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Exercise Sheet 2

The aim of this sheet is to get you thinking about proofs using the inference rules described in the lectures, and working out how to build them. You've seen various types of proof:

- a numbered list of inferences (as in com1002);
- a basic proof-tree

You can use whichever version you prefer for these exercises, but try both of them. How easily can you convert between the different proof styles?

Further info: see p.27 of Huth and Ryan's book.

Notation. The statement "$\alpha \dashv\vdash \beta$" is shorthand for saying "$\alpha \vdash \beta$ and $\beta \vdash \alpha$". So when a question asks you to prove something of this kind, you actually need to do two separate proofs, one in each direction.

Question 2.1 Consider a setting where you wish to do deduction concerning statements on arithmetics of natural numbers. In this setting, we assume some inherent properties of addition and the equality sign: "if $x = y$ then $y = x$" (symmetry of =) and "if $x \leq y$ then $x + z \leq y + z$" (monotonicity of addition). Assume you have succeeded in proving that

"$x + 3 \leq y + 3$ if it is the case that $x \leq y$"

and proving that "$x + 3 = 3 + x$".

List all the statements from above that are

(a) axioms

Answer: There are two axioms: symmetry of = and monotonicity of addition.

(b) assumptions (that are not axioms) that are used in the two proofs

Answer: $x \leq y$ is the only assumption of the first prove (that is not an axiom). The second proof has no additional assumptions.

(c) consequences of the proofs

Answer: $x + 3 \leq y + 3$ is the consequence of the first proof, while $x + 3 = 3 + x$ is the consequence of the second proof.

(d) theorems

Answer:

All the listed axioms are theorems, and $x + 3 = 3 + x$ is a theorem.

Question 2.2 Prove at least five of the following sequents by natural deduction. Solving the rest is good practice! Try to solve one sequent using both tabular and proof tree notation.

Proofs using tabular notation

(a) $p \wedge (q \wedge r) \dashv\vdash (p \wedge q) \wedge r$.

Answer:

1.	$p \wedge (q \wedge r)$	hyp
2.	p	$\wedge E_l$, 1
3.	$q \wedge r$	$\wedge E_r$, 1
4.	q	$\wedge E_l$, 3
5.	r	$\wedge E_r$, 3
6.	$p \wedge q$	$\wedge I$, 2,4
7.	$(p \wedge q) \wedge r$	$\wedge I$, 6,5

The other direction is similar

(b) $p \vee (q \vee r) \dashv\vdash (p \vee q) \vee r$.

Answer:

1.	$p \vee (q \vee r)$	hyp
2.	p	hyp
3.	$p \vee q$	$\vee I_l$, 2
4.	$(p \vee q) \vee r$	$\vee I_l$, 3
5.	$q \vee r$	hyp
6.	q	hyp
7.	$p \vee q$	$\vee I_r$, 6
8.	$(p \vee q) \vee r$	$\vee I_l$, 7
9.	r	hyp
10.	$(p \vee q) \vee r$	$\vee I_r$, 9
11.	$(p \vee q) \vee r$	$\vee E$, 5,6-8,9-10
12.	$(p \vee q) \vee r$	$\vee E$, 1,2-4,5-11

The other direction is similar.

(c) $p \vee p \vdash p$.

Answer:

1.	$p \vee p$	hyp
2.	p	hyp
3.	p	hyp
4.	p	$\vee E$, 1,2,3

(d) $p \wedge (q \vee r) \dashv\vdash (p \wedge q) \vee (p \wedge r)$.

Answer:

1.	$p \wedge (q \vee r)$	hyp
2.	p	$\wedge E_l$, 1
3.	$q \vee r$	$\wedge E_r$, 1
4.	q	hyp
5.	$p \wedge q$	$\wedge I$, 2,4
6.	$(p \wedge q) \vee (p \wedge r)$	$\vee I_l$, 5
7.	r	hyp
8.	$p \wedge r$	$\wedge I$, 2,7
9.	$(p \wedge q) \vee (p \wedge r)$	$\vee I_r$, 8
10.	$(p \wedge q) \vee (p \wedge r)$	$\vee E$, 3,4-6,7-9

1.	$(p \wedge q) \vee (p \wedge r)$	hyp
2.	$p \wedge q$	hyp
3.	p	$\wedge E_l$, 2
4.	q	$\wedge E_r$, 2
5.	$q \vee r$	$\vee I_l$, 4
6.	$p \wedge (q \vee r)$	$\wedge I$, 3,5
7.	$p \wedge r$	hyp
8.	p	$\wedge E_l$, 7
9.	r	$\wedge E_r$, 7
10.	$q \vee r$	$\vee I_r$, 9
11.	$p \wedge (q \vee r)$	$\wedge I$, 8,10
12.	$p \wedge (q \vee r)$	$\vee E$, 1,2-6,7-11

(e) $(p \wedge q) \to r \dashv\vdash p \to (q \to r)$.

Answer:

1.	$(p \wedge q) \to r$	hyp
2.	p	hyp
3.	q	hyp
4.	$p \wedge q$	$\wedge I$,2,3
5.	r	$\to E$, 1,4
6.	$q \to r$	$\to I$, 3-5
7.	$p \to (q \to r)$	$\to I$, 2-6

1.	$p \to (q \to r)$	hyp
2.	$p \wedge q$	hyp
3.	p	$\wedge E_l$, 2
4.	q	$\wedge E_r$, 2
5.	$q \to r$	$\to E$, 1,3
6.	r	$\to E$, 4,5
7.	$(p \wedge q) \to r$	$\to I$, 2-6

(f) $p \to q \vdash p \lor r \to q \lor r$.

Answer:

1.	$p \to q$	hyp
2.	$p \lor r$	hyp
3.	p	hyp
4.	q	$\to E$, 1,3
5.	$q \lor r$	$\lor I_l$, 4
6.	r	hyp
7.	$q \lor r$	$\lor I_r$,6
8.	$q \lor r$	$\lor E$,2,3-5,6-7
9.	$p \lor r \to q \lor r$	$\to I$, 2-8

(g) $\vdash p \to (q \to p)$.

Answer:

1.	p	hyp
2.	q	hyp
3.	p	copy, 1
4.	$q \to p$	$\to I$, 2,3
5.	$p \to (q \to p)$	$\to I$,1-4

(h) $p \vdash \neg\neg p$.

Answer:

1.	p	hyp
2.	$\neg p$	hyp
3.	$\perp$	$\neg E$, 1,2
4.	$\neg\neg p$	$\neg I$ 2-3

(i) $\neg p \lor q \vdash p \to q$.

Answer:

1.	$\neg p \lor q$	hyp
2.	$\neg p$	hyp
3.	p	hyp
4.	$\perp$	$\neg E$, 2,3
5.	q	$\perp E$, 4
6.	$p \to q$	$\to I$, 3-5
7.	q	hyp
8.	p	hyp
9.	q	copy 7
10.	$p \to q$	$\to I$, 8,9
11.	$p \to q$	$\lor E$, 1,2-6,7-9

An alternative proof is

1.	$\neg p \lor q$	hyp
2.	p	hyp
3.	$\neg p$	hyp
4.	$\bot$	$\neg E$, 2,3
5.	q	$\bot E$, 4
6.	q	hyp
7.	q	$\lor E$,1,3-5,6
8.	$p \to q$	$\to I$, 2-7

Sample proofs using proof-tree notation

(a) $p \land (q \land r) \vdash (p \land q) \land r$ (the other direction is similar):

$$\dfrac{\dfrac{\dfrac{p \land (q \land r)}{p}\,(\land E_l) \qquad \dfrac{\dfrac{p \land (q \land r)}{q \land r}\,(\land E_r)}{q}\,(\land E_l)}{p \land q}\,(\land I) \qquad \dfrac{\dfrac{p \land (q \land r)}{q \land r}\,(\land E_r)}{r}\,(\land E_r)}{(p \land q) \land r}\,(\land I)$$

(b) $p \lor (q \lor r) \vdash (p \lor q) \lor r$

The other direction is similar. The labels (a) and (b) are used to show which assumptions are discharged by which inference rules.

$$\dfrac{p \lor (q \lor r) \qquad \dfrac{\dfrac{[p]_b}{p \lor q}\,\lor I_l}{(p \lor q) \lor r}\,\lor I_l \qquad \dfrac{[q \lor r]_b \qquad \dfrac{\dfrac{[q]_a}{p \lor q}\,(\lor I_r)}{(p \lor q) \lor r}\,(\lor I_l) \qquad \dfrac{[r]_a}{(p \lor q) \lor r}\,(\lor I_r)}{(p \lor q) \lor r}\,(\lor E)_a}{(p \lor q) \lor r}\,(\lor E)_b$$

(c) $p \lor p \vdash p$. The discharged instances of p capture the basic observation that "if you assume p you can prove p".

$$\dfrac{p \lor p \qquad [p] \qquad [p]}{p}\,(\lor E)$$

(d) $p \land (q \lor r) \dashv\vdash (p \land q) \lor (p \land r)$.

I've proven it one way round. Can you prove the other?

$$\dfrac{\dfrac{p \land (q \lor r)}{q \lor r}\,(\land E_r) \qquad \dfrac{\dfrac{\dfrac{p \land (q \lor r)}{p}\,(\land E_l) \qquad [q]_a}{(p \land q)}\,(\land I)}{(p \land q) \lor (p \land r)}\,(\lor I_l) \qquad \dfrac{\dfrac{\dfrac{p \land (q \lor r)}{p}\,(\land E_l) \qquad [r]_a}{(p \land r)}\,(\land I)}{(p \land q) \lor (p \land r)}\,(\lor I_r)}{(p \land q) \lor (p \land r)}\,(\lor E)_a$$