Hilbertov prostor

X vektorski prostor nad $\mathbb{R}$ (nad D)
skalarni produkt $\langle\ \rangle$

$$\langle\ \rangle : X \times X \longrightarrow \mathbb{R}$$

1) $\forall x \in X : \langle x, x \rangle \geq 0$

2) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

Pozitivne definitnost

3) $\forall x, y \in X.$

$\langle x, y \rangle = \langle y, x \rangle$ nad $\mathbb{R}$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$ nad $\mathbb{C}$

simetričnost oz antisimetričnost

4) $\forall x, y, z \in X,\ \forall \lambda, \mu \in \mathbb{R}$

$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

linearnost v prvem faktorju

Za izkaz: praktično

Cauchy-Schwarzova neenakost

$\forall x,y \in X$

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}$$

$$\|x\| = \sqrt{\langle x,x\rangle}$$

$$|\langle x,y\rangle| \leq \|x\| \, \|y\|$$

Dokaz nad $\mathbb{R}$

$$t \longmapsto \langle x+ty, x+ty\rangle = s(t) \geq 0$$

$$= \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle$$

$$= \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\| \geq 0$$

$$D \leq 0$$

$$D = 4\langle x,y\rangle - 4\|x\|^2\|y\|^2$$

$$\Rightarrow \langle x,y\rangle^2 \leq \|x\|^2\|y\|^2$$

$$\langle x,y\rangle \leq \|x\|^2\|y\|^2$$

enakost velja natanko tedaj če sta x in y linearno odvisna

Nad $\mathbb{C}$:

x, y

$\exists \alpha \quad |\alpha| = 1$

$\langle x, y\rangle = \alpha |\langle x, y\rangle|$

$\Rightarrow \quad \langle x, \alpha y\rangle = |\langle x, y\rangle|$

$f(t) = \langle x + t\alpha y, x + t\alpha y\rangle =$

$= \|x\|^2 + t\,(\langle \alpha y, x\rangle + \langle x, \alpha y\rangle) + t^2 \langle \alpha y, \alpha y\rangle$

$\qquad\qquad\qquad \| $

$\qquad\qquad 2 \cdot |\langle x, y\rangle|$

$= \|x\|^2 + 2t|\langle x, y\rangle| + t^2\|y\|^2 \geq 0$

$D = 4|\langle x, y\rangle|^2 - 4\|x\|^2\|y\|^2 \leq 0$

<u>Norma</u> nad vektorskem prostorom X

$\| \ \|: X \longrightarrow \mathbb{R}$

1) $\forall x \in X.\ \|x\| \geq 0$

2) $\|x\| = 0 \Leftrightarrow x = 0$

3) $\forall \lambda \in \mathbb{R} \vee \mathbb{C},\ \forall x \in X$

$\|\lambda x\| = |\lambda|\,\|x\|$ homogenost

4) trikotniška neenakost

$\forall x, y \in X.\ \|x+y\| \leq \|x\| + \|y\|$

če je $(X, \langle\ \rangle)$ vektorski prostor s skalarnim produktom je

$\|x\| = \sqrt{\langle x, x \rangle}$ X vektorski prostor z normo

$\|x\| = \sqrt{\langle x, x\rangle}$

1), 2) Skeli iz 1) 2) skalarnega produkta

3) $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{|\lambda|^2 \langle x, x\rangle} =$

$= |\lambda| \, \|x\|$

4)

$\|x+y\|^2 = \langle x+y, x+y\rangle =$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle =$

$= \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2|\langle x, y\rangle|$

$+ \|y\|^2$

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

$(X, \langle\ \rangle) \rightsquigarrow (X, \|\ \|)$

$\rightsquigarrow$ metrični prostor (X, d)

$\forall x, y \in X.\ d(x,y) = \|x - y\|$

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \iff x = y$

3) $d(x,y) = d(y,x)$

4) $\forall x, y, z \in X$

$$d(x,y) + d(y,z) \geq d(x,z)$$

Def: Hilbertov prostor je vektorski prostor s skalarnim produktom ki je v metriki porojeni iz skalarnega produkta poln metrični prostor

Opomba: Banachov prostor je vektorski prostor X z normo $\| \ \|$ ki je v metriki porojeni iz norme poln metrični prostor

Zgledi:

1) $\mathbb{R}^n$, $x = (x_1 \dots x_n)$

$y = (y_1 \dots y_n)$

$$x \cdot y = x_1 y_1 + \dots \pm x_n y_n$$

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$$ d_2 metrika

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$$ je

poln metrični prostor

$(\mathbb{R}^4, \cdot)$ je Hilbertov prostor

$(\mathbb{R}^n, d_\infty)$ $\quad \|\;\|_\infty$ $\quad \max\{|x_1| \dots |x_n|\}$

normi

$(\mathbb{R}^n, d_1)$ $\quad |x_1| + \dots + |x_n|$

$\|\;\|_1$

Banachova prostora

ker sta topološko enaka $(\mathbb{R}^n, d_2)$

$\mathbb{R}^n$ sta Banachova ker sta topološki ekvivalentna $(\mathbb{R}^2, d_2)$

Ampak ne pride iz skalarnega produkta

2) $\mathbb{C}^n$ $z = z_1, \dots z_n$

$w = w_1, \dots w_n$

$z \cdot w = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$

$\| z \| = \sqrt{z_1^2 + \dots + z_n^2}$

$d_2(z,w) = \sqrt{|z_1 - w_1|^2 + \dots + |z_n - w_n|^2}$

$(\mathbb{C}^n, \cdot)$ je hilbertov prostor

Zgled

$[a,b] \subseteq \mathbb{R}$ $a<b$

$X = C([a,b])$ nad $\mathbb{R}$

$\langle f,g\rangle = \int_a^b f(x)g(x)dx$

3) $\langle f,g\rangle = \langle g,f\rangle$

4) $\langle \lambda f + \mu g, h\rangle =$
$\lambda \langle f,h\rangle + \mu\langle g,h\rangle$

1) $\langle f,f\rangle = \int_a^b f^2(x)dx \geq 0$

2) $f=0 \Rightarrow \langle f,f\rangle = 0$

$\int_a^b f^2 dx = 0 \Rightarrow f^2 = 0$ skoraj povsod

$\Rightarrow f^2 = 0$ povsod ker zvezna

$\Rightarrow f=0$ (Naslednja stran)

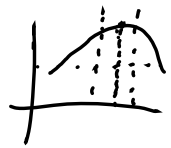
Polnost?

$$\int_a^b f^2(x)\,dx = 0$$

$\bar{c}e$ $\exists\, x_0 \in [a,b]$ $f(x_0) \neq 0 \Longrightarrow$

$\exists\, \delta > 0 : \forall x \in (x_0-\delta, x_0+\delta) \cap [a,b]$.

$$|f(x)| \geq \frac{|f(x_0)|}{2}$$



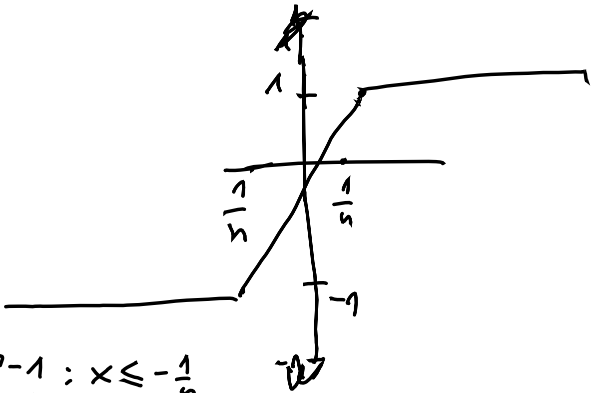
$$\int_a^b f^2(x)\,dx \geq \int_{(x_0-\delta,\, x_0+\delta) \cap [a,b]} f^2(x)\,dx \geq \frac{f(x_0)^2}{2}\,\delta$$

$\bar{c}e$ f [illegible] $\Longrightarrow f = 0$ [illegible]

Trditev: $(C[a,b], \langle\ \rangle)$ ni Hilbertov

Dokaz

$\cancel{f_n(x)}$



$$f_n(x) = \begin{cases} -1 & : x \le -\frac{1}{n} \\ 1 & : x \ge \frac{1}{n} \\ nx & ; -\frac{1}{n} \le x \le \frac{1}{n} \end{cases}$$

$(f_n)_n$ je Cauchyjevo zaporedje v $(C[-1,1], d)$

d-metrika iz skalarnega produkta

$$d^2(f,g) = \int_{-1}^{1} |f(x) - g(x)|^2 \, dx$$

$m > n$

$$d(f_n, f_m)^2 =$$

$$\int_{-1}^{1} (f_n(x) - f_m(x))^2 \, dx =$$

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} ((f_m(x) - f_n(x)))^2 \le \int_{-\frac{1}{n}}^{\frac{1}{n}} 1 \, dx = \frac{2}{n}$$

$$d(f_n, f_m) \le \sqrt{\frac{2}{n}} < \varepsilon$$

$(f_n)_n$ je Cauchyjevo v $(C[-1,1], d)$

$\cancel{\lim f_n}$

Če obstaja lim za $(f_n)_n$ v $C[-1,1]$

mora veljati, da je $f(x) = \begin{cases} 1 & ; 0 < x \le 1 \\ -1 & ; -1 \le x < 0 \end{cases}$

$\Rightarrow f$ ni zvezna v 0

Zgled

$M = (0,1)$

$d_2(x,y) = |x-y|$

~~M~~ M ni poln

$M \cup \{0,1\}$ je poln

Dodali smo možne limite

Napolnitev metričnega prostora

(M,d) napolnimo   lahko napolnemo

$(\tilde{M},\tilde{d})$ poln

1) $M \subseteq \tilde{M}$

2) $\tilde{d}\big/_{M\times M} = d$

3) M je gost v $\tilde{M}$

$L^1(A) = \{ f : A \to \mathbb{R} ; \int_A |f(x)| \, dx < \infty) \}$

$\rightarrow$ kvadratom integrabilne funkcije:

$L^2([a,b]) = \{ f : [a,b] \to \mathbb{R} \; \int_a^b |f|^2 dx < \infty \}$

Zgled:

1) $C\ell[a,b] \subseteq L^2([a,b])$

2) odsekoma zvezne $\subseteq L^2$

3) $f(x) = \frac{1}{\sqrt[4]{x-a}} \in L^2$

4) $g(x) = \frac{1}{\sqrt[2]{x-a}} \notin L^2([a,b])$

Velja: f, g

$|f \cdot g| \leq \frac{|f|^2 + |g|^2}{2} \Rightarrow f \cdot g \in L^1([a,b]$

$\langle f, g \rangle = \int_a^b f(x) \cdot g(x) \, dx$

L^2 je vektorski prostor

$$(f+g)^2 = \underset{\in L^2}{f^2} + \underset{\in L^1}{2fg} + \underset{\in L^2}{g^2}$$

$$\int_a^b (f+g)^2\,dx = \int_a^b f^2(x)\,dx + 2\int f(x)g(x)\,dx + \int_a^b g^2(x) < \infty$$

$$\Rightarrow \quad (f+g) \in L^2$$

$C[a,b] \subseteq L^2([a,b])$

hilbertov

Opomba

$\forall f \in L^2([a,b]) . \exists f_n \in C[a,b] .$

$$\lim_{n\to\infty} f_n = f \iff \lim_{n\to\infty} \|f_n - f\| = 0$$

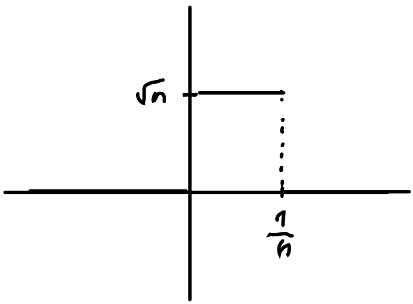
$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x))^2 dx = 0$$

Opomba: nad $\mathbb{C}$: $f = u + i v$ $u,v : [a,b] \to \mathbb{R}$

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx$$

Zgled: na $[0,1]$

$$f_n : \begin{cases} \sqrt{n} & ; \ 0 < x \leq \frac{1}{n} \\ 0 & ; \ \text{sicer} \end{cases}$$

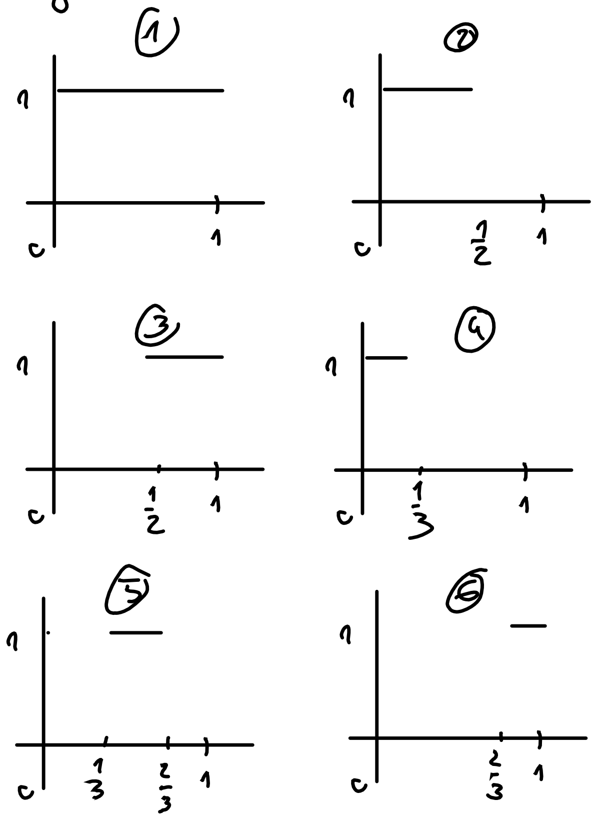


$\lim f_n(x) = 0$ za $x \in [0,1]$

Ali konvergira v L^2?

$$\|f_n - 0\| = \sqrt{\int_0^1 f_n^2(x)\, dx} = \sqrt{\int_0^{\frac{1}{n}} \sqrt{n}^2\, dx} = 1$$

Zgled:



itd

$\lim_{n\to\infty} f_n(x)$ ne obstaja

$\lim f_n(x)$ v L^2 ?

$\int_0^1 f_n^2(x)\,dx \leq \frac{1}{n^2}$ torej limita je 0

Naj bo $(X, \langle \, \rangle)$ vek. prostor s skalarnim produktom.

$A \subseteq X \quad (A \neq \emptyset)$

$x, y \in X$. $\quad x \perp y \Leftrightarrow \langle x, y \rangle = 0$

(simetrična relacija)

$A \subseteq X \quad A^{\perp} = \{ x \in X . \; x \perp a \text{ za } \forall a \in A \}$

ortogonalni komplement

Trditev: $A^\perp$ je vektorski podprostor v X

Dokaz:

$x \in A^\perp \quad \lambda \in \mathbb{R}$ (ali $\mathbb{C}$)

$a \in A$. $\langle \lambda x, a \rangle = \lambda \langle x, a \rangle = \lambda \cdot 0 = 0$

$x, y \in A^\perp$

$\langle x+y, a \rangle = \underbrace{\langle x, a \rangle}_{=0} + \underbrace{\langle y, a \rangle}_{=0} = 0$

Velja: $A \subseteq (A^{\perp})^{\perp}$

Trditev: $v \in X$

$f(x) = \langle x, v \rangle \quad f: X \to \mathbb{R}$

f je zvezna na X

Dokaz:

$x_1, x_2 \in X$

$|f(x_1) - f(x_2)| = |\langle x_1 - x_2, v \rangle| \leq \|x_1 - x_2\| \|v\|$

f je enakomerno zvezna
(celo Lipschitzova zvezna)

Posledica: $A^\perp$ je zaprt vektorski podprostor

$C[a,b] \subseteq L^2[a,b]$ ni zaprt podprostor

Dokaz:

zaporedje x_n

$\lim x_n = x_0 \in X \overset{?}{\Longrightarrow} x_0 \in A^\perp$

$\forall a \in A . \langle x_n, a \rangle = 0$ za $\forall n$

$$\underbrace{\lim \langle x_n, a \rangle}_{=0} = \langle \lim x_n, a \rangle = \langle x_0, a \rangle = 0$$

Opomba: $(X, \langle \rangle)$ hilbertov

$A \leq X$ zaprt podprostor $\Rightarrow A = (A^\perp)^\perp$

Trditev: (Pitagorov izrek)

$(X, \langle\ \rangle)$ vektorski prostor s skalarnim produktom

$x_1 \ldots x_n \in X \quad \forall i \neq j \in [n] .\ x_j \perp x_i$

$(\langle x_j, x_i \rangle = 0)$

Tedaj $\|x_1\|^2 + \ldots + \|x_n\|^2 = \|x_1 + \ldots x_n\|^2$

Dokaz:

$\|x_1 + \ldots + x_n\| = \langle x_1 + x_2 \ldots + x_n, x_n + \ldots + x_n \rangle$

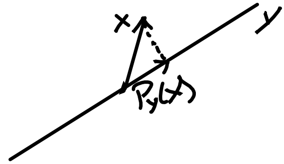
$= \langle \sum_i x_i ; \sum x_j \rangle = \sum_i \sum_j \langle x_i, x_j \rangle =$

$= \|x_1\|^2 + \ldots + \|x_n\|^2$

$(X, \langle\ \ \rangle)$

$Y \leq X$

$x \in X$



Definicija: Pravokotna projekcija vektorja x na podprostor Y (če obstaja) je tak vektor $P_Y(x) \in Y$ da je

$$x - P_Y(x) \in Y^{\perp}$$

Trditev: Če pravokotna projekcija x na Y obstaja, je enolično določena in $P_Y(x)$ je najboljša aproksimacija vektorja X z vektorji iz Y (razdalja je najmanjša)
$$\left(\|x - P_Y(x)\| = \min_{w \in Y} \|x - w\| \right)$$

Dokaz:

$Y \leq X \quad x \in X$

Denimo da $y_1\ y_2$ sta pravokotni projekciji x na Y

$$x - y_1,\ x - y_2 \in Y^{\perp}$$

$$(x-y_1) - (x-y_2) \in Y^{\perp}$$ ker vek. prostor

$$y_2 - y_1 \in Y^{\perp}$$

$$y_2 - y_1 \in Y$$

$$\langle y_2 - y_1, y_2 - y_1 \rangle = 0 \iff$$

$$y_2 - y_1 = 0 \qquad y_2 = y_1$$

$w \in Y$

$$x - w = \underbrace{x - P_Y(x)}_{\in Y^{\perp}} + \underbrace{P_Y(x) - w}_{\in Y}$$

pitagorov izrek:

$$\|x - w\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x) - w\|^2 \geq$$
$$\|x - P_Y(x)\|^2$$

■

Zgled:

$Y = C[a,b]$ $\quad$ $X = L^2[a,b]$

$f \in X - Y$ nima pravokotne projekcije

Tak f nima najboljše aproksimacije z zveznimi funkcijami

Komentar: hilbertov + zaprt podprostor ima projekcije

Opombe:

1) $P_Y^2 = P_Y$

2) $x = \underbrace{x - P_Y(x)} + \underbrace{P_Y(x)}$

$\|x\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x)\|^2$

$\|x\| \geq \|P_Y(x)\|$

3) Če je P_Y definirana na celem X, potem je P_Y linearen in zvezen

$\|P_Y x_1 - P_Y x_2\| = \|P_Y(x_1 - x_2)\| \leq \|x_1 - x_2\|$

P_Y je zvezna

linearnost:

$\lambda x - P_Y(\lambda x)$

Poglejmo:

$$\lambda x - \lambda P_Y(x) = \overbrace{\underbrace{\lambda(x - P_Y(x))}_{\in Y^\perp}}^{\in Y^\perp}$$

$\Rightarrow \lambda P_Y(x) \in Y$

$\lambda x - \underbrace{\lambda P_Y(x)} \in Y^\perp$

To je pravokotna projekcija λx

$x_1, x_2 \in X$

$P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$

$$(x_1 + x_2) - \underbrace{(P_Y(x_1) + P_Y(x_2))}_{\in Y} = \underbrace{x_1 - P_Y(x_1) + x_2 - P_Y(x_2)}_{\in Y^\perp}$$

$\Rightarrow P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$

zaradi: enoličnosti projekcije ■

~~Če P_Y definiran~~

Če je P_Y definiran na X je
Y zaprt prostor

Dokaz:

$\{y_j\} \subseteq Y \quad \lim_{j\to\infty} y_j = y_0$

$P_Y(y_j) = y_j$

$$\underset{\substack{\parallel \\ y_0}}{\lim y_j} = \lim P_Y(y_j) \overset{\text{zveznost}}{=} P_Y(y_0)$$

Če ima X pravokotno projekcijo na Y ima tudi pravokotno projekcijo na $Y^\perp$

x; $P_Y(x)$

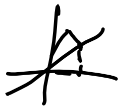
$$P_{Y^\perp}(x) = x - P_Y(x) \in Y^\perp$$

$$x - (x - P_Y) \in (Y^\perp)^\perp$$

$$\| $$

$$P_Y(x) \in Y \subseteq (Y^\perp)^\perp$$

Trditev: Naj bo $Y \leq X$ končno dimen podprostor z ortonormirano bazo $e_1, \dots, e_n$. $\langle e_i, e_j \rangle = \delta_{ij}$



Naj bo $x \in X$. Tedaj

$$P_Y(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$$

Opomba: Vsak končno dimenzionalni podprostor ima pravokotno projekcijo definirano na X

In tudi vsi tisti končne kodimenzije

Dokaz: $P_Y(x) = \sum_{1}^{n} \langle x, e_i \rangle e_i \in Y$

Poglejmo še

$$x - \sum_{1}^{n} \langle x, e_j \rangle e_j \in Y^{\perp}$$

$$\langle x - \sum_{1}^{n} \langle x, e_j \rangle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \sum_{1}^{n} \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$(X, \langle\, \rangle)$

Sistem vektorjev

$(e_j)_{j=1}^{\infty}$

je ortogonalen sistem (OS), če

$\forall i \neq j \; \langle e_i, e_j \rangle = 0$

Tak sistem je ortonormiran (ONS)

$\forall i, j \quad \langle e_i, e_j \rangle = \delta_{ij}$

Trditev: $(X, \langle\ \rangle)$ Naj bo $(e_j)_j$ ONS

Naj bo $x \in X$. Tedaj je $\sum |\langle x, e_j\rangle|^2 \leq \|x\|^2$

(Besselova neenakost)

Opomba: $\langle x, e_j\rangle$ so Fourierovi koeficienti x po ONS $(e_j)_j$

Posledica:

$$\lim_{j \to \infty} \langle x, e_j\rangle = 0$$

Dokaz: $Y_n = \mathcal{L}(\{e_1, \dots, e_n\})$

$x \in X$

$$\exists P_Y(x) = \sum_1^n \langle x, e_j\rangle e_j$$

projekcija je vedno krajša od x

$$\|P_Y(x)\|^2 = \sum_1^n \|\langle x, e_j\rangle e_j\|^2 = \sum_1^n |\langle x, e_j\rangle| \leq \|x\|^2$$

$$\Rightarrow \lim_{n \to \infty} \sum_1^n \|\langle x, e_j\rangle\|^2 = \sum_1^\infty |\langle x, e_j\rangle|^2 \leq \|x\|^2$$

Trditev: Naj bo $(c_j)_{j=1}^{\infty}$ zaporedje števil (ali $\mathbb{R}$ ali $\mathbb{C}$) za katero velja

$$\sum_1^{\infty} |c_j|^2 < \infty$$

Naj bo $(X, \langle\,\rangle)$ Hilbertov prostor $(e_j)_1^{\infty}$ ONS

Tedaj $\exists x \in X$, za katerega velja $c_j = \langle x, e_j \rangle$ za $\forall j$.

$$x = \sum_1^{\infty} c_j e_j = \lim \sum_1^{N} c_j e_j$$

$(X, \langle\ \rangle)$ hilbertov $\quad e_j$ ONB

$x \in X \rightsquigarrow (\langle x, e_j \rangle)_j \quad \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$

$\Rightarrow \quad \exists \tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$

Ali: je $\quad \tilde{x} = x$

Zemuda

20.2

Zgled (pred prejšnjim dokazom)

<u>Modelni hilbertov prostor</u> ℓ^2

Prostor zaporedij

$$\ell^2 = \{ (a_j)_j \; ; \; a_j \in \mathbb{R} \, , \, \sum_1^\infty |a_j|^2 < \infty \}$$

$$\langle (a_j), (b_j) \rangle = \sum a_j b_j \quad \text{nad } \mathbb{R}$$

$$\sum a_j \bar{b}_j \quad \text{nad } \mathbb{C}$$

$$\| a_j \|^2 = \sum |a_j|^2$$

$(X, \langle\,\rangle)$ hilbertov $\quad (e_j)$ KONS

$$x \longmapsto (\langle x, e_j \rangle)_j \in \ell^2$$

$$e_j = (0, \ldots, 0\,1\,0 \ldots 0)$$

Dokaz:

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Rightarrow 6 \Rightarrow 5$

1) $\Rightarrow$ 2)

$x, y \in X \quad x = \sum_1^\infty \langle x, e_j\rangle e_j$

$\langle x,y\rangle = \langle \sum_1^\infty \langle x,e_j\rangle e_j, y\rangle =$

$= \sum_1^\infty \langle x, e_j\rangle \langle e_j, y\rangle$

2) $\Rightarrow$ 3)

$x=y \quad \langle x,y\rangle = \|x\|^2 = \sum_1^\infty |\langle x,e_j\rangle|^2$

3) $\Rightarrow$ 4)

če bi bil vsebovan v strogo večjem ONS

ONS

potem $\exists e_0 \perp e_j$ za $\forall j$ $\|e_0\|=1$

Vstavimo v parcevalovo enakost

$\|e_0\|^2 = 1^2 = \sum_1^\infty |\langle e_0, e_j\rangle|^2 = \sum_1^\infty 0 = 0$

4) $\Rightarrow$ 5)

$x \perp e_j$ za $\forall j$

če $x \neq 0$ tvorimo $e_0 = \frac{x}{\|x\|}$

$(e_0, (e_j)_j)$ je strogo večji KONS, ker ne more biti $\ast$

5) $\Rightarrow$ 1)

$x \in X \quad (\langle x, e_j\rangle)_j$

$\tilde{x} = \sum_1^\infty \langle x,e_j\rangle e_j$ ali sta enaka?

$\vec{v} = x - \tilde{x} = x - \sum_1^\infty \langle x, e_j\rangle e_j$

$\langle v, e_i\rangle = \langle x, e_i\rangle - \sum_1^\infty \langle x,e_j\rangle \langle e_j, e_i\rangle =$

$\langle x,e_i\rangle - \langle x, e_i\rangle = 0$

$\vec{v} = 0 \Rightarrow x = \tilde{x}$

1) $\Rightarrow$ 6) $\forall x = \sum_1^\infty \langle x,e_j\rangle e_j = \lim \sum_1^N \langle x,e_j\rangle e_j$

$\forall \varepsilon > 0 . \exists N_0 \in \mathbb{N}. \|x - \sum_1^{N_0} \langle x,e_j\rangle e_j\| < \varepsilon$

6) $\Rightarrow$ 5)

Naj bo $x \perp e_j$ za $\forall j$

$x = 0$

$\varepsilon > 0$

$\exists$ končna linearna kombinacija $\sum_1^N \lambda_j e_j$

daje $\|x - \sum_1^N \lambda_j e_j\| < \varepsilon$

$\|x\|^2 = \langle x,x\rangle = \langle x - \sum_1^N \lambda_j e_j, x\rangle \leq$

ker pravokotnost

$\leq \|x - \sum_1^N \lambda_j e_j\| \cdot \|x\| < \varepsilon \|x\|$

1) $x=0 \Rightarrow \checkmark$

2) $x \neq 0 \Rightarrow \|x\| < \varepsilon$ za $\forall \varepsilon > 0 \Rightarrow x = 0$ $\ast$

Osredotočimo se na $L^2(-\pi,\pi)$

ONS: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \dots \right.$

$\dots \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx), \dots$

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} f^2(x)}$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \qquad \|f_0\|_2 = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi}} = 1$$

$$\sqrt{\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin^2(nx)\, dx} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\, dx} = 1$$

Opomba: nad $\mathbb{C}$: $\frac{1}{2\pi}e^{inx}$ $n \in \mathbb{Z}$ vzamemo za ONS

$n \neq m$

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{inx} e^{imx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

$$= \frac{1}{2\pi(n-m)} e^{i(n-m)x}\Big|_{-\pi}^{\pi} = 0$$

$f: (-\pi, \pi) \Rightarrow \mathbb{R}$ periodična funkcija s periodo 2π

Klasični furierovi koeficienti

$f \in L^2(-\pi, \pi)$ (Recimo da imamo konos)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad n \in \{0, 1, \dots\}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) = \langle f, \frac{1}{\pi} \cos(nx) \rangle =$$

$$= \frac{1}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \rangle = \sqrt{\pi}\, a_n$$

$$\langle f, \frac{1}{\pi} \sin(nx) \rangle = \sqrt{\pi}\, b_n$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot 1\, dx = \langle f, \frac{1}{\pi} \rangle = \frac{\sqrt{2}}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle =$$

$$\langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{\sqrt{\pi}}{\sqrt{2}} a_0$$

Če je $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x \dots\right\}$ KONS

potem velja parcevalova enakost

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2(x)\,dx = \frac{\pi}{2} a_n + \sum \pi\left(|a_n|^2 + |b_n|^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$

Posledica:

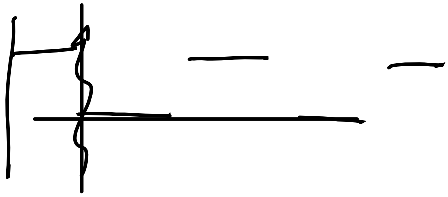
ker je $\left\{\frac{1}{\sqrt{2\pi}} \dots\right\}$ ONS velja

$$\lim_{n\to\infty} a_n = 0 \qquad \lim_{n\to\infty} b_n = 0$$

(Riemann-Lebegueova lema)

Zgled

$$f(x) = \begin{cases} 1; & 0 \leq x \leq \pi \\ 0; & -\pi < x < 0 \end{cases}$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx =$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx$$

$a_0 = 1 \qquad n > 0$

$$a_n = \frac{1}{\pi} \cdot \frac{1}{n} \sin(nx) \Big|_0^{\pi}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx)\, dx = -\frac{1}{\pi n} \cos(nx) \Big|_0^{\pi}$$

$$= \frac{1}{n\pi} \left(1 - (-1)^n\right)$$

$$b_{2k} = 0$$

$$b_{2k+1} = \frac{2}{\pi(2k+1)}$$

KONS:

$$f(x) \overset{L^2}{=} \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x)$$

$=0$

Parsevalova enakost

$$\frac{1}{\pi} \pi = 1 = \frac{1}{2} + \sum_{0}^{\infty} \frac{4}{\pi^2} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\frac{3}{4} S = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}$$

Izrek:

Naj bo f odsekoma zvezna in odsekoma odvedljiva periodična funkcija s periodo 2π

1) Na vsakem intervalu dolžine 2π ima največ končno mnogo točk nezveznosti in v vsaki točki obstajata leva in desna limita

$$x_0 \in \mathbb{R}. \ \exists \lim_{x \uparrow x_0} f(x) = \underbrace{f(x_0 - 0) = f(x_0^-)}_{\text{oznaki}}$$

$$\exists. \lim_{x \downarrow x_0} f(x) = f(x_0 + 0) = f(x_0 +)$$

2) V vsaki točki obstajata levi in desni odvod

$$\lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0^+)}{h}$$

$$\lim_{h \uparrow 0} \frac{f(x_0 + h) - f(x_0^-)}{h} = \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0^-)}{(-h)}$$

Tedaj za $\forall x \in \mathbb{R}$. velja $\frac{f(x+) + f(x-)}{2} =$

$$= \frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

Pomožne trditve

1) Naj bo $f: \mathbb{R} \to \mathbb{R}$ periodična s periodo p odsekoma zvezna.
Tedaj je za $\forall a \in \mathbb{R}$

$$\int_a^{a+p} f(x)\,dx = \int_0^p f(x)\,dx$$

Dokaz:

$$\int_a^{a+p} f(x)\,dx = \int_a^p f(x)\,dx + \int_p^{a+p} f(x)\,dx$$

$$x = t + p$$

$$= \int_a^p f(x)\,dx + \int_0^a \underbrace{f(t+p)}_{= f(t)}\,dt =$$

$$= \int_a^p f(x)\,dx + \int_0^a f(t)\,dt = \int_0^p f(x)\,dx$$

2)

$$\frac{1}{2}+\sum_{1}^{n}\cos(kx)=\frac{1}{2}\frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})}=D_n(x)$$

(Dirichletova jedro)

Dokaz:

$$\frac{1}{2}+\sum_{1}^{n}\cos(kx)=\frac{1}{2}+\sum_{1}^{n}\frac{e^{ikx}+e^{-ikx}}{2}=$$

$$=\frac{1}{2}\sum_{j=-n}^{n}e^{ijx}=\frac{1}{2}e^{-inx}\left(1+e^{ix}+e^{2ix}+\dots+e^{2nix}\right)$$

$$=\frac{1}{2}e^{-inx}\frac{1-e^{(2n+1)ix}}{1-e^{ix}}=$$

$$=\frac{1}{2}\frac{e^{(n+1)ix}-e^{-inx}}{e^{ix}-1}=\frac{1}{2}\frac{e^{i(n+\frac{1}{2})x}-e^{-i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}$$

$$=\frac{1}{2}\frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\frac{x}{2}}$$

3)

$$\int_{-\pi}^{\pi} D_n(x)\,dx = \pi$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$$

4) $D_n(x)$ je soda funkcija, 2π periodična, gladka.

5) $\frac{1}{2}\frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} =$

$$= \frac{1}{2}\frac{\sin(nx)\cos\frac{x}{2} + \cos(nx)\sin(\frac{x}{2})}{\sin\frac{x}{2}} =$$

$$= \frac{1}{2}\left(\sin(nx)\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \cos(nx)\right)$$

Opombe:

f je lihe na $(-\pi,\pi)$

$a_n = 0$

f razvijamo samo po $\{\sin(nx)\}$

f soda na $(-\pi,\pi) \Rightarrow b_n = 0$

f razvijemo le po $\{1, \cos(nx)\}$

f iz $[0,\pi]$ lahko liho razširimo na $(-\pi,\pi)$

$$x \mapsto \begin{cases} f(x) & x>0 \\ -f(-x); & x<0 \end{cases}$$

Tako funkcijo lahko razvijamo le po $\sin(nx)$

$$f(x) = \sum_{1}^{\infty} b_n \sin(nx)$$

nov b_n

$$\int_0^\pi f(x)\sin(nx)\,dx = b_n\int_0^\pi \frac{1-\cos(2nx)}{2} = \frac{\pi}{2}b_n$$

Če f razširimo kot soda funkcija na $(-\pi,\pi)$

$$x \longmapsto \begin{cases} f(x) & x>0 \\ f(-x) & x<0 \end{cases}$$

Tako funkcijo razvijamo po $\{1, \cos(nx)\}$

Opombe 2:

$f \in C^k(\mathbb{R})$ 2π periodična $k \in \mathbb{N}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{u} \underbrace{\cos(nx)\,dx}_{dv} =$$

$$= \frac{1}{\pi}\Big(\underbrace{\frac{1}{n}\sin(nx)\, f(x)\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx \Big)$$

$= \ldots$ naredimo to k krat $\ldots =$

$$a_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx$$

$$a_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx$$

$\vdots$

$$a_n = \frac{\pm 1}{\quad}\int_{-\pi}^{\pi} f^{(k)}(x) \begin{matrix}\cos(nx)\\ \sin(nx)\end{matrix}\, dx$$

omejena

neki od tega

$$a_n, b_n = \mathcal{O}\left(\frac{1}{n^2}\right)$$

velikostni red

večkrat kot je odvedljiva f (kot periodična funkcija) hitreje gredo furierovi koeficienti proti 0

Dokaz

$f \in L^2(-\pi, \pi)$

$\sqrt{\int_{-\pi}^{\pi} |f|^2 dx} = \|f\|_2$

Klasični: furijeovi koeficienti:

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n = 0, 1, 2 \ldots)$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n = 1, 2 \ldots$

$f(x) \overset{L_2}{=} \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$

Parsevalova enakost

$\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx = \frac{|a_0|^2}{2} + \sum_{1}^{\infty} (|a_n|^2 + |b_n|^2)$

f periodična s periodo 2π in odsekoma zvezna

- $\forall x_0 \in \mathbb{R}.\ \exists f(x_0+), f(x_0-)$ limiti in
- odsekoma odvedljiva
- dostajata levi in desni odvod

Potem: $\forall x_0 : \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) =$

$= \frac{f(x_0+) + f(x_0-)}{2}$

Vemo:

- f periodična s periodo p
- $\forall a \in \mathbb{R}.\ \int_{a}^{a+p} f(x) dx = \int_{0}^{p} f(x) dx$
- $\frac{1}{2} + \sum_{1}^{N} \cos(kx) = D_N(x) = \frac{1}{2} \frac{\sin(N + \frac{1}{2})x}{\sin(\frac{1}{2}x)}$

$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n dx = 1 \quad$ soda

$\| $

$\frac{2}{\pi} \int_{0}^{\pi} D_n(x) dx$

Dokaz >>

Dokaz:

$$a_n \cos(nx_0) + b_n \sin(nx_0) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \cdot \cos(nx_0) +$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \ \sin(nx_0) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\cos(nt)\cos(nx_0) + \sin(nt)\sin(nx_0)\right)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n(t-x_0))\,dt$$

(opomba)

$$\boxed{\int_{-\pi}^{\pi} f(t)g(x_0-t)\,dt \ \ldots\ldots \text{konvolucija}}$$

$$t - x_0 = y \quad t = x_0 + y$$

$$= \frac{1}{\pi}\int_{-\pi-x_0}^{\pi-x_0} f(x_0+y)\cos(ny)\,dy =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\cos(ny)\,dy$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0+y)\,dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\cdot\frac{1}{2}\,dy$$

$$\frac{a_0}{2} + \sum_{1}^{n} a_k\cos(kx_0) + b_k\sin(kx_0) = S_n(x_0)$$

$$S_n(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\left(\tfrac{1}{2} + \cos y + \ldots \cos(ny)\right)dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\,D_n(y)\,dy =$$

$$= \frac{1}{\pi}\left(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy + \int_{-\pi}^{0} f(x_0+y)D_n(y)\,dy\right.$$

$$y \rightsquigarrow -y$$

soda funkcija

$$= \frac{1}{\pi}\left(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy + \int_{0}^{\pi} f(x_0-y)D_n(y)\,dy\right.$$

$$\xrightarrow[n\to\infty]{} \frac{f(x_0+)+f(x_0-)}{2}$$

- - - - - - - - - - -

$$\frac{1}{\pi}\int_{0}^{\pi}\left(f(x_0+y)+f(x_0-y)\right)D_n(y)\,dy - \frac{f(x_0+)+f(x_0-)}{2}$$

$$\frac{2}{\pi}\cdot\int_{0}^{\pi} D_n(y)\,dy = 1$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(f(x_0+y)-f(x_0+)\right)D_n(y) + \left(f(x_0-y)-f(x_0-)\right)D_n(y)\,dy$$

Pogledamo sumando

$$\frac{1}{2}\int_{0}^{\pi}\left(f(x_0+y)-f(x_0+)\right)\frac{\sin(n+\frac{1}{2})y}{\sin(\frac{y}{2})}\,dy =$$

$$= \frac{1}{2}\int_{0}^{\pi}\left(f(x_0+y)-f(x_0)\right)\frac{\cos(\frac{y}{2})}{\sin(\frac{y}{2})}\sin(ny) + \cos(ny)\,dy$$

Pogledamo sumand

$$\int_{0}^{\pi} f(x_0+y)-f(x_0^+)\cos(ny)\,dy \xrightarrow[n\to\infty]{} 0$$

- - - - - - - - - - - - - -

$$F(y) = \begin{cases} f(x_0+y) - f(x_0^+) ; & 0 \le y \le \pi \\ 0 ; & -\pi < y < 0 \end{cases}$$

odsekoma zvezna

$$= \int_{-\pi}^{\pi} F(y)\cos(ny)\,dy \xrightarrow[n\to\infty]{\text{Rieman-Lebesgueva lema}} 0$$

Še

$$\int_{0}^{\pi} f(x_0+y) - f(x_0+))\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}\sin(ny)\,dy$$

$$= \int_{0}^{\pi} \frac{f(x_0+y)-f(x_0+)}{y}\ \frac{y}{\sin\frac{y}{2}}\cos(\tfrac{y}{2})\sin(ny)\,dy$$

$$G(y) = \begin{cases} \frac{f(x_0+y)-f(x_0+)}{y}\ \frac{y}{\sin\frac{y}{2}}\cdot\cos\frac{y}{2} ; & 0 < y \le \pi \\ 0 & ; -\pi < y < 0 \end{cases}$$

$$\int_{-\pi}^{\pi} G(y)\sin(ny)\,dy$$

za $y \neq 0$ je f odsekoma zvezna

za $y = 0$ pa obstaja limita

$$\lim_{y\nearrow 0} G(y) = 0$$

$$\lim_{y\searrow 0} G(y) = 2\lim_{y\searrow 0}\frac{f(x_0+y)-f(x_0+)}{y}$$

obstaja po predpostavki.

Rieman-Lebesgueva lema:

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} G(y)\sin(ny)\,dy = 0$$

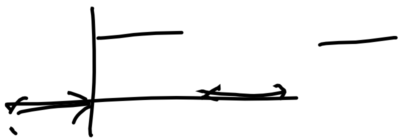
Za drugi sumand uporabimo podobne argumente

∎

Zgled:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi < x < 0 \end{cases}$$

Furierova vrsta: $\frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x$



$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2}$$

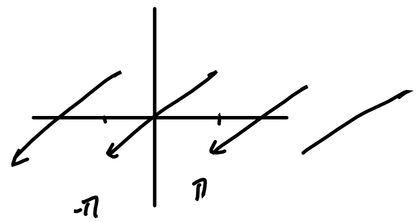
$x = \frac{\pi}{2}$: $\quad 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{1}{2k+1} \sin(2k+1)\frac{\pi}{2}$

+1 ali pa −1

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

zgled

$f(x)=x$



$a_n=0$ za $\forall n$

ker je na $(-\pi,\pi)$ funkcija liha

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$u=x$ $\quad dv=\sin(nx)$ $\quad n\geq 1$

$du=dx$ $\quad v=-\frac{1}{n}\cos(nx)$

$$b_n=\frac{1}{\pi}\left(-\frac{x}{n}\cos(nx)\right)\Big|_{-\pi}^{\pi}+\frac{1}{n}\underbrace{\int_{-\pi}^{\pi}\cos(nx)\,dx}_{=0}$$

$$=\frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi)+\frac{(-\pi)}{n}\cos(n\pi)\right)=$$

$$=-\frac{2}{n}(-1)^n=\frac{2}{n}(-1)^{n+1}$$

$$x\overset{L^2(-\pi,\pi)}{=}\sum_{1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx)$$

Po točkah:

$x=0$: $0=0$

$x=\pi$: $\frac{\pi+(-\pi)}{2}=0$ ✓

$x=\frac{\pi}{2}$: $\frac{\pi}{2}=2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)$

Parsevalova enakost

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\sum_{1}^{\infty}\frac{4}{n^2}=4\sum_{1}^{\infty}\frac{1}{n^2}$$

na kvadrat

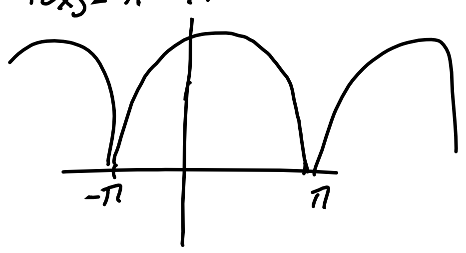
$\|$

$$\frac{1}{\pi}\left(\frac{1}{3}x^3\right)\Big|_{-\pi}^{\pi}=\frac{2}{\pi}\pi^3\cdot\frac{1}{3}=\frac{2}{3}\pi^2=4\sum_{1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}=\sum_{1}^{\infty}\frac{1}{n^2}$$

Zgled.

$f(x) = \pi^2 - x^2$



sode, nenegativno zvezno, v vsaki točki ima levi in desni odvod

sodost $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\,dx = \frac{2}{\pi}\left(\pi^3 - \frac{1}{3}\pi^3\right) =$$

$$= \frac{4}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{(\pi^2 - x^2)}_{u}\underbrace{\cos(nx)dx}_{dv} =$$

$$= \frac{1}{\pi}(\pi^2 - x^2)\frac{1}{n}\sin(x)\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\sin(nx)\,dx =$$

$$= \frac{2}{\pi n}\int_{-\pi}^{\pi} \underset{u}{x}\,\underbrace{\sin(nx)\,dx}_{dv} =$$

$$= \frac{2}{\pi n}\left(x\left(-\frac{1}{n}\cos(nx)\right)\Big|_{-\pi}^{\pi} - \overset{0}{\int_{-\pi}^{\pi}\cos(nx)\,dx}\right) =$$

$$= \frac{2}{\pi n^2}\left(\pi(-1)^{n+1} + (-1)^{n+1}\pi\right) = \frac{4}{n^2}(-1)^{n+1}$$

$$f(x) = \frac{2}{3\pi^2} + \sum_1^\infty \frac{4}{n^2} - (-1)^{n+1}\cos(nx) \quad \forall x \in \mathbb{R}$$

$x = 0 \Rightarrow$

$$f(0) = \pi^2 = \frac{2}{3}\pi^2 + \sum_1^\infty \frac{4}{n^2}(-1)^{n+1}$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$x = \pi \Rightarrow$

$$0 = \frac{2}{3}\pi^2 + \sum_1^\infty \frac{4}{n^2}\overbrace{(-1)^{n+1}(-1)^n}^{-1}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

kons:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)^2\,dx = \frac{1}{2}\frac{16}{9}\pi^4 + \sum_1^\infty \frac{16}{n^4}$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi^4 - 2x^2\pi^2 + x^4)\,dx = \frac{8\pi^4}{9} + 16\sum_1^\infty \frac{1}{n^4}$$

$$\frac{2}{\pi}\left(\pi^5 - \frac{2\pi^5}{3} + \frac{1}{5}x^5\right) = \frac{8\pi^4}{9} + 16S$$

$$\frac{15 - 10 + 3}{15}\pi^4 = \frac{4\pi^4}{9} + 8S$$

$$\| \qquad \frac{8\pi^4}{15}$$

$$8S = \frac{8\pi^4}{15} - \frac{4\pi^4}{9}$$

$$2S = \frac{2\pi^4}{15} - \frac{\pi^4}{9} = \frac{6\pi^4}{45} - \frac{5\pi^4}{45} = \frac{\pi^4}{45}$$

$$S = \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$a_1, a_2, \dots \qquad \lim_{n\to\infty} a_n = a$

$1, -1, 1, -1$

$\frac{1}{1}, \frac{1-1}{2}, \frac{1-1+1}{3} \to 0$

$a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}$

konvergira $\Rightarrow$ konvergira u povprečju

$$S_n^{(x_0)} = \int_{-\pi}^{\pi} f(x_0+y)\, D_n(y)\, dy$$

$$\frac{S_0(x) + S_1(x) + \dots + S_{n-1}(x)}{n} = \sigma_n(x) \xrightarrow[n\to\infty]{} f$$

cesárove delne vsote

če je f zvezna je konvergenca
enakomerna

Fejérjevo jedro: $F_N(x) = \frac{1}{N}\sum_{0}^{N-1} D_n(x)$

Trditev:

1) $F_N(x) = \frac{1}{2N}\left(\frac{\sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^2$
2) F_N je soda
3) $F_N(x) \geq 0$ za $\forall x$
4) $\frac{1}{\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$
5) $\forall a.\ 0<a<\pi.\ \lim_{N\to\infty} F_N(x) = 0$ enakomerno na $a \leq |x| \leq \pi$

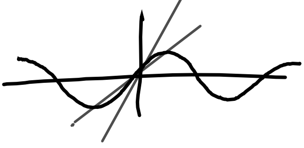
Dokaz:

1) $\Rightarrow$ 2) očitno

1) $\Rightarrow$ 3) očitno

Iz definicije $\Rightarrow$ 4) $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$

1) $\Rightarrow$ 5)



$\forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ velja $\frac{2}{\pi}|y| \leq |\sin y|$

$0 < a \leq |x| \leq \pi$

$0 < \frac{a}{2} \leq \left|\frac{x}{2}\right| \leq \frac{\pi}{2}$

$\frac{2}{\pi}\cdot\frac{|x|}{2} \leq \left|\sin\left(\frac{x}{2}\right)\right|$

$\frac{1}{\left|\sin\frac{x}{2}\right|} \leq \frac{\pi}{|x|} \leq \frac{\pi}{a}$

$F_N(x) = \frac{1}{2N}\frac{\left|\sin\left(\frac{Nx}{2}\right)\right|^2}{\left|\sin\left(\frac{x}{2}\right)\right|^2} \leq \frac{1}{2N}\cdot\frac{\pi^2}{a^2}$

$\lim_{n\to\infty} F_n(x) = 0$ enakomerno na $a \leq |x| \leq \pi$

1) $F_N(x) = \frac{1}{N}\sum_{0}^{N-1} D_n(x) = \frac{1}{N}\sum_{0}^{N-1}\frac{1}{2}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} =$

$= \frac{1}{2N}\frac{1}{\sin^2\left(\frac{x}{2}\right)}\sum_{0}^{N-1}\sin\left(n+\frac{1}{2}\right)x\cdot\sin\left(\frac{x}{2}\right) =$

$= \frac{1}{2N\sin^2\left(\frac{x}{2}\right)}\frac{1}{2}\underbrace{\sum_{0}^{N-1}\left(\cos(nx) - \cos(n+1)x\right)}_{(1-\cos x)+(\cos x-\cos 2x)+(\cos 2x-\cos 3x)+\dots\ =\ 1-\cos(Nx)}$

$= \frac{1-\cos(Nx)}{4N\sin^2\left(\frac{x}{2}\right)} = \frac{1}{2N}\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$

Izrek:
Naj bo f zvezna 2π periodična funkcija
Potem cezárove delne vsote

$$\sigma_N(x) = \frac{1}{N}\left(S_0(x) + \dots + S_{N-1}(x)\right)$$

konvergirajo k f enakomerno na
$[-\pi, \pi]$ oziroma na $\mathbb{R}$

$$S_N(x_0) = a_0 + \sum_{1}^{N}(a_k \cos(kx) + b_k \sin(kx))$$

$\boxed{\text{Trigonometrični polinom... končne kombinacije sinusov in kosinusov}}$

Dokaz:

$$\sigma_N(x) = \frac{1}{N}(S_0(x) + \dots + S_{N-1}(x))$$

$$S(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\, D_N(y)\, dy$$

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y) \cdot F_N(y)\, dy \xrightarrow{\text{enakomerno}} f(x)$$

Naj bo $\varepsilon > 0$. ker je f zvezna in periodična je enakomerno zvezna

$$\exists \delta > 0 .\ \forall y < \delta\ |f(x+y) - f(x)| < \frac{\varepsilon}{2}$$
$$\forall x \in \mathbb{R}$$

$$|\sigma_N(x) - f(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y) F_N(y)\, dy - f(x)\frac{1}{\pi}\int_{-\pi}^{\pi} F_N(y)\, dy\right|$$

$$= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x+y) - f(x)) F_N(y)\, dy\right| \leq$$

$F_N(y) \geq 0$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+y) - f(x)|\, F_N(y)\, dy =$$

$$= \underbrace{\frac{1}{\pi}\int_{-\delta}^{\delta}|f(x+y) - f(x)| F_N(y)\, dy}_{} + \underbrace{\int_{\delta \leq |y| \leq \pi} f(x+y) - f(x)|F_N(y)\, dy}_{}$$

$$< \frac{1}{\pi}\int_{-\delta}^{\delta} F_N(y)\, dy \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2}$$

f je zvezna in periodična torej je omejena

$$\exists M \in \mathbb{R} \quad |f(z)| \leq M \ \forall z \in \mathbb{R}$$
$$\Rightarrow f(x+y) - f(x) \leq 2M \quad \text{za } \forall x \text{ in } \forall y \in \mathbb{R}$$

$$\frac{1}{\pi}\int_{\delta < |y| < \pi} |f(x+y) - f(x)|\, F(y)\, dy \leq 2M \frac{1}{\pi}\int_{\delta < |y| \leq \pi} F_N(y)\, dy$$

Vemo da je limita $\lim_{N \to \infty} F_N(y) = 0$
enakomerno na $\delta \leq |y| \leq \pi$

$$\exists N_0 .\ \forall N \geq N_0 .\ |F_N(y)| \leq \frac{\varepsilon}{2M \cdot 4} = \frac{\varepsilon}{8M}$$

$$\frac{2M}{\pi}\int_{\delta < |y| \leq \pi} F_N(y)\, dy \leq \frac{2M}{\pi} \cdot \frac{\varepsilon}{8M} \cdot 2\pi = \frac{\varepsilon}{2}$$

Za $N \geq N_0$ je

$$|\sigma_N(x) - f(x)| < \varepsilon \quad \text{za } \forall x \in \mathbb{R}$$

Izrek:

$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx);\ n\in\mathbb{N}\right\}$

je KONS v $L^2(-\pi,\pi)$

Dokaz:

$L^2(-\pi,\pi)$ je napolnitev $C([-\pi,\pi])$ v

$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$

ONS je KONS $\Leftrightarrow$ končne lineare kombinacije vektorjev iz ONS goste v prostoru

---

OPOMBA: KONČNE LINEARE KOMBINACIJE $\sin(nx)$ $\cos(nx)$ $n\in\mathbb{N}_0$ so trigonometrični polinomi.

Ali so trigonometrični polinomi gosti v $L^2(-\pi,\pi)$?

1) $C(-\pi,\pi)$ so goste v $L^2(-\pi,\pi)$ (Privzamemo)

$f\in L^2(-\pi,\pi)$. $\varepsilon>0$. $\exists\, \tilde{f}\in C[-\pi,\pi]$. $\|f-\tilde{f}\|<\frac{\varepsilon}{2}$

2) Ali so trigonometrični polinomi gosti v $(C[-\pi,\pi], d_2)$

če da:

$\exists T(x)$ trigonometrični polinom

$T(x) = \lambda_0 + \sum_{1}^{N} \lambda_n \cos(nx) + \mu_n \sin(nx)$

$\|\tilde{f}-T\|_2 < \frac{\varepsilon}{2} \Rightarrow \|f-T\|_2 < \varepsilon$

OPOMBA.

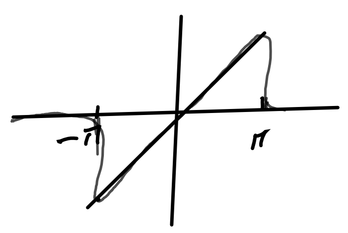
$g_n \in L^2[-\pi,\pi] \xrightarrow[\text{na } [-\pi,\pi]]{\text{enakomerno}} f$

$\overset{?}{\Rightarrow} \lim_{n\to N} g_n = f$ v d_2

$d_2(f,g_n)^2 = \int_{-\pi}^{\pi}(f-g_n)^2 \xrightarrow{n\to N} 0$

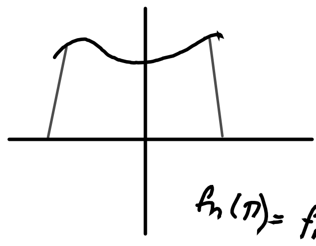
Vemo: f zvezna 2π periodična $\Rightarrow$ jo lahko poljubno dobro enakomerno aproksimiramo s trigonometričnimi polinomi:

$\|f-\sigma_N\|_\infty < \varepsilon$



Ni: zvezne ali periodične funkcije

$f_n(x) = \begin{cases} f(x) \\ -n(f(\pi-\frac{1}{n}))(x-\pi); & \pi-\frac{1}{n} < x \le \pi \\ n f(-\pi+\frac{1}{n})(x+\pi) & -\pi \le x < -\pi+\frac{1}{n} \end{cases}$



$f_n(\pi) = f_n(-\pi) = 0$

zvezna 2π periodična funkcija

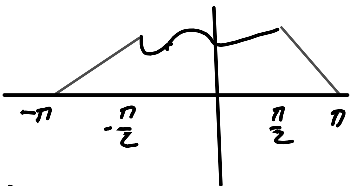
$\|f-f_n\|_2^2 = \int_{-\pi}^{\pi}(f_n(x)-f(x))^2\,dx \le \frac{2}{n}(2\pi)^2 \xrightarrow{n\to\infty} 0$

Izrek: (Weierstrassov)

Naj bo f zvezna na $[a,b]$
Naj bo $\varepsilon > 0$. Potem $\exists$ polinom p da je
$\|f-p\|_\infty < \varepsilon$

Dokaz: Dovolj je opazovati $a=-\frac{\pi}{2}$ $b=\frac{\pi}{2}$

f zvezna na $[-\frac{\pi}{2}, \frac{\pi}{2}]$
f zvezno razširimo na $[-\pi, \pi]$



$$\tilde{f} = \begin{cases} f(x) & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -\frac{2}{\pi} f(\frac{\pi}{2})(x-\pi) & \frac{\pi}{2} < x \leq \pi \\ \frac{2}{\pi} f(-\frac{\pi}{2})(x+\pi) & -\pi \leq x < -\frac{\pi}{2} \end{cases}$$

$\tilde{f}$ je zvezna 2π periodična.

Za $\varepsilon > 0$ $\exists$ T trigonometrični polinom, da je $\|\tilde{f} - T\|_\infty < \varepsilon/2$

$\cos(nx), \sin(nx)$ jih je končno

Aproksimiramo s Taylorjevimi polinomi enakomerno.

Vektorska analiza

(v $\mathbb{R}^2$ ali $\mathbb{R}^3$)

$D^{odp} \subseteq \mathbb{R}^3$ $\quad u: D \to \mathbb{R}$ zvezna funkcija

jo imenujemo <u>skalarno polje</u>

vsaki točki v D priredi skalar

$D \subseteq \mathbb{R}^3$ $\quad \vec{R}: D \longrightarrow \mathbb{R}^3$ zvezna

<u>vektorsko polje</u> vsaki točki priredi

vektor $\vec{R}(t)$

$\mathbb{R}^3$

$u(x,y,z) = x$ v bazi $\vec{e}_1, \vec{e}_2, \vec{e}_3$ $(\vec{i}, \vec{j}, \vec{k})$

Baza: $\vec{p} = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_3$

$$\vec{q} = -\frac{1}{\sqrt{3}} e_1 + \frac{1}{\sqrt{3}} e_2 + \frac{1}{\sqrt{3}} e_3$$

$$\vec{r} = \frac{1}{\sqrt{6}} e_1 + \frac{2}{\sqrt{6}} e_2 - \frac{1}{\sqrt{6}} e_3$$

Točka (x,y,z) ima v bazi $(\vec{p}, \vec{q}, \vec{r})$

koordinate (α, β, γ) $\qquad$ ONB

$(\alpha, \beta, \gamma) \longmapsto \alpha$ ni ista funkcija

ONB je pozitivno orientirana, če je

$$[\vec{p}, \vec{q}, \vec{r}] > 0 \; (\Leftrightarrow \vec{r} = \vec{p} \times \vec{q})$$

oziroma negativno orientirana kot

$$[\vec{p}, \vec{q}, \vec{r}] < 0 \; (\Leftrightarrow \vec{r} = -\vec{p} \times \vec{q}$$

(oznake od zdaj naprej za mešani produkt bo $(\vec{p}, \vec{q}, \vec{r})$ najbrž

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} =: U$$

(x, y, z) $(e_1, \dots, e_n)$

(α, β, γ) $(\vec{p}, \vec{q}, \vec{r})$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$U^T U = I$$

$$U^{-1} = U^T$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$u(x, y, z)$ v (e_1, e_2, e_3) koordinatah

$$\tilde{u}(\alpha, \beta, \gamma) = u\left(U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}\right)$$

Vektorska polja

$$\vec{R}(x,y,z) = \left(X(x,y,z),\ Y(x,y,z),\ Z(x,y,z)\right)$$

v (α,β,γ) koordinatah

$$\tilde{R}(\alpha,\beta,\gamma) = \left(U^T \circ \vec{R} \circ U\right)\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

$$\tilde{u} = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$\vec{R}(x,y,z) = (x+2y+3z,\ x+2y+3,\ x+2y+3z)$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

$$x = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$y = \frac{\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$

$$z = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{3}} - \frac{\gamma}{\sqrt{6}}$$

$$x+2y+3z = \frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma = c$$

to je $(R\circ U)\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = (c,c,c)$

$$U\left(R\circ U\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}\right) = \left(\frac{2}{\sqrt{2}},\ \frac{1}{\sqrt{3}},\ \frac{2}{\sqrt{6}}\right)c$$

$D^{odp} \subseteq \mathbb{R}^3$ ponovitev:

$u: D \to \mathbb{R}$ C^1

$u \longmapsto \vec{\nabla} u = \operatorname{grad} u = (u_x, u_y, u_z)$

$\vec{R} = (X, Y, Z)$

$\vec{R}: D \to \mathbb{R}^3$

$\vec{R} \longmapsto \operatorname{div} \vec{R} = \vec{\nabla} \cdot \vec{R} = X_x + Y_y + Z_z$

$\vec{R} \longmapsto \operatorname{rot} \vec{R} = \vec{\nabla} \times \vec{R} = (Z_x - Y_z, X_z - Z_x, Y_x - X_y)$

$u \in C^2(D)$, $\vec{R}: D \to \mathbb{R}^3$ C^2

$\operatorname{rot} \operatorname{grad} u = \vec{0}$

$\overset{\shortparallel}{\vec{\nabla}} \times \vec{\nabla} u = \vec{0}$

$\operatorname{div} \operatorname{rot} \vec{R} = 0$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{R}) = 0$

$\operatorname{div} \operatorname{grad} u = \vec{\nabla} \cdot \vec{\nabla} u = u_{xx} + u_{yy} + u_{zz} = \Delta u$

Izrek:

D zvezdasto ($\exists$ točka, ki vidi vse ostale točke )

$\vec{R}: D \to \mathbb{R}^3$ vek polje

1) $\operatorname{rot} \vec{R} = \vec{0} \Rightarrow \exists u \in C^2(D).\ \vec{R} = \vec{\nabla} u$

2) $\operatorname{div} \vec{R} = 0 \Rightarrow \exists \vec{G}: D \to \mathbb{R}^3$ C^2 vek polje da je $\vec{R} = \operatorname{rot} \vec{G}$

Zgled

ad 1) ne velja obratno

$\mathbb{R}^3 - z\text{-os}$ D ni zvezdasto

$\vec{R} = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

$\arctan \frac{y}{x} = u$ ni dobro definirana na $\mathbb{R}^3 - z\text{-os}$

•) če ima $\vec{R}$ potencial u, potem so vsi potenciali oblike $u + c$

(D povezan)

~~Zgled~~

$\vec{R}: D \longrightarrow \mathbb{R}^3$ vec. polje

$\operatorname{div} \vec{R} = f \quad f \in C^1(D)$

$\exists u$ na D. $\Delta u = f$

$\operatorname{div} \operatorname{grad} u = \Delta u = f = \operatorname{div} \vec{R}$

$$\operatorname{div}(\underbrace{\vec{R} - \operatorname{grad} u}_{= \operatorname{rot}(\vec{G})}) = 0$$

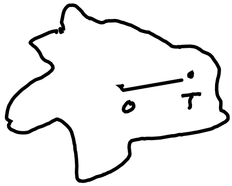
$$\vec{R} = \operatorname{grad} u + \operatorname{rot} \vec{G}$$

Dokaz (izreka dve strani nazaj)

1) $\vec{R} : D \to \mathbb{R}^3$ $\quad C^1(D)$

$$\operatorname{rot} \vec{R} = \vec{0} = (Z_y - Y_z, X_z - Z_x, Y_x - X_y) = \vec{0}$$

BŠZS. je D zvezdasto glede na 0



če $(x,y,z) \in D$.
$\forall t \in [0,1].\ (t\cdot x, t\cdot y, t\cdot z) \in D$

Definiramo:

$$u(x,y,z) = \int_0^1 \big(X(tx,ty,tz)\cdot x + Y(tx,ty,tz)\,y + Z(tx,ty,tz)\cdot z\big)\,dt$$

<u>u je potencial za $\vec{R}$</u>

$u_x = X \quad u_y = Y \quad u_z = Z$

$$u_x = \int_0^1 \Big(X(tx,ty,tz) + X_x\cdot t\cdot x + \underset{\overset{\|}{X_y}}{Y_x}\cdot ty + \underset{\overset{\|}{X_z}}{Z_x}\cdot tz\Big)\,dt$$

$$= \int_0^1 \underbrace{\big(X(tx,ty,tz) + tx\cdot X_x + ty\,X_y + tz\cdot X_z\big)}_{= \frac{d}{dt}\left(t\cdot X(tx,ty,tz)\right)}\,dt$$

$$= tX(tx,ty,tz)\Big|_0^1 = X(x,y,z)$$

Podobno $U_y = Y$ iz $U_z = Z$

Dokaz

2)

$\operatorname{div}\vec{R} = \vec{0} = X_x + Y_y + Z_z$

(D je zvezdasta glede na 0 BŠZS)

$\alpha(x,y,z) = \int_0^1 t\, X(tx, ty, tz)\, dt$

$\beta(x,y,z) = \int_0^1 t\, Y(tx, ty, tz)\, dt$

$\gamma(x,y,z) = \int_0^1 t\, Z(tx, ty, tz)\, dt$

$\operatorname{div}(\alpha, \beta, \gamma) = 0$ ker:

$$\left.\begin{aligned} \alpha_x &= \int_0^1 t^2 X_x(tx, ty, tz)\, dt \\ \beta_y &= \int_0^1 t^2 Y_y(tx, ty, tz)\, dt \\ \gamma_z &= \int_0^1 t^2 Z_z(tx, ty, tz)\, dt \end{aligned}\right\} +$$

$= 0$

$\vec{G} = \begin{pmatrix} \alpha & \beta & \gamma \\ x & y & z \end{pmatrix} = (\alpha, \beta, \gamma) \times (x, y, z)$

$\vec{G} = (z\beta - y\gamma,\ x\gamma - z\alpha,\ y\alpha - x\beta)$

$\vec{R} = \operatorname{rot}\vec{G}$?

1 komponenta od $\operatorname{Rot}\vec{G}$:

$(y\alpha - x\beta)_y - (x\gamma - z\alpha)_z =$

$= \alpha + y\alpha_y - x\beta_y - x\gamma_z + \alpha + z\alpha_z$

$= 2\alpha + \alpha_x + y\alpha_y + z\alpha_z$

uporabimo predpostavko $\alpha_x + \beta_y + \gamma_z = 0$

$= \int_0^1 \underbrace{\left(2t X(tx, ty, tz) + x t^2 X_x + y t^2 X_y + z t^2 X_z\right)}_{\frac{d}{dt}\left(t^2 X(tx, ty, tz)\right)} dt$

$t^2 X(tx, ty, tz)\Big|_0^1 = X(x,y,z)$

Podobno še drugi dve komponenti

zgled:

$$\vec{R}=(y^2z^3+2,\ 2xyz^3+1,\ 3xy^2z^2)$$

$$\operatorname{rot}\vec{R}\ (\underbrace{6xyz^2-6xyz^2}_{0},\ \underbrace{3y^2z^2-3y^2z^2}_{0},\ 2yz^3-2yz^3)=\vec{0}$$

$$u(x,y,z)=\int_0^1 X(tx,ty,tz)\,x+\dots)\,dt$$

u potencial obstaja: $u_x=y^2z^3+2$

$u_y=2xyz^3+1$

$u_z=3xy^2z^2$

$$u(x,y,z)=xy^2z^3+2x+C(y,z)$$

$\downarrow$

$$u_y=2xyz^3+C_y=2xyz^3+1$$

$$C_y=1$$

$$C(y,z)=y+D(z)$$

$$u(x,y,z)=xy^2z^3+2x+y+D(z)$$

$\downarrow$

$$u_z=3xy^2z^2+D'(z)=3xy^2z^2$$

$$D'=0$$

$D=D_0$ konstanta

$$u(x,y,z)=xy^2z^3+2x+y+D_0$$

(Rot $\vec{R}\neq\vec{0}$ tedaj ni potenciala)

$$u=xy^2z^3+2xt+A(yz)$$

$\downarrow$

$$2xyz^3+A_y=2yz^3+x$$

$A_y=x$ * ker je A funkcija y in z ja nexa

Zgled

$$\vec{R} = (2y-1\ ,\ -1\ ,\ 4x-2xy)$$

$$\operatorname{div}\vec{R} = (0,0,0) = 0$$

iščemo $\vec{G} = ?$ $\operatorname{rot}\vec{G} = \vec{R}$

$$\alpha(x,y,z) = \int_0^1 t(2ty-1)\,dt = \frac{2}{3}y - \frac{1}{2}$$

$$\beta(x,y,z) = \int_0^1 t(-1)\,dt = -\frac{1}{2}$$

$$\gamma(x,y,z) = \int_0^1 t(4tx - 2xyt^2)\,dt = \frac{4x}{3} - \frac{1}{2}xy$$

$$\vec{G} = \begin{pmatrix} \frac{2}{3}y - \frac{1}{2} & -\frac{1}{2} & \frac{4}{3}x - \frac{1}{2}xy \\ x & \cancel{y} & z \end{pmatrix} =$$

$$= \left(-\frac{1}{2}z - \frac{4}{3}xy + \frac{1}{2}xy^2,\ \frac{4}{3}x^2 - \frac{1}{2}x^2y - \frac{2}{3}zy + \frac{1}{2}z,\ \frac{2}{3}y^2 - \frac{1}{2}y + \frac{1}{2}x\right)$$

$$\operatorname{rot}\vec{G} = \Big(\underbrace{\frac{4}{3}y - \frac{1}{2} + \frac{2}{3}y - \frac{1}{2}}_{=2y-1},\ \underbrace{-\frac{1}{2} - \frac{1}{2}}_{=-1},\ \underbrace{\frac{8}{3}x - xy + \frac{4}{3}x - xy}_{=4x-2xy}\Big)$$

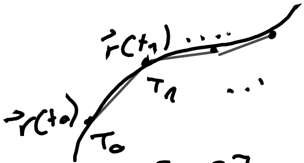
Krivuljni integral

Dolžina krivulje:

Γ naj bo C^1 krivulja v $\mathbb{R}^3$

reg. Parametrizacija $\vec{r}(t) : \vec{r}[\alpha, \beta] \longrightarrow \Gamma$

$\dot{\vec{r}}(t) \neq 0$



D delitev $[\alpha, \beta]$

$$\sum_1^n d(T_{j-1}, T_j) = l(D)$$

$$l(D) = \sum_1^n \sqrt{(x(t_j) - x(t_{j-1}))^2 + \dots + (z(t_j) - z(t_{j-1}))^2}$$

$$\overset{\text{Lagrange}}{=} \sum_1^n \sqrt{\dot{x}^2(\tau_j') + \dot{y}^2(\tau_j'') + \dot{z}^2(\tau_j''')}$$

$$\doteq \sum_1^n \sqrt{\dot{x}(\tau_j) + \dot{y}^2(\tau_j) + \dot{z}^2(\tau_j'')} =$$

$$= R(\sqrt{\dot{x} + \dot{y}^2 + \dot{z}^2}, D, \tau) \xrightarrow[\max \Delta t_j \to \infty]{}$$

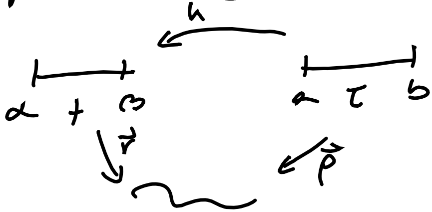
$$\longrightarrow \int_\alpha^\beta \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Naj bo Γ C^1 krivulja v $\mathbb{R}^3$ z regularno parametrizacijo $\vec{r}:[\alpha,\beta)\to\Gamma$

Definiramo dolžino krivulje

$$l(\Gamma)=\int_\alpha^\beta|\dot{\vec{r}}|\,dt=\int_\alpha^\beta\sqrt{\dot x^2+\dot y^2+\dot z^2}\,dt$$

Ali je ta definicija neodvisna od regularne parametrizacije



$$\vec{h}:[a,b]\xrightarrow{Dif}[\alpha,\beta]$$

$$\vec{\rho}=\vec{r}\circ\vec{h}$$

$$\vec{\rho}'=\dot{\vec{r}}(h)\,h'$$

$$\underline{\int_a^b|\rho'|\,d\tau=\int_\alpha^\beta|\dot{\vec{r}}|\,dt}$$

$$\int_\alpha^\beta|\dot{\vec{r}}(t)|\,dt=\int_{[\alpha,\beta]}|\dot{\vec{r}}(t)|\,dt=\int_{[a,b)}|\dot{\vec{r}}(h(\tau))|\,|h'(\tau)|\,d\tau$$

$t=h(\tau)$

$$=\int_{[a,b]}|\vec{\rho}'|\,d\tau$$

Γ nej bo C^1 krivulja $\vec{r}(t)$ reg. param.

$$s(t) = \int_\alpha^t |\dot{\vec{r}}(\tau)|\, d\tau \qquad s: [\alpha, \beta] \to [0, l(\Gamma)]$$

= dolžina Γ med $\vec{r}(\alpha)$ in $\vec{r}(t)$

Odvod: $\dot{s} = |\dot{\vec{r}}(t)| > 0$

s je strogo naraščajoča od $[\alpha, \beta]$ v $[0, l(\Gamma)]$

torej ima inverz:

$$T: [0, l(\Gamma)] \to [\alpha, \beta]$$

$$s \mapsto T(s) \quad \text{inverz od } s$$

Poglejmo parametrizacijo Γ:

$$s \mapsto \vec{r}(T(s))$$

Vemo $s \circ T = id_{[0, l(\Gamma)]}$ in $T \circ s = id_{[\alpha, \beta]}$

odvajamo:

$$\dot{s}(T(s)) \cdot T'(s) = 1$$

izračunajmo velikost parametrizacije

$$s \mapsto \vec{r}(T(s))$$

$$\frac{d}{ds}(\vec{r}(T(s))) = \dot{\vec{r}}(T(s)) \cdot T'(s) =$$

$$= \frac{\dot{\vec{r}}(T(s))}{\dot{s}(T(s))} = \frac{\dot{\vec{r}}(T(s))}{|\dot{\vec{r}}(T(s))|}$$

velikost hitrosti je 1

Tako parametrizirana krivulja je naravno parametrizirana in s je naravni parameter

Zgled:

vijačnica:

$a>0$

$b\neq 0$

$$t \longmapsto (a\cos t, a\sin t, bt) = \vec{r}(t)$$

$$|\dot{\vec{r}}| = \sqrt{a^2+b^2}$$

$$s(t) = \int_0^t \sqrt{a^2+b^2}\, d\tau = \sqrt{a^2+b^2} \cdot t = s$$

$$T(s) = \frac{s}{\sqrt{a^2+b^2}}$$

naravno parametrizirana vijačnica

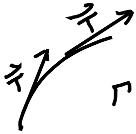
$$s \longmapsto \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right)$$

$$\shortparallel$$

$$\vec{r}_N(s)$$

$$|\dot{\vec{r}}_N(s)| = 1$$

Orientacija krivulje



Γ gladka C^1 krivulja

<u>orientacija</u> Γ je zvezen izbor enotskega tangentnega vektorja vzdolž Γ: $\vec T$

$|\vec T| = 1$

Γ povezana $\Rightarrow$ Γ ima dve orientaciji

$\vec T$ in $-\vec T$

$\vec r : [\alpha, \beta] \to \Gamma$ reg. parametrizacija

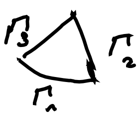
$\vec T = \frac{\dot{\vec r}}{|\dot{\vec r}|}$ en možni izbor

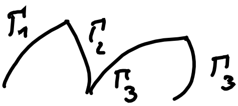
V tem primeru sta orientacija in parametrizacija usklajeni:

$(\Gamma, \vec T)$ ali $\vec \Gamma$

Vsaka krivulja je orientabilna

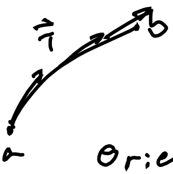
Odsekoma gladka krivulja





$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$

Γ z robnima točkama a, b



(krivulja z robom $\partial\Gamma = \{a, b\}$)

Naj bo $\vec{T}$ orientacija Γ

Orientacija Γ porodi orientacijo $\partial\Gamma$

Ena od teh dveh točk je prva oz začetna Γ (Z_Γ) in druga je končna točka Γ (K_Γ)

Glede na sliko je $a = Z_\Gamma$ $\quad b = K_\Gamma$

S tem je rob Γ orientiran skladno z orientacijo Γ

Naj bo Γ odsekoma gladka krivulja.

$\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$

1) Γ_j je gladka krivulja z robom

2) $\Gamma_j \cap \Gamma_l = \emptyset \Leftarrow |j-l| \bmod n \geq$

3) $\Gamma_j \cap \Gamma_{j+1} = \{P_j\}$ $\quad j \in \{1, \ldots, n-1\}$ $\quad P_j$ robna točka

$\leftarrow$ ena sama točka za Γ_j in Γ_{j+1}

Γ_n in Γ_1 se lahko sekata ali pa ne

Orientacija Γ je tak izbor orientacij $\Gamma_1 \ldots \Gamma_n$, da je $K_{\Gamma_j} = P_j = Z_{\Gamma_{j+1}}$

če se Γ_1 in Γ_n sekata je $K_{\Gamma_n} = Z_{\Gamma_1}$

Krivuljni integral

Dve vrsti krivuljnih integralov:

1) Integral skalarnega polja po krivulji Γ
(Orientacije tu ne potrebujemo)

Γ gladka krivulja omejena

$u: \Gamma \longrightarrow \mathbb{R}$ vzemo skalarno polje
$\vec{r}: [\alpha, \beta] \longrightarrow \Gamma$ regularna parametrizacija

Definiramo $\int\limits_{\Gamma} u \, ds := \int\limits_{\alpha}^{\beta} u(\vec{r}(t)) |\dot{\vec{r}}(t)| \, dt$

Opombe:

1) $u = 1 \Rightarrow$ je to dolžina Γ

2) Ta vrednost je neodvisna od reg. parametrizacije

3) $u =$ dolžinska gostota $\Rightarrow$ dobimo maso Γ

Če je $u = \rho_0$ konstanta, je Γ homogena

Če je $\Gamma_1 \cup \dots \cup \Gamma_n$ odsekoma gladka

je $\int\limits_{\Gamma} u \, ds = \sum\limits_{j=1}^{n} \int\limits_{\Gamma_j} u \, ds$

2) Krivuljni integral vektorskega polja po orientirani krivulji

>> >>

Zgled: Γ homogena polkrožnica

$$\Gamma = \{(x,y) \in \mathbb{R}^2, \ x^2+y^2=a^2, \ y \geq 0\}$$

Lega težišča: ?

$$m(\Gamma) = \int_\Gamma \rho_0 \, ds \qquad x_T(\Gamma) = \frac{1}{m_\rho} \int_\Gamma x \rho_0 \, ds$$

$$y_T(\Gamma) = \frac{1}{m(\Gamma)} \int_\Gamma y \rho_0 \, dy$$

$$m(\Gamma) = \int_\Gamma \rho_0 \, ds = \pi a \rho_0$$

Parametrično:

$$x(t) = a\cos t \qquad y(t) = a \sin t$$

$$ds = |\dot{\vec{r}}| dt$$

$$\dot{x} = -a \sin t \qquad x^2+y^2=a^2$$
$$\dot{y} = a \cos t$$

$$m(\Gamma) = \int_0^\pi \rho_0 a \, dt = \rho_0 a \, dt$$

$$x_T(\Gamma) = \frac{1}{m(\Gamma)} \int_0^\pi \arctan t \, \rho_0 a \, dt = 0$$

$$\frac{1 \rho_0 a}{\rho_0 a \pi} \int_0^\pi \sin t \, a \, dt = \frac{a}{\pi}(-\cos t \Big|_0^\pi =$$

$$\frac{2}{\pi} \cdot a$$

Vztrajnostni moment okoli ~~x~~ y osi

$$J_y = \int_\Gamma (x^2+z^2)\rho \, ds = \int_0^\pi a^2 \cos^2 t \cdot a \, dt = \rho_0 a^3 \int_0^\pi \frac{1+\cos(2t)}{2} dt$$

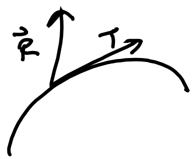
$$= \frac{\pi}{2} \rho_0 a^3 = \frac{1}{2} m(\Gamma) \cdot a^2$$

2) Krivuljni integral vektorskega polja po orientirani krivulji

(Delo sile vzdolž $\vec{\Gamma}$)

$\vec{R} : \vec{\Gamma} \to \mathbb{R}^3$ zvezno vek. polje

$\vec{R} = (X, Y, Z$



če je Γ gladka in

$\vec{r} : [\alpha, \beta] \to \Gamma$ param. ki je vsklajena z orientacijo Γ

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

skalarni produkt

Definiramo $\int_\Gamma \vec{R}\, d\vec{r} = \int_\alpha^\beta \vec{R}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\, dt$

$$= \int_\alpha^\beta \vec{R}(\vec{r}(t)) \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \cdot |\dot{\vec{r}}|\, dt = \int_\Gamma (\vec{R} \cdot \vec{T})\, ds$$

Rezultat je neodvisen od parametrizacije Γ, ki je vsklajena z orientacijo

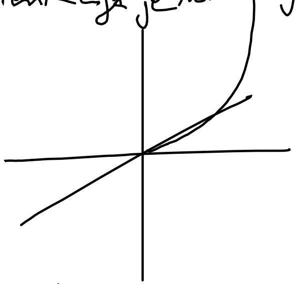
$$\int_{-\vec{\Gamma}} \vec{R}\, dr = -\int_{\vec{\Gamma}} \vec{R}\, dr$$

Zgled:

$\vec{R}(x,y,z) = (xy, z, x-z)$

$r(t) = (t, t, \frac{1}{2}t^2) \qquad t \in [0,1]$

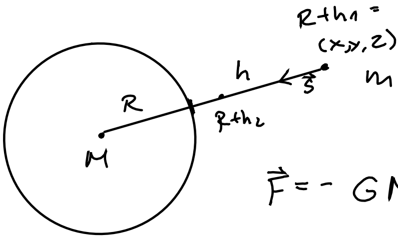
orientacija je tista ki je usklajena z $\vec{r}$



$$\int_{\vec{r}} \vec{R}\, dr = \int_0^1 \left(t^2, \frac{1}{2}t^2, t - \frac{1}{2}t^2\right) \cdot (1, 1, t)\, dt =$$

$$= \int_0^1 \left(\frac{3}{2}t^2 + t^2 - \frac{1}{2}t^2\right) dt = \frac{5}{6} - \frac{1}{8} = \frac{17}{24}$$

Zgled:



$|\vec{s}|=1$

$$\vec{F} = -GM\frac{m}{|(x,y,z)|^3}(x,y,z) =$$

$$= -GM\frac{m}{r^3}\cdot\vec{r}$$

$t \longmapsto \vec{s}\cdot t$ $\quad$ t teče od $R+h_1$ do $R+h_2$

$$\int_\Gamma \vec{F}\,d\vec{r} = \int_{R+h_1}^{R+h_2} -GM\frac{m}{t^3}(t\cdot\vec{S})\cdot\vec{s}\,dt =$$

$$= -GMm\int_{R+h_1}^{R+h_2}\frac{dt}{t^2} = -GMm\,\frac{1}{t}\Big|_{R+h_1}^{R+h_2} = -GmM\left(\frac{1}{R+h_2}-\frac{1}{R+h_1}\right)$$

$$= \frac{GM(h_1-h_2)m}{(R+h_1)(R+h_2)} \doteq \underbrace{\frac{GM}{R^2}}_{g}\,m\cdot\Delta h = mg\Delta h$$

Opomba

•) $\int\limits_{\vec{\Gamma}} \vec{R}\, d\vec{r} = \int\limits_{\vec{\Gamma}} \underbrace{X\,dx + Y\,dy + Z\,dz}_{\text{diferencialna forma}}$

••) če je Γ sklenjena $(z_\Gamma = k_\Gamma)$

$$\oint\limits_{\vec{\Gamma}} \vec{R}\, d\vec{r}$$

Naj bo $\vec{R}$ potencialno vektorsko polje

Naj bo Γ C^1 krivulja s parametrizacijo $\vec{r}(t)$

Orientacija usklajena z $\vec{r}$ $\quad \vec{T} = \frac{|\dot{\vec{r}}|}{|\dot{\vec{r}}|}$

$$\int_{\vec{\Gamma}} \vec{R}\, dr = \int_{\vec{\Gamma}} \operatorname{grad} u\, d\vec{r} = \int_{\alpha}^{\beta} (u_x, u_y, u_z)(r(t)) \cdot \dot{\vec{r}}(t)\, dt =$$

$$= \int_{\alpha}^{\beta} \vec{r}(t) = (x(t), y(t), z(t)) = \int_{\alpha}^{\beta} \underbrace{(u_x \cdot \dot{x} + u_y \dot{y} + u_z \dot{z})}_{\frac{d}{dt}\left(u(x(t), y(t), z(t))\right)} dt =$$

$$= u(\vec{r}(\beta)) - u(\vec{r}(\alpha)) = u(K_{\vec{\Gamma}}) - u(Z_{\vec{\Gamma}})$$

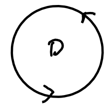
Trditev: Če je vek. polje $\vec{R}$ potencialno, je vrednost integrala $\vec{R}$ po orientirani krivulji $\vec{r}$ enaka razliki potenciala polja $\vec{R}$ med končno točko $\vec{r}$ in začetno točko $\vec{r}$

Posledica: Če je $\vec{R}$ potencialno polje in je r sklenjena je integral $\oint_r \vec{R}\, d\vec{r} = 0$

Zgled:

$$\vec{R} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\vec{\Gamma} = \{ (x,y,0);\ x^2+y^2 = R^2;\ R>0 \}$$



$$x = R\cos t \qquad t \in [0, 2\pi)$$
$$y = R\sin t$$
$$z = 0$$

$$\dot{x} = -R\sin t$$
$$\dot{y} = R\cos t$$
$$\dot{z} = 0$$

$$\int_{\Gamma} \vec{R}\,d\vec{r} = \int \left(-\frac{1}{R^2} R\sin t, \frac{1}{R^2} R\cos t, 0 \right) \cdot (-R\sin t, R\cos t, 0)\,dt =$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi \neq 0$$

$\Rightarrow$ To polje ni potencialno v $\mathbb{R}^3 - \{z\text{-os}\}$

(Vemo pa da je $\operatorname{rot}\vec{R} = \vec{0}$

Izrek (karakterizacija potencialnih vek. polj)

Naj bo $D \subseteq \mathbb{R}^3$ odp in $\vec{R} : D \to \mathbb{R}^3$ zvezno vektorsko polje:

Naslednje tri izjave so ekvivalentne:

1) R je potencialno na D
($\exists u \in C^1(D).\ \vec{R} = \operatorname{grad} u$)

2) Integral $\vec{R}$ po orientiranih krivuljah je neodvisen od krivulj z isto začetno in isto končno točko

3) Integral po sklenjenih krivuljah je enak 0

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