CS 10C: INTRO TO DATA STRUCTURES AND ALGORITHMS

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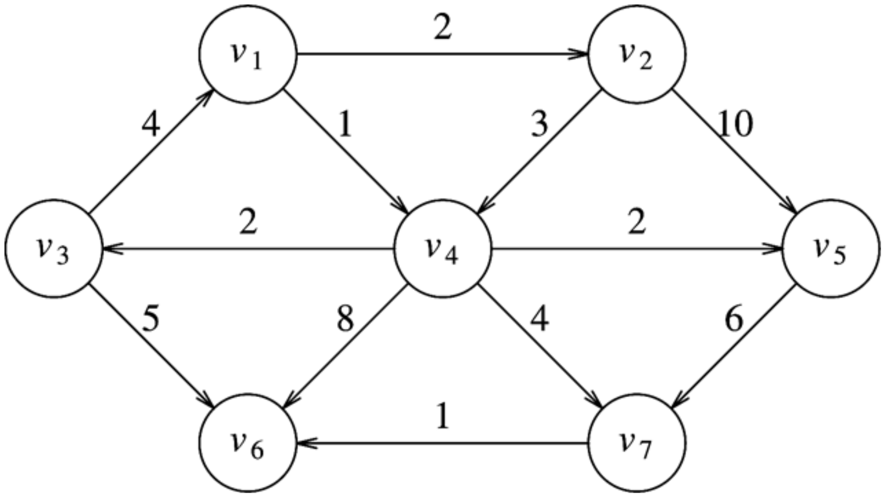
Graphs



Graph

- A Graph $G = (V,E)$
  - A set V of vertices and a set E edges
- A graph is a way representing connections/relationships between pairs of objects in V
- Each edge is a pair (v,w), where $v, w \in V$.
- Edges are either directed or undirected
  - Directed referred to as *ordered*
    - Directed graphs are called Digraphs
    - Directed graphs with no cycles (acyclic) are called DAGs
  - Undirected referred to as unordered

Graph

- An edge (v,v) is a loop
- Vertices can have incoming and outgoing edges
  - indegree – number of incoming edges of a vertex v
  - outdegree – number of outgoing edges of a vertex v
- A path in a graph is a sequence of vertices $w_1, w_2, w_3, \ldots, w_N$ such that $(w_i, w_{i+1}) \in E$ for $1 \leq i < N$
  - A **simple path** is a path where all vertices are distinct except possibly first/last
  - **path length:** $N-1$ for N vertices
- Edges can have an associated cost called the **weight**.

Graphs - Definitions

- An **undirected graph** is **connected** if there is a path from every vertex to every other vertex
- A **directed graph** that is connected is called **strongly connected.**
- **Weakly connected** - If directed graph is not strongly connected, but removing direction makes graph connected
- **Complete graph** – edge between every pair of vertices.

Graph Representations

- Adjacency matrix – $|V|^2$ matrix, with $\mathbf{1}$ if there is an edge between two vertices, $\mathbf{0}$ otherwise.
  - Good for dense graphs.
  - Wasted space if not dense.
  - If graph is sparse, adjacency list is better.

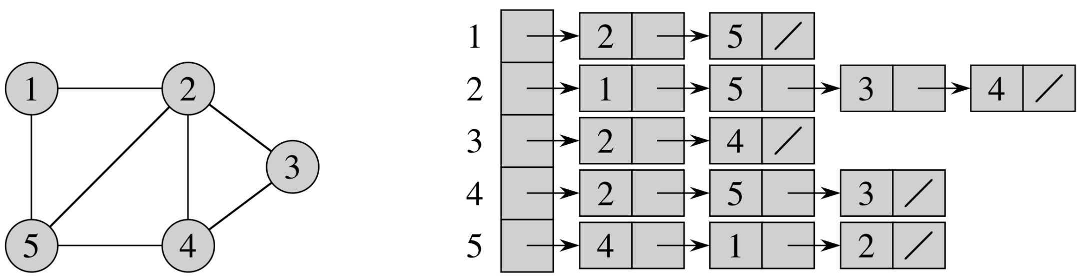
$$|E| = \Theta(|V|^2)$$

- Adjacency list – for each vertex, store a list of vertices that share an edge
  - Example…

Adjacency list

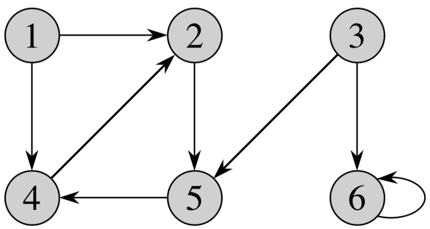
1	2, 4, 3
2	4, 5
3	6
4	6, 7, 3
5	4, 7
6	(empty)
7	6

Undirected Graph

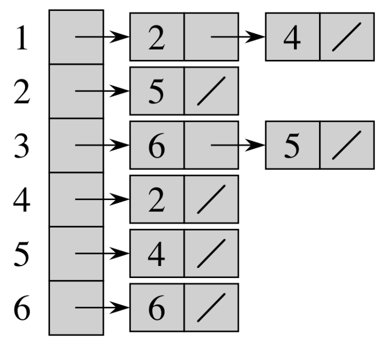


	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Directed Graph



(a)

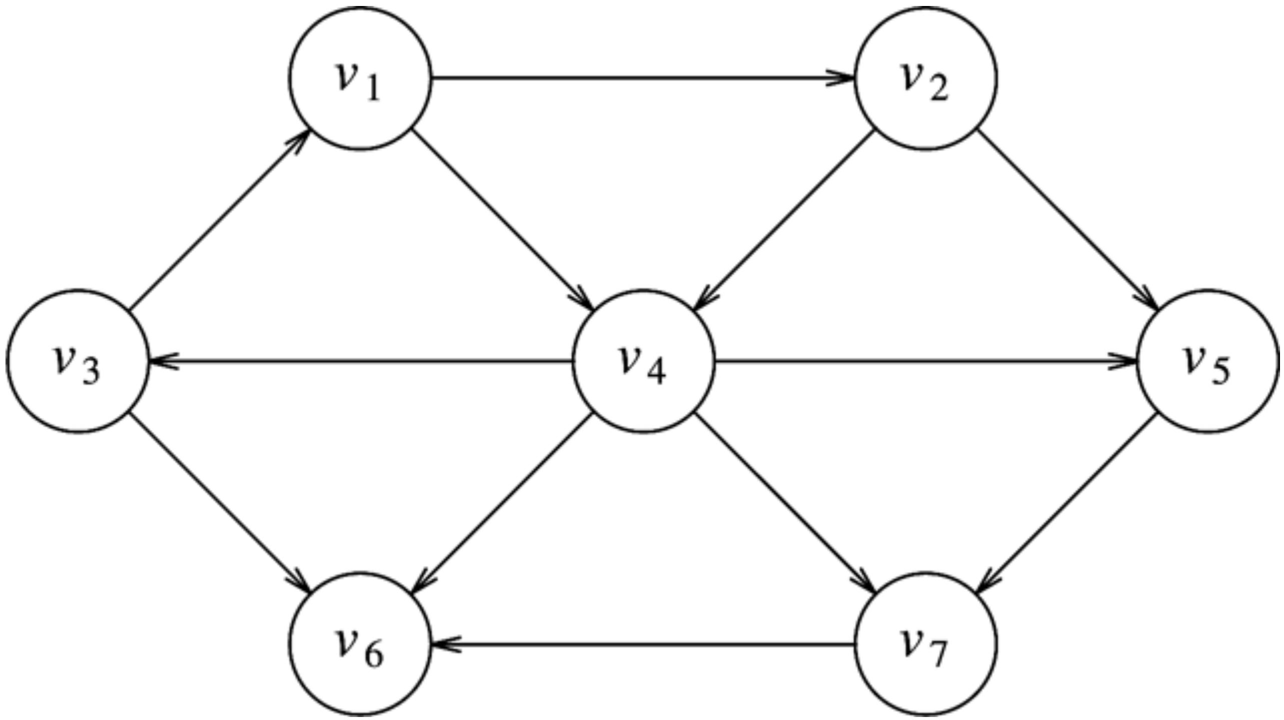


(b)

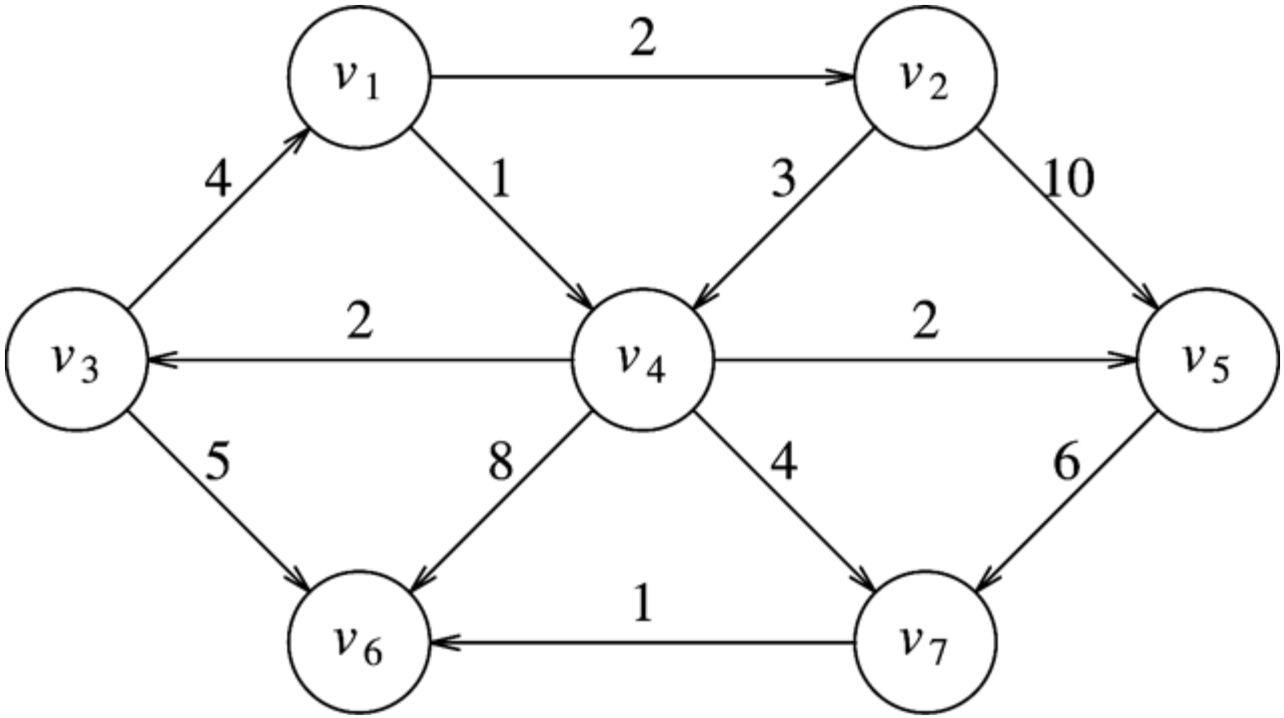
	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

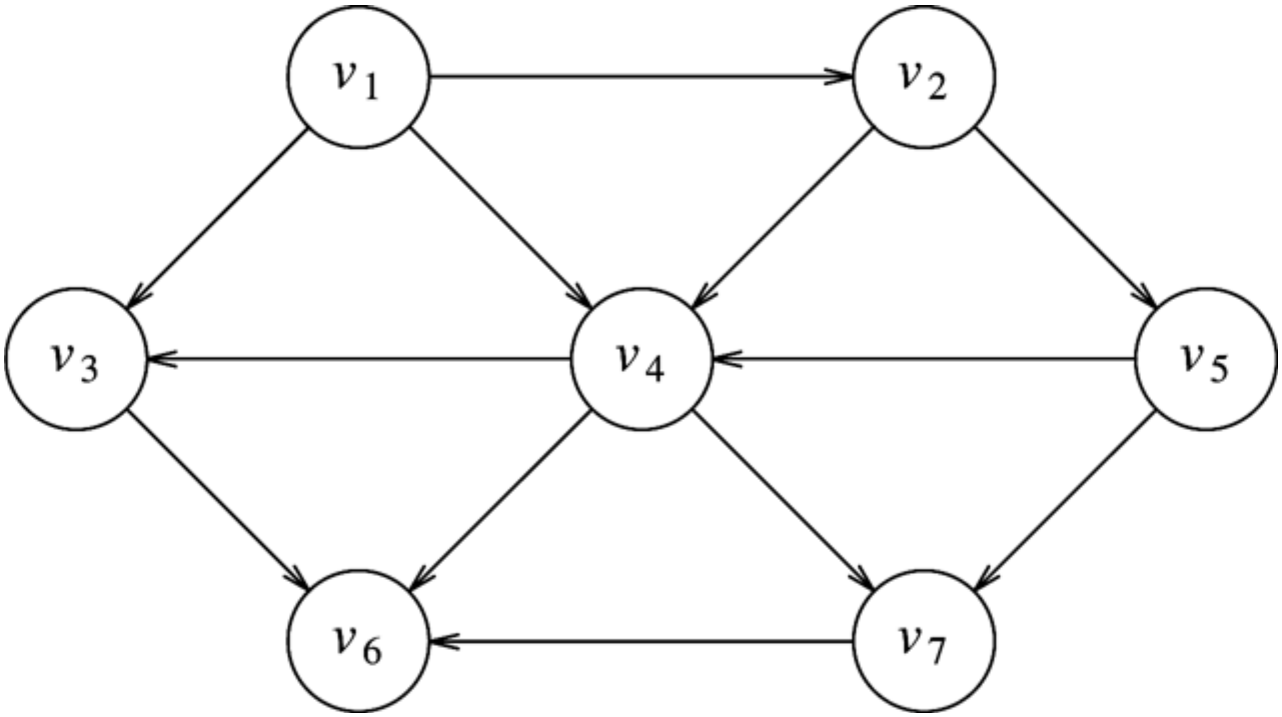
Directed Graph – Unweighted

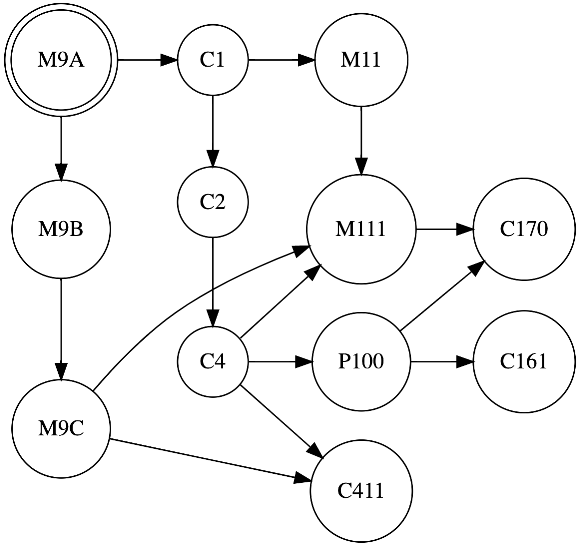


Directed Graph - Weighted



Directed Acyclic Graph

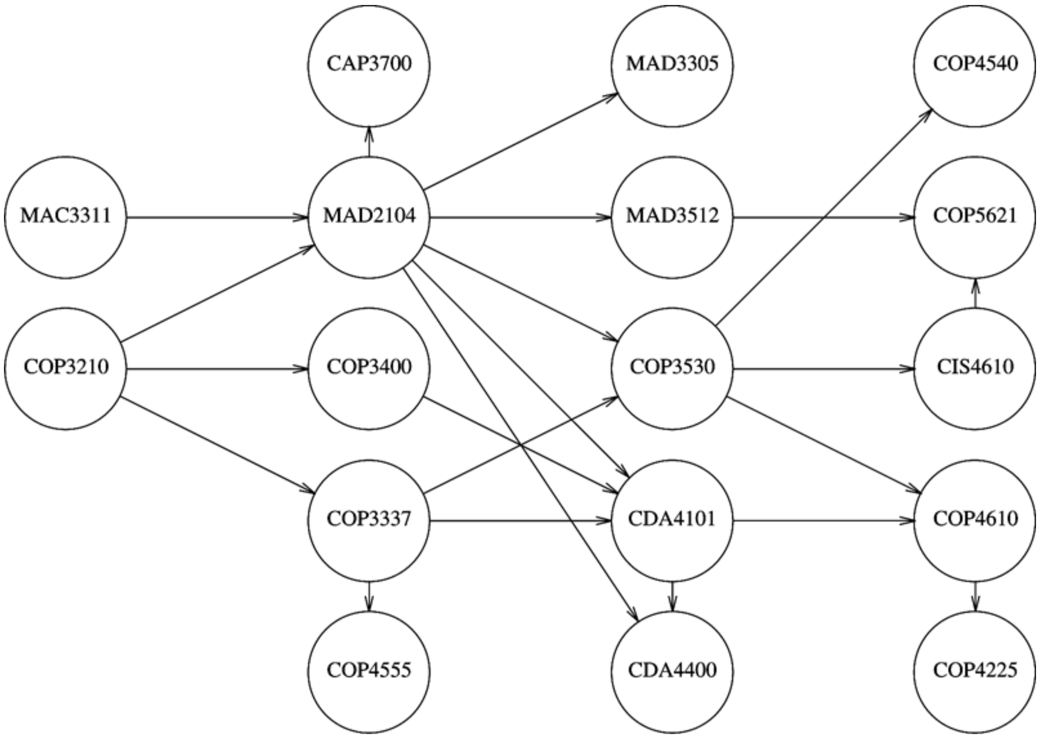



M9A
C1
M11
M9B
C2
M111
C170
C4
P100
C161
M9C
C411

Topological Sort

- Linear ordering of vertices for a DAG (Directed Acyclic Graph).
- Every vertex comes before all vertices to which it has outgoing edges.
- Every DAG has at least 1 topological sort
  - 1 indicates a Hamiltonian path
  - 2 or more, no Hamiltonian path
  - Hamiltonian path – every vertex is visited exactly once.
- Topological Sort - used in scheduling jobs or tasks

Scheduling

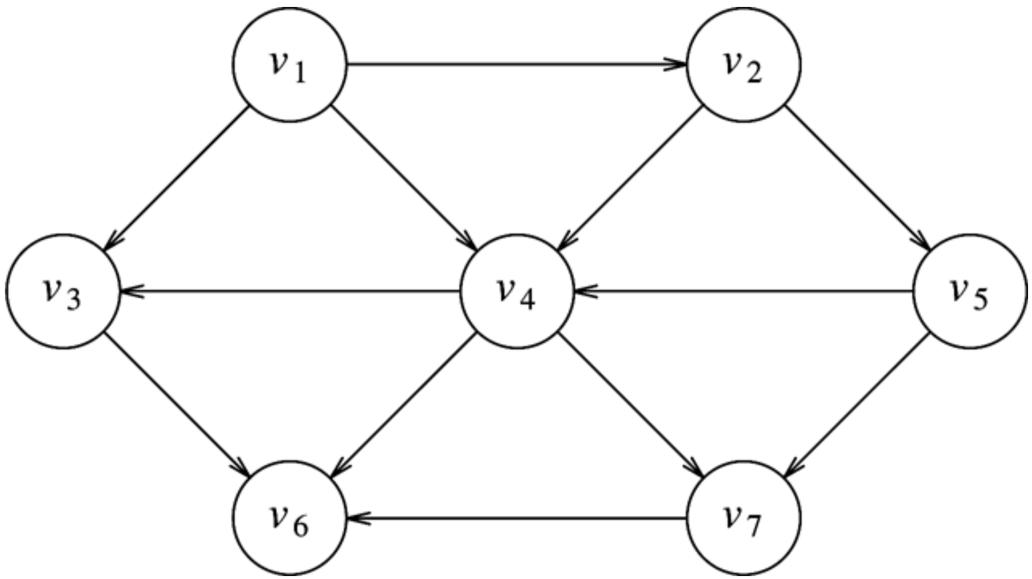

CAP3700
MAD3305
COP4540
MAC3311
MAD2104
MAD3512
COP5621
COP3210
COP3400
COP3530
CIS4610
COP3337
CDA4101
COP4610
COP4555
CDA4400
COP4225

Topological Sort

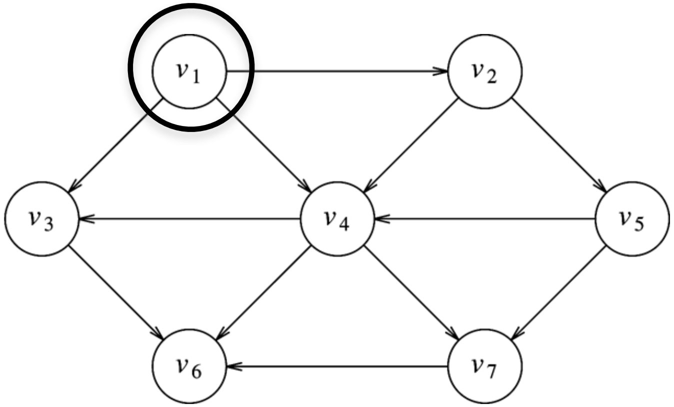
- **Simple algorithm:**
  1. Find an initial vertex v with no incoming edges
  2. Print v
  3. Remove v from graph
  4. Remove *v's* outgoing edges from graph, updating effected vertices
  5. Iterate (steps 1-4) over the remaining graph

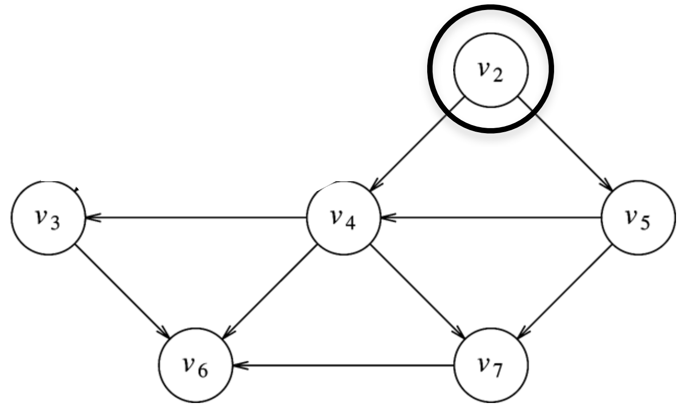
  Running Time : $O(|V|^2)$
  - Finding a vertex with in-degree zero, linear scan of V
  - There are $|V|$ calls to do this
- Speed up:
  - Use a Queue to store vertices with in-degree zero
  - Only have to remove the front of Queue, skip linear scan of V
  - **Running Time:** $O(|E| + |V|)$

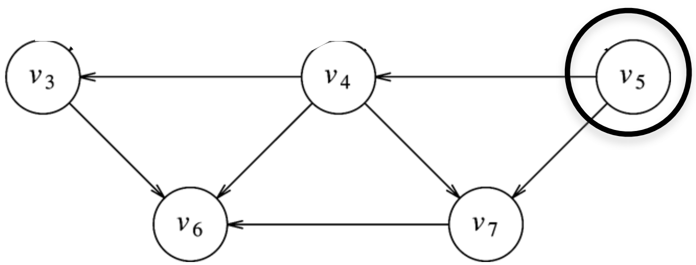
Topological Sort

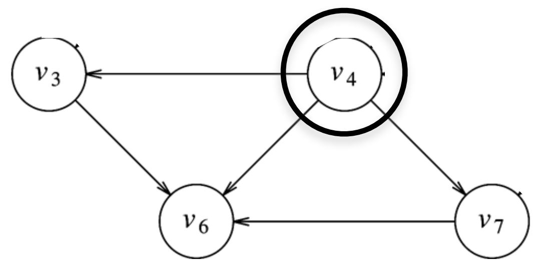


v1, v2, v5, v4, v3, v7, v6



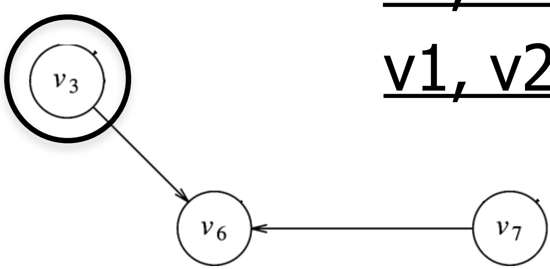


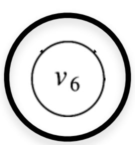




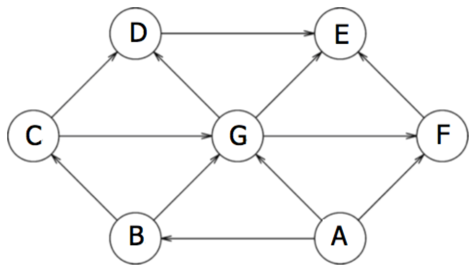
<u>v1, v2, v5, v4, v3, v7, v6</u>

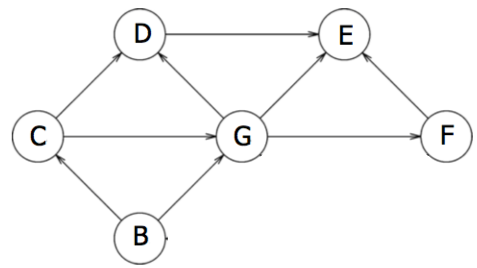
<u>v1, v2, v5, v4, v7, v3, v6</u>

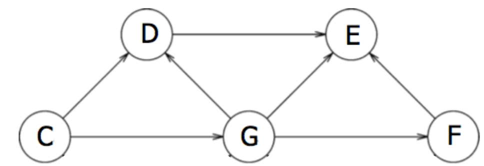


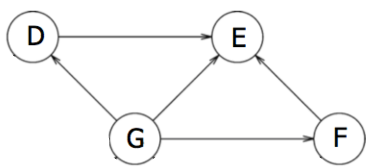


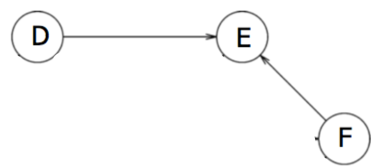
Topological Sort





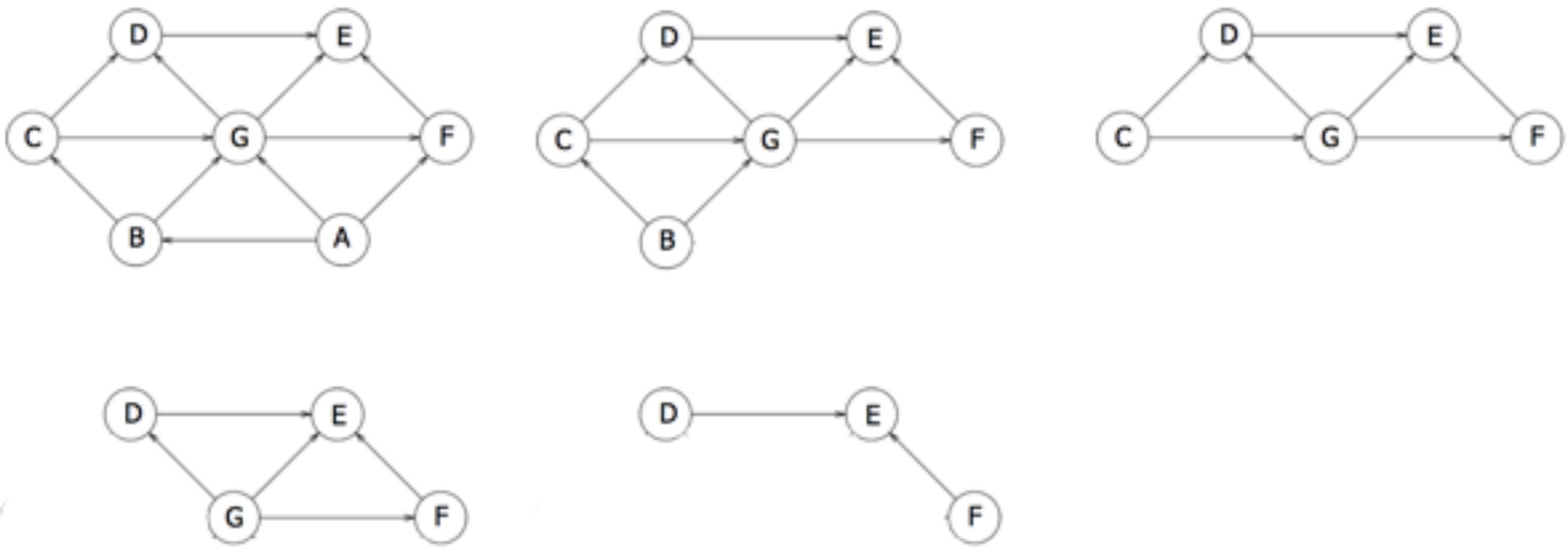






A, B, C, G, D, E, F

Topological Sort

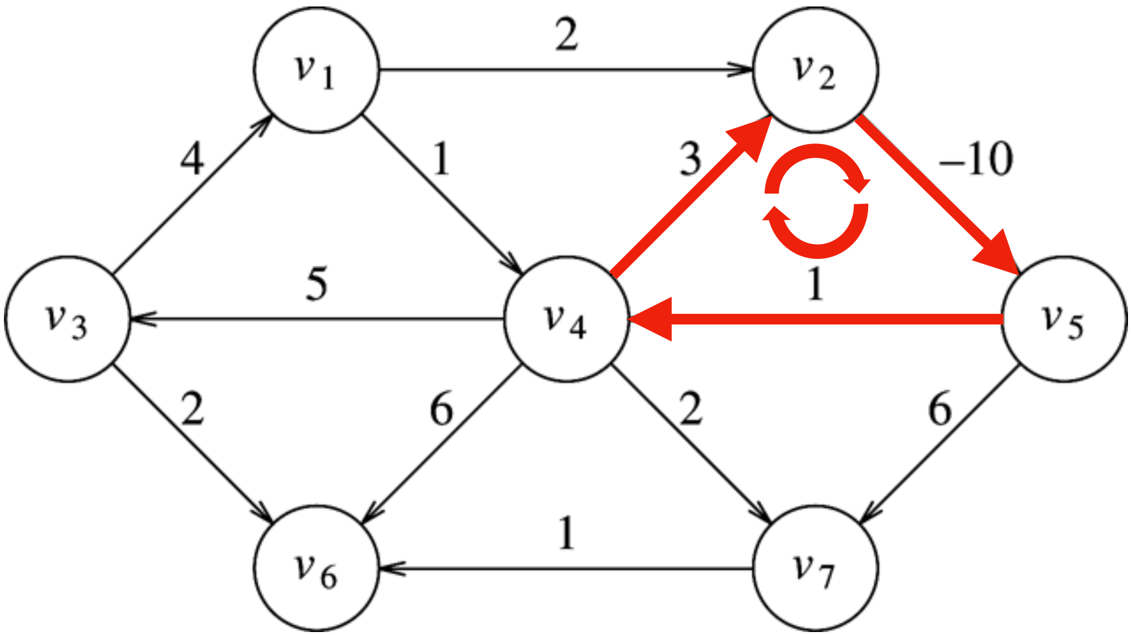


A, B, C, G, D, E, F

Single-Source Shortest Path

- **Problem:** find the path between two vertices that minimizes the number of edges or minimizes the sum of the weights of the edges.
  - Unweighted Graphs – number of edges
  - Weighted Graphs – sum of the costs
- **BFS** – solves for unweighted graphs
- **Dijkstra's** – solves for directed graph with positive weights
- **Bellman Ford** – solves for graphs that can include negative weights

Negative Cost Cycle



- Shortest Path is NOT defined.
- Reason: because you could complete cycle repeatedly and lower the cost.
- This is true for Dijkstra's and Bellman-Ford.

Single Pair Shortest Path Initialization Step

for each $u \in V - \{s\}$
$\quad u.d = \infty$
$s.d = 0$
$Q = \emptyset$
$\text{ENQUEUE}(Q, s)$

- Set all distances to infinite, since a path from source not yet found.
- Set distance to source equal to zero (no cost to go from a node to itself).

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.

$\text{BFS}(V, E, s)$

for each $u \in V - \{s\}$

$\quad u.d = \infty$

$s.d = 0$

$Q = \emptyset$

$\text{Enqueue}(Q, s)$

while $Q \neq \emptyset$

$\quad u = \text{Dequeue}(Q)$

$\quad$ **for** each $v \in G.Adj[u]$

$\quad\quad$ **if** $v.d == \infty$

$\quad\quad\quad v.d = u.d + 1$

$\quad\quad\quad \text{Enqueue}(Q, v)$

(u, v)
if v.d == infinity
then update v.d to u.d + 1

(u, v)
if v.d == infinity
then update v.d to 0 + 1

(u, v)
if v.d == infinity
then update v.d to 1 + 1

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.

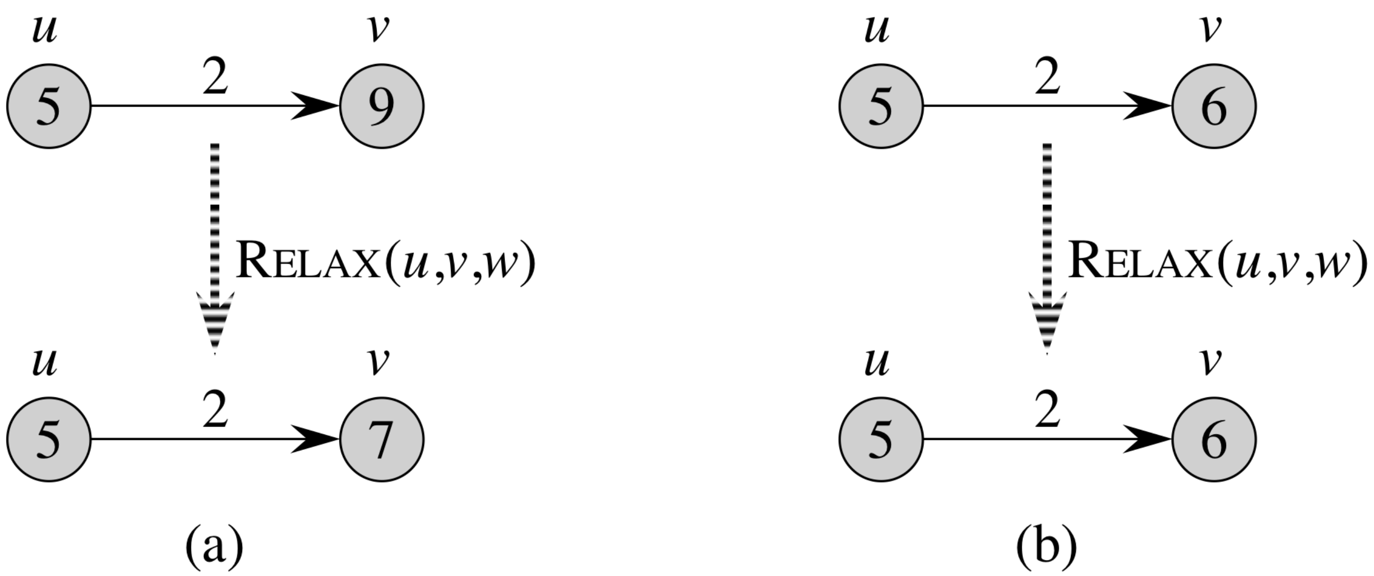
Dijkstra's - RELAX

$$
\begin{aligned}
&\textsc{Relax}(u, v, w) \\
&\quad \textbf{if } v.d > u.d + w(u, v) \\
&\qquad v.d = u.d + w(u, v) \\
&\qquad v.\pi = u
\end{aligned}
$$

- Single-pair shortest path update distance step for Dijkstra's is similar to BFS.
- Main difference is that the weights of the edges, e.g. $w(u, v)$, are positive values.
- Edge weights can be larger than 1.
- Dijkstra's - edge weights cannot be negative.

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.

Dijkstra's - RELAX



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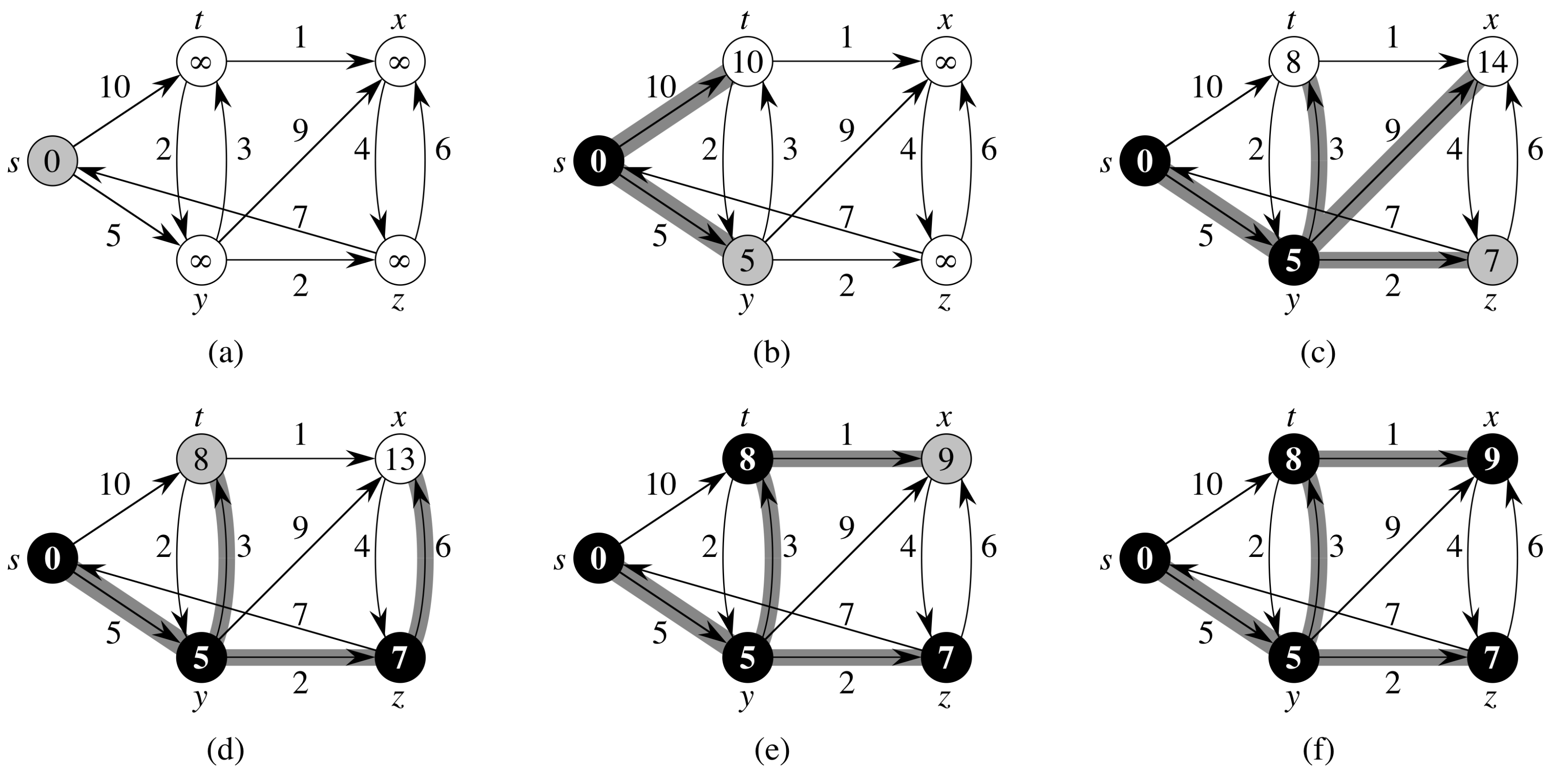
Dijkstra's - Example



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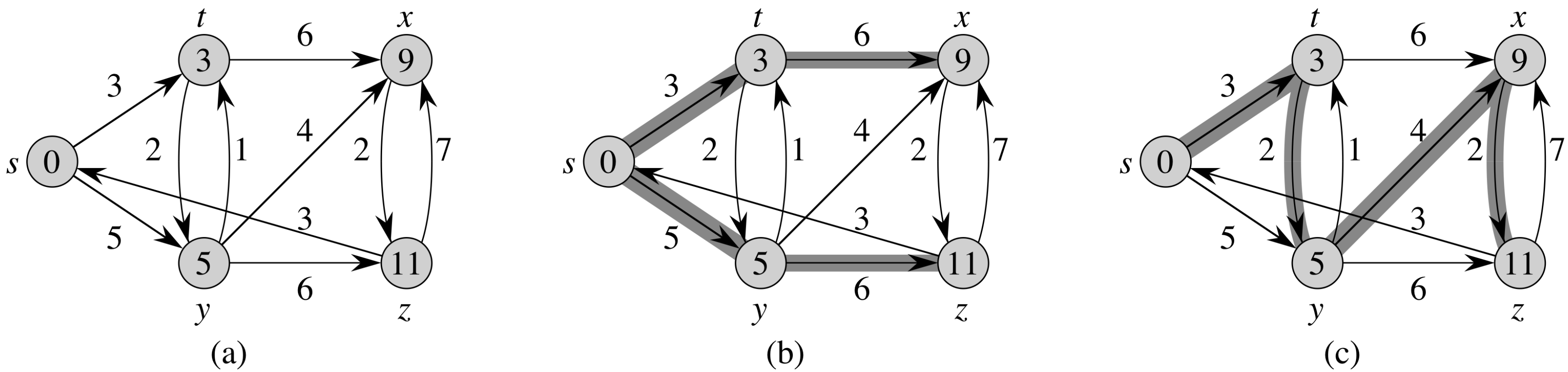
Non-Unique Shortest Path



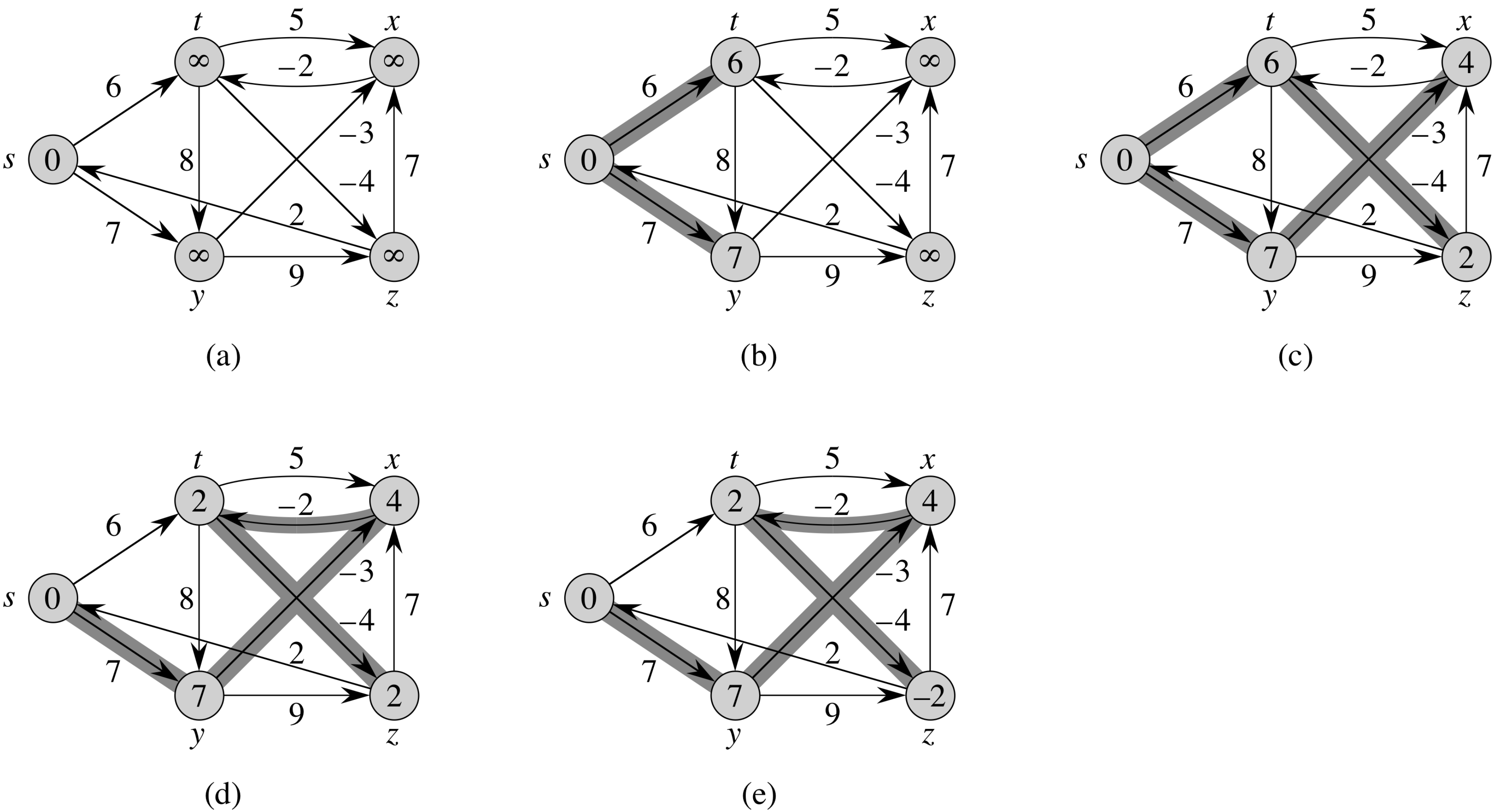
Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.

Bellman-Ford - CLRS

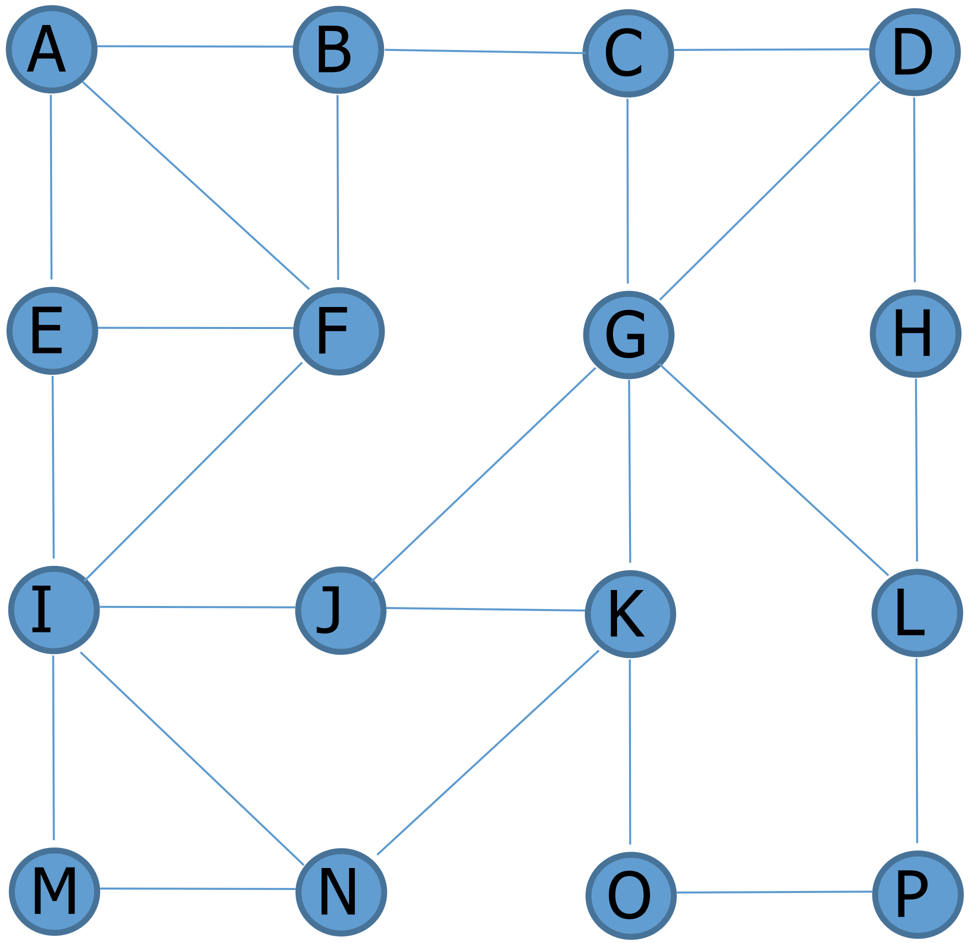
```
BELLMAN-FORD(G, w, s)
  INIT-SINGLE-SOURCE(G, s)
  for i = 1 to |G.V| - 1
      for each edge (u, v) ∈ G.E
          RELAX(u, v, w)
  for each edge (u, v) ∈ G.E
      if v.d > u.d + w(u, v)
          return FALSE
  return TRUE
```

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.

Bellman Ford - Example

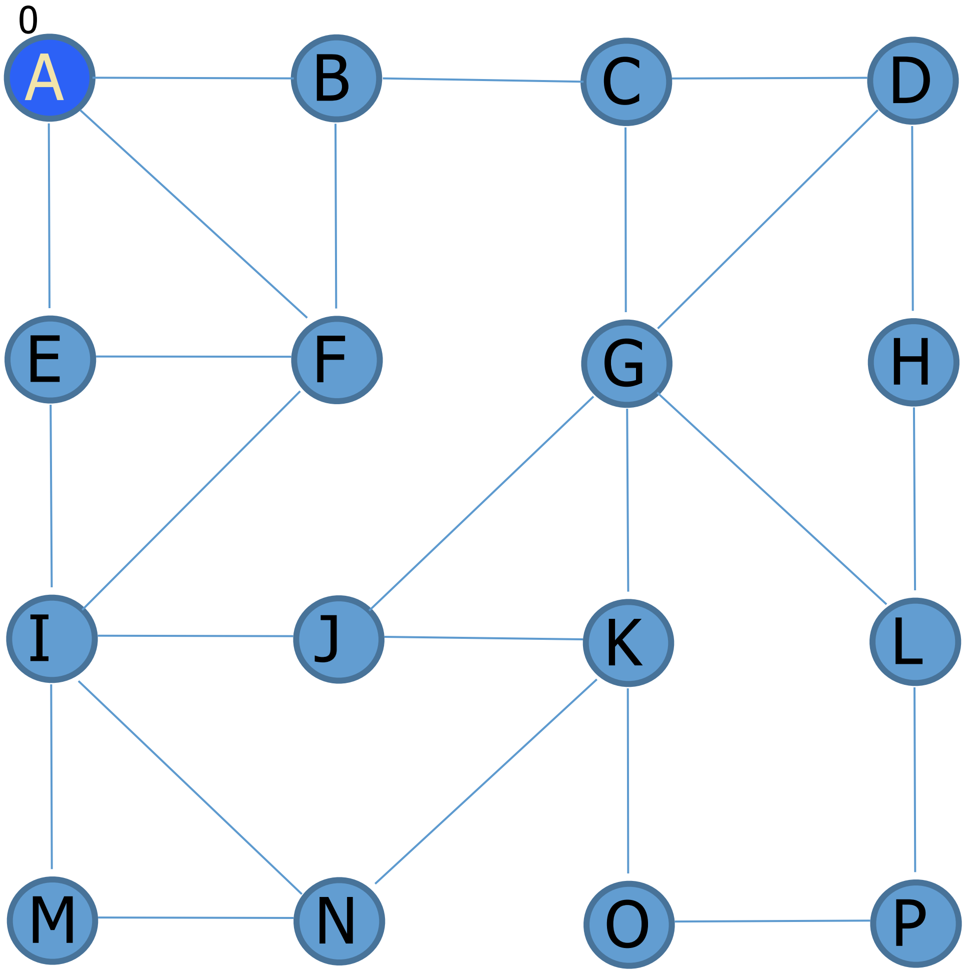


Breadth First Search (BFS)



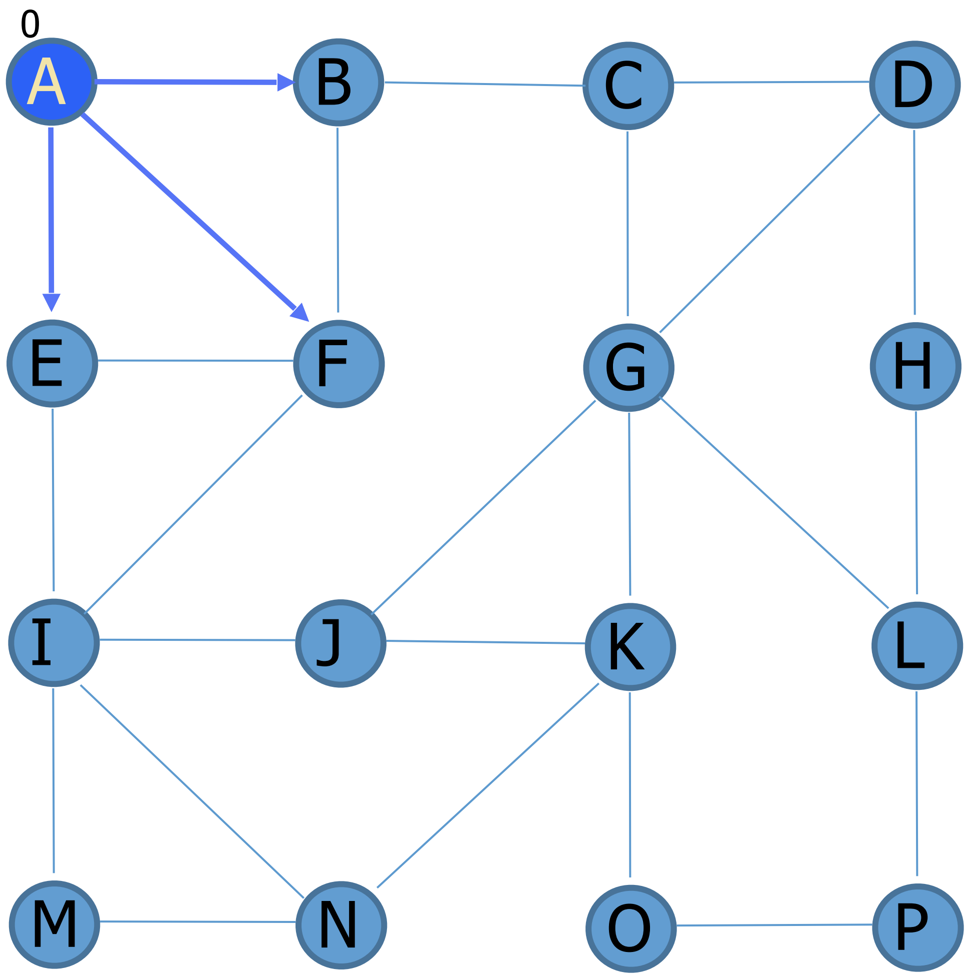
BFS

A visited



BFS

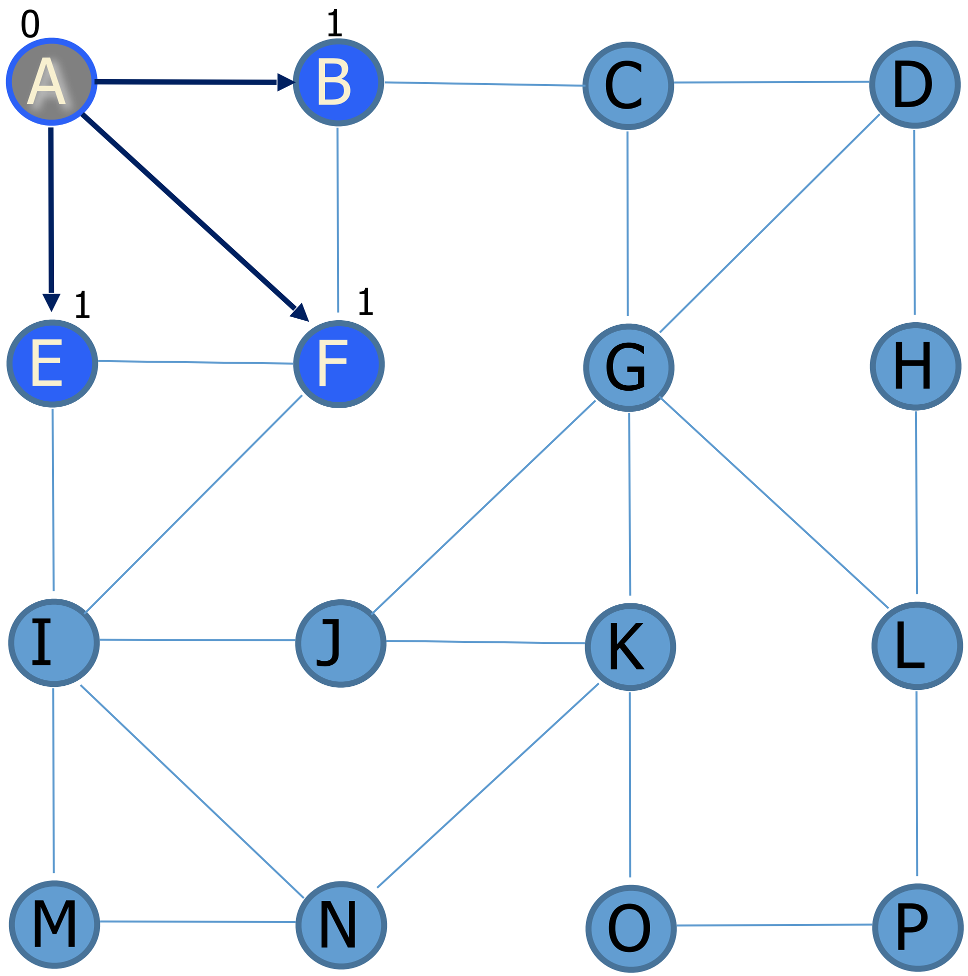
Explore A's neighbors



BFS

B, E, F visited

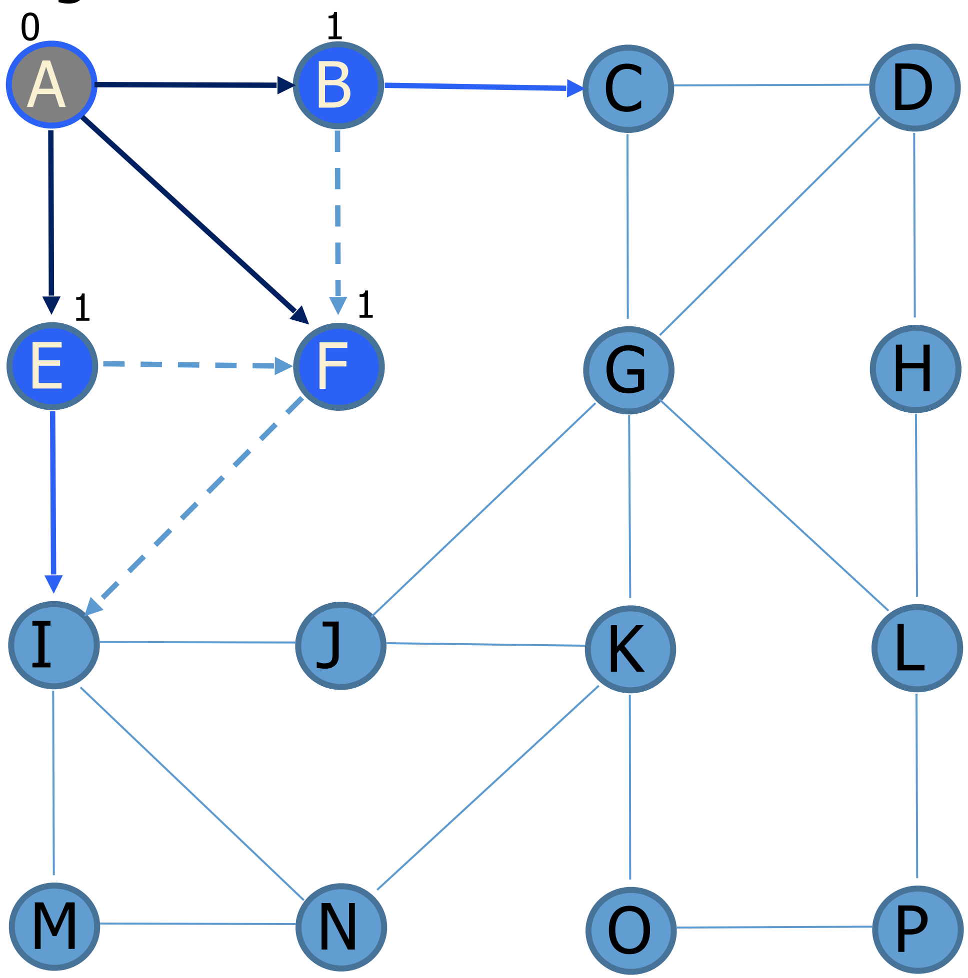
A is processed



BFS

Explore B's neighbors

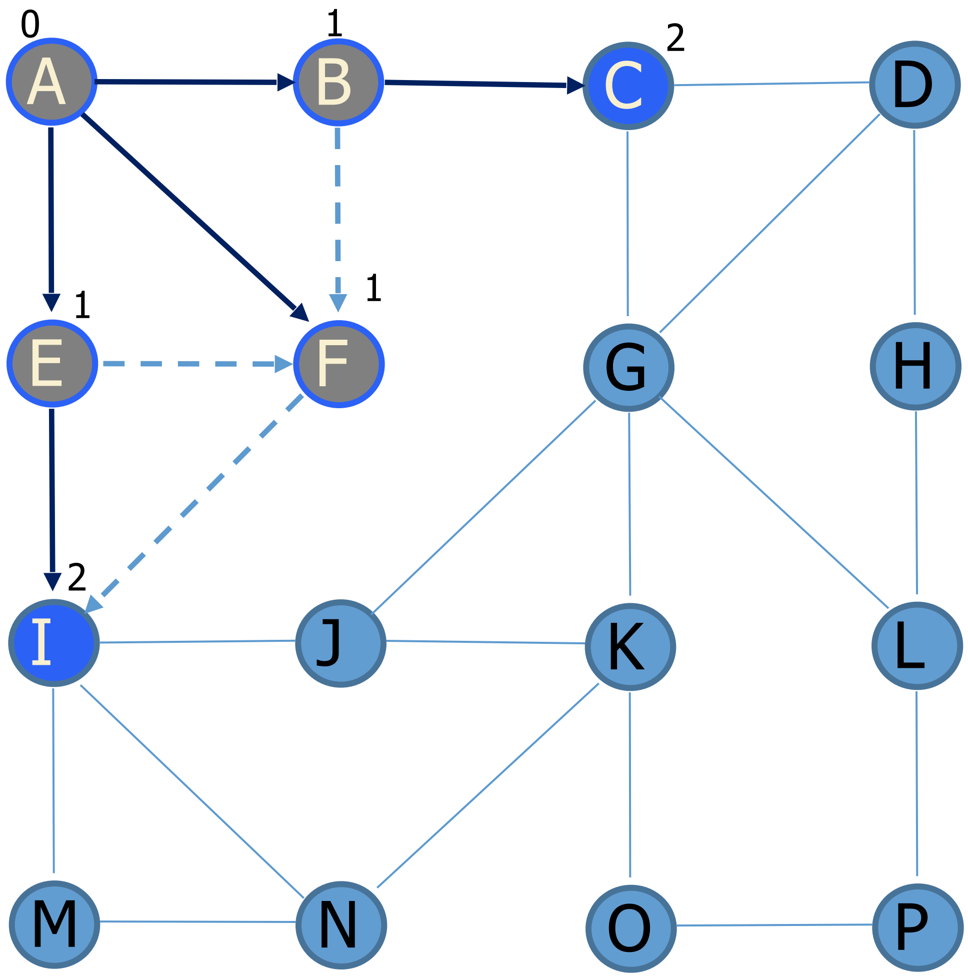
Explore E then F's neighbors



BFS

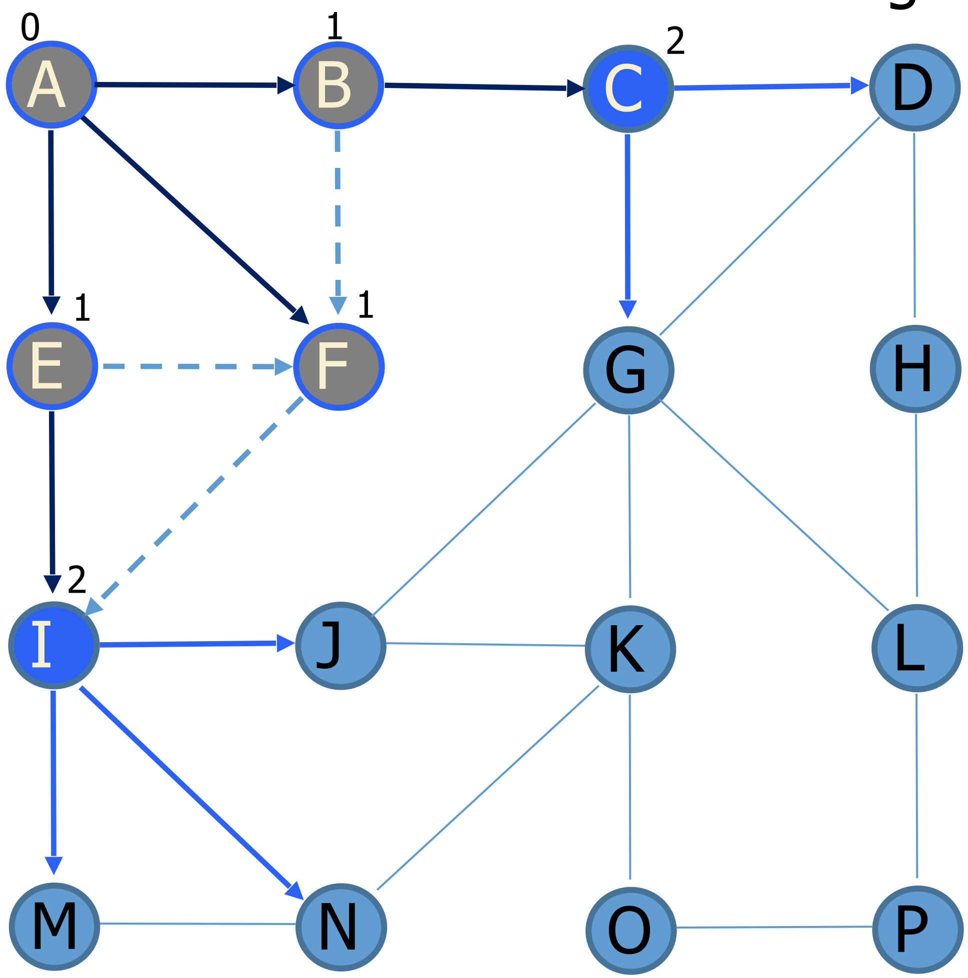
C, I
visited

B,E,F are
processed



BFS

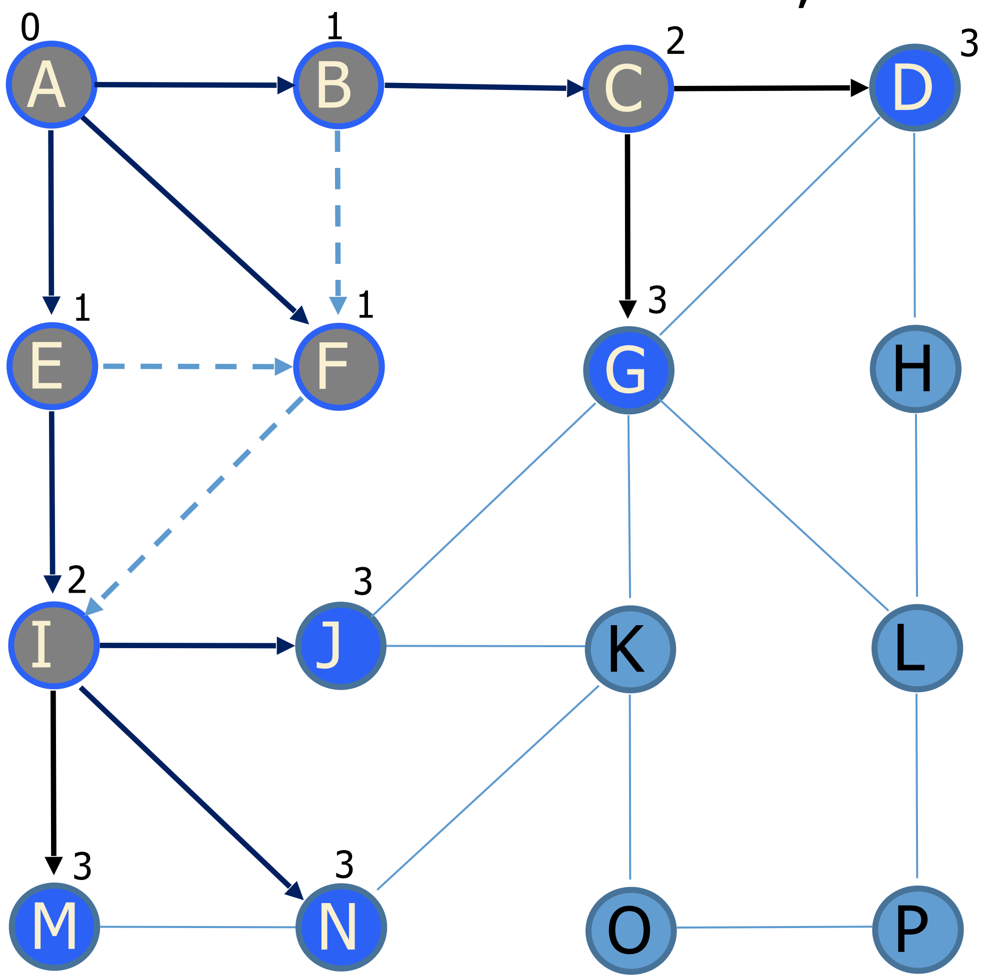
Explore C's neighbors



Explore I's neighbors

BFS

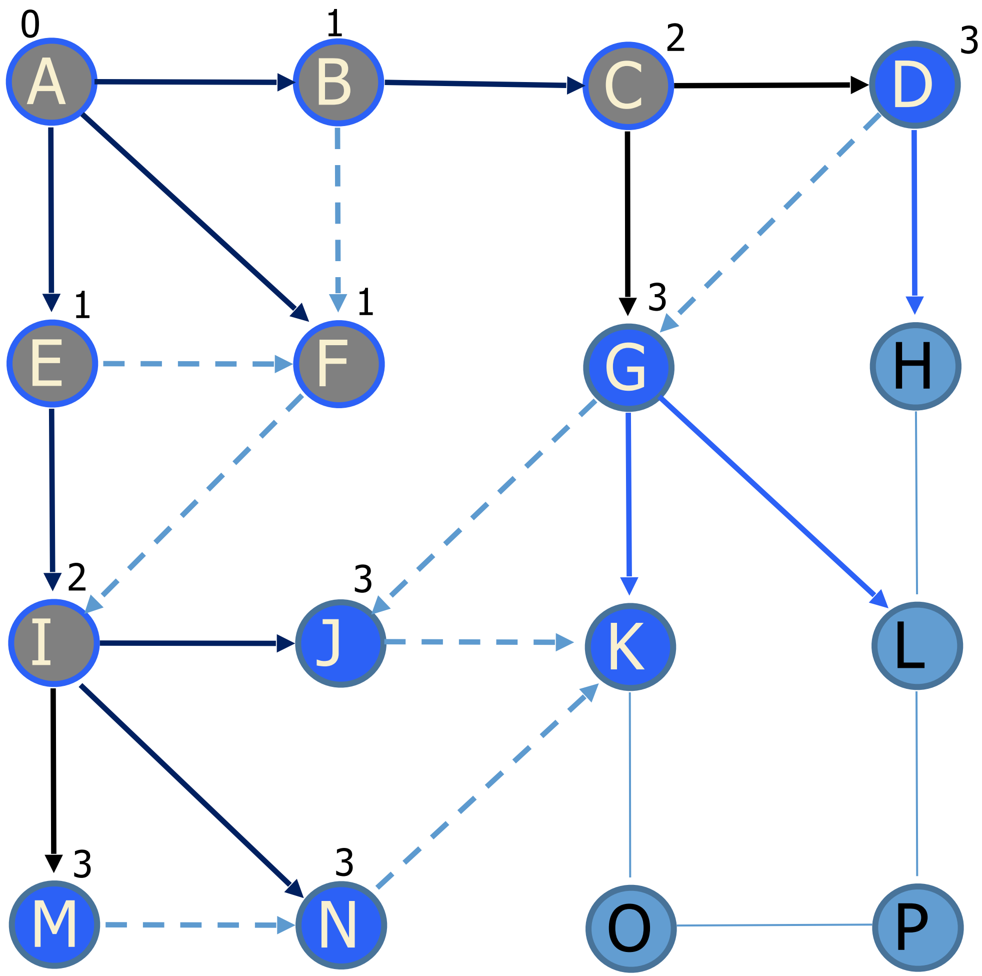
C processed
D,E visited



I processed
J,M,N
visited

BFS

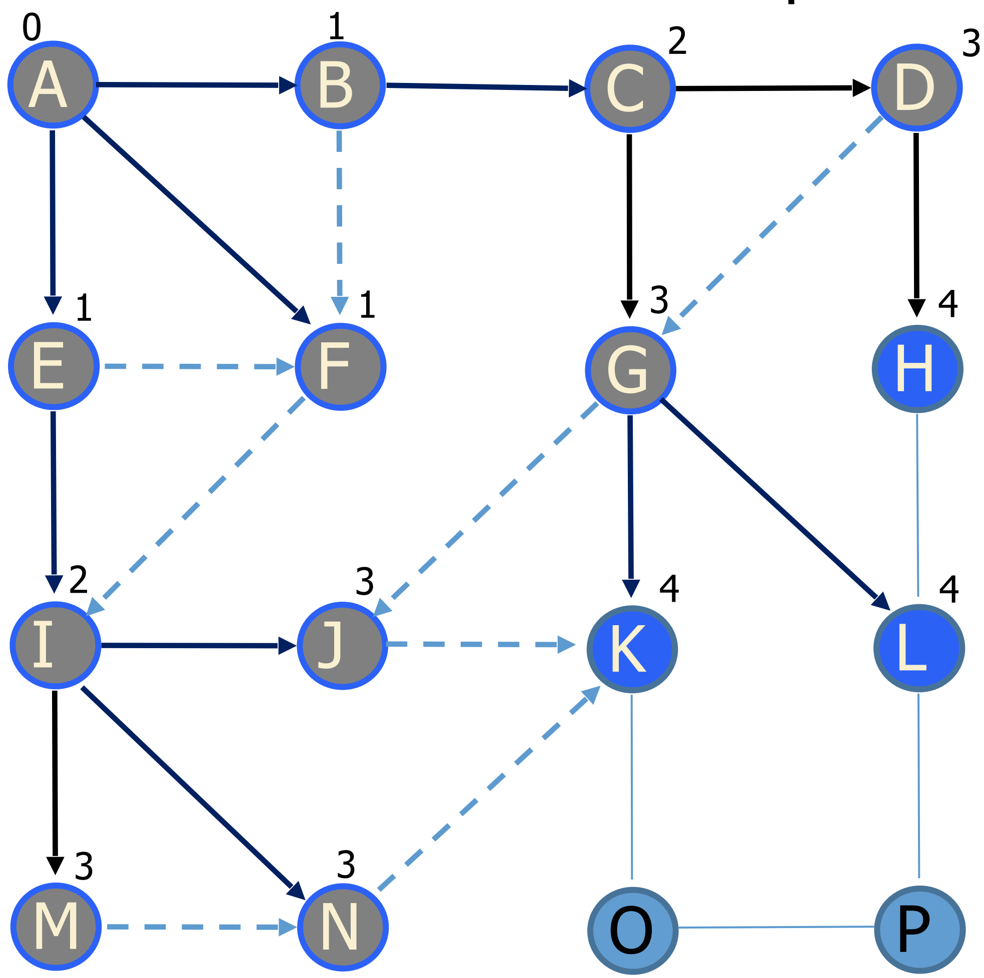
Explore D's,G's neighbors



Explore J's, M's, N's neighbors

BFS

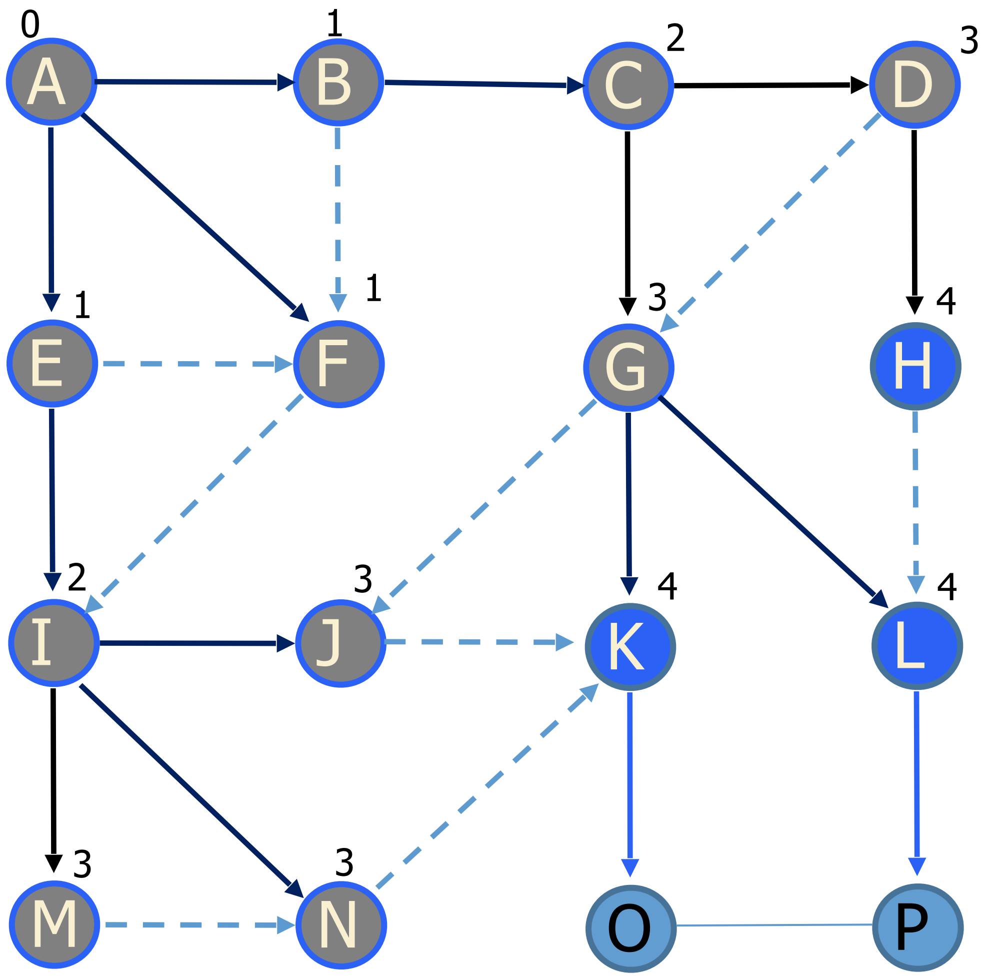
D, G processed



H, K, L visited

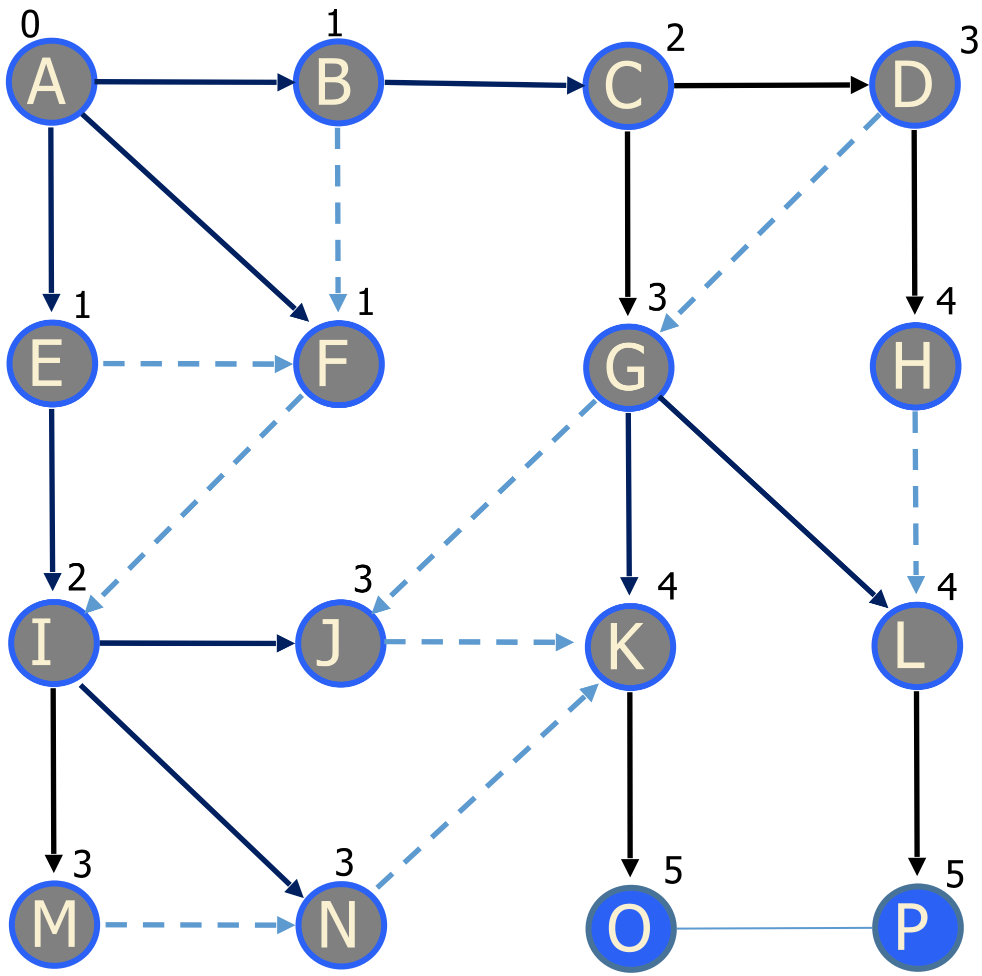
J, M, N processed

BFS



Explore H, K, L neighbors

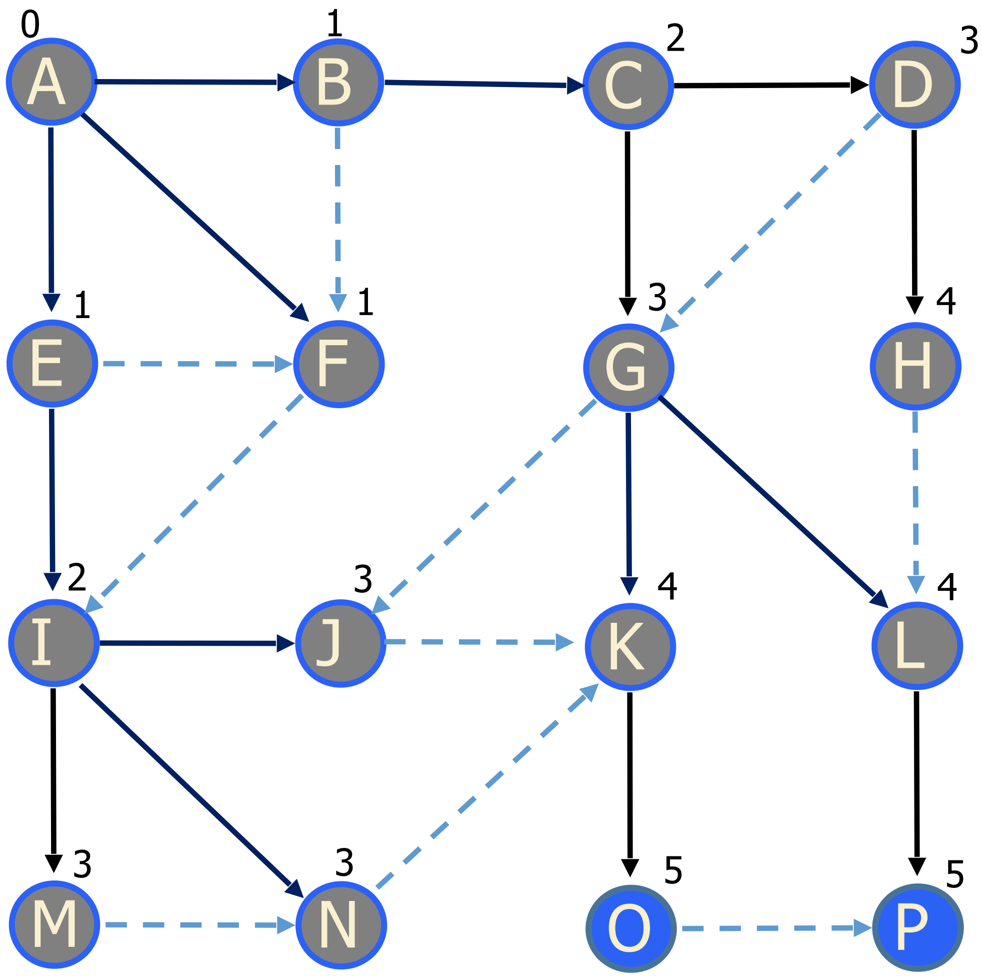
BFS



H, K, L processed

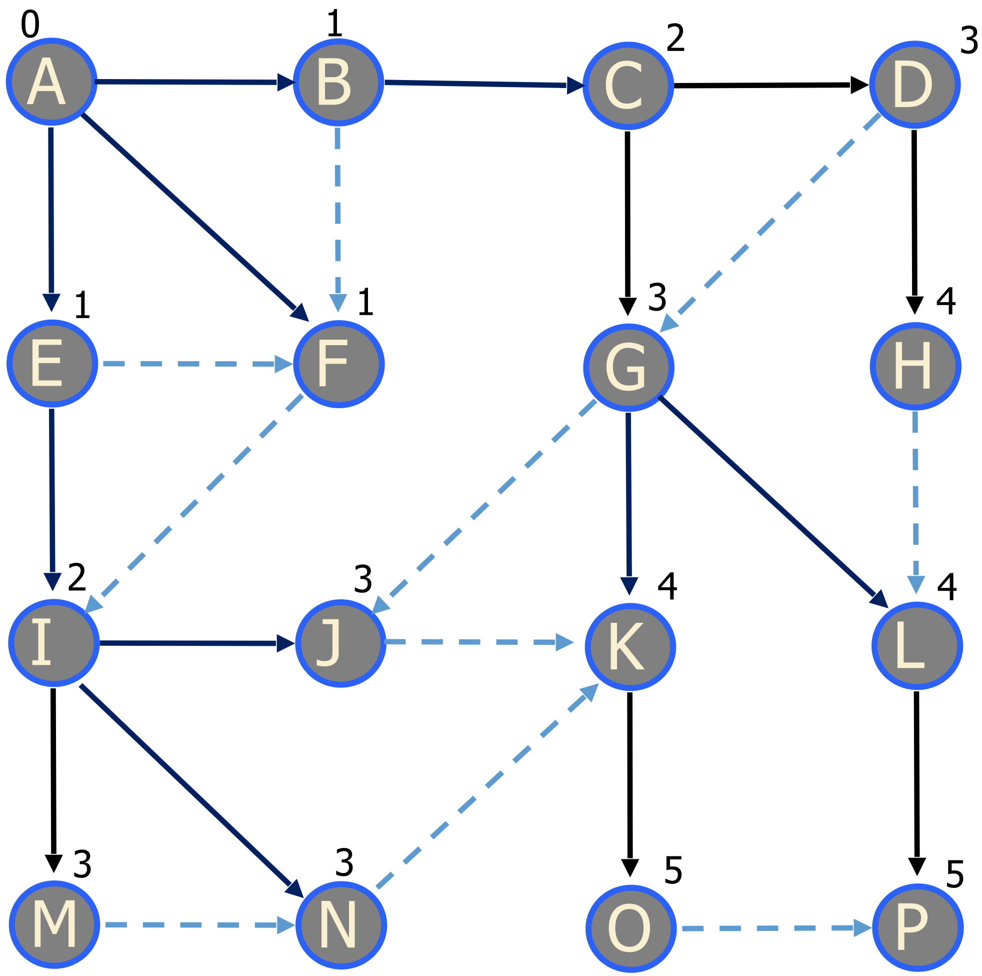
O,P visited

BFS



Explore O's neighbor

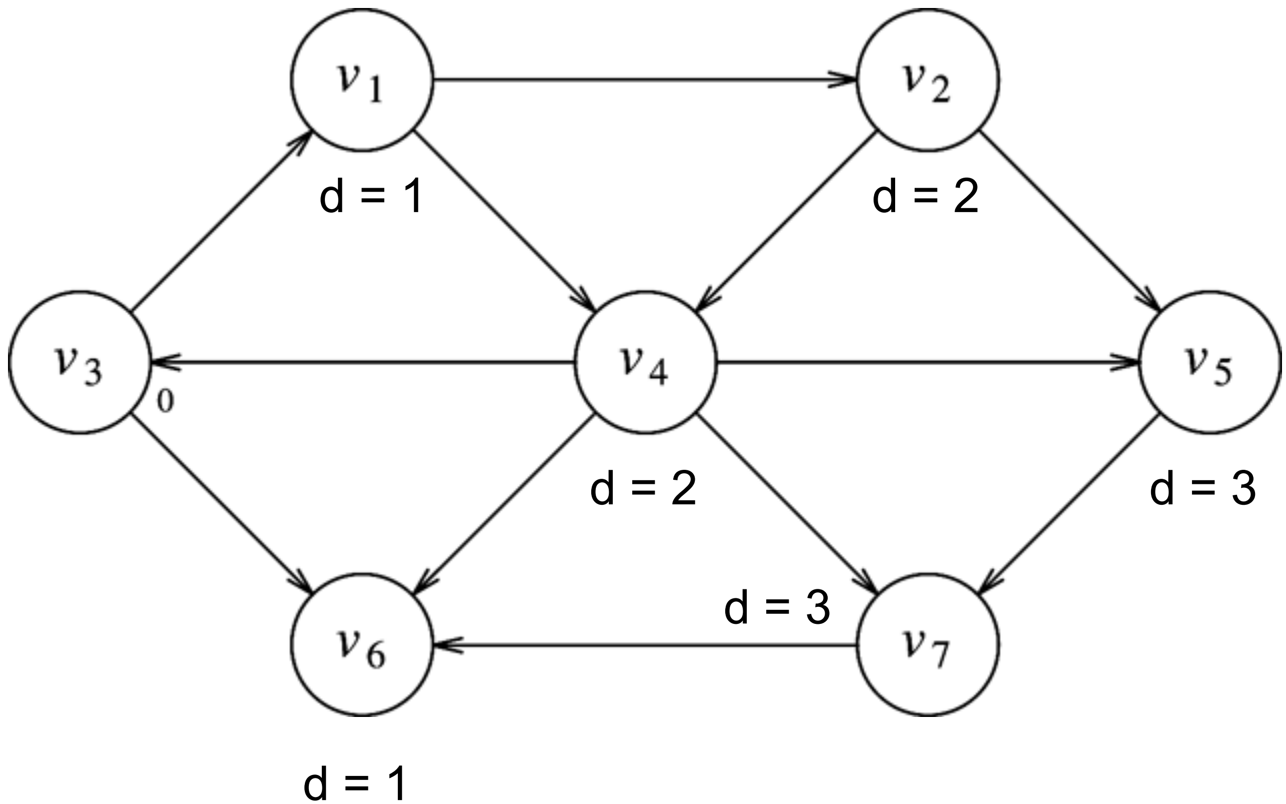
BFS



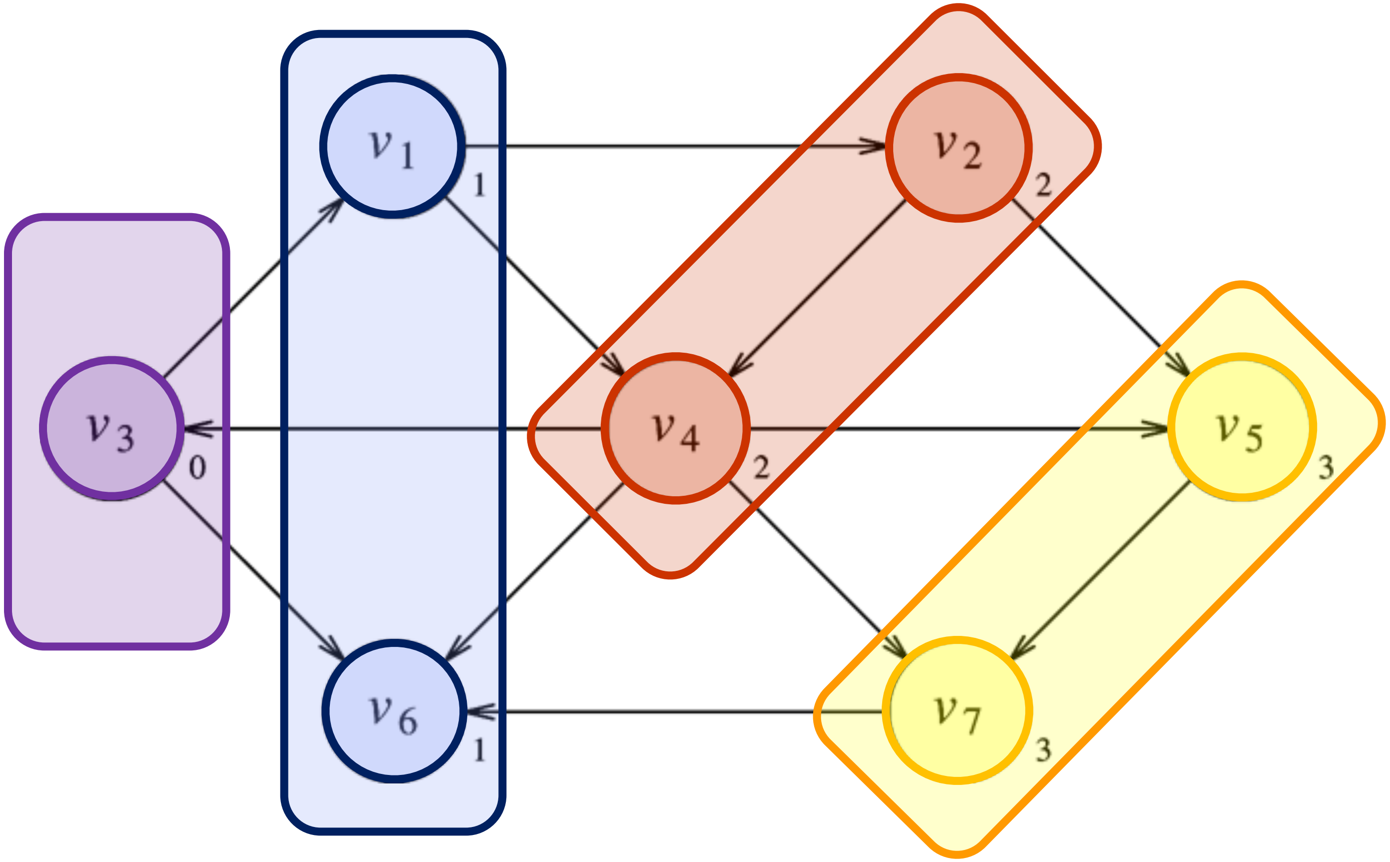
Done

Running Time = $O(|E|+|V|)$

BFS – In Class Activity

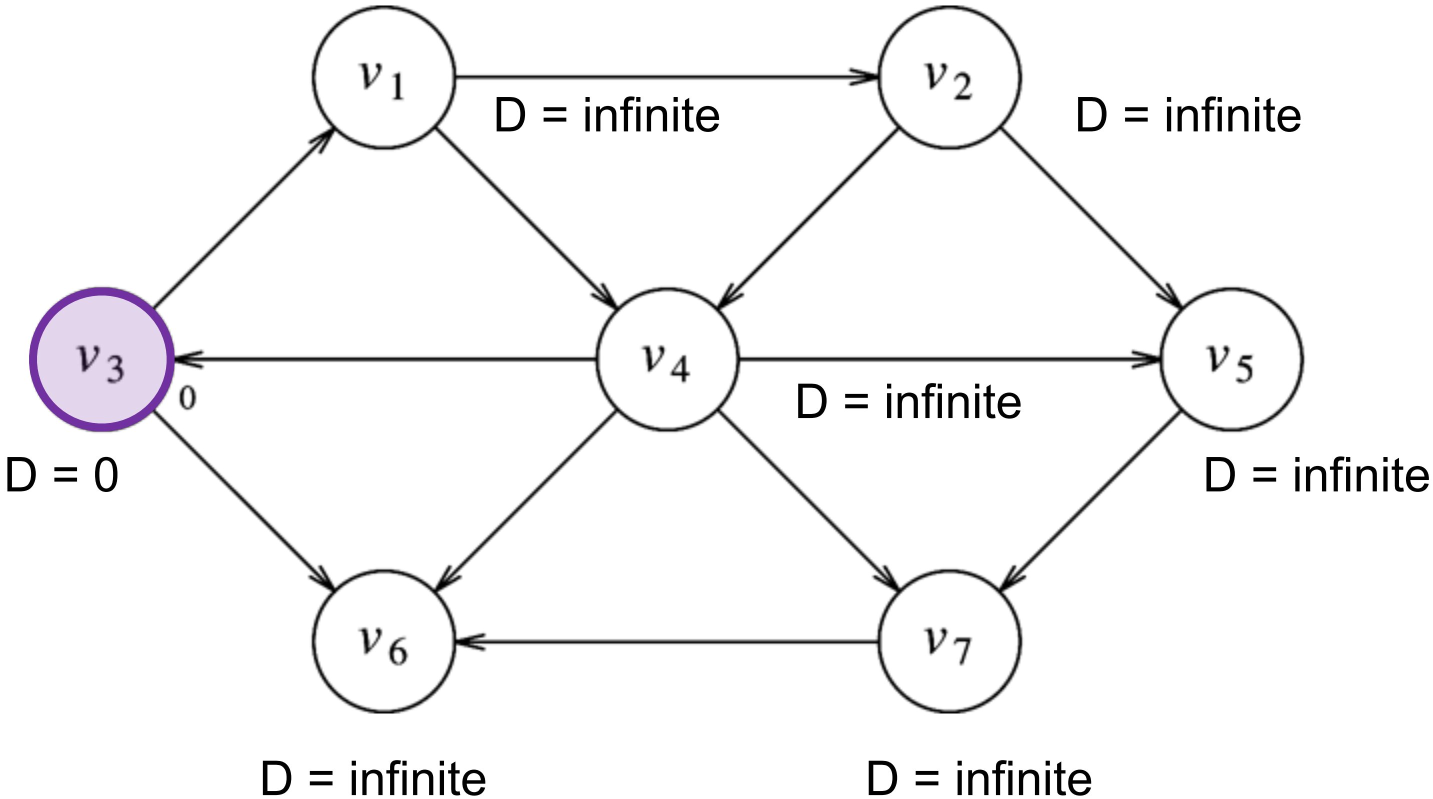


BFS – In Class Activity

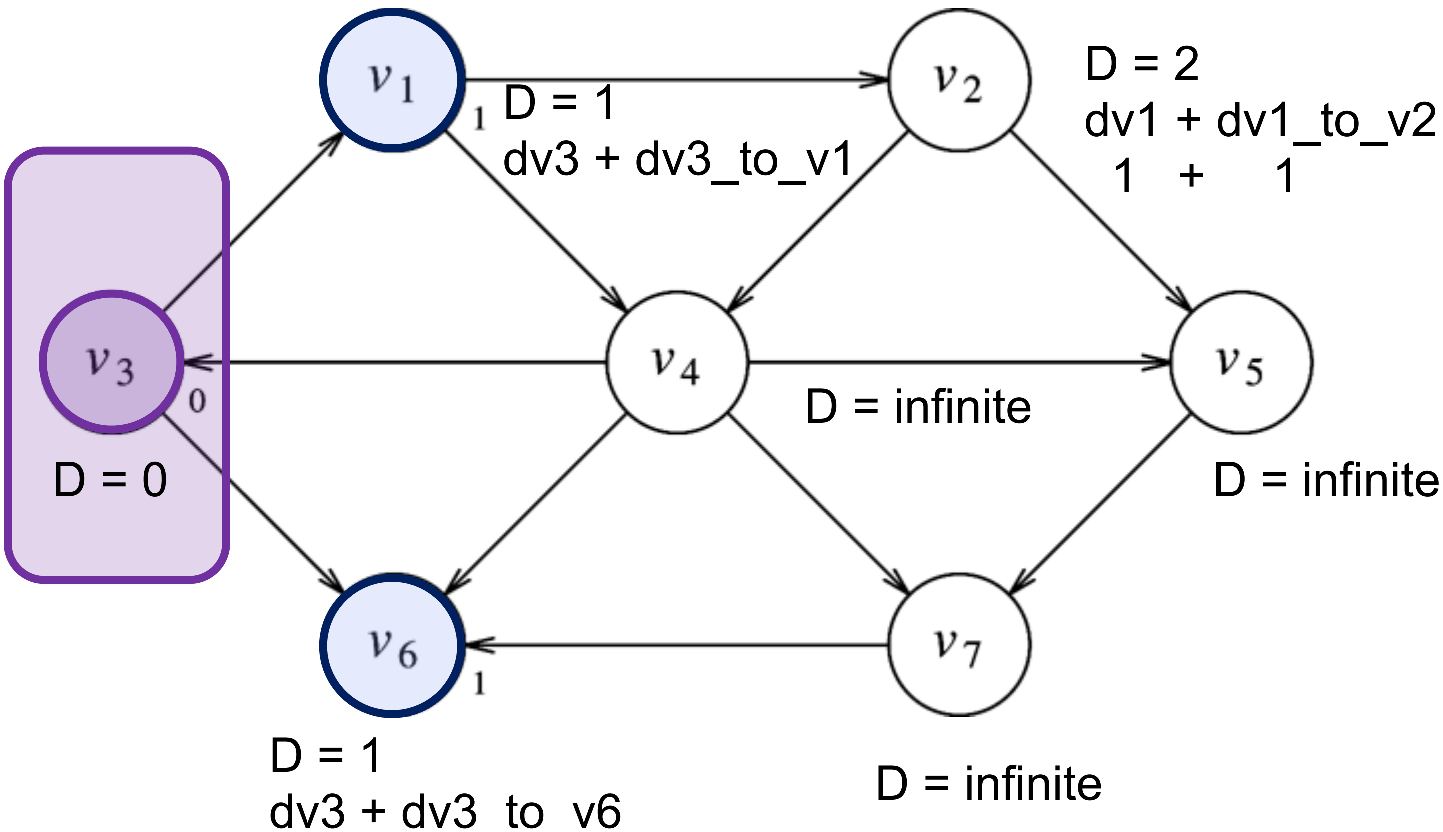


Running Time = $O(|E|+|V|)$, every V is explored, all E in worst case.

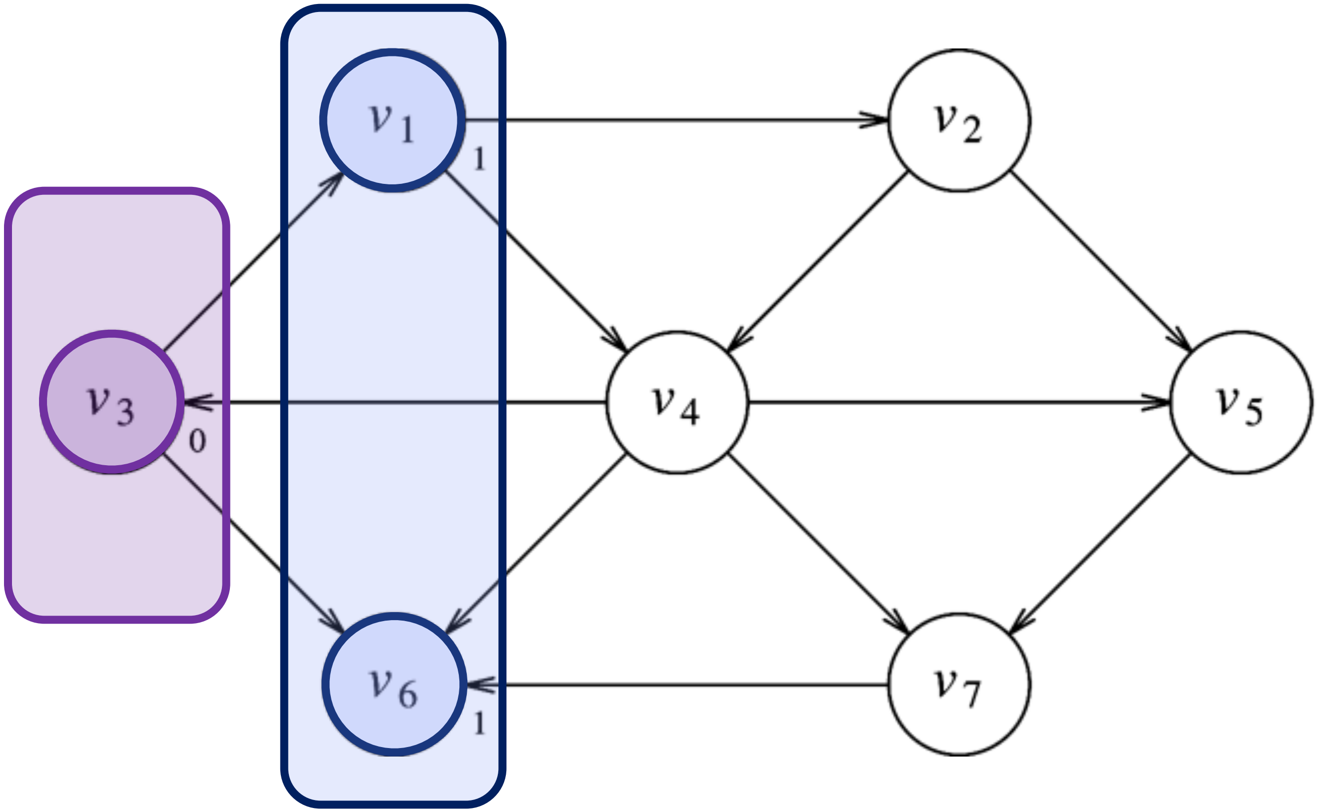
BFS – In Class Activity



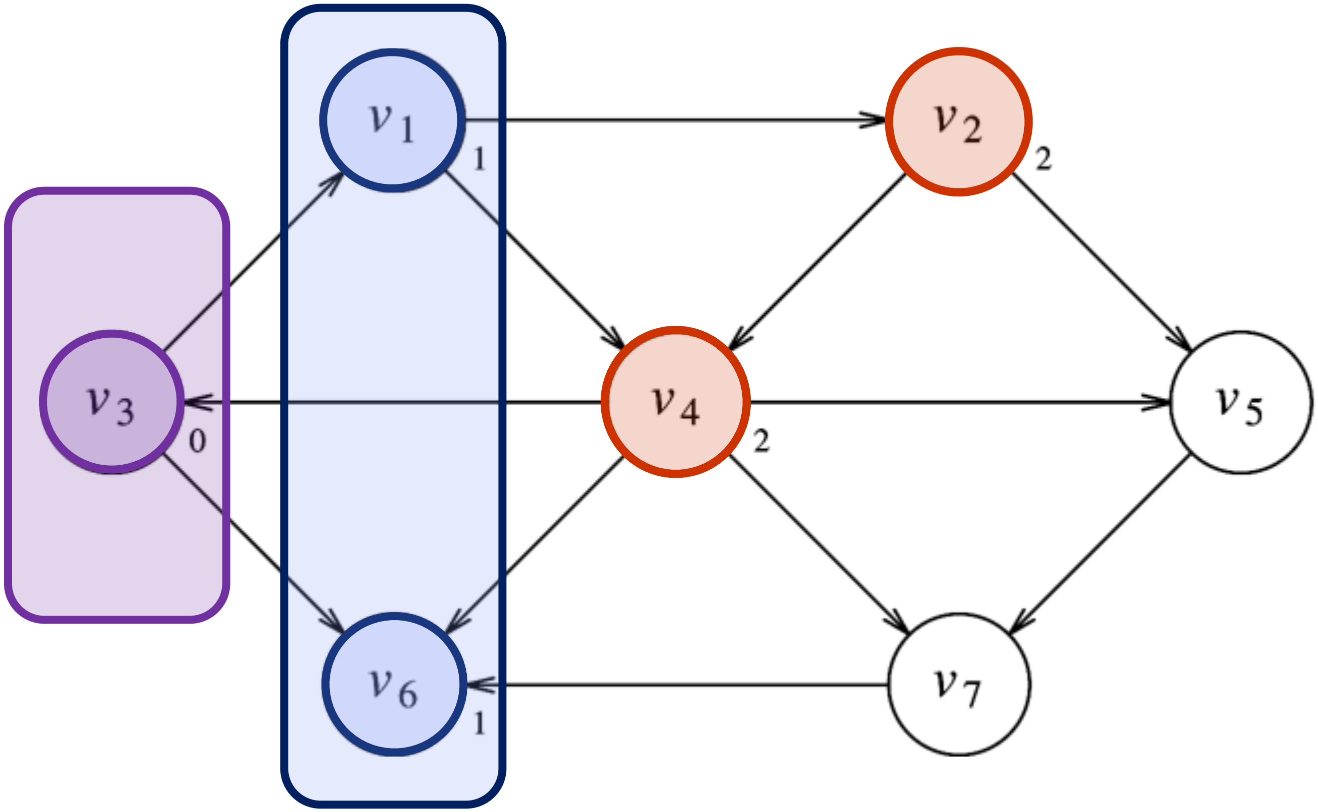
BFS – In Class Activity



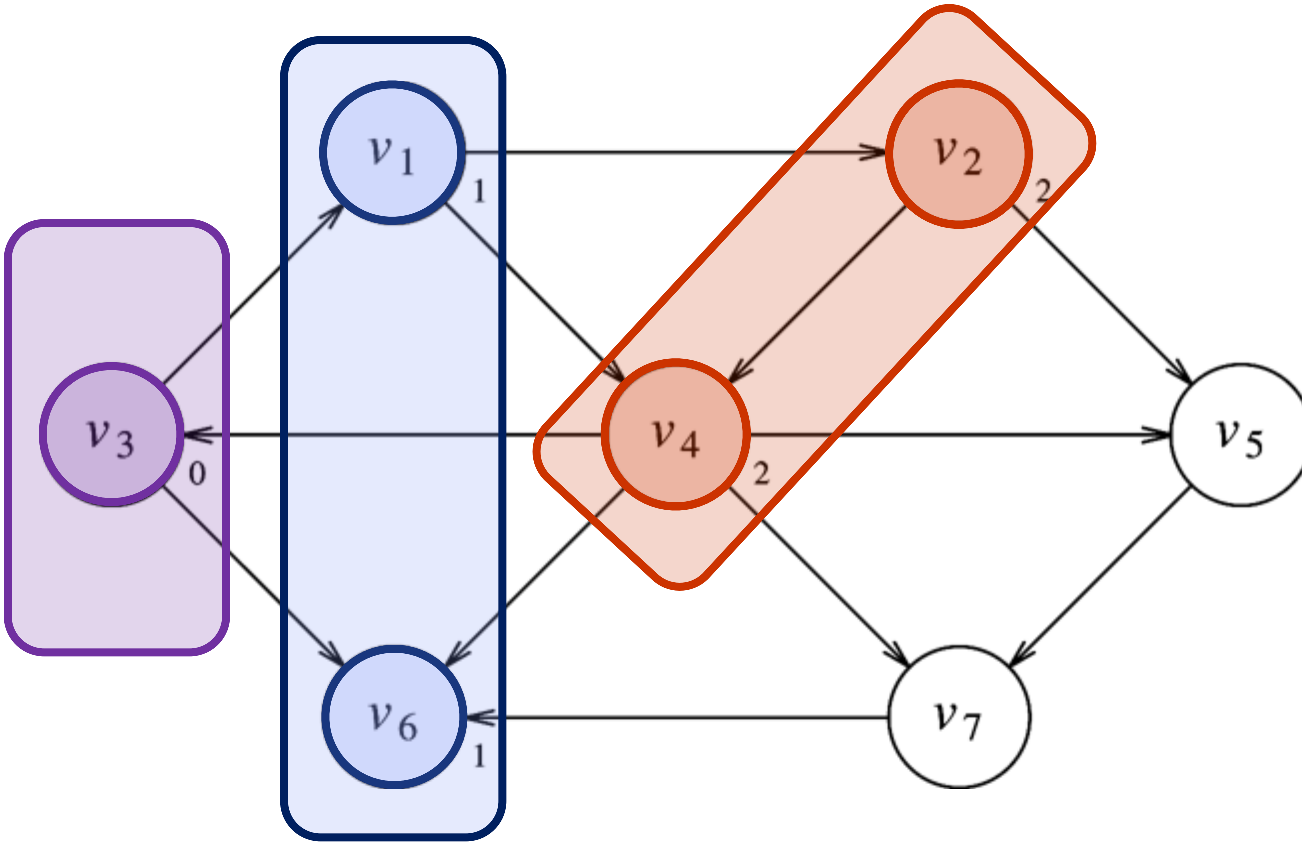
BFS – In Class Activity



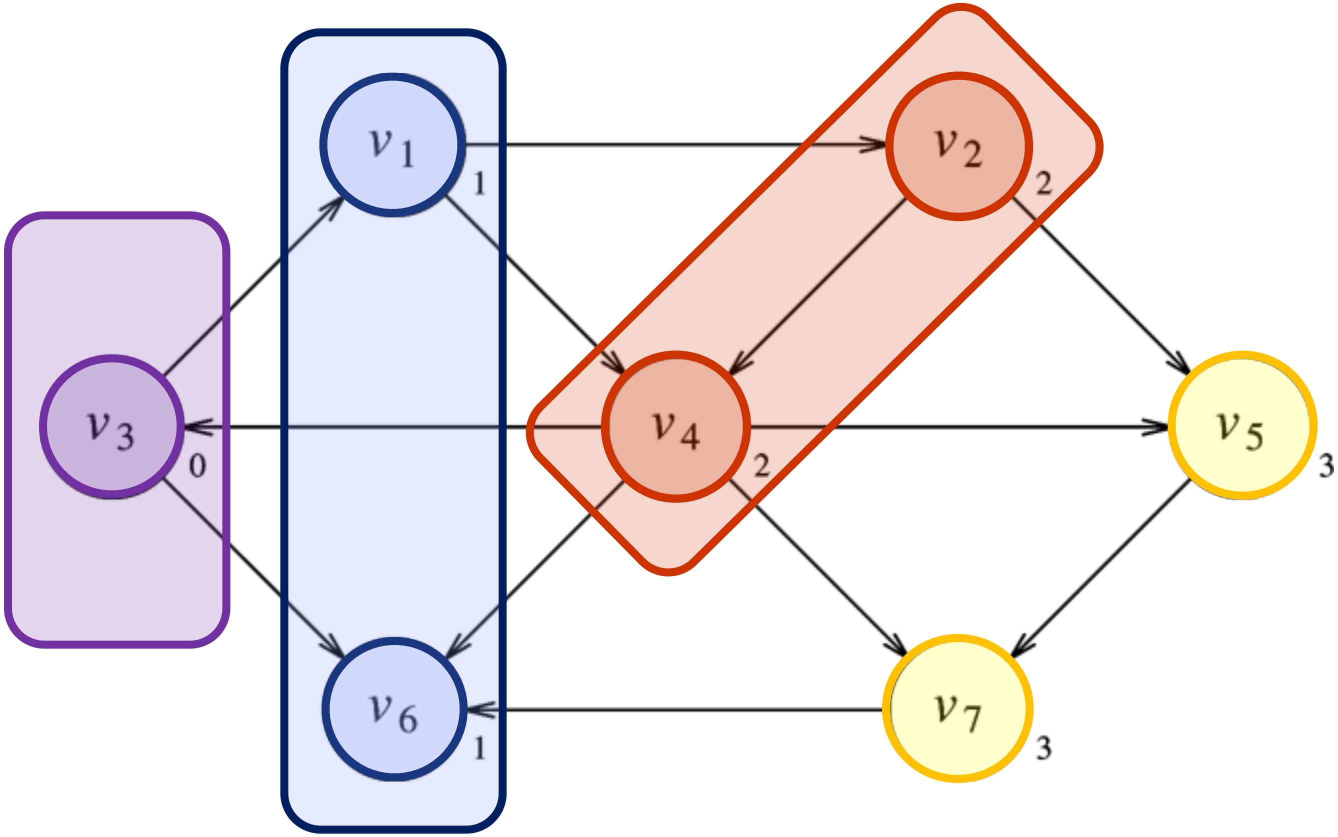
BFS – In Class Activity



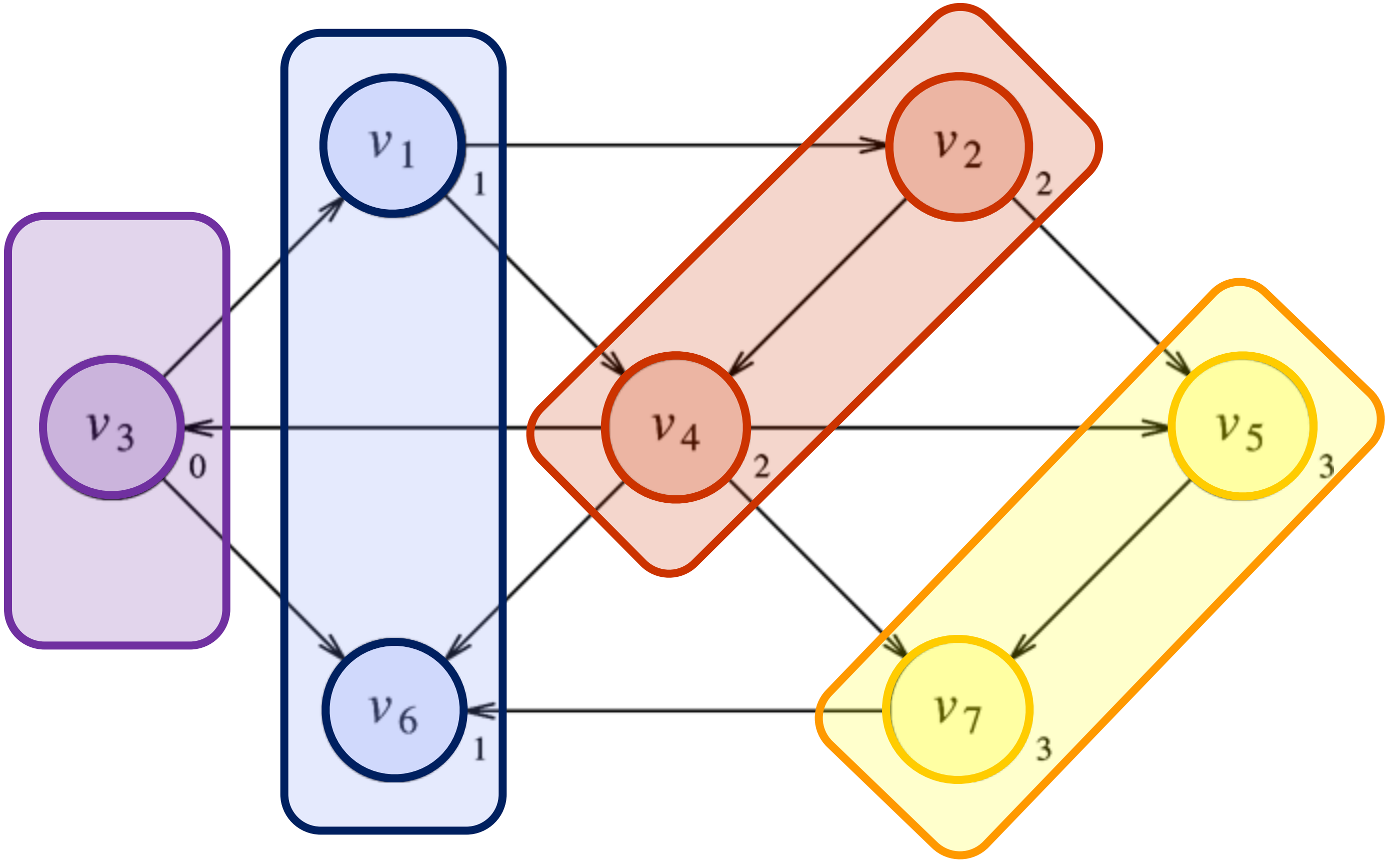
BFS – In Class Activity



BFS – In Class Activity



BFS – In Class Activity



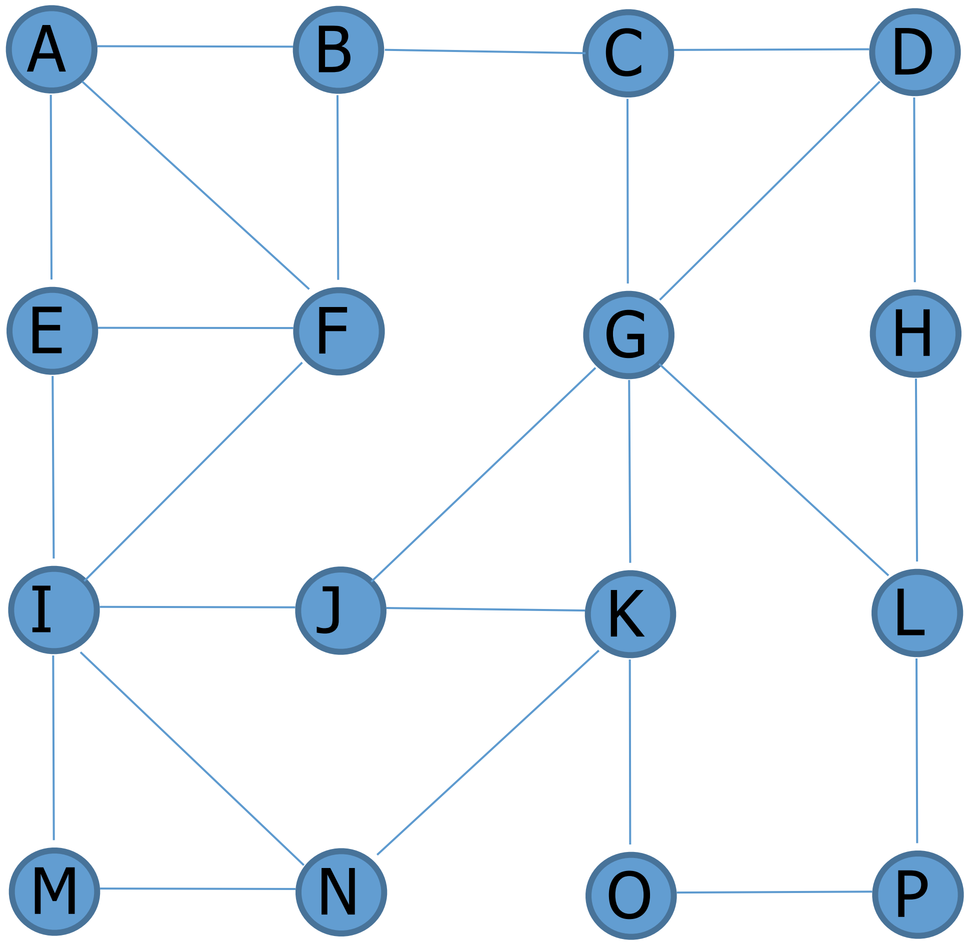
Running Time = $O(|E|+|V|)$

DFS - Depth First Search

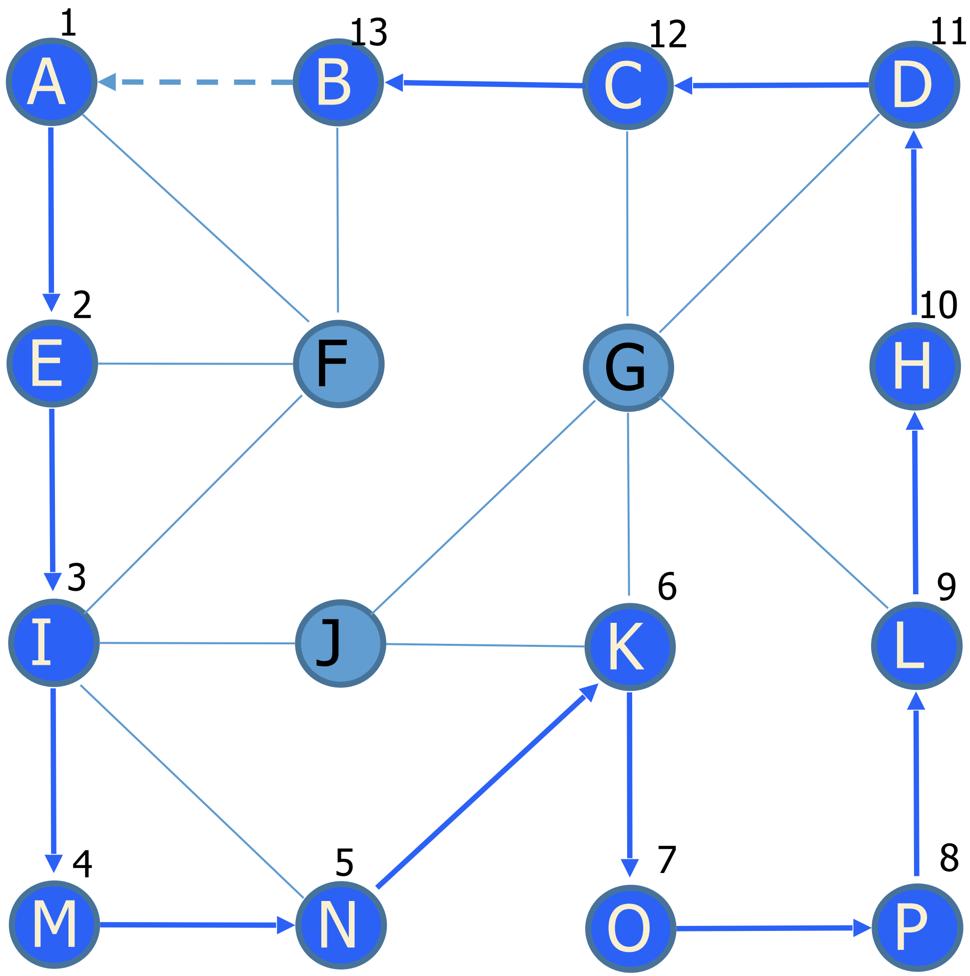
Generalization of a preorder traversal.

1. Start at an arbitrary vertex s.
2. Traverse (visit) as many descendants of s as possible without visiting a previously visited vertex.
3. Once a previously visited vertex is encountered, back out until a vertex v_i with unexplored edges is found.
4. Depth first search on $v_{i.}$
5. Repeat (steps 3-4) until all vertices visited.

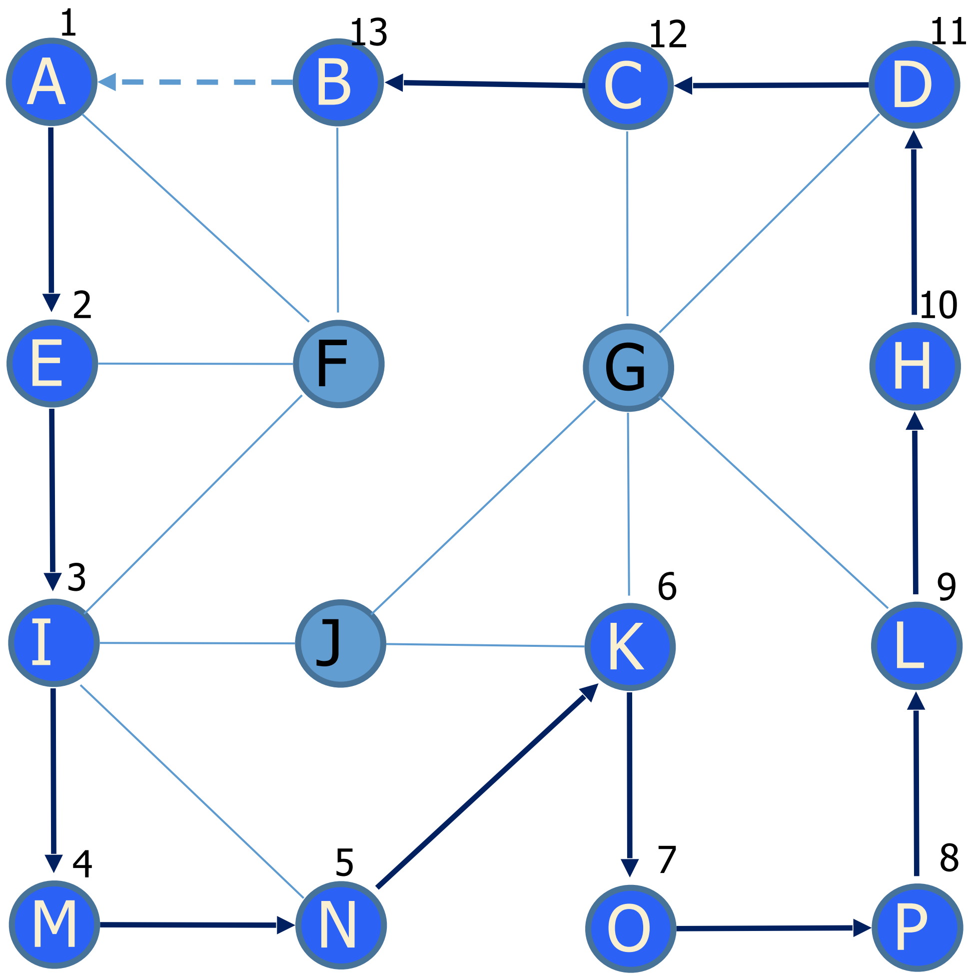
DFS - Depth First Search



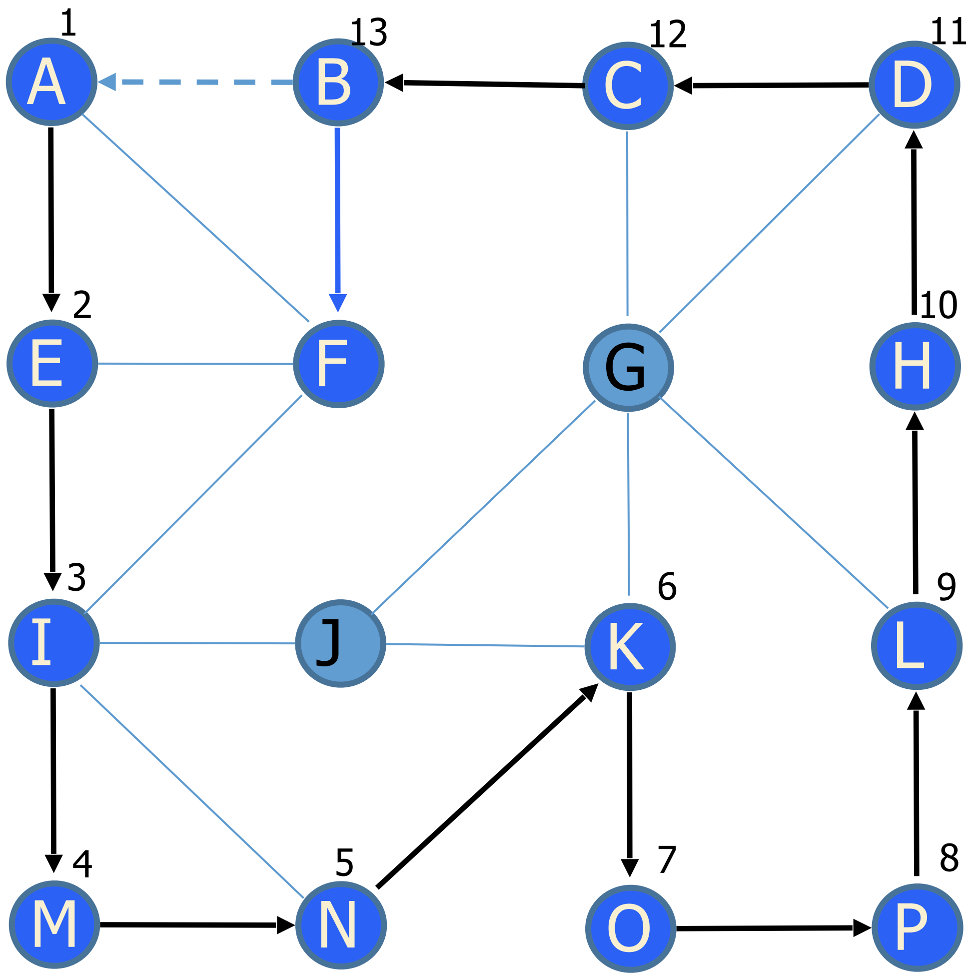
DFS - Depth First Search



DFS - Depth First Search



DFS - Depth First Search



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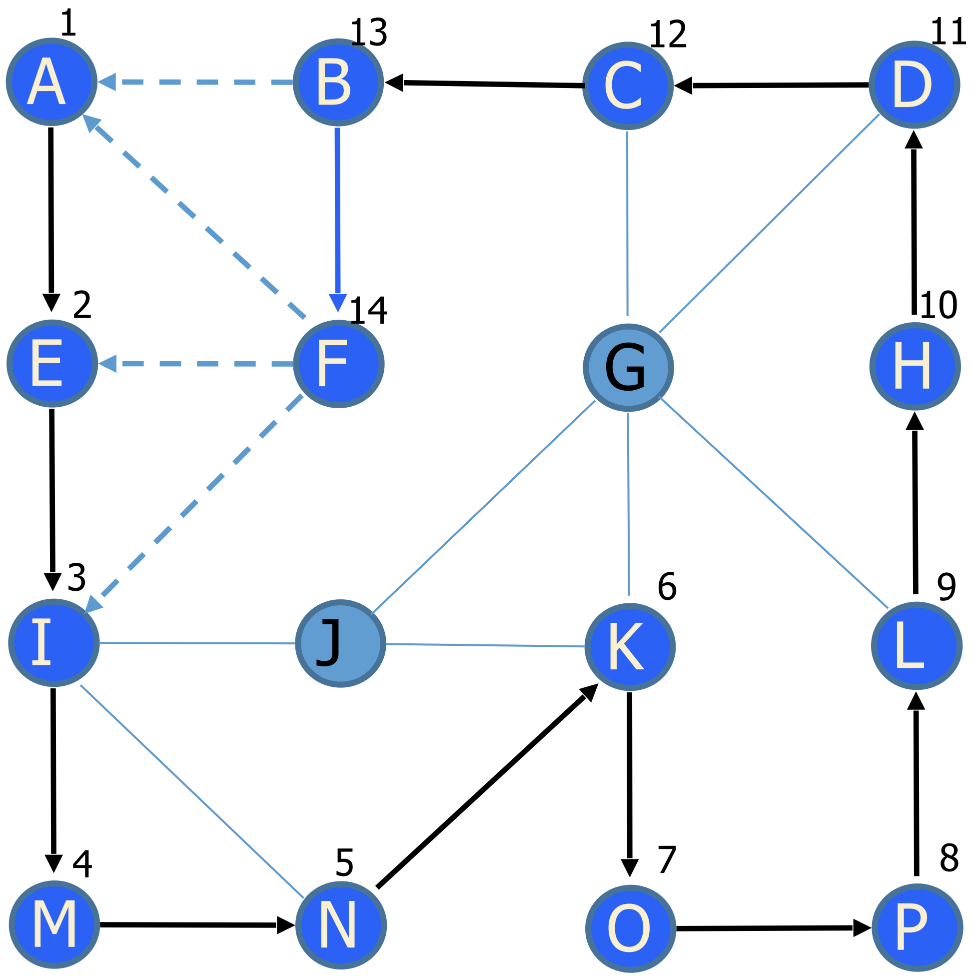

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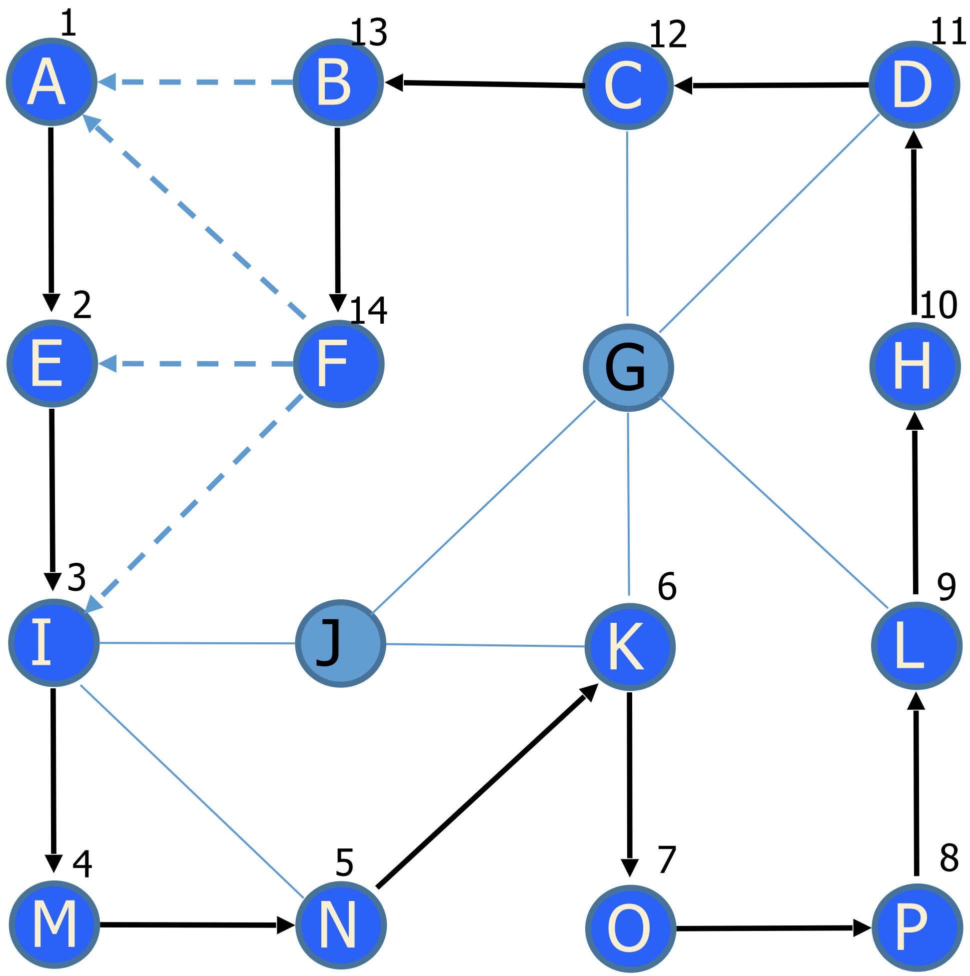
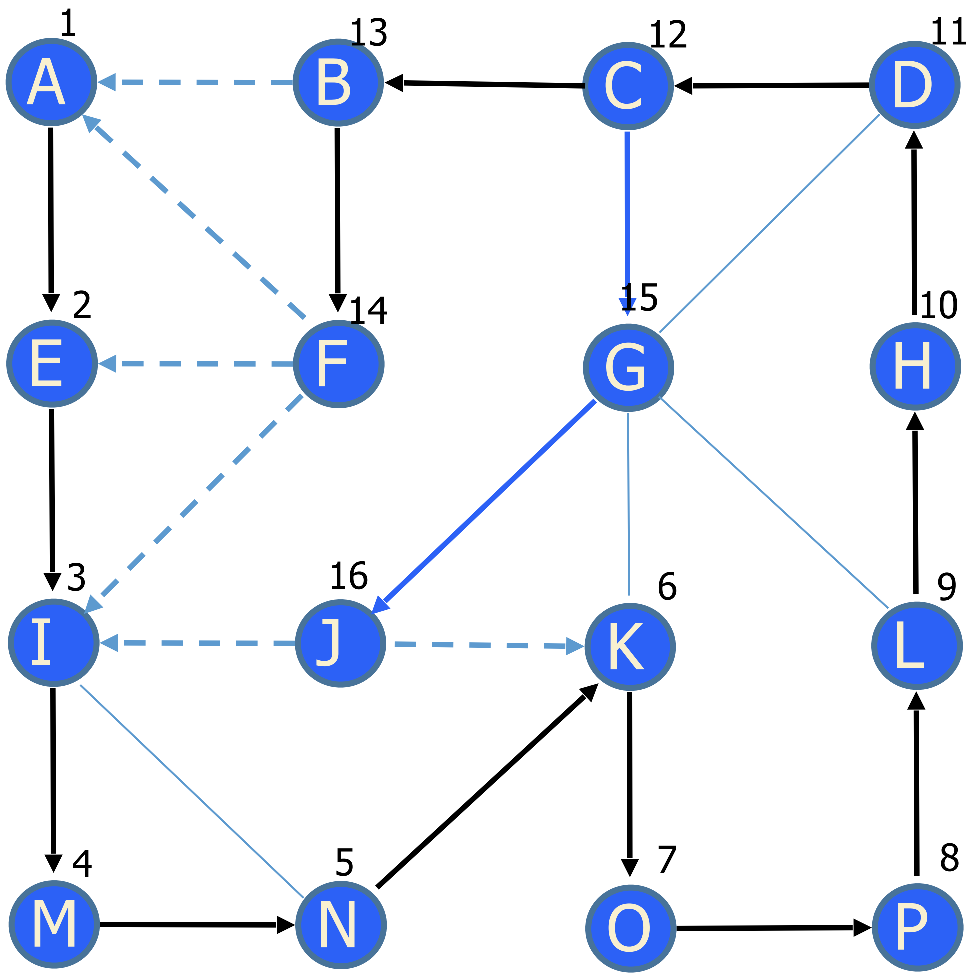
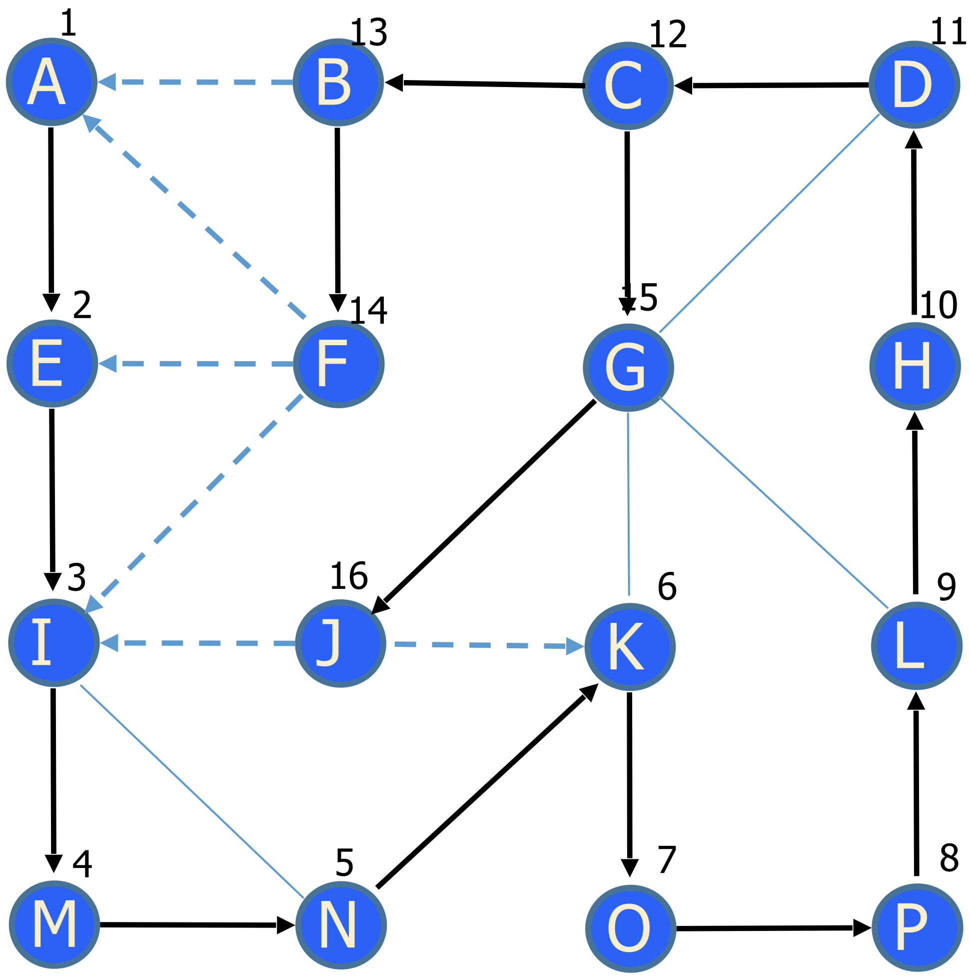


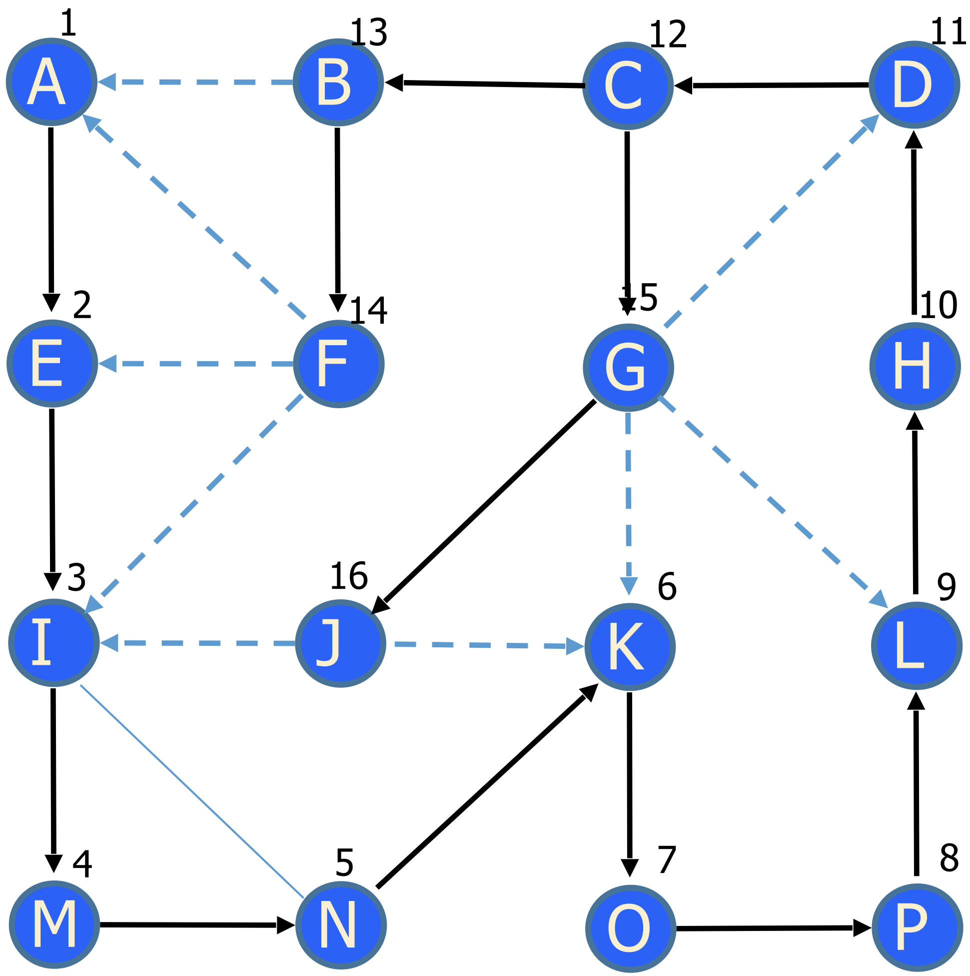
DFS - Depth First Search



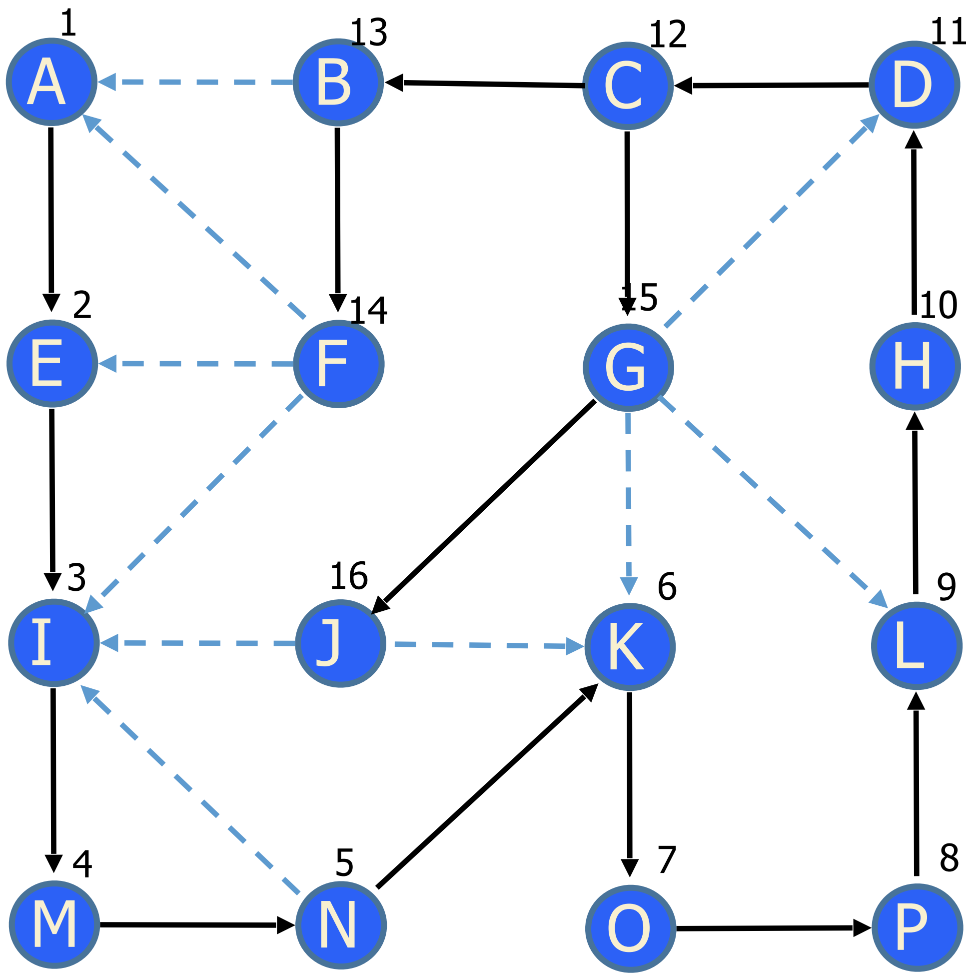
DFS - Depth First Search



DFS - Depth First Search

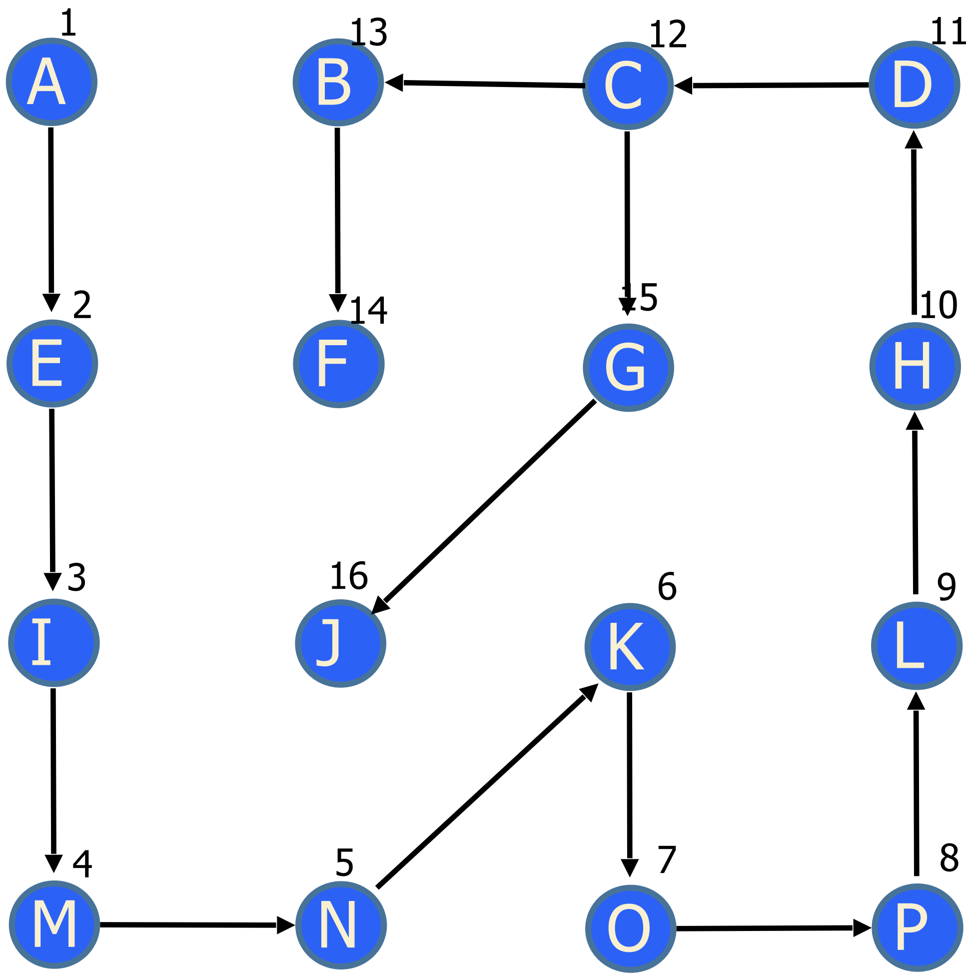


DFS - Depth First Search

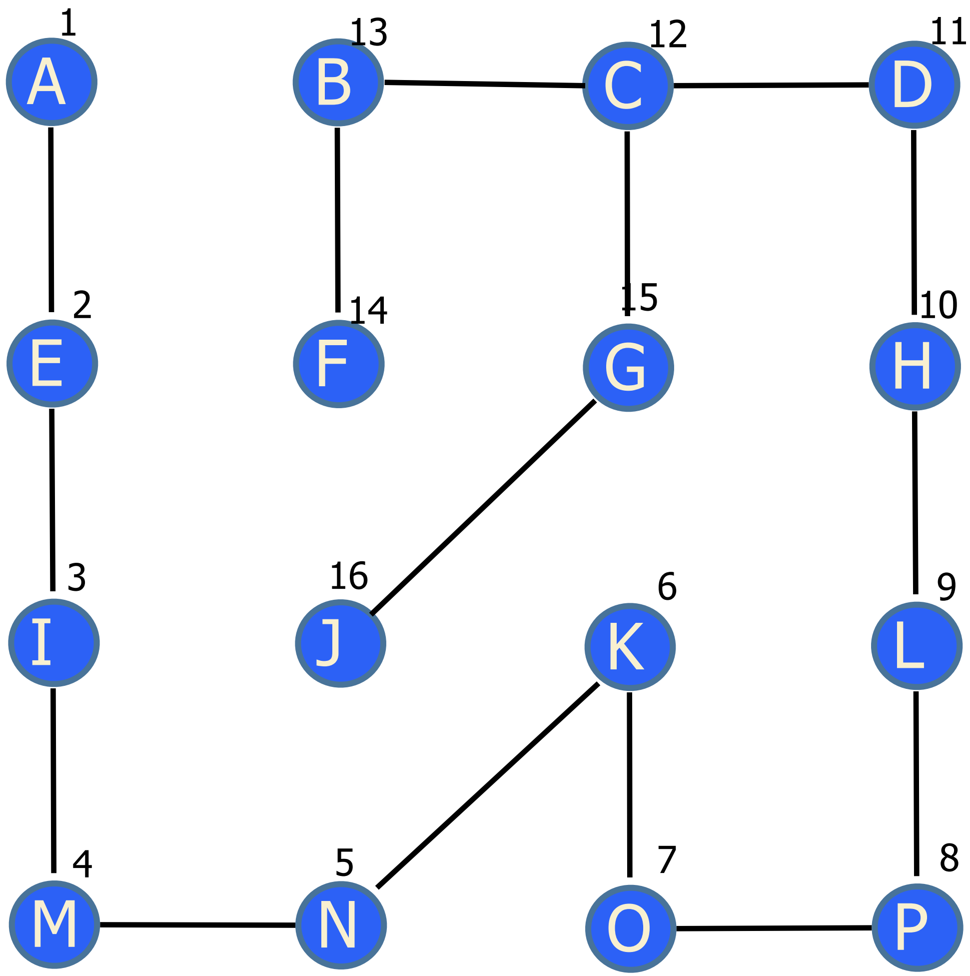


Running Time = $O(|E|+|V|)$

Spanning Tree



Spanning Tree



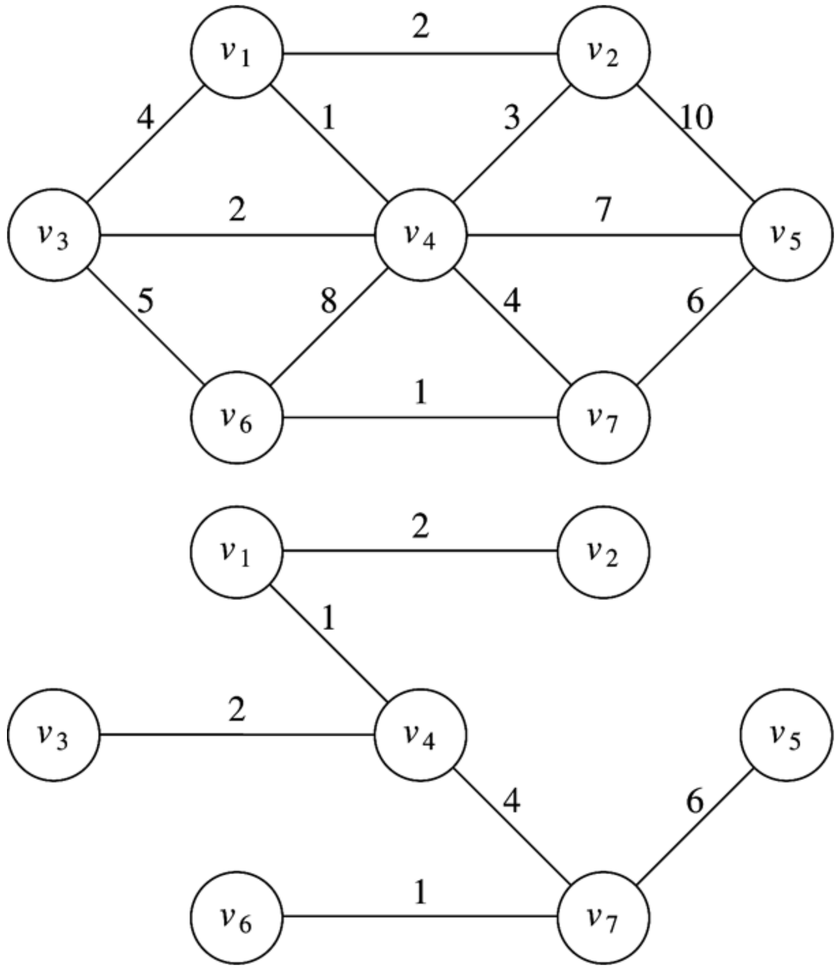
MST – Minimum Spanning Tree

- **Spanning Tree** – Given an undirected, connected graph G, a spanning tree T is a connected acyclic subgraph (tree) that contains all vertices of the graph.
- **Minimum Spanning Tree** – the spanning tree of the smallest weight, where the weight is the sum of the weights on all T's edges.
- **MST Problem** – Find the minimum spanning tree for a given weighted connected graph.
- Not always a unique MST. There can be more than 1 MST with a different set of edges, perhaps some shared, in 1 or more MSTs.

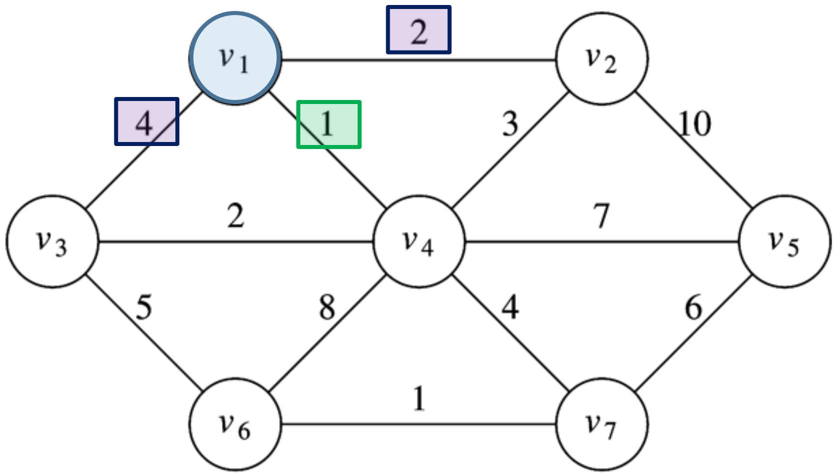
MST

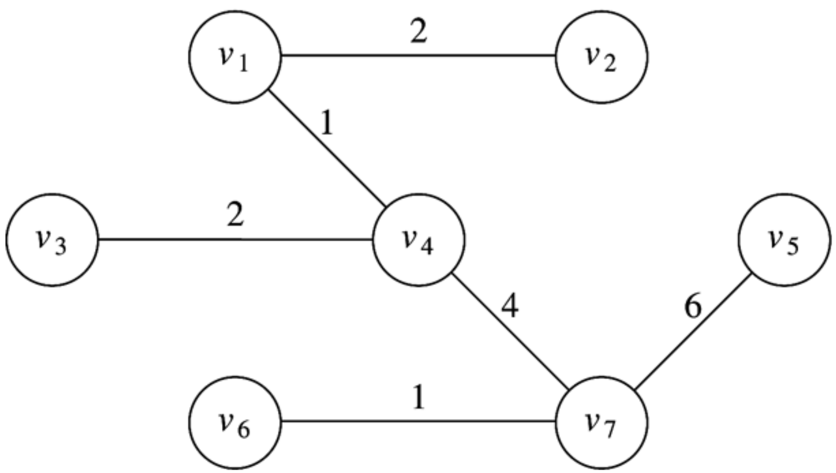
- useful for minimizing the amount of wiring needed for:
  - phone lines
  - cable lines
- used in wireless sensor networks
- used in network routing algorithms

Prim's Algorithm

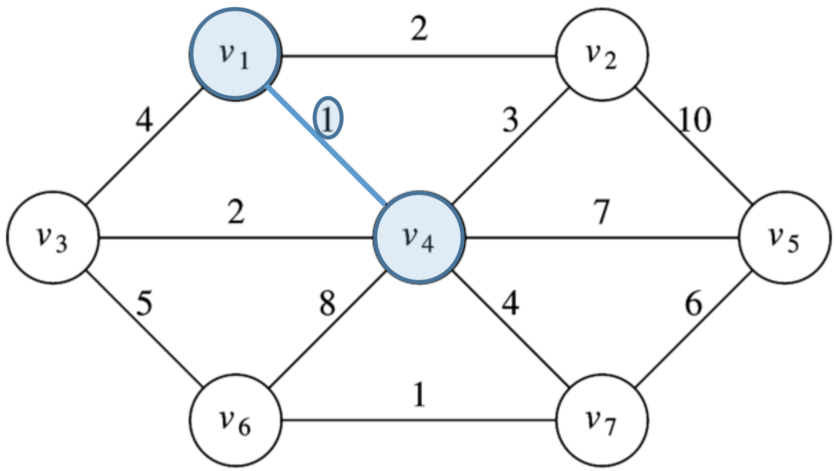


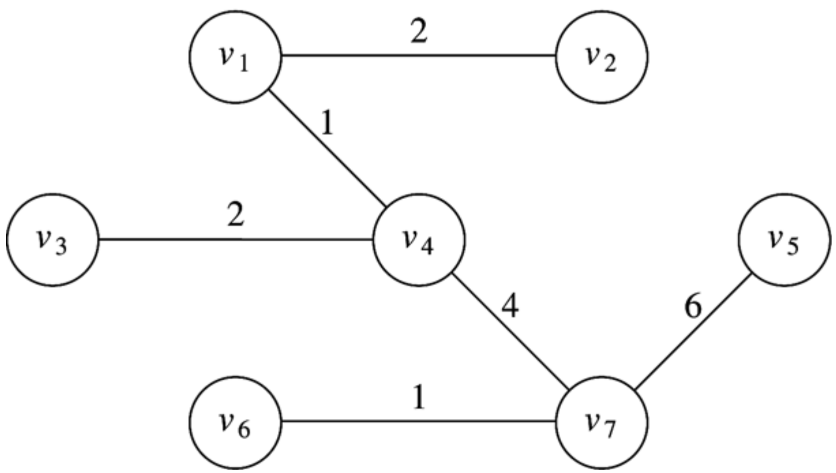
Prim's Algorithm





Prim’s Algorithm





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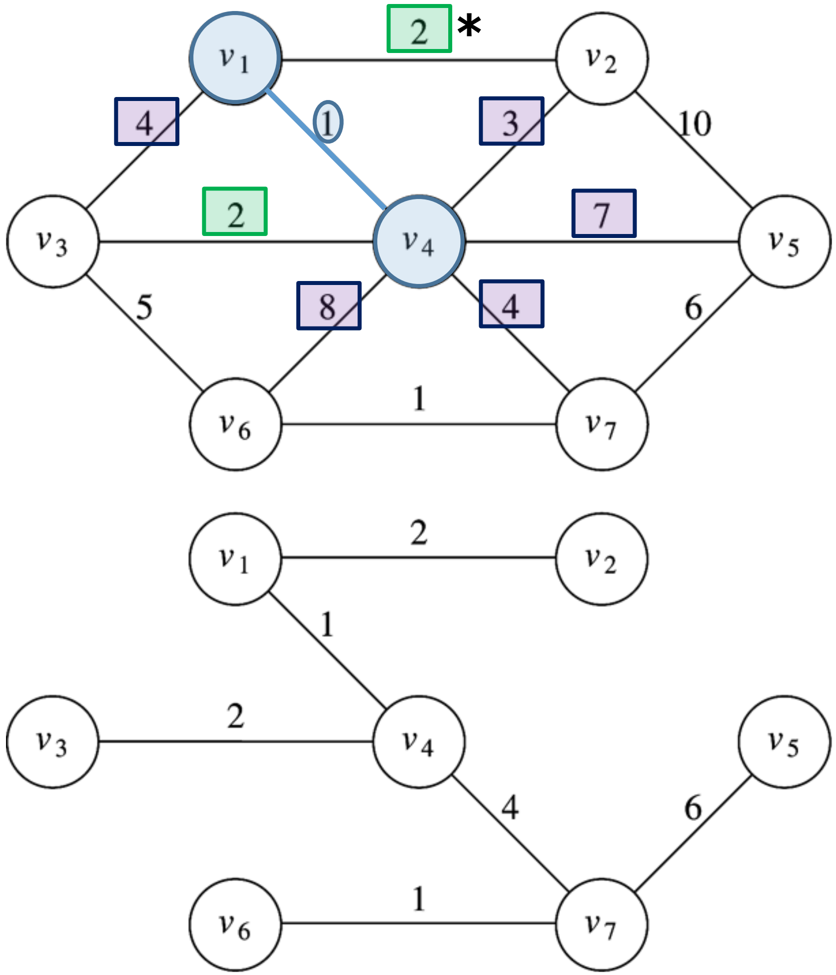

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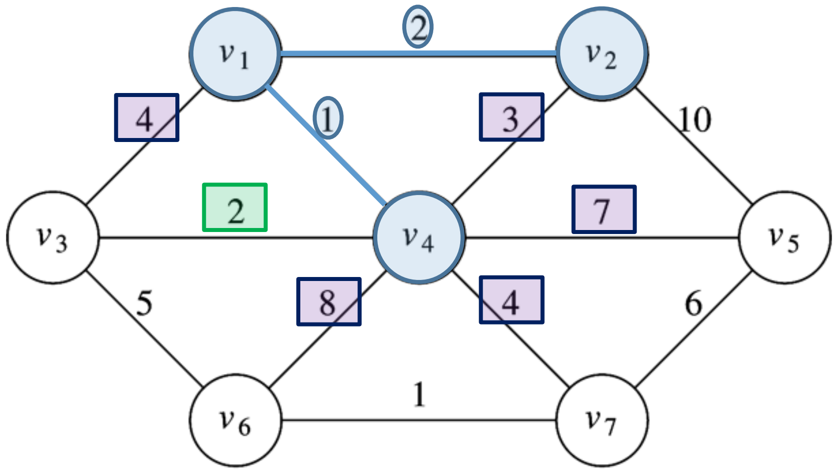

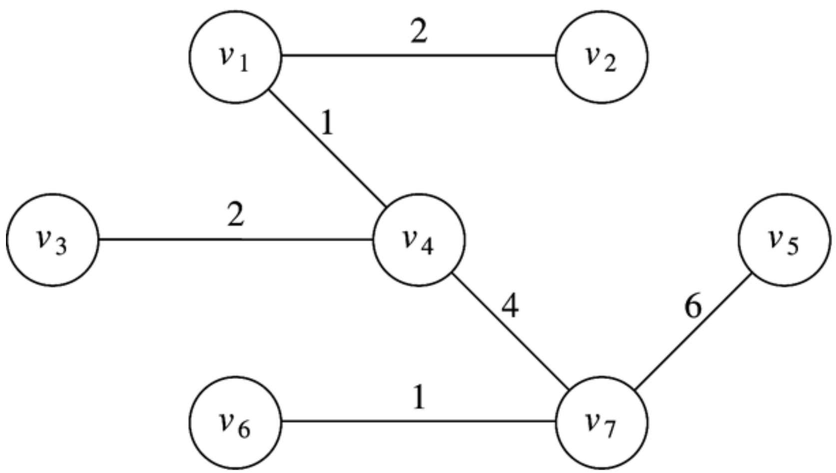

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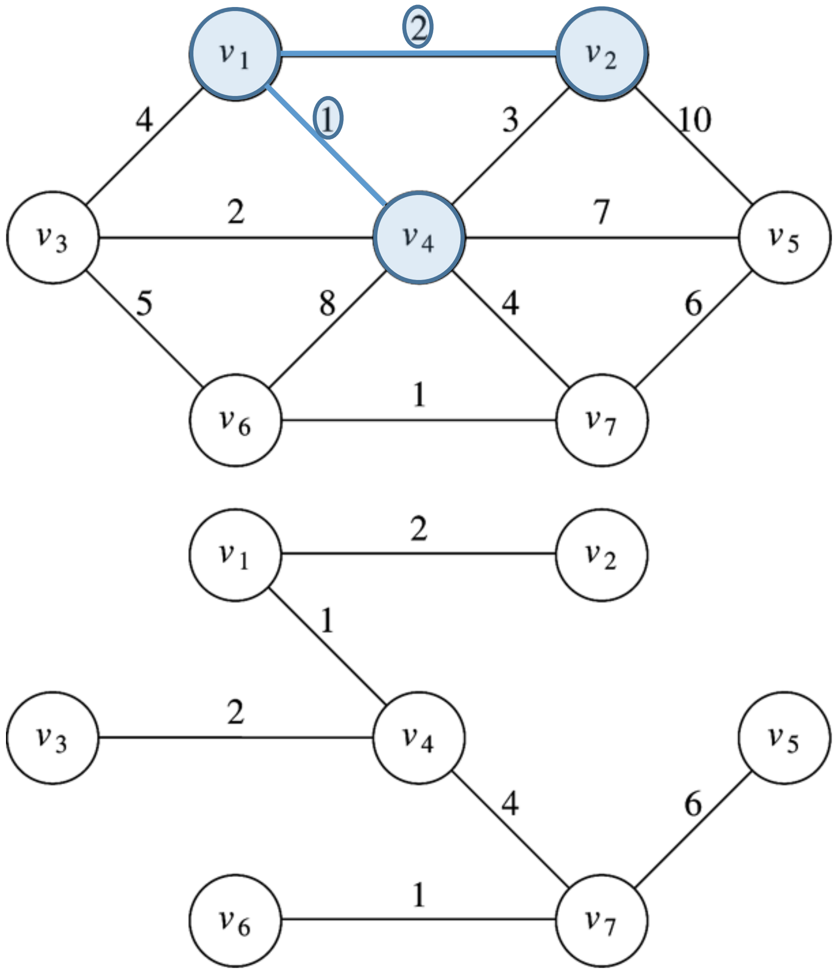

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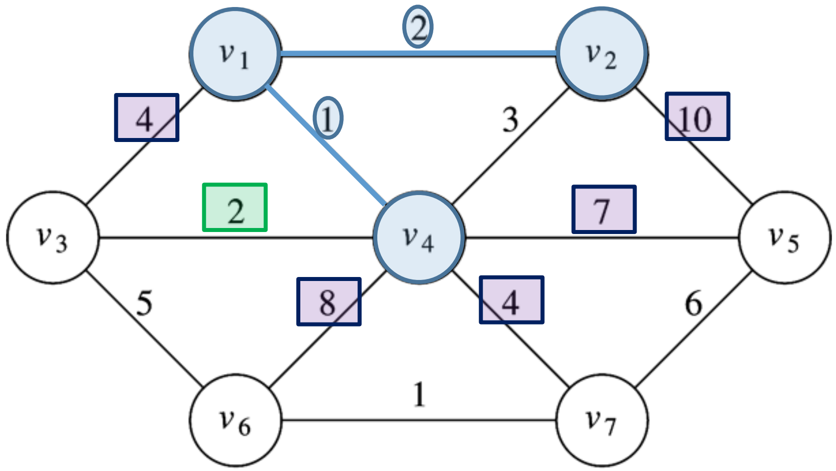
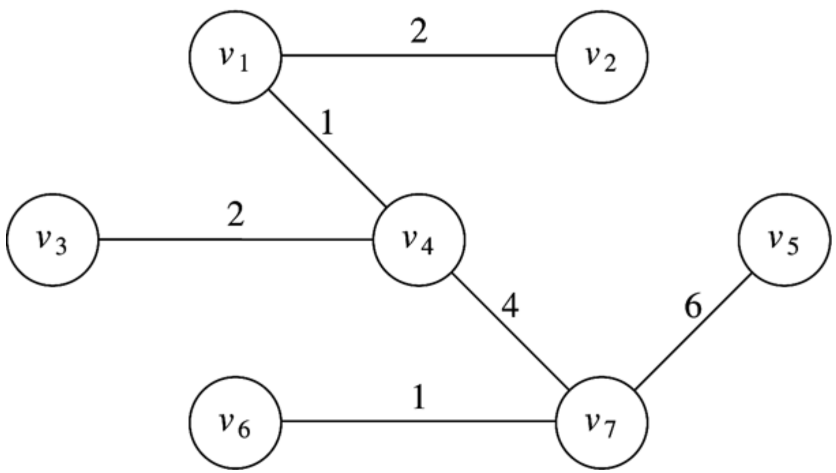
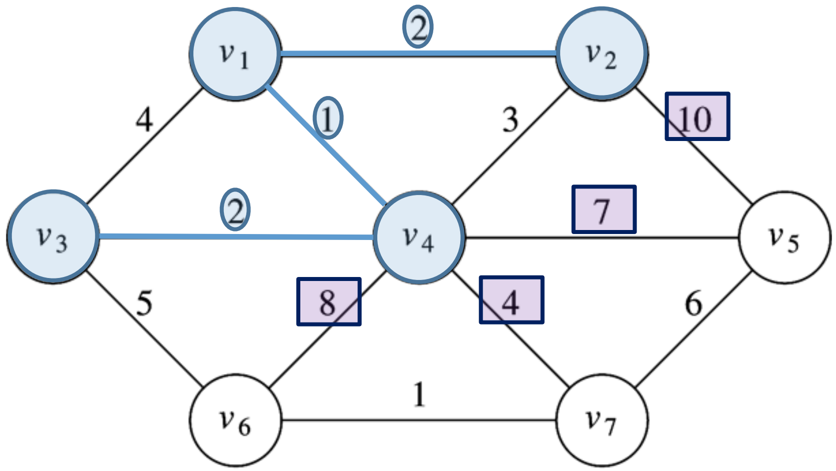
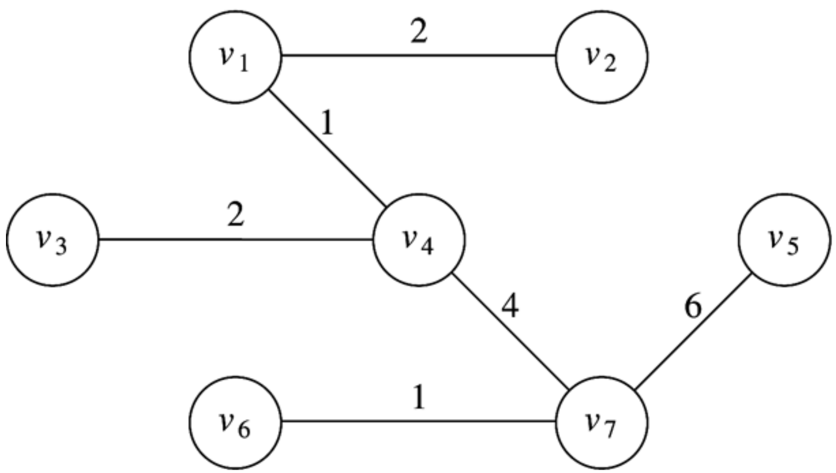




Prim's Algorithm





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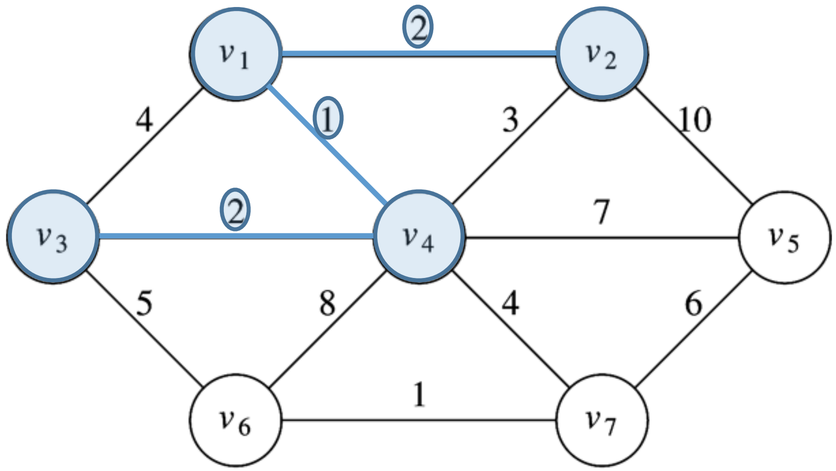
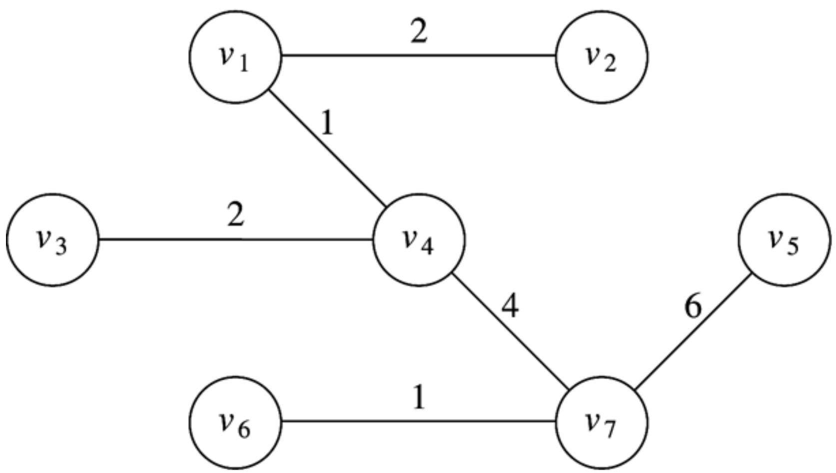
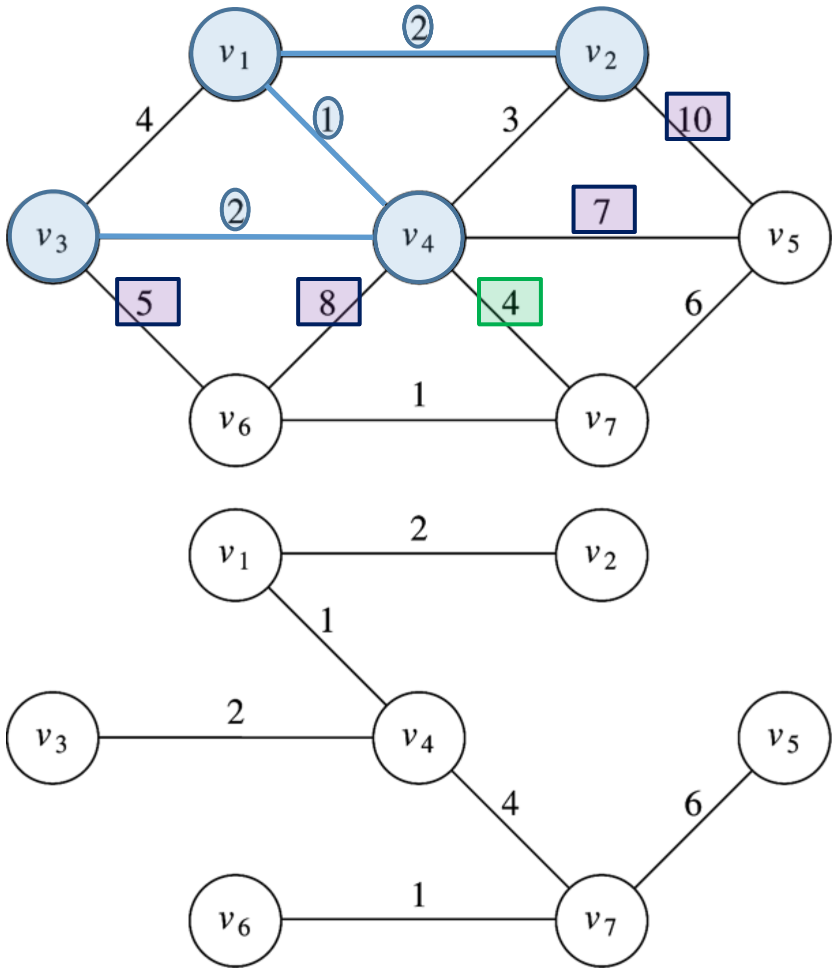
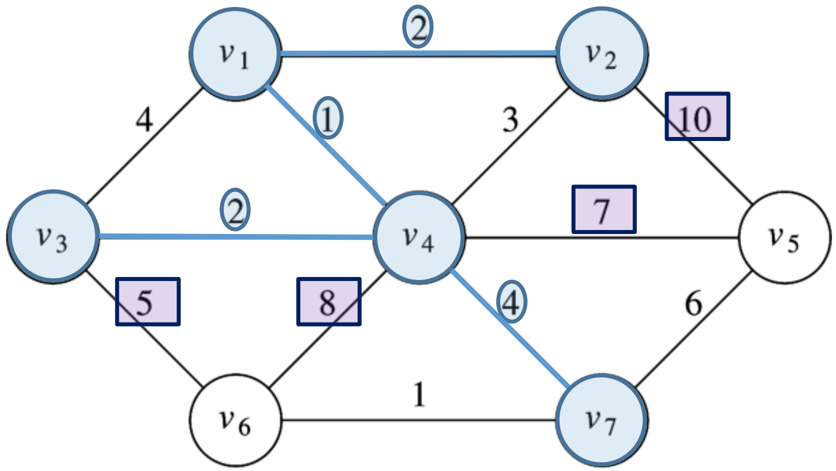


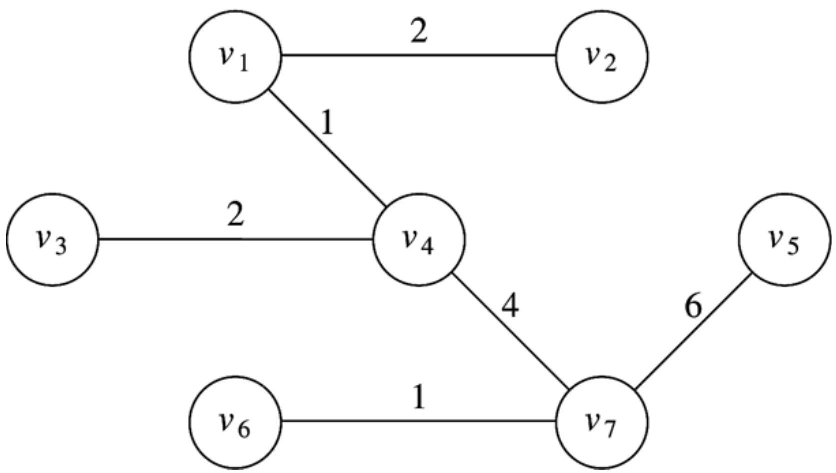


Prim’s Algorithm



Prim's Algorithm





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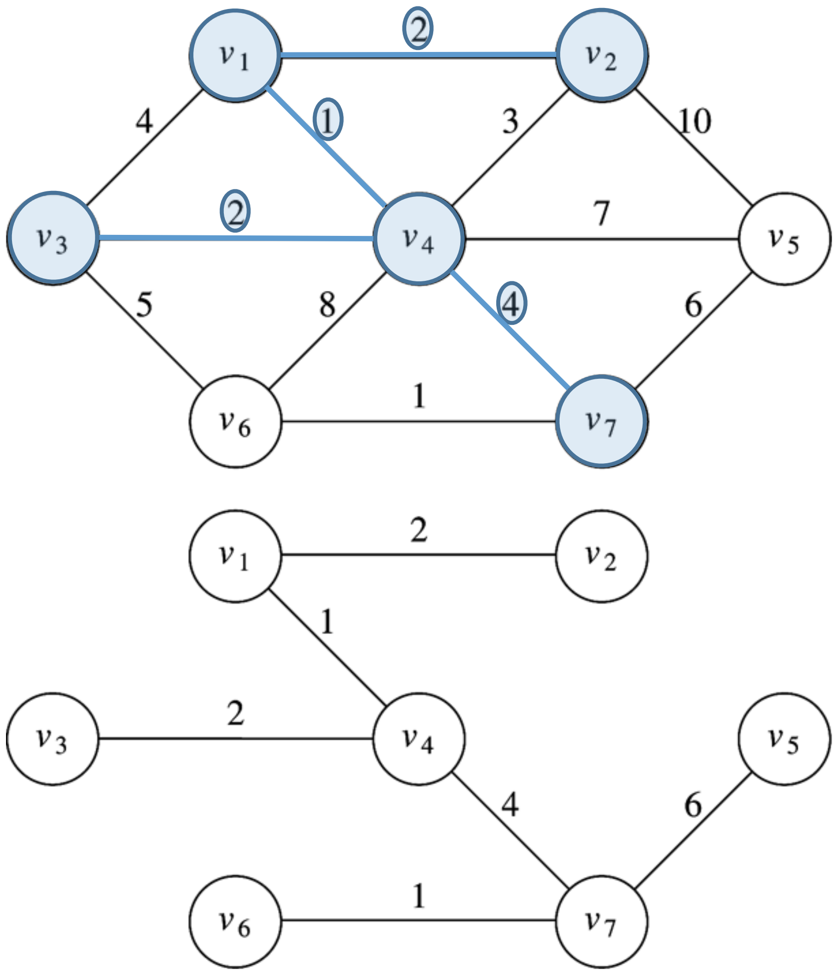

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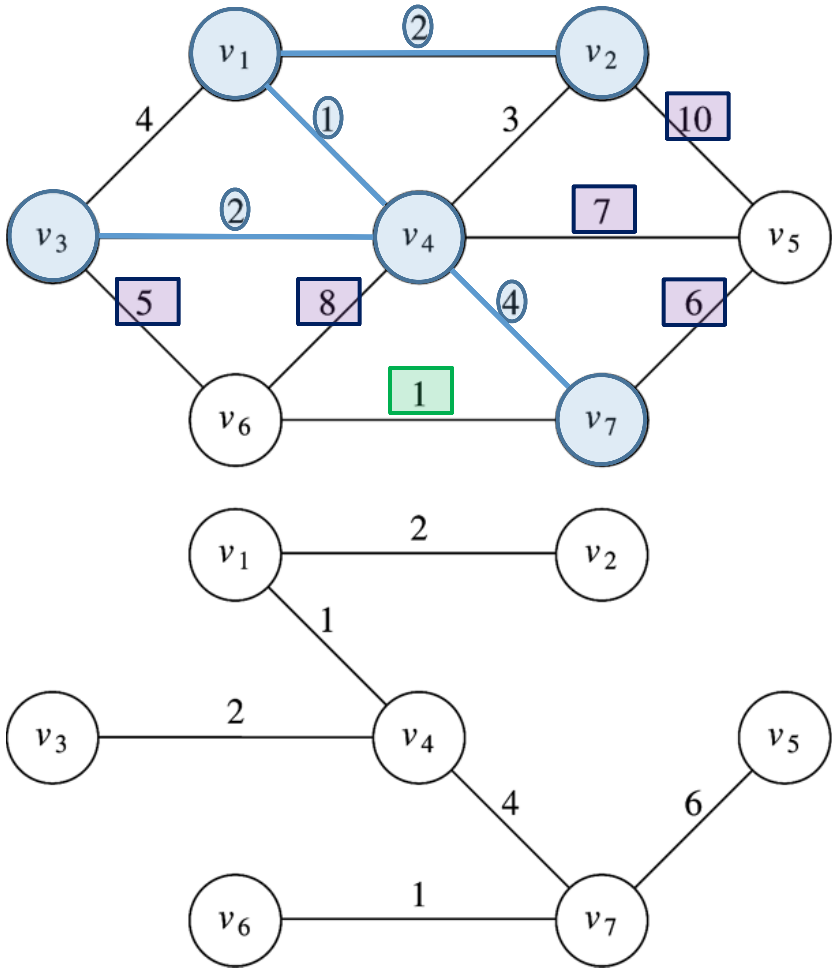

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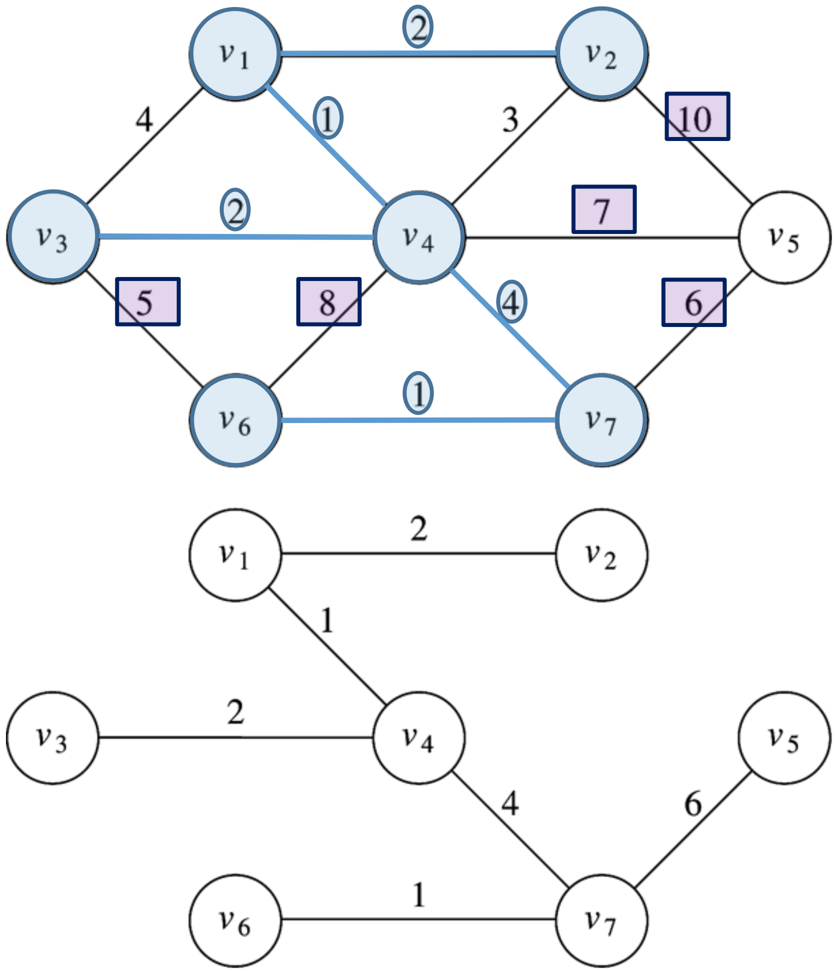

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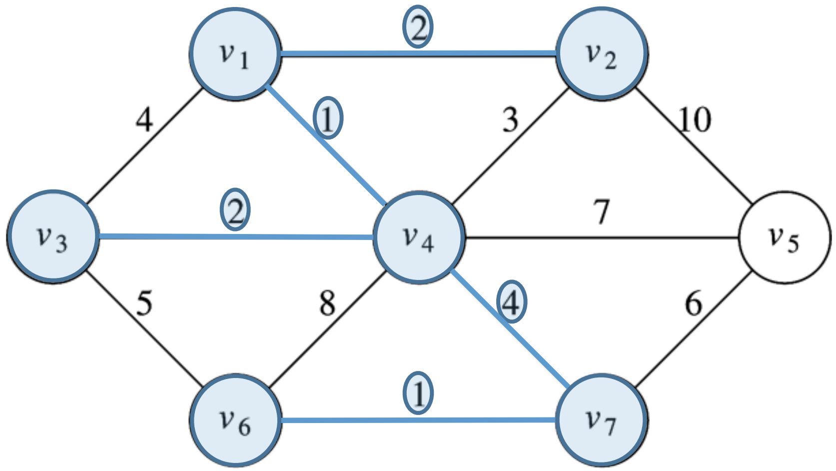

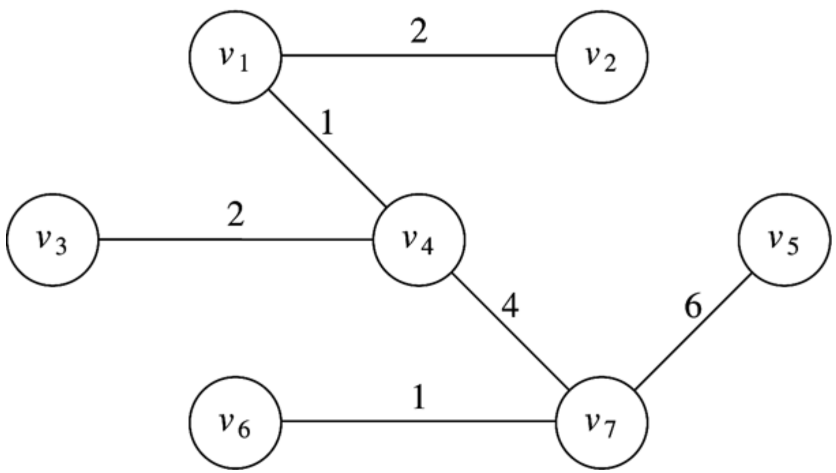

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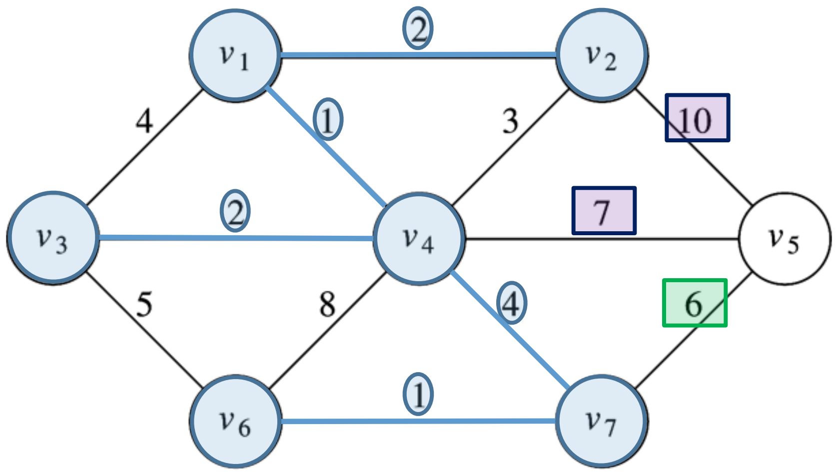

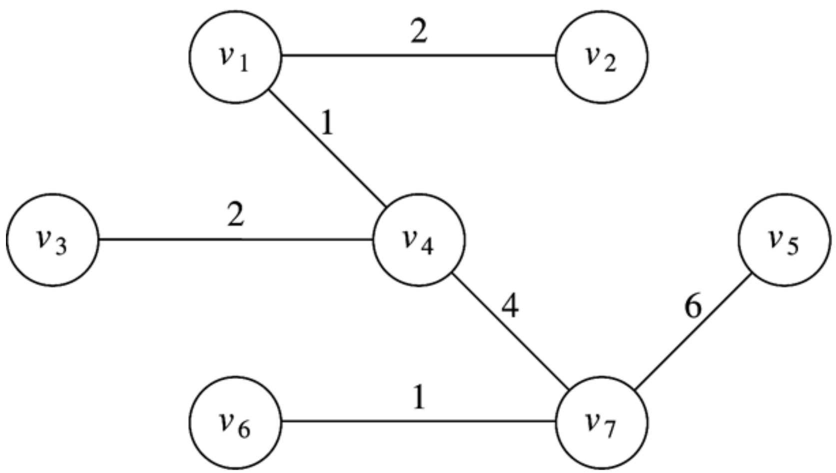

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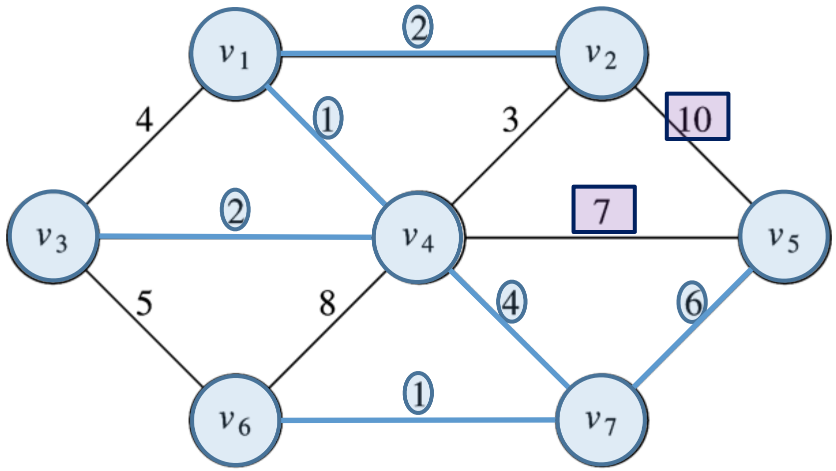

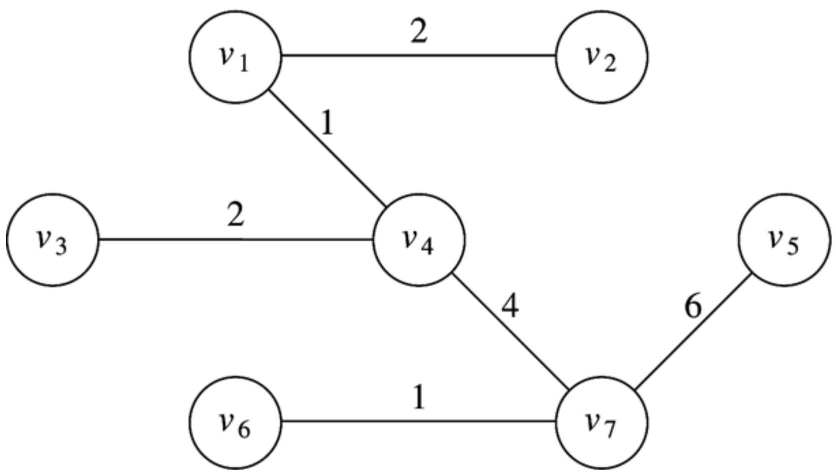

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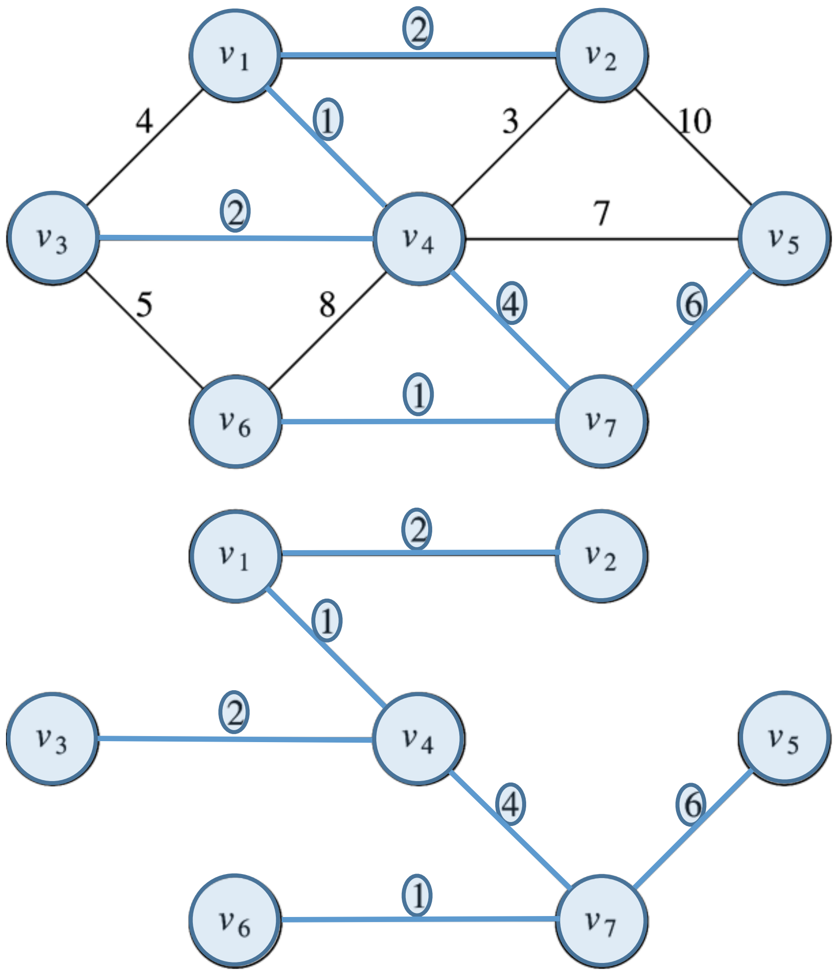
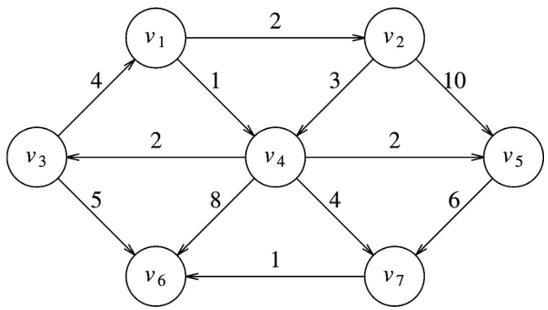
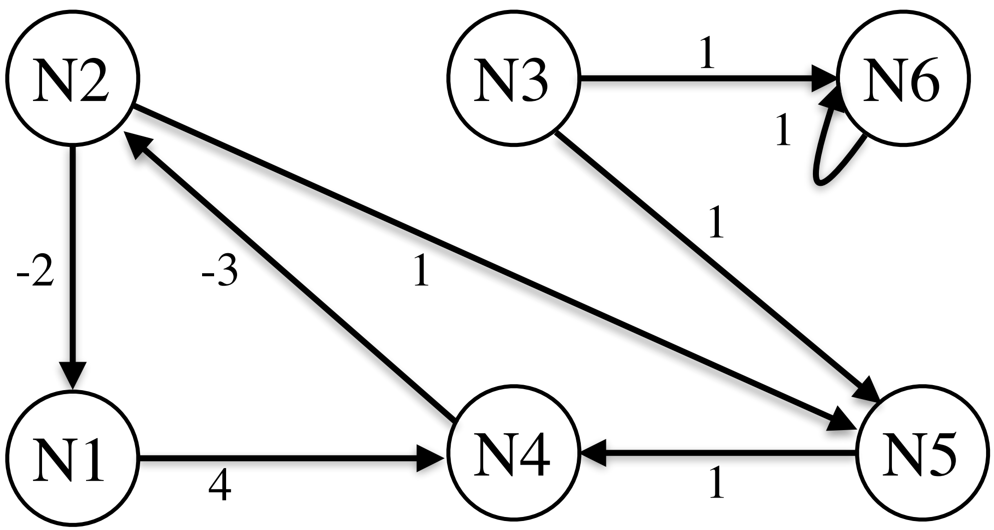
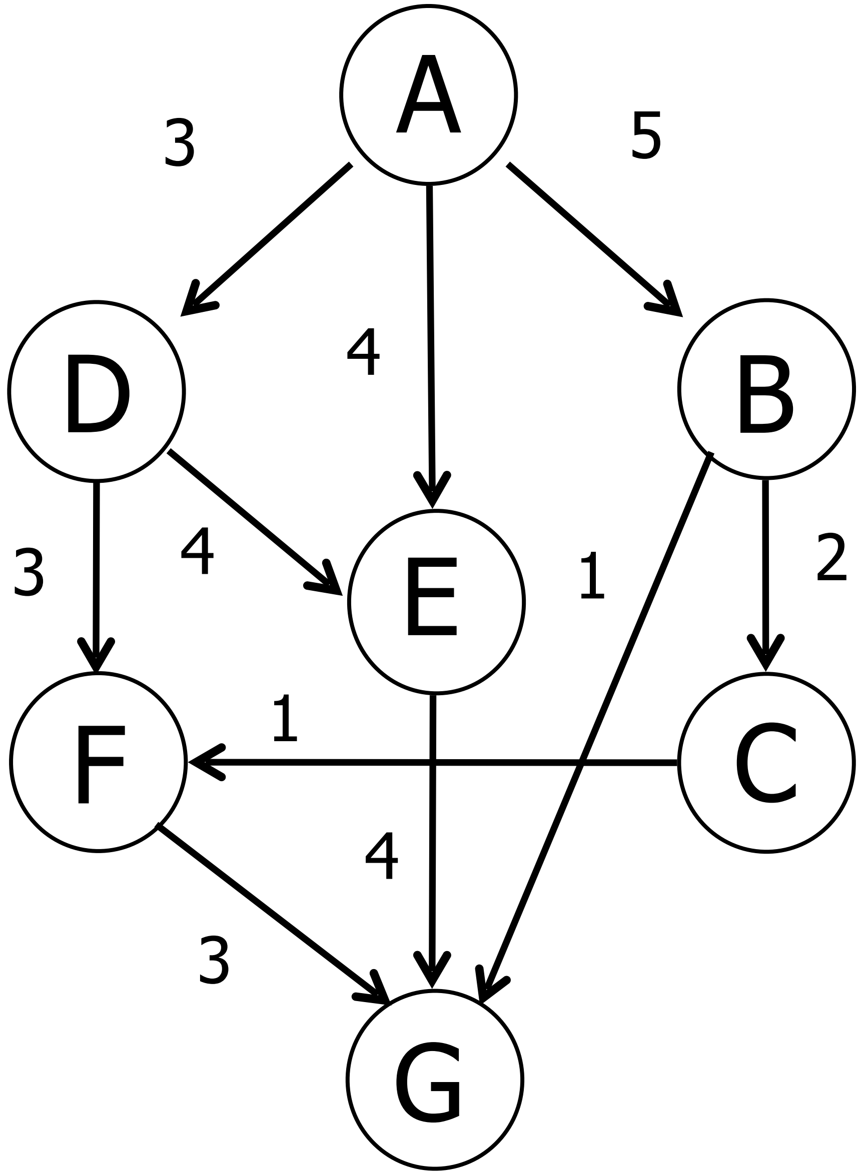


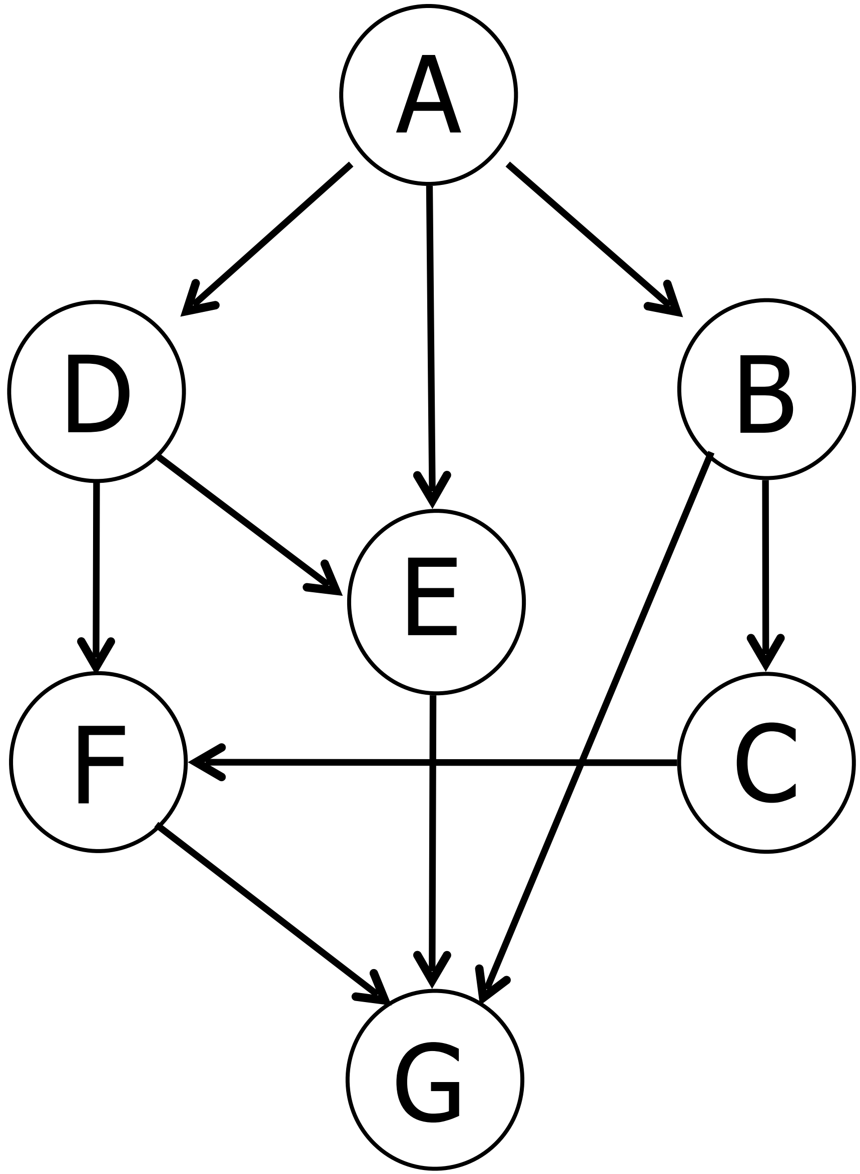
- $O(|V|^2)$ Linear scan to find min
- $O(|E|\log|V|)$ Binary heaps

Directed Graph - Weighted






A
B
C
D
E
F
G
3
5
4
4
3
1
2
1
4
3


A
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