Introduction to Mobile Robotics and Robot Operating System (ROS)

Lecture 3. Kinematic models of wheeled robots.

Probabilistic motion models

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Outline

1. Kinematic models of wheeled robots
    a. Differential drive
    b. Tricycle
    c. Ackermann principle
    d. Omni- and mecanum-wheels
2. Probabilistic motion models
    a. Odometry-based model
    b. Speed control based model

RECURSIVE BAYESIAN POSE ESTIMATION

$$p(\mathbf{x_t}|map, \mathbf{z_t}, \mathbf{u_t}) = C \cdot p(\mathbf{z_t}|\mathbf{x_t}, map) \int_S p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}}) p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}}) d\mathbf{x_{t-1}}$$

C — normalization coefficient

S — the probabilistic space of robot poses

$p(\mathbf{z_t}|\mathbf{x_t}, map)$ — observation (measurement) model

$p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}})$ — motion model

$p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}})$ — previous system state (robot pose)

KALMAN FILTER

Prediction:

$$\hat{\mathbf{x}}_t = \mathbf{F}_t \hat{\mathbf{x}}_{t-1} + \mathbf{B}_k \vec{\mathbf{u}_t}$$

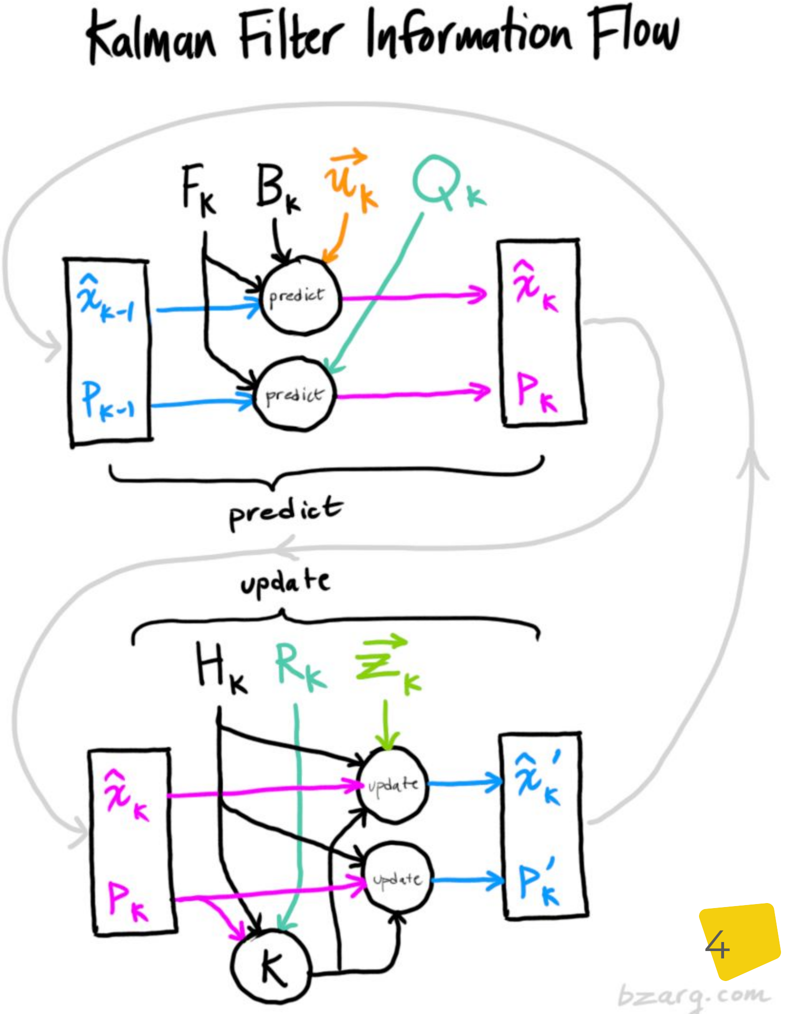
$$\hat{\Sigma}_t = \mathbf{F}_t \Sigma_{t-1} \mathbf{F}_t^T + \mathbf{Q}_t$$

Correction:

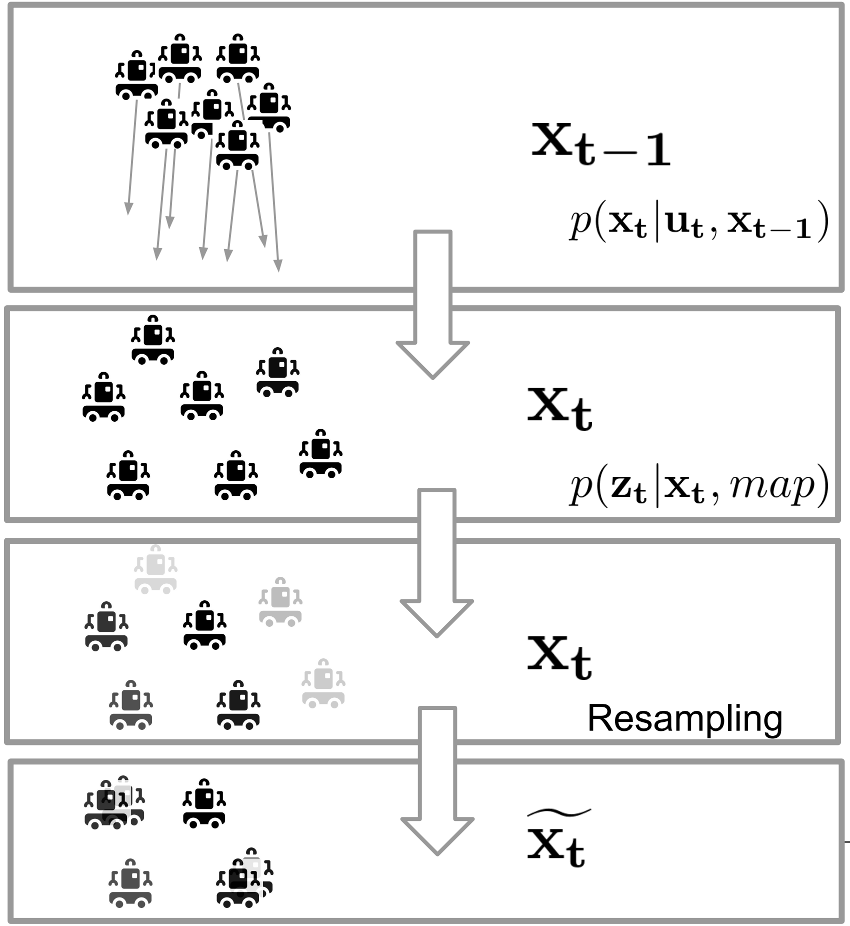
$$\mathbf{K}' = \hat{\Sigma}_t \mathbf{H}_t^T (\mathbf{H}_t \hat{\Sigma}_t \mathbf{H}_t^T + \mathbf{R}_t)^{-1}$$

$$\mathbf{x}'_t = \mathbf{H}_t \hat{\mathbf{x}}_t + \mathbf{K}' (\vec{\mathbf{z}_t} - \mathbf{H}_t \hat{\mathbf{x}}_t)$$

$$\Sigma'_t = \hat{\Sigma}_t - \mathbf{K}' \mathbf{H}_t \hat{\Sigma}_t$$



PARTICLE FILTER



Algorithm 1 Generic Monte-Carlo localization algorithm

procedure MCL($\mathbf{x_{t-1}}, m, \mathbf{u_t}, \mathbf{z_t}$)
  $\{\mathbf{x_t^n}\} = \{\widetilde{\mathbf{x_t^n}}\} = \emptyset$
  for $n = 1$ to N do
    sample $x_t^n \sim p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}^n})$ — Motion model
    $w_t^n = p(\mathbf{z_t}|\mathbf{x_t^n}, map)$ — Observation model
    $\{\widetilde{\mathbf{x_t^n}}\} = \{\widetilde{\mathbf{x_t^n}}\} + \langle x_t^n, w_t^n \rangle$
  end for
  for $n = 1$ to N do — Resampling
    draw i with probability $\propto \widetilde{w_t^i}$
    $\{\mathbf{x_t^n}\} = \{\mathbf{x_t^n}\} + \langle \widetilde{x_t^i}, \widetilde{w_t^i} \rangle$
  end for
  return $\{\mathbf{x_t^n}\}$
end procedure

RECURSIVE BAYESIAN POSE ESTIMATION

$$p(\mathbf{x_t}|map, \mathbf{z_t}, \mathbf{u_t}) = C \cdot p(\mathbf{z_t}|\mathbf{x_t}, map) \int_S p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}}) p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}}) d\mathbf{x_{t-1}}$$

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RECURSIVE BAYESIAN POSE ESTIMATION

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Kinematic models of wheeled robots

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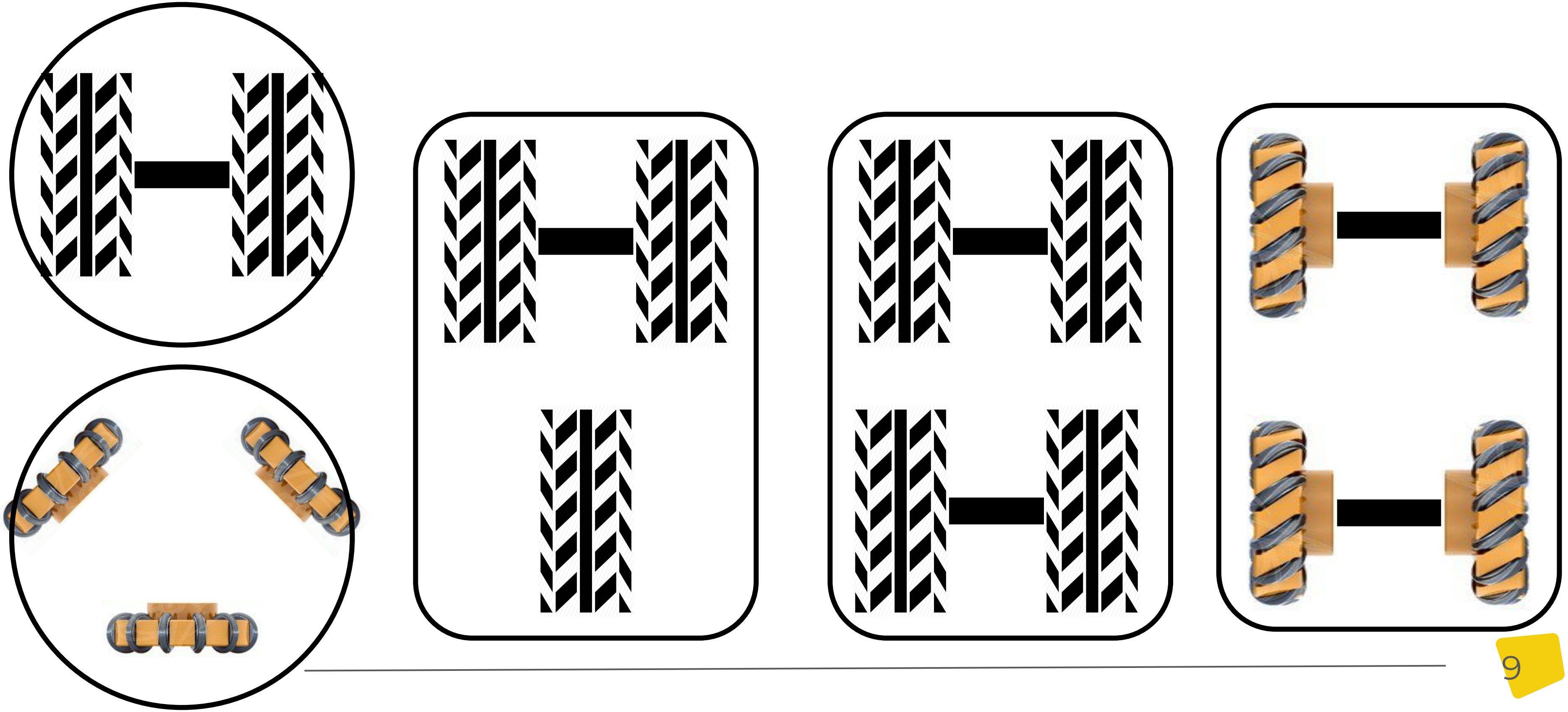
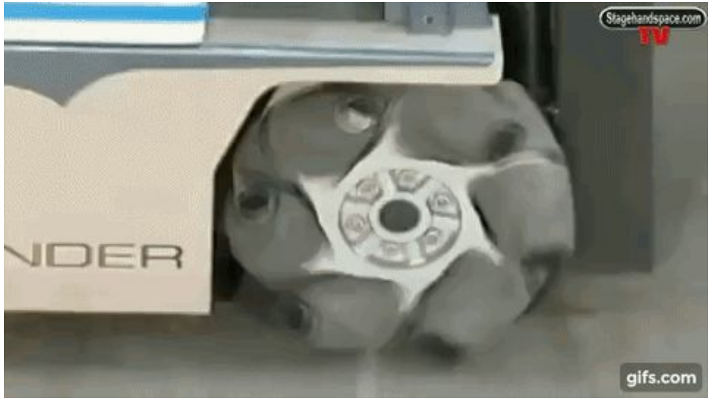
01

(some) TYPES OF WHEELED KINEMATICS

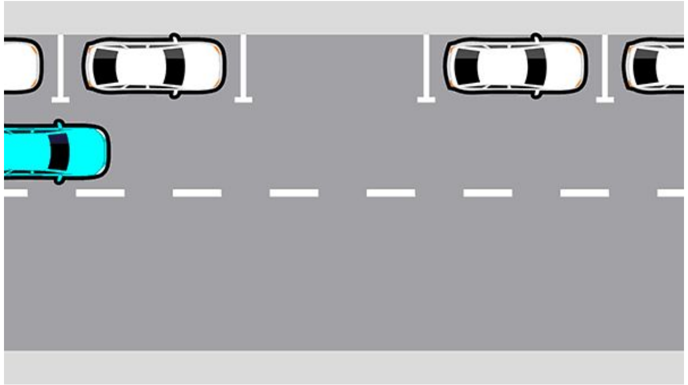


HOLONOMIC SYSTEMS

- A robot is called **holonomic** if the number of **controlled** degrees of freedom = the **total** number of degrees of freedom.



- A **nonholonomic system** is a mechanical system on which, in addition to geometric ones, kinematic constraints are also superimposed.
- Mathematically, nonholonomic constraints are expressed by non-integrable equations.



HOLONOMIC SYSTEMS

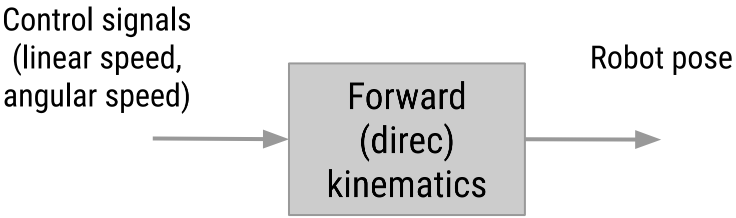
- **Holonomic constraints** limit the allowed state space (geometry).
- For instance, if there is a truck and a trailer, **not all angles** between them **are possible**. This is a **holonomic constraint**.

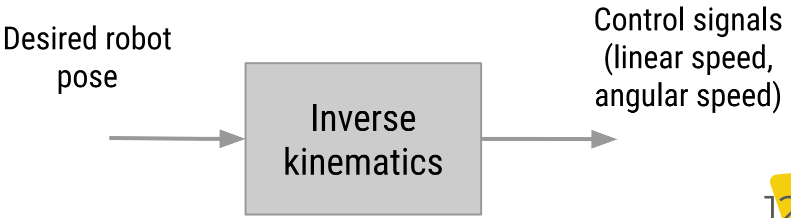
- **Nonholonomic constraints** limit the control space relative to the current state.
- For instance, a car **can not move sideway**.

DIRECT AND INVERSE KINEMATICS PROBLEMS

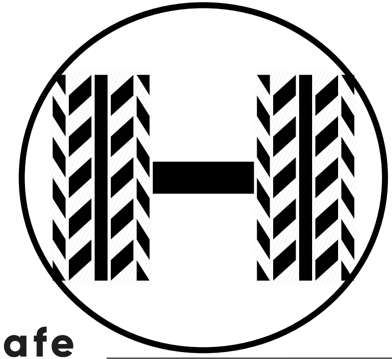
- The **direct kinematics problem** — having control parameters (for example, wheel speeds) and motion time, find the position into which the robot has moved.

- The **inverse kinematics problem** is to find the control parameters that move the robot into a given position in a given time.





DIFFERENTIAL DRIVE

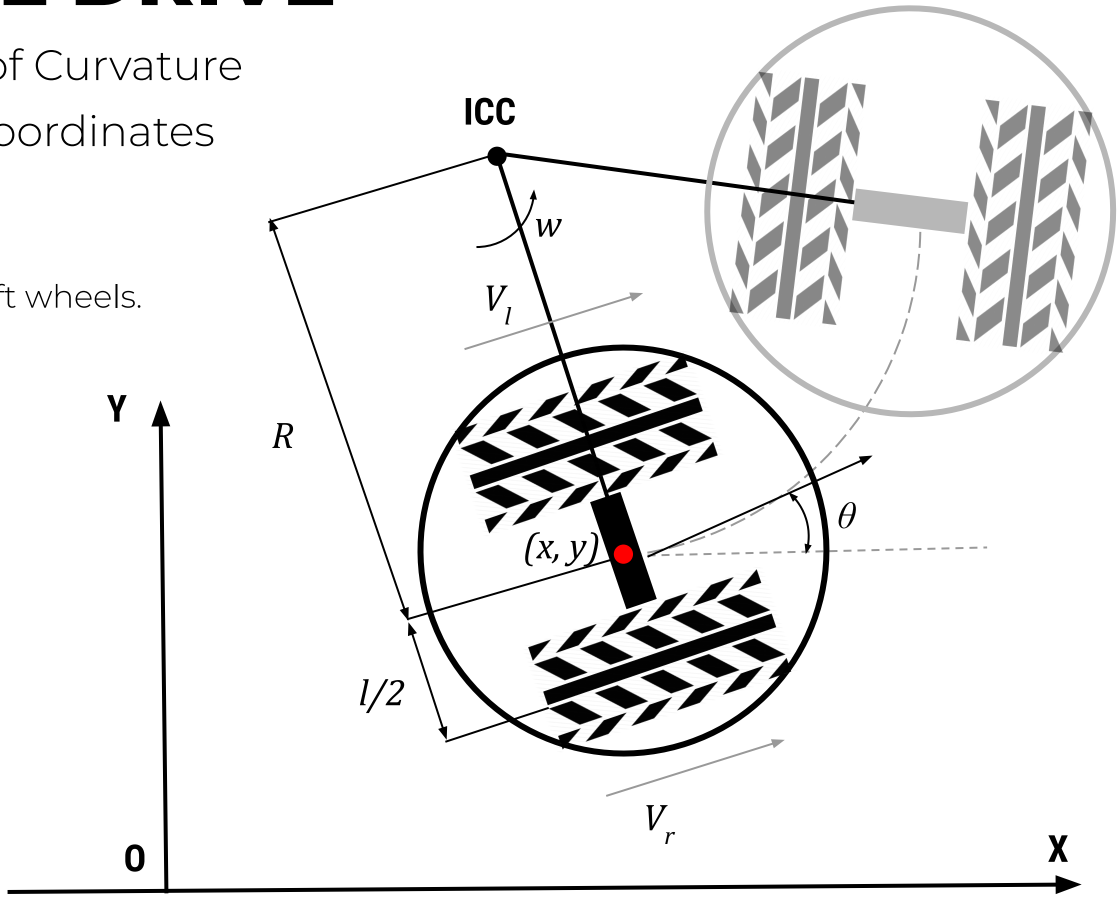


DIFFERENTIAL DRIVE

ICC – Instantaneous Center of Curvature

(x, y, θ) – wheel axle center coordinates

$\left.\begin{matrix} V_r \\ V_l \end{matrix}\right\}$ – the speed of the right and left wheels. Controlled parameters.



DIFFERENTIAL DRIVE

ICC – Instantaneous Center of Curvature

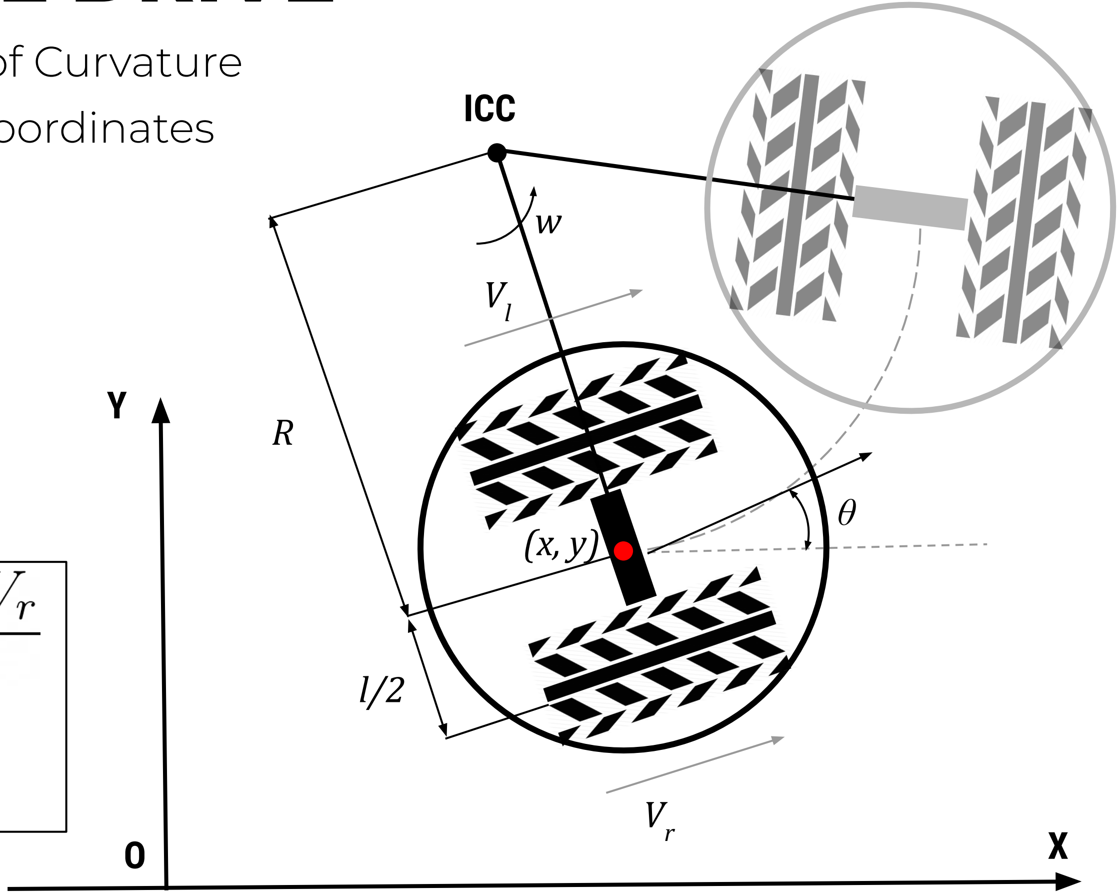
(x, y, θ) – wheel axle center coordinates

$$w(R + \frac{l}{2}) = V_r$$

$$w(R - \frac{l}{2}) = V_l$$

$$w = \frac{V_r - V_l}{l} \qquad V = \frac{V_l + V_r}{2}$$

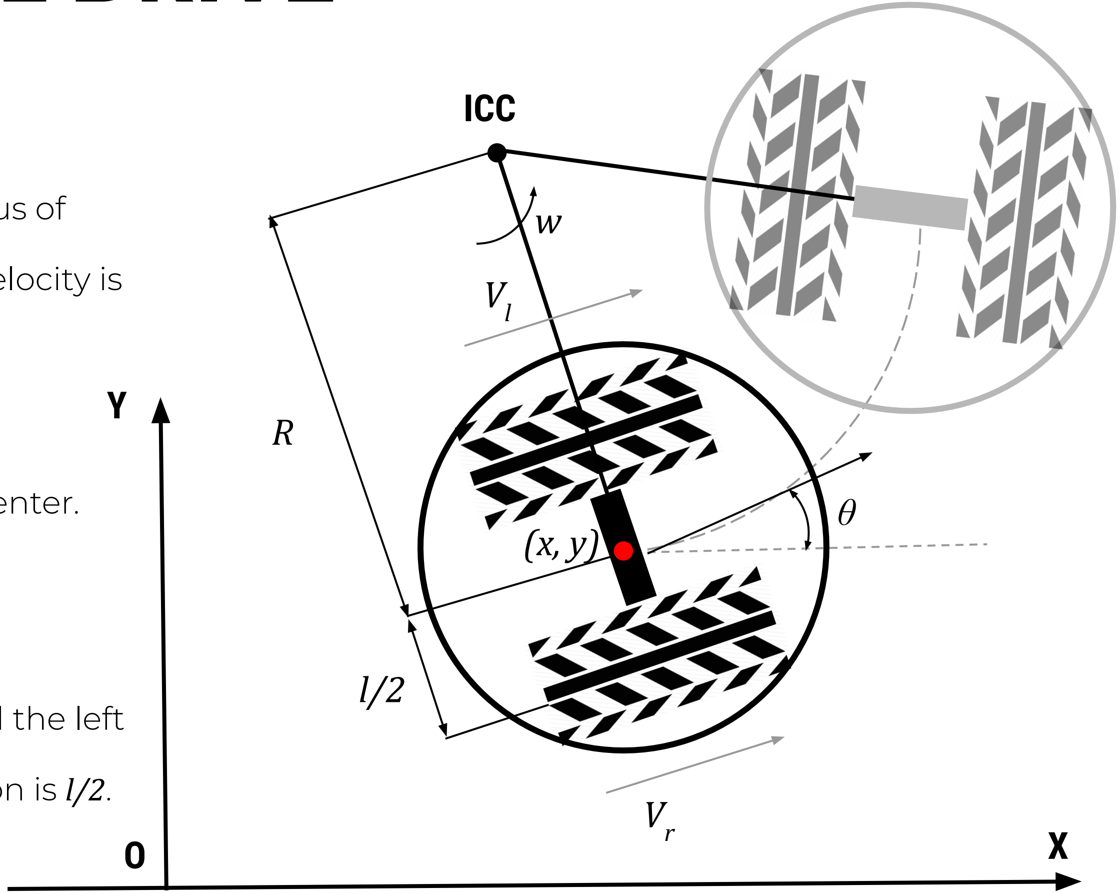
$$R = \frac{l}{2}\frac{V_r + V_l}{V_r - V_l}$$



DIFFERENTIAL DRIVE

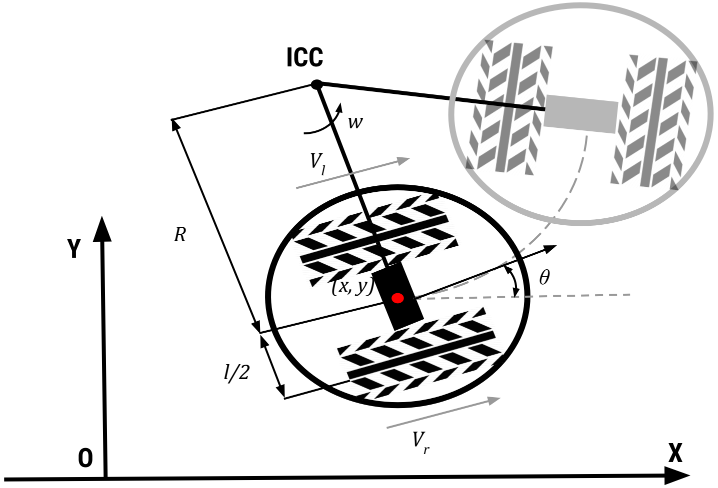
3 types of motion:

- ❑ $V_l = V_r$ — linear motion. The radius of rotation is **infinity**. The angular velocity is **zero**.
- ❑ $V_l = -V_r$ — rotation around the center. The radius of rotation is zero.
- ❑ $V_l = 0$ ($V_r = 0$) — rotation around the left (right) wheel. The radius of rotation is $l/2$.



DIFFERENTIAL DRIVE FORWARD KINEMATICS

$$ICC = [x - R\sin(\theta), y + R\cos(\theta)]$$

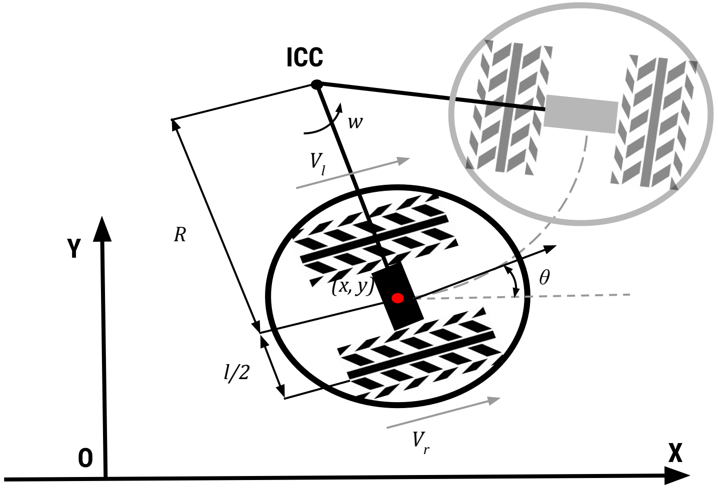


At the time moment **t+δt** the robot pose is defined as:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} cos(\omega\delta t) & -sin(\omega\delta t) & 0 \\ sin(\omega\delta t) & cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - ICC_x \\ y - ICC_y \\ \theta \end{bmatrix} + \begin{bmatrix} ICC_x \\ ICC_y \\ \omega\delta t \end{bmatrix}$$

DIFFERENTIAL DRIVE FORWARD KINEMATICS

$$x(t) = \int_0^t V(t)cos[\theta(t)]dt$$
$$y(t) = \int_0^t V(t)sin[\theta(t)]dt$$
$$\Theta(t) = \int_0^t \omega(t)dt$$

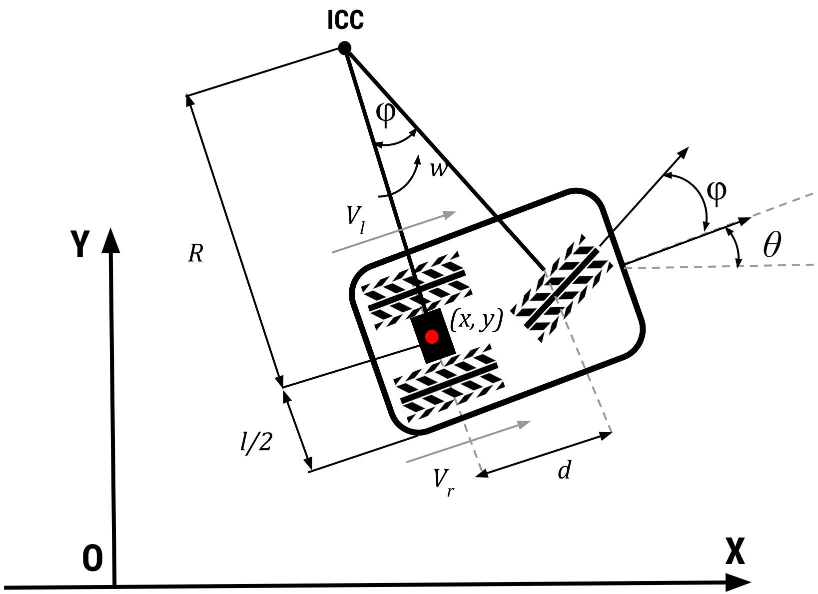


$$x(t) = \frac{1}{2}\int_0^t [v_r(t) + v_l(t)]cos[\theta(t)]dt$$
$$y(t) = \frac{1}{2}\int_0^t [v_r(t) + v_l(t)]sin[\theta(t)]dt$$
$$\Theta(t) = \frac{1}{l}\int_0^t [v_r(t) - v_l(t)])dt$$

TRICYCLE

$$ICC = [x - R\sin(\theta), y + R\cos(\theta)]$$

$$R = \frac{d}{\tan \ \varphi}$$



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TRICYCLE

Features:

- Can not rotate in place
- When using 4 wheels, a differential for the rear wheels and an Ackermann steering geometry for the steering wheels is required

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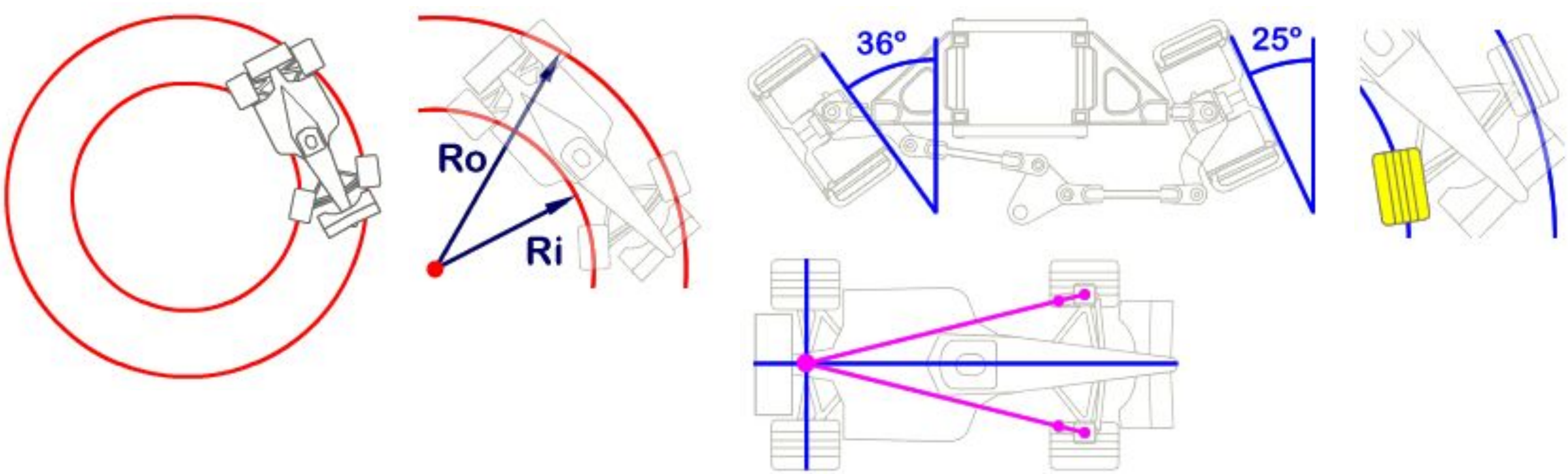
ACKERMANN STEERING PRINCIPLE

Steering geometry principle designed to allow steering wheels to go around circles of different radii and to avoid wheel slip.



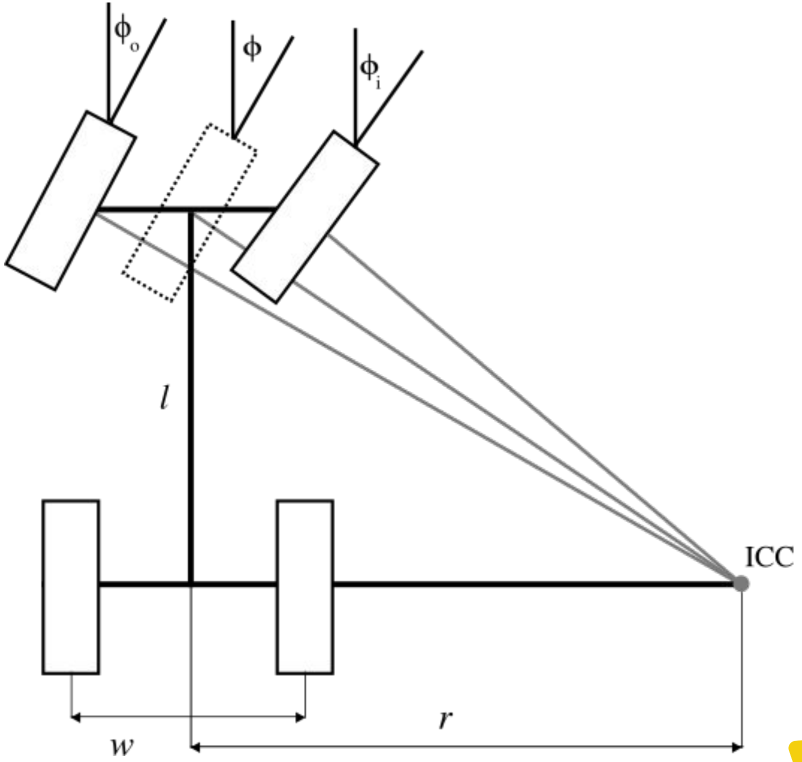
Source: http://www.rc-auto.ru/articles_tuning/id/445/

ACKERMANN STEERING PRINCIPLE

$$\tan(\phi) = \frac{l}{r}$$

$$\tan(\phi_i) = \frac{l}{r - \frac{w}{2}}$$

$$\tan(\phi_o) = \frac{l}{r + \frac{w}{2}}$$



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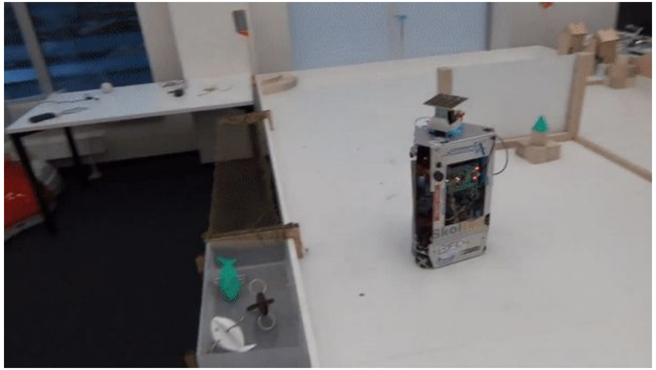
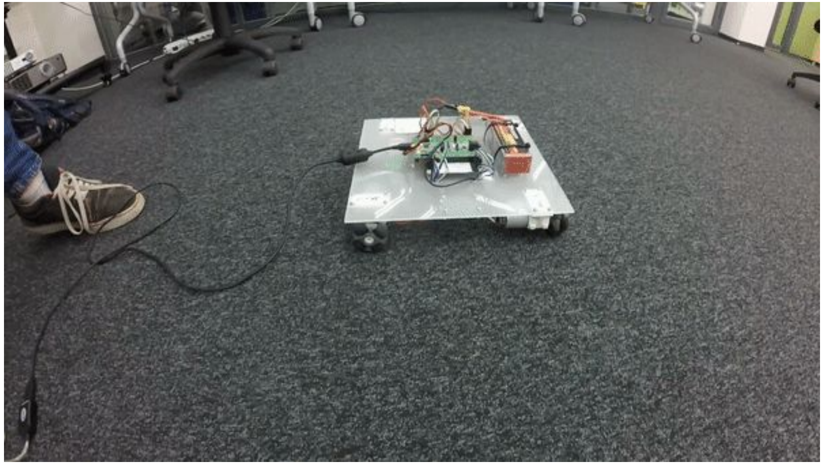
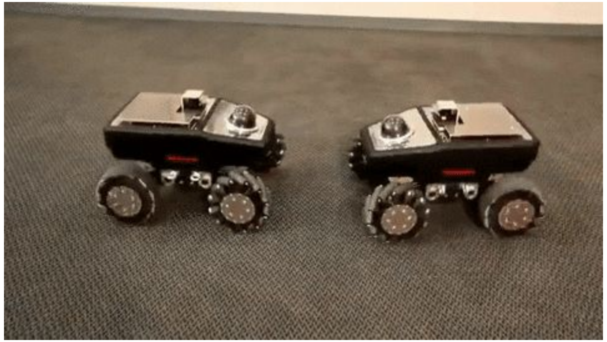
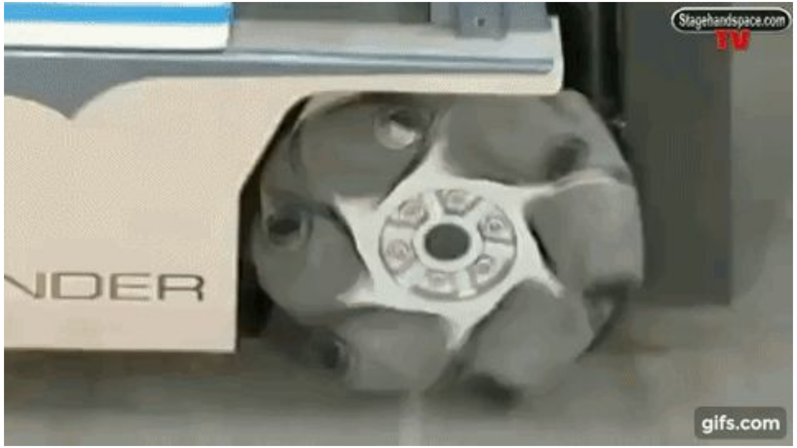
OMNIDIRECTIONAL WHEELS





MECANUM WHEELS (ILON WHEEL, SWEDISH WHEEL)

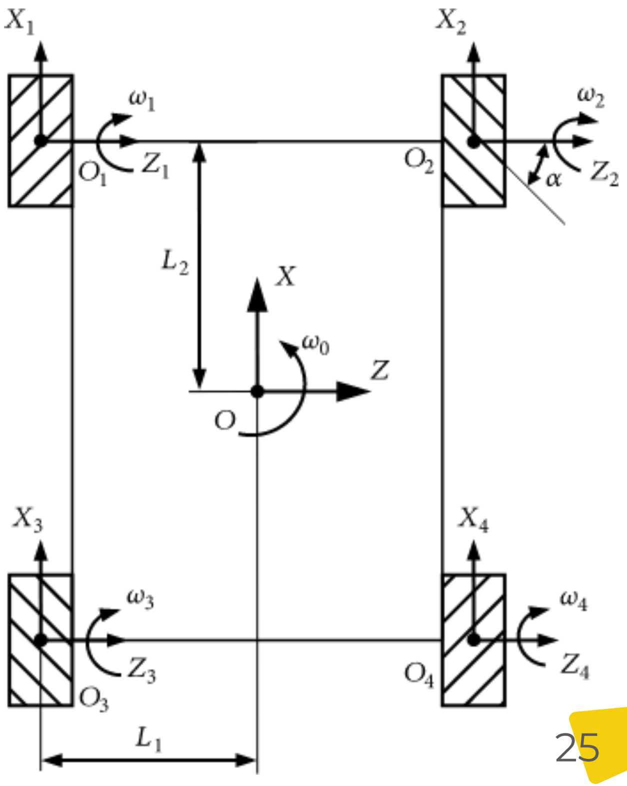




MECANUM WHEELS (ILON WHEEL, SWEDISH WHEEL)

$$\begin{bmatrix} v_x \\ v_z \\ \omega_0 \end{bmatrix} = \frac{r}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -\frac{1}{L_1 + L_2} & \frac{1}{L_1 + L_2} & -\frac{1}{L_1 + L_2} & \frac{1}{L_1 + L_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

Motion type	ω_1	ω_2	ω_3	ω_4
Straight	ω	ω	ω	ω
Perpendicular	ω	$-\omega$	$-\omega$	ω
45º motion	0	ω	ω	0
In place rotation	ω	$-\omega$	ω	$-\omega$



Li, Yunwang, et al. "Modeling and kinematics simulation of a Mecanum wheel platform in RecurDyn." *Journal of Robotics* 2018 (2018).

Probabilistic motion models

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02

RECURSIVE BAYESIAN POSE ESTIMATION

$$p(\mathbf{x_t}|map, \mathbf{z_t}, \mathbf{u_t}) = C \cdot p(\mathbf{z_t}|\mathbf{x_t}, map) \int_S p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}})p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}})d\mathbf{x_{t-1}}$$

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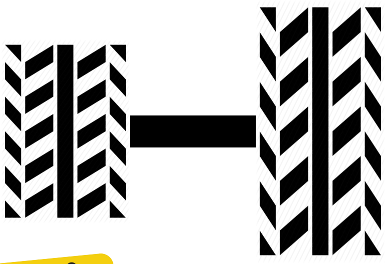
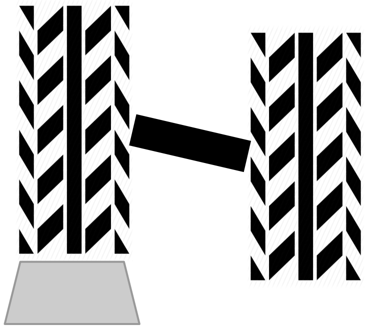
WHY DO WE NEED PROBABILISTIC MOTION MODELS?

- Actuators, like sensors, are not absolutely accurate.
- External factors also affect the precision of motion.

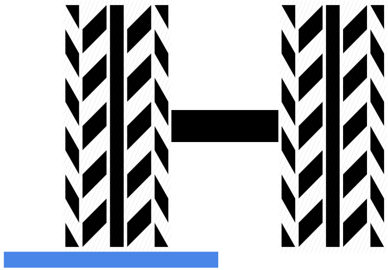
Difference in wheel diameters



Obstacles



Slippage on various surfaces



PROBABILISTIC MOTION MODELS

In practice, there are 2 types of motion models:

- ❑ **Odometry**-based
- ❑ **Speed** control based (dead reckoning)

Historically was used in ships navigation

- ❑ Odometry-based models are used when the robot is equipped with wheels encoders.
- ❑ Speed-based models are used when there are no encoders. They are based on calculating the traveled distance given the speed and travel time.

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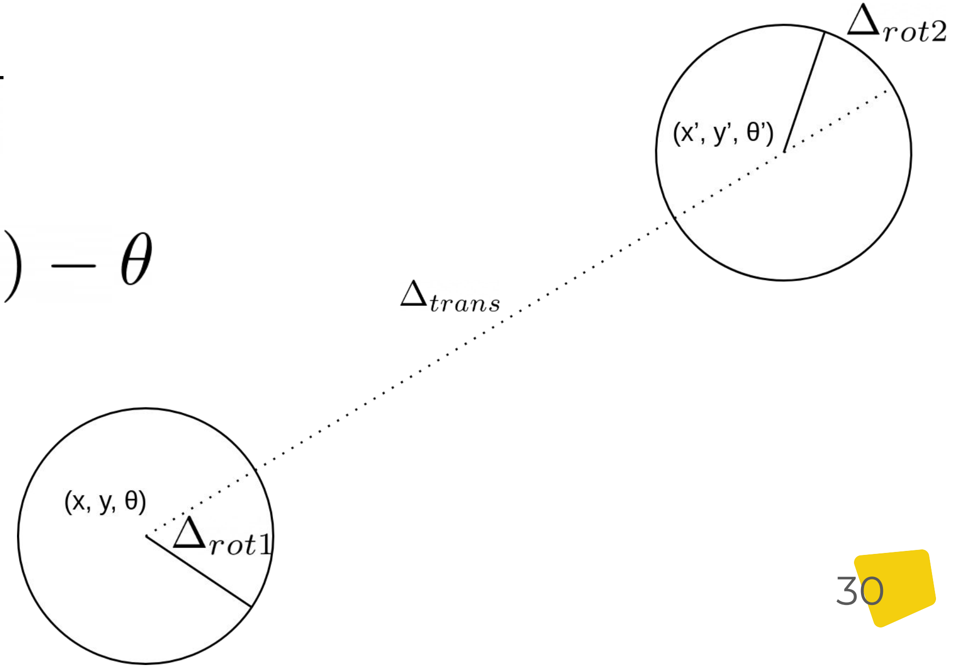
ODOMETRY-BASED MODEL

- The robot is moving from (x, y, θ) to (x', y', θ')
- Encoders provide the following information: $u_t = (\Delta_{trans}, \Delta_{rot1}, \Delta_{rot2})$

$$\Delta_{trans} = \sqrt{(x - x')^2 + (y - y')^2}$$

$$\Delta_{rot1} = atan2(y' - y, x' - x) - \theta$$

$$\Delta_{rot2} = \theta' - \theta - \Delta_{rot1}$$



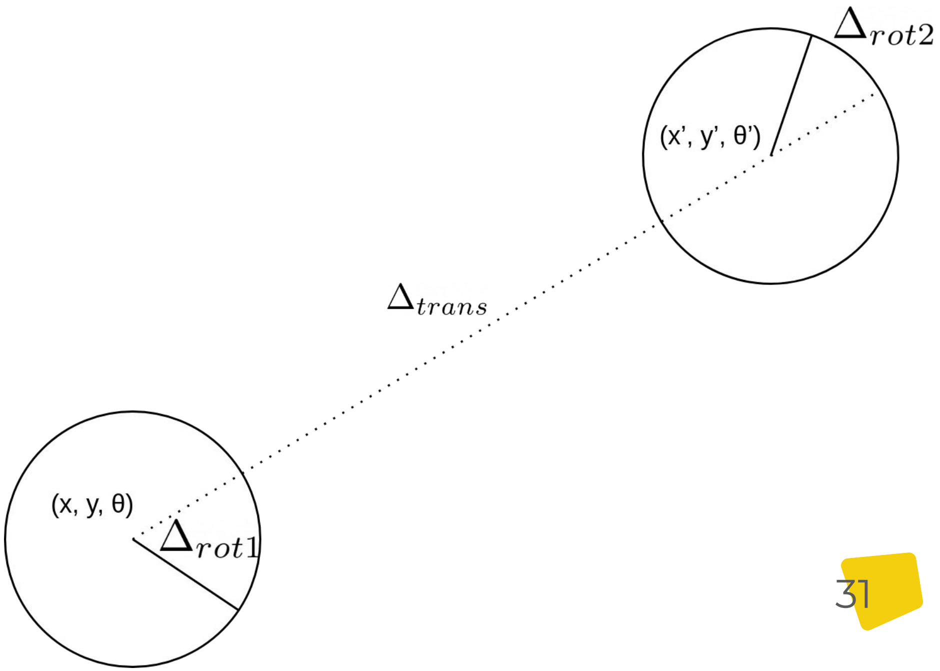
NOISE MODEL

Real motion is prone to error (noise):

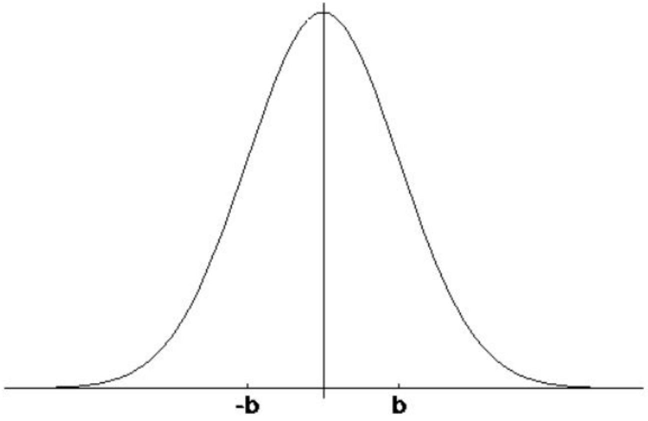
$$\hat{\Delta}_{trans} = \Delta_{trans} + \eta_1$$

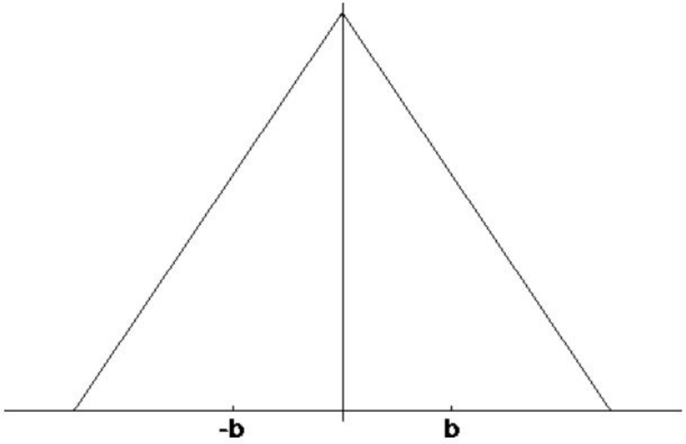
$$\hat{\Delta}_{rot1} = \Delta_{rot1} + \eta_2$$

$$\hat{\Delta}_{rot2} = \Delta_{rot2} + \eta_3$$



NOISE MODEL





$$\varepsilon_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

$$\varepsilon_{\sigma^2}(x) = \begin{cases} 0 \text{ if } |x| > \sqrt{6\sigma^2} \\ \dfrac{\sqrt{6\sigma^2} - |x|}{6\sigma^2} \end{cases}$$

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NOISE MODELING

1. Algorithm **prob_normal_distribution**(a,b):

2. return $\frac{1}{\sqrt{2\pi\, b^2}} \exp\left\{-\frac{1}{2}\frac{a^2}{b^2}\right\}$

1. Algorithm **prob_triangular_distribution**(a,b):

2. return $\max\left\{0, \frac{1}{\sqrt{6}\, b} - \frac{|a|}{6\, b^2}\right\}$

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SAMPLING FROM NOISE MODEL

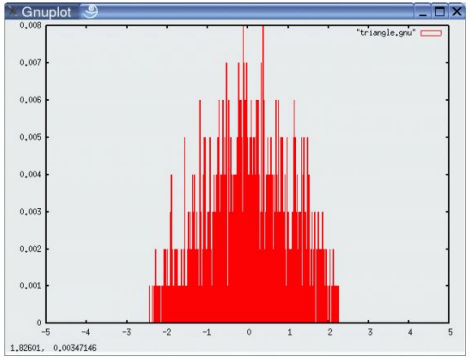
1. Algorithm **sample_normal_distribution**(b):

2. return $\frac{1}{2}\sum_{i=1}^{12} rand(-b, b)$

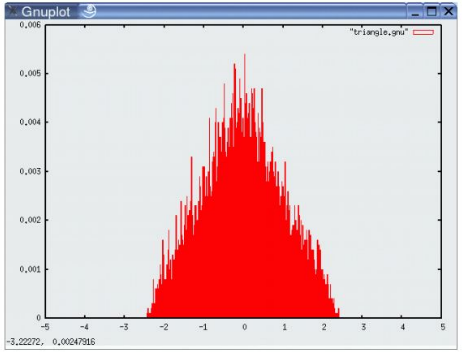
1. Algorithm **sample_triangular_distribution**(b):

2. return $\frac{\sqrt{6}}{2}\,[\mathrm{rand}(-b, b) + \mathrm{rand}(-b, b)]$

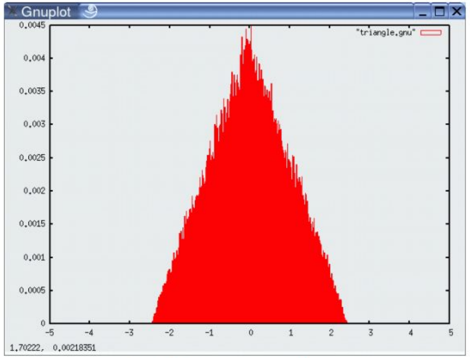
SAMPLING FROM NOISE MODEL



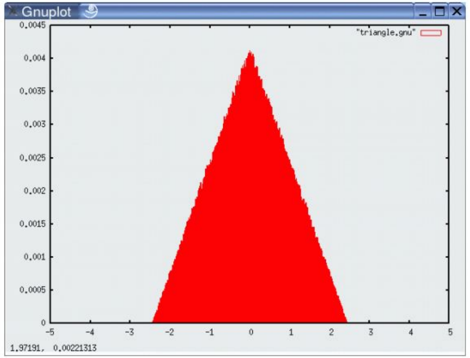
10^3 samples



10^4 samples



10^5 samples



10^6 samples

POSE POSTERIOR DISTRIBUTION ESTIMATION

hypotheses odometry

1. Algorithm **motion_model_odometry**$(\boxed{x, x'}, \boxed{\bar{x}, \bar{x}'})$
2. $\delta_{trans} = \sqrt{(\bar{x}'-\bar{x})^2 + (\bar{y}'-\bar{y})^2}$
3. $\delta_{rot1} = \text{atan2}(\bar{y}'-\bar{y}, \bar{x}'-\bar{x}) - \bar{\theta}$
4. $\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$
5. $\hat{\delta}_{trans} = \sqrt{(x'-x)^2 + (y'-y)^2}$
6. $\hat{\delta}_{rot1} = \text{atan2}(y'-y, x'-x) - \theta$
7. $\hat{\delta}_{rot2} = \theta' - \theta - \hat{\delta}_{rot1}$
8. $p_1 = \text{prob}(\delta_{\text{rot1}} - \hat{\delta}_{\text{rot1}}, \alpha_1 |\delta_{\text{rot1}}| + \alpha_2 \delta_{\text{trans}})$
9. $p_2 = \text{prob}(\delta_{\text{trans}} - \hat{\delta}_{\text{trans}}, \alpha_3 \delta_{\text{trans}} + \alpha_4 (|\delta_{\text{rot1}}| + |\delta_{\text{rot2}}|))$
10. $p_3 = \text{prob}(\delta_{\text{rot2}} - \hat{\delta}_{\text{rot2}}, \alpha_1 |\delta_{\text{rot2}}| + \alpha_2 \delta_{\text{trans}})$
11. return $p_1 \cdot p_2 \cdot p_3$

odometry params (**u**)

values of interest (**x**,**x'**)

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SAMPLING FROM MOTION MODEL

1. Algorithm **sample_motion_model**(u, x):

$u = \langle \delta_{rot1}, \delta_{rot2}, \delta_{trans} \rangle, x = \langle x, y, \theta \rangle$

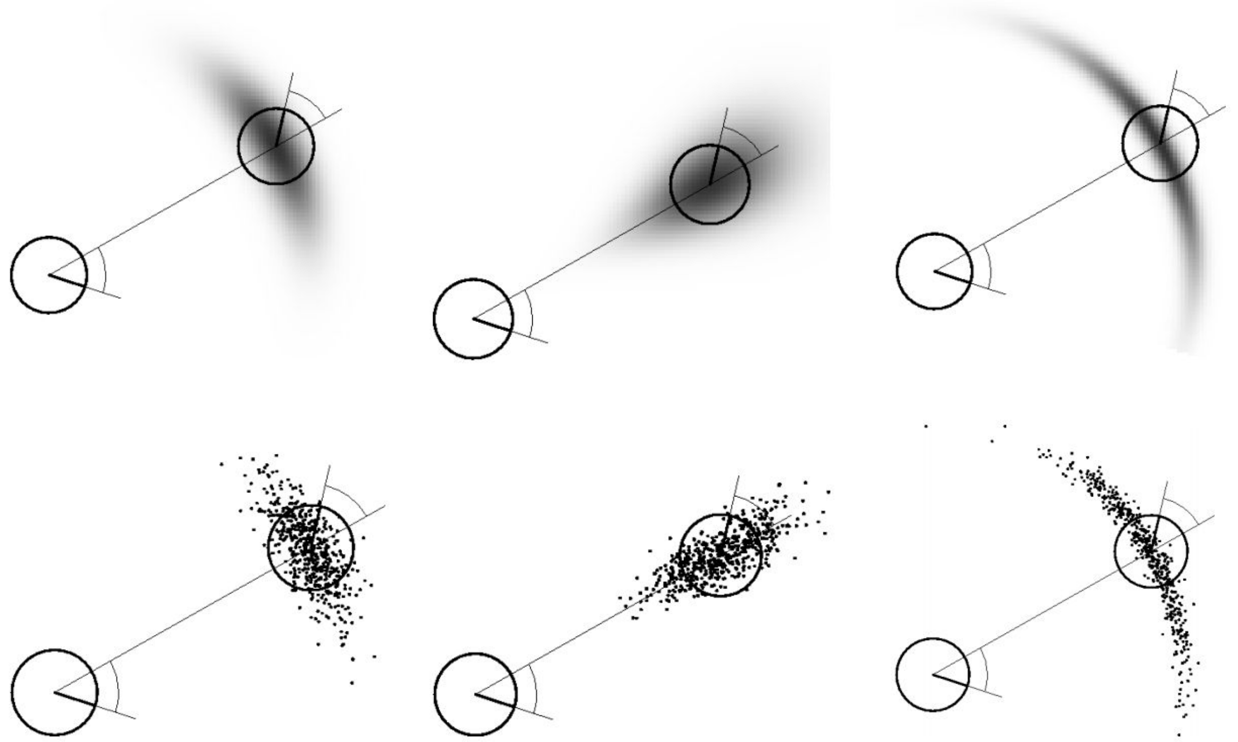
1. $\hat{\delta}_{rot1} = \delta_{rot1} + \text{sample}(\alpha_1 |\delta_{rot1}| + \alpha_2\, \delta_{trans})$
2. $\hat{\delta}_{trans} = \delta_{trans} + \text{sample}(\alpha_3\, \delta_{trans} + \alpha_4\, (|\delta_{rot1}| + |\delta_{rot2}|))$
3. $\hat{\delta}_{rot2} = \delta_{rot2} + \text{sample}(\alpha_1 |\delta_{rot2}| + \alpha_2\, \delta_{trans})$

4. $x' = x + \hat{\delta}_{trans} \cos(\theta + \hat{\delta}_{rot1})$
5. $y' = y + \hat{\delta}_{trans} \sin(\theta + \hat{\delta}_{rot1})$
6. $\theta' = \theta + \hat{\delta}_{rot1} + \hat{\delta}_{rot2}$

7. Return $\langle x', y', \theta' \rangle$

sample_normal_distribution

EXAMPLE OF ODOMETRY-BASED MOTION MODEL

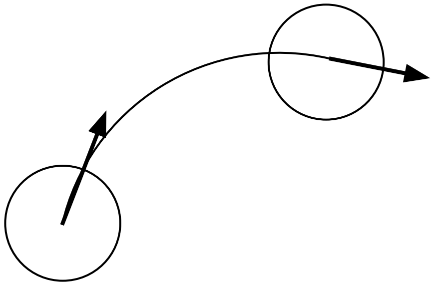


SPEED-BASED MOTION MODEL

- Such a model assumes that we control the parameters of the robot's motion — linear and angular velocity
- The robot moves along a circular arc
- Control signals (speeds) are subject to noise

$$\hat{v} = v + \varepsilon_{\alpha_1|v|+\alpha_2|\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3|v|+\alpha_4|\omega|}$$



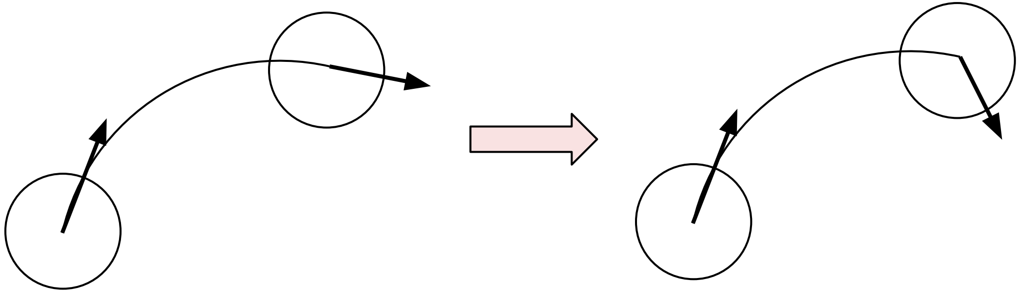
SPEED-BASED MOTION MODEL

- To allow the final turn, a third motion parameter is introduced

$$\hat{v} = v + \varepsilon_{\alpha_1|v|+\alpha_2|\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3|v|+\alpha_4|\omega|}$$

$$\hat{\gamma} = \varepsilon_{\alpha_5|v|+\alpha_6|\omega|}$$



SPEED-BASED MOTION MODEL

$$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}} \ sin(\theta + \hat{\omega}\Delta t)$$

$$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}} \ cos(\theta + \hat{\omega}\Delta t)$$

$$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$$

SAMPLING FROM SPEED-BASED MOTION MODEL

Algorithm sample_motion_model_velocity(u_t, x_{t-1})**:**

$\hat{v} = v + \textbf{sample}(\alpha_1 v^2 + \alpha_2 \omega^2)$

$\hat{\omega} = \omega + \textbf{sample}(\alpha_3 v^2 + \alpha_4 \omega^2)$

$\hat{\gamma} = \textbf{sample}(\alpha_5 v^2 + \alpha_6 \omega^2)$

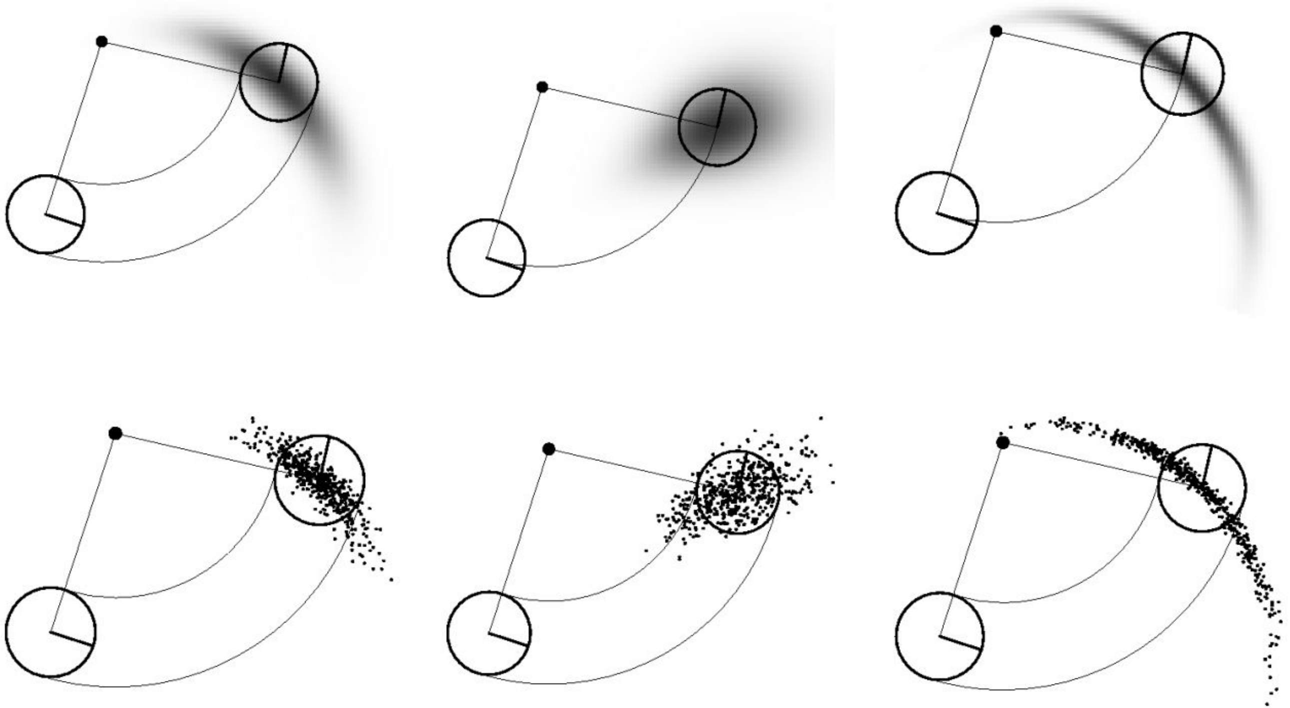
$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}}\ sin(\theta + \hat{\omega}\Delta t)$

$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}}\ cos(\theta + \hat{\omega}\Delta t)$

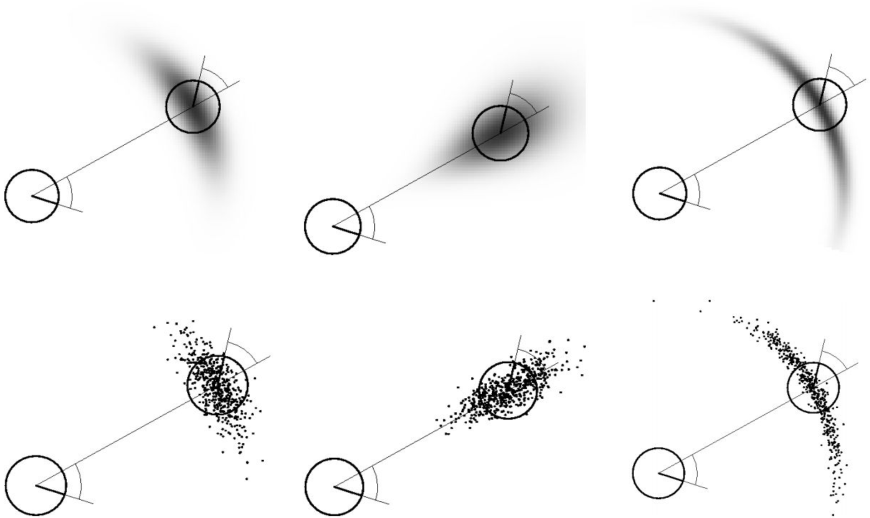
$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$

return $x_t = (x', y', \theta')^T$

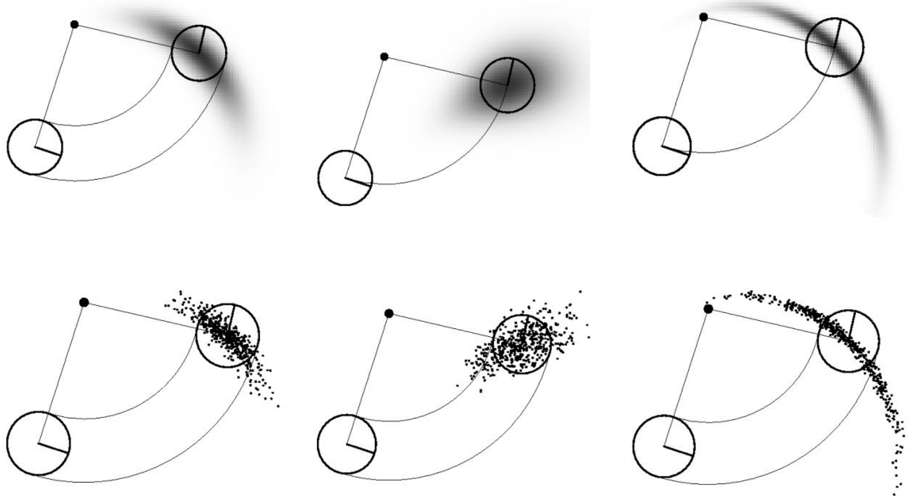
EXAMPLE OF SPEED-BASED MOTION MODEL



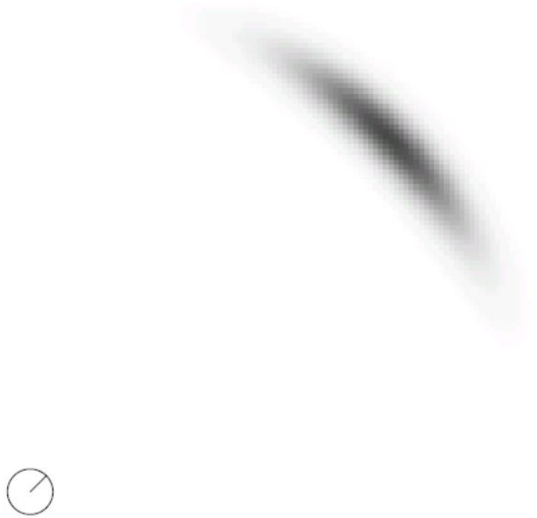
ODOMETRY-BASED MODEL

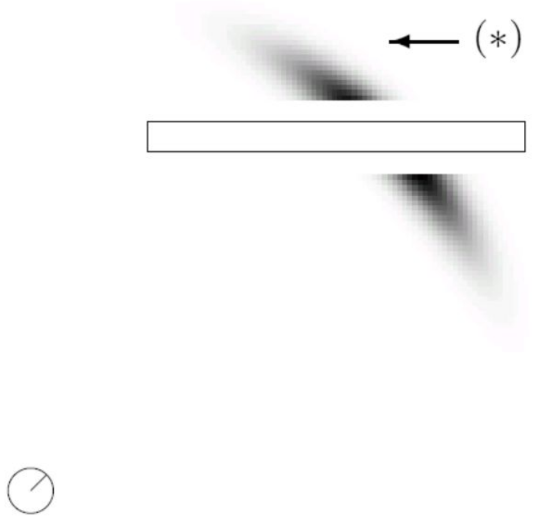


SPEED-BASED MODEL



MOTION MODELS ACCOUNTING FOR ENVIRONMENT





$p(x' \mid u, x) \quad \neq \quad p(x' \mid u, x, m)$

Approximation: $p(x' \mid u, x, m) = \eta \; p(x' \mid m) \; p(x' \mid u, x)$

ADDITIONAL RESOURCES

1. Differential Drive Kinematics
2. Probabilistic Robotics (in Notion). Chapter 5.
3. Mobility: wheels and whegs

Thanks for attention!

Questions? Additions? Welcome!