Osnove Newtnove mehanike

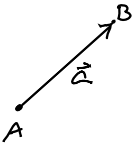
Def: Afini prostor $\mathcal{A}$ nad vektorskim prostorom V je množica $\mathcal{A}$ z binarno operacijo $\mathcal{A}\times V \longrightarrow \mathcal{A}$: $(A,\vec{a}) \longmapsto A+\vec{a}$ z lastnostmi:

i) $(A+\vec{a})+\vec{b} = A+(\vec{a}+\vec{b})$

ii) $\forall\, A,B\in\mathcal{A}\ \exists!\, \vec{a}\in V.\ B=A+\vec{a}$

$\dim\mathcal{A}=\dim V$

Primeri:



Def: Definiramo operacijo $\mathcal{A}\times\mathcal{A}\longrightarrow V$.
s predpisom $B-A=\vec{a} \Leftrightarrow \vec{B}=A+\vec{a}$

Trditev:

i) $A-A=\vec{0}$

ii) $(A-B)+(B-A)=\vec{0}$

iii) $(A-B)+(B-C)+(C-A)=\vec{0}$

iv) $(A-B)+\vec{a}=(A+\vec{a})-B$

v) $(A+B)-C=B+(A-C)$

Dokaz:

i) $A-A=\vec{a} \Leftrightarrow A=A+\vec{a} \Leftrightarrow A=(A+\vec{a})+\vec{a}=A+2\vec{a}$

$\Rightarrow \vec{a}=2\vec{a}$ (ker je $\vec{a}$ natanko določen)

$\Rightarrow \vec{a}=\vec{0}$

v) $A-C=\vec{a} \Rightarrow A=C+\vec{a}$

$B+(A-C)=B+\vec{a}=C+(\vec{b}+\vec{a})$

$B-C=\vec{b} \Rightarrow B=C+\vec{b}$

$$(A+B)-C=\big(\underbrace{(C+\vec{a})}_{\text{vektor}}+(C+\vec{b})\big)-C=$$, zato lahko zamenjamo oklepaje

$$=C+\vec{a}+(\underbrace{C+\vec{b}-C}_{\text{vektor}})=\vec{C}+\vec{a}+\vec{b}$$

$(\mathcal{A}, V)$ $(\mathcal{A}', V')$

$g : \mathcal{A} \to \mathcal{A}'$

Def: Preslikava $g : \mathcal{A} \to \mathcal{A}'$ je afina če obstaja $d_g \in \mathcal{L}(V, V')$ tako da velja $g(A) - g(B) = d_g(A-B)$ za $\forall A, B \in \mathcal{A}$

$$g(A) = g(B) + d_g(A-B)$$

$g(A) = g(O) + d_g(A-O)$; O je pol afine preslikave

Izbira pola je poljubn

$$\tilde{g}(A) = g(\tilde{O}) + d_g(A-\tilde{O}) = g(O) + d_g(\tilde{O}-O) + d_g(A-\tilde{O})$$
$$= g(O) + d_g(\underbrace{(\tilde{O}-O) + (A-\tilde{O})}_{A-O}) = g(A)$$

Izberimo $O \in \mathcal{A}$ $\quad \vec{a} = A - O$

$$A \in \mathcal{A} \; ; \; A = O + (A-O) = O + \vec{a}$$

$\forall A \in \mathcal{A}$ lahko identificiramo z vektorjem $\vec{a} \in V$

Definicija: Galilejeva struktura G je trojica $(\mathcal{A}, \tau, d)$, kjer je $\mathcal{A}$ štirirazsežen afini prostor, $\tau \in \mathcal{L}(V, \mathbb{R})$ in d evklidska razdalja nad prostorom istočasnih dogodkov.

linearna preslikava

Funkcionalu τ pravimo časovnost. Dogodka $A, B \in \mathcal{A}$ sta istočasna, če $A-B \in \ker \tau$

Definicija: Galilejevi strukturi $G(\mathcal{A}, \tau, d)$ in $\tilde{G}(\tilde{\mathcal{A}}, \tilde{\tau}, \tilde{d})$ sta ekvivalentni, če obstaja afina bijekcija $g: \mathcal{A} \to \tilde{\mathcal{A}}$, ki ohranja časovnost in razdaljo istočasnih dogodkov

$$\tilde{\tau}(g(A)-g(B)) = \tau(A-B)$$

A, B istočasna $\Leftrightarrow$ $g(A), g(B)$ istočasna

$$d(A,B) = \tilde{d}(g(A), g(B))$$

Definicija
$\mathbb{R}\times E$ afini prostor, kjer je E trorazsežni evklidski prostor

Na $\mathbb{R}\times E$ vpeljemo naravno Galilejevo strukturo.

$$A\in \mathbb{R}\times E \Rightarrow A=(t,P) \quad t\in\mathbb{R}\,,\, P\in E$$

ima normo porojeno s skalarnim produktom

$$\tau(A_2-A_1)=t_2-t_1$$
$$d(A_1,A_2)=\|P_1-P_2\|$$

Tej strukturi pravimo naravna Galilejeva struktura

Definicija: Koordinatni sistem na afinem prostoru $\mathcal{A}$ je bijektivna preslikava $\varphi:\mathcal{A}\to\mathbb{R}$

$$A\mapsto \varphi(A)=(\pi_t\varphi(A),\pi_p\varphi(A))$$
$$= \varphi_t(A),\varphi_p(A))$$

za katero je φ_t afina preslikava

$$(\mathbb{R}\times E,\ t,\ \|\ \|)$$

$$\tau(A-B)=t(\varphi_t(A)-\varphi_t(B))$$
$$d(A,B)=\|\varphi_p(A)-\varphi_p(B)\|$$

Lahko merimo razdaljo tudi med neistočasnimi dogodki

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi"] \arrow[dr, "\tilde{\varphi}"'] & \mathbb{R}\times E \arrow[d, leftrightarrow] \\
 & \mathbb{R}\times E
\end{tikzcd}$$

Kdaj sta dve naravni galilejevi strukturi ekvivalentni

$g: \mathbb{R}\times E \longrightarrow \mathbb{R}\times E$

$$A = \begin{bmatrix} t \\ P \end{bmatrix} \longmapsto g(A) = \begin{bmatrix} t' \\ P' \end{bmatrix} = g(O) + d_g(A - O)$$

$$= \begin{bmatrix} t_0' \\ P_0' \end{bmatrix} + \underbrace{\left[\begin{array}{c|c} \alpha & \vec{a}^T \\ \hline \vec{c} & Q \end{array}\right]}_{4\times 4 \text{ matrika}} \begin{bmatrix} t - t_0 \\ P - P_0 \end{bmatrix}$$

$$O = \begin{bmatrix} t_0 \\ P_0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} t_1 \\ P_1 \end{bmatrix} \quad A_2 = \begin{bmatrix} t_2 \\ P_2 \end{bmatrix}$$

$$t(g(A_2) - g(A_1)) = t(A_2 - A_1)$$

$$g(A_2) - g(A_1) = \begin{bmatrix} \alpha & \vec{a}^T \\ \vec{c} & Q \end{bmatrix} \begin{bmatrix} t_2 - t_1 \\ P_2 - P_1 \end{bmatrix} = \begin{bmatrix} \alpha(t_2 - t_1) + \vec{a}(P_2 - P_1) \\ (t_2 - t_1)\vec{c} + Q(P_2 - P_1) \end{bmatrix}$$

$$\Rightarrow \alpha(t_2 - t_1) + \vec{a}(P_2 - P_1) = t_2 - t_1 \Rightarrow \alpha = 1, \ \vec{a} = \vec{0}$$

To velja da se ohranja časovnost

ohranjanje razdalje med istočasnimi dogodki:

$$d(g(A_1), g(A_2)) = d(A_2, A_1) \quad \text{za } t_1 = t_2$$

$$\| \quad \swarrow t_2 - t_1 = 0 \qquad\qquad \|$$

$$\|Q(P_2 - P_1)\| \qquad\qquad \|P_2 - P_1\|$$

$$\Rightarrow Q \in O(3)$$

$\uparrow$ ortogonalna matrika 3x3

Definicija: Preslikava, ki ohranja Galilejevo strukturo pravimo Galilejeva preslikava

Trditev: Galilejeve preslikave med naravna Galilejevima strukturama $\mathbb{R}\times E$ je oblike

$$\begin{bmatrix} t \\ P \end{bmatrix} \longmapsto \begin{bmatrix} t' \\ P' \end{bmatrix} = \begin{bmatrix} t_0' + t - t_0 \\ P_0' + \vec{c}(t-t_0) + Q(P-P_0) \end{bmatrix} =$$

kjer je $Q \in O(3)$, $\vec{c}$ poljuben vektor, t_0 poljubno število in P_0' poljubna točka

$$= \begin{bmatrix} \tilde{t}_0 + t \\ \tilde{P}_0 + \vec{c}t + Q(P-P_0) \end{bmatrix}$$

$t_0' - t_0 = \tilde{t}_0$

Dva opazovalca: $\varphi(t,P)$, $\varphi'(t',P')$

gibanje $t \mapsto P(t)$ terakterija tocke

$\vec{v} = \frac{dP}{dt}$ vektor hitrosti $\lim_{h\to 0}\frac{P(t+h)-P(t)}{h}$

$\ddot{P} = \dot{\vec{v}} = \vec{a} = \frac{d\vec{v}}{dt}$ vektor pospeška

$|\vec{v}| = v$ brzina

$$P'(t') = P_0' + \vec{c}(t'-t_0') + \overbrace{Q(P(t)-P_0)}^{Q(P(t'-t_0')-P_0)}) \text{ terakterija v } \varphi'$$

$$\vec{v}' = \frac{dP'}{dt'} = \vec{c} + Q\frac{dP}{dt}(t'-t_0') = \vec{c} + QP(t)$$

$$\| $$

$$\dot{P}' \qquad \dot{P}'(t') = \vec{c} + Q\dot{P}(t)$$

najprej črtica, nato odvod

$$\ddot{P}'(t') = \vec{a}' = \frac{dv'}{dt'} = Q\ddot{P}(t)$$

	φ	φ'
	P	$P' = P_0' + \vec{c}\,t + Q(P-P_0)$
$\vec{a} = P_2 - P_1$		$\vec{a}' = P_2' - P_1' = Q(P_2 - P_1) = Q\vec{a}$

Definicija: vektor $a = P_2 - P_1$ je koordinatno neodvisen

$A \in \mathcal{L}(V, V)$ $A' = Q^T A Q$

λ $\lambda' = \lambda$

Sistem materialnih točk $(P_1, \dots P_n) = \underline{P}$

$$\underline{P}' = \underline{P_o}' + \underline{\vec{c}}\,t + Q\,(\underline{P} - \underline{P_o})$$

$$\underline{P_o}' = (P_o', \dots P_o') \qquad Q(\underline{P} - \underline{P_o}) = \big(Q\,(P_1 - P_o), \dots Q(P_n - P_o)\big)$$

$$\underline{\vec{c}} = (\vec{c}, \dots \vec{c})$$

$$\Rightarrow \underline{\dot{P}}' = \vec{c} + Q\,\underline{\dot{P}}$$

$$\underline{\ddot{P}}' = Q\,\underline{\ddot{P}}$$

Princip determiniranosti

V danem KS (koordinatni sistem) je trektorija sistema materialnih točk natanko določena z začetnim položajem in začetno hitrostjo.

To specialno pomeni, da obstaja funkcija interakcije $\underline{\vec{f}}$ tako da je $\underline{\ddot{P}} = \underline{\vec{f}}\,(t, \underline{P}, \underline{\dot{P}})$

$\left(\ddot{P}(t) = \vec{f}\,(t, \underline{P(t)}, \dot{P}(t)) \right.$ na dolgo$)$

8.10

Princip relativnosti Obstaja razred koordinatnih sistemov v katerem je funkcija interakcije invariantna za Galilejeve transformacije. Koordinatnim sistemom iz tega pravimo **inercialni koordinatni sistem**

$(t, P) \xrightarrow{GT} (t', P')$ Galilejeva tranformacija.

Potem je $\underline{f} = \underline{f}'$ $(\ddot{P} = \ddot{P}')$

$$\underset{\parallel}{\ddot{\underline{P}}'} = \vec{\underline{f}}(t', P', \dot{P}')$$

$$Q\ddot{\underline{P}} = Q\vec{\underline{f}}(t, \underline{P}, \underline{\dot{P}}) = \vec{\underline{f}}(t_0' + t, P_0' + \vec{c}t + Q(P - P_0), \vec{c} + Q\dot{\underline{P}})$$

i) $\vec{c} = \vec{0}$ $\quad Q = I$ $\quad P_0' = P_0$

$$\vec{\underline{f}}(t, \underline{P}, \underline{\dot{P}}) = \vec{\underline{f}}(t_0' + t, \underline{P}, \underline{\dot{P}}) \text{ za } \forall t_0'$$

$\Rightarrow f$ ni eksplicitno odvisna od časa (t)

(**homogenost časa**)

ii) $\vec{c} = 0$, $Q = I$, $\dot{P}' = P_0 + \vec{a}$

$$\vec{f}(P, \dot{P}) = \vec{f}(\underset{\parallel}{\underline{P + \vec{a}}}, \dot{P}) \quad \text{za } \forall \vec{a}$$

$$(P_1 + \vec{a}, \ldots\ldots P_n + \vec{a})$$

$\Rightarrow \underline{\vec{f}}$ je odvisna samo od relativnih položajev

$$\vec{f}(P, \dot{P}) = \vec{f}(\underset{i \neq j \text{ ..}}{P_i - P_j}, \underline{\dot{P}})$$

vse kombinacije tega

(**homogenost prostora**)

iii) $Q = I$

$$f(P_i - P_j, \dot{P}) = f(P_i - P_j, \vec{c} + \dot{P}) \Rightarrow \vec{f}(P_i - P_j, \dot{P}_k - \dot{P}_l)$$

(**homogenost prostora hitrost**)

iv) Q poljuben

$$Q\vec{f}(P_i - P_j, \dot{P}_k - \dot{P}_l) = f(Q(P_i - P_j), Q(\dot{P}_k - \dot{P}_l))$$

f je **izotropična funkcija**

$$Q\vec{g}(\vec{a}) = \vec{g}(Q\vec{a}) \text{ za } \forall Q \in O(3)$$

Posebni primer : $N=1$ (izolirana točka)

$\ddot{P} = \vec{f}()$ ← brez argumentov (konstanta

še vedno velja izotropičnost $Q\ddot{P} = Q\vec{f} = \vec{f} \Rightarrow \vec{f} = 0$

$\Rightarrow \ddot{P} = \vec{0}$

v IKS (inercialni koordinatni sistem) se izolirana materialna točka giblje premočrtno s konstantno hitrostjo. $P = v_0 t + P_0 (t=0)$

$N=2 \Rightarrow$

$\ddot{P}_1 = \vec{f}_1 (P_1 - P_2, \dot{P}_1 - \dot{P}_2)$

$\ddot{P}_2 = \vec{f}_1 (P_2 - P_1, \dot{P}_2 - \dot{P}_1)$

če se točke gibljejo po premici proti drugi bosta vedno ostali na tej premici

(Podobno tri: točke ostanejo na ravnini:)

$\ddot{P}_1 = f_1(P_1 - P_2, P_1 - P_3, \underline{\underline{P_2 - P_3}}, \ldots$

← tudi to je pomembno

Definicija: Interakcija $\ddot{P}_i = f_i(\ldots)$ je parska če je odvisna samo od relativnih položajev in hitrosti glede na P_i in je delovanje točk autonomno

$$\vec{f}_i(\underset{j \neq i}{P_i - P_j}, \underset{k \neq i}{\dot{P}_i - \dot{P}_k})$$

$$\vec{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{f}_{ji}(P_i - P_j, \dot{P}_i - \dot{P}_j)$$

Def: Interakcija $\vec{f}_{ji} = \vec{f}_{ji}(P_i - P_j, \dot{P}_i - \dot{P}_j)$ je lokalna če velja $\lim\limits_{|P_i - P_j| \to \infty} f_{ji} = \vec{0}$

Princip sorazmernosti

V IKS za sistem materialnih točk $P_1 \dots P_N$ obstajajo natanko določene konstante α_{ij} tako, da ne glede na interakcije $\vec{f}_i$ velja $\vec{f}_i = -\sum_{j=1}^{N} \alpha_{ij} \vec{f}_j$

Lema: Za konstante α_{ij} velja

i) $\alpha_{ij} \alpha_{ji} = 1$

ii) $\alpha_{ij} \alpha_{jk} \alpha_{ki} = 1$

Dokaz:

$\vec{f}_i = \vec{f}_i (P_i - P_k) = \vec{f}_i (P_i - P_1, P_i - P_2 \dots P_i - P_N)$

in naj bodo f_i lokalni

P_k pošljemo proti ∞ razen P_i in P_j

$$\vec{f}_i = -\sum_{\substack{k=1 \\ k \neq i}}^{N} \alpha_{ki} \vec{f}_k = -\alpha_{ji} \vec{f}_j$$

$$\vec{f}_j = -\sum_{\substack{k=1 \\ k \neq j}}^{N} \alpha_{kj} \vec{f}_k = -\alpha_{ij} f_i$$

$$f_i = -\alpha_{ji} f_j = \alpha_{ji} \alpha_{ij} f_i \Rightarrow \alpha_{ji} \alpha_{ij} = 1$$

P_i, P_j, P_k ostanejo, ostale gredo proti ∞

$$\vec{f}_i = -\alpha_{ji} f_j - \alpha_{ki} f_k = -\alpha_{ji}(-\alpha_{ij} f_i - \alpha_{kj} \vec{f}_k) - \alpha_{ki} f_k =$$

$$\vec{f}_j = -\alpha_{ij} \vec{f}_i - \alpha_{kj} f_k$$

$$f_i = \underbrace{\alpha_{ji} \alpha_{ij}}_{1} f_i + \alpha_{ji} \alpha_{kj} f_k - \alpha_{ki} f_k$$

$$\alpha_{ki} f_k = \alpha_{ji} \alpha_{kj} \vec{f}_k \quad / \alpha_{ik}$$

$$f_k = \alpha_{ji} \alpha_{kj} \alpha_{ik} f_k \Rightarrow \alpha_{ik} \alpha_{kj} \alpha_{ji} = 1$$

Lema: Naj za pozitivna števila α_{ij} velja

i) $\alpha_{ij} \cdot \alpha_{ji} = 1$

ii) $\alpha_{ij} \alpha_{jk} \alpha_{ki} = 1$.

Potem $\exists$ pozitivna števila m_i, tako da velja da je

$\alpha_{ji} = \frac{m_j}{m_i}$

Števila m_i so določena do sorazmernostnega faktorja natančnost. Številom m_i pravimo

inercijske mase

Dokaz: $\alpha_{ii} = 1$

$\alpha_{ij} \alpha_{jj} \alpha_{ji} = 1 \Rightarrow \alpha_{jj} = 1$

$l_{ij} = \log \alpha_{ij}$ $\quad$ $l_{ij} + l_{jk} + l_{ki} = 0$

$l_{ij} = -l_{ji}$

$l_{i_0 j} + l_{jk} + l_{k i_0} = 0$

$l_{ij} - l_{i_0 j} + l_{ki} - l_{k i_0} = 0 \Rightarrow$

$l_{ij} - l_{i_0 j} = -l_{ki} + l_{k i_0} = l_{ik} - l_{i_0 k} \quad \forall j,k$

$\Rightarrow l_{ij} - l_{i_0 j} = n_{i i_0}$

$\left.\begin{array}{l} l_{ij} = n_{i i_0} + l_{i_0 j} \\ j = 1 \end{array}\right\} \Rightarrow 0 = n_{i i_0} + l_{i i} \Rightarrow l_{i i_0} = n_{i i_0}$

$l_{ij} = l_{i i_0} - l_{j i_0}$

$l_{i i_0} = \log m_i$ ← definiram

$\Rightarrow l_{ij} = \log m_i - \log m_j = \log \frac{m_i}{m_j} \Rightarrow \alpha_{ij} = \frac{m_i}{m_j}$

Izrek: V inercialnem koordinatnem sistemu velja

$$m_1\ddot{P}_1+m_2\ddot{P}_2+\dots+m_n\ddot{P}_n=\vec{0}$$

Dokaz:

$$f_i=-\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_{ji}f_j=-\sum_{j=1}^{N}\frac{m_j}{m_i}f_j$$

$$\ddot{P}_i=-\sum_{j=1}^{N}\frac{m_j}{m_i}\ddot{P}_j \quad /m_i$$

$$\sum_{i=1}^{N}m_i\ddot{P}_i=\vec{0}$$

$$P_*=\frac{1}{m}\sum_{i=1}^{N}m_iP_i \quad , m=\sum_{i=1}^{N}m_i$$ masno središče

seštevanje točk:

$$\sum_{i=1}^{N}P_i=\sum\Big(\underbrace{m_i(P_i-O)}_{\text{vektor}}+m_iO\Big)=\sum_{i=1}^{N}m_i(P_i-O)+\underbrace{\sum_{i=1}^{N}m_iO}_{O}$$

$$P_*=O+\frac{1}{m}\sum_{i=1}^{N}m_i(P_i-O)$$

$$m\ddot{P}_*=\vec{0}$$

Definicija: sila na P_i je produkt interakcije f_i z maso m_i:

$$m_i \ddot{P}_i = m_i f_i(\ldots) =: \vec{F}_i$$

Princip o masi: masa materialnih točk je enaka v vseh koordinatnih sistemih

$$\ddot{P}_1 = f_1(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \qquad m_1 \ddot{P}_1 = F_1$$

$$\ddot{P}_2 = f_2(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \qquad m_2 \ddot{P}_2 = F_2$$

$$m_1 \ddot{P}_1 + m_2 \ddot{P}_2 = F_1 + F_2 = 0$$

$$F_2 = -F_1$$

$$\vec{F}_1 = \vec{F}_{21} \qquad \vec{F}_{12} = -\vec{F}_{21}$$

$$F_2 = \vec{F}_{12}$$

Trditev: Če so vse sile parske in lokalne velja zakon akcije in reakcije (3NZ)

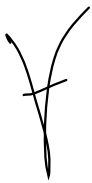
Dokaz: $f_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} f_{ji}(p_j - p_i, \dot{p}_j - \dot{p}_i)$

$$\Big\} \; p_j \to \infty \quad j \notin \{i,k\}$$

$$m_i f_i = m_i \ddot{p}$$

$$m_i f_{ji} + m_j f_{ij} = \vec{0}$$

$$\vec{F}_{ji} = -\vec{F}_{ij}$$



10.10
zamudila prva uro

odvod vzdolž trakterije $\frac{d}{dt} E(t, P(t), \dot{P}(t)) =$

$$\frac{\partial E}{\partial t} + \left(\frac{\partial E}{\partial P}\right)^T \dot{P} + \left(\frac{\partial E}{\partial \dot{P}}\right)^T \ddot{P}$$

Izrek o ohranjenju: Naj bo sila $\vec{F}$ potencialna s potencialom $U(t, P, \dot{P})$. Potem je vsota kinetične in potencialne energije konstanta gibanja (= njen odvod vzdolž trakterije je 0) $\iff$ moč sil je nasprotno enaka odvodu potencialne energije vzdolž trakterije

Dokaz: $\Rightarrow$ $E_0 = T + U = \frac{1}{2} m |\dot{P}|^2 + U$

$$0 = \frac{\partial E_0}{\partial t} = m \ddot{P} \dot{P} + \dot{F} \dot{P} + \frac{dU}{dt} \Rightarrow \frac{dU}{dt} = -F \dot{P}$$

$\Leftarrow$ $A = T_2 - T_1$

$$A = \int_{t_1}^{t_2} \vec{F} \cdot \dot{\vec{P}}\, dt = -\int_{t_1}^{t_2} \frac{dU}{dt} dt = \underset{U_1}{U(t_1)} - \underset{U_2}{U(t_2)}$$

$$\Rightarrow U_1 + T_1 = U_2 + T_2 = E_0$$

Sila je konzervativna, če je potencialna in odvisna samo od položaja

Posledica: če je sila konzervativna velja izrek o energiji

Dokaz: $$\frac{d}{dt}U(P(t)) = \left(\frac{\partial U}{\partial P}\right)^T \dot{P} = -F\dot{P}$$

$$\vec{F}\dot{P} = -\frac{dU}{dt}$$

$$F'\dot{P}' = QF(\vec{c} + Q\dot{P}) = \underset{\overset{\|}{F'}}{QF\cdot\vec{c}} + \underbrace{Q\vec{F}\cdot Q\dot{P}}_{\vec{F}\cdot\dot{P} = -\frac{dU}{dt}} = -\frac{dU}{dt} + m\ddot{P}'\vec{c} =$$

$$= -\frac{d}{dt}(U - m\dot{P}'\vec{c})$$

ortogonalna preslikava ohranja skalarni produkt

$$\tilde{U} = U - m\dot{P}'\vec{c}$$

$$\tilde{U}(P', \dot{P}') = U'(P') - m\dot{P}'\vec{c}$$

Izrek: Izrek o energiji je invarianten za Galilejeve transformacije

$$E_0' = T' + \tilde{U}' = \frac{1}{2} m |\vec{c}|^2 + m c Q \dot{P} + T + \tilde{U} - m \dot{P}' \vec{c} =$$

$$= E_0 + \frac{1}{2} m |\vec{c}|^2 + m \vec{c} (Q \dot{P} - \dot{P}') = E_0 - \frac{1}{2} m |\vec{c}|^2$$

$$T' = \frac{1}{2} m \dot{P}' \dot{P}' = \frac{1}{2} m (\vec{c} + Q \dot{P}) (\vec{c} + Q \dot{P}) = \frac{1}{2} m |\vec{c}|^2 + m \vec{c} Q \dot{P} + \frac{1}{2} m |\dot{P}|^2$$

$$\dot{P}' = \vec{c} + Q \dot{P}$$

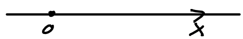
Premočrtno gibanje

= pospešek ima konstantno smer

$\vec{a} = a(t)\,\vec{c}$

$\vec{v} = \int_{t_0}^{t} \vec{a}(t)\,dt + \vec{v}(t_0) = \left(\int_{T_0}^{t} a(t)\,dt\right)\vec{e} + \vec{c}(t_0)$

Obstaja KS v katerem tir leži na premici



$p = x \quad \dot{p} = \dot{x} \quad \ddot{p} = \ddot{x}$

$v = \dot{x}$ je lahko zdaj tudi negativen

$m\ddot{x} = f(t, x, \dot{x})$

omejimo se ko bo f konzervativna

$m\ddot{x} = f(x)$

kdaj je sila v odvisnosti položaja potencialna?

(potencialna ⇒ konzervativ

$f = -\frac{dU}{dx} \Rightarrow U = \int_{x_0}^{x} f(\xi)\,d\xi + U_0$

sila $f(x)$ je potencialna če je f zvezna

če je $f(x)$ zvezna ⇒ velja izrek o energiji

$$m\ddot{x} = f(x)$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = E_0$$

$$\dot{x}^2 = \frac{2}{m}(E_0 - U(x))$$

$$\frac{dx}{dt} = \dot{x} = \pm\sqrt{\frac{2}{m}(E_0 - U(t))}$$

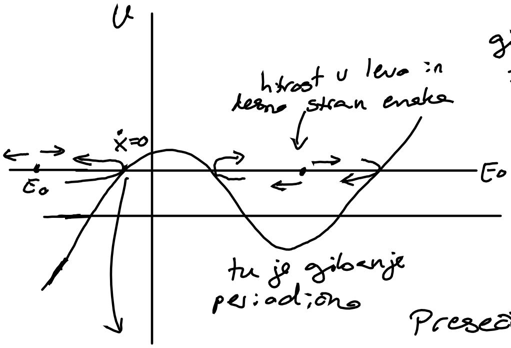
$$\pm\frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(t)}} = dt$$

$$\operatorname{sgn}\dot{x}\int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x)}} = \int_{t_0}^{t} dt = t - t_0 \Rightarrow t = t(x) \Rightarrow x = x(t)$$

Lahko to remo ko $\dot{x} \neq 0$ ($\operatorname{sgn}\dot{x}$ je konstanten)

$$t = t_0 \pm \int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

Kvalitativna obravnava gibanja



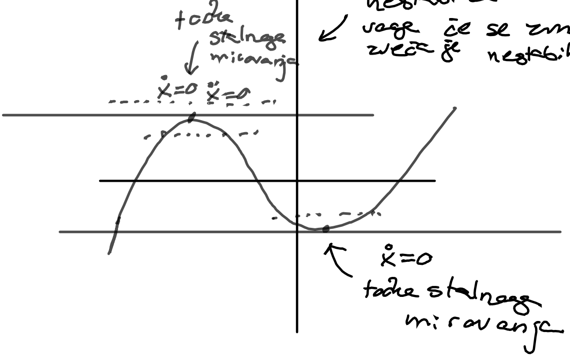
gibanje je možno tam kjer je energijska nivojnica nad potencialom

E_0 nad grafom

Presečišče je točka trenutnega mirovanja (točka obrata gibanja

$$0 = \dot{x} = \frac{f(x)}{m} = -\frac{dU}{dx}$$

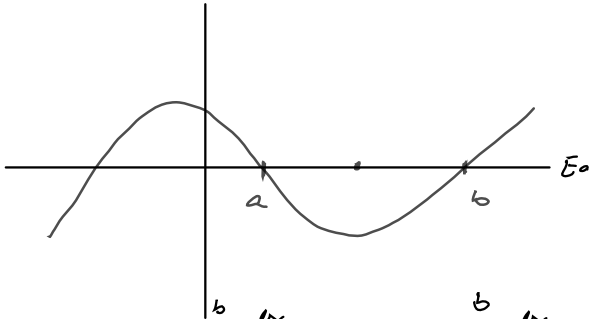
nestabilna ravnovesna vaga če se zmanjša ali zveča je nestabilno



Lokalni minimum je stabilna ravnovesna lega (če se energija malo spremeni pride do majhnega periodičnega gibanja (majhen odmik od ravnovesne vage)

Prevoj je tudi nestabilna lega

Kakšna je perioda periodičnega gibanja



gibanje: $t_1 = \int_{x_0}^{b} \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}}$ $\quad t_2 = -\int_{a}^{b} \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}}$

$$t_3 = \int_{a}^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}}$$

$$T = t_1 + t_2 + t_3 = 2\int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}} = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0-U(x)}}$$

↑ perioda

Ali je res periodično in se ne ustavi prej?

$$\int_a^{a+d} \frac{dx}{\sqrt{E_0-U(x)}} \qquad U(x) = U(a) + \frac{dU}{dx}(\xi)(x-a)$$

$\frac{dU}{dx}(a) < 0$ a je presečišče

$$\exists \delta > 0 .\ \xi \in [a, a+\delta) \Rightarrow 2\frac{dU}{dx}(\xi) < \frac{1}{2}\frac{dU}{dx}(a)$$

$$E_0 - U(x) = -\frac{dU}{dx}(\xi)(x-a) < -2\frac{dU}{dx}(a)(x-a)$$

$\Rightarrow$ integral je konvergenten

ker $\int_a^{a+\delta} \frac{dx}{\sqrt{E_0-U(x)}} \leq \frac{1}{\sqrt{-\frac{1}{2}U'(x)(a)}} \int_a^{a+\delta} \frac{dx}{\sqrt{x-a}} < \infty$

Če se energijska nivojnica dotika lokalnega maksimuma, potem ne pridemo do dotikališča v končnem času

Lema $$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi$$

Dokaz: $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$

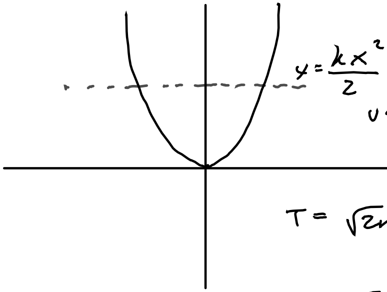
$$b-x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$
$$x-a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$
$$(b-x)(x-a) = \left(\frac{1}{2}(b-a)\right)^2(1-\cos^2\vartheta) \overset{\sin^2\vartheta}{=} \left(\frac{1}{2}(b-a)\right)^2$$
$$dx = -\frac{1}{2}(b-a)\sin\vartheta\, d\vartheta$$
$$I = -\int_\pi^0 \frac{\frac{1}{2}(b-a)\sin\vartheta\, d\vartheta}{\frac{1}{2}(b-a)\sin\vartheta} = -\int_\pi^0 d\vartheta = \pi$$

Primer:

Harmonični oscilator $\quad U = \frac{1}{2}kx^2$

$F = -kx$

$$m\ddot{x} = -kx$$
$$\ddot{x} + \frac{k}{m}x = 0 \qquad \frac{k}{m} = \omega^2$$
$$\ddot{x} + \omega x = 0$$
$$x = A\cos(\omega t - \delta)$$



vsa gibanja so periodična

$$T = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0 - \frac{1}{2}kx^2}} =$$
$$= \frac{\sqrt{2m}}{\sqrt{\frac{1}{2}k}}\int_a^b \frac{dx}{\sqrt{\frac{2E_0}{k} - x^2}} = 2\sqrt{\frac{m}{k}}\,\pi = \frac{2\pi}{\omega}$$

T perioda je neodvisna od energije (izokronično gibanje)

Primer: $U = ax^2 + bx^{-2}$

$[U]$ dimenzija enota

$$[U] = [a x^2] = [a][x^2] = [a]\overset{\text{enota}}{L^2}$$

$$[U] = [b x^{-2}] = [b]\, L^{-2}$$

$$\Rightarrow \quad [a] L^2 = [b] L^{-2}$$

$$L^4 = \frac{[b]}{[a]} \quad \Rightarrow \quad L = \sqrt[4]{\frac{[b]}{[a]}}$$

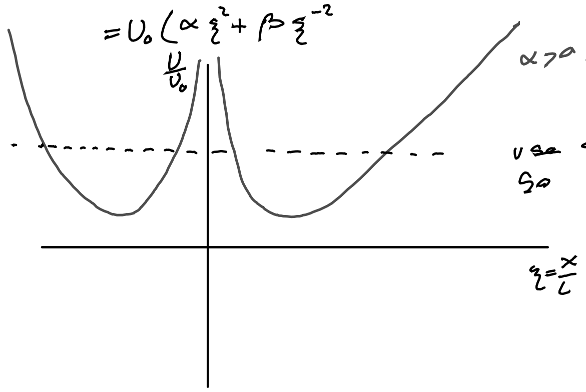
$$x = L\xi \qquad L = \left(\frac{b}{a}\right)^{\frac{1}{4}}$$

ξ brezdimenzijski

$$U = a\sqrt{\frac{|b|}{|a|}}\,\xi^2 + b\sqrt{\frac{|a|}{|b|}}\,\xi^{-2} = \underbrace{\sqrt{|a||b|}}_{U_0}\left(\alpha\xi^2 + \beta\xi^{-2}\right) =$$

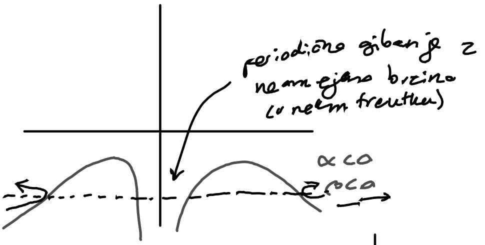
$$a = \underbrace{\frac{a}{|a|}}_{\alpha}|a| \qquad b = \underbrace{\frac{b}{|b|}}_{\beta}|b|$$

$$= U_0\left(\alpha\xi^2 + \beta\xi^{-2}\right)$$

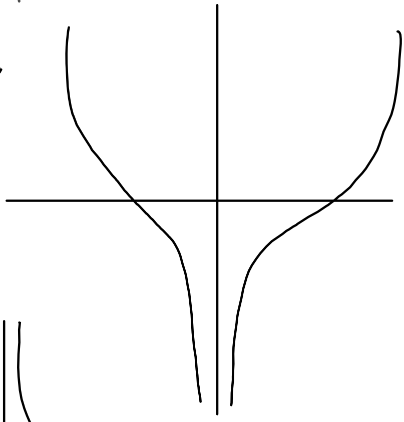


$\alpha > 0,\ \beta > 0$

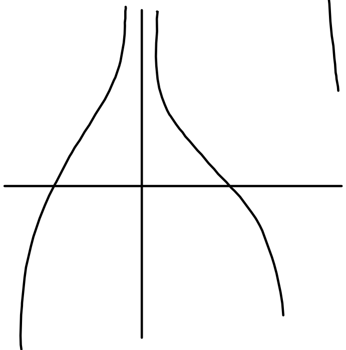
vsa gibanja so periodična



$\alpha > 0\ \beta < 0$



vsa gibanja so periodična



$\alpha < 0$
$\beta > 0$

vsa gibanja so neomejena

Računajmo periodo za $\alpha, \beta > 0$

$$U = \sqrt{|a||b|}\,(\xi^2 + \xi^{-2})$$
$\sqrt{|a||b|} = U_0$

$E_0 = U_0 e_0$ ← število

$$T = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - (ax^2 + bx^{-2})}} = \sqrt{2m}\int_{\xi_1}^{\xi_2} \frac{L\,d\xi}{\sqrt{U_0 e_0 - U_0(\xi^2 - \xi^{-2})}} =$$

$$= \sqrt{2m}\frac{L}{\sqrt{U_0}}\int_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{e_0 - \xi^2 - \xi^{-2}}} d\xi = \frac{\sqrt{2m}}{\sqrt{U_0}} L \int \frac{\xi\,d\xi}{\sqrt{\xi^2 e_0 - \xi^4 - 1}} d\xi$$

$u = \xi^2$

$$= \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1^2}^{\xi_2^2} \frac{\frac{1}{2}\,du}{\sqrt{-u^2 + ue_0 - 1}} = \sqrt{2m}\frac{L}{\sqrt{U_0}}\frac{1}{2}\pi$$

v krajiščih je ničla in imamo kvadratni koren z $a_2 = -1$

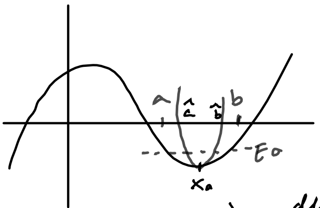
$$U_0 = \sqrt{|a||b|}$$

$$\frac{L}{\sqrt{U_0}} = \frac{|b|^{\frac{1}{4}}}{|a|^{\frac{1}{4}}\,|a|^{\frac{1}{4}}|b|^{\frac{1}{4}}} = \sqrt{\frac{1}{|a|}}$$

$$T = \sqrt{\frac{m}{2|a|}}\,\pi$$ opet je izohronično (neodvisno od energije)

in notri ni b

Harmonična optimizacija periode



$$T = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

$$U(x) = U(x_0) + \overbrace{\frac{dU}{dx}(x_0)}^{=0 \text{ ker je lok. minimum}}(x - x_0) + \frac{1}{2}\frac{dU}{dx}(x_0)(x-x_0)^2$$

$$T \approx \sqrt{2m}\int_{\hat{a}}^{\hat{b}} \frac{dx}{\sqrt{E_0 - U_0 - \frac{1}{2}U''(x_0)(x-x_0)^2}} =$$

$$= \sqrt{2m}\,\frac{1}{\sqrt{\frac{1}{2}U''(x_0)}}\,\pi = 2\pi\sqrt{\frac{m}{U''(x_0)}}$$

$$E_0 = U(x) + \frac{1}{2} m g(x) \dot{x}^2$$

$$u(x) = \int_{x_1}^{x} \sqrt{g(x)}\, dx \qquad \dot{u} = \dot{x}\sqrt{g(x)}$$

$$\dot{u}^2 = \dot{x}^2 g(x)$$

$$\frac{1}{2} m \dot{u}^2 + U(x) = E_0$$

$$\hat{U}(u) = U(x(u))$$

$$U(x) = \hat{U}(u)$$

$$U'(x) = \frac{d\hat{U}}{du}\underbrace{\frac{du}{dx}}_{=\sqrt{g(x)}} \qquad \left.\frac{d\hat{U}}{du}\right|_{u_0} = 0$$

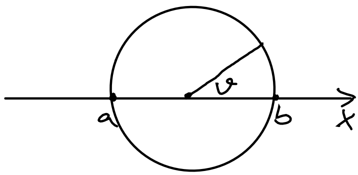
$$U''(x_0) = \frac{d^2\hat{U}}{du^2}\left(\frac{du}{dx}\right)^2 + 0 = \frac{d^2\hat{U}}{du^2}(u_0)\cdot g(x_0)$$

$\leadsto$ harmoniczna aproksymacja

$$T = 2\pi\sqrt{}$$

Libracijska aproksimacija periode

$$t-t_0 = \operatorname{sgn}(\dot x)\int_{x_0}^{x} \frac{dx}{\sqrt{(E_0-U(x))\cdot\frac{2}{m}}}$$



$$x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$$

$$b - x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$

$$x - a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$

$$E_0 - U(x) = (b-x)(x-a)\,\chi(x)$$

vemo da ima ničle v a in b, ostane pa še $\chi(x)$

$$= \operatorname{sgn}\dot x \int \frac{-\frac{1}{2}(b-a)\sin\vartheta\, d\vartheta}{\sqrt{\left(\frac{1}{2}(b-a)\right)^2 \sin^2 \chi(\vartheta)}}$$

$$= -\operatorname{sgn}(\dot x)\int_{\vartheta_0}^{\vartheta} \frac{\sin\vartheta\, d\vartheta}{|\sin\vartheta|\sqrt{\chi(\vartheta)}}$$

$$\vartheta \in (0,\pi) \underset{\vartheta}{\Rightarrow} \dot x < 0, \quad \sin\vartheta > 0$$

$$= \int_{\vartheta_0} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$$

$$\vartheta \in (\pi, 2\pi) \Rightarrow \dot x > 0 \quad \sin\vartheta < 0$$

$$= \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$$

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}} \Rightarrow T = 2\sqrt{\frac{m}{2}} \int_0^{T} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}} \doteq$$

trapezna formula

$$\doteq 2\sqrt{\frac{m}{2}}\ \frac{1}{2}\pi\left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}}\right) = \pi\sqrt{\frac{m}{2}}\left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}}\right)$$

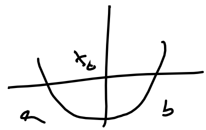
$$U(a) = E_0$$

$$U(b) = E_0$$

$$\chi(x) = \frac{E_0 - U(x)}{(b-x)(x-a)}$$

$$\chi(a) = \lim_{x \to a} \chi(x) = -\frac{U'(a)}{b-a}$$

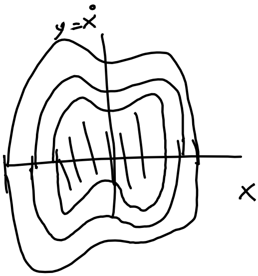
$$\chi(b) = \frac{U'(b)}{b-a}$$



$$T = \pi\sqrt{\frac{m}{2}}\sqrt{b-a}\left(\frac{1}{\sqrt{-U'(a)}} + \frac{1}{\sqrt{U'(b)}}\right) \approx 2\pi\sqrt{\frac{m}{U''(x_0)}}$$

$$U'(a) = U'(x_0) + U''(x_0)(b-a)$$

Fazni portret



$$\frac{1}{2} m \dot{x}^2 + U(x) = E_0 \rightsquigarrow$$

$$\frac{1}{2} m y^2 + U(x) = E_0$$

nivojnica v fazni ravnini

$$y = \pm \sqrt{\frac{2}{m} (E_0 - U(x))}$$

$$\frac{dy}{dx} = \pm \sqrt{\frac{2}{m}} \cdot \frac{1}{2} \frac{1 - (-U'(x))}{\sqrt{E_0 - U(x)}}$$

ploščina

$$A(E) = 2 \int_{a(E)}^{b(E)} \sqrt{\frac{2}{m} (E_0 - U(x))}$$

$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{}}$$

vstavimo b v integral oziroma a

$$\frac{dA}{dE} = 2 b'(E) \cdot 0 - 2 a'(E) \cdot 0 + 2 \sqrt{\frac{2}{m}} \int_a^b \frac{1}{2} \frac{dx}{\sqrt{E - U(x)}} = \frac{1}{m} T$$

$$\Rightarrow T = m \frac{dA}{dt}$$

Primer: Harmonični oscilator

$$U = \frac{1}{2}kx^2 \quad \text{(hookov potencial)}$$

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 = E_0$$

$$\frac{y^2}{\left(\sqrt{\frac{2E}{m}}\right)^2} + \frac{x^2}{\sqrt{\frac{2E_0}{k}}^2} = 1$$

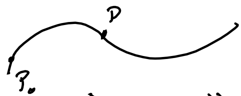
elipsa ~~s pol.~~
s polosema

ploščina
↓

$$A = \pi \cdot \frac{2E_0}{\sqrt{km}} \Rightarrow \frac{dA}{dE_0} = \frac{2\pi}{\sqrt{km}} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

izohronična
perioda

Gibanje po krivulji



$s = s(t)$

$\vec r = \vec r(s(t))$

$$\vec v = \frac{d\vec s}{dt} = \left(\frac{dr}{ds}\right)\dot s = \dot s\,\vec e_t \qquad |\vec v| = |\dot s|$$

$$\frac{dr}{ds} = \vec e_t$$

$$\vec a = \frac{d}{dt}\left(\dot s\,\vec e_t\right) = \ddot s\, e_t + \dot s\,\frac{de_t}{ds} = \dot s$$

$$\frac{d\vec e_t}{ds} = \varkappa\,\vec e_n$$

$$\frac{de_t}{ds} \perp \vec e_t$$

ukrivljenost

$$\varkappa = \frac{1}{R} \qquad R \ldots \text{krivinski polmer}$$

$$\frac{de_t}{ds} = \frac{\frac{de_t}{ds}}{\left|\frac{de_t}{ds}\right|}\left|\frac{de_t}{ds}\right|$$

normalni vektor

$$|\vec v \times \vec a| = \varkappa\,|\dot s|^3 \Rightarrow \varkappa = \frac{|\vec v \times \vec a|}{|\dot s|^3}$$

$$\vec a = \ddot s\, e_t + \varkappa\,\dot s^2\,\vec e_n$$ tangentni + normalni pospešek

$\vec{e}_b = \vec{e}_t \times \vec{e}_n$ ... binormala

$m\vec{a} = F$ $\quad$ $m\vec{a} = \vec{F} + \vec{S}$ ← rezultanta vezi

rezultanta aktivnih sil

$m\ddot{s} = \vec{F}\vec{e}_t + \vec{S}\vec{e}_t$

$m\kappa\dot{s}^2 = F\vec{e}_n + \vec{S}\vec{e}_n$ + konstruiramo mo zveze sila vezi

$0 = F\vec{e}_b + \vec{S}\vec{e}_b$

Gibanje po krivulji brez trenja: $\vec{S}\cdot e_t = 0$

Idealna krivulja, gladka krivulja

$\vec{S}\cdot e_t = -\frac{\vec{v}}{|v|} k \sqrt{|\vec{S}\cdot e_n|^2 + |\vec{S}\cdot e_b|^2} \cdot e_t$

Gibanje po gladki krivulji

$\vec{s} \cdot \vec{e}_t = 0$

$m\ddot{s} = \vec{F} \cdot \vec{e}_x$ $\qquad F = -\operatorname{grad} U$

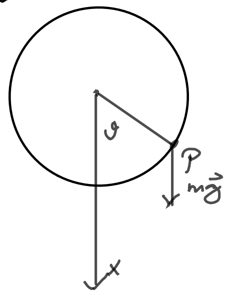
$U(s) = U(\vec{r}(s))$

$\frac{dU}{ds} = \operatorname{grad} U \frac{d\vec{s}}{ds} = -F \vec{e}_t$

$m\ddot{s}\dot{s} = \vec{F} \cdot \vec{e}_x \dot{s} \rightarrow \frac{1}{m}\dot{s}^2 + U(s) = E_0$

gibanje po gladki krivulji je integrabilno

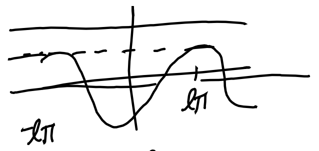
Primer:
Matematično nihalo = gibanje po gladki krožnici pod vplivom sile teže



$\vec{F}_g = m\vec{g}$

$U = -m\vec{g}\vec{r} = -m g\vec{i}\, l(\cos\vartheta\, \vec{i} + \sin\vartheta\, \vec{j}) = -m g l \cos\vartheta$

$s = l\vartheta \quad U(s) = mgl\cos(\frac{s}{l})$



Perioda matematičnega nihanja

$T = \sqrt{2m}\int_{-s_0}^{s_0} \frac{ds}{\sqrt{E_0 - U(s)}} = \qquad E_0 = -mgl\cos(\frac{s_0}{l})$

$= 2\sqrt{2m}\frac{1}{\sqrt{mgl}}\int_0^{s_0} \frac{ds}{\sqrt{\cos(\frac{s}{l}) - \cos(\frac{s_0}{l})}} \qquad s = l\vartheta$

$= \frac{2\sqrt{2}l}{\sqrt{gl}}\int_0^{\vartheta_0} \frac{l\, d\vartheta}{\sqrt{\cos\vartheta - \cos\vartheta_0}} = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\vartheta} \frac{d\vartheta}{\sqrt{\cos\vartheta - \cos\vartheta_0}}$

$1 = \cos^2\frac{\vartheta}{2} + \sin^2\frac{\vartheta}{2}$

$\cos\vartheta = \cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2}$

$1 - \cos\vartheta = 2\sin^2\frac{\vartheta}{2}$

$= 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\vartheta_0} \frac{d\vartheta}{\sqrt{2\sin^2\frac{\vartheta_0}{2} - 2\sin^2\frac{\vartheta}{2}}} =$

$u = \frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta_0}{2}}$

$\sin^2\frac{\vartheta}{2} = \sin^2\frac{\vartheta_0}{2} u^2$

$du = \frac{1}{\sin\frac{\vartheta_0}{2}}\cos\frac{\vartheta}{2}\cdot\frac{1}{2}\, d\vartheta$

$= 2\sqrt{\frac{l}{g}}\int_0^1 \frac{2\, du \cdot \cancel{\sin\frac{\vartheta_0}{2}}}{\cos\frac{\vartheta}{2}\sqrt{\cancel{\sin^2\frac{\vartheta_0}{2}}(1-u^2)}} =$

$\sin\frac{\vartheta_0}{2} = k$

$= 4\sqrt{\frac{l}{g}}\int_0^1 \frac{du}{\sqrt{1 - \sin^2\frac{\vartheta_0}{2}u^2}\sqrt{1-u^2}} =$

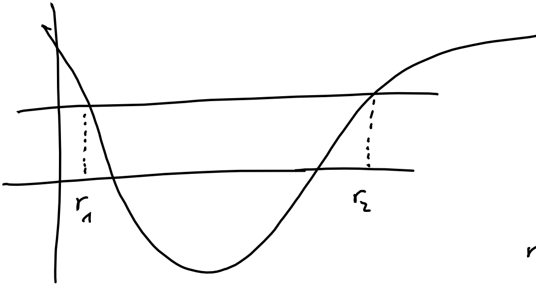
$4\sqrt{\frac{l}{g}}\int_0^1 \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}}$ $\qquad$ to je popolni eliptični integral prve vrste

$= K(k^2)$

$T = 4\sqrt{\frac{l}{g}}\, K(k^2)$

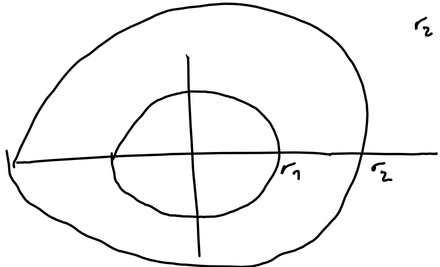
5.10

ena ura + ena ura poslušanje



r_1 pericenter

r_2 apocenter



$\vec{r} = \vec{r}(\vartheta$

$\frac{d\vec{r}}{dt} = \dot{r} e_r + r\dot{\vartheta} e_\vartheta$ Tarej se dotika dvorih krožnic

tir je simetričen

Mozgoše Filipovi zapichu ???

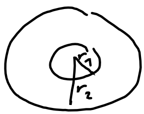
12.11.25

$\frac{\Delta\varphi}{\pi} \in \mathbb{Q} \iff$ tir je zaprt

periodično gibanje koordinate r

Trditev: Nezaprti tir "omejenega" gibanja je gosta množica nad absolutnim radijem

Dokaz:



Dovolj je dokazati, da so presečišča tira s vsako krožnico med absidnima ~~radijama~~ radijema gosta množica te krožnice

Bertrandov izrek:

Edina potenciala za katera so vsi tiri v okolici krožnega tira zaprti sta geometrijski potencial in Hookov potencial

↪ $U = \frac{1}{2}kr^2$

??
Ne vem kaj je ta beseda

Relativno gibanje

Koordinatni sistem $\varphi(t,P)$ se giblje glede na koordinatni sistem $\varphi'(t',P')$, če $\exists$ trojica

$(P_0 \in E,\ P_0' : \mathbb{R} \to E,\ U : \mathbb{R} \to SO(3))$ take da velja

$t = t'$

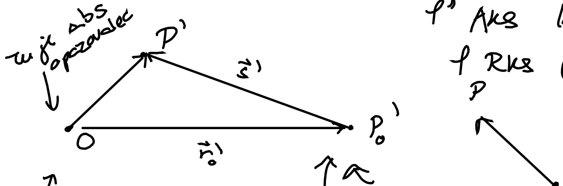
$P' = P_0'(t) + Q(t)(P - P_0)$ ← pri Galileju je bil Q konstanten

translatorno gibanje ($P_0'(t)$)

rotacijsko gibanje ($Q(t)$)

φ' Aks (absolutni KS)

φ Rks (relativni KS)



EM KS

Drug KS

relativni opazovalec, kot ga vidi absolutni opazovalec

$P_0 = 0$ za rel. opazovalca

absolutni opazovalec ve več in lahko zapiše enačbo

to vidi relativni opazovalec

Trditev: Rotacijski del gibanja je neodvisen od trajice

Dokaz: $\tilde{P}_0, \tilde{P}_0', \tilde{Q}$

$$P_0' + Q(P-P_0) = \tilde{P}_0 + \tilde{Q}(P-\tilde{P}_0) \quad \text{za } \forall P$$

$$P=P_0 \Rightarrow P_0' = \tilde{P}_0 + Q(P_0-\tilde{P}_0)$$

$$\tilde{P}_0' + \tilde{Q}(P-\tilde{P}_0) + Q(P-P_0) = \tilde{P}_0' + \tilde{Q}(P-\tilde{P}_0)$$

$$\tilde{P}_0 + \tilde{Q}(P_0-\tilde{P}_0) + Q(P-P_0) = \tilde{P}_0' + Q(P-\tilde{P}_0) \Rightarrow$$

$$Q(P-P_0) = \tilde{Q}(P-\tilde{P}_0) - \tilde{Q}(P_0-\tilde{P}_0) =$$

$$\tilde{Q}\left((P-\tilde{P}_0) - (P_0-\tilde{P}_0)\right) = \tilde{Q}(P-P_0)$$

$$\Rightarrow Q = \tilde{Q}$$

$P=P(t) \Rightarrow$

$$\dot{P}' = \dot{P}_0' + \dot{Q}(P-P_0) + Q(\dot{P})$$

$\vec{V}_{rel} = \dot{P}$ relativna hitrost

rotacijska hitrost

V_0' translatorna hitrost

$$Q^TQ = I = QQ^T$$

$$(Q^T)^{\cdot}Q + Q^T\dot{Q} = 0$$

$$\dot{Q}^TQ + Q^T\dot{Q} = 0$$

$$W = Q^T \cdot \dot{Q}$$

$$W^T = \dot{Q}^TQ$$

$W^T = -W \Rightarrow W$ je poševno simetrični tenzor

Linearna preslikava = Tenzor (drugega reda)

Trditev $W = Q^T\dot{Q}$ je poševno simetrični tenzor (p.s.t)

Trditev: Naj bo W poševno simetrični tenzor definiran na trirazsežnem vektorskem prostoru. Potem obstaja vektor w da je $W\vec{a} = \vec{\omega}\times\vec{a}$ za $\forall \vec{a}$

Dokaz:

$$W\vec{p} = \lambda\vec{p}$$

$$\vec{p}\cdot W\vec{p} = \lambda|\vec{p}|^2$$

$$W^T\vec{p}\cdot\vec{p} = -W\vec{p}\cdot\vec{p} = -\lambda|\vec{p}|^2 = \lambda|\vec{p}|^2 \Rightarrow \lambda = 0$$

$\{\vec{p},\vec{q},\vec{r}\}$ ONB

$$W\vec{p} = \vec{0}$$

$$W\vec{q} = \underbrace{(W\vec{q}\cdot p)}_{=0}\vec{p} + \underbrace{(W\vec{q}\cdot\vec{q})}_{0}\vec{q} + (W\vec{q}\cdot\vec{r})\vec{r}$$

$\vec{q}\,W\vec{p}$

$$W\vec{a} = W((\vec{a}\vec{p})\vec{p} + (\vec{a}\vec{q})\vec{q} + (\vec{a}\vec{r})\vec{r}) = \vec{a}\vec{q}\,(W\vec{q}\cdot\vec{r})\vec{r} + \vec{a}\vec{r}\,(W\vec{r}\cdot\vec{q})\vec{q}$$

$\cancel{a_q W\vec{p} + a_q Wq +}$

$p\times q = r$ in $\vec{r}\times\vec{p} = q$ ker ONB

$$= (W\vec{q}\cdot\vec{r})[\vec{a}\vec{q}\,(\vec{p}\times\vec{q}) - (\vec{a}\vec{r})(\vec{r}\times\vec{p})] =$$

$$= (W\vec{q}\cdot\vec{r})\;\vec{p}\times\underbrace{((\vec{a}\vec{q})\vec{q} + (\vec{a}\vec{r})\vec{r}]}_{\vec{a}} = (W\vec{q}\;\vec{r})(\vec{p}\times\vec{a})$$

$$(\vec{a}\cdot\vec{p})\vec{p} = 0$$

$$\vec{\omega} = (W\vec{q}\cdot\vec{r})$$

Vektorju $\vec{w}$ pravimo osni vektor p.s.t W

$$\vec{w} = \vec{w}(W)$$

$$e_{ijk} = \begin{cases} 1 & ijk \text{ je soda permutacija števil } 123 \\ -1 & ijk \text{ je liha} \\ 0 & \text{če ni permutacija} \end{cases}$$

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^{3} e_{ijk} a_j b_k \vec{e}_i = \underbrace{e_{ijk} a_j b_k e_i}_{\text{pogost zapis brez vsote}}$$

Sumacijska konvencija
če se indeks ponovi, se sumira

$$\vec{a} = \sum a_i \vec{e}_i = a_i \vec{e}_i$$

$$\vec{a} \cdot \vec{b} = a_i b_i$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{\underline{ii}} = 1$$

$$W_{ij} = -e_{ijk} w_k$$ seštevanje samo po k

$$W\vec{a} = \underbrace{W_{ij} a_j}_{} e_i \qquad w \times \vec{a} = \underbrace{e_{ijk} w_j a_k}_{} e_i$$

$$W_{ij} a_j = e_{ijk} w_j a_k \quad \text{za } \forall a$$

$$W_{ij} a_j = e_{ikj} w_k a_j$$ j in k zamenjamo

$$\Rightarrow W_{ij} = e_{ikj} w_k = -e_{ijk} w_k$$

$$e_{ijk} e_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$ velja (dokaza ne bo)

$$e_{ijk} e_{mjk} = \delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm} = 3 \cdot \delta_{im} - \delta_{im} = 2\delta_{im}$$

$$e_{ijk} e_{ijk} = 6$$

$$e_{ijp} W_{ij} = -e_{ijp} e_{ijk} \omega_k = -2\delta_{pk}\omega_k = -2\omega_p$$

$$\omega_p = -\frac{1}{2} e_{ijp} w_{ij}$$

Trditev $A \in \mathcal{L}(U,V)$

$$\underline{A^*(\vec{a} \times \vec{b}) = (A\vec{a}) \times (A\ b)}$$

A^* adjungiranka A-ja $\quad A^{-1} = \frac{1}{\det A}(A^*)^T$

Dokaz: $\det A \neq 0 \qquad \vec{a}, b, \vec{c}$ lin neod

$$\det A = \frac{[A\vec{a}, A\vec{b}, A\vec{c}]}{[\vec{a}, \vec{b}, \vec{c}]}$$

$$(A\vec{a} \times A\vec{b})\vec{c} = (A\vec{a} \times A\vec{b})(AA^{-1}\vec{c}) = [A\vec{a}, A\vec{b}, AA^{-1}\vec{c}] =$$

$$= \det A\, [\vec{a}, \vec{b}, A^{-1}\vec{c}] = \det A (\vec{a} \times \vec{b}) A^{-1}\vec{c} =$$

$$= (\det A) A^{-T}(\vec{a} \times \vec{b})\vec{c} \Rightarrow A\vec{a} \times A\vec{b} = \det(A) A^{-T}(\vec{a} \times \vec{b})$$

$$\cancel{A\vec{a} \times A} \quad A\vec{a} \times A\vec{b} = A^*(\vec{a} \times \vec{b})$$

$$\det A = 0 \Rightarrow \forall \varepsilon > 0 . \exists A_\varepsilon .\ \det A_\varepsilon = 0 .\ |A - A_\varepsilon| < \varepsilon$$

$$\lim_{\varepsilon \to 0} A_\varepsilon \vec{a} \times A_\varepsilon \vec{b} = A_\varepsilon^*(a \times \vec{b})$$

$$A_\varepsilon \to A \qquad A_\varepsilon^* \xrightarrow{?} A^*$$

če pa so lin odvisni

$$[\vec{a}, \vec{b}, \vec{c}] = \det A\, [A\vec{c}, A\vec{b}, A\vec{c}]$$ trdi velja

$A = Q \in O(3)$

$Q^* = \det(Q)(Q^{-1})^T = \det Q \, Q$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

$\quad\quad\quad\quad\quad\quad\quad (Q^T)^T = Q$

$Q \in SO(3) \quad \Rightarrow Q(\vec{a} \times \vec{b}) = (Q\vec{a}) \times (Q\vec{b})$

$W = Q^T\dot{Q}$ asnemu vektorju p.s.t pravimo rotacijski vektor

Vektorju $\vec{w}' = Q\vec{u}$ pa pravimo vektor kotne hitrosti rotacije Q

$Q^T\dot{Q}\vec{a} = \omega \times \vec{a}$

$\dot{Q}\vec{a} = Q(\omega \times \vec{a}) = \vec{\omega}' \times Q\vec{a}$

Trditev
Osni vektor p.s.t $W = \dot{Q}Q^T$ je vektor kotne hitrosti rotacije Q

Dokaz: $\dot{Q}Q^T\vec{a} = \vec{y} \times \vec{a}$

$$\left.\begin{aligned} &\dot{Q}\,\vec{a} = \vec{\omega}' \times Q\vec{a} \\ &\dot{Q}Q^T\vec{a} = \vec{\omega}' \times Q(Q^T\vec{a}) = \vec{\omega} \times \vec{a} \end{aligned}\right\} \Rightarrow \vec{y} = \vec{\omega}$$

$Q \in SO(3)$

$W_1 W_2$ ni poševno simetričen tenzor

$[W_1, W_2] = W_1W_2 - W_2W_1$ je poševno sim. tenz

komutator

$(W, +, [\,])$ je algebra

množica p.s.t

$(V, +, \times)$ je tudi algebra (Lijeva algebra)

prostor 3D vektorjev

Trditev: $\omega([W_1, W_2]) = \omega(W_1) \times \omega(W_2)$

$\omega \longmapsto W(\omega)$ je homomorfizem algeber

Dokaz $[W_1, W_2]\vec{a} = (W_1 W_2 - W_2 W_1)\vec{a} =$

$$\vec{\omega}_1 \times (\vec{\omega}_2 \times \vec{a}) - \vec{\omega}_2 \times (\vec{\omega}_1 \times \vec{a}) =$$

$$= (\vec{\omega}_1 \vec{a})\vec{\omega}_2 - \cancel{(\omega_1 \omega_2)}\vec{a} - ((\omega_2 \vec{a})\omega_1 - \cancel{(\omega_2 \omega_1)}\vec{a}) =$$

$$= (\omega_1 \times \omega_2) \times \vec{a}$$

Izrek: Naj bo W p.s.t z enotskim osnim vektorjem $\hat{e}$.

Potem je $e^{\vartheta W}$ rotacija okrog osi $\hat{e}$ za kot ϑ