5.12

Formule

$\#$ permutacij $= |\{\sigma : A \xrightarrow{\sim} A\}| = n!$

$B \subseteq A \Rightarrow |A - B| = |A| - |B|$

$|A \times B| = |A||B|$

Št neurejenih izborov podmnožice k od $n = \binom{n}{k}$

multinomski: n_1 tipa 1 n_2 tipa 2, ... n_k tipa k

$\#$ permutacij neurejeno $= \dfrac{n!}{n_1! \cdot \ldots \cdot n_k!}$

① $J = \{ \underbrace{\text{juha, solata}}_{\text{predjed}}, \underbrace{\overset{\text{meso}}{\text{riba}}, \text{zelenjava}}_{\text{glavna jed}}, \underbrace{\text{sladoled, torta}}_{\text{sladica}} \}$

$2 \cdot 3 \cdot 2 = 12$ je št možnih kosil

② kup kart 52 kart. karte razdelimo na 4 igralce vsak dobi 13

1. kolko načinov lahko karte tako razdelimo
2. kolko da ima vsak igralec karte samo enega tipa (pik, srce, križ, kara)

$$4! \cdot \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} =$$

$$= \frac{52! \; (52-13)! \cdots}{13!\,(52-13)!\; 13! \cdots \;13} = \frac{52!}{(13!)^4} 4!$$

2. 1, 4!

③ Kostanjevica na Krki ima 702 preb.

Dokaži: Najmanj dva prebivalca imata enaki začetnici (25 črk)

$$\begin{array}{r} 25\cdot 25 = 625 \\ \underline{50} \\ \underline{125} \\ 623 \end{array}$$

$702 - 625 > 0$

④ imamo 10 kart

2 rdeči, 3 zelene, 5 modrih

1). # možnosti da je prva rdeča karta izvlečena pred [illegible]

1. 0
2. $\dfrac{8!}{1!2!5!}$
3. $\dfrac{7!}{2!5!} + 2\cdot\dfrac{7!}{4!1!2!}$
4.

R Z Z Z

$$\frac{\binom{10}{4}\cdot 6}{2} = \frac{10\cdot\cancel{8}\cdot 9\cdot 7}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot 2}\cdot\cancel{6} = \frac{1260}{2} = 630$$

modre

$$\binom{10}{5}4 = \frac{10!}{5!5!}4 = \frac{10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2}\cdot 4 =$$

$=$

ii) možnosti da bo prva rdeča krogla izvlečena pred zadnjo zeleno

$$\frac{4\cdot 3}{2} = 6 \quad \_ \; \_ \; \_ \; \_ \; \_ \; \_$$

$$\binom{10}{5}\cdot\binom{4}{2}$$

mmmau

zelena

$$\binom{10}{5}6+\binom{10}{5}3$$

$$\frac{10!}{3!5!2!} - \binom{10}{5} = \frac{10!}{5!}\left(\frac{1}{3!2!} - \frac{1}{5!}\right) =$$

$$= \frac{10!}{5!}\left(\frac{5! - 12}{5!2!3!}\right) =$$

$$= \frac{10!}{5!}\,\frac{4\cdot 5 - 2}{5!2!} = \frac{10!}{5!5!}\,\frac{18}{2}$$

$$= \binom{10}{5}9$$

13.10

1) 3 rdeče
1 zelena
5 belih

izberemo 2

i) $\Omega = ?$

$$\Omega = \binom{V}{2} \qquad V = \{r, r, r, z, b, b, b, b, b\}$$

$$\Omega = \{(r,r), (r,b), (r,z), (z,b), (z,r), (b,b)\ (b,r), (b,z)\}$$

ii) najprej rdeča: $\frac{3}{9} = \frac{1}{3}$

iii) $R_1 = \frac{1}{3}$ druga zelena: $\frac{8}{9} \cdot \frac{1}{8} = \frac{1}{9}$

$$P(R_1 \cup Z_2) = P(R_1) + P(Z_2) - P(R_1 \cap Z_2)$$

$$P(R_1 \cap Z_2): \frac{3}{9} \cdot \frac{1}{8}$$

$$P(\ldots) = \frac{4}{9} - \frac{3}{8 \cdot 9} = \frac{29}{8 \cdot 9}$$

2) 2 černí n bílých

i) $\bar{C}_{j,k}$ = {nejdřív j-tém a k-tém kroku zvolena koule}

$P(\bar{C}_{j,k}) = ?$ funkce j, k, n

$$\bar{C}_{1,k} = \left(\frac{n}{n+2}\right)^{j-1} \frac{2}{n+2} \left(\frac{n+1}{n+2}\right)^{k-j-1} \left(\frac{1}{n+2}\right) =$$

$$= \frac{2 \cdot n^{j-1}}{(n+2)^{j+k-j-1+1}} (n+1)^{k-j-1} = 2n^{j-1} \frac{(n+1)^{k-j-1}}{(n+2)^k}$$

$X = k$

$$P(X,k) = P\left(\bigcup_{i=1}^{k-1} \bar{C}_{i,k}\right) = \sum_{i=1}^{k-1} P(\bar{C}_{i,k}) = \frac{2}{(n+2)^k} \sum_{k=1}^{k} n^{j-1} (n+1)^{k-j-1}$$

$$n^{j-1} (n+1)^{k-j-1} = \frac{n^j}{n(n+1)^j} (n+1)^{k-1} = \left(\frac{n}{n+1}\right)^j \frac{(n+1)^{k-1}}{n}$$

$$= \frac{2(n+1)^{k-1}}{n(n+2)^k} \quad \frac{1-\left(\frac{n}{n+1}\right)^k}{1-\frac{n}{n+1}} = \frac{2\left((n+1)^k - n^k\right)}{n(n+2)^k}$$

ii) $P(\bar{C}_1 | X=k)$

$$P\left(\bigcup_{i=2}^{k} \bar{C}_{i,k}\right) = P(X=k) - P(\bar{C}_{1,k}) = \frac{2(n+1)^{k-1} - n^{k-1})}{(n+2)^k} -$$

$$2\frac{(n+1)^k}{(n+2)^k} =$$

$$= \frac{2}{n(n+2)^k}\left((n+1)^k(n+1) - n^k\right) = \frac{1}{n(n+2)^k}(n+1)^{k+1} - n^k)$$

cz: $\frac{2}{\cancel{(n+2)}^k}\left((n+1)^{k-1} \cancel{(n+2)}^k - n^{k-1}\right) =$

$$2(n+1)^{k-1} - \frac{n^{k-1}}{(n+2)^k}$$

3)

K .... konzervativec $\rightsquigarrow P(A \mid K) = 1$

L .... levičar $\rightsquigarrow P(A \mid L) = r$

$r = 0 \Rightarrow P(A) = 1$

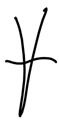
$$P(A) = P(A \mid K)P(K) + P(A \mid L)P(L) =$$
$$1 \cdot p + (1-p)(1-r)$$

$$P(V_3 \mid V_2) = \frac{P(V_3 \cap V_2)}{P(V_2)} = \frac{P(V_3)}{P(V_2)}$$

$$P(V_2) = P(V_2 \mid C)P(C) + P(V_2 \mid L)P(L) =$$
$$= 1 \cdot p + (1-p)(1-r)$$

$$P(V_3) = P(V_3 \mid C)p + P(V_3 \mid L)(1-p) =$$
$$1 \cdot p + (1-r)^2(1-p)$$

$$\Rightarrow P(V_3 \mid V_2) = \frac{p + (1+r)^2(1-p)}{p + (1+r)(1-p)}$$



20.10

1) Mečemo kovanec, dokler ne pade najmanj eno število in en grb

$P(\{grb\}) = p \quad 0<p<1$

Naj bo $\{X=k\} = \{$ prvič pade najmanj 1 število in 1 grb na koraku $k\}$

Meti so sabaj neodvisni

Izračunajte $P(X \geq k)$

$$P(X \geq k) = \sum_{k=n}^{\infty} P(X=n) = \sum_{\cancel{k=1}\; n=k}^{\infty} \left(p^{n-1}(1-p) + (1-p)^{n-1} p \right) =$$

$$= \frac{(1-p)}{p^k} \sum_{k=0}^{\infty} p^n + \frac{p}{(1-p)^k} \sum_{n=0}^{\infty} (1-p)^n =$$

$$\frac{\cancel{(1-p)}}{p^{k-1}} \frac{1}{\cancel{1-p}} + \frac{\cancel{p}}{(1-p)^{k-1}} \frac{1}{\cancel{1-1+p}} =$$

$$= \frac{1}{p^{k-1}} + \frac{1}{(1-p)^{k-1}}$$

v prvih $n-1$ metih ~~...~~ je samo isto

$$\cancel{p(k-1)}\; p^{k-1} + (1-p)^{k-1}$$

$p=0 \vee p=1 \Rightarrow \quad P(X>k) = 0$

$$= \frac{1}{p^{k-1}} \frac{(1-p)^{k-1} + p^{k-1}}{(1-p^{(k-1)})} = \frac{(1-p)^{k-1} + p^{k-1}}{p^{k-1} - p^{2k-1}} \quad ???$$

2)

Mečemo kovanec dokler ne dobimo ali

m grbov ali na števil

max metov: m+n min metov: min{m,n}

$P(\{grb\}) = p$

$P(\{X=k\}) =$ $\{X=k\} =$ pri č. k dobimo ali m grbov ali m števil

$P(\{X=$ ~~grb~~ pri ... m grbov$\}) = p^{m}(p-1)^{k-m}$

$+ p^{k-m}(p-1)^{m}$

~~k - metov~~ m ~~jih je~~ neke vrste : $k < 2m$

$p^{m}(p-1)^{k-m} + p^{k-m}(p-1)^{m}$

$P(X=\{\ \}) =$

$P(X=\{k-1$ tem mestu pade $m-1$ grbov$\})$

$\{G=k\} = \{$ pade $m-1$ grbov v $k-1$ korakih, potem pa spet pade grb $\}$

$\{S=k\} = \{$ Na $k-1$ korakih pade $m-1$ števil potem pa spet število $\}$

$$\binom{k-1}{m-1}\left(p^m (1-p)^{k-m} + p^{k-m}(1-p)^m\right)$$

3)

$$X \sim \mathrm{Bin}\,(n, \underset{p}{\underset{\parallel}{\tfrac{1}{2}}})$$

$$P(X=k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{n-k} = \binom{n}{k}\frac{1}{2^n}$$

Definiramo slučajni spremenljivki

Y za $k=0,\dots,n$ velja $P(Z) \sim P(X,Y)$

$$P(X=k,\ Y=k+1) = P(X=k)\frac{n-k}{n} \qquad (a)$$

in
$$P(X=k,\ Y=k-1) = P(X=k)\frac{k}{n} \qquad (b)$$

in $P(X=k, Y=l) = 0$ če $|k-l|>1$ ali $l=k$

Računaj te porazdelitev slučajne spremenljivke Y

Rešitev:

Popolna verjetnost

$$P(Y=l) = \sum_{k=0}^{n} P(X=k, Y=l) =$$

$$= P(X=l-1, Y=l) + P(X=l+1, Y=l)$$

$$P(Y=l) = \binom{n}{l-1}\frac{1}{2^n}\frac{n-l+1}{n} + \binom{n}{l+1}\frac{1}{2^n}\frac{l+1}{n} =$$

$$= \frac{1}{2^n}\left(\frac{n!}{(l-1)!\,(n-(l-1))!}\cdot\frac{n-(l-1)}{n} + \frac{n!}{(l+1)!\,(n-(l+1))!}\cdot\frac{l+1}{n}\right)$$

$$= \frac{1}{2^n}\left(\binom{n-1}{l-1} + \binom{n-1}{l}\right) = \frac{1}{2^n}\binom{n}{l}$$

k)

Naj bodo $X \sim B(\frac{1}{2}) \Rightarrow X=0$ z verjetnostjo $\frac{1}{2}$ in

$X=1$ z verjetnostjo $\frac{1}{2}$

$Y \sim B(\frac{1}{2})$

$Z = Z(X,Y)$

$$P(Z) = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad \text{"} \in \mathbb{R}^4 \text{"}$$

$P(X=1, Y=0)$

Velja: $*\ 0 \leq p_{ij} \leq 1$

$*\ \sum_{0 \leq i,j \leq 1} p_{ij} = 1$

Recimo da $P(Z) = \begin{pmatrix} \frac{1}{4}+\delta & \frac{1}{4}-\delta \\ \frac{1}{4}-\delta & \frac{1}{4}+\delta \end{pmatrix}$

kdaj sta X in Y neodvisna od δ

$-\frac{1}{4} \leq \delta \leq \frac{1}{4}$

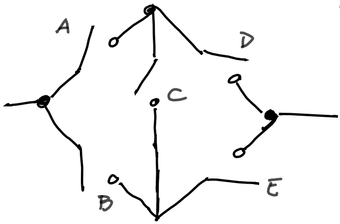
neodvisni

Dejstvo: $X \perp Y \iff \det(p_{ij}) = 0$

$\det(p_{ij}) = \left((\frac{1}{4}-\delta)^2 - (\frac{1}{4}-\delta)^2 \right) = \delta$

$\Rightarrow X \perp Y$ če je $\delta = 0$

5) V vezju



Vsako stikalo prepušča el. tok z verjetnostjo $\frac{1}{3}$ in posamezna stikala so med seboj neodvisna

A = dogodek da tok gre preko A

Kolikšna je verjetnost da vezje prepušča?

A

$$\frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}\right)+\frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}\right) \quad ??$$

D C E B E C D

A

$$\frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3}\right)+\frac{1}{3}\cdot\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9} \quad ??$$

A D

odštejem

$$\frac{2}{3}\left(\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}\right)+\frac{1}{3}(\ldots)$$

zaprto

št grbov pg

$$P(X=k) = P(H_c)\cdot P(X=k|H_c) + P(X=k|H_{gg})P(H_{gg}) + P(H_{gc})P(X=k|H_{gc})$$

$$P(X=0|H_{gg}) = 1$$

$$P(X=k|H_{gg}) = 0$$

$$P(X=0|H_{gc}) = 0$$

$$P(X=k|H_{gc}) = P(X=k-1)$$

$$P(X=0|H_c)$$

$k=0 \Rightarrow$

$$P(X=0) = (1-p)P(X=0) + 1\cdot p^2 + p(1-p)\cdot 0$$

$$P(X=0)(1-1+p) = p^2$$

$$P(X=0) = p$$

$k\neq 0 \Rightarrow$

$$P(X=k) = P(X=k)(1-p) + \cancel{P(X=k-1)p^2} + p(p-1)P(X=k-1)$$

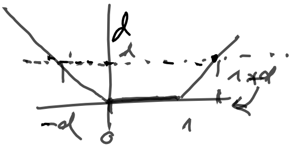
∀ españa

Naj bo $X \in U[-3,3]$ in definiramo

$$\{D=d\} = \{\text{mera}(X,[0,1]) = d\}$$

Izračunajte kumulativno porazdelitveno funkcijo na $F_d(D)$ in narišite njen graf

$\boxed{F_D(d) = P(D \le d)}$ $\quad D = g(X)$ za $g(x) = d(X,[0,1])$



$$g(x) = \begin{cases} -x & ; x \le 0 \\ 0 & ; 0 \le x \le 1 \\ x-1 & ; 1 \le 0 \end{cases}$$

gostota:

$$f_X(x) = \begin{cases} \frac{1}{3-(-3)} = \frac{1}{6} & ; x \in [-3,3] \\ 0 & ; \text{sicer} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-3}^{x} f_X(x)\,dx = \begin{cases} \frac{x+3}{6} & ; -3 \le x \le 3 \\ 0 & ; x < -3 \\ 1 & ; x > 3 \end{cases}$$

Kaj so dogodki $\{D \le d\}$?

$d < 0 \Rightarrow \{D \le d\} = \{\}$

$d = 0 \Rightarrow \{D \le d\} = [0,1]$ $\quad F_D(0) = P(X \in [0,1]) = F_X(1) - F_X(0)$

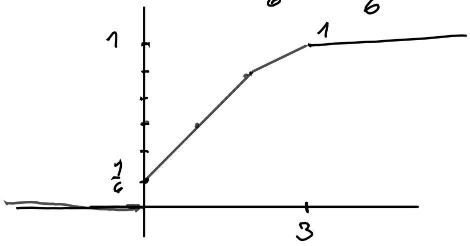
$d \in [0,2] \Rightarrow \{D \le d\} = [-d, 1+d]$ $\quad P_D(d) = F_X(1+d) - F(-d)$

$d \in [2,3] \Rightarrow \{D \le d\} = [-d, 3]$ $\quad P_D(d) = F_X(3) - F(-d)$

$$F_D(0) = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

$$F_D(d \in [0,2]) = \frac{d+4}{6} - \frac{-d+3}{6} = \frac{2d+1}{6}$$

$$F_D(d \in [2,3]) = \frac{6}{6} - \frac{-d+3}{6} = \frac{3+d}{6}$$



$$f_d = F_D'(d) = \begin{cases} 0 & : d < 0 \\ \frac{1}{3} & d \in [0,2] \\ \frac{d}{6} & d \in [2,3] \\ 0 & d > 0 \end{cases}$$

Definirajmo $X \sim \text{Cauchy}$

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

a) Izračunajte $Y = \frac{1}{1+X^2}$

Namig: $F_Y(y) = P(Y \leq y) = P\left(\frac{1}{1+X^2} \leq y\right) =$

$$= P\left(\frac{1}{y} \leq 1+X^2\right) = P\left(\sqrt{\frac{1}{y}-1} \leq X\right) = \int_{-\infty}^{x} f_X(t)\,dt$$

$-X \leq$

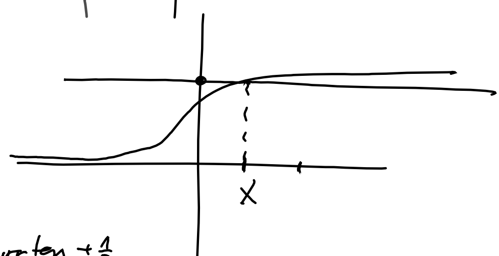
X

$$P\left(\sqrt{\frac{1}{y}-1} \leq X\right) = \frac{1}{\pi}\arctan(x)\Big|_{-\infty}^{x} = \frac{1}{\pi}\arctan x + \frac{1}{2}$$

$-X \leq$

$$P\left(\sqrt{\frac{1}{y}-1} \leq X\right) - P\left(-\sqrt{\frac{1}{y}-1} \leq X\right) =$$

$\cancel{\frac{1}{\pi}\arctan x + \frac{1}{2}}$



$y = \frac{1}{\pi}\arctan + \frac{1}{2}$

$\tan\left(\pi x - \frac{\pi}{2}\right)$

$$P(\cancel{Y} \leq y) = P_X\left(X^2 \geq \frac{1}{y} - 1\right) = \qquad y = \frac{1}{1+\cancel{X}^2}$$

$$= \underset{x \geq 0}{P}\left(x \geq \sqrt{\frac{1-y}{y}}\right) + \underset{x<0}{P}\left(x \leq -\sqrt{\frac{1-y}{y}}\right) =$$

$$= 2\, P\left(x \geq \sqrt{\frac{1-y}{y}}\right) = 2\left(1 - F_X\left(\sqrt{\frac{1-y}{y}}\right)\right) = F_Y(y)$$

det

simetrija

$$f_X(y) = \frac{d}{dx} F_X(x)$$

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

iščemo $f_Y(y)$

$$f_Y(y) = F_Y'(y)$$

$$\underline{f_Y(y)} = \left(2 - 2F_X\left(\sqrt{\frac{1-y}{y}}\right)\right)' = \qquad \sqrt{\frac{1}{y}-1}$$

$$= -2\, f_X\left(\sqrt{\frac{1-y}{y}}\right) \frac{-\frac{1}{y^2}}{2\sqrt{\frac{1}{y}-1}} =$$

$$= -2 \frac{1}{\pi\left(1 + \frac{1-y}{y}\right)} \cdot \frac{-1}{2y^2\sqrt{\frac{1-y}{y}}} = \frac{1}{\pi} \frac{\sqrt{y}}{y^2\sqrt{1-y}}$$

$$Y = \frac{1}{1+X^2}$$

$$F_Y(y) = P_Y(Y \leq y) = P_X\left(\frac{1}{1+\cancel{X}^2} \leq y\right) =$$

$$\overset{?}{\underset{\checkmark}{=}} P_X\left(X^2 \geq \frac{1}{y} - 1\right) = \quad \text{Deluje, ker je } y>0$$

ker $Y: \Omega \to (0, 1]$

in ker je $1+x^2>0$

$$= P\left(X \geq \sqrt{\frac{1-y}{y}}\right) + P\left(X \leq -\sqrt{\frac{1-y}{y}}\right) \underset{\text{simetrija}}{=} 2P\left(\cancel{X} \geq \sqrt{\frac{1-y}{y}}\right) =$$

$$= 2\left(1 - F_X\left(\sqrt{\frac{1-y}{y}}\right)\right) \qquad f_X(x) = \frac{1}{\pi(1+x^2)}$$

$F_X'(y) = f_X(y)$, če obstaja

$$\Rightarrow f_Y(y) = F_Y'(y) = -2 F_X'\left(\sqrt{\frac{1-y}{y}}\right)$$

$$= -2 f_X\left(\sqrt{\frac{1-y}{y}}\right) \frac{1}{2}\sqrt{\frac{y}{1-y}} \cdot \frac{-y-1+y}{y^2} =$$

$$= f_X\left(\sqrt{\frac{1-y}{y}}\right) \sqrt{\frac{y}{1-y}} \frac{1}{y^2} =$$

$$= \frac{1}{\pi} \frac{1}{1+\frac{1-y}{y}} \sqrt{\frac{y}{1-y}} \frac{1}{y^2} =$$

$$= \frac{1}{\pi} \frac{y}{1} \frac{1}{y^2} \sqrt{\frac{y}{1-y}} = \frac{1}{\pi y}\sqrt{\frac{y}{1-y}} = \frac{1}{\pi\sqrt{y(1-y)}}$$

Deth

	No	yes	
White	141	19	160
Black	149	17	166
	290	36	326

Definiramo $R : \{white, black\} \longrightarrow \mathbb{R}$

$R(white) = 0$

$R(black) = 1$

$D : \{\overset{No}{deth}, \overset{yes}{deth}\}$

$\underline{X} = (R, D)$

$\underline{U} = \{$ število $X = (i, j))$

Vprašanja?

1. $P(D=1 \mid R=i)$; $i \in \{0, 1\}$
2. $D \perp R$? (Ali sta neodvisni spremenljivki ?)

Lahko ocenimo $\hat{P}_{ij} = \dfrac{\#(X; X=(i,j)))}{N = 136}$

$$\hat{P} = \underbrace{\begin{pmatrix} \frac{141}{326} & \frac{19}{326} \\ \frac{149}{326} & \frac{17}{326} \end{pmatrix}}_{P(D=0) \quad P(D=1)}$$

$\hat{P}_{00} + \hat{P}_{01}$ = vrejetnost da je bel = $P(R=0)$

$= P(R=1)$

1) $P(D=1 \mid R=0) = \dfrac{P_{01}}{P_{00} + P_{01}} = 0{,}11$

$\dfrac{P_{11}}{P_{10} + P_{11}} = 0{,}10$