Part 1: Propositional Calculus (a)

COM2107 Logic in Computer Science

Slides by Mike Stannett

Adapted by Jonni Virtema

School of Computer Science
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Main Themes

Today we'll be looking at:

- Syntax (reminder of material from com1002)
- Natural deduction (more formally than in com1002)
- Proof system for PL (more formally than in com1002)
- Principle of harmony

Coming up over in the rest of Part 1:

- Semantics (the meaning of a logical formula)
- Soundness, completeness and compactness

Georg's Notes

Reading: 2.1–2.4.

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Basics

A proposition is a statement that is either true or false.

- "aliens once visited Sheffield"
- "it never rains in southern California"
- "$1 + 1 = 2$"
- "$p \to q$"

Deciding whether a statement is true or false requires a judgment to be made.

How do you judge whether something is true?

In this module, the source of judgements is irrelevant!

Examples

- you meet someone who saw it happen
- you thought about it and it makes sense
- you saw a formal proof in a book
- you overheard someone say it on the bus
- someone wrote it on Facebook
- ...

Evidence and reason

In logic we base judgments on evidence and reason:

- Axioms: are the fundamental universal truths that you accept
  - "if $x = y$ and $y = z$, then $x = z$" (transitivity of identity)
  - "it is raining or it is not raining" (law of excluded middle)
- Assumptions: things you are told
  - "I shot the sheriff"
  - "but it was in self-defence"
- Consequences: things you deduce
  - "not guilty"
- Theorems: things you deduce without any assumptions (that are not axioms)

The validity of consequences depends on

- whether the evidence is reliable (we have no control over this)
- whether your reasoning is sensible

Formal logic

Formal logic captures the rules of reasoning:
how consequences are derived from assumptions?

How can we decide if our logical description of reasoning works as intended?

If the assumptions (the basic evidence we've been given) are wrong, the conclusions may well be wrong as well. There's nothing we can do about that. However:

- consistency: if the assumptions are true, then the things we deduce from them should also be true;
- completeness: if something follows from assumptions, it should be possible to formally deduce it.

We'll see later that we can't always achieve both of these goals at the same time.

Proofs, axioms, theorems, rules, theories

- Proof: A step-by-step explanation of how we deduced something.
- Assumptions: The things we're told are true (if any).
- Inference Rules: The rules we follow when building a proof.
- Axioms: Inference rules that do not have assumptions!
- Theorems: The things we can prove without extra assumptions. Simple theorems are sometimes called lemmas. Axioms are theorems as well!
- Theory: The collection of theorems that can be deduced.

Examples

The "theory of relativity" is a body of knowledge (collection of theorems) in mathematical physics, based on axioms involving things like the speed of light.

The language of logic

To describe a theory fully you also need to say what language things are expressed in.

Examples

- propositional calculus (Prop): true, false, variables, connectives
- predicate calculus (Pred) adds: quantifiers
- arithmetic adds: numbers, addition, multiplication, etc.
- analysis adds: integration, differentiation, “continuous function”, etc.

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Syntax

https://en.wiktionary.org/wiki/syntax

syntax means:

1. A set of rules that govern how words are combined to form phrases and sentences.
2. (*computing*) The formal rules of formulating the statements of a computer language.
3. (*linguistics*) The study of the structure of phrases, sentences, and language.

Well-formed formulae

What strings of characters make sense as logical formulæ?

- These are OK:
  - $p \wedge q$
  - $(p \vee q) \rightarrow p$
  - $\varphi \vee \psi$ (technically a shorthand)
- These aren't:
  - $\neg 7$
  - $)p\wedge \rightarrow \$$
  - $(p \rightarrow)q$

We call the ones that are OK "well-formed formulas" (wffs for short).

But how do we define them?

Formation rules for Prop

1. The following are wffs:
   - propositional variables: p, q, r, $\ldots$
   - logical constant for true: $\top$ ("true", "top")
   - logical constant for false: $\bot$ ("false", "bottom")
2. If φ and ψ are wffs, then so are:
   - negation: $\neg\varphi$ ("not φ")
   - conjunction: $\varphi \wedge \psi$ ("φ and ψ")
   - disjunction: $\varphi \vee \psi$ ("φ or ψ")
   - implication: $\varphi \to \psi$ ("if φ then ψ")

(3. bracketing: if φ is an wff then (φ) is as well)

4. Nothing else is a wff.

Notation

We can also write these rules pictorially, like this:

$$\frac{\varphi \text{ prop} \qquad \psi \text{ prop}}{\varphi \wedge \psi \text{ prop}}\ (\wedge F)$$

Here,

- "φ prop" is shorthand for "φ is a proposition"
- the horizontal line separates things we assume (above the line) from things we deduce.
- the symbols to the right of the line are the name of the rule

This rule is called $(\wedge F)$ ("$\wedge$-Formation"). It says...

If φ and ψ are propositions, then so is $\varphi \wedge \psi$.

The basic formation rules

The rules tell us how to construct propositions

$$\frac{}{\top \text{ prop}}(\top F) \qquad \frac{}{\bot \text{ prop}}(\bot F) \qquad \frac{}{\text{p prop}}(\mathrm{VAR}F)$$

$$\frac{\varphi \text{ prop} \qquad \psi \text{ prop}}{\varphi \to \psi \text{ prop}} \to F \qquad \frac{\varphi \text{ prop}}{\neg\varphi \text{ prop}} \neg F$$

$$\frac{\varphi \text{ prop} \qquad \psi \text{ prop}}{\varphi \wedge \psi \text{ prop}} \wedge F \qquad \frac{\varphi \text{ prop} \qquad \psi \text{ prop}}{\varphi \vee \psi \text{ prop}} \vee F$$

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Each time we draw one of these diagrams (we omit writing "true")

$$\frac{\varphi_1 \qquad \dots \qquad \varphi_n}{\psi} \qquad\qquad \frac{\varphi_1 \text{ true} \qquad \dots \qquad \varphi_n \text{ true}}{\psi \text{ true}}$$

we are stating a hypothetical judgment. It says

IF we know φ_1 (is true) and ... and we know φ_n (is true),
THEN we can conclude ψ (is true).

We can also write it using sequent notation:

$$\varphi_1, \dots, \varphi_n \vdash \psi$$

Example

$$\frac{\textit{If I'm at home then I'm asleep}, \qquad \textit{I'm at home}}{\textit{I'm asleep}}$$

or equivalently,

$$\textit{If I'm at home then I'm asleep}, \quad \textit{I'm at home} \vdash \textit{I'm asleep}$$

This proof makes sense (it's an instance of *modus ponens*), but it's based on *hypotheses*. It doesn't actually tell you I'm asleep, just that I ought to be if the statements above the line are correct.
In fact, they're not: I'm at work and I'm awake.

Inference rules

Natural deduction is a form of reasoning that uses a small, well-defined, collection of these rules. Each inference rule looks like this

$$\frac{\varphi_1 \qquad \cdots \qquad \varphi_n}{\psi}\ \text{(name of rule)}$$

where there are 0 or more things listed above the line, and 1 listed underneath. Rules that have no entries above the line represent axioms – they say "the thing below the line (ψ) is definitely true; you don't need to make any assumptions ($\varphi_1 \dots \varphi_n$) to prove it".

Gluing rules together

If using one rule lets you deduce something that's used as a hypothesis by another one, you can glue the two rules together to form a proof-tree.

Examples

You can combine these rules (I'll tell you their names later)

$$\frac{\varphi \to \psi \qquad \varphi}{\psi} \quad \text{and} \quad \frac{\psi \qquad \gamma}{\psi \wedge \gamma}$$

to generate this proof-tree

$$\frac{\dfrac{\varphi \to \psi \qquad \varphi}{\psi} \qquad \gamma}{\psi \wedge \gamma}$$

Proof-trees

What does a proof-tree prove?

$$\dfrac{\dfrac{\varphi \to \psi \qquad \varphi}{\psi} \qquad \gamma}{\psi \wedge \gamma}$$

In this proof we are no longer *assuming* that ψ is true – we've proven it using two other hypotheses. Overall, this proof-tree shows that:

$$\varphi \to \psi, \varphi, \gamma \vdash \psi \wedge \gamma.$$

Hiding intermediate steps

If we know a proof exists but don't need to write it all out, we can write

$$\begin{array}{c} \varphi \to \psi \qquad \varphi \qquad \gamma \\ D \\ \psi \wedge \gamma \end{array}$$

where we're using D (short for "deduction" – you can use another letter if you prefer) as a name for the missing bit of the picture.

Requiring intermediate steps

If someone wants you to find a proof, they may write

$$\begin{array}{c}\Gamma \\ \vdots \\ \varphi\end{array}$$

- "Capitalised Gamma" Γ is a collection of hypotheses.
- The vertical dots indicate that the steps in the middle still need to be found.
- There's no guarantee that the gap can actually be filled – someone may be asking you to do the impossible.

The proof won't be complete until you provide the details (if one exists) of the missing deduction.

proof-trees

You can keep combining diagrams in this way. The picture you obtain is a proof that given the things at the leaves of the tree, you can deduce the thing at its root.

Examples

This proof-tree:

$$\dfrac{\dfrac{p \vee q \qquad p \to r \qquad q \to r}{r} \qquad r \to z}{z}$$

proves this sequent (hypothetical judgment):

$$p \vee q, p \to r, q \to r, r \to z \vdash z$$

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Prop

The language of propositional calculus ("Prop") was given earlier. We have:

- $p, q, r, \ldots$ ("propositional variables")
- $\top, \bot$ ("true", "false")
- $\neg\varphi$ ("not")
- $\varphi \land \psi$, $\varphi \lor \psi$, $\varphi \to \psi$ ("and", "or", "implies")

I'm using $\varphi, \psi, \gamma, \ldots$ as schematic variables. They stand for "any proposition".

Using hypotheses

We often write Γ and Δ for collections of hypotheses to save space. Given this notation, the *structural rules* for proving things are:

- anything assumed can be proven ("it's true because you said it was")

$$\Gamma, \varphi \vdash \varphi$$

- you don't need to use *all* of the hypotheses available to you

$$\Gamma \vdash \psi \;\Rightarrow\; \Gamma, \varphi \vdash \psi$$

- you can use hypotheses more than once

$$\Gamma, \varphi, \varphi \vdash \psi \;\Rightarrow\; \Gamma, \varphi \vdash \psi$$

- you can use hypotheses in any order

$$\Gamma, \varphi_1, \varphi_2, \Delta \vdash \psi \;\Rightarrow\; \Gamma, \varphi_2, \varphi_1, \Delta \vdash \psi$$

Conjunction introduction

Conjunction-introduction ($\wedge I$) is the rule

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\,(\wedge I)$$

Apart from its name ($\wedge I$), it looks very like the formation rule ($\wedge F$), but it means something very different. It tells us how to introduce (I) the conjunction symbol ($\wedge$) into a proof

$$\varphi, \psi \vdash \varphi \wedge \psi$$

if you can prove φ and you can prove ψ,
then you can deduce $\varphi \wedge \psi$

Combining proofs (tabular example)

The two proofs

proof 1			proof 2		
1.	p	hyp	1.	p	hyp
2.	$p \wedge p$	$(\wedge I)$ (1,1)	2.	q	hyp
			4.	$(p \wedge q)$	$(\wedge I)$ (1,2)

can be combined to give a conjunction proof:

1.	p	hyp
2.	$p \wedge p$	$(\wedge I)$ (1,1)
3.	p	hyp
4.	q	hyp
5.	$(p \wedge q)$	$(\wedge I)$ (3,4)
6.	$(p \wedge p) \wedge (p \wedge q)$	$(\wedge I)$ (2,5)

1.	p	hyp
2.	q	hyp
3.	$p \wedge p$	$(\wedge I)$ (1,1)
4.	$(p \wedge q)$	$(\wedge I)$ (1,2)
5.	$(p \wedge p) \wedge (p \wedge q)$	$(\wedge I)$ (3,4)

Combining proofs (proof-tree example)

$$\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\,(\wedge I) \qquad\qquad \dfrac{\dfrac{\psi}{\psi \vee \neg\gamma}\,(\vee I_l) \qquad (\psi \vee \neg\gamma) \to (\gamma \to \psi)}{\gamma \to \psi}\,(\to E)$$

can be "paired" to give

$$\dfrac{\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\,(\wedge I) \qquad \dfrac{\dfrac{\psi}{\psi \vee \neg\gamma}\,(\vee I_l) \qquad (\psi \vee \neg\gamma) \to (\gamma \to \psi)}{\gamma \to \psi}\,(\to E)}{(\varphi \wedge \psi) \wedge (\gamma \to \psi)}\,(\wedge I)$$

Conjunction elimination

If you can prove $\varphi \wedge \psi$, you can infer both φ and ψ separately:

$$\frac{\varphi \wedge \psi}{\varphi}\,(\wedge E_l) \qquad \frac{\varphi \wedge \psi}{\psi}\,(\wedge E_r)$$

or in other words:

$\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$

Implication elimination (modus ponens)

Implication ($\to$) elimination (E) looks like this:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ (\to E)$$

$$\varphi, \varphi \to \psi \vdash \psi$$

if you can prove φ, and you can prove $\varphi \to \psi$,
then you can deduce ψ

This is often called *modus ponens*. Some authors use the shorthand "mp" instead of ($\to E$).

Implication introduction

This corresponding introduction rule looks much more complicated:

$$\cfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\ (\to I)$$

It says "if assuming φ enables you to prove ψ, then you can deduce $\varphi \to \psi$".

Discharging an assumption

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\ (\to I)$$

The square brackets say that the assumption φ is temporary ("local"). Once we finish using the assumption to deduce $\varphi \to \psi$, that particular instance of φ is no longer relevant. It has been discharged by $(\to I)$, and is longer available in the rest of the proof.

Disjunction introduction

The inference rules for *disjunction introduction* capture the idea that, if we want to prove $\varphi \vee \psi$, it's enough to prove φ or to prove ψ. Proving either one of them will do; you don't need to prove both.

$$\frac{\varphi}{\varphi \vee \psi}\,(\vee I_l) \qquad \frac{\psi}{\varphi \vee \psi}\,(\vee I_r)$$

Disjunction elimination

This rule tells us how to remove a disjunction symbol from a proof.

$$\frac{\varphi \vee \psi \qquad \varphi \rightarrow \gamma \qquad \psi \rightarrow \gamma}{\gamma} \; (\vee E)$$

If we know that $\varphi \vee \psi$ can be proven,
and we know that each of φ and ψ is enough on its own to entail γ,
then either way, we can infer γ.

Another way to write $(\vee E)$

Georg's notes show $(\vee E)$ written a bit differently:

$$\dfrac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\vee E)$$

Can you see why the two versions are interchangeable?
You can use either version when answering questions.

False and True

The inference rules for $\perp$ and $\top$ are

$$\frac{}{\top}\,(\top I) \qquad \frac{\perp}{\varphi}\,(\perp E)$$

There is only one rule for each.

Negation

We can think of negation as a special case of implication:

$$\neg\varphi \equiv (\varphi \to \bot)$$

Using $(\to I)$ and $(\to E)$ then tells us what the rules for negation should be:

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\neg\varphi}\,(\neg I) \qquad\qquad \dfrac{\varphi \qquad \neg\varphi}{\bot}\,(\neg E)$$

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Harmony: over to you...

Georg's notes mention several examples of *harmony*, the idea that elimination rules act as inverses to introduction rules.

Examples

$$\dfrac{\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\,(\wedge I)}{\varphi}\,(\wedge E_l) \qquad\qquad \dfrac{\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\,(\wedge I)}{\psi}\,(\wedge E_r)$$

If you start with φ and ψ separately and use $(\wedge I)$ to combine them, you can subsequently use $(\wedge E_l)$ and $(\wedge E_r)$ to separate them out again.

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Natural deduction at work

Recap: Intuitionistic or constructive inference rules

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\wedge I \qquad \frac{\varphi \wedge \psi}{\varphi}\wedge E_l \qquad \frac{\varphi \wedge \psi}{\psi}\wedge E_r$$

$$\frac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\to I \qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}\to E \qquad \frac{\begin{matrix}[\varphi]\\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\neg I \qquad \frac{\varphi \qquad \neg\varphi}{\bot}\neg E$$

$$\frac{\varphi}{\varphi \vee \psi}\vee I_l \qquad \frac{\psi}{\varphi \vee \psi}\vee I_r \qquad \frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi]\\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\vee E$$

$$\frac{}{\top}\top I \qquad \frac{\bot}{\varphi}\bot E$$

Proofs

inference rules allow us to construct proofs

$$
\dfrac{\begin{matrix} D_1 \\ \varphi \end{matrix} \qquad \begin{matrix} D_2 \\ \psi \end{matrix}}{\varphi \wedge \psi} \wedge I
\qquad\qquad
\begin{matrix} \dfrac{\varphi \wedge \psi}{\varphi} \wedge E_l \\ D \end{matrix}
\qquad\qquad
\begin{matrix} D_1 \\ \dfrac{\varphi \wedge \psi}{\varphi} \wedge E_l \\ D_2 \end{matrix}
$$

elimination rules decompose judgments top-down

- they are typically used towards beginning of proof

introduction rules decompose judgments bottom-up

- they are typically used towards end of proof

Example

proof of $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$

Examples

$$p \rightarrow r$$

Example

proof of $p \to q, q \to r \vdash p \to r$

Examples

$$p$$

$$\frac{r}{p \to r} \to I$$

Example

proof of $p \to q, q \to r \vdash p \to r$

Examples

$$\dfrac{\dfrac{p \to q \qquad p}{q}\;{\to}E}{\dfrac{r}{p \to r}\;{\to}I}$$

Example

proof of $p \to q, q \to r \vdash p \to r$

Examples

$$\dfrac{\dfrac{q \to r \qquad \dfrac{p \to q \qquad p}{q}\,{\to}E}{r}\,{\to}E}{p \to r}\,{\to}I$$

Example

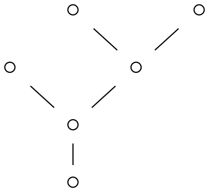
proof of $p \to q, q \to r \vdash p \to r$

Examples

$$\dfrac{\dfrac{q \to r \qquad \dfrac{p \to q \qquad [p]}{q}\,{\to}E}{r}\,{\to}E}{p \to r}\,{\to}I$$

Example

proof tree has emerged



Example

more generally $\varphi \to \psi, \psi \to \gamma \vdash \varphi \to \gamma$

Examples

$$\dfrac{\dfrac{q \to r \qquad \dfrac{p \to q \qquad [p]}{q}\,{\to}E}{r}\,{\to}E}{p \to r}\,{\to}I$$

Example

more generally $\varphi \to \psi, \psi \to \gamma \vdash \varphi \to \gamma$

Examples

$$
\dfrac{\dfrac{\psi \to \gamma \qquad \dfrac{\varphi \to \psi \qquad [\varphi]}{\psi} \to E}{\gamma} \to E}{\varphi \to \gamma} \to I
$$

Example

We (and computers) often prefer linear proofs with

- local subproofs indicated by boxes
- premise/conclusion relationships by line numbers/labels

Examples

1.	$\psi \to \gamma$	hyp
2.	$\varphi \to \psi$	hyp

$\varphi \to \gamma$

Example

We (and computers) often prefer linear proofs with

- local subproofs indicated by boxes
- premise/conclusion relationships by line numbers/labels

Examples

1.	$\psi \to \gamma$	hyp
2.	$\varphi \to \psi$	hyp
3.	φ	hyp
	γ	
	$\varphi \to \gamma$	$\to I$

Example

We (and computers) often prefer linear proofs with

- local subproofs indicated by boxes
- premise/conclusion relationships by line numbers/labels

Examples

1.	$\psi \to \gamma$	hyp
2.	$\varphi \to \psi$	hyp
3.	φ	hyp
4.	ψ	$\to E$,2,3
	γ	
	$\varphi \to \gamma$	$\to I$

Example

We (and computers) often prefer linear proofs with

- local subproofs indicated by boxes
- premise/conclusion relationships by line numbers/labels

Examples

1.	$\psi \to \gamma$	hyp
2.	$\varphi \to \psi$	hyp
3.	$\boxed{\varphi}$	hyp
4.	ψ	$\to E$,2,3
5.	γ	$\to E$,1,4
	$\varphi \to \gamma$	$\to I$

Example

We (and computers) often prefer linear proofs with

- local subproofs indicated by boxes
- premise/conclusion relationships by line numbers/labels

Examples

1.	$\psi \to \gamma$	hyp
2.	$\varphi \to \psi$	hyp
3.	φ	hyp
4.	ψ	$\to E$,2,3
5.	γ	$\to E$,1,4
6.	$\varphi \to \gamma$	$\to I$,3-5

Example

we can now use $\varphi \to \psi, \psi \to \gamma \vdash \varphi \to \gamma$ as a lemma in subsequent proofs

or as derived inference rule

$$\frac{\varphi \to \psi \qquad \psi \to \gamma}{\varphi \to \gamma}$$

Rewriting Sequents

We proved

$$(\varphi \to \psi), (\psi \to \gamma) \vdash (\varphi \to \gamma)$$

The proof of

$$\vdash (\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma))$$

contains the same proof (see lecture notes). And so does the proof of

$$\vdash ((\varphi \to \psi) \land (\psi \to \gamma)) \to (\varphi \to \gamma)$$

"Deduction Theorem" (we won't prove this)

We can always rewrite sequents

$$\varphi_1, \ldots, \varphi_n \vdash \varphi \qquad \text{as} \qquad \vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \varphi) \ldots)$$
$$\text{or} \qquad \vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$$

and vice versa.

The first form is often quicker to deal with.

Theorems

Technically, a theorem is a sequent of the form $\vdash \varphi$, with no hypotheses to the left of the turnstile symbol. The deduction theorem says we can always think of sequents in Prop as describing theorems.

Examples

The sequent

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

can be thought of as the theorem

$$\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots)$$

Equivalent statements and biconditionals

If $\varphi \vdash \psi$ and $\psi \vdash \varphi$ both hold, we write

$$\varphi \dashv\vdash \psi$$

and say that φ, ψ are (provably) equivalent. This is the same as saying that

$$\vdash \varphi \rightarrow \psi \quad \text{and} \quad \vdash \psi \rightarrow \varphi$$

are both theorems. We also have

$$\varphi \dashv\vdash \psi \quad \text{if and only if} \quad \vdash \varphi \leftrightarrow \psi$$

where we define the biconditional arrow ($\leftrightarrow$) by

$$\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

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What have we learned?

Today and last week:

- Basic terminology for any proof calculi
- Natural deduction for constructive propositional logic

Next week:

- Natural deduction for classical propositional logic
- Semantics and structural induction
- Soundness, completeness, and compatness