Introduction to Mobile Robotics and Robot Operating System (ROS)

Lecture 3. Kinematics models of wheeled robots. Probabilistic motion models
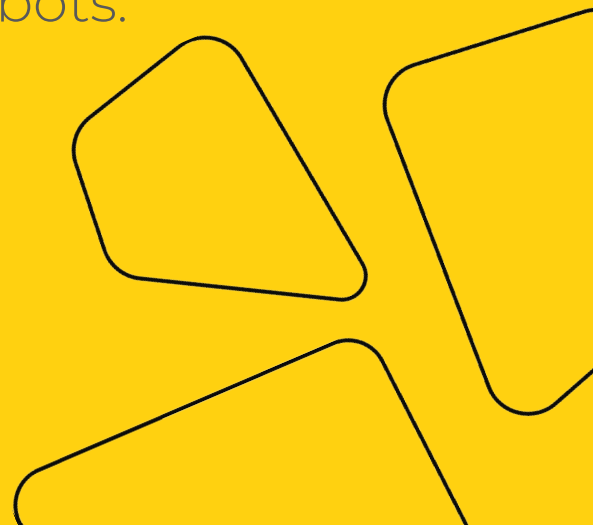
Oleg Shipitko,

March 2021

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Outline

1. Kinematic models of wheeled robots
   a. Differential drive
   b. Tricycle
   c. Ackermann principle
   d. Omni- and mecanum-wheels
2. Probabilistic motion models
   a. Odometry-based model
   b. Speed control based model

RECURSIVE BAYESIAN POSE ESTIMATION

$$p(\mathbf{x_t}|map, \mathbf{z_t}, \mathbf{u_t}) = C \cdot p(\mathbf{z_t}|\mathbf{x_t}, map) \int_S p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}}) p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}}) d\mathbf{x_{t-1}}$$

C — normalization coefficient

S — the probabilistic space of robot poses

$p(\mathbf{z_t}|\mathbf{x_t}, map)$ — observation (measurement) model

$p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}})$ — motion model

$p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}})$ — previous system state (robot pose)

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KALMAN FILTER

Prediction:

$$\hat{\mathbf{x}}_t = \mathbf{F}_t \hat{\mathbf{x}}_{t-1} + \mathbf{B}_k \vec{\mathbf{u}}_t$$

$$\hat{\mathbf{\Sigma}}_t = \mathbf{F}_t \mathbf{\Sigma}_{t-1} \mathbf{F}_t^T + \mathbf{Q}_t$$
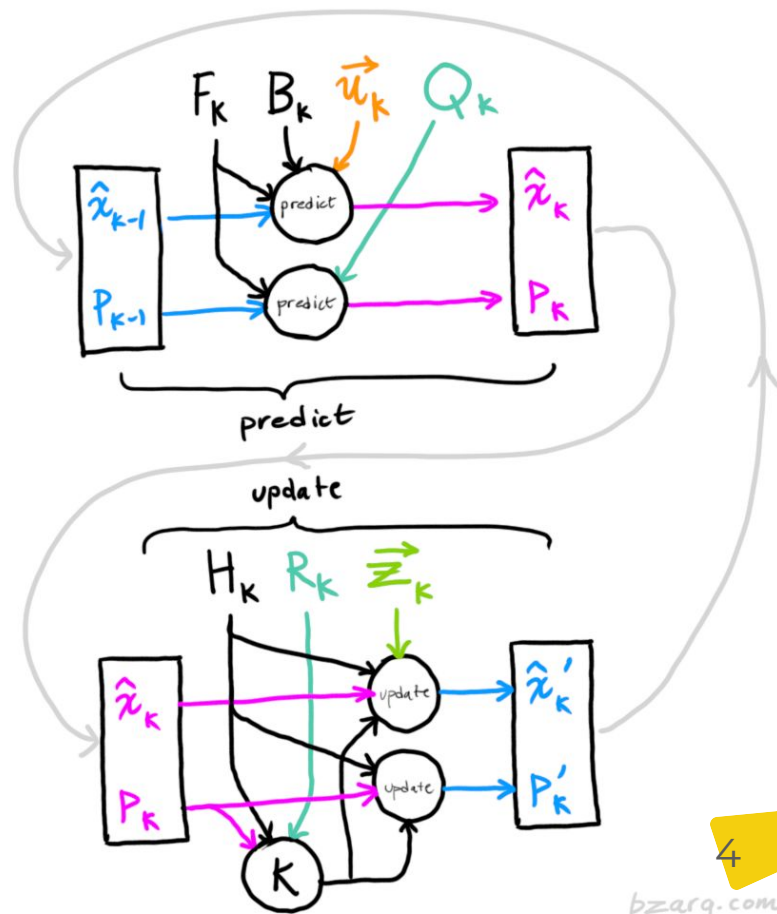
Correction:

$$\mathbf{K}' = \hat{\mathbf{\Sigma}}_t \mathbf{H}_t^T (\mathbf{H}_t \hat{\mathbf{\Sigma}}_t \mathbf{H}_t^T + \mathbf{R}_t)^{-1}$$
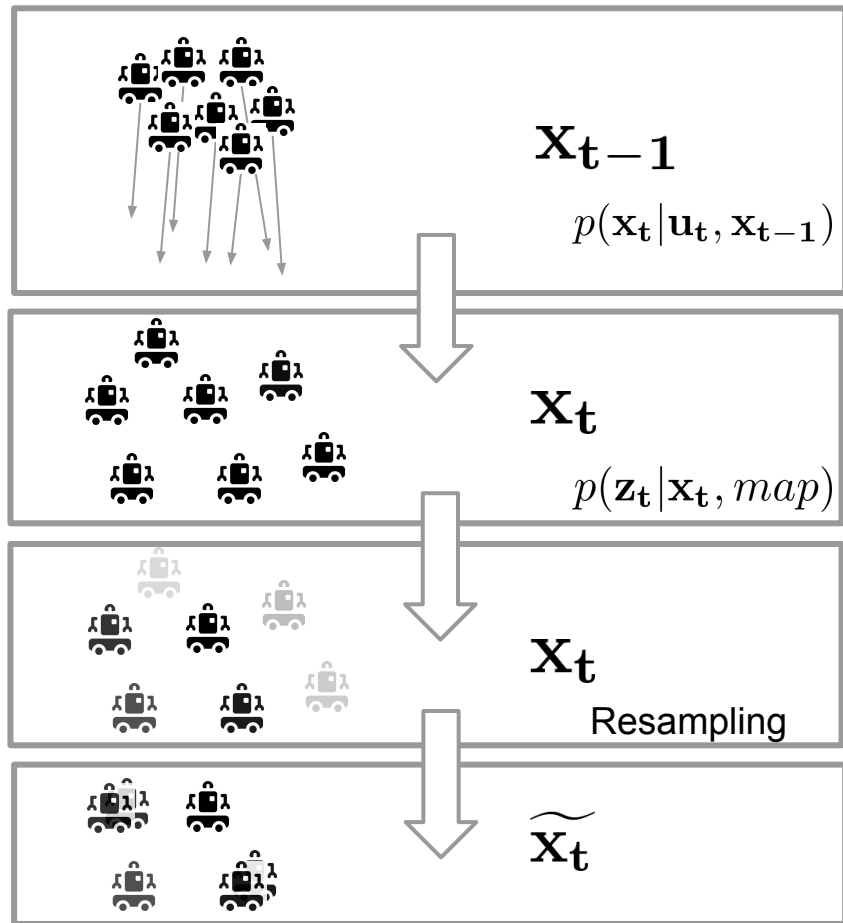
$$\mathbf{x}'_t = \mathbf{H}_t \hat{\mathbf{x}}_t + \mathbf{K}'(\vec{\mathbf{z}}_t - \mathbf{H}_t \hat{\mathbf{x}}_t)$$

$$\mathbf{\Sigma}'_t = \hat{\mathbf{\Sigma}}_t - \mathbf{K}' \mathbf{H}_t \hat{\mathbf{\Sigma}}_t$$

Kalman Filter Information Flow

bzarg.com

PARTICLE FILTER

$\mathbf{x_{t-1}}$

$p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}})$

$\mathbf{x_t}$

$p(\mathbf{z_t}|\mathbf{x_t}, map)$

$\mathbf{x_t}$

Resampling

$\widetilde{\mathbf{x_t}}$

Algorithm 1 Generic Monte-Carlo localization algorithm

1: procedure MCL($\mathbf{x_{t-1}}, m, \mathbf{u_t}, \mathbf{z_t}$)
2:    $\{\mathbf{x_t^n}\} = \{\widetilde{\mathbf{x_t^n}}\} = \emptyset$
3:    for $n = 1$ to N do
4:        sample $x_t^n \sim p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}^n})$ Motion model
5:        $w_t^n = p(\mathbf{z_t}|\mathbf{x_t^n}, map)$ Observation model
6:        $\{\widetilde{\mathbf{x_t^n}}\} = \{\widetilde{\mathbf{x_t^n}}\} + \langle x_t^n, w_t^n \rangle$
7:    end for
8:    for $n = 1$ to N do
9:        draw i with probability $\propto \widetilde{w_t^i}$ Resampling
10:       $\{\mathbf{x_t^n}\} = \{\mathbf{x_t^n}\} + \langle \widetilde{x_t^i}, \widetilde{w_t^i} \rangle$
11:    end for
12:    return $\{\mathbf{x_t^n}\}$
13: end procedure

RECURSIVE BAYESIAN POSE ESTIMATION

$$p(\mathbf{x_t}|map, \mathbf{z_t}, \mathbf{u_t}) = C \cdot p(\mathbf{z_t}|\mathbf{x_t}, map) \int_S p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}}) p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}}) d\mathbf{x_{t-1}}$$

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RECURSIVE BAYESIAN POSE ESTIMATION

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Kinematic models of wheeled robots

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01

(some) TYPES OF WHEELED KINEMATICS
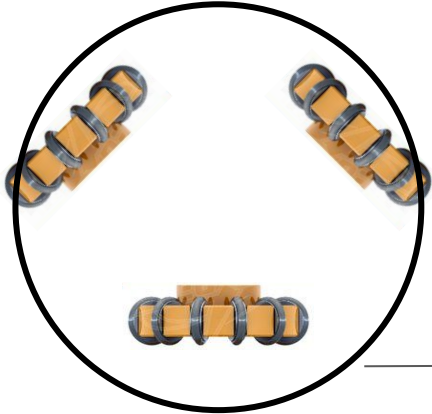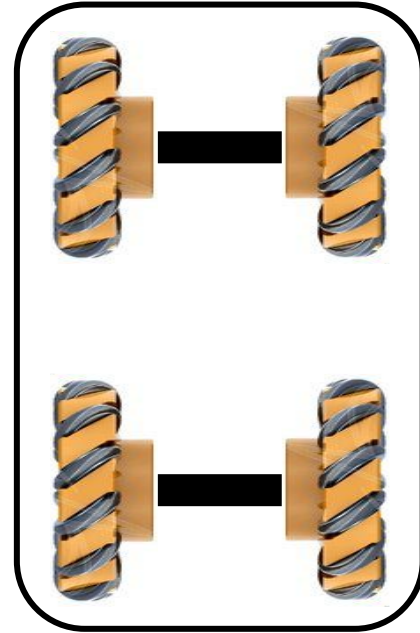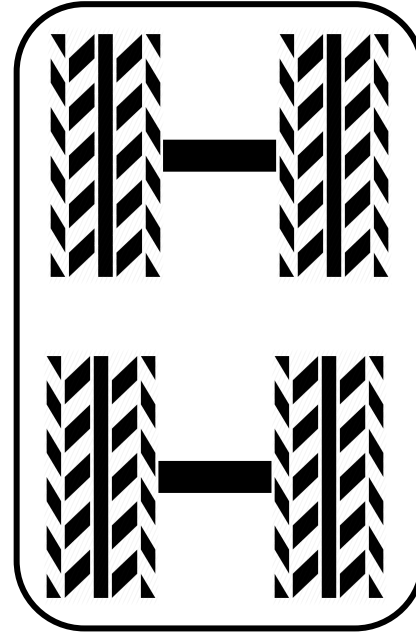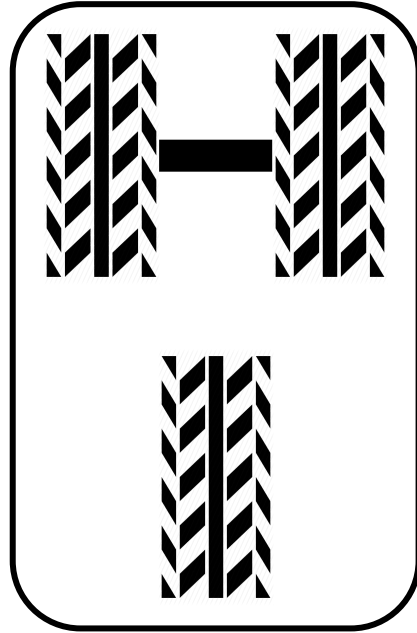
HOLONOMIC SYSTEMS

❏ A robot is called **holonomic** if the number of **controlled** degrees of freedom = the **total** number of degrees of freedom.

❏ A **nonholonomic system** is a mechanical system on which, in addition to geometric ones, kinematic constraints are also superimposed.

❏ Mathematically, nonholonomic constraints are expressed by non-integrable equations.
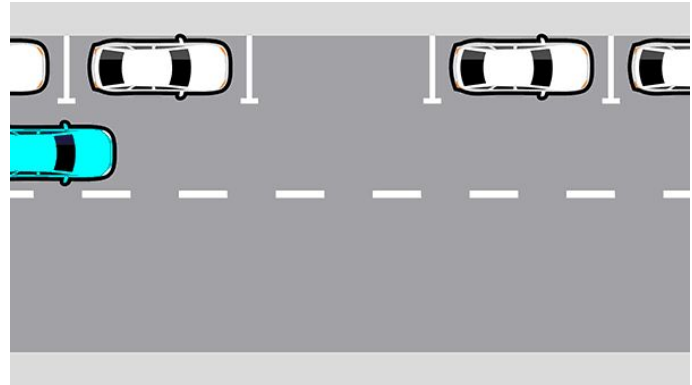
HOLONOMIC SYSTEMS

- **Holonomic constraints** limit the allowed state space (geometry).

- For instance, if there is a truck and a trailer, **not all angles** between them **are possible**. This is a **holonomic constraint**.
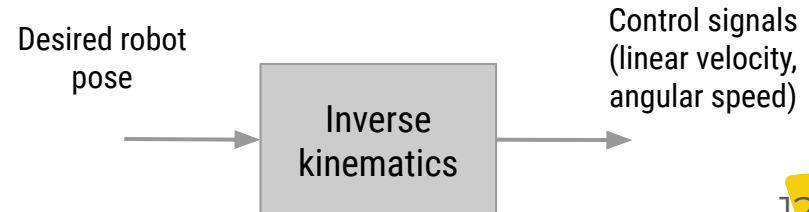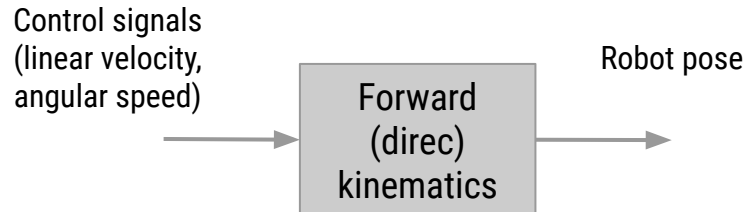
- **Nonholonomic constraints** limit the control space relative to the current state.

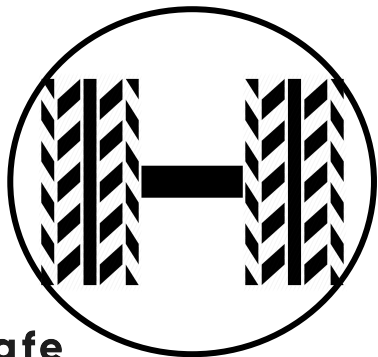- For instance, a car **can not move sideway**.

DIRECT AND INVERSE KINEMATICS PROBLEMS

❑ The **direct kinematics problem** — having control parameters (for example, wheel speeds) and motion time, find the position into which the robot has moved.

❑ The **inverse kinematics problem** is to find the control parameters that move the robot into a given position in a given time.

Control signals
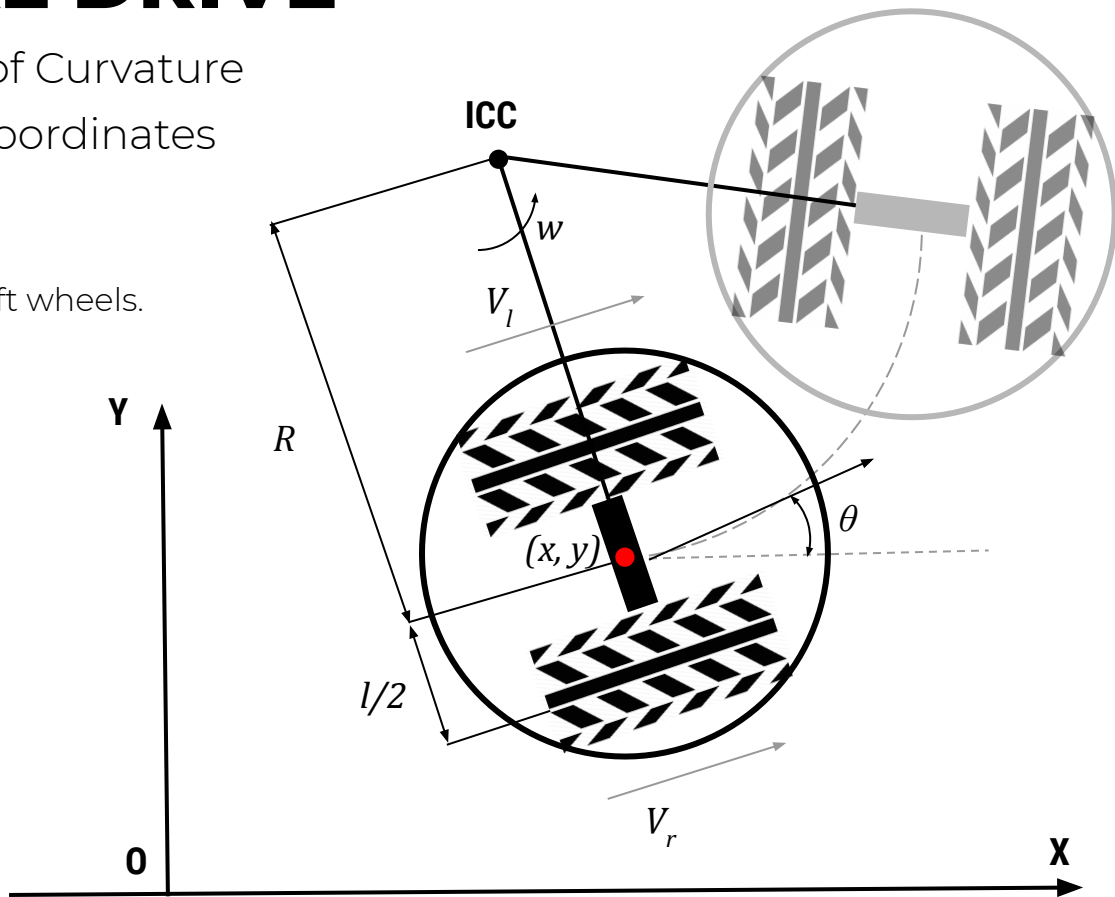(linear velocity,
angular speed)

Robot pose

Forward
(direc)
kinematics

Desired robot
pose

Control signals
(linear velocity,
angular speed)

Inverse
kinematics

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DIFFERENTIAL DRIVE

DIFFERENTIAL DRIVE

ICC − Instantaneous Center of Curvature

(x, y, θ) − wheel axle center coordinates

$$V_r$$
$$V_l$$

− the speed of the right and left wheels. Controlled parameters.

ICC

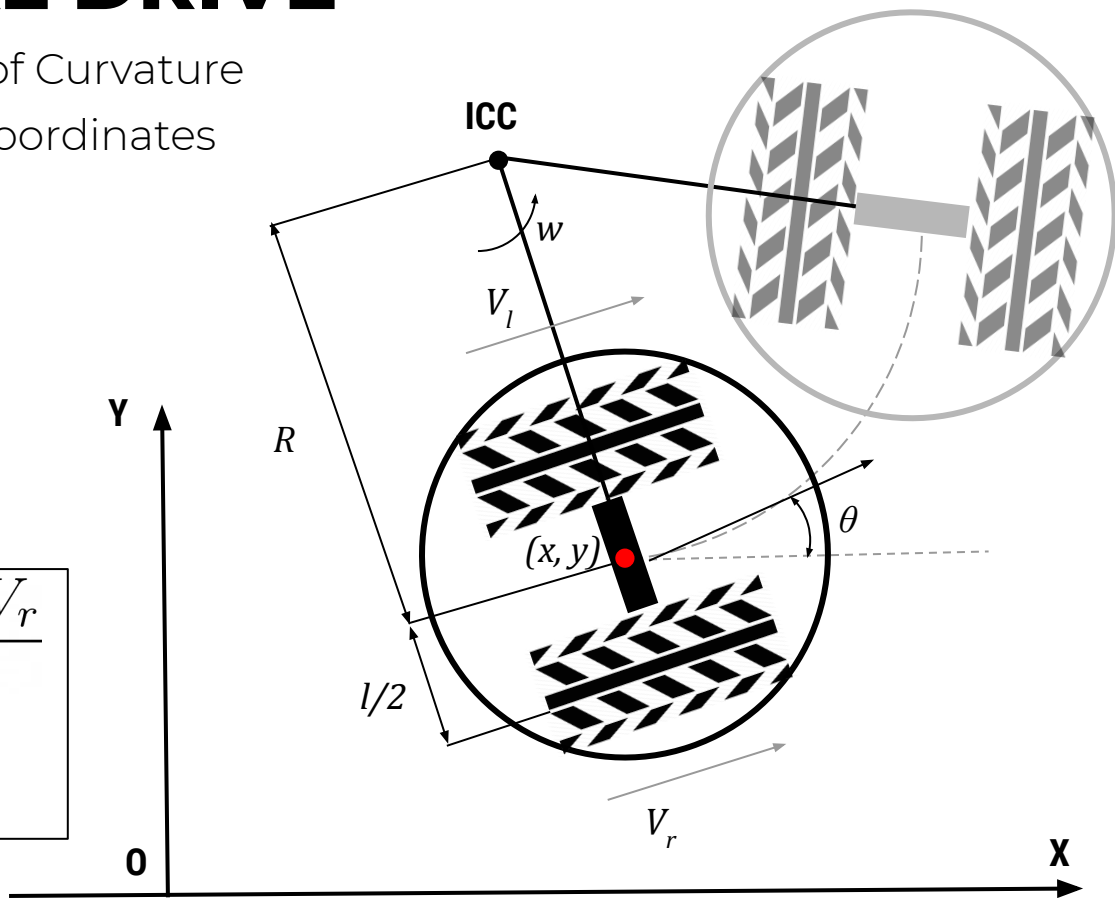w

V_l

R

Y

(x, y)

$θ$

$l/2$

V_r

0

X

DIFFERENTIAL DRIVE

ICC − Instantaneous Center of Curvature
(x, y, θ) − wheel axle center coordinates

$$w\left(R + \frac{l}{2}\right) = V_r$$

$$w\left(R - \frac{l}{2}\right) = V_l$$

$$w = \frac{V_r - V_l}{l} \quad V = \frac{V_l + V_r}{2}$$
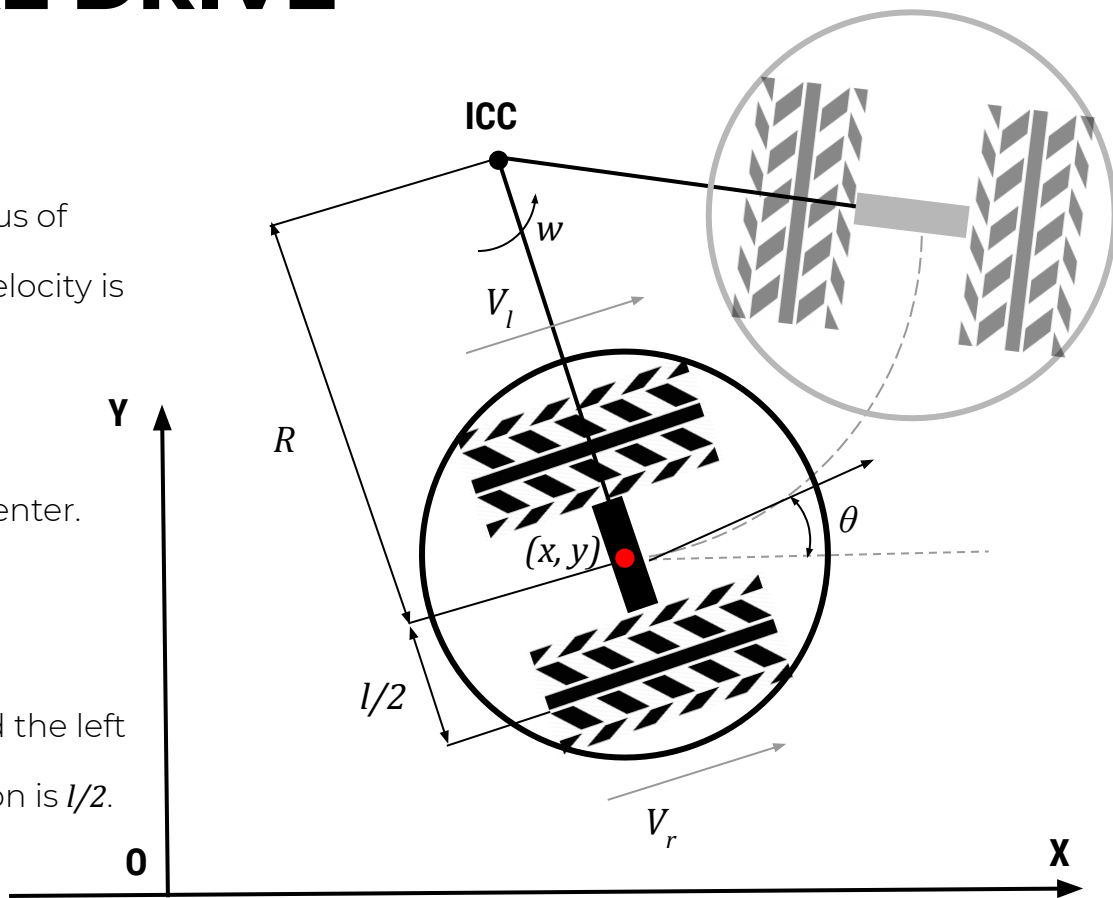
$$R = \frac{l}{2}\frac{V_r + V_l}{V_r - V_l}$$

DIFFERENTIAL DRIVE

3 types of motion:

❏ $V_l=V_r$ — linear motion. The radius of rotation is **infinity**. The angular velocity is **zero**.
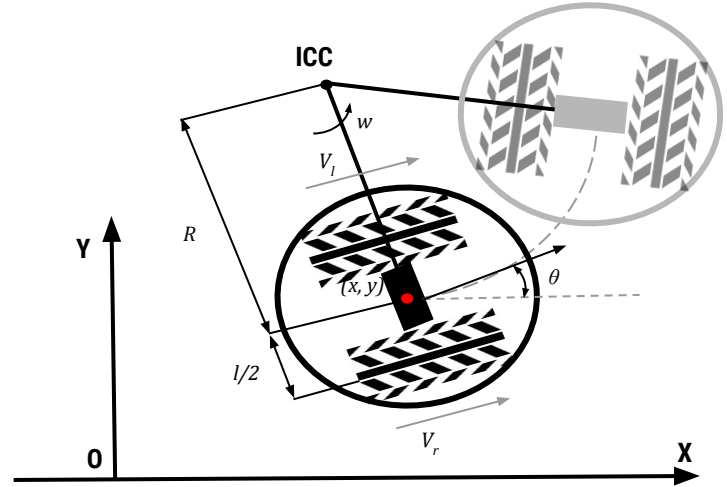
❏ $V_l= -V_r$ — rotation around the center. The radius of rotation is zero.

❏ $V_l= 0 \ (V_r= 0)$ — rotation around the left (right) wheel. The radius of rotation is $l/2$.

DIFFERENTIAL DRIVE FORWARD KINEMATICS

$$ICC = [x - R\sin(\theta), y + R\cos(\theta)]$$

At the time moment **t+δt** the robot pose is defined as:

$$
\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} =
\begin{bmatrix} cos(\omega\delta t) & -sin(\omega\delta t) & 0 \\ sin(\omega\delta t) & cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} x - ICC_x \\ y - ICC_y \\ \theta \end{bmatrix} +
\begin{bmatrix} ICC_x \\ ICC_y \\ \omega\delta t \end{bmatrix}
$$

DIFFERENTIAL DRIVE FORWARD KINEMATICS
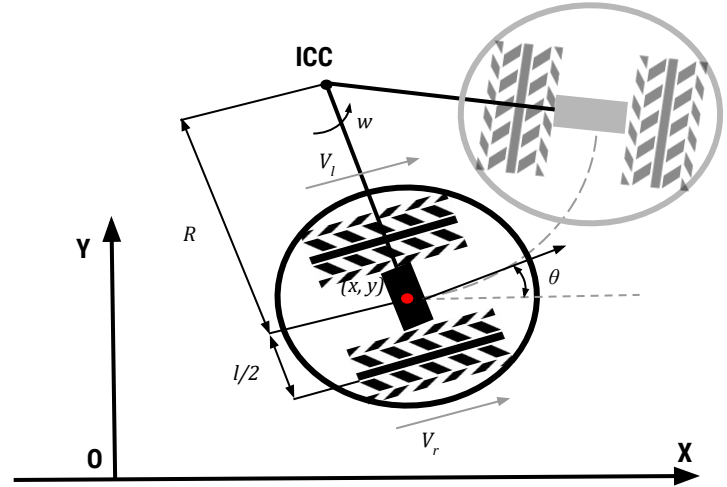
$$x(t) = \int_0^t V(t)cos[\theta(t)]dt$$

$$y(t) = \int_0^t V(t)sin[\theta(t)]dt$$

$$\Theta(t) = \int_0^t \omega(t)dt$$

$$x(t) = \frac{1}{2}\int_0^t [v_r(t) + v_l(t)]cos[\theta(t)]dt$$

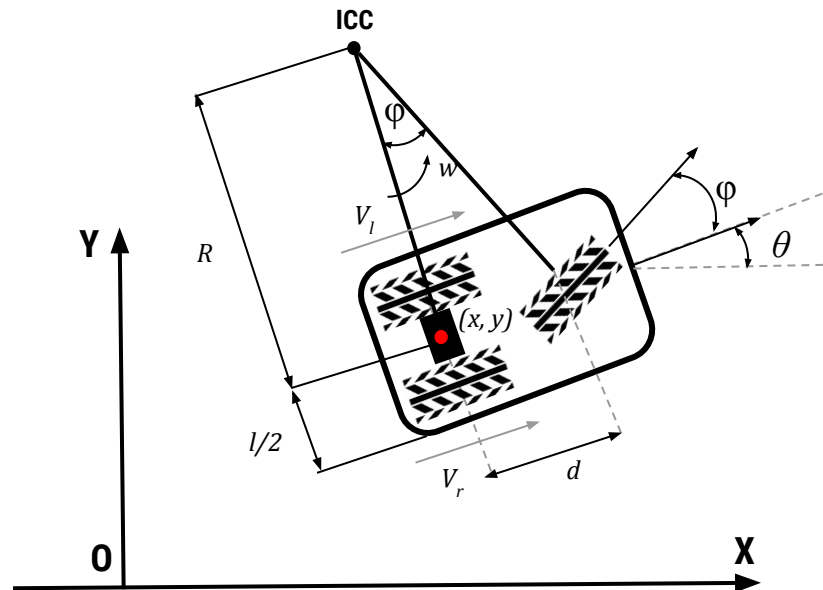$$y(t) = \frac{1}{2}\int_0^t [v_r(t) + v_l(t)]sin[\theta(t)]dt$$

$$\Theta(t) = \frac{1}{l}\int_0^t [v_r(t) - v_l(t)])dt$$

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TRICYCLE

$$ICC = [x - R\sin(\theta), y + R\cos(\theta)]$$

$$R = \frac{d}{\tan \varphi}$$

At the time moment **t+δt** the robot pose is defined as:

$$
\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} cos(\omega\delta t) & -sin(\omega\delta t) & 0 \\ sin(\omega\delta t) & cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - ICC_x \\ y - ICC_y \\ \theta \end{bmatrix} + \begin{bmatrix} ICC_x \\ ICC_y \\ \omega\delta t \end{bmatrix}
$$

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TRICYCLE

Features:

- ❏ Can not rotate in place

- ❏ When using 4 wheels, a differential for the rear wheels and an Ackermann steering geometry for the steering wheels are required

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ACKERMANN STEERING PRINCIPLE

Steering geometry principle designed to allow steering wheels to go around circles of different radii and to avoid wheel slip.
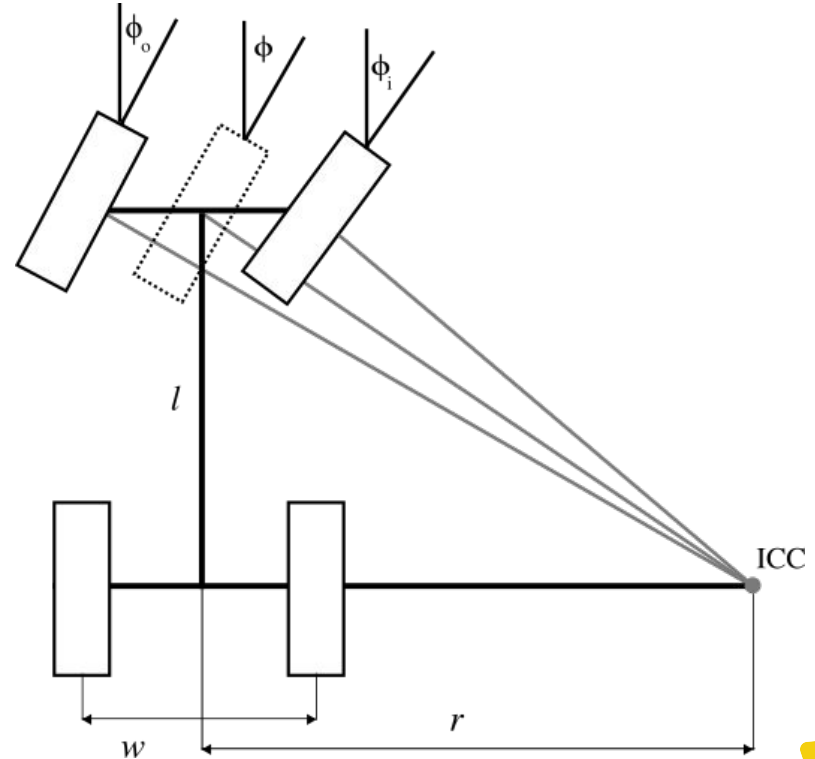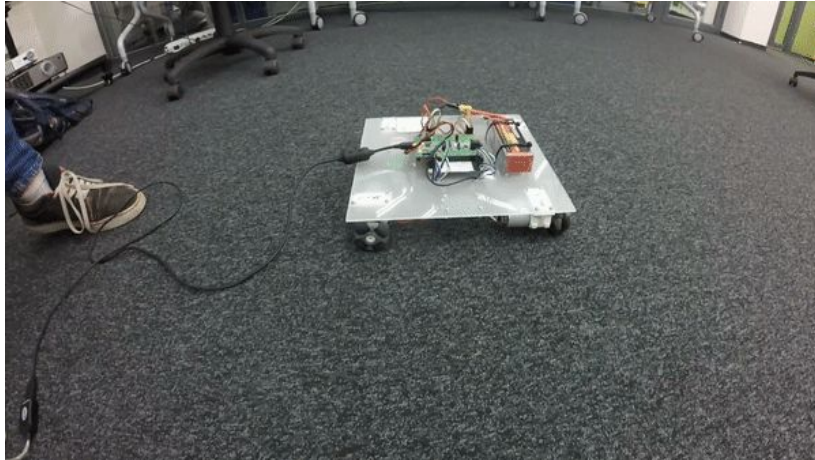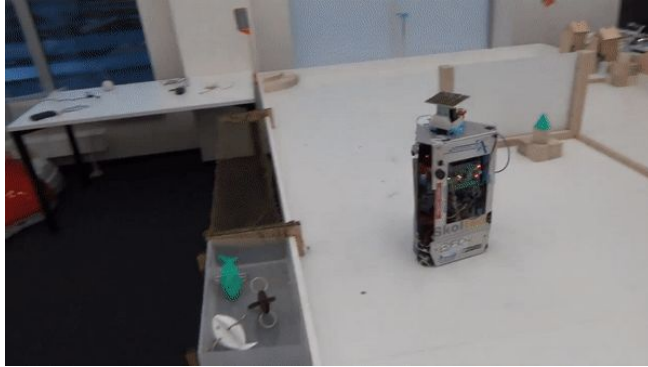
ACKERMANN STEERING PRINCIPLE

$$tan(\phi) = \frac{l}{r}$$

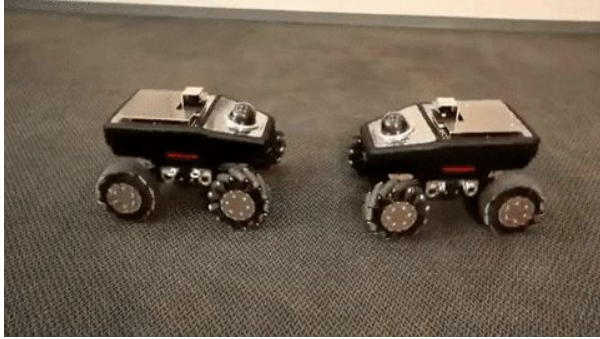$$tan(\phi_i) = \frac{l}{r - \frac{w}{2}}$$
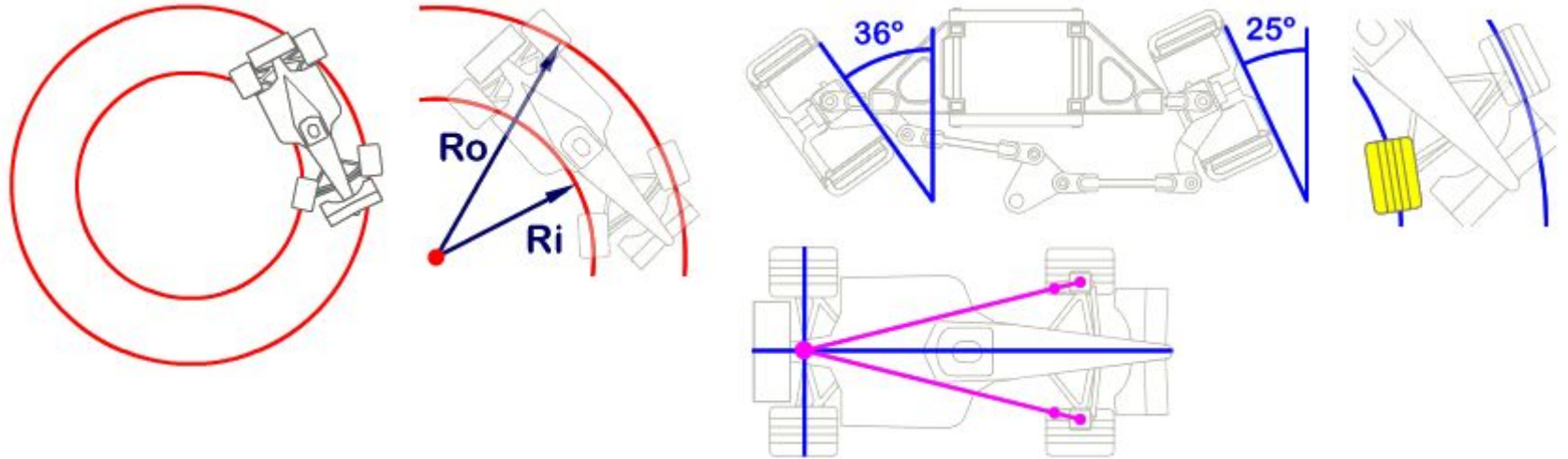
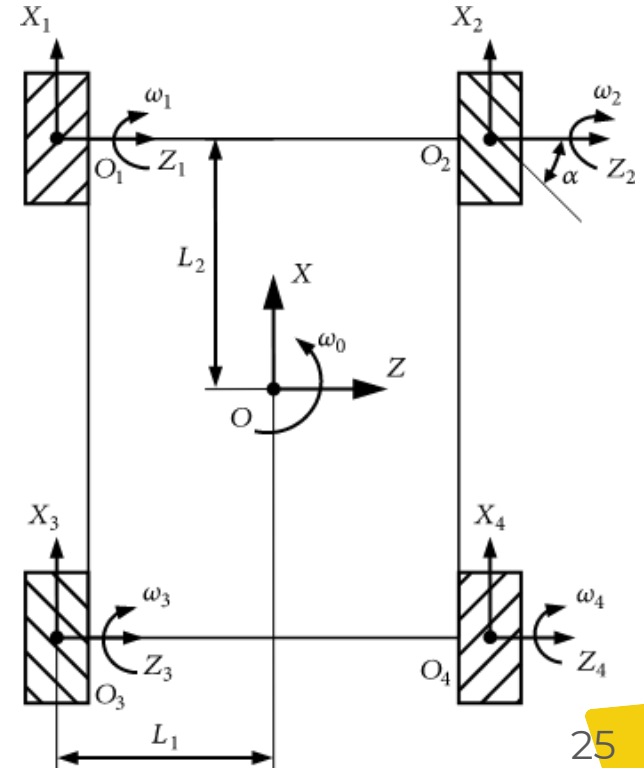$$tan(\phi_o) = \frac{l}{r + \frac{w}{2}}$$

OMNIDIRECTIONAL WHEELS

MECANUM WHEELS (ILON WHEEL, SWEDISH WHEEL)

MECANUM WHEELS (ILON WHEEL, SWEDISH WHEEL)

$$\begin{bmatrix} v_x \\ v_z \\ \omega_0 \end{bmatrix} = \frac{r}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -\dfrac{1}{L_1 + L_2} & \dfrac{1}{L_1 + L_2} & -\dfrac{1}{L_1 + L_2} & \dfrac{1}{L_1 + L_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

Motion type	ω_1	ω_2	ω_3	ω_4
Straight	ω	ω	ω	ω
Perpendicular	ω	$-\omega$	$-\omega$	ω
45º motion	0	ω	ω	0
In place rotation	ω	$-\omega$	ω	$-\omega$

Probabilistic motion models

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02

RECURSIVE BAYESIAN POSE ESTIMATION

$$p(\mathbf{x_t}|map, \mathbf{z_t}, \mathbf{u_t}) = C \cdot p(\mathbf{z_t}|\mathbf{x_t}, map) \int_S p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}}) p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}}) d\mathbf{x_{t-1}}$$

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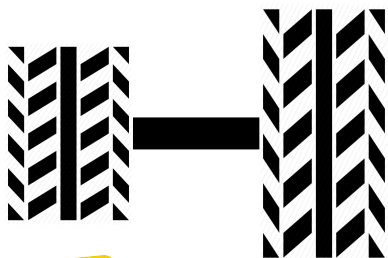
$p(\mathbf{x_t}|\mathbf{u_t}, \mathbf{x_{t-1}})$ — motion model

$p(\mathbf{x_{t-1}}|map, \mathbf{z_{t-1}}, \mathbf{u_{t-1}})$ — previous system state (robot pose)
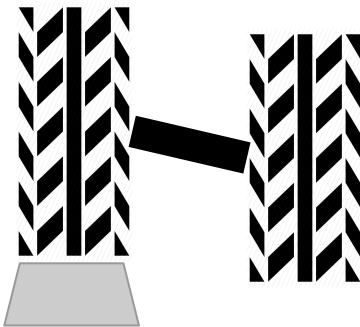
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WHY DO WE NEED **PROBABILISTIC** MOTION MODELS?

- ❏ Actuators, like sensors, are not absolutely accurate.
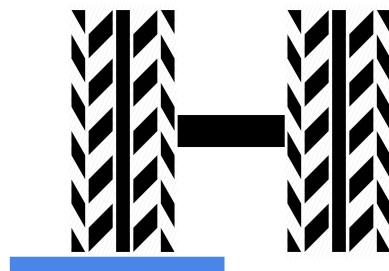- ❏ External factors also affect the precision of motion.
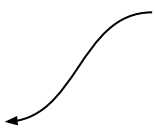
Difference in wheel diameters

Obstacles

Slippage on various surfaces

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PROBABILISTIC MOTION MODELS

In practice, there are 2 types of motion models:

❏ **Odometry**-based

❏ **Speed** control based (dead reckoning)

Historically was used
in ships navigation

❏ Odometry-based models are used when the robot is equipped with wheels encoders.

❏ Speed-based models are used when there are no encoders. They are based on calculating the traveled distance given the speed and travel time.
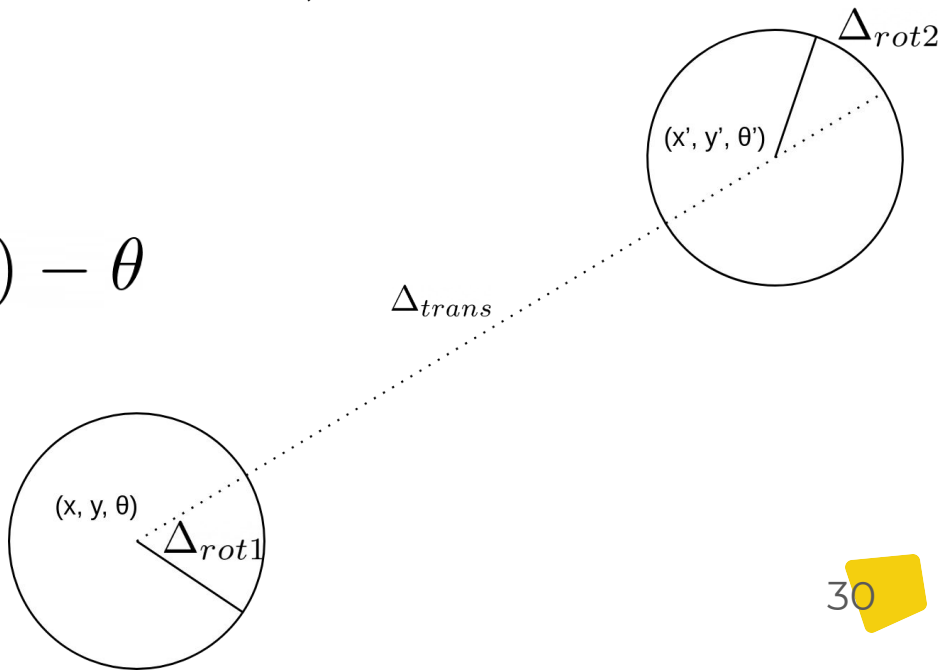
ODOMETRY-BASED MODEL

❏ The robot is moving from **(x, y, θ)** to **(x', y', θ')**

❏ Encoders provide the following information: $u_t = (\Delta_{trans,} \Delta_{rot1}, \Delta_{rot2})$

$$\Delta_{trans} = \sqrt{(x - x')^2 + (y - y')^2}$$

$$\Delta_{rot1} = atan2(y' - y, x' - x) - \theta$$

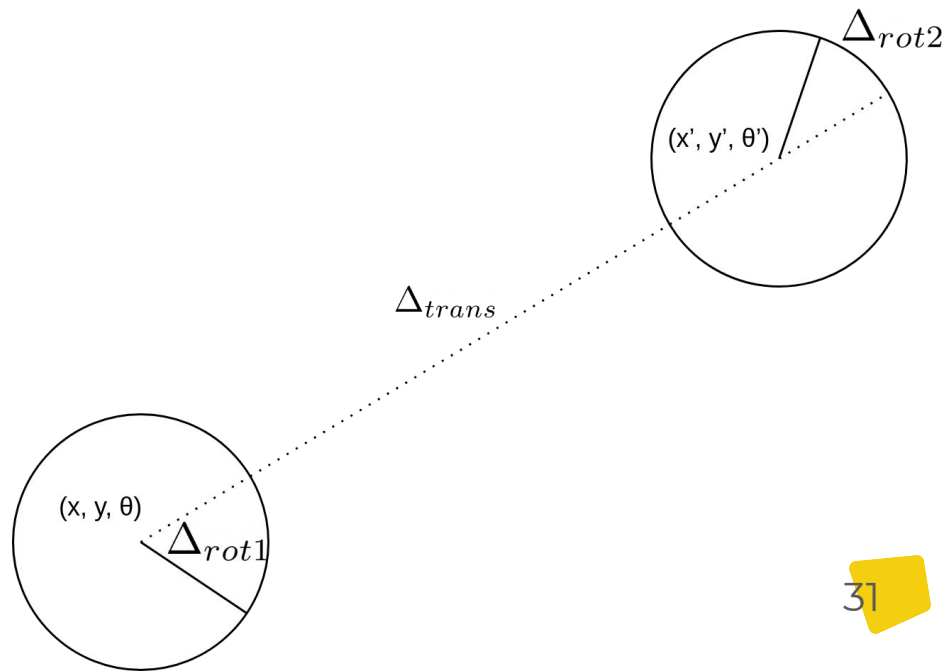$$\Delta_{rot2} = \theta' - \theta - \Delta_{rot1}$$

NOISE MODEL

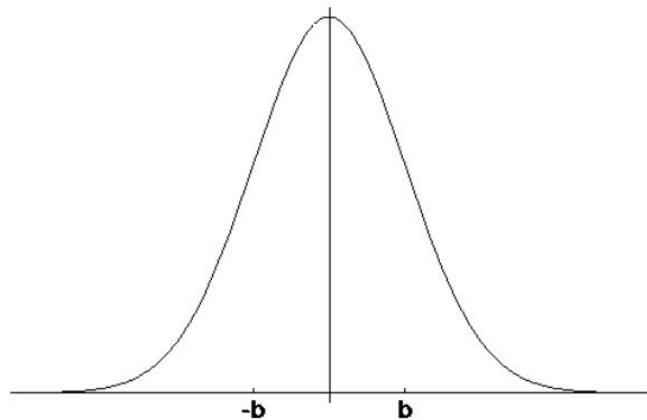Real motion is prone to error (noise):

$$\hat{\Delta}_{trans} = \Delta_{trans} + \eta_1$$

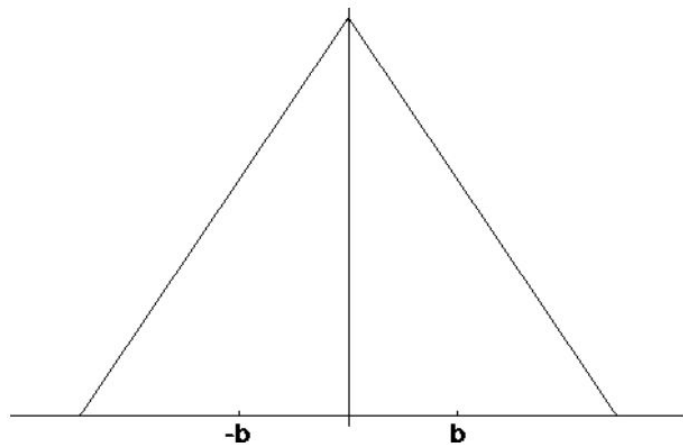$$\hat{\Delta}_{rot1} = \Delta_{rot1} + \eta_2$$

$$\hat{\Delta}_{rot2} = \Delta_{rot2} + \eta_3$$

Δ_{rot2}

(x', y', θ')

Δ_{trans}

(x, y, θ)

Δ_{rot1}

NOISE MODEL

$$\varepsilon_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

$$\varepsilon_{\sigma^2}(x) = \begin{cases} 0 \text{ if } |x| > \sqrt{6\sigma^2} \\ \dfrac{\sqrt{6\sigma^2} - |x|}{6\sigma^2} \end{cases}$$

NOISE MODELING

1. Algorithm **prob_normal_distribution**(a,b):

2. return $\dfrac{1}{\sqrt{2\pi\, b^2}} \exp\left\{ -\dfrac{1}{2}\dfrac{a^2}{b^2} \right\}$

1. Algorithm **prob_triangular_distribution**(a,b):

2. return $\max\left\{ 0, \dfrac{1}{\sqrt{6}\, b} - \dfrac{|a|}{6\, b^2} \right\}$

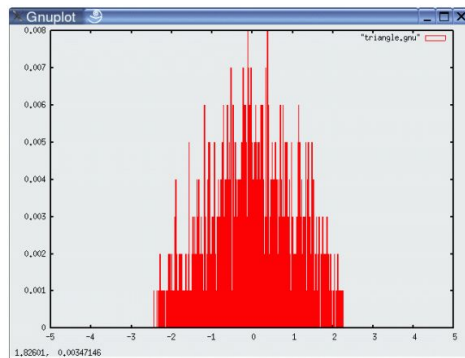girafe
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SAMPLING FROM NOISE MODEL

1.   Algorithm **sample_normal_distribution**(b):

2.   return $\dfrac{1}{2} \sum\limits_{i=1}^{12} rand(-b, b)$

1.   Algorithm **sample_triangular_distribution**(b):

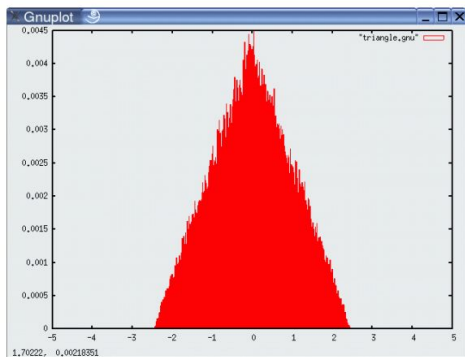2.   return $\dfrac{\sqrt{6}}{2} \left[\mathbf{rand}(-b, b) + \mathbf{rand}(-b, b) \right]$

SAMPLING FROM NOISE MODEL

10^3 samples
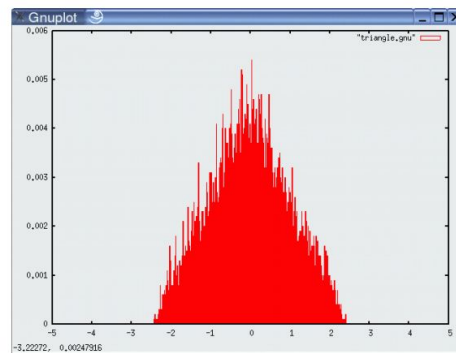
10^4 samples

10^5 samples

10^6 samples

35
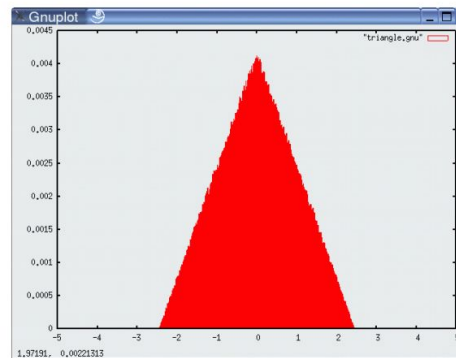
POSE POSTERIOR DISTRIBUTION ESTIMATION

hypotheses   odometry

1. Algorithm **motion_model_odometry**$(\boxed{x, x'}, \boxed{\bar{x}, \bar{x}'})$

2. $\delta_{trans} = \sqrt{(\bar{x}'-\bar{x})^2 + (\bar{y}'-\bar{y})^2}$

3. $\delta_{rot1} = \mathrm{atan2}(\bar{y}'-\bar{y}, \bar{x}'-\bar{x}) - \bar{\theta}$ $\longleftarrow$ odometry params (**u**)

4. $\delta_{rot2} = \bar{\theta}'-\bar{\theta} - \delta_{rot1}$

5. $\hat{\delta}_{trans} = \sqrt{(x'-x)^2 + (y'-y)^2}$

6. $\hat{\delta}_{rot1} = \mathrm{atan2}(y'-y, x'-x) - \dot{\theta}$ $\longleftarrow$ values of interest (**x,x'**)

7. $\hat{\delta}_{rot2} = \theta'-\theta - \hat{\delta}_{rot1}$

8. $p_1 = \mathrm{prob}(\delta_{\mathrm{rot1}} - \hat{\delta}_{\mathrm{rot1}}, \alpha_1 \,|\, \delta_{\mathrm{rot1}} \,|\, + \alpha_2 \delta_{\mathrm{trans}})$

9. $p_2 = \mathrm{prob}(\delta_{\mathrm{trans}} - \hat{\delta}_{\mathrm{trans}}, \alpha_3 \delta_{\mathrm{trans}} + \alpha_4(|\,\delta_{\mathrm{rot1}}\,| + |\,\delta_{\mathrm{rot2}}\,|))$

10. $p_3 = \mathrm{prob}(\delta_{\mathrm{rot2}} - \hat{\delta}_{\mathrm{rot2}}, \alpha_1 \,|\, \delta_{\mathrm{rot2}} \,|\, + \alpha_2 \delta_{\mathrm{trans}})$

11. return $p_1 \cdot p_2 \cdot p_3$

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SAMPLING FROM MOTION MODEL

1. Algorithm **sample_motion_model**(u, x):

$$u = \langle \delta_{rot1}, \delta_{rot2}, \delta_{trans} \rangle, x = \langle x, y, \theta \rangle$$

1. $\hat{\delta}_{rot1} = \delta_{rot1} + \text{sample}(\alpha_1 \mid \delta_{rot1} \mid + \alpha_2 \ \delta_{trans})$

2. $\hat{\delta}_{trans} = \delta_{trans} + \text{sample}(\alpha_3 \ \delta_{trans} + \alpha_4 \ (\mid \delta_{rot1} \mid + \mid \delta_{rot2} \mid))$

3. $\hat{\delta}_{rot2} = \delta_{rot2} + \text{sample}(\alpha_1 \mid \delta_{rot2} \mid + \alpha_2 \ \delta_{trans})$

4. $x' = x + \hat{\delta}_{trans} \cos(\theta + \hat{\delta}_{rot1})$
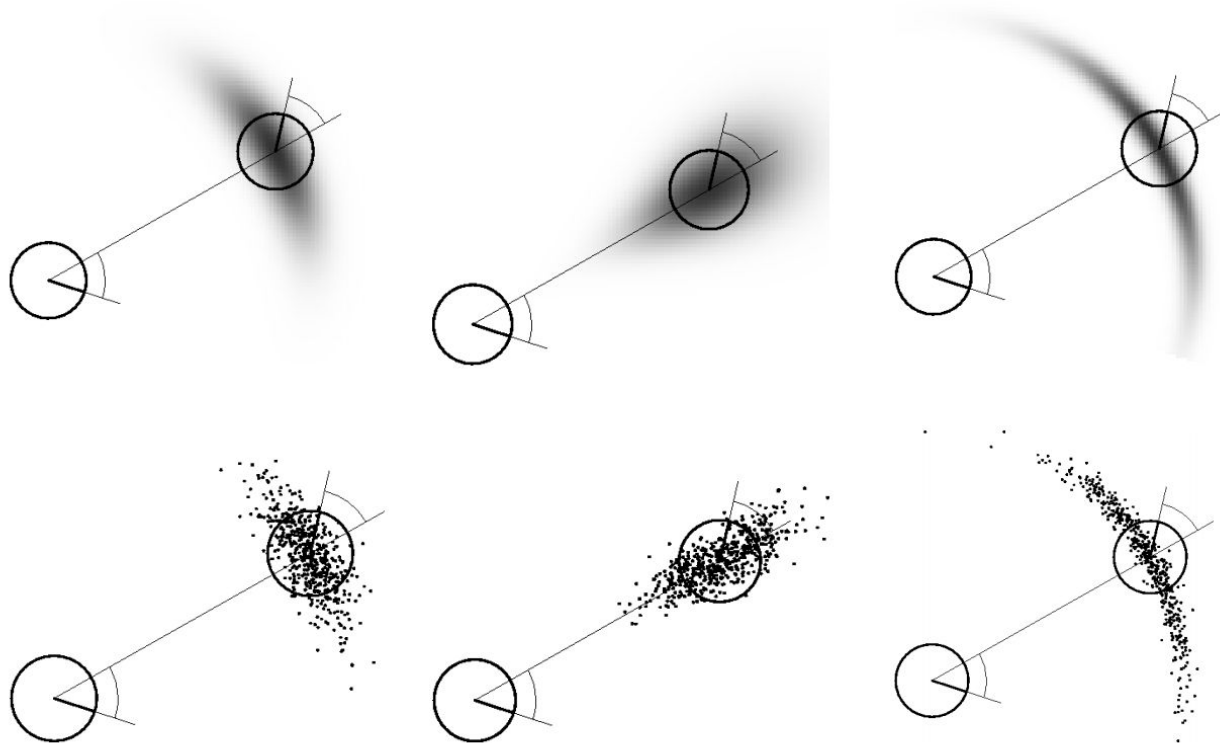
5. $y' = y + \hat{\delta}_{trans} \sin(\theta + \hat{\delta}_{rot1})$

6. $\theta' = \theta + \hat{\delta}_{rot1} + \hat{\delta}_{rot2}$

sample_normal_distribution

7. Return $\langle x', y', \theta' \rangle$

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EXAMPLE OF ODOMETRY-BASED MOTION MODEL

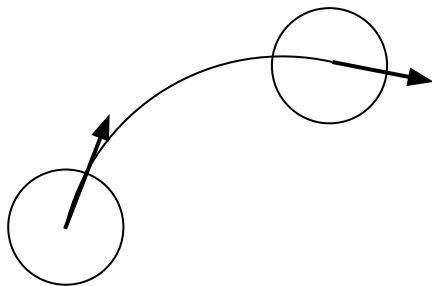# SPEED-BASED MOTION MODEL

❏ Such a model assumes that we control the parameters of the robot's motion — linear and angular velocity

❏ The robot moves along a circular arc
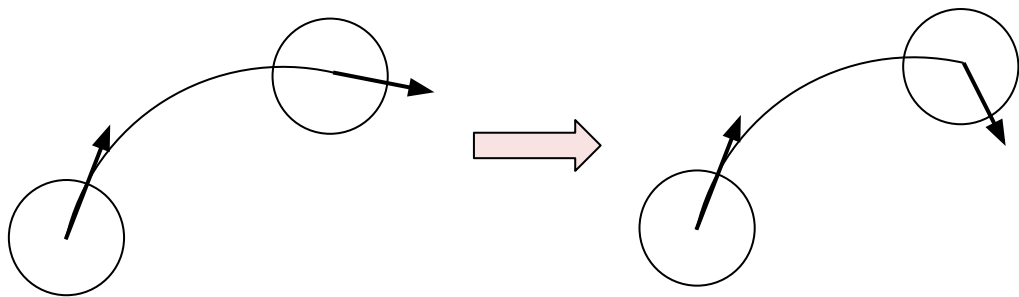
❏ Control signals (speeds) are subject to noise

$$\hat{v} = v + \varepsilon_{\alpha_1 |v| + \alpha_2 |\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3 |v| + \alpha_4 |\omega|}$$

SPEED-BASED MOTION MODEL

❑ To allow the final turn, a third motion parameter is introduced

$$\hat{v} = v + \varepsilon_{\alpha_1|v|+\alpha_2|\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3|v|+\alpha_4|\omega|}$$

$$\hat{\gamma} = \varepsilon_{\alpha_5|v|+\alpha_6|\omega|}$$

SPEED-BASED MOTION MODEL

$$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin \theta + \frac{\hat{v}}{\hat{\omega}} sin(\theta + \hat{\omega}\Delta t)$$

$$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos \theta - \frac{\hat{v}}{\hat{\omega}} cos(\theta + \hat{\omega}\Delta t)$$

$$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$$

SAMPLING FROM SPEED-BASED MOTION MODEL

1:       **Algorithm sample_motion_model_velocity(u_t, x_{t-1}):**

2:       $\hat{v} = v + \textbf{sample}(\alpha_1 v^2 + \alpha_2 \omega^2)$

3:       $\hat{\omega} = \omega + \textbf{sample}(\alpha_3 v^2 + \alpha_4 \omega^2)$

4:       $\hat{\gamma} = \textbf{sample}(\alpha_5 v^2 + \alpha_6 \omega^2)$
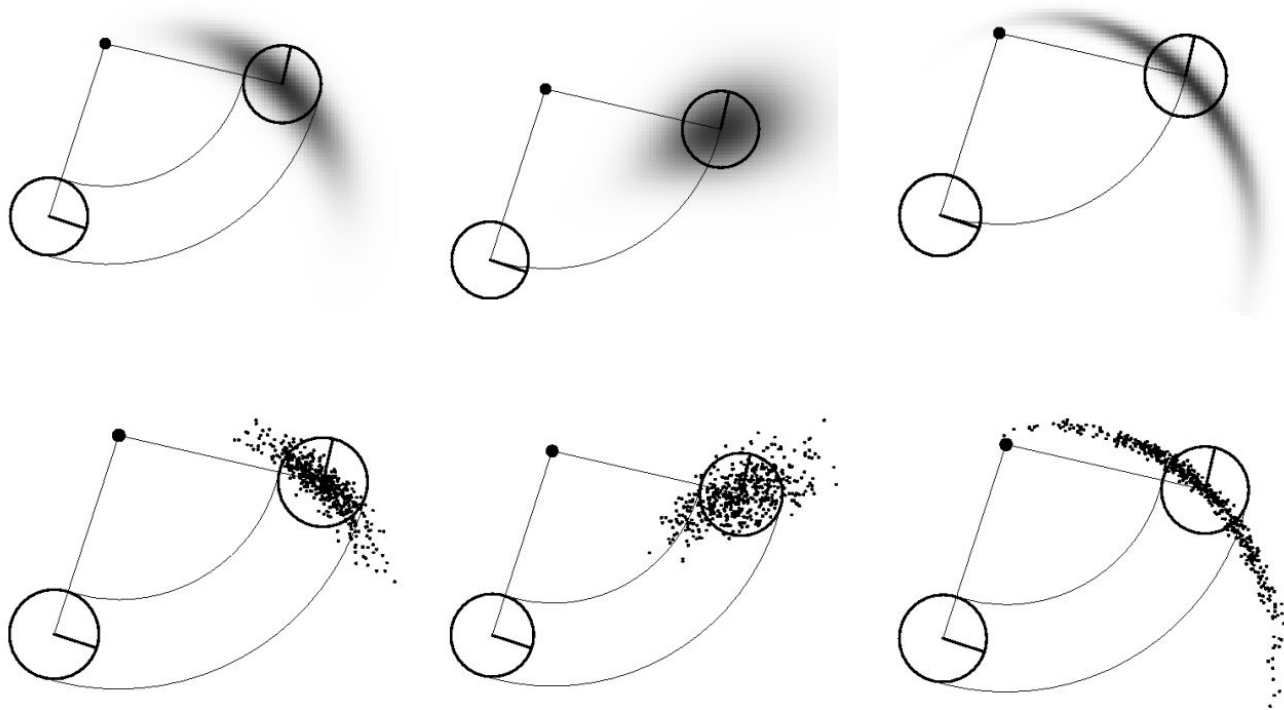
5:       $x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}} \, sin(\theta + \hat{\omega}\Delta t)$

6:       $y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}} \, cos(\theta + \hat{\omega}\Delta t)$
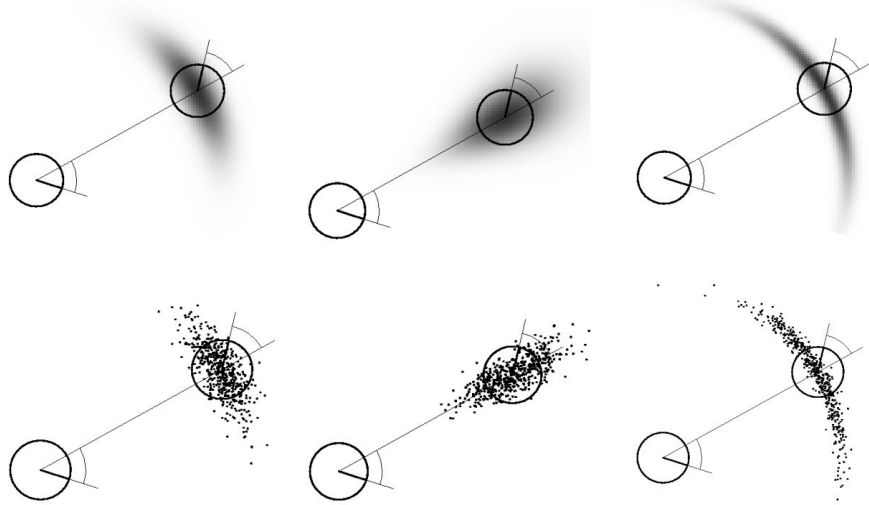
7:       $\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$
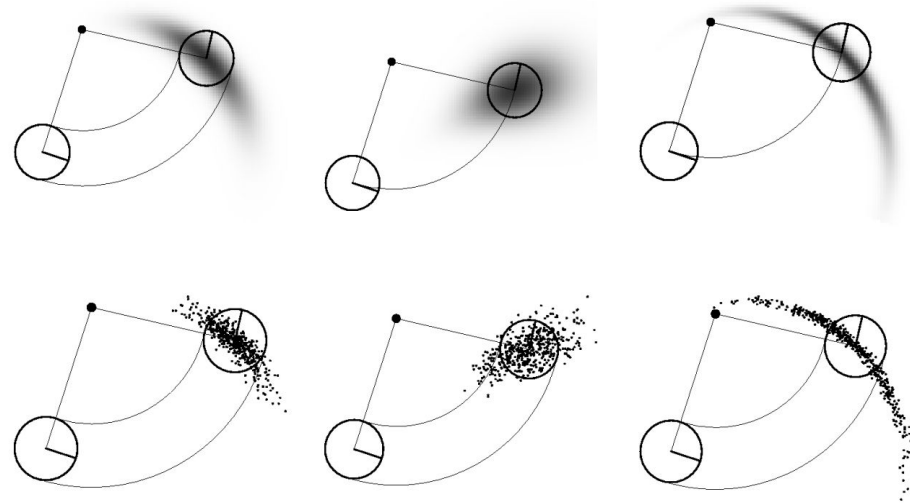
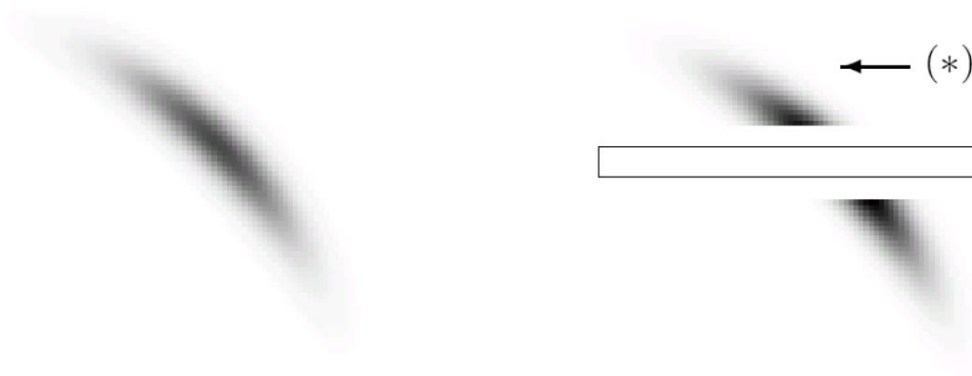8: $\textbf{return } x_t = (x', y', \theta')^T$

EXAMPLE OF SPEED-BASED MOTION MODEL

ODOMETRY-BASED MODEL

SPEED-BASED MODEL

MOTION MODELS ACCOUNTING FOR ENVIRONMENT

$(*)$

$p(x'|u,x) \qquad \neq \qquad p(x'|u,x,m)$

Approximation: $p(x'|u,x,m) = \eta\, p(x'|m)\, p(x'|u,x)$

ADDITIONAL RESOURCES

1. [Differential Drive Kinematics](#)

2. [Probabilistic Robotics](#) (in Notion). Chapter 5.

3. [Mobility: wheels and whegs](#)

Thanks for attention!

Questions? Additions? Welcome!

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