

# Newton's method for approximating zeros of functions

Let  $g$  be a continuously differentiable function on  $\mathbb{R}$ .

Suppose that  $g(x_*) = 0$ ,  $g'(x_*) \neq 0$ , and  $x_0$  is sufficiently close to  $x_*$ .

For  $n \geq 0$ , let  $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ , that is

$$x_{n+1} = F(x_n), \text{ where } F(x) = x - \frac{g(x)}{g'(x)}.$$

Then the sequence  $x_n = F^n(x_0)$  converges to  $x_*$ .

- The same applies to a function  $g : \mathbb{C} \rightarrow \mathbb{C}$ .

Question. Given a zero  $z_*$  of  $g$ , what does the set of numbers  $z \in \mathbb{C}$  such that  $F^n(z) \rightarrow z_*$  looks like?

Example 1.  $g(z) = z^2 - 1$  has two zeros,  $-1$  and  $1$ .

$$F(z) = z - \frac{g(z)}{g'(z)} = z - \frac{z^2-1}{2z} \text{ for } z \neq 0.$$

If  $z = bi \neq 0$ , then  $F(z) = di$  and  $F^n(z)$  does not converge to  $1$  or  $-1$ .

If  $z = x + iy$  with  $x > 0$ , then  $F^n(z) \rightarrow 1$ .

If  $z = x + iy$  with  $x < 0$ , then  $F^n(z) \rightarrow -1$ .

Example 2.  $g(z) = z^3 - 1$  has three zeros:

$$z_1 = 1, \quad z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$F(z) = z - \frac{g(z)}{g'(z)} = z - \frac{z^3-1}{3z^2} \text{ for } z \neq 0.$$

One might guess that the complex plane is divided into three sectors, and in each one  $F^n(z)$  converges to the corresponding zero.

*But instead ...*