Fourierova vrsta

$$FV(f)(x) = \underset{\substack{\uparrow \\ \frac{a_0}{2} \text{ na predavanjih}}}{a_0} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x\,dx$$

$FV(f)$ konvergira k f, če je f zvezna v x, če pa ni, pa konvergira k $\dfrac{f(x^-) + f(x^+)}{2}$

$$\boxed{\begin{aligned} \int x\cos(nx)\,dx &= \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C \\ \int x\sin(nx)\,dx &= -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C \end{aligned}}$$

f soda $\Rightarrow b_n = 0$

f liha $\Rightarrow a_n = 0$

$$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$$

$$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$$

6.3

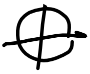
②

$f(x) = |x|$ razvij v Fourierovo vrsto in seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{\pi}\int_{0}^{\pi}x\,dx = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)\,dx \underset{\text{sodost}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi}x\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Bigg|_{0}^{\pi} =$$



$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx) \overset{\text{lihost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{-2}{(2k+1)^2}\cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi}\cdot\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{1}\frac{1}{(2k+1)^2} = -\frac{\pi}{2}\cdot\frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \overset{?}{=} S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

S ... lih.

S' ... ostalo

$$= S + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) =$$

$$= S + \frac{1}{4}\Sigma = \Sigma$$

$$S = \frac{3}{4}\Sigma$$

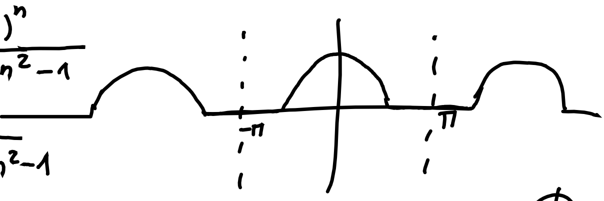
$$\Sigma = \frac{4}{3}S = \frac{\pi^2}{6}$$

③

$f(x) = \max(\cos x, 0)$

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$



⊕

$$a_0 = 2\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(x) = \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos(nx) + \cancel{2\int_{\frac{\pi}{2}}^{\pi} \cos(nx)\,dx}$$

ničelnost $f(x)$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right) =$$

$n \neq 1$

$$\frac{1}{\pi}\left[\frac{1}{n+1}\sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1}\sin\left((n-1)\frac{\pi}{2}\right)\right]$$

$$n=4k: \frac{1}{\pi}\frac{1}{n+1}\cdot 1 + \frac{1}{n-1}(-1)\frac{1}{\pi} = \frac{-2}{n^2-1}\cdot\frac{1}{\pi} \quad \frac{-2}{16k^2-1}\cdot\frac{1}{\pi}$$

n lih: 0

⊕

$$n = 4k+2: \frac{1}{\pi}\frac{1}{4k+3}\sin\frac{3\pi}{2} + \frac{1}{\pi}\frac{1}{4k+1}\sin\frac{\pi}{2} =$$

$$= \frac{1}{\pi}\left(-\frac{1}{4k+3} + \frac{1}{4k+1}\right)$$

$$= \frac{1}{\pi}\begin{cases} 0 & ;\ n \text{ liho} \\ \frac{(-1)^{m+1}\cdot 2}{(2n)^2-1} & ;\ n=2m \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{1}{\pi} + \frac{\sin(2x)}{2}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}$$

$b_n = 0$ ker f soda

$$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cdot 2}{(2m)^2-1}\cos(2mx)$$

$$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}S_1 \Rightarrow$$

$$S_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)\cdot\frac{\pi}{2}(-1) = \frac{1}{2} - \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^2-1}\cdot\underset{(-1)^m}{\underset{\parallel}{\cos(m\pi)}} =$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\underset{S_2}{\underset{\parallel}{\sum_{m=1}^{\infty}\frac{1}{4m^2-1}}}$$

$$S_2 = -\frac{1}{\pi}\cdot\frac{\pi}{2} = -\frac{1}{2}$$

ⓦ

$f(x) = x^2 \qquad f: [0,\pi] \longrightarrow \mathbb{R}$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v Sinusno FV in skiciraj njen graf

1) če f razširimo do sode funkcije

$f_s : [-\pi, \pi] \longrightarrow \mathbb{R}$

$x < 0 \Rightarrow f_s(x) = f(-x)$

$FV_{\cos}(f)(x) = FV(f)(x)$

2) $f_l : [-\pi, \pi] \longrightarrow \mathbb{R} \qquad f_l(x) = -f(-x)$

$FV_{\sin}(f)(x) = FV(f_l)(x)$

1) $f_S(x) = x^2$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_{0}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

$b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

Poglejmo:

$$\int x^2 \cdot e^{inx} =$$

$$\boxed{\int x e^{inx} = -\frac{i \cdot x \cdot e^{ix}}{n} + \frac{e^{inx}}{n^2}}$$

$u = x^2 \qquad dv = e^{inx} dx$

$du = 2x\, dx \qquad v = \frac{1}{in} e^{inx}$

$$= x^2 \cdot \frac{e^{inx}}{in} - \int 2x \frac{e^{inx}}{in} dx = \text{od prej}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \frac{2}{in}\left(-\frac{ixe^{inx}}{n} + \frac{e^{inx}}{n^2}\right) + C$$

$$= \underset{\substack{\shortparallel \\ \cos nx + i \sin nx}}{e^{inx}}\left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3}\right)$$

$$\int x^2 \cos(nx)\, dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$$

$$\int x^2 \sin(nx)\, dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$$

$$a_n = \frac{1}{\pi}\left(\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) + \frac{2}{n^3}\sin(nx)\right)\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right) = \frac{4(-1)^n}{n^2}$$

$$FV\cos(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{n^2} \cos(x$$

b) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{soda}}{f_p(x)} \sin(x) = \frac{2}{\pi} \underbrace{\int_0^{\pi} f_p(x)\sin(dx)}_{}$

$= f(x)$ na $[0,\pi] = x^2$

$$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2}{n^2}\sin nx + \frac{2}{33}(\cos(nx))\right]\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^2}(-1)^n - \frac{2}{n^3}\right)$$

$$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left((-1)^{n+1}\frac{\pi^2}{2} + \frac{2(1-(-1)^n)}{n^3}\right)\sin(nx)$$

$f(x) = x(\pi - x)$ فوریه ۲

⑥

$f(x) = \sin^3 x$ razvij v FV

Pred premislek:

$f(x) = \sin 2x$ je že FV

$b_2 = 1$, ostali so 0 0

$$f(x) = \sin^2 x = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right) =$$

$$= \frac{1}{2}\left(\cos(0) - \cos(2x)\right) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2}\sin x - \frac{1}{2}\cos 2x \sin x =$$

$$= \frac{1}{2}\sin x - \frac{1}{2}\cdot\frac{1}{2}\left(\sin(3x) - \sin(x)\right) =$$

$$= \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

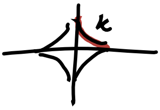
polinom

$\forall p(\sin x, \cos x)$ ima končno furierovo vrsto

$p \in \mathbb{R}[x, y]$

$a>0$ Izračunaj težišče homogenega

loka astroide



$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

$$x_T=\frac{\int_K x\,dm}{m(K)}=\frac{\int_K x\,\rho\,ds}{\int_K \rho\,ds}\quad = l(K)$$

$\vec{r}(t)$ ... parametrizacija

$$ds=|\dot{\vec{r}}(t)|dt=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt$$

u ... skalarno polje

$$\int_K u\,ds=\int_a^b u(\vec{r}(t))|\dot{\vec{r}}(t)|dt$$

$x=a\cos^3 t$ $\qquad t\in[0,\frac{\pi}{2}]$ ← ker je vse pozitivno (gledamo pravi lok)

$y=a\sin^3 t$

$$\dot{\vec{r}}(t)=(-3a\cos^2 t\cdot\sin t,\ 3a\sin^2 t\cos t)$$

$$|\dot{\vec{r}}|=3a\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}=$$

$$=3a\cos t\sin t$$

$$l(K)=\int_0^{\frac{\pi}{2}}|\dot{\vec{r}}(t)|dt=\int_0^{\frac{\pi}{2}}3a\cos t\sin t\,dt=$$

$u=\sin t\,dt$

$du=\cos t\,dt$

$$=3a\int_0^1 u\,du=\frac{3}{2}a$$

$$\int_K x\,ds=\int_0^{\frac{\pi}{2}}a\cos^3 t\ 3a\cos t\sin t\,dt=$$

$\cos t=u \qquad du=-\sin t$

$$=3a^2\int_0^1 u^4\,du=\frac{3}{5}a^2$$

$$y_T=\frac{3}{5}a^2$$

$a>0 \quad \alpha \in [0, 2\pi]$

$K = S(0,a) \cap \Pi$

$\Pi : y = x \tan\alpha$

$I : \int_K (y-z) + (z-x)dy + (x-y)dz$

$\vec{f}(x,y,z) = (y-z,\ z-x,\ x-y)$ vektorsko polje

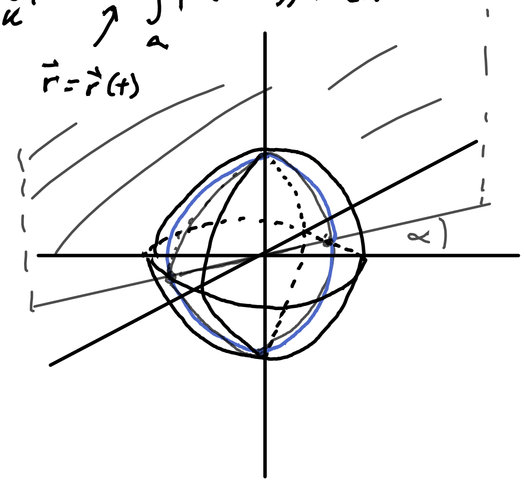
$I = \int_K \vec{f}\, d\vec{r}$

$\vec{r} = (x,y,z) \Rightarrow d\vec{r}$

$$\int_K \vec{f}\, d\vec{r} = \int_a^b \vec{f}(\vec{r}(t))\, \dot{\vec{r}}(t)\, dt$$

$\nearrow$

$\vec{r} = \vec{r}(t)$



Parametrizacja K

$x = a\cos\alpha \cdot \cos\vartheta$

$y = a\sin\alpha \cos\vartheta$

$z = a\sin\vartheta$

$r(\vartheta) = (a\cos\vartheta\cos\alpha, a\cos\vartheta\sin\alpha, a\sin\vartheta)$

$\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\dot{\vec{r}}(\vartheta) = (-a\sin\vartheta\cos\alpha, -a\sin\vartheta\sin\alpha, a\cos\vartheta)$

$\vec{f}(\vec{r}(\vartheta)) = a(\cos\vartheta\sin\alpha - \sin\vartheta,\ \sin\vartheta - \cos\vartheta\cos\alpha,$
$\cos\vartheta\sin\alpha - \cos\vartheta\cos\alpha)$

$\vec{f} \cdot \dot{\vec{r}} = a^2(-\cos\alpha\cancel{\cos\vartheta\sin\alpha\sin\vartheta} + \sin^2\vartheta\cos\alpha\, \cancel{\cos\alpha}$

$- \sin^2\vartheta\sin\alpha + \cos\vartheta\cancel{\cos\alpha\sin\vartheta}\sin\alpha +$

$+ \cos^2\vartheta\sin\alpha - \cos^2\vartheta\cos\alpha) =$

$= a^2(\cos 2\vartheta \sin\alpha - \cos 2\vartheta\cos\alpha)$ ✗

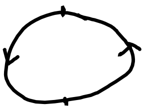
Ne

$= a^2(\cos\alpha - \sin\alpha$

$$\int_K \vec{f}\, d\vec{r} = \int_0^{2\pi} a^2(\cos\alpha - \sin\alpha)\, d\vartheta =$$

$2\pi a^2(\cos\alpha - \sin\alpha)$

$\uparrow$ pri orientaciji



$a, b, c \in \mathbb{R}$

$$I = \int_S a\,dy\,dz + b\,dx\,dz + c\,dx\,dy$$

$$S = \{(x,y,z) \,;\, x^2+y^2+z^2=1 \,,\; x,y,z \geq 0\}$$

$$I = \int \vec{f}\, d\vec{S}$$

$$\vec{f} = (a,b,c)$$

$\swarrow$ enotske normale

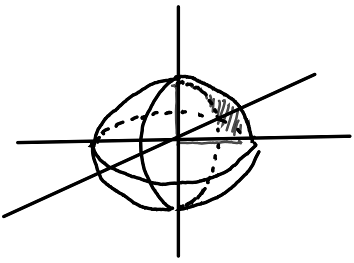
$$\int_S (\vec{f} \cdot \vec{N})\, dS$$

orientacija ploskve je usmerjena s smerjo normale

$$\int_S \vec{f}\, d\vec{S} = \int_D \vec{f}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$\nearrow$ $r = \vec{r}(u,v)$

$\nwarrow$ smer se ujema s predpisano orientacijo



Sferične koordinate

$x = \cos\vartheta \cos\varphi \qquad \vartheta \in [0, \frac{\pi}{2}]$

$y = \cos\vartheta \sin\varphi \qquad \varphi \in [0, \frac{\pi}{2}]$

$z = \sin\vartheta$

$$\vec{r}(\vartheta, \varphi) = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, \sin\vartheta)$$

$$\vec{r}_\vartheta = (-\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\vartheta)$$

$$\vec{r}_\varphi = (-\cos\vartheta\sin\varphi, \cos\vartheta\cos\varphi, 0)$$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta\cos^2\varphi - \cos\vartheta\sin\vartheta\sin^2\varphi) =$$

$$= (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta) =$$

$$= -\cos\vartheta\,(\underbrace{\cos\vartheta\cos\varphi, -\cos\vartheta\sin\varphi, \sin\vartheta}_{\Rightarrow})$$

$$\int_S \vec{f}\, d\vec{S} = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (-a\cos^2\vartheta\cos\varphi + b\cos^2\vartheta\sin\varphi - c\sin\vartheta\cos\vartheta)\, d\vartheta$$

$$= \int_0^{\frac{\pi}{2}} \int \left((-a\cos\varphi + b\sin\varphi)\cos^2\vartheta \overset{-}{\cancel{+}} \frac{c}{2}\sin 2\vartheta\right) d\vartheta =$$

Prevedemo na B funkcijo

$$= \int_0^{\frac{\pi}{2}} (-a\cos\varphi + b\sin\varphi)\frac{\pi}{4} - \frac{c}{2}) =$$

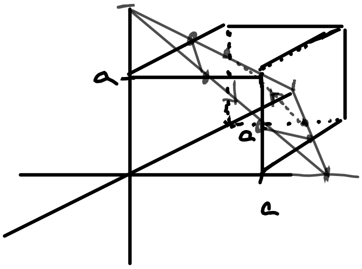
$$= \left(-\frac{\pi a}{4}\sin\varphi + \frac{\pi}{4} b\cos\varphi - \frac{c}{2}\varphi\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4} a - \frac{\pi}{4} b - \frac{\pi}{4} c = -\frac{\pi}{4}(a+b+c)$$

$a>0$

$\vec{f}(x,y,z)=(y^2-z^2, z^2-x^2, x^2-y^2)$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z=\frac{3a}{2}$



krivulja 6 kotnik

cirkulacija.... integral vektorskega polja vzdolž sklenjene krivulje

K_1:

$x=x$

$y=0$

$z=\frac{3a}{2}-x$

$\vec{r}(x)=(x,0,\frac{3a}{2}-x)$

$$\int_{K_1} \vec{f}\,dr$$

$\dot{\vec{r}}(x)=(1,0,-1)$

$\vec{f}(r(x)= (0-(\frac{3}{2}-x)^2, (\frac{3}{2}a-x)^2-x^2, x^2-0)$

$\vec{f}\cdot\dot{\vec{r}} = -(\frac{3}{2}a-x)^2-x^2 = -2x^2+3ax-\frac{9}{4}a^2$

$$\int_{\frac{a}{2}}^{a}(-(\frac{3}{2}a-x)^2-x^2)\,dx =$$

$$-\frac{1}{3}(\frac{3}{2}a-x)^3\Big|_{\frac{a}{2}}^{a} - \frac{1}{3}x^3\Big|_{\frac{a}{2}}^{a} =$$

$$= -\cancel{\frac{1}{3}\frac{a^3}{8}} + \cancel{\frac{1}{3}a^3} - \cancel{\frac{1}{3}a^3} + \cancel{\frac{1}{3}\frac{a^3}{8}} = 0$$

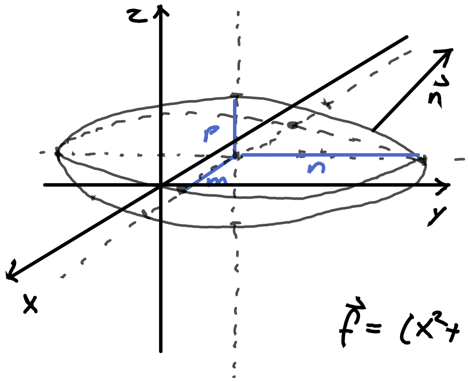
$m, n, p > 0$

$a, b, c \in \mathbb{R}$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanja stran te ploskve



$\vec{f} = (x^2 + y^2 + z^2)$

$$\boxed{I = \int_{\underset{\underset{\partial D}{\|}}{S}} \vec{f} dS = \int_D \operatorname{div} \vec{f} dV}$$

gaussov izrek

Normala mora biti zunanja

$$\int_D (2x + 2y + 2z) dV = 2x_T + 2 \cdot y_T + 2z_T$$

$$x_T = \int_D x dV = x_T \cdot V(D) = a V(D)$$

enotska krogla B: $V(B) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

$$\Downarrow$$

$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3}\pi$$

D dobimo če B raztegnemo za faktor m v smeri: x, n, p v drugih dveh

B opišemo:

$x = r \cos\vartheta \cos\varphi$

$y = r \cos\vartheta \sin\varphi$

$z = r \sin\vartheta$

$r \in [0, 1]$

D opišemo:

$x = m r \cos\varphi \cos\vartheta$

$y = n r \cos\vartheta \sin\varphi$

$z = p r \sin\vartheta$

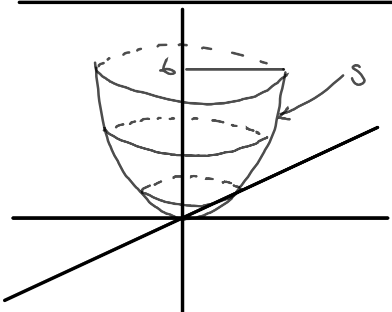
$$I = 2 \frac{4}{3}\pi m \cdot n \cdot p (a + b + c)$$

$\vec{f}(\vec{r}) = |\vec{r}|^2 \cdot \vec{r} \quad b > 0$

izračunaj pretok $\vec{f}$ skozi

a) rob območja $D = \{x,y,z;\ 2z \geq x^2+y^2;\ z \leq b$

b) ploskev $2z = x^2+y^2;\ z \leq b$



Pretok skozi S:

$$\int_S \vec{f}\, d\vec{S}$$

a)

$$\int_{\partial D} \vec{f}\, dS = \int_D \operatorname{div} \vec{f}\, dV$$

$$\vec{f}(x,y,z) = (x^2+y^2+z^2)(x,y,z) =$$

$$(x(x^2+y^2+z^2),\ y(x^2+y^2+z^2),\ z(x^2+y^2+z^2)) =$$

$$(x^3+y^2x+z^2x,\ x^2y+y^3+z^2y,\ x^2z+y^2z+z^3)$$

$$\operatorname{div}\vec{f} = 3x^2+y^2+z^2 + x^2+3y^2+z^2+ y^2+x^2+3z^2$$

$$= 5x^2+5y^2+5z^2$$

$$\int_D \operatorname{div} \vec{f}\, dV = 5\int_D (x^2+y^2+z^2)\, dV$$

$x = r\cos\varphi \qquad r^2 = x^2+y^2 = 2z$

$y = r\sin\varphi \qquad z = \frac{r^2}{2}$

$z = z =$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^{b} (r^2+z^2)\, r\, dz$$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \frac{1}{3}z^3\right)\Big|_{\frac{r^2}{2}}^{b} dr$$

$$= 10\pi \int_0^{\sqrt{2b}} \left(br^3 + \frac{1}{3}b^3 - \frac{r^5}{2} - \frac{1}{6}r^6\right) dr =$$

$$= 10\pi \left(\frac{1}{4}br^4 + \frac{1}{3}b^3 r - \frac{1}{12}r^6 - \frac{1}{6\cdot 7}r^7\right)\Big|_0^{\sqrt{2b}} =$$

$$= 10\pi \left(\frac{1}{4}b^3\cdot 4 + \frac{1}{3}b^3\sqrt{2b} - \frac{1}{12}\cdot 8b^3 - \frac{1}{6\cdot 7}\cdot 8b^3\sqrt{2b}\right)$$

nerabe

$$= 10\pi\left(b^3 + \frac{b^4}{3} - \frac{2}{3}b^3 - \frac{b^4}{12}\right) =$$

$$10\pi\left(\frac{b^3}{3} + \frac{b^4}{4}\right)$$

b) $\partial D = S \cup S_0$

$$\int_S \vec{f}\, dS = \int_{\partial D} \vec{f}\, d\vec{S} - \int_{S_0} \vec{f}\, dS$$

$$\int_{S_0} \vec{f}\, dS = \int_{S_0} (\vec{f}\cdot\vec{n})\, dS = \qquad \vec{n} = (0,0,1)$$

$$= \int_{S_0} (2x^2+2y^2+z^3)\, dS = \int_{S_0} (bx^2+by^2+b^3)\, dS =$$

$x = r\cos\varphi$

$y = r\sin\varphi$

$$= b\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r(r^2+b^{\overset{2}{\cancel{3}}})\, dr = b\left(\int_0^{2\pi} d\varphi \int_a^{\sqrt{2b}} r^3 dr + b^2\underbrace{\int_{S_0} \cancel{b^4}\, dS}_{P(S_0) = 2\pi b}\right)$$

$$= b\left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_a^{\sqrt{2b}} r^3 dr\right) =$$

$$= b\left(2\pi b^3 + \cancel{\int_a^{2\pi}} 2\pi\, \frac{1}{4}\cdot 4b^2\right) =$$

$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \cancel{\int dS} \int_S \vec{f}\, d\vec{S} = \frac{10}{3}\pi b^3 + \frac{10}{3}\pi b^4 - 2\pi b^4 - 2\pi b^3$$

Izračunaj $I = \int_K \frac{x\,dy - y\,dx}{x^2+y^2}$

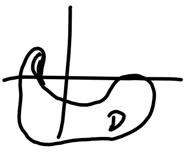
če je $K \subseteq \mathbb{R}^2$ sklenjena krivulja;

a) ki ne obkroži izhodišča

b) ki obkroži izhodišče

Omejimo se na primer: $K = \partial D$ za D ^{2}odsekoma gladkim robom

a)



$$\boxed{\int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx\,dy}$$

↑ pozitivna orientacija

↗ greenova formula

$P = \frac{-y}{x^2+y^2}$ $\quad Q = \frac{x}{x^2+y^2}$

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^4} = \frac{y^2-x^2}{(x^2+y^2)^4}$$

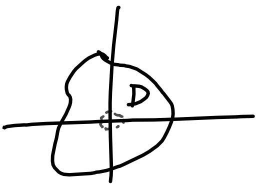
$$P_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\int_D (Q_x - P_y)\,dx\,dy = \int_D 0\,dx\,dy = 0$$

∂D ne obkroži $(0,0) \Leftrightarrow (0,0) \notin D$

$(x^2+y^2) \neq 0$

b)



iz D izrežemo dovolj majhen krog $K(0,\varepsilon)$

$$D' = D - K(0,\varepsilon)$$

Uporabimo greenovo formulo na D'

$$\iint_{D'} (Q\cancel{x} + P_{dy}\cancel{dx}) = \int_{\partial D'} (Q_x - P_y)\,dx\,dy = 0$$

$$\partial D' = \partial D^+ + \partial K(0,\varepsilon)^-$$

$$0 = \int_{\partial D^+} P\,dx + Q\,dy + \int_{\partial K^-} P\,dx + Q\,dy$$

Pozor orientacije obratna kot ponavadi

$$\Longrightarrow \int_{\partial D^+} \dots = \int_{\partial K^+} \dots$$

$x = \varepsilon \cos\varphi$ $\quad y = \varepsilon \sin\varphi$

~~$d\varepsilon = \varepsilon$~~

$dx = -\varepsilon \sin\varphi$

$dy = \varepsilon \cos\varphi$

$$\int_{\partial K} \dots = \frac{1}{\varepsilon}\int_0^{2\pi} \cos^2\varphi + \sin^2\varphi = 2\pi$$

$r_1 \dots r_n \in \mathbb{R}^3$ različni

$e_1 \dots e_n \in \mathbb{R}$

$$f(\vec{r}) = \sum_{i=1}^{n} \operatorname{grad}\left(\frac{-e_i}{4\pi |r-r_i|}\right.$$

izračunaj pretok $\vec{f}$ skozi zaključeno ploskev S, ki objame točke $r_1 \dots r_n$'

$r_1 \dots r_n \in \text{Int} D) \quad S = \partial D$

$$\int_{\partial D} \vec{f}\, d\vec{S} = \int_{D} \operatorname{div} \vec{f}\, dV$$

$\uparrow$

gaussov izrek vrjetno ne velja,

ker   ker ima f singularne točke v D

$$\operatorname{div}(\vec{f}(r)) = \sum_{i=1}^{n} \operatorname{div}\left(\operatorname{grad}\left(\frac{-e_i}{4\pi |r-r_i|}\right)\right)$$

Spomnimo se: $\operatorname{grad} \frac{1}{|\vec{r}-\vec{a}|} = -\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3}$

in $\operatorname{div}\left(\operatorname{grad} \frac{1}{|\vec{r}-\vec{a}|}\right) = 0$

Iz D izrežemo krogle $K_i = K(r_i, \underset{\underset{0}{\vee}}{\varepsilon})$

Tako da $\overline{K}_i \subset \text{Int}(D)$ in $K_i \cap K_j = \emptyset$ za $i \neq j$

$D' = D - (\bigcup_{i=1} K_i)$

Na D' uporabimo gaussov izrek

$$\int_{\partial D'} f'(\vec{r})\, dS = \int_{D'} \operatorname{div} \vec{f}\, dV = 0$$

$$\partial D' = \partial D^{+} \cup \left(\bigcup_{j=1}^{n} \partial K_i^{-}\right) \Rightarrow \int_{\partial D'} = \int_{\partial D^{+}} \dots + \sum_{i=1}^{n} \int_{\partial K_i^{-}} \dots = 0$$

$$\Rightarrow \int_{\partial D^{+}} \dots = \sum_{i=1}^{n} \int_{\partial K_i^{+}} \dots$$

$$\int_{\partial K_i} \vec{f}\, dS = \int_{K_i} \sum_{j=1}^{n} \operatorname{grad}\left(\frac{-e_j}{4\pi |r-r_j|}\right) \cancel{dS}$$

$$\sum_{j=1}^{n} \int_{\partial K_j} \left(\frac{+e_j}{4\pi} \frac{r-r_j}{|\vec{r}-\vec{r}_j|^3}\right) dS$$

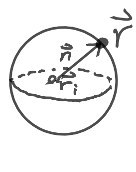
$$\int_{\partial K_j} \frac{\vec{r}-r_j}{|\vec{r}-\vec{r}_j|^3}\, dS \xRightarrow{i \neq j} r_j \in \overline{K}_i \Rightarrow$$

lahko uporabimo gaussov izrek

$\Longrightarrow \quad = 0$

$$\sum_{j=1}^{n} \int_{\partial K_i} \frac{e_j}{4\pi} \frac{\vec{r}-\vec{r}_j}{|\vec{r}-\vec{r}_j|^3}\, d\vec{S} = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}\, d\vec{S}$$

$d\vec{S} = \vec{n}\, dS$



$$\vec{n} = \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|}$$

$$= \frac{e_i}{4\pi} \int_{\partial K_i} \frac{|\vec{r}-\vec{r}_i|^2}{|\vec{r}-\vec{r}_i|^4}\, dS = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{dS}{|r-r_i|^2} =$$

$\nearrow$

na ∂K_i je

$|r-r_i| = \varepsilon$

$$= \frac{e_i}{4\pi} \int \frac{dS}{\varepsilon^2} =$$

$$\frac{e_i}{4\pi\varepsilon^2} P(\partial K_i) = \frac{e_i}{4\pi\varepsilon^2} 4\pi\varepsilon^2 = e_i$$

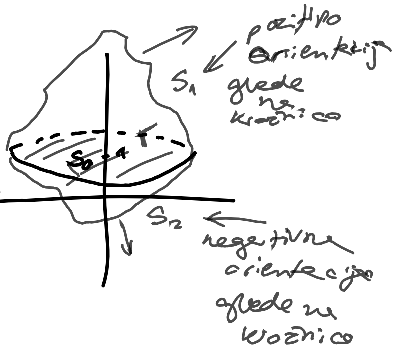
$$\Rightarrow \int f\, d\vec{S} = \sum_{i=1}^{n} e_i$$

$$I = \int_S (1+x^2) f(x)\, dy\,dz - 2xy\, f(x)\, dz\,dx - 3z\, \overset{dx\,dy}{~~dx\,dy~~}$$

Določi: $f \in C^1$ da I enak za vse ploskve S, katerih rob je krožnica $\{(\cos t, \sin t, 1)\}$
Določi to vrednost za I

$$\vec{f} = ((1+x^2)f(x),\ 2xy\, f(x),\ -3z)$$

$$\Longrightarrow I = \int_S \vec{f}\, d\vec{S}$$



$S_1 \cup S_2$ je sklenjena ploskva

$S_1^+ \cup S_2^- = \partial D^+$

Gauss na D:

$$\int_{\partial D^+} \vec{f}\, d\vec{S} = \int_D \operatorname{div} \vec{f}\, dV$$

$$\partial D = \int_{S_1} - \int_{S_2} = 0 \Longleftarrow \operatorname{div} \vec{f}\, dV = 0$$

$\vec{f}$ je solenoidalno $(\operatorname{div} \vec{f} = 0) \Longleftrightarrow f(x)$ je oblike $3\arctan x + C$

$$\int_S \vec{f}\, d\vec{S} = \int_S ((1+x^2)(3\arctan x + C),\ -2xy(3\arctan x + C),\ -3z)\, d\vec{S}$$

$$= \int_{S_0} \vec{f}\, dS =$$

samo krog: $d\vec{S} = \vec{n} \cdot dS \quad \vec{n} = (0,0,1)$

$$= \int_{S_0} -3z\, dS = -3 \int_{S_0} z\, dS = -3 z_T\, P(S_0) =$$

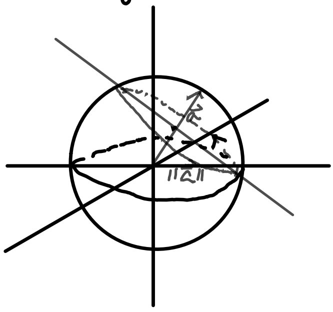
$$= -3 \cdot 1 \cdot \pi$$

Opomba: Lahko bi dokazali, da je tudi $\operatorname{div}\vec{F}$ potreben pogoj

$\vec{a} \neq \vec{0}$,
$\vec{b} \in \mathbb{R}$

$$\vec{F}(\vec{r}) = (\vec{r}-\vec{a}) \times (\vec{r}-\vec{b})$$

izračunaj cirkulacijo $\vec{F}$ vzdolž krivulje K: $|\vec{r}| = |\vec{a}| \wedge \vec{r}\cdot\vec{a} = \frac{|\vec{a}|^2}{2}$



enačba ravnine:

$$\vec{r}\cdot\vec{n} = d$$
$\nwarrow$ normala

ravnina z normalo $\vec{a}$ in točka $\vec{r} = \frac{\vec{a}}{2}$

K je krožnica

B naj bo krog z robom ki $K = \partial B$

$$\int_{\partial B} \vec{f}\, d\vec{r} \underset{\uparrow}{=} \int_{B} \operatorname{rot}\vec{f}\, d\vec{S}$$

Stokesov izrek

normala: B: $\vec{n} = \frac{\vec{a}}{\|\vec{a}\|}$

$$\operatorname{rot}\vec{f} = \operatorname{rot}(\underbrace{\vec{r}\times\vec{r}}_{0} - a\times r - \vec{r}\times\vec{b} + \vec{a}\times\vec{b}) =$$

$$\boxed{\text{Vemo: } \operatorname{rot}(\vec{r}\times\vec{c}) = -2\vec{c}}$$

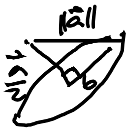
$$= -2\vec{a} + 2\vec{b} + 0 = 2(\vec{b}-\vec{a})$$

$$I = \int_B 2(\vec{b}-\vec{a})\, d\vec{S} = \frac{2}{\|\vec{a}\|}\int_B (\vec{b}-\vec{a})\cdot\vec{a}\, dS =$$

$$= \frac{2(\vec{b}-\vec{a})\cdot\vec{b}}{\|\vec{a}\|} \cdot P(B)$$

$P(B) = ?$

polmer:



$$b = \sqrt{\|a\|^2 - \frac{\|a\|^2}{4}} = \frac{\sqrt{3}}{2}\|a\|$$

$$P(B) = \pi b^2 = \pi\cdot\frac{3}{4}\|a\|$$

$$I = 3\pi(\vec{b}-\vec{a})\vec{b}\|a\|$$

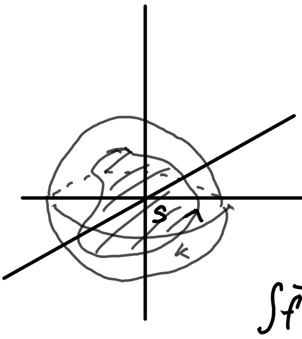
$\nwarrow$ ni isti $\vec{b}$ kot b

3.4

K zaključena krivulja
na sferi $x^2+y^2+z^2=1$

Izračunaj $I = \int_K \frac{dx+dy+dz}{(x^2+y^2+z^2)^2} = \int_K \vec{f}\, d\vec{r}$

$$\vec{f} = \left(\frac{1}{(x^2+y^2+z^2]^2}, \frac{1}{(x^2+y^2+z^2]^2}, \frac{1}{(x^2+y^2+z^2]^2}\right)$$



$\vec{f}(\vec{r}) = (1,1,1)$ za $r \in \vec{S}$

$K = \partial S$

$$\int_K \vec{f}\, d\vec{r} = \int_S \operatorname{rot} \vec{f}\, d\vec{S} = \ldots$$

obstaja boljši način

$$\int_K \vec{f}\, d\vec{r} = \int_K \underbrace{(1,1,1)}_{\vec{g}(\vec{r})} d\vec{r} = \int_S \operatorname{rot}(\vec{g}(\vec{r}))\, d\vec{S} =$$

$$= \int_S 0\, d\vec{S} = 0$$

Kaj če $K \neq \partial S$

Če je polje $\vec{f}$ potencialno ($\vec{f} = \operatorname{grad} u$) in K krivulja z začetkom v a in koncem v b $\Rightarrow \int_K \vec{f}\, d\vec{r} = u(b) - u(a)$

Posledica: če je K sklenjena je integral vzdolž K od $\int_K \operatorname{grad} u\, d\vec{r} = \vec{0}$

$\vec{f}$ ni potencialno, ampak $\vec{g}$ je potencialno (ker $\operatorname{rot} \vec{g} = 0$)

$\vec{g} = \operatorname{grad}(x+y+z)$ $I = \int_K \vec{g}\, d\vec{r} = 0$

Opomba: $\int_K h(x^2+y^2+z^2)(dx+dy+dz)$

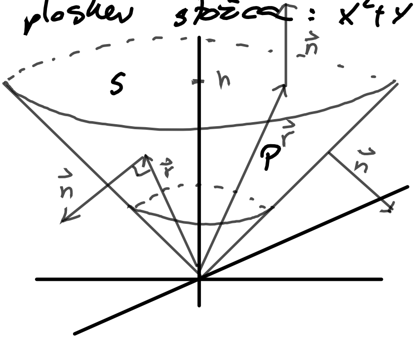
$\Big\downarrow$ na S,

konstantno polje

$h > 0$

$\vec{f}(\vec{r}) = \vec{r}$

Izračunaj pretok $\vec{f}$ skozi plašč in osnovno ploskev stožca: $x^2+y^2 \leq z^2 \quad z \in [0, h]$



$\vec{r} \cdot \vec{n} = 0$

$$\int_P \vec{f}\, dS = \int_P \vec{f}\, \vec{n}\, dS = \int_P \vec{r}\, \vec{n}\, dS = \int 0\, dS = 0$$

$$\int_S \vec{f}\, d\vec{S} = \int_S \vec{r}\, \vec{n}\, dS = \int_S z\, dS = z_T \cdot P(S) =$$

$\vec{n} = (0,0,1) \qquad = h \cdot \pi h^2 = \pi h^3$

2. način (z gaussom)

$$\int \vec{f}\, dS = \int_D \operatorname{div} \vec{f}\, dV = \int_D (1+1+1)\, dV = 3P(V) =$$

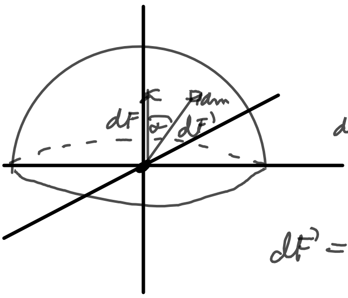
$$= 3 \cdot \frac{\pi h^2 \cdot h}{3} = \pi h^3$$

$$\int_{\partial D} = \int_S + \int_P$$

$a > 0$

Določi privlačno silo med točkastim telesom z maso m_0, ki je v izhodišče in ploskvijo $x^2+y^2+z^2=a^2$ ≥ 0 z maso M

↑ homogena



$$F = \frac{m_1 m_2}{r^2} G$$

$$dm = \rho dS$$

$$dF' = \frac{m_0 dm}{r^2} G = \frac{m_0 \rho dS}{r^2} G$$

Rezutanta bo kazala gor torej nas zanima samo z komponenta

$$dF = dF \cos\alpha$$

$$x = a\cos\vartheta\cos\varphi \qquad \alpha = \frac{\pi}{2} - \vartheta$$
$$y = a\cos\vartheta\sin\varphi$$
$$z = a\sin\vartheta$$

$$dF = \frac{\overset{\}}{G} m_0 \rho}{r^2} \cos\left(\frac{\pi}{2} - \vartheta\right) dS$$

$$dS = a^2 \cos\vartheta \, d\varphi \, d\vartheta$$

↑ od prej z $\sqrt{EG-F^2} = a^2 \cos\vartheta$

$$F = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \frac{G m_0 \rho}{a^2} \sin\vartheta \, a^2 \cos\vartheta \, d\vartheta =$$

$$= G m_0 \rho \, 2\pi \int_0^{\frac{\pi}{2}} \sin\vartheta \cos\vartheta \, d\vartheta =$$

$$G m_0 \rho \, \cancel{2}\pi \int_0^{\frac{\pi}{2}} \sin 2\vartheta \, d\vartheta = \frac{\pi}{2} G m_0 \rho \cos 2\vartheta \Big|_0^{\frac{\pi}{2}} = \quad \oplus$$

$$= \pi G m_0 \rho = \frac{G m_0 \cdot \pi}{2a^2}$$

↑

$$\rho = \frac{M}{P(S)} = \frac{M}{2\pi a^2}$$

$D \subseteq \mathbb{R}^3$ gaussasto območje (območje z gladkim robom) in prostornino V

$a \in \mathbb{R}^3$

Izračunaj:

$$\int_{\partial D} (\vec{r} \times \vec{a}) \times d\vec{S} = I$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{y} \cdot \vec{z})\vec{x}$$

$$I = \int_{\partial D} (\vec{r} \cdot d\vec{S})\vec{a} - (\vec{a} \cdot d\vec{S})\vec{r} =$$

$$d\vec{S} = \vec{n} \cdot dS$$

$a = (a_1, a_2, a_3)$

$r = (x, y, z)$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n})\vec{a} - (\vec{a} \cdot \vec{n})\vec{r})\, dS =$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n})dS)(a_1, a_2, a_3) - (\vec{a}\vec{n}x, \vec{a}\vec{n}y, \vec{a}\vec{n}z)dS$$

$$= \left(\int_{\partial D} (\vec{r} \cdot \vec{n})\, dS\right)\vec{a} - \underbrace{\int(\underbrace{\vec{a}\vec{n}x\, dS}_{x\vec{a} \cdot \vec{n} dS}, \vec{a}\vec{n}y\, dS, \vec{a}\vec{n}z\, dS)}$$

$$\left(\int(x\vec{a} \cdot \vec{n}\, dS, \int y\vec{a}\vec{n}\, dS, z\vec{a}\vec{n}\, dS\right)$$

$$\int_{\partial D} (\vec{r}\vec{n})\, dS = \int_{\partial D} \vec{r}\, d\vec{S} = \int_D \operatorname{div} \vec{r}\, dV =$$

$$= 3\int_D dV = 3V$$

$$\int_{\partial D} x\vec{a}\vec{n}\, dS = \int_{\partial D} x\vec{a}\, d\vec{S} = \int_D \operatorname{div}(x\vec{a})\, dV = a_1 V$$

$$\operatorname{div} x\vec{a} = \nabla(xa_1, xa_2, xa_3) = a_1$$

Podobno y in

$$I = 3V\vec{a} - (a_1V, a_2V, a_3V) = 2\vec{a}V$$

Dokaži da je za

$$\vec{f} = (2x\cos y - y^2\sin x, 2y\cos x - x^2\sin y, 4)$$

$\int_K \vec{f}\,d\vec{r}$ enak za vse krivulje med $(0,0,0)$ in $(5,3,\pi)$

in izračunaj

$\int_{K_0} \vec{f}\,d\vec{r}$ za $K_0: \vec{r}(t) = (\cos t, \sin t, t)$; $t \in [-2\pi, 2\pi]$

Dokazujemo: $\vec{f}$ je potencialno

(smo že naredili na začetku)

$u = x^2\cos y + y^2\cos x + 4z$

$\uparrow$ potencial za $\vec{f}$

$$\int_{K_0} \vec{f}\,d\vec{r} = u(b) - u(a) = u(1, 0, 2\pi) - u(1, 0, -2\pi$$

$$= 4\cdot 2\pi - (-4\cdot 2\pi) = 16\pi$$

10. 9

$$I = \int_K u(x,y)(y\,dx + x\,dy)$$

$$\vec{f}(x,y) = y, x)$$

$$\vec{g} = u \cdot \vec{f} = u(x,y) \cdot y,\ u(x,y)\, x)$$

Radi bi da je $\vec{g}$ potencialno polje

$$\vec{g} = \operatorname{grad} v = (v_x,\ v_y$$

$$v_x = u(x,y) \cdot y \qquad v_y = u(x,y) \cdot x$$

če odvajamo
eno po x enkrat pa po y mora priti enako

$$v_{xy} = u(x,y) + u_y y \qquad v_{yx} = u_x x + u(x,y)$$

$$\rightsquigarrow \quad y u_y = x u_x$$

- Ta pogoj lahko dobimo z greenom
- vlozimo v ravnine v $\mathbb{R}^3$

$$t = x \cdot y$$
$$s = \frac{x}{y}$$

$$I = \int_K yu\,dx + xu\,dy + 0\,dz$$

$\vec{h} = (yu, xu, 0)$ je potencialno $\Leftrightarrow$ $\operatorname{rot}\vec{h} = 0$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} = y\frac{\partial}{\partial t} + \frac{1}{y}\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial y} = x\frac{\partial}{\partial t} - \frac{x}{y^2}\frac{\partial}{\partial s}$$

$$x\left(yu_t + \frac{1}{y}u_s\right) - y\left(xu_t - \frac{x}{y^2}u_s\right) = 0$$

$$tu_t + su_s - tu_t + su_s = 2su_s \qquad s > 0$$

$u_s = 0 \Rightarrow u = u(t)$ ... samo od t odvisen

$u = h(t)$

$u = h(x \cdot y)$ kjer $h \in C^1$

$u, l \in C^2$ $\quad D \subseteq \mathbb{R}^3$ z odsekoma gladkim robom

$\vec{e} = \mathbb{R}^3$ enotski

$$\boxed{\text{Smerniodvod: } \frac{\partial u}{\partial \vec{e}} = \operatorname{grad} u \cdot \vec{e}}$$

Opomba: $\frac{\partial u}{\partial (1,0,0)} = \frac{\partial u}{\partial x} \cdot \vec{e}$

$\vec{n}$ ...... zunanja enotska normala za ∂D

a) Dokaži:

$$\int_{\partial D} u \cdot \frac{\partial v}{\partial \vec{n}} \, dS = \int_D \operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v) \, dV$$

b) $$\int_{\partial D} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \right) dS = \int_D (u \Delta v - v \Delta h) \, dV$$

$$\int_{\partial D} u \frac{dv}{d\vec{n}} \, dS = \int_{\partial D} u \operatorname{grad} v \underbrace{\vec{n} \, dS}_{d\vec{S}} = \int_{\partial D} \underbrace{u \operatorname{grad}}_{\vec{F}} \, d\vec{S}$$

$$\operatorname{div} \vec{f} = \operatorname{div}(u \operatorname{grad} v) = \operatorname{div}(u v_x, u v_y, u v_z) =$$

$$= (u v_x)_x + (u v_y)_y + (u v_z)_z =$$

ko razpišemo

$$= \operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v$$

b)

$$\int_{\partial D} \left(u \frac{dv}{d\vec{n}} - v \frac{du}{\partial \vec{n}} \right) dS = \int_{\partial D} u \frac{dv}{d\vec{n}} \, dS - \int_{\partial D} v \frac{du}{d\vec{n}} \, dS =$$

$$= \int_D \cancel{\operatorname{grad} u \cdot \operatorname{grad}} v + u \Delta v - \cancel{\operatorname{grad} v \cdot \operatorname{grad} u} - v \Delta u$$

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ harmonična $(\Delta f = 0$

Dokaži da za $a>0$ velja:

$$\frac{1}{4\pi a^2}\int_{S(0,a)} f\,dS = f(0,0,0) = I(a)$$

Nasvet: Parametriziraj $S(0,a)$ in nato s pomočjo odvajanja in gaussovega izreka dokaži trditev.

$x = a\cos\vartheta\cos\varphi \qquad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = a\cos\vartheta\sin\varphi \qquad \varphi \in [0, 2\pi]$

$z = a\sin\vartheta \qquad \rightsquigarrow \vec{r}(\varphi,\vartheta)$

$$dS = |r_\varphi \times r_\vartheta|\,d\varphi d\vartheta = a^2\cos\vartheta\,d\varphi\,d\vartheta$$

$$I(a) = \frac{1}{4\pi \cancel{a^2}}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cancel{a^2}\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))}_{g(a,\varphi,\vartheta)}\,d\vartheta$$

$$I'(a) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g}{\partial a}\,d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\,\left(f_x(\vec{r}(\varphi,\vartheta))\cdot x_a + f_y\cdot y_a + f_z\cdot z_a\right)d\vartheta =$$

$$f(x,y,z) = f(a\cos\varphi\cos\vartheta, a\cos\varphi\sin\vartheta, a\sin\vartheta)$$

v $\vec{r}$ se skriva a

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\,(f_x\underline{\cos\varphi\cos\vartheta} + f_y\underline{\cos\vartheta\sin\varphi} + f_z\underline{\sin\vartheta})\,d\vartheta$$

komponente enotske normale $\vec{n}$ za $S(0,1)$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underline{\cos\vartheta}\,\mathrm{grad} f\cdot\vec{n}\,\underline{d\vartheta} = \qquad dS = a^2\cos\vartheta\,d\vartheta\,d\varphi$$

dS za $a=1$

$$= \frac{1}{4\pi}\int_{S(0,1)} \mathrm{grad} f\,\vec{n}\cdot dS = \frac{1}{4\pi}\int_{S(0,1)} \mathrm{grad} f\,d\vec{S} =$$

$$= \frac{1}{4\pi}\int_{K(0,1)} \overbrace{\mathrm{div}(\mathrm{grad} f)}^{\Delta f}\,dV = 0$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

$I'(a)$ je $0 \Rightarrow I(a)$ je konstantna

$$I(a) = \frac{1}{4\pi \cancel{a^2}}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cancel{a^2}\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))d\vartheta$$

$$\lim_{a\to 0} I(a) = \frac{1}{4\pi}\lim_{a\to 0}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\,d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{a\to 0}\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\,d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, f(0,0,0)\,d\vartheta =$$

$$= \frac{1}{4\pi} f(0,00)\cdot \overset{\swarrow P(S(0,1))}{4\pi} = f(0,0,0)$$

ker je $\cos\vartheta$ determinant

Holomorfne funkcije

$\overset{odp}{D \subseteq \mathbb{C}}$ območje $\quad f \in \mathcal{O}(D)$.

Katere od f_1, f_2 in f_3 so holomorfne

$$f_1(z) = \overline{f(z)}$$
$$f_2(z) = f(\bar{z})$$
$$f_3(z) = \overline{f(\bar{z})}$$

holomorfnost: $\forall a \in D.\ \exists \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} =: f'(a)$

$$f(x+iy) = u(x,y) + iv(x,y) \qquad u = \mathrm{Re} f \quad v = \mathrm{Im} f$$

$$f \in \mathcal{O}(D) \iff u_x = v_y \wedge u_y = -v_x$$

$u, v \in C^1(D)$

$$f_1(x+yi) = \underbrace{u(x,y)}_{u_1(x,y)} \underbrace{-iv(x,y)}_{v_1(x,y)}$$

$u_{1x} = u_x \qquad u_{1y} = u_y$

$v_{1x} = -v_x \qquad v_{1y} = -v_y$

$u_{1x} = u_x = v_y = -v_{1y}$

$u_{1y} = -v_y$

$\Longrightarrow \quad v_y = -v_y$ in ...

$\qquad v_x = -v_x$

$\Downarrow$

$v_x = 0 = v_y$

$u_x = u_y = 0$

u, v konstantni

? ? ?

f_1 ni holomorfne

$f_2(x+yi) = u_2(x,y) + v_2(x,y)$; $\Big\}$ f_2 na $\overline{D}$

$\qquad\qquad\qquad ||$

$\qquad\qquad u(x,-y) + v(x,-y)$

~~$u_{2x} = u_x$ $\qquad u_{2y} = u_y \cdot (-1)$~~ $= -u_y$

~~$v_{2x} = v_x$ $\qquad v_{2y} = -v_y$~~

$u_{2x}(x,y) = u_x(x,-y) \qquad u_{2y}(x,y) = -u_y(x,-y)$

$v_{2x}(x,y) = v_x(x,-y) \qquad v_{2y}(x,y) = -v_y(x,-y)$

$u_{2x}(x,y) = u_x(x,-y) = v_y(x,-y) = -v_{2y}(x,y)$

Torej f_2 ni holomorfna, razen če so $\overset{u,v}{\text{~~dve~~}}$ konstanti

$f_3(x+yi) = u_3(x,y) + iv(x,y) \Rightarrow$

$u_3(x,y) = u(x,-y)$

$v_3(x,y) = -v(x,-y)$

$(u_3)_x(x,y) = u_x(x,-y) \qquad v_{3x}(x,y) = v_x(x,-y)$

$(u_3)_y(x,y) = u_y(x,-x) \qquad v_{3y}(x,y) = +v_y(x,-y)$

$(u_3)_x = (v_3)_y \Leftrightarrow$

$\qquad u_x(x-y) = +v_y(x,-y) \Leftrightarrow u_x = v_y$ ✓

$u_{3y} = -v_{3x} \Leftrightarrow$

$-u_y(x,-y) = -v_x(x,-x) \Leftrightarrow v_y = u_x$ $\qquad$ ....

je holomorfna

Dokaži da obstaj $f \in \mathcal{O}(\mathbb{C})$ za katero je $u(x,y) = x^3 - 3xy^2$ in jo določi

$$u_x = 3x^2 - 3y^2 = v_y$$
$$u_y = -6xy = -v_x$$

$$v = 3\int(x^2 - y^2)\,dy = 3x^2y - y^3 + C(x)$$
$$v = 6\int xy\,dx = 3x^2y + D(y)$$

$$v = 3x^2y - y^3 + C$$ "unija členov"

$$f(x+yi) = x^3 - 3xy^2 + i(3xy^2 - y^3 + C)$$
$f(z) = z^3 + iC \leftarrow$ z ugibanjem

Brez navdiha:

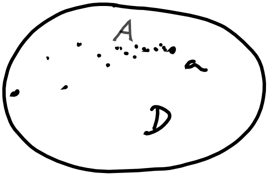
$z = x + yi \Rightarrow x = \frac{z + \bar{z}}{2}$ $\quad y = \frac{z - \bar{z}}{2i}$

DN: vstavi notri

Imenitna trditev:

$D \subseteq \mathbb{C}$ neko območje, $f, g \in \mathcal{O}(D)$

funkciji ki se ujemata na množici A s stekališčem v D



v A obstaja zaporedje

$(a_n)_n$ $a_n \in A$ $a_n \neq a$

$\lim_{n \to \infty} a_n = a$

$\Longrightarrow f = g$

$u = x^3 - 3xy^2$ $v = 3x^3y - y^3 + iC$

če najdemo $g \in \mathcal{O}(C)$ ki se z f ujema na $\mathbb{R}$, po trditvi T sledi $f = g$

$x \in \mathbb{R}$ $f(x) = f(x + i0) = u(x,0) + v(x,0) = x^3 + iC$

$\Rightarrow f(z) = z^3 + iC$

Še imenitnejši premislek:

$$u(x,y) \rightsquigarrow f = ?$$

$$x \in \mathbb{R}: \; f(x) = \underset{\checkmark}{u(x,0)} + i\,\underset{?}{v(x,0)} \overset{x \to z}{\rightsquigarrow} f(z) \; ☺$$

$$v(x,0) = \int v_x(x,0)\,dx \underset{\overset{\uparrow}{\text{CRS}}}{=} -\int u_y(x,0)\,dx$$

$$u_y = -6xy$$

$$v(x,0) = -\int -6x\cdot 0\,dx = +C$$

$$\Longrightarrow \quad f(x) = x^3 + iC \quad \Longrightarrow \quad f(z) = z^3 + iC$$

Podobno

$$u(x,0) = \int u_x(x,0)\,dx = \int v_y(x,0)\,dx$$

$$u(x,y) = e^x(\cos(ky) + \sin(ky))$$

Določi k da je $u = \mathrm{Re} f$ za $f \in \mathcal{O}(\mathbb{C})$

Če je $u = \mathrm{Re} f \Longrightarrow$

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \end{aligned} \quad \Longrightarrow u_{xx} = -u_{yy}$$

Podobno

$$v_{xx} = -v_{yy}$$

$$\Longrightarrow \Delta u = 0$$

$$u_{xx} = e^x(\cos(ky) + \sin(ky)) = u$$

$$u_y = e^x(-k\sin(ky) + k\cos(ky))$$

$$u_{yy} = k^2 e^x(-\cos(ky) - \sin(ky)) = -k^2 v$$

$$\Delta u = e^x(1-k^2)(\cos(ky) + \sin(ky)) = 0 \text{ za } \forall x,y$$

$$\Rightarrow k = \pm 1$$

$$v(x,0) = -\int u_y(x,0)\,dx = -\int e^x k\,dx = -ke^x + C$$

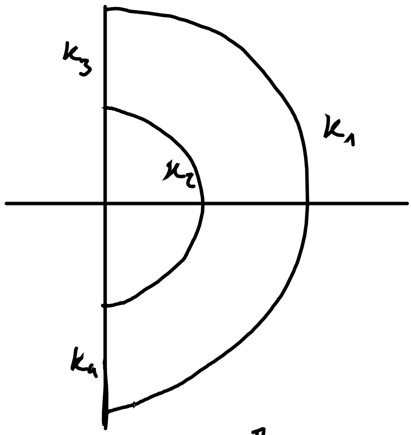
$$f(x) = u(x,0) + iv(x,0) = e^x - ike^x + C_i = e^x(1-ik) + iC$$

$$f(z) = e^z(1 \pm i) + iC$$

12.4

Integrali holomorfne funkcije

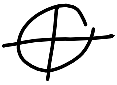
$f(z)=\frac{z}{\bar{z}}$ $\quad D=\{ z ; |z| \in [1,2] \; \mathrm{Re}(z) > 0 \}$



$$\int_{K_1} f(z)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2e^{i\varphi}}{2e^{-i\varphi}} 2ie^{\varphi} d\varphi =$$

$z = 2e^{i\varphi}$

$dz = 2ie^{\varphi}$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2ie^{3i\varphi} d\varphi = \frac{2}{3}i \, e^{3i\varphi} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$\frac{2}{3}(-i-i) = -\frac{2}{3}i$ $\quad$ (kam je šel −)

$$\int_{K_2} f(z)dz = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} i\frac{e^{i\varphi}}{e^{-i\varphi}} e^{i\varphi} d\varphi = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{1}{3}i \, e^{i3\varphi} =$$

$= \frac{1}{3}(i+i) = \frac{2i}{3}$

$$\int_{K_3 \cup K_4} f(z)dz = \quad \cancel{dz} \quad \begin{matrix} z = iy \\ dz = i\,dy \end{matrix}$$

$$= \int_{-1}^{-2} \frac{\cancel{iy}}{-\cancel{iy}} i\,dy + \int_{2}^{1} \frac{\cancel{iy}}{-\cancel{iy}} i\,dy = -i(-2+1) + i(1-2) =$$

$= i(2-1-1+2) \; .... \; = 2i$

$f(z) = \overline{z}$

$D \overset{df}{\subseteq} \mathbb{C}$ s kosoma gladkim robom

$$\int_{\partial D} f(z)\,dz = ?$$

$z = x + iy$

$dz = dx + idy$

$$I = \int_{\partial D} (x - iy)(dx + idy) =$$

$$= \int_{\partial D} (x\,dx + y\,dy) + i(x\,dy - y\,dx) =$$

$$\overset{green}{=} \int_{D} (0 - 0) + i \int_{D} (1 - (-1))\,dx\,dy =$$

$$= 2i \cdot P(D)$$

$D^{odp} \subseteq \mathbb{C}$ kosana gładka

$f \in \sigma(D) \cap C^1(\overline{D})$

$$\int_{\partial D} f(z)\,dz = \int_{\partial D} (u+iv)(dx+idy) =$$

$$= \int_{\partial D} (u\,dx - v\,dy) + i(v\,dx + u\,dy) =$$

$$= \int_D \underbrace{(-v_x - u_y)}_{0} dx\,dy + i\int \underbrace{(u_x - v_y)}_{0} dx\,dy = 0$$

$u_x = v_y$

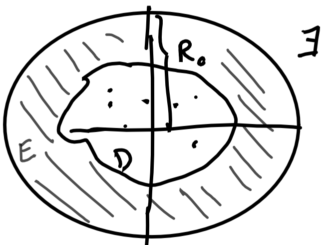
$u_y = -v_x$

$n > 1$

$z_1, \ldots, z_n \in \mathbb{C}$ poljubne neujno različne

$D^{odp} \subseteq \mathbb{C}$ omejeno ki vsebuje $z_1, \ldots, z_n$

Dokaži $\int_{\partial D} \frac{dz}{(z-z_1)\cdots(z-z_n)} = 0$



$\exists R_0 > 0.\ \overline{D} \subseteq \Delta(0,R)$ za $\forall R > R_0$

za $R > R_0$

$$\boxed{\int_{\partial\Delta(0,R)} f(z)dz = \int_{\partial D} f(z)dz}$$

$E = \Delta(0,R) - \overline{D}$

f je holomorfna na E

$\Rightarrow \int_{\partial E} f(z)dz = 0$

$$\int_{\partial E} f(z)dz = \int_{\partial\Delta(0,R)} f(z)dz) - \int_{\partial D} f(z)dz = 0$$

$$\int_{K_R} \frac{dz}{(z-z_1)\cdots(z-z_n)} = \int_{\partial D} \frac{dz}{(z-z_1)\cdots(z-z_n)}$$

$$\int_{\partial E} \cdots = 0$$

$\int_{K_1} f(z)dz = \int_{K_2} f(z)dz$ če krivulji skupaj tvorita E in je f holomorfna na okolici E

$\int_{K_R}$: $|z| = Re^{i\varphi}$

$dz = Ri\, e^{i\varphi}$

$$I = \int_0^{2\pi} \frac{iRe^{i\varphi} d\varphi}{(Re^{i\varphi}-z_1 \cdots (Re^{i\varphi}-z_n)}$$

$$|I| \leq \int_0^{2\pi} \left| \frac{iRe^{i\varphi}}{(Re^{i\varphi}-z_1)\cdots(Re^{i\varphi}-z_n)} \right| d\varphi$$

$$= \left| \frac{R}{(Re^{i\varphi}-z_1)\ldots(Re^{i\varphi}-z_n)} \right| \leq \frac{R}{(R-|z_1|)\cdots(R-|z_n|)}$$

$$|Re^{i\varphi} - z_k| \geq ||Re^{i\varphi}| - |z_k|| = |R-|z_k|| = R-|z_k|$$

$$|I| \leq \int_0^{2\pi} \frac{R}{(R-|z_1|)\cdots(R-|z_n|)} d\varphi = \frac{2\pi R}{(R-|z_1|)\cdots(R-|z_n|)}$$

$$\xrightarrow{R\to\infty} 0 \Rightarrow I = 0$$

če bi bil $n=1$

$$\int_{\partial D} \frac{dz}{z-z_1} \Rightarrow I = 2\pi i \text{ kr}$$

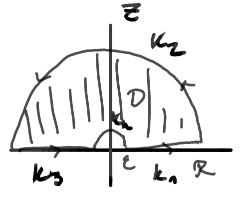
$$\boxed{\int_{\partial D} \frac{f(z)dz}{z-z_0} \overset{f\in\mathcal{O}(D)}{=} 2\pi i f(z_0)}$$

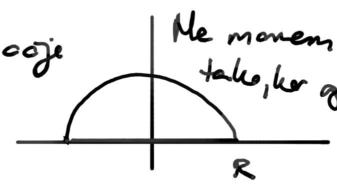
s pomočjo kompleksne integracije 2u.4
izračunaj

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

Če mamo $\sin x$ ali $\cos x$ v integralu, potem ju zamenjamo z e^x
Drugače pa samo zamenjamo x z z

$f(z) = \frac{e^{iz}}{z}$ območje





Ne moremo tako, ker gre čez $z=0$

R bomo pošlali v ∞ in ε proti 0

$$\int_{\partial D} f(z)dz = 0$$
$\uparrow$
$f \in \mathcal{O}(\bar{D})$

∂D razdelimo na gladke kose:

$$\int_{\partial D} f(z)dz = \int_{K_1} + \int_{K_2} + \int_{K_3} + \int_{K}$$

K_1: $z=x$ $dz=dx$ $x\in[\varepsilon,R]$

$$\int_{K_1} f(z)dz = \int_{\varepsilon}^{R} \frac{e^{ix}}{x}dx = \int_{\varepsilon}^{R} \frac{\cos x + i\sin x}{x} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} =$$

$$= \underbrace{\int_0^\infty \frac{\cos x}{x}dx}_{} + i\overset{I}{\overset{\shortparallel}{\int_0^\infty \frac{\sin x}{x}dx}}$$

ne obstaja ☹

"Rešitev": združimo K_1 in K_3

$$\int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\cos x}{dx}dx + i\int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\sin x}{x}dx \xrightarrow[\varepsilon\to 0]{R\to\infty} 2iI$$

liha funkcija

sode funkcije

K_2: $z=Re^{i\varphi}$

$dz = Rie^{i\varphi}d\varphi$

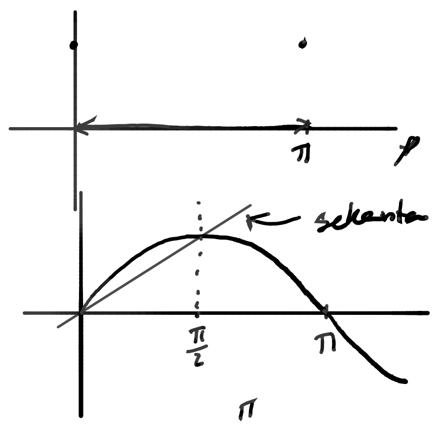
$$\int_{K_2} \dots = \int_0^\pi \frac{e^{Re^{i\varphi}}}{Re^{i\varphi}} Rie^{i\varphi}d\varphi =$$

$$= i\int_0^\pi e^{iRe^{i\varphi}}d\varphi = \int_0^\pi e^{iR(\cos\varphi + i\sin\varphi)}d\varphi$$

$$\left|\int_{K_2}\dots\right| \le \int_0^\pi \left|e^{iR(\cos\varphi + i\sin\varphi)}\right|d\varphi =$$

$$= \left|e^{-R\sin\varphi}\cdot e^{i\dots}\right| = e^{-R\sin\varphi}$$

$$= \int_0^\pi e^{-R\sin\varphi} \xrightarrow{R\to\infty}$$



$$\int_0^\pi e^{-R\sin\varphi}d\varphi = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\varphi}d\varphi$$

$e^{-R\sin\varphi} \le$ $\sin\varphi \in [0,1]$ za $\varphi \in$

$-R\sin\varphi \le 0$

$e^{-R\sin\varphi} \le 1$ ☹

$\sin\varphi \ge \frac{2\varphi}{\pi}$

$$\int_0^\pi e^{-R\sin\varphi}d\varphi \le 2\int_0^{\frac{\pi}{2}} e^{-\frac{R2\varphi}{\pi}}d\varphi =$$

$$= -2\frac{\pi}{2R}\left(e^{-R}-1\right) = \frac{\pi}{R} - \frac{\pi}{R}e^{-R} =$$

$$\frac{\pi}{R}\left(1-e^{-R}\right) \xrightarrow{R\to\infty} 0$$

$\int_{K_4}$: $z=\varepsilon e^{i\varphi}$

...

isto kot pri K_2

$$\int_{K_4} = i\int_\pi^0 e^{i\varepsilon(\cos\varphi + i\sin\varphi)}d\varphi \rightsquigarrow i\int_\pi^0 1d\varphi = -i\pi$$
$\downarrow$
$\varepsilon\to 0$
$\rightsquigarrow 1$

Iz ana 2: $\int_0^\pi g(\varepsilon,\varphi)d\varphi \xrightarrow{\varepsilon\to 0} \int_0^\pi g(0,\varphi)d\varphi$ vstavimo $\varepsilon=0$

zvezna na $\mathbb{R}\times[0,\pi]$

$$0 = \int_{\partial D} f(z)dz = \int_{K_1} + \dots + \int_{K_4} \xrightarrow[\varepsilon\to 0]{R\to\infty} 2iI + 0 - i\pi$$

$$\rightsquigarrow \quad I = \frac{i\pi}{2i} = \frac{\pi}{2}$$

$a>0 \quad p\in(0,1) \quad 0<\varepsilon<R$

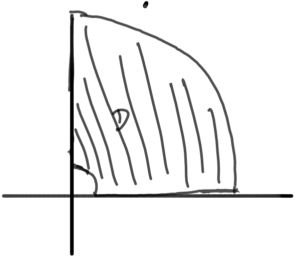
S pomočjo integracije po ∂D izračunaj

$$I_1=\int_0^\infty x^{p-1}\cos(ax)\,dx$$

$$I_2=\int_0^\infty x^{p-1}\cos(bx)\,dx$$

$D=\{z;\ |z|\in(\varepsilon,R) \wedge \operatorname{Re} z>0,\ \operatorname{Im} z>0$

$f(z)=z^{p-1}e^{iaz}$



holomorfnost:

$e^{iaz}\in\mathcal{O}(\mathbb{C})$

$p=\frac{1}{2}:\ z^{p-1}=z^{-\frac{1}{2}}=\frac{1}{\sqrt{z}}$

$\sqrt{-1}=\pm i$

$$\boxed{\begin{array}{l}\alpha\in\mathbb{C},\ z\neq0\\ z^\alpha=e^{\alpha\ln z}\end{array}}$$

Definicija kompleksne potence

$\ln z$ je nedoločen do konstante $2k\pi i$:

$z^\alpha=e^{\alpha(\overset{\text{neka veja}}{\ln_0(z)}+2k\pi i)}=(z^\alpha)_0\,e^{2\alpha k\pi i}$

če $\alpha\in\mathbb{Z}$ je z^α enolično definirana

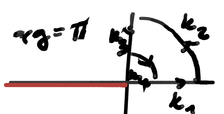
z^α je holomorfno definirana, ko se odločimo za vejo $\ln(z)$

$\hookrightarrow \arg z\in(\varphi_0,\varphi_0+2\pi)$

$\mathbb{C}$ prerezemo v smeri φ_0

Naš primer: ni vključeno, ker ne bi bilo zvezno v $\arg=\pi$

$\arg z\in(-\pi,\pi)$



Izberemo si rez v smeri π, da ne preseka našega domena

Produkt holomorfnih je holomorfna

z^{p-1} holomorfna

$\mathbb{C}-(-\infty,0]$

$$\int_{k_1}f(z)\,dz=\int_{\underset{z=x}{\varepsilon}}^{R}x^{p-1}e^{aix}\,dx=\int_\varepsilon^R x^{p-1}(\cos ax+i\sin ax)\,dx=$$

$$\xrightarrow[\varepsilon\to0^+]{R\to\infty} I_1+iI_2$$

k_3: $z=ix \quad dz=i\,dx \quad \ln(ix)=\ln y+i\frac{\pi}{2}$

$$z^{p-1}=e^{(p-1)\ln(z)}=e^{(p-1)(\ln x+\frac{\pi}{2}i)}=$$

$$=e^{(p-1)\ln x}\cdot e^{(p-1)i\frac{\pi}{2}}=x^{p-1}\cdot e^{(p-1)i\frac{\pi}{2}}$$

$$\int_R^\varepsilon x^{p-1}e^{(p-1)i\frac{\pi}{2}}\cdot e^{iaix}=\int_R^\varepsilon x^{p-\cancel{ax}-1}e^{i(p-1)\frac{\pi}{2}}=\dots$$

$$=-e^{i(p-1)\frac{\pi}{2}}\int_\varepsilon^R x^{p-1}e^{-ax}\,dx\xrightarrow[\varepsilon\to0]{R\to\infty}$$

$$=-e^{i(p-1)\frac{\pi}{2}}\int_0^\infty x^{p-1}e^{-ax}=\dots\frac{1}{a}\int_0^\infty\frac{t^{p-1}}{a^{p-1}}e^{-t}\,dt$$

$t=ax$
$dt=a\,dx$

$$=-\frac{e^{i(p-1)\frac{\pi}{2}}}{a^p}\Gamma(p)$$

$\int_{k_2}$: $z=Re^{i\varphi}$

~~$(z_1\cdot z_2)^\alpha=z_1^\alpha\cdot z_2^\alpha$~~ :(

(Velja če si izbereš ustrezne veje)

$z^{p-1}=(Re^{i\varphi})^{p-1}=$

$e^{(p-1)\ln(Re^{i\varphi})}$ $\qquad dz=iRe^{i\varphi}d\varphi$

$\ln(Re^{i\varphi})=\ln R+i\varphi$

$$=e^{\ln R(p-1)+(p-1)i\varphi}=R^{p-1}\cdot e^{i(p-1)\varphi}$$

$$\int_{k_2}\dots\int_0^{\frac{\pi}{2}}R^{p-1}e^{i(p-1)\varphi}e^{ia\,R(\cos\varphi+i\sin\varphi)}\,iRe^{i\varphi}\,d\varphi$$

$$\Big|\int_{k_2}\Big|\le\int_0^{\frac{\pi}{2}}|\dots|\,d\varphi=$$ — abs vrednost $=1$

$$R^p\int_0^{\frac{\pi}{2}}e^{-aR\sin\varphi}d\varphi\le\int_0^{\frac{\pi}{2}}R^p e^{-\frac{2\varphi}{\pi}aR}d\varphi=$$

$\sin\varphi\ge\frac{2\varphi}{\pi}$

$$=\underbrace{-R^p\frac{\pi}{2aR}}_{0}(\underbrace{e^{-aR}-1}_{-1})\longrightarrow0$$

$\alpha p<1)$