Hilbertov prostor

X vektorski prostor nad $\mathbb{R}$ (nad $\mathbb{C}$)
skalarni produkt $\langle\ \rangle$

$$\langle\ \rangle : X \times X \longrightarrow \mathbb{R}$$

1) $\forall x \in X : \langle x, x \rangle \geq 0$

2) $\langle x, x \rangle = 0 \iff x = 0$

pozitivna definitnost

3) $\forall x, y \in X$

$\langle x, y \rangle = \langle y, x \rangle$ nad $\mathbb{R}$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$ nad $\mathbb{C}$

simetričnost oz antisimetričnost

4) $\forall x, y, z \in X, \ \forall \lambda, \mu \in \mathbb{R}$

$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

linearnost v prvem faktorju

Za sklani: praktično

Cauchy-Schwarzova neenakost

$\forall x,y \in X$

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}$$

$$\|x\| = \sqrt{\langle x,x\rangle}$$

$$|\langle x,y\rangle| \leq \|x\| \, \|y\|$$

Dokaz nad $\mathbb{R}$

$$t \longmapsto \langle x+ty, x+ty\rangle = s(t) \geq 0$$

$$= \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle$$

$$= \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\| \geq 0$$

$$D \leq 0$$

$$D^2 = 4\langle x,y\rangle - 4\|x\|^2\|y\|^2$$

$$\Rightarrow \langle x,y\rangle^2 \leq \|x\|^2\|y\|^2$$

$$\langle x,y\rangle \leq \|x\|^2\|y\|^2$$

enakost velja natanko tedaj če sta x in y linearno odvisna

Nad $\mathbb{C}$:

x, y

$\exists \alpha \quad |\alpha| = 1$

$$\langle x,y\rangle = \alpha |\langle x,y\rangle|$$

$$\Rightarrow \quad \langle x, \alpha y\rangle = |\langle x,y\rangle|$$

$$f(t) = \langle x + t\alpha y, x + t\alpha y\rangle =$$

$$= \|x\|^2 + t\underbrace{\left(\langle \alpha y, x\rangle + \langle x, \alpha y\rangle\right)}_{= 2\cdot|\langle x,y\rangle|} + t^2\langle \alpha y, \alpha y\rangle$$

$$= \|x\|^2 + 2t|\langle x,y\rangle| + t^2\|y\|^2 \geq 0$$

$$D = 4|\langle x,y\rangle|^2 - 4\|x\|^2\|y\|^2 \leq 0$$

Norma nad vektorskem prostorom X

$\| \ \|: X \longrightarrow R$

1) $\forall x \in X. \ \|x\| \geq 0$

2) $\|x\| = 0 \Leftrightarrow x = 0$

3) $\forall \lambda \in \mathbb{R} \vee \mathbb{C}, \forall x \in X$

$\|\lambda x\| = |\lambda| \, \|x\|$ homogenost

4) trikotniška neenakost

$\forall x, y \in X. \ \|x+y\| \leq \|x\| + \|y\|$

če je $(X, \langle \rangle)$ vektorski prostor s skalarnim produktom je

$\|x\| = \sqrt{\langle x, x \rangle}$ X vektorski prostor z normo

$\|x\| = \sqrt{\langle x, x\rangle}$

1), 2) sledi iz 1) 2) skalarnega produkta

3) $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{|\lambda|^2 \langle x, x\rangle} =$

$= |\lambda| \|x\|$

4)

$\|x+y\|^2 = \langle x+y, x+y\rangle =$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle =$

$= \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

$(X, \langle\ \rangle) \rightsquigarrow (X, \|\ \|)$

$\rightsquigarrow$ metrični prostor (X, d)

$\forall x, y \in X .\ d(x,y) = \|x - y\|$

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \iff x = y$

3) $d(x,y) = d(y,x)$

4) $\forall x, y, z \in X$

$$d(x,y) + d(y,z) \geq d(x,z)$$

Def: Hilbertov prostor je vektorski prostor s skalarnim produktom ki je v metriki porojeni iz skalarnega produkta poln metrični prostor

Opomba: Banachov prostor je vektorski prostor X z normo $\| \|$ ki je v metriki porojeni iz norme poln metrični prostor

Zgledi:

1) $\mathbb{R}^n$, $x = (x_1 \dots x_n)$
$y = (y_1 \dots y_n)$

$$x \cdot y = x_1 y_1 + \dots \pm x_n y_n$$

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \qquad d_2 \text{ metrika}$$

$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$ je poln metrični prostor

$(\mathbb{R}^4, \cdot)$ je Hilbertov prostor

$(\mathbb{R}^n, d_\infty)$ $\quad \|\,\|_\infty$ $\quad \max\{|x_1| \ldots |x_n|\}$

normi

$(\mathbb{R}^n, d_1)$ $\quad |x_1| + \ldots + |x_n|$

$\|\,\|_1$

Banachova prostora

ker sta topološko enaka $(\mathbb{R}^n, d_2)$

$\mathbb{R}^n$ Sta Banachova ker sta topološki ekvivalentna $(\mathbb{R}^2, d_2)$

Ampak ne prideta iz skalarnega produkta

2) $\mathbb{C}^n$ $\quad z = z_1, \dots z_n$

$w = w_1, \dots w_n$

$$z \cdot w = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

$$\|z\| = \sqrt{z_1^2 + \dots + z_n^2}$$

$$d_2(z,w) = \sqrt{|z_1 - w_1|^2 + \dots + |z_n - w_n|^2}$$

$(\mathbb{C}^n, \cdot)$ je Hilbertov prostor

Zgled

$[a,b] \subseteq \mathbb{R} \qquad a<b$

$X = C([a,b])$

nad $\mathbb{R}$

$$\langle f,g \rangle = \int_a^b f(x)\, g(x)\, dx$$

3) $\langle f,g \rangle = \langle g,f \rangle$

4) $\langle \lambda f + \mu g, h \rangle =$
$\lambda \langle f,h \rangle + \mu \langle g,h \rangle$

1) $\langle f,f \rangle = \int_a^b f^2(x)\, dx \geq 0$

2) $f = 0 \Rightarrow \langle f,f \rangle = 0$

$\int_a^b f^2\, dx = 0 \Rightarrow f^2 = 0$ skoraj povsod

$\Rightarrow f^2 = 0$ povsod ker zveznost

$\Rightarrow f = 0$ (Naslednja stran)

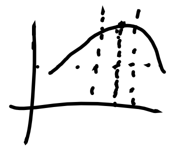
Polnost?

$$\int_a^b f^2(x)dx = 0$$

ce $\exists x_0 \in [a,b]$ $f(x) = 0 \Longrightarrow$

$\exists \delta > 0 : \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a,b].$

$$|f(x)| \geq \frac{|f(x_0)|}{2}$$



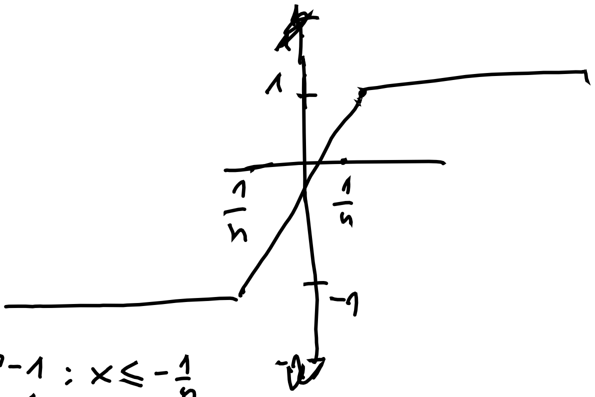
$$\int_a^b f^2(x)\,dx \geq \int_{(x_0-\delta,\, x+\delta) \cap [a,b]} f^2(x)dx) \geq \frac{f(x_0)^2}{2}\delta$$

ce f wem $\Longrightarrow f = 0$ pased

Trditev: $(C[a,b], \langle\,\rangle)$
ni Hilbertov

Dokaz

~~$f_n(x)$~~



$$f_n(x) = \begin{cases} -1 & ; x \leq -\frac{1}{n} \\ 1 & ; x \geq \frac{1}{n} \\ nx & ; -\frac{1}{n} \leq x \leq \frac{1}{n} \end{cases}$$

$(f_n)_n$ je Cauchyjeva zaporedje v $(C[-1,1], d)$
d-metrika iz skalarnega produkta

$$d^2(f,g) = \int_{-1}^{1} |f(x) - g(x)|^2 \, dx$$

$m > n$

$$d(f_n, f_m)^2 =$$

$$\int_{-1}^{1} (f_n(x) - f_m(x))^2 \, dx =$$

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} (f_m(x) - f_n(x))^2 \leq \int_{-\frac{1}{n}}^{\frac{1}{n}} 1 \, dx = \frac{2}{n}$$

$$d(f_n, f_m) \leq \sqrt{\frac{2}{n}} < \varepsilon$$

$(f_n)_n$ je Cauchyjevo v $(C[-1,1], d)$

~~$\lim f_n$~~

Če obstaja $\lim$ ~~f_n~~ $(f_n)_n$ v $C[-1,1]$

mora veljati, da je $f(x) = \begin{cases} 1 & ; 0 < x \leq 1 \\ -1 & ; -1 \leq x < 0 \end{cases}$

$\Rightarrow f$ ni zvezna v 0

$\times$

Zgled

$M = (0,1)$

$d_2(x,y) = |x-y|$

~~M~~ M ni poln

$M \cup \{0,1\}$ je poln

Dodali smo možne limite

<u>Napolnitev metričnega prostora</u>

(M,d) napolnimo   lahko napolnemo

$(\tilde{M},\tilde{d})$ poln

1) $M \subseteq \tilde{M}$

2) $\tilde{d}|_{M \times M} = d$

3) M je gost v $\tilde{M}$

$L^1(A) = \{ f : A \to \mathbb{R} ; \int_A |f(x)| \, dx < \infty) \}$

$\to$ kvadratom integrabilne funkcije :

$L^2([a,b]) = \{ f : [a,b] \to \mathbb{R} \ \int_a^b |f|^2 dx < \infty \}$

Zgled :

1) $C\ell[a,b] \subseteq L^2([a,b])$

2) odsekoma zvezne $\subseteq L^2$

3) $f(x) = \dfrac{1}{\sqrt[4]{x-a}} \in L^2$

4) $g(x) = \dfrac{1}{\sqrt[2]{x-a}} \notin L^2([a,b])$

Velja : f, g

$|f \cdot g| \leq \dfrac{|f|^2 + |g|^2}{2} \Rightarrow f \cdot g \in L^1([a,b])$

$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$

L^2 je vektorski prostor

$$(f+g)^2 = \underset{\in L^2}{f^2} + \underset{\in L^1}{2fg} + \underset{\in L^2}{g^2}$$

$$\int_a^b (f+g)^2\,dx = \int_a^b f(x)\,dx + 2\int f(x)g(x)\,dx + \int_a^b g^2(x) < \infty$$

$$\Rightarrow \quad (f+g) \in L^2$$

$C[a,b] \subseteq L^2([a,b])$

hilbertov

Opomba

$\forall f \in L^2([a,b]) . \exists f_n \in C[a,b] .$

$$\lim_{n\to\infty} f_n = f \iff \lim_{n\to\infty} \|f_n - f\| = 0$$

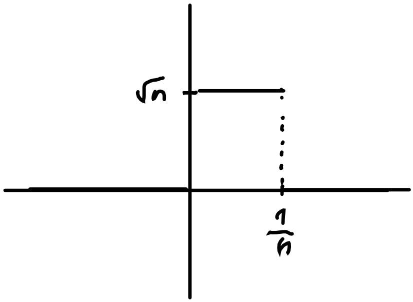
$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x))^2 dx = 0$$

Opomba: nad $\mathbb{C}$: $f = u + i v \quad u, v : [a,b] \to \mathbb{R}$

$$\int_a^b f(x) dx = \int_a^b u(x) dx + i \int_a^b v(x) dx$$

Zgled: na $[0,1]$

$$f_n: \begin{cases} \sqrt{n} & ; \ 0 < x \leq \frac{1}{n} \\ 0 & ; \ \text{sicer} \end{cases}$$

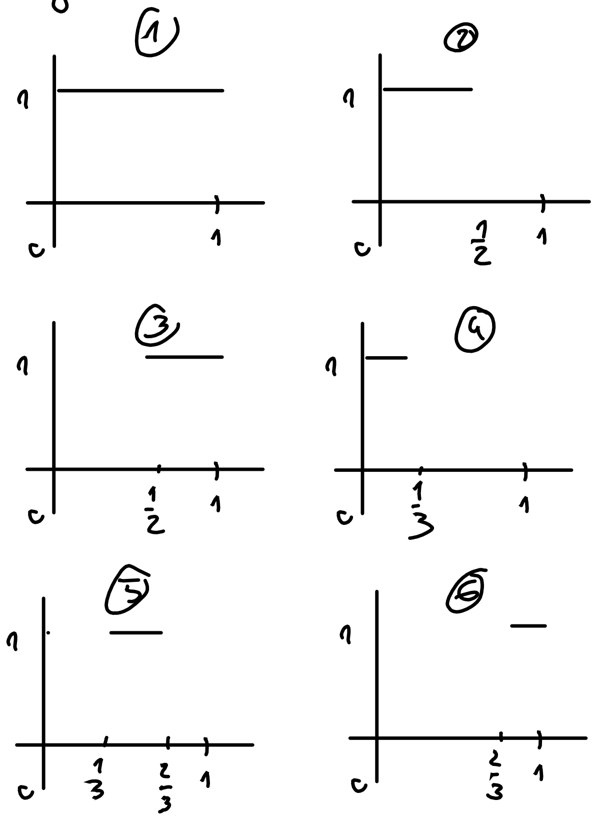


$\lim f_n(x) = 0$ za $x \in [0,1]$

Ali konvergira v L^2?

$$\|f_n - 0\| = \sqrt{\int_0^1 f_n^2(x)\, dx} = \sqrt{\int_0^{\frac{1}{n}} \sqrt{n}^2 dx} = 1$$

Zgled:



itd

$\lim\limits_{n\to\infty} f_n(x)$ ne obstaja

$\lim f_n(x)$ v L^2 ?

$\int_0^1 f_n^2(x)\,dx \leq \frac{1}{n^2}$ torej limita je 0

Naj bo $(X, \langle \ \rangle)$ vek. prostor s skalarnim produktom.

$A \subseteq X \quad (A \neq \emptyset)$

$x, y \in X. \quad x \perp y \Leftrightarrow \langle x, y \rangle = 0$

(simetricna relacija)

$A \subseteq X \quad A^{\perp} = \{ x \in X.\ x \perp a \text{ za } \forall a \in A \}$

Ortogonalni komplement

Trditev: $A^\perp$ je vektorski podprostor v X

Dokaz:

$x \in A^\perp \quad \lambda \in \mathbb{R}$ (ali $\mathbb{C}$)

$a \in A$. $\langle \lambda x, a \rangle = \lambda \langle x, a \rangle = \lambda \cdot 0 = 0$

$x, y \in A^\perp$

$$\langle x+y, a \rangle = \underbrace{\langle x, a \rangle}_{=0} + \underbrace{\langle y, a \rangle}_{=0} = 0$$

Velja: $A \subseteq (A^{\perp})^{\perp}$

Trditev: $v \in X$

$f(x) = \langle x, v \rangle \quad f: X \to \mathbb{R}$

f je zvezna na X

Dokaz:

$x_1, x_2 \in X$

$|f(x_1) - f(x_2)| = |\langle x_1 - x_2, v \rangle| \leq \|x_1 - x_2\| \|v\|$

f je enakomerno zvezna

(celo Lipschitzova zvezna)

Posledica: $A^\perp$ je zaprt vektorski podprostor

$C[a,b] \subseteq L^2[a,b]$ ni zaprt podprostor

Dokaz:

zaporedje x_n

$\lim x_n = x_0 \in X \overset{?}{\Longrightarrow} x_0 \in A^\perp$

$\forall a \in A . \langle x_n, a \rangle = 0$ za $\forall n$

$$\underbrace{\lim \langle x_n, a \rangle}_{=0} = \langle \lim x_n, a \rangle = \langle x_0, a \rangle = 0$$

Opomba: $(X, \langle\rangle)$ hilbertov

$A \leq X$ zaprt podprostor $\Rightarrow A = (A^\perp)^\perp$

Trditev: (Pitagorov izrek)

$(X, \langle\ \rangle)$ vektorski prostor s skalarnim produktom

$x_1 \dots x_n \in X \quad \forall i \neq j \in [n] .\ x_j \perp x_i$

$(\langle x_j, x_i \rangle = 0)$

Tedaj $\|x_1\|^2 + \dots + \|x_n\|^2 \quad \|x_1 + \dots x_n\|^2$

Dokaz:

$\|x_1 + \dots + x_n\| = \langle x_1 + x_2 \dots + x_n, x_n + \dots + x_n \rangle$

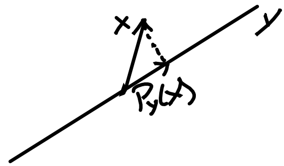
$= \langle \sum_i x_i ; \sum x_j \rangle = \sum_i \sum_j \langle x_i, x_j \rangle =$

$= \|x_1\|^2 + \dots + \|x_n\|^2$

$(X, \langle\ \rangle)$

$Y \leq X$

$x \in X$



Definicija: <u>Pravokotna projekcija</u> vektorja x na podprostor Y (če obstaja) je tak vektor $P_Y(x) \in Y$ da je

$$x - P_Y(x) \in Y^{\perp}$$

Trditev: če pravokotna projekcija x na Y obstaja, je enolično določena in $P_Y(x)$ je najboljša aproksimacija vektorja X z vektorji iz Y (razdalja je najmanjša)

$$\left(\|x - P_Y(x)\| = \min_{w \in Y} \|x - w\|\right)$$

Dokaz:

$Y \leq X \quad x \in X$

Denimo da y_1 y_2 sta pravokotni projekciji x na Y

$x - y_1,\ x - y_2 \in Y^\perp$

ker vek. prostor

$(x - y_1) - (x - y_2) \in Y^\perp$

$y_2 - y_1 \in Y^\perp$

$y_2 - y_1 \in Y$

$\langle y_2 - y_1, y_2 - y_1 \rangle = 0 \iff$

$y_2 - y_1 = 0 \quad y_2 = y_1$

$w \in Y$

$$x - w = \underbrace{x - P_Y(x)}_{\in Y^\perp} + \underbrace{P_Y(x) - w}_{\in Y}$$

pitagorov izrek:

$$\|x - w\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x) - w\|^2 \geq$$

$$\|x - P_Y(x)\|^2$$

■

Zgled:

$Y = C[a,b]$ $\quad X = L^2[a,b]$

$f \in X - Y$ nima pravokotne projekcije

Tak f nima najboljše aproksimacije z zveznimi funkcijami

Komentar: hilbertov tzaprt ptor ima projekcije

Opombe:

1) $P_y^2 = P_y$

2) $x = \underbrace{x - P_y(x)} + \underline{P_y(x)}$

$\|x\|^2 = \|x - P_y(x)\|^2 + \|P_y(x)\|^2$

$\|x\| \geq \|P_y(x)\|$

3) Če je P_y definirana na celem X, potem je P_y linearen in zvezen

$\|P_y x_1 - P_y x_2\| = \|P_y(x_1 - x_2)\| \leq \|x_1 - x_2\|$

P_y je zvezna

linearnost:

$\lambda x - P_y(\lambda x)$

Poglejmo:

$\lambda x - \lambda P_y(x) = \underbrace{\lambda(x - P_x(x))}_{\in Y^\perp}$ $\in Y^\perp$

$\Rightarrow \lambda P_y(x) \in Y$

$\lambda x - \underbrace{\lambda P_y(x)} \in Y^\perp$

To je pravokotna projekcija λx

$x_1, x_2 \in X$

$P_y(x_1 + x_2) = P_y(x_1) + P_y(x_2)$

$(x_1 + x_2) - \underbrace{(P_y(x_1) + P_y(x_2))}_{\in Y} = x_1 - P_y(x_1) + x_2 - P_y(x_2)$

$\in Y^\perp$

$\Rightarrow P_y(x_1 + x_2) = P_y(x_1) + P_y(x_2)$

zaradi: enoličnosti projekcije ■

~~Če P_y definiran~~

Če je P_Y definiran na X je
Y zaprt prostor

Dokaz:

$\{y_j\} \subseteq Y \quad \lim_{j\to\infty} y_j = y_0$

$P_Y(y_j) = y_j$

$$\lim y_j = \lim P_Y(y_j) \overset{\text{zveznost}}{=} P_Y(y_0)$$

$\lim y_j = y_0$

Če ima X pravokotno projekcijo na Y ima tudi pravokotno projekcijo na $Y^\perp$

$$x;\ P_Y(x)$$

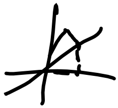
$$P_{Y^\perp}(x) = x - P_Y(x) \in Y^\perp$$

$$x - (x - P_Y) \in (Y^\perp)^\perp$$

$$\| $$

$$P_Y(x) \in Y \subseteq (Y^\perp)^\perp$$

Trditev: Naj bo $Y \leq X$ končno dimen podprostor z ortonormirano bazo $e_1, \dots, e_n$. $\langle e_i, e_j \rangle = \delta_{ij}$



Naj bo $x \in X$. Tedaj

$$P_Y(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$$

Opomba: Vsak končno dimenzionalni podprostor ima pravokotno projekcijo definirano na X

In tudi vsi tisti končne kodimenzije

Dokaz: $P_Y(x) = \sum_{1}^{n} \langle x, e_i \rangle e_i \in Y$

Poglejmo še

$$x - \sum_{1}^{n} \langle x, e_j \rangle e_j \in Y^{\perp}$$

$$\langle x - \sum_{1}^{n} \langle x, e_j \rangle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \sum_{1}^{n} \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$(X, \langle\ \rangle)$

Sistem vektorjev

$(e_j)_{j=1}^{\infty}$

je ortogonalen sistem (OS), če

$\forall i \neq j \quad \langle e_i, e_j \rangle = 0$

Tak sistem je ortonormiran (ONS)

$\forall i, j \quad \langle e_i, e_j \rangle = \delta_{ij}$

Trditev: $(X, \langle\,\rangle)$ Naj bo $(e_j)_j$ ONS

Naj bo $x \in X$. Tedaj je $\sum |\langle x, e_j \rangle|^2 \leq \|x\|^2$

(Besselova neenakost)

Opomba: $\langle x, e_j \rangle$ so Fourierovi koeficienti x po ONS $(e_j)_j$

Posledica:

$$\lim_{j \to \infty} \langle x, e_j \rangle = 0$$

Dokaz: $Y_n = \mathcal{L}(\{e_1, \dots, e_n\})$

$x \in X$

$$\exists P_Y(x) = \sum_1^n \langle x, e_j \rangle e_j$$

projekcija je vedno krajša od x

$$\|P_Y(x)\|^2 = \sum_1^n \|\langle x, e_j \rangle e_j\|^2 = \sum_1^n |\langle x, e_j \rangle| \leq \|x\|^2$$

$$\Longrightarrow \lim_{n \to \infty} \sum_1^n \|\langle x, e_j \rangle\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

Trditev: Naj bo $(c_j)_{j=1}^{\infty}$ zaporedje števil (ali $\mathbb{R}$ ali $\mathbb{C}$) za katero velja

$$\sum_1^{\infty} |c_j|^2 < \infty$$

Naj bo $(X, \langle\,\rangle)$ Hilbertov prostor $(e_j)_1^{\infty}$ ONS

Tedaj $\exists x \in X$, za katerega velja $c_j = \langle x, e_j \rangle$ za $\forall j$.

$$x = \sum_1^{\infty} c_j e_j = \lim \sum_1^{N} c_j e_j$$

$(X, \langle\ \rangle)$ hilbertav e_j ONB

$x \in X \rightsquigarrow (\langle x, e_j \rangle)_j \quad \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$

$\Rightarrow \exists \tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$

A1: je $\tilde{x} = x$

Zemuda

20.2

Zgled (pred prejšnjim dokazom)

<u>Modelni hilbertov prostor</u> ℓ^2

Prostor zaporedij

$$\ell^2 = \{(a_j)_j \,;\, a_j \in \mathbb{R} \,,\, \sum_1^\infty |a_j|^2 < \infty\}$$

$$\langle (a_j), (b_j) \rangle = \sum a_j b_j \quad \text{nad } \mathbb{R}$$

$$\sum a_j \bar{b}_j \quad \text{nad } \mathbb{C}$$

$$\|a_j\|^2 = \sum |a_j|^2$$

$(X, \langle\,\rangle)$ hilbertov (e_j) KONS

$$x \longmapsto (\langle x, e_j \rangle)_j \in \ell^2$$

$$e_j = (0, \dots, 0\, 1\, 0 \dots 0)$$

Dokaz:

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Rightarrow 6 \Rightarrow 5$

$1) \Rightarrow 2)$

$x, y \in X \quad x = \sum_1^\infty \langle x, e_j \rangle e_j$

$\langle x, y \rangle = \langle \sum_1^\infty \langle x, e_j \rangle e_j, y \rangle =$

$= \sum_1^\infty \langle x, e_j \rangle \langle e_j, y \rangle$

$2) \Rightarrow 3)$

$x = y \quad \langle x, y \rangle = \|x\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2$

$3) \Rightarrow 4)$

če bi bil ONS vsebovan v strogo večjem ONS

potem $\exists e_0 \perp e_j$ za $\forall j$ $\|e_0\| = 1$

Vstavimo v parcevalovo enakost

$\|e_0\|^2 = 1^2 = \sum_1^\infty |\langle e_0, e_j \rangle|^2 = \sum_1^\infty 0 = 0$

$4) \Rightarrow 5)$

$x \perp e_j$ za $\forall j$

če $x \neq 0$ tvorimo $e_0 = \frac{x}{\|x\|}$

$(e_0, (e_j)_j)$ je strogo večji KONS, kar ne more biti $\times$

$5) \Rightarrow 1)$

$x \in X \quad \langle x, e_j \rangle_j$

$\tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$ ali sta enaka?

$\vec{v} = x - \tilde{x} = x - \sum_1^\infty \langle x, e_j \rangle e_j$

$\langle v, e_i \rangle = \langle x, e_i \rangle - \sum_1^\infty \langle x, e_j \rangle \langle e_j, e_i \rangle =$

$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$

$\vec{v} = 0 \Rightarrow x = \tilde{x}$

$1) \Rightarrow 6) \quad \forall x = \sum_1^\infty \langle x, e_j \rangle e_j = \lim \sum_1^N \langle x, e_j \rangle e_j$

$\forall \varepsilon > 0 . \exists N_0 \in \mathbb{N} . \|x - \sum_1^{N_0} \langle x, e_j \rangle e_j\| < \varepsilon$

$6) \Rightarrow 5)$

Naj bo $x \perp e_j$ za $\forall j$

$\underline{x = 0}$

$\varepsilon > 0$

$\exists$ končna linearna kombinacija $\sum_1^N \lambda_j e_j$

daje $\|x - \sum_1^N \lambda_j e_j\| < \varepsilon$

$\|x\|^2 = \langle x, x \rangle = \langle x - \sum_1^N \lambda_j e_j, x \rangle \leq$

ker pravokotnost

$\leq \|x - \sum_1^N \lambda_j e_j\| \cdot \|x\| < \varepsilon \|x\|$

1) $x = 0 \Rightarrow \checkmark$

2) $x \neq 0 \Rightarrow \|x\| < \varepsilon$ za $\forall \varepsilon > 0 \Rightarrow x = 0$ $\times$

Osredotočimo se na $L^2(-\pi,\pi)$

ONS: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \dots \right.$

$\dots \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx), \dots$

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} f^2(x)}$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \qquad \|f_0\|_2 = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi}} = 1$$

$$\sqrt{\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin^2(nx)\, dx} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\, dx} = 1$$

Opomba: nad $\mathbb{C}$: $\frac{1}{2\pi} e^{inx}$ $n \in \mathbb{Z}$ vzamemo za ONS

$n \neq m$

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{inx} e^{imx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

$$= \frac{1}{2\pi(n-m)} e^{i(n-m)x}\Big|_{-\pi}^{\pi} = 0$$

$f: (-\pi, \pi) \Rightarrow \mathbb{R}$ periodična funkcija s periodo 2π

Klasični furierovi koeficienti

$f \in L^2(-\pi, \pi)$ (Recimo da imamo kanos)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \qquad n \in \{0, 1, \dots\}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) = \langle f, \frac{1}{\pi} \cos(nx) \rangle =$$

$$= \frac{1}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \rangle = \sqrt{\pi}\, a_n$$

$$\langle f, \frac{1}{\pi} \sin(nx) \rangle = \sqrt{\pi}\, b_n$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot 1\, dx = \langle f, \frac{1}{\pi} \rangle = \frac{\sqrt{2}}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle =$$

$$\langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{\sqrt{\pi}}{\sqrt{2}} a_0$$

Če je $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x \ldots\}$ KONS

potem velja parcevalova enakost

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2(x)\,dx = \frac{\pi}{2}a_n + \sum \pi(|a_n|^2 + |b_n|^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$

Posledica:

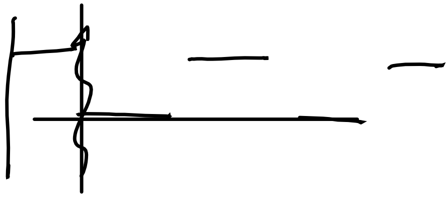
ker je $\{\frac{1}{\sqrt{2\pi}} \ldots\}$ ONS velja

$$\lim_{n\to\infty} a_n = 0 \qquad \lim_{n\to\infty} b_n = 0$$

(Riemann-Lebegueova lema)

Zgled

$$f(x) = \begin{cases} 1; & 0 \leq x \leq \pi \\ 0; & -\pi < x < 0 \end{cases}$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx =$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos(nx)\, dx$$

$$a_0 = 1 \qquad n > 0$$

$$a_n = \frac{1}{\pi} \cdot \frac{1}{n} \sin(nx) \Big|_0^{\pi}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx)\, dx = -\frac{1}{\pi n} \cos(nx) \Big|_0^{\pi}$$

$$= \frac{1}{n\pi} (1 - (-1)^n)$$

$$b_{2k} = 0$$

$$b_{2k+1} = \frac{2}{\pi(2k+1)}$$

KONS:

$$f(x) \overset{L^2}{=} \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x)$$

$$= 0$$

Parsevalova enakost

$$\frac{1}{\pi} \pi = 1 = \frac{1}{2} + \sum_0^{\infty} \frac{4}{\pi^2} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$= \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\frac{3}{4} S = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}$$

Izrek:

Naj bo f odsekoma zvezna in odsekoma odvedljiva periodična funkcija s periodo 2π

1) Na vsakem intervalu dolžine 2π ima največ končno mnogo točk nezveznosti in v vsaki točki obstajata leva in desna limita

oznaki

$$x_0 \in \mathbb{R}. \ \exists \lim_{x \uparrow x_0} f(x) = f(x_0 - 0) = f(x_0^-)$$

$$\exists. \lim_{x \downarrow x_0} f(x) = f(x_0 + 0) = f(x_0^+)$$

2) V vsaki točki obstajata levi in desni odvod

$$\lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0^+)}{h}$$

$$\lim_{h \uparrow 0} \frac{f(x_0 + h) - f(x_0^-)}{h} = \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0^-)}{(-h)}$$

Tedaj za $\forall x \in \mathbb{R}$. velja $\frac{f(x+) + f(x-)}{2} =$

$$= \frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

Pomožne trditve

1) Naj bo $f: \mathbb{R} \to \mathbb{R}$ periodična s periodo p odsekoma zvezna
Tedaj je za $\forall a \in \mathbb{R}$

$$\int_a^{a+p} f(x)\,dx = \int_0^p f(x)\,dx$$

Dokaz:

$$\int_a^{a+p} f(x)\,dx = \int_a^p f(x)\,dx + \int_p^{a+p} f(x)\,dx$$

$$x = t+p$$

$$= \int_a^p f(x)\,dx + \int_0^a \underbrace{f(t+p)}_{= f(t)}\,dt =$$

$$= \int_a^p f(x)\,dx + \int_0^a f(t)\,dt = \int_0^p f(x)\,dx$$

2)

$$\frac{1}{2}+\sum_{1}^{n}\cos(kx)=\frac{1}{2}\frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})}=D_n(x)$$

(Dirichletova jedro)

Dokaz:

$$\frac{1}{2}+\sum_{1}^{n}\cos(kx)=\frac{1}{2}+\sum_{1}^{n}\frac{e^{ikx}+e^{-ikx}}{2}=$$

$$=\frac{1}{2}\sum_{j=-n}^{n}e^{ijx}=\frac{1}{2}e^{-inx}(1+e^{ix}+e^{2ix}+\dots+e^{2nix})$$

$$=\frac{1}{2}e^{-inx}\frac{1-e^{(2n+1)ix}}{1-e^{ix}}=$$

$$=\frac{1}{2}\frac{e^{(n+1)ix}-e^{-inx}}{e^{ix}-1}=\frac{1}{2}\frac{e^{i(n+\frac{1}{2})x}-e^{-i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}$$

$$=\frac{1}{2}\frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}$$

3)

$$\int_{-\pi}^{\pi} D_n(x)\,dx = \pi$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$$

4) $D_n(x)$ je soda funkcija, 2π periodična, gladka.

5) $\frac{1}{2}\ \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} =$

$$= \frac{1}{2}\ \frac{\sin(nx)\cos\frac{x}{2} + \cos(nx)\sin(\frac{x}{2})}{\sin\frac{x}{2}} =$$

$$= \frac{1}{2}\left(\sin(nx)\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \cos(nx)\right)$$

Opombe:

f je lihe na $(-\pi,\pi)$

$a_n = 0$

f razvijamo samo po $\{\sin(nx)\}$

f soda na $(-\pi,\pi) \Rightarrow b_n = 0$

f razvijemo le po $\{1, \cos(nx)\}$

f iz $[0,\pi]$ lahko liho razširimo na $(-\pi,\pi)$

$$x \mapsto \begin{cases} f(x) & x>0 \\ -f(-x); & x<0 \end{cases}$$

Tako funkcijo lahko razvijamo le po $\sin(nx)$

$$f(x) = \sum_1^\infty b_n \sin(nx)$$

$$\int_0^\pi f(x)\sin(nx)\,dx = b_n\int_0^\pi \frac{1-\cos(2nx)}{2} = \frac{\pi}{2}\underbrace{b_n}_{\text{novi } b_n}$$

Če f razširimo kot soda funkcija na $(-\pi,\pi)$

$$x \longmapsto \begin{cases} f(x) & x>0 \\ f(-x) & x<0 \end{cases}$$

Tako funkcijo razvijamo po $\{1, \cos(nx)\}$

Opombe 2:

$f \in C^k(\mathbb{R})$ 2π periodična $k \in \mathbb{N}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{u} \underbrace{\cos(nx)\,dx}_{dv} =$$

$$= \frac{1}{\pi}\left(\underbrace{\frac{1}{n}\sin(nx) f(x)\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx \right)$$

$= \ldots$ naredimo to k krat $\ldots =$

$$a_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx$$

$$a_n = \frac{-1}{n\pi}\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx$$

$\vdots$

$$a_n = \frac{\pm 1}{\quad}\int_{-\pi}^{\pi} f^{(k)}(x) \begin{matrix}\cos(nx) \\ \sin(nx)\end{matrix}\, dx$$

omejena

nekaj od tega

$$a_n, b_n = \mathcal{O}\left(\frac{1}{n^2}\right)$$

velikostni red

večkrat kot je odvedljiva f (kot periodična funkcija) hitreje gredo furierovi koeficienti proti 0

Dokaz

$f \in L^2(-\pi, \pi)$

$$\sqrt{\int_{-\pi}^{\pi} |f|^2 dx} = \|f\|_2$$

Klasični furijejovi koeficienti:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \quad n = 1, 2, \dots$$

$$f(x) \stackrel{L_2}{=} \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Parsevalova enakost

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx = \frac{|a_0|^2}{2} + \sum_1^{\infty} (|a_n|^2 + |b_n|^2)$$

f periodična s periodo 2π in odsekoma zvezna

- $\forall x_0 \in \mathbb{R}.\ \exists f(x_0+), f(x_0-)$ limiti in
- odsekoma odvedljiva
- dostajata levi in desni odvod

Potem: $\forall x_0$: $\frac{a_0}{2} + \sum_1^{\infty} (a_n \cos(nx) + b_n \sin(nx)) =$

$$= \frac{f(x_0+) + f(x_0-)}{2}$$

Vemo:

- f periodična s periodo p
- $\forall a \in \mathbb{R}.\ \int_a^{a+p} f(x)\,dx = \int_0^p f(x)\,dx$
- $\frac{1}{2} + \sum_1^N \cos(kx) = D_N(x) = \frac{1}{2} \frac{\sin(N + \frac{1}{2})x}{\sin(\frac{1}{2}x)}$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x)\,dx = 1 \quad \text{soda}$$

$$\| $$

$$\frac{2}{\pi} \int_0^{\pi} D_n(x)\,dx$$

Dokaz >>

Dokaz:

$$a_n\cos(nx_0)+b_n\sin(nx_0) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt\cdot\cos(nx_0) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt\ \sin(nx_0) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\big(\cos(nt)\cos(nx_0)+\sin(nt)\sin(nx_0)\big)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n(t-x_0))\,dt$$

(opomba) $\int_{-\pi}^{\pi} f(t)g(x_0-t)\,dt$ ..... konvolucija

$t-x_0=y \qquad t=x_0+y$

$$= \frac{1}{\pi}\int_{-\pi-x_0}^{\pi-x_0} f(x_0+y)\cos(ny)\,dy =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\cos(ny)\,dy$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0+y)\,dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\cdot\frac{1}{2}\,dy$$

$$\frac{a_0}{2}+\sum_{1}^{n} a_k\cos(kx_0)+b_k\sin(kx_0) = S_n(x_0)$$

$$S_n(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\left(\tfrac{1}{2}+\cos y+\dots\cos(ny)\right)dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)D_n(y)\,dy =$$

$$= \frac{1}{\pi}\left(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy + \int_{-\pi}^{0} f(x_0+y)D_n(y)\,dy\right)$$

$y \rightsquigarrow -y$

$$= \frac{1}{\pi}\left(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy + \int_{0}^{\pi} f(x_0-y)D_n(y)\,dy\right.$$

soda funkcija

$$\xrightarrow{n\to\infty} \frac{f(x_0+)+f(x_0-)}{2}$$

- - - - - - - - - - -

$$\frac{1}{\pi}\int_{0}^{\pi} \big(f(x_0+y)+f(x_0-y)\big)D_n(y)\,dy - \frac{f(x_0+)+f(x_0-)}{2}$$

$$\frac{2}{\pi}\cdot\int_{0}^{\pi} D_n(y)\,dy = 1$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \big(f(x_0+y)-f(x_0+)\big)D_n(y) + \big(f(x_0-y)-f(x_0-)\big)D_n(y)\,dy$$

Pogledamo sumando

$$\frac{1}{2}\int_{0}^{\pi} \big(f(x_0+y)-f(x_0+)\big)\frac{\sin(n+\frac{1}{2})y}{\sin(\frac{y}{2})}\,dy =$$

$$= \frac{1}{2}\int_{0}^{\pi} \big(f(x_0+y)-f(x_0)\big)\frac{\cos(\frac{y}{2})}{\sin(\frac{y}{2})}\sin(ny)+\cos(ny)\,dy$$

Pogledamo sumand

$$\int_{0}^{\pi} f(x_0+y)-f(x_0^+)\cos(ny)\,dy \xrightarrow[n\to\infty]{} 0$$

- - - - - - - - - - - - -

$$F(y) = \begin{cases} f(x_0+y)-f(x_0^+) \ ; & 0\le y\le\pi \\ 0 \ ; & -\pi<y<0 \end{cases}$$

odsekoma zvezna

$$= \int_{-\pi}^{\pi} F(y)\cos(ny)\,dy \xrightarrow[n\to\infty]{\text{Rieman-Lebesgueova lema}} 0$$

Še

$$\int_{0}^{\pi} f(x_0+y)-f(x_0+))\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}\sin(ny)\,dy$$

$$= \int_{0}^{\pi} \frac{f(x_0+y)-f(x_0+)}{y}\ \frac{y}{\sin\frac{y}{2}}\cos(\tfrac{y}{2})\sin(ny)\,dy$$

$$G(y) = \begin{cases} \frac{f(x_0+y)-f(x_0+)}{y}\ \frac{y}{\sin\frac{y}{2}}\cdot\cos\frac{y}{2} \ ; & 0<y\le\pi \\ 0 \ ; & -\pi<y<0 \end{cases}$$

$\int_{-\pi}^{\pi} G(y)\sin(ny)\,dy$ za $y\neq 0$ je f odsekoma zvezna

za $y=0$ pa obstaja limita

$$\lim_{y\uparrow 0} G(y) = 0$$

$$\lim_{y\downarrow 0} G(y) = 2\lim_{y\downarrow 0}\frac{f(x_0+y)-f(x_0+)}{y}$$

obstaja po predpostavki.

Riemann-Lebesgueova lema:

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} G(y)\sin(ny)\,dy = 0$$

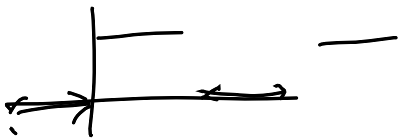
Za drugi sumand uporabimo podobne argumente

■

Zgled:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi < x < 0 \end{cases}$$

Furierova vrsta: $\frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x$



$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2}$$

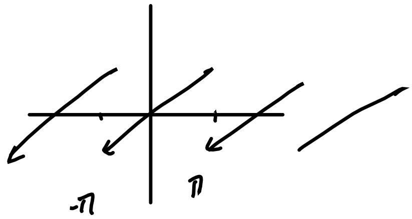
$x = \frac{\pi}{2}$: $\quad 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{1}{2k+1} \sin(2k+1)\frac{\pi}{2}$

+1 ali pa −1

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

zgled

$f(x) = x$



$a_n = 0$ za $\forall n$

ker je na $(-\pi, \pi)$ funkcija liha

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\,dx$$

$u = x \quad dv = \sin(nx) \quad n \geq 1$

$du = dx \quad v = -\frac{1}{n}\cos(nx)$

$$b_n = \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx)\right)\Big|_{-\pi}^{\pi} + \underbrace{\frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx}_{=0}$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + \frac{(-\pi)}{n}\cos(n\pi)\right) =$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$x \overset{L^2(-\pi,\pi)}{=} \sum_{1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx)$$

Po točkah:

$x = 0: \quad 0 = 0$

$x = \pi: \quad \frac{\pi + (-\pi)}{2} = 0$ ✓

$x = \frac{\pi}{2}: \quad \frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$

Parsevalova enakost

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{1}^{\infty}\frac{4}{n^2} = 4\sum_{1}^{\infty}\frac{1}{n^2}$$

$\parallel$ na kvadrat

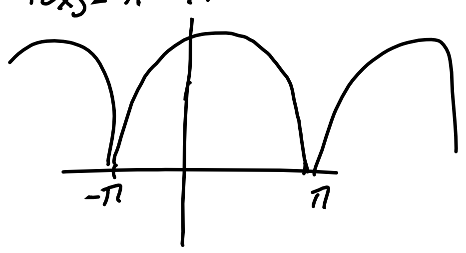
$$\frac{1}{\pi}\left(\frac{1}{3}x^3\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi}\pi^3\cdot\frac{1}{3} = \frac{2}{3}\pi^2 = 4\sum_{1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{1}^{\infty}\frac{1}{n^2}$$

Zgled.

$f(x) = \pi^2 - x^2$

soda, nenegativna zvezna, v vsaki točki ima levi in desni odvod



sodost $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\,dx = \frac{2}{\pi}\left(\pi^3 - \frac{1}{3}\pi^3\right) =$$

$$= \frac{4}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{(\pi^2 - x^2)}_{u}\,\underbrace{\cos(nx)\,dx}_{dv} =$$

$$= \frac{1}{\pi}(\pi^2 - x^2)\frac{1}{n}\sin(x)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} 2x\sin(nx)\,dx =$$

$$= \frac{2}{\pi n}\int_{-\pi}^{\pi}\underbrace{x}_{u}\,\underbrace{\sin(nx)\,dx}_{dv} =$$

$$= \frac{2}{\pi n}\left(x\left(-\frac{1}{n}\cos(nx)\right)\Big|_{-\pi}^{\pi} - \overbrace{\int_{-\pi}^{\pi}\cos(nx)\,dx}^{0}\right) =$$

$$= \frac{2}{\pi n^2}\left(\pi(-1)^{n+1} + (-1)^{n+1}\pi\right) = \frac{4}{n^2}(-1)^{n+1}$$

$$f(x) = \frac{2}{3\pi^2} + \sum_{1}^{\infty}\frac{4}{n^2} - (-1)^{n+1}\cos(nx) \quad \forall x \in \mathbb{R}$$

$x = 0 \Rightarrow$

$$f(0) = \pi^2 = \frac{2}{3}\pi^2 + \sum_{1}^{\infty}\frac{4}{n^2}(-1)^{n+1}$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$x = \pi \Rightarrow$

$$0 = \frac{2}{3}\pi^2 + \sum_{1}^{\infty}\frac{4}{n^2}\overbrace{(-1)^{n+1}(-1)^n}^{-1}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

kons:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)^2\,dx = \frac{1}{2}\frac{16}{9}\pi^4 + \sum_{1}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{\pi}\int_{0}^{\pi}(\pi^4 - 2x^2\pi^2 + x^4)\,dx = \frac{8\pi^4}{9} + 16\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left(\pi^5 - \frac{2\pi^5}{3} + \frac{1}{5}x^5\right) = \frac{8\pi^4}{9} + 16S$$

$$\frac{15 - 10 + 3}{15}\pi^4 = \frac{4\pi^4}{9} + 8S$$

$$\|$$

$$\frac{8\pi^4}{15}$$

$$8S = \frac{8\pi^4}{15} - \frac{4\pi^4}{9}$$

$$2S = \frac{2\pi^4}{15} - \frac{\pi^4}{9} = \frac{6\pi^4}{45} - \frac{5\pi^4}{45} = \frac{\pi^4}{45}$$

$$S = \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$a_1, a_2, \ldots$ $\lim_{n\to\infty} a_n = a$

$1, -1, 1, -1$

$\frac{1}{1}, \frac{1-1}{2}, \frac{1-1+1}{3} \to 0$

$a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}$

konvergira $\Rightarrow$ konvergira u prosjeku

$$S_n^{(x_0)} = \int_{-\pi}^{\pi} f(x_0+y)\, D_n(y)\, dy$$

$$\frac{S_0(x) + S_1(x) + \ldots + S_{n-1}(x)}{n} = \sigma_n(x) \xrightarrow[n\to\infty]{} f$$

cesárove delne vsote

če je f zvezna je konvergenca enakomerna

Fejérjevo jedro: $F_N(x) = \frac{1}{N}\sum_0^{N-1} D_n(x)$

Trditev:

1) $F_N(x) = \frac{1}{2N}\left(\frac{\sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^2$

2) F_N je soda

3) $F_N(x) \geq 0$ za $\forall x$

4) $\frac{1}{\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$

5) $\forall a.\ 0<a<\pi.\ \lim_{N\to\infty} F_N(x) = 0$

enakomerno na $a \leq |x| \leq \pi$

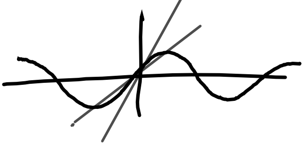
Dokaz:

1) $\Rightarrow$ 2) očitno

1) $\Rightarrow$ 3) očitno

Iz definicije $\Rightarrow$ 4) $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$

1) $\Rightarrow$ 5)



$\forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ velja $\frac{2}{\pi}|y| \leq |\sin y|$

$0 < a \leq |x| \leq \pi$

$0 < \frac{a}{2} \leq |\frac{x}{2}| \leq \frac{\pi}{2}$

$\frac{2}{\pi}\cdot\frac{|x|}{2} \leq |\sin\left(\frac{x}{2}\right)|$

$\frac{1}{|\sin\frac{x}{2}|} \leq \frac{\pi}{|x|} \leq \frac{\pi}{a}$

$F_N(x) = \frac{1}{2N}\frac{|\sin\left(\frac{Nx}{2}\right)|^2}{|\sin\left(\frac{x}{2}\right)|^2} \leq \frac{1}{2N}\cdot\frac{\pi^2}{a^2}$

$\lim_{n\to\infty} F_n(x) = 0$ enakomerno na $a \leq |x| \leq \pi$

1) $F_N(x) = \frac{1}{N}\sum_0^{N-1} D_n(x) = \frac{1}{N}\sum_0^{N-1}\frac{1}{2}\frac{\sin(n+\frac{1}{2})x}{\sin\left(\frac{x}{2}\right)} =$

$= \frac{1}{2N}\frac{1}{\sin^2\left(\frac{x}{2}\right)}\sum_0^{N-1}\sin(n+\frac{1}{2})x\cdot\sin\left(\frac{x}{2}\right) =$

$= \frac{1}{2N\sin^2\left(\frac{x}{2}\right)}\frac{1}{2}\underbrace{\sum_0^{N-1}(\cos(nx) - \cos(n+1)x)}$

$(1-\cos x) + (\cos x - \cos 2x) + (\cos 2x - \cos 3x) + \dots$

$= 1 - \cos(Nx)$

$= \frac{1-\cos(Nx)}{4N\sin^2\left(\frac{x}{2}\right)} = \frac{1}{2N}\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$

Izrek:
Naj bo f zvezna 2π periodična funkcija
Potem cezárove delne vsote

$$\sigma_N(x) = \frac{1}{N}(S_0(x)+\dots+S_{N-1}(x))$$

konvergirajo k f enakomerno na
$[-\pi,\pi]$ oziroma na $\mathbb{R}$

$$S_N(x_0) = a_0 + \sum_{1}^{N}(a_k\cos(kx) + b_k\sin(kx))$$

Trigonometrični polinom... končne kombinacije sinusov in kosinusov

Dokaz:

$$\sigma_N(x) = \frac{1}{N}(S_0(x)+\dots+S_{N-1}(x))$$

$$S(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\,D_N(y)\,dy$$

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y)\cdot F_N(y)\,dy \xrightarrow{\text{enakomerno}} f(x)$$

Naj bo $\varepsilon > 0$. ker je f zvezna in periodična je enakomerno zvezna

$$\exists\delta>0.\ \forall y<\delta\ |f(x+y)-f(x)| < \frac{\varepsilon}{2}$$
$$\forall x\in\mathbb{R}$$

$$|\sigma_N(x) - f(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y)F_N(y)\,dy - f(x)\frac{1}{\pi}\int_{-\pi}^{\pi}F_N(y)\,dy\right|$$

$$= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x+y)-f(x))F_N(y)\,dy\right| \leq$$

$F_N(y)\geq 0$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+y)-f(x)|\,F_N(y)\,dy =$$

$$= \underbrace{\frac{1}{\pi}\int_{-\delta}^{\delta}|f(x+y)-f(x)|F_N(y)\,dy}_{} + \int_{\delta\leq|y|\leq\pi}|f(x+y)-f(x)|F_N(y)\,dy$$

$$< \frac{1}{\pi}\int_{-\delta}^{\delta}F_N(y)\,dy\cdot\frac{\varepsilon}{2} < \frac{\varepsilon}{2}$$

f je zvezna in periodična torej je omejena

$$\exists M\in\mathbb{R}\quad |f(z)|\leq M\ \forall z\in\mathbb{R}$$

$$\Rightarrow f(x+y)-f(x)\leq 2M \quad \text{za } \forall x \text{ in } \forall y \in\mathbb{R}$$

$$\frac{1}{\pi}\int_{\delta<|y|<\pi}|f(x+y)-f(x)|\,F(y)\,dy \leq 2M\frac{1}{\pi}\int_{\delta<|y|\leq\pi}F_N(y)\,dy$$

Vemo da je limita $\lim_{N\to\infty}F_N(y) = 0$
enakomerno na $\delta\leq|y|\leq\pi$

$$\exists N_0.\ \forall N\geq N_0.\ |F_N(y)|\leq\frac{\varepsilon}{2M\cdot 4} = \frac{\varepsilon}{8M}$$

$$\frac{2M}{\pi}\int_{\delta<|y|\leq\pi}F_N(y)\,dy \leq \frac{2M}{\pi}\cdot\frac{\varepsilon}{8M}\cdot 2\pi = \frac{\varepsilon}{2}$$

Za $N\geq N_0$ je

$$|\sigma_N(x) - f(x)| < \varepsilon \quad \text{za } \forall x\in\mathbb{R}$$

Izrek:

$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx);\ n \in \mathbb{N}\}$

je KONS v $L^2(-\pi,\pi)$

Dokaz:

$L^2(-\pi,\pi)$ je napolnitev $C([-\pi,\pi])$ v

$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$

ONS je KONS $\Leftrightarrow$ končne linearne kombinacije vektorjev iz ONS goste v prostoru

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OPOMBA: KONČNE LINEARE KOMBINACIJE $\sin(nx)$ $\cos(nx)$ $n \in \mathbb{N}_0$ so trigonometrični polinomi.

Ali so trigonometrični polinomi gosti v $L^2(-\pi,\pi)$?

1) $C(-\pi,\pi)$ so goste v $L^2(-\pi,\pi)$ (privzamemo)

$f \in L^2(-\pi,\pi)$. $\varepsilon > 0$. $\exists\ \tilde{f} \in C[-\pi,\pi]$. $\|f-\tilde{f}\| < \frac{\varepsilon}{2}$

2) Ali so trigonometrični polinomi gosti v $(C[-\pi,\pi], d_2)$

če da:

$\exists T(x)$ trigonometrični polinom

$T(x) = \lambda_0 + \sum_1^N \lambda_n \cos(nx) + \mu_n \sin(nx)$

$\|\tilde{f} - T\|_2 < \frac{\varepsilon}{2} \Rightarrow \|f-T\|_2 < \varepsilon$

OPOMBA:

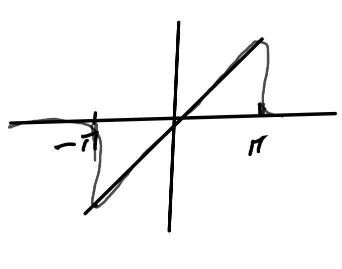
$g_n \in L^2[-\pi,\pi] \xrightarrow[\text{na } [-\pi,\pi]]{\text{enakomerno}} f$

$\overset{?}{\Rightarrow} \lim_{n\to N} g_n = f$ v d_2 ✓

$d_2(f,g_n)^2 = \int_{-\pi}^{\pi}(f-g_n)^2 \xrightarrow{n\to N} 0$

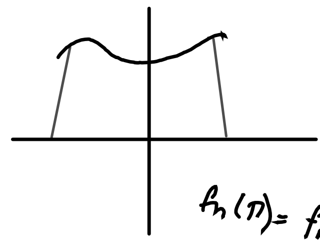
Vemo: f zvezna 2π periodična $\Rightarrow$ jo lahko poljubno dobro enakomerno aproksimiramo s trigonometričnimi polinomi:

$\|f - \sigma_N\|_\infty < \varepsilon$



Ni zvezne 2π periodične funkcije

$f_n(x) = \begin{cases} f(x) \\ -n(f(\pi-\frac{1}{n})(x-\pi); & \pi - \frac{1}{n} < x \le \pi \\ nf(-\pi+\frac{1}{n})(x+\pi) & -\pi \le x < -\pi + \frac{1}{n} \end{cases}$



$f_n(\pi) = f_n(-\pi) = 0$

zvezna 2π periodična funkcija

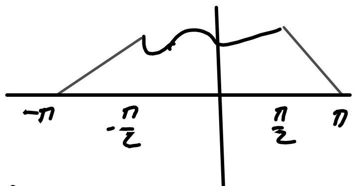
$\|f-f_n\|_2^2 = \int_{-\pi}^{\pi}(f_n(x)-f(x))^2\,dx \le \frac{2}{n}(2\pi)^2 \xrightarrow{n\to\infty} 0$

Izrek: (Weierstrassov)

Naj bo f zvezna na $[a,b]$.
Naj bo $\varepsilon > 0$. Potem $\exists$ polinom p da je
$\|f-p\|_\infty < \varepsilon$

Dokaz: Dovolj je opazovati $a = -\frac{\pi}{2}$ $b = \frac{\pi}{2}$

f zvezna na $[-\frac{\pi}{2}, \frac{\pi}{2}]$
f zvezno razširimo na $[-\pi, \pi]$



$$\tilde{f} = \begin{cases} f(x) & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -\frac{2}{\pi} f(\frac{\pi}{2})(x-\pi) & \frac{\pi}{2} < x \leq \pi \\ \frac{2}{\pi} f(-\frac{\pi}{2})(x+\pi) & -\pi \leq x < -\frac{\pi}{2} \end{cases}$$

$\tilde{f}$ je zvezna 2π periodična.

Za $\varepsilon > 0$ $\exists$ T trigonometrični polinom, da je $\|\tilde{f} - T\|_\infty < \frac{\varepsilon}{2}$

$\cos(nx)$, $\sin(nx)$ jih je končno

Aproksimiramo s Taylorjevimi polinomi enakomerno.

$\blacksquare$

Vektorska analiza
(v $\mathbb{R}^2$ ali $\mathbb{R}^3$)

$D^{odp} \subseteq \mathbb{R}^3$ $\quad u: D \to \mathbb{R}$ zvezna funkcija

jo imenujemo <u>skalarno polje</u>

vsaki točki v D priredi skalar

$D \subseteq \mathbb{R}^3$ $\quad \vec{R}: D \longrightarrow \mathbb{R}^3$ zvezna

<u>vektorsko polje</u> vsaki točki priredi

vektor $\vec{R}(t)$

$\mathbb{R}^3$

$u(x,y,z) = x$ v bazi $\vec{e}_1, \vec{e}_2, \vec{e}_3$ $(\vec{i}, \vec{j}, \vec{k})$

Baza: $\vec{p} = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_3$

$$\vec{g} = -\frac{1}{\sqrt{3}} e_1 + \frac{1}{\sqrt{3}} e_2 + \frac{1}{\sqrt{3}} e_3$$

$$\vec{r} = \frac{1}{\sqrt{6}} e_1 + \frac{2}{\sqrt{6}} e_2 - \frac{1}{\sqrt{6}} e_3$$

Točka (x,y,z) ima v Bazi $(\vec{p}, \vec{g}, \vec{r})$

ONB

koordinate (α, β, γ)

$(\alpha, \beta, \gamma) \longmapsto \alpha$ ni ista funkcija

ONB je pozitivno orientirana, če je
$[\vec{p}, \vec{q}, \vec{r}] > 0 \; (\Leftrightarrow \vec{r} = \vec{p} \times \vec{q})$

oziroma negativno orientirana če
$[\vec{p}, \vec{q}, \vec{r}] < 0 \; (\Leftrightarrow \vec{r} = -\vec{p} \times \vec{q}$

(oznaka od zdej naprej za mešani produkt bo $(\vec{p}, \vec{q}, \vec{r})$ najbrž

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} =: U$$

$(x, y, z) \; (e_1 \ldots e_n)$

$(\alpha, \beta, \gamma) \; (\vec{p}, \vec{q}, \vec{r})$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$U^T U = I$$

$$U^{-1} = U^T$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$u(x, y, z)$ v (e_1, e_2, e_3) koordinatah

$$\tilde{u}(\alpha, \beta, \gamma) = u\left(U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}\right)$$

Vektorska polja

$$\vec{R}(x,y,z) = (X(x,y,z),\ Y(x,y,z),\ Z(x,y,z))$$

$V(\alpha,\beta,\gamma)$ koordinatah

$$\tilde{R}(\alpha,\beta,\gamma) = (U^T \circ \vec{R} \circ U)\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$$

$$\tilde{u} = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$\vec{R}(x,y,z) = (x+2y+3z,\ x+2y+3,\ x+2y+3z)$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$$

$$x = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$y = \frac{\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$

$$z = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{3}} - \frac{\gamma}{\sqrt{6}}$$

$$x+2y+3z = \frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma = c$$

to je $(R \circ U)\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = (c,c,c)$

$$U(R \circ U\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}) = \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}\right)c$$