ has exactly m pre-images.

In multiplicative notations, $S^1 = \{z : |z| = 1\} = \{e^{2\pi i x} : x \in \mathbb{R}\}$,
and $E_m(z) = z^m$.

E_m is a group homomorphism from $S^1 = \mathbb{R}/\mathbb{Z}$ into itself, i.e. it is an endomorphism of S^1. Indeed, $E_m(x+y) = E_m(x) + E_m(y) \mod 1$.

If $d(x,y)$ on the circle is sufficiently small, say, if $d(x,y) < \frac{1}{2m}$, then $d(E_m(x), E_m(y)) = m\, d(x,y)$.
This equality holds if the arc $E_m(\Delta)$ has length $< \frac{1}{2}$, which is guaranteed if $|\Delta| < \frac{1}{2m}$. So E_m is expanding.



It follows that for any $x \neq y$ in S^1 there is $n \in \mathbb{N} \cup \{0\}$ such that
$d(E_m^n(x), E_m^n(y)) \geq \frac{1}{2m}$.

<u>Def.</u> A continuous map [a homeomorphism] $f : X \to X$ is called <u>expansive</u> if there exists a constant $\delta > 0$ such that for any $x \neq y$ in X there is $n \in \mathbb{N} \cup \{0\}$ [$n \in \mathbb{Z}$] s.t. $d(f^n(x), f^n(y)) \geq \delta$.
Equivalently, if $d(f^n(x), f^n(y)) < \delta$ for all n, then $x = y$.

<u>Ex.</u> E_m is expansive.

<u>Fixed points of E_m</u>

$E_m(0) = 0$. The graph of the map on the interval intersects the diagonal $m-1$ times, so there are $m-1$ fixed pts. Let us find x in $[0,1)$
$E_m(x) = x \iff mx = x \mod 1 \iff (m-1)x = k \in \mathbb{Z} \iff x = \frac{k}{m-1}$, $k = 0, 1, \dots, m-2$.

<u>Periodic points of E_m of period n</u>

$E_m^n(x) = x \iff m^n x = x \mod 1 \iff x = \frac{k}{m^n - 1}$, $k = 0, 1, \dots, m^n - 2$.

<u>Ex</u> $m = 3$, $n = 2$, $m^n - 1 = 8$ $\quad E_3^2\left(\frac{5}{8}\right) = 3^2 \cdot \frac{5}{8} = \frac{45}{8} = \frac{5}{8} \mod 1$

Note that periodic points of E_m are dense in S^1.

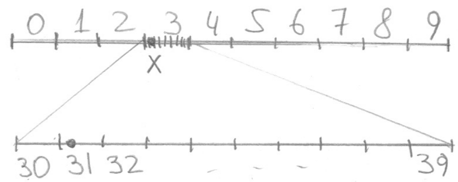
? Is E_m top. transitive? Yes.

Pf Let U, V be non-empty open sets in S^1. Since U is open, it contains an open interval, and it follows that $E_m^n(U) = S^1$ for some $n \in \mathbb{N}$ (in fact, for all sufficiently large n). Hence $E_m^n(U) \cap V \neq \emptyset$.

Let us find a point with dense orbit.
To do this, we will write numbers in $[0,1]$ in base m.

Base 10

We divide $[0,1]$ into 10 closed intervals (they share endpoints)



$x = 0.3\ldots \iff x$ is in interval 3

$x = 0.31\ldots \iff$ —— " —— 31.

...

$x = 0.a_1a_2a_3\ldots$ means that $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$

Multiplication by 10 shifts the sequence; and so $\{10x\} = 0.a_2a_3\ldots$

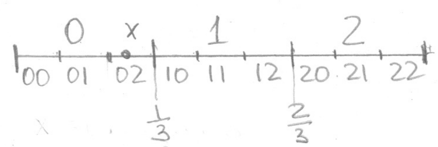
For endpoints of the intervals, there are two decimal expansions:

$$0.a_1\ldots a_{k-1}\underset{\neq 9}{a_k}999\ldots = 0.a_1\ldots a_{k-1}(a_k+1)000\ldots$$

We can view such points as belonging to either of the two intervals.

Base m (similar)

For example, let $m=3$



$x = 0.0\ldots_3$ means that $x \in [0, \frac{1}{3}]$

$x = 0.02\ldots_3$ —— " —— $x \in [\frac{2}{9}, \frac{1}{3}]$

$x = 0.a_1a_2a_3\ldots_3$ means that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$

The endpoints of the intervals have two expansions in base 3, for example, $\frac{1}{3} = 0.02222\ldots_3 = 0.1000\ldots_3$

Every other point has exactly one expansion.

Multiplying by 3 shifts the sequence: $\{3x\} = 0.a_2a_3\ldots$

A way to look at it: $x = 0.a_1a_2a_3\ldots_3 \Rightarrow x$ is in the interval a_1

$E_3(x)$ —— " —— a_2

$E_3^2(x)$ —— " —— a_3

...

We observe that the orbit of x is dense in $S^1 \iff$
$\iff$ it visits each of the intervals $[\frac{k}{3^n}, \frac{k+1}{3^n}]$, $n \in \mathbb{N}$, $0 \le k < 3^n$.

Give a base 3 expansion of a pt. with dense orbit.