6.10

kateri elementi kolobarja $\mathbb{Z}[i]$

so asociirani $m+ni$

$$x = u(m+ni) = um + uni$$

kateri so obrnljivi:

$$(x+yi)(a+bi) = 1$$

$$xa - yb + i(ay+xb) = 1$$

$$ay + xb = 0 \qquad xa - yb = 1$$

$$x = \frac{-ay}{b} \qquad \frac{-a^2y}{b} - yb = 1$$

$$-\frac{y}{b}(a^2+b^2) = 1$$

$$y = \frac{-b}{a^2+b^2}$$

$$x = \frac{a}{a^2+b^2} \qquad |a| \leq |a^2| \Rightarrow$$

$$a^2+b^2 \in \{0,1\} \Rightarrow a^2+b^2 = 1$$

$\Rightarrow a \in \{0,1\}$

$b \in \{0,1\}$ obrnljivi $\{1,-1,i,-i\}$

asoc: $\{m+ni, -m-ni, n-mi, -n+mi$

$d \in \mathbb{Z}$

$\mathbb{Z}[\sqrt{d}] = \{m + n\sqrt{d};\ m, n \in \mathbb{Z}\}$

1) Pokaži $\mathbb{Z}[\sqrt{d}]$ je podkolobar $\mathbb{C}$

2) Množica $\mathbb{Q}[\sqrt{d}] = \{g + r\sqrt{d};\ g, r \in \mathbb{Q}\}$ je podpolje $\mathbb{C}$ generirano z $\mathbb{Z}[\sqrt{d}]$

---

1) $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{C}$

zaprtost za seštevanje, množenje, enke

za množenje:

$$(m+n\sqrt{d})(x+x\sqrt{d}) = mx + nyd + \sqrt{d}(nx+ym)$$

enka $= 1 = 1 + 0\cdot\sqrt{d}$

seštevanje: $(m+n\sqrt{d}) - (x+y\sqrt{d}) = (m-x) + (n-y)\sqrt{d}$

2) invertiranje

je tu

$$\frac{m+n\sqrt{d}}{x+x\sqrt{d}} = \frac{(m+n\sqrt{d})(x-y\sqrt{d})}{x^2-y^2d} = \frac{c}{x^2-y^2d} + \frac{e\sqrt{d}}{x^2-y^2d} \checkmark$$

Avtomorfizmi $\mathbb{Z}[\sqrt{d}] = \{id;\ \underbrace{\sqrt{d} \mapsto -\sqrt{d}}_{\sigma}\}$

Norma $N(x) = x\sigma(x)$

$$N(g+r\sqrt{d}) = (g+r\sqrt{d})(g-r\sqrt{d}) = g^2 - r^2d$$

3) $\forall x,y \in \mathbb{Z}[\sqrt{d}]: N(xy) = N(x)N(y)$

$$N(x,y) = xy\sigma(x,y) = xy\sigma(x)\sigma(y) = x\sigma(x)y\sigma(y)$$
$$= N(x)\cdot N(y)$$

4) $x \in \mathbb{Z}[\sqrt{d}]$ je obrnljiv $\Leftrightarrow$ $N(x) = \pm 1$

$$xy = 1 \Rightarrow N(xy) = N(1)$$
$$N(x)N(y) = 1$$
$$N(x) = \frac{1}{N(y)} \qquad N(y) \in \mathbb{Z} \wedge N(x) \in \mathbb{Z}$$
$$\Rightarrow N(y) \in \{\overset{-1}{\cancel{0}}, 1\}$$
$$\Rightarrow N(x) \in \{-1, 1\}$$

6) $p \in \mathbb{P}$; $N(x) = \pm p \Rightarrow x$ nerazcepen

recimo da x razcepen ~~in $N(x) = \pm p$~~ a,b neobrnljiva

$x = ab$

$$N(x) = N(a)N(b) = \pm p \Rightarrow N(a) = 1 \vee N(b) =$$

4) $\Leftarrow$ $N(x) = 1$, $x\sigma(x) = \pm 1 \Rightarrow x^{-1} = \pm\sigma(x)$

5) naprej

Recimo da je $N(a) \pm 1 \Rightarrow a$ obrnljiv.

$d < -1 \Rightarrow 1, -1$ sta edina obrnljiva v $\mathbb{Z}[\sqrt{d}]$

x obrnljiv $\Rightarrow N(x)\, N(y) = \pm 1$

$N(a+b\sqrt{d}) = (a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - b^2 d = a^2 + b^2 \underset{\overset{\vee}{1}}{|d|}$

če je $b^2 > 1 \Rightarrow N > 1$

$\Rightarrow b = 0$

a^2 mora biti $1 \Rightarrow a = \pm 1$ sta edini

namesti

in vemo da sta dalljna to rej ke je rezultat

Pokaži da so $1+i$, $7+8i$, 3 nerazcepni v $\mathbb{Z}[i]$

Recimo da so razcepni

$1+i = x\, y \quad N(xy) = x\sigma(x)\, y\sigma(y) = 2$

praštevilo

$p \to$ nerazcepne

$N(7+8i) = 49 + 64 =$ velik $= 113$ praštevilo

~~Nje predeuda~~ Norma je 9

$3 = xy \quad N(xy) = x\sigma(x)\, y\sigma(y) = 9$

vsaj 2 morata biti 1

~~$x\sigma(x)$~~ $= N(x) = N(\sigma(x)) = \sigma(x)\sigma\sigma(x) = $ ~~x~~ $x\sigma(x)$

$\underline{N(x) = n} \Rightarrow \underline{n}$ razcepen

$N(x) = n = x\sigma(x)$

6) Poišči vse delitelje elementa 2 v $\mathbb{Z}[i]$

$x|2 \iff 2=xy$ za nek y

$N(x\cdot y) = 4$ $\quad N(x)N(y)=4$

$x\sigma(x)y\sigma(y) = 4 = 2\cdot 2 = xy\sigma(xy) = 2\sigma(xy)$

$\Rightarrow \sigma(xy)=2 = \sigma(x)\sigma(y) \Rightarrow$

$N(x)=2$ $\quad x=\pm 1 \pm i$

$x=\pm 2$ $\quad$ so vse možnosti

$x=\pm 2i$

1)

$a = 3+4i \qquad \delta = m^2+n^2$

$b = 1-3i$

$|a| = 9+16 =$ ~~[illegible]~~ 25

$|b| = 1+9 = 10$ [illegible]

$a = kb + r$

$$\underset{\in \mathbb{Z}[i]}{\frac{a}{b} = k} + \underset{\in \mathbb{Q}[i]}{\frac{r}{b}}$$

$a = 3-4i$

$b = 1-3i$

$$\frac{a}{b} = \frac{(3-4i)(1+3i)}{10} = \frac{3+12+i(9-4)}{10} = 1+\frac{1}{2}+\frac{1}{2}i$$

$$\frac{1}{2}(1+i)(1-3i) = \frac{1}{2}(1+3+i(1-3)) = 2-i$$

$$a = b + 2-i \qquad /:(2-i)$$

$$\frac{a}{2-i} = \frac{b}{2-i} + 1 \quad \Rightarrow$$

$$\frac{a}{2-i} - \frac{b}{2-i} = 1 \quad \Longrightarrow \quad a-b = 2-i$$

2)

$a=6$ $\qquad \mathbb{Z}[\sqrt{-5}]$

$b=2+2\sqrt{-5}$

$c=2 \xRightarrow{N(a)} \frac{36}{2}=3 \quad \frac{N(b)}{2} = 4+4\cdot5 = 4\cdot 6 = 1+\sqrt{5}$

$c|a$ ~~$N(c)|N(b)$~~ $\Rightarrow$ ~~$\frac{a}{c}$~~ , $d|\frac{a}{c}$, $d|b \Rightarrow 6\cdot 2\cdot 3$

$\delta(c) = 1+5=6$

$6\cdot2\cdot3 \qquad 6\cdot2\cdot3$

$\parallel \qquad \parallel$

$6^2 \qquad 6\cdot4$

$\parallel \qquad \parallel$

$N(gcd(a,b)) = gcd(N(a), N(b)) = gcd(36, 24) = 12 = x^2+5y^2$

$y^2 \in \{1, 2\}$

$\Downarrow \qquad \Downarrow$

$x^2 = 23$ X $\qquad x^2 = $ X

Tak ne obstaja

3) UFD ⇒ gcd obstaja

$a = \prod_{i \in I} p_i^{k_i}$

$d = \gcd(a,b) = \prod_{i \in I} p_i^{\min\{k_i, l_i\}}$

$b = \prod_{i \in I} p_i^{l_i}$

$d \mid a \wedge d \mid b$

Naj bo $c = \prod_{i \in I} p_i^{m_i}$ $\quad c \mid a \wedge c \mid b$

$a = c\alpha \qquad \alpha = \prod p_i^{n_i}$

$a = \prod p_i^{n_i + m_i} = \prod p_i^{k_i} \Rightarrow m_i \leq k_i$

podobno $m_i < l_i \Rightarrow \forall i$

$c \mid d$

$\mathbb{Z}[\sqrt{-2}]$ euclídeo

a) $\forall a,b \quad b\neq 0 \quad \exists q,r \in K. \quad a = qb + r \quad \wedge \; r = 0 \vee \delta(r) < \delta(b)$

b) $\delta(a) \leq \delta(a,b) \quad \forall a,b \in K - \{0\}$

$\delta : m + \sqrt{-2}n = m^2 + 2n^2$

Veno de b valga

$a,b \in \mathbb{Z}[\sqrt{-2}]$ [illegible] $a = x + \sqrt{-2}y \quad b = m + \sqrt{-2}n$

$$\frac{a}{b} = g_1 + g_2\sqrt{-2}$$

$$a = ([g_1] + [g_2]\sqrt{-2})b + (g_1 - [g_1] + (g_2 - [g_2])\sqrt{2})b$$

DN:

Naj bo $K[X]$ UFD

Potem ima polinom n^{EK} enolično faktorizacijo

Torej je tud $K[$[illegible]$]$ UFD

Nej bo K UFP

Naj bo pos poljuben polinom

Razcepimo $p_{OS} = \prod_{k=1}^{r}\left(\sum_{i=1}^{n} a_{ik}X^{n}\right) =$

$$= \prod_{k=1}^{r}\left(a_{nk}X^{n} + \ldots + a_{0k}\right) = X^{nr}\underbrace{\left(\prod_{k=1}^{r} a_{nk}\right)} + \ldots$$

$\text{cont}(fg) = \text{cont}(f)\text{cont}(g)$

$p \mid \text{cont}(f,g) \Rightarrow \gcd(a_0 \ldots\ldots a_n) = p$

$fg = 0$ v $K/p[X]$

$\Rightarrow f = 0 \vee g = 0$

$(ax+b)(cx+d) = acx^2 + x(cb+ad) + bd$

$\Rightarrow p \mid \text{cont} f \vee p \mid \text{cont} g$

Recimo da f razcepen $F[X]$

potem lahko zapišemo $af = p(x)q(x) \in K[X]$

Najprej skelo $\mathrm{cont}(af)$ ker $af = pq$ / cont

$$\mathrm{cont}(af) = \mathrm{cont}(\underline{q})\,\mathrm{cont}(p) = a\,\mathrm{cont} f$$

$$= a \cdot \text{enota}$$

enolično skupaj delitelji

$$a \mid \mathrm{cont}(pq) \Rightarrow \frac{p}{x} \cdot \frac{q}{y} = \frac{fa}{a} = f$$

$$\parallel$$

$$\underbrace{x}_{\vee p} ; \underbrace{y}_{\vee q}$$

$f \mid pq \Rightarrow f \mid p \ \vee f \mid q$

$kf = pq \qquad \mathrm{cont} f\, \mathrm{cont} k = \mathrm{cont}\, p\, \mathrm{cont}\, q$

pq imata razcep $\Rightarrow f \mid p \vee f \mid q$

$\Rightarrow \forall\, p_1(x) \dots p_n(x)$

2) $K[X]$ glavni kolobar $\Leftrightarrow$ K je polje

$a \in K - \{0\}$

$\exists_b \ \underline{ab = 1}$

(a) je ideal $(a) = \{af \ ; f \in K[x]\}$

20.10

1) $1+\sqrt{-3}$ nerazcepen v $\mathbb{Z}[\sqrt{-3}]$, ampak ni praelement

$1+\sqrt{3}$ nerazcepen

Recimo kajo $(1+\sqrt{-3}) = \alpha \cdot \beta$

$$N(\alpha \cdot \beta) = N(a) \cdot N(b) = 1+3 = 4$$

$$N(\alpha) = N(\beta) = \pm 2$$

če bi bila 1 bi bila obrnljiv

$$N(a+\sqrt{-3}b) = a^2 + 3b^2 = \pm 2$$

$$b = 0$$

$a^2 = \pm 2$ neobstaja

Ni praelement

~~$(a+b\sqrt{-3}$~~$)(1+\sqrt{-3}) =$ ~~$a+b\sqrt{-3}+a\sqrt{-3}-$~~ $b =$

Vsak element deli svojo normo

$$(1+\sqrt{-3})(1-\sqrt{-3}) = 4 = 2 \cdot 2$$

$$(1+\sqrt{-3}) \nmid 2$$

Recimo da $k(1+\sqrt{-3}) = 2$

$N(2) = 4 \Rightarrow$
$N(k) = \pm 1$
$\Rightarrow k = \pm 1$

Domača naloga

1)

$\omega = \frac{(1+\sqrt{-19})}{2}$ $\mathbb{Z}[\omega] = \{a+b\omega : a,b \in \mathbb{Z}\}$

$N(a+b\omega) = |a+b\omega|^2 = a^2+ab+5b^2$

2 in 3 nerazcepna

$N(2) = 4$

$2 = ab$; a,b neobr. $\Rightarrow N(a) = N(b) = 2$

$a =\cdot\, a+b\omega$

$N(a) = a^2+ab+5b^2 = 2$

$b \neq 0 \Rightarrow 5b^2 > 2 \Rightarrow N(a) > 2$ $\times$

$N(3) = 9$ $9 = x \cdot y$ $9 = 3 \cdot 3 \Rightarrow N(x) = N(y) = 3$

$N(x) = 3$ $x = a+b\omega$

$a^2+ab+5b^2 = 3$

$b \neq 0 \Rightarrow 5b^2 > 3 \Rightarrow N(x) > 3$ $\times$

$N(x) = 1 \Rightarrow x \in \{1,-1\}$

$a^2+ab+5b^2 = 1$

$b \neq 0 \Rightarrow 5b^2 > 1$ $\times$ $\Rightarrow b = 0$

$a^2 = 1 \Rightarrow a \in \{1,-1\} \Rightarrow x \in \{1,-1\}$

Moduli

$K =$ {zgornje trikotne matrike nad F}

$M = \begin{bmatrix} F \\ F \end{bmatrix}$ je levi K-modul

iščemo podmodule od M

$KM \subseteq M$

$N = \{\lambda \begin{bmatrix} u \\ v \end{bmatrix} ; \lambda \in F\}$ za $u, v \in F$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} au+bv \\ cv \end{bmatrix} \in N$$

$$\lambda au + \lambda bv = \mu u$$

$$\lambda cv = \mu v \quad \Rightarrow \quad \lambda c = \mu$$

$$\lambda au + \lambda bv = \lambda cu$$

$$au - cu + bv = 0$$

$$bv = (c-a)u$$

DN naprej

$\mathbb{Z}[\omega]$ ni evklidski

$(\exists\, \delta : \mathbb{Z}[\omega] - \{0\} \to \mathbb{N} \cup \{0\})$

$\omega = gx + r \quad \Rightarrow \quad r \in \{1, -1, 0\}$

$r = 0 \Rightarrow \omega = gx \Rightarrow x$ asociiren z ω

$N(x) = N(\omega) = 5$

$2 = g'x \pm 1 \Rightarrow x \in \{\pm 2, \pm 3\} \Rightarrow N(x) \in \{4, 9\}$

$r = 1 \Rightarrow \omega = gx + 1$ $\times$

$-1 + \omega = gx \Rightarrow 1 - 1 + 5 = N(g)N(x) = 5$

$N(x) = 5$ $\times$

$r = 1 \Rightarrow 1 + \omega = gx \Rightarrow 7 = N(g)N(x) \Rightarrow N(x) = 7$ $\times$

$\mathbb{Z}[\omega]$ je glavni

$$\left(\forall \alpha \in \mathbb{C} - \mathbb{Z}[\omega].\ \exists p, g \in \mathbb{Z}[\omega].\ 0 < |p\alpha - g| < 1\right)$$

$$\Longrightarrow \mathbb{Z}[\omega] \text{ glavni kolobar}$$

Naj bo I poljuben ideal. Naj bo $b \in I$ tak da

$\forall x \in I.\ N(x) \geq N(b)$

Naj bo $a \in I$ poljuben $\alpha = \frac{a}{b}$ $a = \alpha b$

$\alpha \in \mathbb{Z}[\omega] \Rightarrow b | a$ za $\forall a \in I \Longrightarrow I = (b)$

$\alpha \in \mathbb{Z}[\omega]$ in pogoj $\Longrightarrow 0 < |p\frac{a}{b} - g|^2 < 1$ $\quad / \cdot |b|$

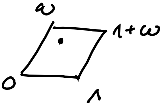
$$0 < |\underbrace{pa}_{\in I} - \underbrace{gb}_{\in I}|^2 < |b|^2 \quad \times$$

$\underline{\forall \alpha \in \mathbb{C}-\mathbb{Z}[\omega]\ \exists \rho, g \in \mathbb{Z}[\omega]\ .\ 0<|\rho\alpha - g|<1}$

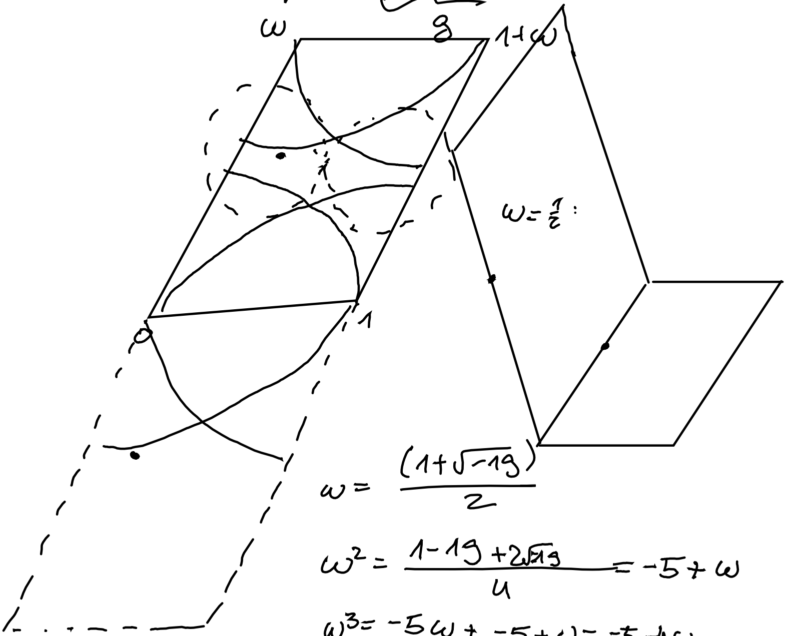
$\alpha \in \mathbb{C}-\mathbb{Z}[\omega]$ poljuben

$\alpha = a+b\omega \quad a,b\in\mathbb{R}$

$\beta = \alpha - (m+\omega n) \quad \beta \in$



$$\rho\alpha - g = \rho\beta + \underbrace{\rho(\ldots) - g}$$



$$\omega = \frac{(1+\sqrt{-19})}{2}$$

$$\omega^2 = \frac{1-19+2\sqrt{-19}}{4} = -5+\omega$$

$$\omega^3 = -5\omega + -5+\omega = -5-4\omega$$

DM ze teche:

$d = 4k+1$

$(a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - db^2 = 4(2-k)$

↑

$b^2=1$

$a^2=9$

$b^2=1$:

a^2-4k-1

$a=3$

$8-4k=4(2-k)$

$$\left\{ (x-1)p(x) \right\} = \sum a_n = 0$$

Tisto je ideal

$$\sum_{n=0}^{k} a_n x^n$$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 : (x-1) = a_n x^{n-1}$$

$$\frac{- a_n x^n + a_n x^{n-1} \quad \cancel{+a_n}}{(a_n + a_{n-1}) x^{n-1} + \dots + a_0}$$

ōc geta

[illegible] pd [illegible]

ind. predp.

$$ax - a = a(x-1)$$ ✓

Because [illegible] [illegible] f.g. $\exists\ B = \{f_1 \dots f_n\}$

st every element [illegible] expressed

$g_1 f_1 + \dots + g_n f_n$

$\sqrt{f_1^2 + \dots + f_n^2}$

$\sqrt{(g_1 f_1 + \dots + g_n f_n)^2} \leq \sqrt{(g_1^2 + \dots)} \sqrt{f_1^2 + \dots + f_n^2}$

$= h_1 f_1 + \dots + h_n f_n$

$M = {}_{\mathbb{Z}}\mathbb{Z}_{12}$

$\mathbb{Z}_2 = \{0, 6\}$ enostaven

$\mathbb{Z}_3 = \{0, 4, 8\}$ enostave

$\mathbb{Z}_4 = \{0, 3, 6, 9\} \supseteq \mathbb{Z}_2$

$\mathbb{Z}_6 = \{0, 2, 4, 6, 8, 10\} \supseteq \mathbb{Z}_2, \mathbb{Z}_3$

3) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{12})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_4)$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{12}, \mathbb{Z}_{12})$

$\mathbb{Z} \longrightarrow \mathbb{Z}_{12}$

$0 \longmapsto 0$ 1 natanko določi

$1 \longmapsto$ karkoli $\Longrightarrow$ imamo 12 homomorfizmov

$$\boxed{\begin{array}{l} K \text{ kolobar, } {}_KM \text{ modul} \Rightarrow \\ \mathrm{Hom}(K, M) = \{x \longmapsto xm : m \in M\} \cong {}_KM \end{array}}$$

$\mathbb{Z}_3 \to \mathbb{Z}_4$

$0 \mapsto 0$

1.	2.	3.
$1 \mapsto 1$	$1 \mapsto 2$	$n \mapsto 0$
$2 \mapsto 2$	$2 \mapsto 1$	
$3 \mapsto 0$	$3 \mapsto 0$	
$4 \mapsto 1$	$4 \mapsto 2$	

$\mathbb{Z}_{12} \mapsto \mathbb{Z}_{12}$

$0 \mapsto 0$

$1 \mapsto a$

$n \mapsto na$, mod 12

K kolobar. K enostaven $\Leftrightarrow$ K obseg

$\Rightarrow$ Recimo da je enostaven, $a \in K$ poljuben

a obrnljiv

$Ka = K \Rightarrow \exists k \in K . \; ka = 1$

$\Leftarrow$ M podkolobar ima

Vaje za nazaj

1 Naloga M K-modul $M \neq \{0\}$ enostaven $\Leftrightarrow M = Km$ $\forall m \in M$

Naj bo M enostaven. Edina podmodula sta $\{0\}$ in M

$\Longrightarrow$ Naj bo $m \in M$ poljuben $\underline{Km \text{ je podmodul}}$

$\alpha, \beta \in K \quad \alpha k m - \beta h m = (\alpha k - \beta h) m \in Km$

$\Longleftarrow$ $M = Km$ za $\forall m \in M$.

Naj bo N podmodul.

$\forall n \in N$. $Kn = M$ torej $N = M$

2. naloga

levi K modul ${}_K M$ enostaven $\Leftrightarrow$ $M \cong K/{}_K I$, kjer je ${}_K I \leq {}_K$ maksimalen levi ideal

$\Rightarrow$ Naj bo M enostaven, $a \in M$

$\varphi: K \longrightarrow M$ $\qquad$ φ surjektiven po prejšnji nalogi

$k \longmapsto ka$

$\underline{\ker\varphi = \{k;\ ka = 0\} = {}_K I}$ maksimalen

$I \subsetneq J \subsetneq K$ $\qquad$ $\varphi_*(J) = M \Rightarrow m = ja$ $\forall m \in M$

$\qquad$ ker $J \neq \ker\varphi$ $\qquad$ $\Rightarrow j = ja \Rightarrow j(1-a) = 0$

$K \longrightarrow M$ $\qquad$ $0 \subsetneq \varphi_*(J) \underline{\subsetneq} M$ $\quad \times \quad$ $1 - a \in \ker\varphi$

$\Rightarrow$

$J \xrightarrow{sur} \varphi_*(J)$ $\qquad$ $1 - a \in J$

$(1-a) + a = 1 \in J$

$\Longrightarrow J = I \vee J = K$ $\qquad$ $\Rightarrow J = K$

$\times$

$I \longrightarrow 0$

$\Leftarrow$ Naj bo ${}_K M \cong K/{}_K I$; I maksimalen levi ideal

$$\begin{tikzcd}
0 & < & N & < & M \\
I \arrow[u] & < & J \arrow[u] & < & K \arrow[u]
\end{tikzcd}$$

φ surj

$J = \varphi^*(N)$

$N \subseteq \varphi_*(J) = \varphi_*(\varphi^*(N)) \Longrightarrow \varphi_*(J) = N$

$K/I \longrightarrow M$ $\qquad$ $\varphi: K \longrightarrow M$

$1 + I \longmapsto e \in M - \{0\}$ $\quad \Longrightarrow \quad$ $k \longmapsto ke$

$x + I = x(1 + I) \longmapsto xe$

$\underline{\text{Menostaven}}$ $\qquad$ $\underline{I = J} \vee \underline{J = K}$

$$\begin{tikzcd}
I \arrow[d] & < & J_2 \arrow[d] & < & J_1 \arrow[d] & < & K \arrow[d] \\
0 & < & \varphi_*(J_2) & = & \varphi_*(J_1) & < & M
\end{tikzcd}$$

$/ : \varphi(J_2)$

$$0 = \frac{\varphi_*(J_1)}{\varphi_*(J_2)} < M/\varphi(J_2)$$

??

Tema neloge

esparta

Od zadnjič:

$$\mathrm{End}\,(M\oplus N)\cong\begin{bmatrix}\mathrm{End}(M) & \mathrm{Hom}(N,M)\\ \mathrm{Hom}(M,N) & \mathrm{End}(N)\end{bmatrix}$$

1) določi kolobarje endomorfizmov $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_2\oplus\mathbb{Z}_3)$, $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_6\oplus\mathbb{Z}_3)$

$$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_2\oplus\mathbb{Z}_3)=\begin{bmatrix}\mathrm{End}\,\mathbb{Z}_2 & \mathrm{Hom}\,(\mathbb{Z}_3,\mathbb{Z}_2)\\ \mathrm{Hom}\,(\mathbb{Z}_2,\mathbb{Z}_3) & \mathrm{End}\,\mathbb{Z}_3\end{bmatrix}$$

$$\mathrm{End}\,\mathbb{Z}_2:\quad \begin{matrix}0\mapsto 0\\ 1\mapsto 1\end{matrix}\qquad\begin{matrix}1\mapsto 0\\ 0\mapsto 1\end{matrix}\qquad\cong\mathbb{Z}_2$$

$$\boxed{\text{Splošno: } \mathrm{End}\,\mathbb{Z}_p\cong\mathbb{Z}_p}\qquad\Rightarrow \mathrm{End}\,\mathbb{Z}_3\cong\mathbb{Z}_3$$

$1\mapsto$ red elementa 3 ne obstaja in

$$\mathrm{End}\,(\mathbb{Z}_2\oplus\mathbb{Z}_3)\cong\begin{bmatrix}\mathbb{Z}_2 & 0\\ 0 & \mathbb{Z}_3\end{bmatrix}\cong\mathbb{Z}_2\times\mathbb{Z}_3\cong\mathbb{Z}_6$$

$$\mathrm{End}\,(\mathbb{Z}_6\oplus\mathbb{Z}_3):\quad \mathrm{End}\,(\mathrm{End}\,(\mathbb{Z}_2\oplus\mathbb{Z}_3)\oplus\mathbb{Z}_3)=$$

$$=\begin{bmatrix}\mathrm{End}\,(\mathbb{Z}_2\oplus\mathbb{Z}_3) & \mathrm{Hom}\\ \mathrm{Hom} & \mathrm{End}\,\mathbb{Z}_3\end{bmatrix}=\begin{bmatrix}\mathbb{Z}_6 & \mathrm{Hom}\\ \mathrm{Hom} & \mathbb{Z}_3\end{bmatrix}$$

$$\mathrm{Hom}\,(\mathbb{Z}_6,\mathbb{Z}_3)\underset{\mathbb{Z}}{\cong}\mathrm{Hom}\,(\mathbb{Z}_2\oplus\mathbb{Z}_3,\mathbb{Z}_3)\cong$$

$$\cong\underbrace{\mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_3)}_{0}\oplus\mathrm{Hom}(\mathbb{Z}_3,\mathbb{Z}_3)\cong\mathbb{Z}_3$$

$$\mathrm{End}\,(\mathbb{Z}_6\oplus\mathbb{Z}_3)=\begin{bmatrix}\mathbb{Z}_6 & \mathbb{Z}_3\\ \mathbb{Z}_3 & \mathbb{Z}_3\end{bmatrix}$$

$$\overset{1\mapsto 2}{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}\begin{bmatrix}0 & 0\\ 2 & 0\end{bmatrix}\overset{?}{=}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\quad\text{se ne izide}$$

$x\mapsto(2, x \bmod \mathbb{Z}_3)$

$$\begin{bmatrix}0 & \begin{matrix}\mathbb{Z}_3\to\mathbb{Z}_2\oplus\mathbb{Z}_3\to\mathbb{Z}_6\\ 1\mapsto(1,0)\mapsto 4\end{matrix}\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ \begin{matrix}\mathbb{Z}_6\to\mathbb{Z}_2+\mathbb{Z}_3\to\mathbb{Z}_3\\ (1,0)\mapsto 2\\ x\mapsto 2x \bmod 3\end{matrix} & 0\end{bmatrix}$$

$$=\begin{bmatrix}x\mapsto 2x\mapsto(2x,0)\overset{\bmod 3}{=}4x\end{bmatrix}$$

~~2 ↦ 8 mod 3 = 2~~

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\qquad\qquad\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$$

$$\updownarrow\ \mathbb{Z}_6\to\mathbb{Z}_3\qquad\qquad\updownarrow$$

$$\begin{bmatrix}0 & x\mapsto 4x\\ 0 & 0\end{bmatrix}\underset{\mathbb{Z}_3\to\mathbb{Z}_6}{\begin{bmatrix}0 & 0\\ x\mapsto 2x & 0\end{bmatrix}}\begin{bmatrix}x\mapsto 8x & 0\\ 0 & 0\end{bmatrix}$$

$k\quad \mathbb{Z}_3\longrightarrow\mathbb{Z}_6$

$1:=\quad 1\mapsto(1,0)\longrightarrow 4$

$2=\quad 1\mapsto(2,0)\mapsto 2$

$k=\quad 1\mapsto(k,0)\mapsto\ldots$

$\mathbb{Z}_6\longrightarrow\mathbb{Z}_3$

$k=\quad 1\mapsto k \bmod 3$

Hitreje:

$$\mathbb{Z}_{18} = \mathbb{Z}_6 \oplus \mathbb{Z}_3 = \mathbb{Z}_2 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3) \quad \mathbb{Z}$$

$$\operatorname{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_2 & \operatorname{Hom}(\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_2) \\ \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3) & \operatorname{End}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \end{bmatrix} =$$

$$\|$$

$$\operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \oplus H \mathbb{Z}_2 \oplus \mathbb{Z}_3)$$

$$= \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \cong$$

$$\underline{\underline{\cong \mathbb{Z}_2 \oplus M_2(\mathbb{Z}_3)}}$$

Bimoduli

$K^{op} = \{x^{op} \mid x \in K\}$ $x^{op} \cdot y^{op} = (yx)^{op}$

Grupe: $G \cong G^{op}$
$x^{op} = x^{-1}$

$_KM \cong M_{K^{op}}$ vsak levi modul lahko predstavimo kot desni K modul

Bimodul je $_KM_S$, povezuje pa ga asociativnostni zakon
K, S kolobarja $_KM$ je levi K-modul
M_S je desni S-modul

$x \in K$, $y \in S$

$m \in M$ $\underbrace{(x \cdot m)}_{\in M} y = x(m \cdot y)$

Primeri

1) $_KK_K$ je bimodul

2) $_KM_{K^{op}}$ ni bimodul (razen če K komut.

$m \cdot x^{op} = xm$ $x(my^{op}) = xym$

$\shortparallel$

$(xm)y^{op} = yxm$

3) $_KM_{End(M)^{op}}$ je bimodul

$$(xm)\varphi^{op} = \varphi(xm) = x\varphi(m) = x(m\varphi^{op})$$

2) Naj bosta $_KM_S$ in $_KN_R$ bimodula (K, S, R kolobarji)

Premisli, da imajo homomorfizmi: $M \to N$ naravno strukturo (S, R)-bimodula

← homomorfizmi kot levih K modula

Zgubi strukturo K modula

$$_S\mathrm{Hom}(M,N)_R$$

$s \in S \quad s\varphi := m \mapsto \varphi(ms)$

$r \in R \quad \varphi r := m \mapsto \varphi(m)r$

Hom(M,N) je levi S modul

$$(s+t)\varphi = (m \mapsto \varphi(m(s+t))) = (m \mapsto \varphi(ms) + m \mapsto \varphi(mt))$$

$$\|$$

$$\varphi(ms) + \varphi(mt) \qquad = s\varphi + t\varphi$$

$$1 \cdot \varphi = (m \mapsto \varphi(m)) = \varphi$$

$s(\varphi + \psi) =$ ✓

Hm. $(st)\varphi\ (m) = \varphi(mst) = \varphi((ms)t) =$

$= (t\varphi)(ms) = s(t\varphi(m)) = s(t\varphi)(m)$

↗ ali je to vredu $(t\varphi)(xm) = x((t\varphi)(m))$

$$(t\varphi)(xm) = \varphi((xm)t) = \varphi(x(mt)) = x\varphi(mt)$$

Podobno za desni R modul

$$s(\varphi r)(m) = (\varphi r)(ms) = \varphi(ms)r = (s\varphi(m))r$$

Lev K-modul ${}_K M$. Dual je $M^*_K = Hom({}_{\cancel{K}} M, {}_{\cancel{K}} K_K)$

Ker je domena bimodul, je tudi M^*_K ← Funkcional desni K modul!

$$M = \begin{bmatrix} F \\ F \end{bmatrix}$$

3) Dobi $\left({}_{M_2(F)} M\right)^*$

$$M^*_{M_2(F)} = Hom\left({}_{M_2(F)} M, M_2(F)_{M_2(F)}\right) \cong$$

$$\begin{bmatrix} F & F \\ F & F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \cong M \oplus M$$

$$\cong Hom(M, M \oplus M) = End(M) \oplus End(M) \cong F^2$$

To sta obsega (Shurova lema)

$M=\begin{bmatrix} F \\ F \end{bmatrix}$ $\qquad M \overset{?}{=} \begin{bmatrix} F & F \end{bmatrix}$

$K=\begin{bmatrix} F & F \\ F & F \end{bmatrix}$ $\qquad K \cong M \oplus M$ vrstica + vrstica

$$\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \oplus \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$$

struktura desnega K-modula

$$M^* = \operatorname{Hom}(M,K)_K \cong \operatorname{Hom}(M,M) \oplus \operatorname{Hom}(M,M) \cong \underline{F \oplus F}$$

ni desni K modul. Izgubimo strukturo modula

(*) $\underline{F \cong \operatorname{Hom}(M,M)} = \operatorname{End} M$

End(M) obseg, če M enostaven

$\underline{F \cong \operatorname{End}(M)}$

2-dim vek prostor

$\supseteq$ V posebnem: $\varphi \in \operatorname{Hom}_K(M,M) \leq \operatorname{Hom}_F(M,M) \cong M_2(F)$

$\varphi(Ax) = A\varphi(x) \qquad \varphi = B \Rightarrow$

$BAx = ABx$ za $\forall A \in K$ $\forall x \in M. \Rightarrow BA = AB \Leftrightarrow \forall A \in K.$

$$B \in Z(M_2(F)) = F \cdot I$$

diagonalne matrike

$\cong F$

$\subseteq$ očitno

(**) $\underline{M^*_K}$ torej $(F \oplus F)_K$

$\varphi \in \operatorname{Hom}(M,K)$

$\varphi(m) = (\varphi_1(m), \varphi_2(m)) = \left(\varphi(m) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \varphi(m) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

stolpci

$$\varphi(m) = \left[\begin{array}{c:c} \varphi_1(m) & \varphi_2(m) \end{array}\right]$$

$$\varphi(Am) = \left[\begin{array}{c:c} \varphi_1(Am) & \varphi_2(Am) \end{array}\right]$$

$$A\varphi(m) = \left[\begin{array}{c:c} A\varphi_1(m) & A\varphi_2(m) \end{array}\right]$$

$\varphi_{1,2}$ skalarna večkratnika identitete

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \varphi \cdot A \overset{\text{v } M^*}{=} (m \longmapsto \varphi(m) \cdot A)$ dano po definiciji

$[\varphi_1, \varphi_2] \cdot A = \varphi_1 \cdot a + \varphi_2 \cdot b + \varphi_1 \cdot c + \varphi_2 \cdot d = \varphi_1(a+c) + \varphi_2(b+d)$

kako sklepamo da je

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \varphi(m) = [\varphi_1(m), \varphi_2(m)] * \qquad m = \begin{bmatrix} x \\ y \end{bmatrix}$

skalarji

$$= \begin{bmatrix} \lambda_1 m & \lambda_2 m \end{bmatrix} = \begin{bmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y & \lambda_2 y \end{bmatrix}$$

$$\varphi(m) \cdot A = \begin{bmatrix} a\lambda_1 x + b\lambda_2 x & c\lambda_1 x + d\lambda_2 x \\ a\lambda_1 x + b\lambda_2 y & c\lambda_1 y + d\lambda_2 y \end{bmatrix} =$$

$$= \begin{bmatrix} x(a\lambda_1 + b\lambda_2), & \overset{x}{\cancel{}}(c\lambda_1 + d\lambda_2) \\ y(a\lambda_1 + b\lambda_2) & \overset{y}{\cancel{}}(c\lambda_1 + d\lambda_2) \end{bmatrix}$$

$\varphi = [\lambda_1 \ \lambda_2] \qquad \varphi \cdot A = [\lambda_1 a + \lambda_2 b, \ c\lambda_1 + d\lambda_2]$

Lažje: $\quad F \oplus F \longrightarrow \operatorname{Hom}(M,K)$

$$[\lambda, \mu] \longmapsto \left(\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} [\lambda, \mu]\right)$$

$$(\varphi \cdot A)\begin{bmatrix} x \\ y \end{bmatrix} := \Big(\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} [\lambda, \mu]}_{\varphi}\Big) A = \begin{bmatrix} x \\ y \end{bmatrix} ([\lambda, \mu] A)$$

1) Določi dual ${}_{\mathbb{Z}}\mathbb{Z}_n^*$

$= Hom(\mathbb{Z}_n, \mathbb{Z}) = 0$

$1 \mapsto x$ red 1 | red x $\Rightarrow x = 0$

2) K cel kolobar (brez del. nič, komut)

za ${}_KM$ K-modul definiramo

$tor(M) - \{m \in M ; xm = 0 , x \in K - \{0\}\}$

$tor(\bigoplus_{n=2}^{\infty} \mathbb{Z}_n) = \bigoplus_{n=2}^{\infty} \mathbb{Z}_n$; ampak ne obstaja element, ki bi uničil vse

Dokaži da je $tor(M) \leq M$ podmodul in

$\varphi : M \to N$

$\varphi_*(tor(M)) \leq tor(N)$

a) $xm = 0$ $yn = 0$ $m, n \in M$ $x, y \in K$

$\underbrace{xy}_{\neq 0}(m+n) = xym + x\underbrace{yn}_{0} = yxm = 0$

$\alpha \in K$ $xm = 0$

$x\,\alpha m = \alpha x m = 0$

b) φ homomorfizem

$x\varphi(m) = \varphi(xm) = 0 \rightarrow m \in tor(M) \Rightarrow \varphi(m) \in tor(N)$

$\Rightarrow \varphi_*(tor(M)) \subseteq tor(N)$

je podmodul: $xy(\alpha\varphi(m) + \beta\varphi(n)) =$

$\alpha y \underbrace{\varphi(xm)}_{0} + \beta x \underbrace{\varphi(n)}_{0} = 0$

$tor\left(M/tor(M)\right) = 0$ DN

$$x\,(m + torM) = \underset{\sim}{xm} + torM$$

$= 0 \Rightarrow m \in torM \rightarrow m + torM = 0$

$\neq 0 \Rightarrow m \notin torM$

$3.\ M$ je prost $\overset{K \text{ cel}}{\Rightarrow} \operatorname{tor}(M)=0$

$$\varphi : M \xrightarrow{\cong} \bigoplus_{i=I} K$$

$$\varphi(\operatorname{tor} M) \leq \operatorname{tor}\Big(\bigoplus_{i=I} K\Big)$$

φ inj $\Rightarrow$ dovolj je dokazat $\operatorname{tor}\Big(\bigoplus_{i=I} K\Big)=0$

↑

(Dovolj je inj vložitev, ne izomorfizem)

$$\bigoplus_{i\in I} K = \{ (x_i)_{i\in I} \ ; x_i \in K \quad x_i = 0 \text{ za vse razen končno mnogo} \}$$

$$(x_1, x_2, \dots \dots x_i, \dots)$$

$$(x_i)_{i\in I} \in \operatorname{tor}\Big(\bigoplus K\Big)$$

ne velja za noben element razen 0

$$y(x_i)_{i\in I} = (yx_i)_{i\in I} \Longleftrightarrow \overset{\overset{0}{\neq}}{y} x_i = 0 \ \forall i$$

$$\Rightarrow x_i = 0 \ \forall i$$

4) Ali je $\mathbb{Z}_n$ prost $\mathbb{Z}$-modul?

NE, ker ima torzijo

Izrek: M končno generiran $\mathbb{Z}$-modul.

M prost $\Leftrightarrow$ M torzijska prost

5) Pokaži

${}_{\mathbb{Z}}\mathbb{Q}$ ni prost $\mathbb{Z}$-modul

$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$

$(c\,b)\frac{a}{b} - (a d)\frac{c}{d} = 0$ netrivialna linearna kombinacija

Ali je lahko baza ?

$\mathbb{Q} = (\frac{a}{b})$ $\frac{1}{b^2}$ ni oblike $\alpha\frac{a}{b}$

Ali je $(\mathbb{Q}^*, \cdot)$ prost $\mathbb{Z}$ modul

$-1 \in \mathbb{Q}^*$ $(-1)(-1) = 1 \Rightarrow (-1) \in T(\mathbb{Q}^*)$

$\Rightarrow$ ima torzijo, torej ni prost

kaj pa $(\mathbb{Q}_+, \cdot)$

$\sum \left(\frac{a_i}{b_i}\right)^{n_i} > 0 \neq 0$

Eksaktna zaporedja

kratke ≡ kvocienti

$$0 \xrightarrow{0} L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{0} 0$$

$\operatorname{im}\varphi = \ker\psi \quad \operatorname{im}\psi = \ker 0 = N$

$0 = \operatorname{im} 0 = \ker\varphi$

← ψ surjektivna

← φ injektivna

$$M/\varphi(L) \cong N$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

$x \longmapsto 2x$

$-2x \quad x \longmapsto x \bmod 2$

inj

← edina možna preslikava

$\ker\psi = 2\mathbb{Z} = \operatorname{im}\varphi$

← enotska krožnica

$$0 \xrightarrow{\text{inj ✓}} \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{T} \xrightarrow{\text{sur ✓}} 0$$

$$n \longmapsto \boxed{\begin{matrix} n \\ x \end{matrix}} \begin{matrix} \longmapsto 1 \\ \longmapsto e^{2\pi i x} \end{matrix}$$

P je projektiven če ∀ eksaktno zaporedje $0 \to L \to M \to P \to 0$ razpade

kratko eksaktno zaporedje razpade če je ekvivalentno $0 \longrightarrow L \longrightarrow L \oplus N \longrightarrow N \longrightarrow 0$

P je projektiven $\iff$ 1) $\iff$ 2)

$\iff$ 1)

$$\begin{tikzcd}
 & P \arrow[d, "\forall f"] \arrow[dl, dashed, "\exists g"'] & \\
A \arrow[r, "sur"'] & B \arrow[r] & 0
\end{tikzcd}$$

$\iff$ 2) P je direktni sumand prostega modula $K^{(I)}$

$$\exists N .\ P \oplus N \cong \underset{\overset{\nearrow}{\text{kolobar}}}{K}^{(I)} = \bigoplus_{i \in I} K$$

1. naloga 7. 12

$\begin{bmatrix}F\\F\end{bmatrix}$ je projektiven in ne prost nad $M_2(F)$

2) $M_2(F) \cong \begin{bmatrix}F\\F\end{bmatrix} \underset{\substack{\uparrow\\ \text{zunanja}}}{\oplus} \begin{bmatrix}F\\F\end{bmatrix} \cong \begin{bmatrix}F&0\\F&0\end{bmatrix} \underset{\substack{\uparrow\\ \text{notranja}}}{\oplus} \begin{bmatrix}0&F\\0&F\end{bmatrix}$

$$\begin{bmatrix}x&z\\y&w\end{bmatrix} \mapsto \left(\begin{bmatrix}x\\y\end{bmatrix}, \begin{bmatrix}z\\w\end{bmatrix}\right) \mapsto \begin{bmatrix}x&0\\y&0\end{bmatrix} \oplus \begin{bmatrix}0&z\\0&w\end{bmatrix}$$

je očiten izomorfizem

$\Rightarrow$ je projektiven

Recimo da je prost, potem

$\begin{bmatrix}F\\F\end{bmatrix} = \begin{cases} M_2(F) & = \dim 4 \\ M_2(F) \oplus M_2(F) & = \dim 8 \\ \vdots \end{cases}$ vse dimenzije so prevelike

$\Rightarrow \begin{bmatrix}F\\F\end{bmatrix}$ ni prost

2. način:

P proj $\Rightarrow P$ orz prost

$\varphi \quad P \longrightarrow M$ homomorfizem

$\varphi(\mathrm{tor}(P)) \subseteq \mathrm{tor}(M)$

2) P proj $\underset{(K \text{ cel})}{\Longrightarrow}$ P torz. prost

Velja: $\varphi : P \to M$ homo.

$\varphi(\operatorname{tor}(P)) \subseteq \operatorname{tor}(M)$

$P \oplus N \cong K^{(I)}$

$P \longrightarrow K^{(I)}$

$\varphi(\operatorname{tor}) \subseteq \operatorname{tor}(K^{(I)}) \overset{\text{ker je } K^{(I)} \text{ prost}}{=} 0$

Zaradi injektivnosti φ je je tudi $\operatorname{tor}(P) = 0$

$$\begin{tikzcd} \text{prost} \arrow[r, Rightarrow] & \text{proj.} \arrow[r, Rightarrow, "K \text{ cel}"] & \text{torz. prost} \arrow[ll, Rightarrow, bend left, "K \text{ glavni } V"] \end{tikzcd}$$

končnogeneriran modul

Kategorije

Top, Grp, Ab, Vect_F, K-Mod, ...

K mod:

objekti: K - moduli

morfizmi: homomorfizmi K-modulov

$$\mathrm{Hom}_K(P,\_): K\ \mathrm{mod} \longrightarrow \mathrm{Ab}$$

$${}_K M \longmapsto \mathrm{Hom}(P,M)$$

$$(M \xrightarrow{f} N) \longmapsto \mathrm{Hom}(P,M) \longrightarrow \mathrm{Hom}(P,N)$$

$$\varphi \longmapsto f\circ\varphi$$

3. naloga

$\mathrm{Hom}(P,\_)$ je funktor

- ohranja kompozitum
- ohranja identiteto

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$F(g\circ f) \underset{(x)}{=} (\varphi \longmapsto \varphi\circ g\circ f)$$

$$F(g)\circ F(f)(x) = (\varphi \mapsto \varphi\circ g)(\varphi \mapsto \varphi\circ f)(x) =$$

$$= (\varphi \longmapsto \varphi\circ g)\ x\circ f = x\circ f\circ g = (\varphi \mapsto \varphi\circ(\varphi\circ g)) = F(f\circ g)$$

ups, zamenjan vrstni red

$$F(\mathrm{id}) = (\varphi \longmapsto \mathrm{id}\,\varphi) = \mathrm{id}$$

P proj $\iff$ $\underbrace{\mathrm{Hom}(P,-)}$ eksakten

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(P,L) \xrightarrow{F(\varphi)} \mathrm{Hom}(P,M) \xrightarrow{F(\psi)} \mathrm{Hom}(P,N) \longrightarrow 0$$

$\mathrm{Hom}(P,-)$ je eksakten če slika eksaktna zaporedja v eksaktna

$\Longrightarrow$ 1) $P = K$ $\quad \mathrm{Hom}(K, \underline{{}_K M}) \cong {}_K M$

$f \mapsto \varphi \circ f \qquad f \longmapsto f(1)$

$$\begin{tikzcd}
\mathrm{Hom}(P,L) \arrow[r, "\overset{\shortparallel}{F(\varphi)}"] \arrow[d, phantom, "\wr\!\shortparallel"] & \mathrm{Hom}(P,M) \arrow[d, phantom, "\wr\!\shortparallel"] & f \arrow[d] \\
L \arrow[r, "\varphi"'] & M & f(1)
\end{tikzcd}$$

2) P je prost $\quad P \cong K^{(I)}$

$$(\mathrm{Hom}, -) = \mathrm{Hom}(\bigoplus_I K, -) = \bigoplus_I \mathrm{Hom}(K,-)$$

$$0 \to \bigoplus \mathrm{Hom}(K,L) \to \bigoplus \mathrm{Hom}(K,M) \to \dots$$

$$(l_1 \dots l_n) \longmapsto (\varphi(l_1), \varphi(l_2) \dots)$$

8. 12

1. ura

Tenzorski produkt

ni nujno, ampak je bolj enostavno če je

${}_KM$, ${}_KN$ modula, K komutativen

$$M\otimes_K N = \left\{ \sum_{i=1}^{l} m_i\otimes n_i \;\middle|\; m_i\in M \ n_i\in N \ l\in\mathbb{N}_0 \right\}$$

1) bilinearnost

$$(m+m')\otimes n = m\otimes n + m'\otimes n$$

$$m\otimes(n+n') = m\otimes n + m\otimes n'$$

2) biaditivnost $x\in K$

$$x\sum m_i\otimes n_i = \sum x(m_i\otimes n_i) = \sum (xm_i)\otimes n_i = \sum m\otimes(xn)$$

$$\begin{tikzcd}
\varphi\colon {}_KM\times{}_KN \arrow[r, "\text{bilin}"] \arrow[ddd] & {}_KL \\
(m,n) \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(m,n) \\
m\otimes n & \\
M\otimes N \arrow[uur, dashed, "\exists!\bar{\varphi}"'] &
\end{tikzcd}$$

bilinearne
$\varphi(xm,n) = x\varphi(m,n) = \varphi(m,xn)$

U,V vek. prostora $\quad U=\mathrm{span}\{\overset{\text{baza}}{e_i}\}\quad V=\mathrm{span}\{f_j\}$

$$U\otimes V = \mathrm{span}\{e_i\otimes f_j\}$$

1. Zapiši

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(1,1)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(0,1)+(-1,0)\otimes\left(\tfrac{1}{2},1\right)\in\mathbb{R}^2\otimes\mathbb{R}^2$$

kot enostavni tenzor $(u\otimes v)$

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(1,0)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(0,1)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(0,1)+\tfrac{1}{2}(-1,0)\otimes(1,0)+$$

$$(-1,0)\otimes(0,1)=$$

$$=\left(0,\tfrac{1}{2}\right)\otimes(1,0)+(0,1)\otimes(0,1)=$$

$$=\underline{\underline{(0,1)\otimes\left(\tfrac{1}{2},1\right)}}$$

2.

Poišči primer vek. prost. U, V in tenzorja $x \in U \otimes V$, ki ni enostaven

$U, V = \mathbb{R}^2$

$U', V' = \mathbb{R}$

U', V' $\qquad \sum (x_i \otimes y_i) = \sum x_i y_i (1 \otimes 1) = \cancel{\square}$ $(1 \otimes \sum x_i y_i\, 1 \otimes 1)$

enostavn.

$$\boxed{M \otimes_k k \cong M}$$

$$(1,0) \otimes (1,0) + (0,1) \otimes (0,1) = x \otimes y$$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$$x \otimes y = \big(x_1(1,0) + x_2(0,1)\big) \otimes \big(y_1(\overset{1}{\cancel{0}}, \cancel{1}) + y_2(0,1)\big) =$$

$$= x_1 y_1 (1,0) \otimes (1,0) + x_2 y_1 (0,1) \otimes (1,0) + x_1 y_2 (1,0) \otimes (0,1)$$

$$+ x_2 y_2 (0,1) \otimes (0,1)$$

$x_1 y_1 \neq 0 \quad \Rightarrow x_1 y_1 \neq 0$

$x_2 y_1 = 0 \quad \rightarrow x_2 = 0$

$x_1 y_2 = 0$

$x_2 y_2 \neq 0 \quad \rightsquigarrow \overset{x_2 \neq 0}{\times}$