Circle homeomorphisms and diffeomorphisms.

Rotation number.

Ex Consider a circle rotation R_α. Let $F:\mathbb{R}\to\mathbb{R}$ be its lift.
Then $F(x)=x+\alpha+k$, where $k\in\mathbb{Z}$, and
$\lim\limits_{n\to\infty}\frac{1}{n}(F^n(x)-x)=\lim\limits_{n\to\infty}\frac{1}{n}(\cancel{x}+n\alpha+nk\cancel{-x})=\alpha+k=\alpha \mod 1$.

Let $f:S^1\to S^1$ be an orientation-preserving homeomorphism.
Then the degree of f is 1. Let $F:\mathbb{R}\to\mathbb{R}$ be a lift of f,
i.e. F is continuous and $f\circ\pi=\pi\circ F$. Then

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \pi\downarrow & & \downarrow\pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

F is strictly increasing;
for any $x\in\mathbb{R}$, $F(x+1)-F(x)=1$ and hence $F(x+k)-F(x)=k$ for any $k\in\mathbb{Z}$;
for any $x,y\in\mathbb{R}$ with $|x-y|\le 1$, $|F(x)-F(y)|\le 1$;
for any $x\in\mathbb{R}$, $F(x+1)-(x+1)=F(x)+1-x-1=F(x)-x$,
thus $F(x)-x$ is 1-periodic and hence bounded (above and below) on $\mathbb{R}$.
These properties also hold for F^n, $n\in\mathbb{N}$. To see this, we can either show by induction that F^n is strictly increasing and $F^n(x+1)-F^n(x)=1$, or observe that f^n is also an orientation-preserving circle homeomorphism and show that F^n is its lift.

<u>Proposition</u> Let $f:S^1\to S^1$ be an orientation-preserving homeomorphism, and let $F:\mathbb{R}\to\mathbb{R}$ be a lift of f. Then for every $x\in\mathbb{R}$
the limit $\tau(F)=\lim\limits_{n\to\infty}\frac{1}{n}(F^n(x)-x)$ exists and is independent of x.
If F_1,F_2 are lifts of f, then $\tau(F_1)-\tau(F_2)=F_1-F_2\in\mathbb{Z}$.

<u>Def</u> $\tau(f)=\pi(\tau(F))$ is called the <u>rotation number</u> of f.
It is defined mod 1, and we can choose a representative in $[0,1)$.
The rotation number is also denoted by $\rho(f)$

<u>Ex</u> $\tau(R_\alpha)=\alpha$.

Proof of the proposition

- Existence. Let $x \in \mathbb{R}$. Consider the sequence $a_n = F^n(x) - x$.

Lemma Suppose that for a sequence (a_n) of real numbers there is $L \in \mathbb{R}$ s.t.
(*) $a_{m+n} \le a_m + a_n + L$ for all $m, n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists in $\mathbb{R} \cup \{-\infty\}$.

Pf Let $a = \liminf \frac{a_n}{n}$. Then $a \in \mathbb{R} \cup \{-\infty\}$ since $\frac{a_n}{n} \le \frac{n a_1 + nL}{n} = a_1 + L$

Let $c > a$. We will show that $\limsup \frac{a_n}{n} \le c$.
We take a large $n \in \mathbb{N}$ s.t. $\frac{a_n}{n} + \frac{L}{n} < c$. For any $\ell > n$ we write $\ell = kn + r$, $0 \le r < n$, and applying (*) obtain $a_\ell \le k a_n + a_r + kL$
Then $\frac{a_\ell}{\ell} \le \frac{k a_n + a_r + kL}{\ell} \le \frac{a_n}{n} + \frac{a_r}{\ell} + \frac{L}{n} < c + \frac{a_r}{\ell}$.
Since $\{a_r : 0 \le r < n\}$ is bdd, $\frac{a_r}{\ell} \to 0$ as $\ell \to \infty$.
It follows that $\limsup \frac{a_n}{n} \le c$ for any $c > a$. Thus $\lim\limits_{n\to\infty} \frac{a_n}{n} = a$. □

To prove the existence of the limit of $\frac{1}{n}(F^n(x) - x)$ in $\mathbb{R}$, we show that $a_n = F^n(x) - x$ satisfies $a_{m+n} \le a_m + a_n + 1$, and $\frac{a_n}{n}$ is bounded below.
Let $x_n = F^n(x)$, so that $a_n = x_n - x$, and let $k = k_n = \lfloor a_n \rfloor$. Then we have

$$a_{m+n} = F^{m+n}(x) - x = F^m(x_n) - x_n + x_n - x =$$
$$= \underbrace{(F^m(x+k) - (x+k))}_{= F^m(x) - x = a_m} + \underbrace{(x_n - x)}_{a_n} + \underbrace{(F^m(x_n) - F^m(x+k))}_{\le 1 \text{ since } |x_n - x - k| \le 1} - \underbrace{(x_n - x - k)}_{a_n - \lfloor a_n \rfloor \ge 0} \le a_m + a_n + 1$$

The sequence $\frac{a_n}{n}$ is bdd below since $\frac{a_n}{n} = \frac{1}{n}(F^n(x) - x) =$
$$= \frac{1}{n} \sum_{i=0}^{n-1} (F^{i+1}(x) - \underbrace{F^i(x)}_{x_i}) = \frac{1}{n} \sum_{i=0}^{n-1} (F(x_i) - x_i) \ge \min_{0 \le y \le 1} (F(y) - y).$$
Therefore, $\frac{a_n}{n} = \frac{1}{n}(F^n(x) - x)$ converges to a limit in $\mathbb{R}$.

- Independence of x. It suffices to consider $x, y \in [0,1)$ (why?)
We have: $\left| \frac{1}{n}(F^n(x) - x) - \frac{1}{n}(F^n(y) - y) \right| \le \frac{1}{n}\Big(\underbrace{|F^n(x) - F^n(y)|}_{\le 1 \text{ as } |x-y| \le 1} + \underbrace{|x-y|}_{\le 1}\Big) \le \frac{2}{n}$
Hence $\lim \frac{1}{n}(F^n(x) - x) = \lim \frac{1}{n}(F^n(y) - y)$.

- Let F_1, F_2 be two lifts of f. Then $F_1 = F_2 + k$, and it follows that $\tau(F_1) = \tau(F_2) + k$ (Show this) □

Questions:

- Is $\tau(f)$ an invariant of top. conjugacy? Yes, for orientation-preserving h.
- Let $\tau(f) = \alpha$. Is f top. conjugate to R_α? If not, is there a semiconjugacy? The answers depend on $\alpha \in, \notin \mathbb{Q}$, and for $\alpha \notin \mathbb{Q}$ on top. transitivity of f.