$x = 0, 1$

Pokaži da velja *

$$x = \sum_{i=1}^{\infty} (2^{-4i} + 2^{-4i-1}$$

b) binarni zapis za x

c) zapiši v IEEE formatu

IEEE754

enojna natančnost $P(2, 24, -125, 128)$

$$(-1)^{\sigma}(1+m)\, 2^{\tilde{e}-127}$$

m dolžine 23
$\tilde{e}$ dolžine 8
σ dolžine 1

dvojna natančnost $P(2, 53, -1021, 1024)$

$$(-1)^{\sigma}\, (1+m) 2^{\tilde{e}-1023}$$

m 52
$\tilde{e}$ 11
σ 1

a) $x = \sum_{i=1}^{\infty} 2^{-4i} + \dots = \frac{\frac{1}{16}}{1-\frac{1}{16}} + \frac{1}{2} \frac{\frac{1}{16}}{1-\frac{1}{16}} = \frac{\cancel{3}}{2 \cdot \cancel{16}} \frac{\cancel{16}}{\cancel{15}\,5} = \frac{1}{10}$

$= 0,1$

b) $0,0001100110011 = 0,0\overline{0011}_{(2)}$

c)

$1,1\overline{0011} \cdot 2^{-4} =$

$1 + 0,1\overline{0011} \cdot 2^{-4}$

$0,100110 \ldots\ldots 001101$

$\tilde{e} - 127 = -4 \Rightarrow \tilde{e} = 123 = 1111011$

$123 : 2 = 61$	1
$61 : 2 = 30$	1
$30 : 2 = 15$	0
$15 : 2 = 7$	1
$7 : 2 = 3$	1
$3 : 2 = 1$	1
$1 : 2 = 0$	1

$x = 2^{-1} + 2^{-k} + 2^{-t}$

$y = 2^{-1} + 2^{-k}$

$k = \frac{t}{2} + 1$

$t = 2k - 2$

$x^2 + y^2$ izračunamo z izrazom

$x \cdot x - y \cdot y$

z obravnavo relativne napake pokažite da izračuni direktno stabilen

$$x^2 = (2^{-1} + 2^{-k} + 2^{-t})^2 = 2^{-2} + 2^{-2k} + 2^{-2t} + 2 \cdot 2^{-1-k} + 2 \cdot 2^{-k-t} + 2 \cdot 2^{-t-1} =$$

$$\cancel{2^{-2}} + \cancel{2^{-2k}} + \underset{\substack{\| \\ 2^{-2t}}}{2^{-4k+4}} + \cancel{2^{-k}} + 2^{-k-t+1} + 2^{-t}$$

$$y^2 = (2^{-1} + 2^{-k})^2 = \cancel{2^{-2}} + \cancel{2 \cdot 2^{-k}} + \cancel{2^{-2k}}$$

$$x^2 - y^2 = 2^{-4k+4} + 2^{-k-t+1} + 2^{-t}$$

$$= 2^{-2t} + 2^{-t} + 2^{-k-t+1}$$

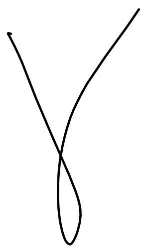
$$fl(x^2) = 0{,}0\overset{-2}{1}\ldots\overset{-k}{1}\ldots\overset{-t}{1}\,0\,\overset{-2k}{1}\ldots\overset{-t-k+1}{1}\;\ldots\;=$$

$$= 0{,}01\ldots1\ldots\overset{-t}{1}1$$

$$fl(y^2) = 0{,}01\ldots\overset{-k}{1}\ldots\overset{-t}{0}0\,1 = 0{,}01\ldots\overset{-k}{1}\ldots01\overset{-t}{0}$$

ni več denormalizirani

zato izberemo število s sodo zadnjo štenko, ker je 1 iz mesta -2^k zadnja ničla v ~~[illegible]~~ zapisu



22.10

$g(x) = -x^2 + 8x \cancel{+} - 12 \qquad x_{r+1} = g(x_r)$

$\lim\limits_{r\to\infty} x_r = 4$ za $\forall x_0 \in (3,5)$

$\Longleftrightarrow$

$\lim\limits_{r\to\infty} |x_r - 4| = 0$

$= |x_{r+1} - 4|$

$\lim |x_{r+1} - 4| = \lim\limits_{r\to\infty} |-x_r^2 + 8x_r - 16| =$

$= \lim\limits_{r\to\infty} |x_r - 4|^2 = \lim\limits_{r\to\infty} |x_{r-1} - 4|^4 = \dots = \lim\limits_{r\to\infty} |x_0 - 4|^{2^{r+1}}$

konvergira za $|x_0 - 4| < 1 \Rightarrow x_0 \in (3,5)$

proti 0

Red konvergence

α .... negibna točka

$$g'(\alpha) = g''(\alpha) = \dots = g^{(p-1)}(\alpha) = 0 \qquad g^{(p)} \neq 0$$

potem je red konvergence enak p

a) pokažite da lahko $\sqrt{\ }$ pozitivnega števila $a>$ izračunamo z iteracijo $x_{r+1} = \underbrace{x_r \cdot \frac{x_r^2+3a}{3x_r^2+a}}_{g(x_r)}$

b) določite red konvergence

c) dokažite da iteracija konvergira k $\sqrt{a}$ za $\forall$ začetni približek $x_0 > 0$

1. Preverimo če je $\sqrt{a}$ negibna točka iteracijske funkcije

$$\sqrt{a} \cdot \frac{\sqrt{a}^2+3a}{3\sqrt{a}^2+a} = \sqrt{a} \cdot \frac{4a}{4a} = \sqrt{a}$$

1. negibnost
2. privlačnost

2. preverjamo privlačnost $g'(\sqrt{a}) < 1$

$$g(x) = \frac{x^3+3ax}{3x^2+a}$$

$$g'(x) = \frac{(3x^2+3a)(3x^2+a) - 6x(x^3+3ax)}{(3x^2+a)^2} =$$

$$g'(\sqrt{a}) = \frac{(3a+3a)(3a+a) - 6a^2 - 18a\cdot a}{(3a+a)^2} =$$

$6a \cdot 3a + 6a^2$

$$= 0 < 1$$

b) $$g''(x) = \left(3\frac{(x^2+a)(3x^2+a) - 2x^2(x^2+3a)}{(3x^2+a)^2}\right)' =$$

$$= 3\frac{\big((2x+a)(3x^2+a) + 6x(x^2+a) - 4x(x^2+3a) - 4x^3\big)(3x^2+a)^2}{}$$

$$- \frac{2(\cancel{3x^2+a})\big((x^2+a)(3x^2+a) - 2x^2(x^2+3a)\big)}{(3x^2+a)^{\cancel{4}\,3}}$$

$$= -3\frac{(6x^3+3ax^2+2ax+a+6x^3+6a-4x^3+12xa-4x^3)(3x^2+a)^2}{}$$

$$- \frac{}{}$$

$$g'' = \frac{48xa}{(3x^2+a)}(x^2+a)$$

h_1, h_2

$$g''' = h_1' h_2 + h_1 h_2' = h_1'(x^2+a) + h_1(x)2x$$

$$g'''(\sqrt{a}) = h_1(\sqrt{a}) \cdot 0 + h_1(\sqrt{a})2\sqrt{a}$$

$$h_1\sqrt{a} = \frac{48a\sqrt{a}}{3a+a}$$

$$g'''\sqrt{a} = \frac{48 \cdot 2a^2}{(4a)^3} = \frac{3}{2a} \quad >0 \neq 0$$

$\Rightarrow$ red konvergence $=3$

c) $g(x) = x\,\frac{x^2+3a}{3x^2+a}$

locimo primere ko $x_0 < \sqrt{a}$

$x_1 > x_0$?

$x_1 > \sqrt{a}$ ali $x_1 < \sqrt{a}$?

$$x_1 = \cancel{x_0}\,\frac{x_0^2+3a}{3x_0^2+a} > \cancel{x_0}$$

$$x_0^2+3a > 3x_0^2+a$$

$$2a > 2x_0^2 \quad ✓$$

$$\sqrt{a} > x_0 \quad ✓$$

$x_1 < \sqrt{a}$

$$x_0\,(x_0^2+3a) < \sqrt{a}\,(3x_0^2+a)$$

$$\Leftarrow$$

$$\sqrt{a}\,(x_0^2+3a) < \sqrt{a}\,(3x_0^2+a)$$

$$2a < 2x_0$$

Sledi: $x_1 < \sqrt{a}$

torej x_r naraščajo in so omejeni

konvergirajo nekam

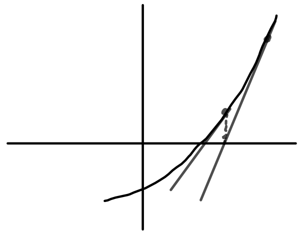
konvergira k $\sqrt{a}$?

$x_0 \in (\sqrt{a}, \infty)$

pokažemo da imamo padajoče

navzdol omejeno zaporedje

Tangentna metoda



$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$x_{r+1} = g(x_r)$$

Babilonska metoda za računanje $\sqrt{a}$ $a>0$ temelji na iteraciji

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{a}{x_r}\right)$$

a) Preverite, da iteracija ustreza tangentni metodi za funkcijo $f(x) = x^2 - a$

b) določite red konvergence

c) Dokažite da iteracija konvergira k $\sqrt{a}$ za $\forall x_0 > 0$

a)
$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \overset{?}{=} x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - a}{2x} =$$
$$= \frac{2x^2 - x^2 + a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right) \checkmark$$

b)
$$g'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) \rightsquigarrow g'(\sqrt{a}) = 0$$
$$g''(x) = \frac{2}{2}\left(\frac{a}{x^3}\right) = \frac{a}{x^3} \qquad \frac{1}{\sqrt{a}} > 0$$

red konvergence je 2

c) $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$

g naraščajoča za $x \in (\sqrt{a}, \infty)$

$\sqrt{a} < x_{r+1} < x_r$

še za

$\sqrt{a} < \frac{1}{2}\left(x_r + \frac{a}{x_r}\right) < x_r$

$2\sqrt{a}x_r < x_r^2 + a < 2x_r^2$

$a < x_r^2$ ✓

$x_r^2 - 2\sqrt{a}x_r + a > 0$

$(x_r - \sqrt{a})^2 > 0$

padajoče navzdol omejeno zaporedje

$x_0 \in (0, \sqrt{a}) \Rightarrow x_1 \in (\sqrt{a}, \infty)$

$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) \overset{?}{>} \sqrt{a}$

$x_0^2 + a > 2x_0\sqrt{a}$

$x_0^2 - 2x_0\sqrt{a} + a > 0$

$(x_0 - \sqrt{a})^2 > 0$ ✓

Naj bo $f \in \mathcal{C}^2$ α njena enostavna ničla

a) Dokažite, da metoda

$$x_{r+1} = x_r - \frac{2f(x_r)\cdot f'(x_r)}{2f'(x_r)^2 - f(x_r)f''(x_r)}$$ ustreza tangentni

metodi za funkcijo $F(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$

b) poenostavite metodo za $f(x) = x^2 - a$

a) $$F'(x) = \frac{f'(x)\sqrt{|f'(x)|} - \left(\sqrt{|f'(x)|}\right)'}{|f'(x)|} =$$

$$\left(\sqrt{|f'(x)|}\right)' = \frac{f''(x)}{2\sqrt{|f'(x)|}}\,\frac{f'(x)}{|f'(x)|}$$

$$= \frac{\frac{f'(x)}{\sqrt{|f'(x)|}}\left(1 - \frac{f''(x)}{2|f'(x)|}\right)}{|f'(x)|} = \frac{\operatorname{sgn} f\,(1 - \frac{1}{2}\operatorname{sgn} f)}{\sqrt{|f'(x)|}} =$$

$$= \frac{\operatorname{sgn} f - \frac{1}{2}}{\sqrt{|f'(x)|}}$$

$$\frac{F(x)}{F'(x)} = \frac{f(x)}{\cancel{\sqrt{|f'(x)|}}}\,\frac{\cancel{\sqrt{|f'(x)|}}}{(\operatorname{sgn} f - \frac{1}{2})} = \underline{\underline{\frac{f(x)}{\operatorname{sgn} f - \frac{1}{2}}}}$$

1.1

$P(2,3,-1,3)$ $\qquad \pm m \cdot b^e \quad L \leq e \leq U$

$P(b, t, L, U)$

baza

0	e: -1	1	2	3
0,100	0,01	1	10	100
0,101	0,0101	1,01	10,1	101
0,110	0,011	1,1	11	110
0,111	0,0111	1,11	11,1	111

1.2

$P(2, 9, -10, -10)$

$x = 47{,}712$

$47 : 2 = 23$ in 1 $\qquad 1 + 2 + 2^2 + 2^3 + 2^5 =$

07

1 ost

$23 : 2 = 11$ in 1

$11 : 2 = 5$ in 1

$5 : 2 = 2$ in 1

$2 : 2 = 1$ in 0

$1 : 2 = 1$ in 1

$$\begin{array}{r} 32 \\ 8 \\ 4 \\ 3 \\ \hline 47 \end{array}$$

10111

$$\frac{0{,}712 \cdot 2}{1{,}424} \quad 1$$

$0{,}101111010 \cdot 10^6$

$$\frac{0{,}424 \cdot 2}{0{,}848} \quad 0$$

$$\frac{0{,}848 \cdot 2}{1{,}696} = 1$$

$$\frac{0{,}696 \cdot 2}{1{,}392} \quad 1$$

$$\frac{0{,}392 \cdot 2}{0{,}784} \quad 0$$

$$0{,}1 = \sum_{i=1}^{\infty} \left(2^{-4i} + 2^{-4i-1}\right)$$

$$\sum_{i=1}^{\infty} (2^{-4})^i = \frac{1}{1-2^{-4}} = \frac{2^4}{2^4-1}$$

$$\frac{1}{2} \sum_{i=1}^{\infty} (2^{-4})^i = \frac{2^3}{2^4-1}$$

$$+ \quad = \frac{2^4+2^3}{15} = \frac{24}{15} = \frac{8}{5} = \frac{16}{10} = 1{,}6$$

$$\frac{3}{2} \, \frac{16}{15} = \frac{8}{5} =$$

1 2.3

$f(x) = x^5 - 10x + 1$

$f(0) = 1$

$f(0,2) = 0,2^5 - 2 + 1 < 0$

ima vsaj eno ničlo

če ima več, ma stacionarno točko

$f'(x) = 5x^4 - 10 = 0$

$x^4 = 2$

$x = \pm\sqrt[4]{2}$ ni med 0 in 0,2

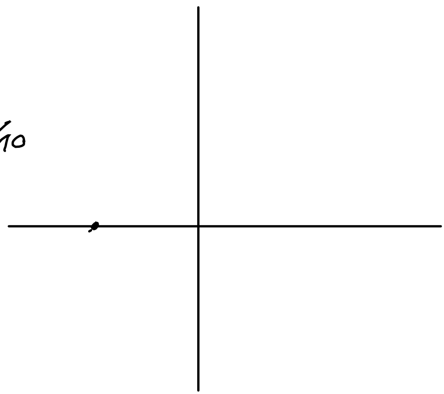
b)

$x_0 = 0$

$x_{r+1} = g(x_r)$

$g(x) = (x^5+1)/10$

$x_{r+1} = \frac{(x_r^5+1)}{10}$



$\lim_{r\to\infty} f(x_r)$

$g'(x) = \frac{5x^4}{10}$ $g'(0) = 0 <$

$g(g(0)) = g\left(\frac{1}{10}\right) = \frac{\frac{1}{10^5}+1}{10} = \frac{10^5+1}{10^6}$

$g(x) = -x^2 + 8x - 12$

$g(x) = x$

$-x^2 + 8x - 12 - x = 0$

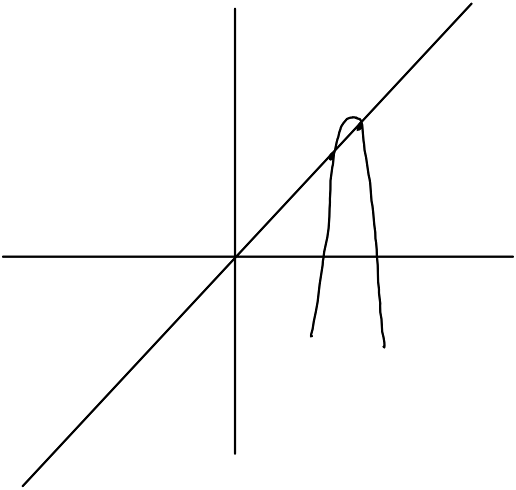
$-x^2 + 7x - 12 = 0$

$-(x-3)(x-4) = 0$

$g'(x) = -2x + 8$ $\qquad |-2x+8| < 1$

$g'(3) = 2$ [illegible] $\qquad \in \left(\frac{7}{2}, \frac{9}{2}\right)$

$g'(4) = 0$ [illegible]



$b_n = \frac{10}{3} b_{n-1} - b_{n-2}$

$b_{n-2} = \frac{10}{3} b_{n-1} - b_n$

2,7 naloge $\quad f(x) = (x^2 - a)x = x^3 - ax$

$$g(x) = x\,\frac{x^2+3a}{3x^2+a} \quad \text{je}$$

negibne točke x: $\sqrt{a}, -\sqrt{a}, 0$

za $x_0 > 0$ konvergira k $\sqrt{a}$

$$x_r - \sqrt{a} = g(x_{r-1}) - \sqrt{a} = \frac{(x_{r-1} - \sqrt{a})^3}{\underbrace{3x_{r-1}^2 + a}_{>0}} < 0$$

$x_0 - \sqrt{a} < 0$

$\Longrightarrow x_{r-1} - \sqrt{a} < 0$

ali je strogo naraščajoče? $\quad x_r < \sqrt{a}$

2.19 naloga

$$f(x)=x^2-a \quad a>0$$

$\boxed{\text{Hallyjeva metoda} \quad g(x)=x-\frac{2f(x)f'(x)}{2f'(x)^2-f(x)f''(x)}}$

$$f'(x)=2x \qquad f''(x)=2$$

$$g(x)=x-\frac{2(x^2-a)2x}{2(2x)^2-(x^2-a)2}=$$

$$=x-\frac{4x^3-4ax}{8x^2-2x^2+2a}=\frac{2x^3+6ax}{6x^2+2a}=$$

$$=\frac{x(x^2+3a)}{3x^2+a}$$

2.15 naloga

$$x_{r+1} = \frac{x_{r-1}x_r + a}{x_{r-1}+x_r}$$

$x_0, x_1 > \sqrt{a}$

$$\boxed{\begin{array}{l}\text{sekantna metoda}\\ x_{r+1} = x_r - \dfrac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}\end{array}}$$

$f(x) = x^2 - a$

$$x_{r+1} = x_r - \frac{(x_r^2-a)(x_r - x_{r-1})}{x_r^2 - \cancel{a} - x_{r-1}^2 + \cancel{a}} =$$

$$= \frac{\cancel{x_r^3} - x_{r-1}^2 x_r - \cancel{x_r^3} + a x_r + x_{r-1}x_r^2 - a x_{r-1}}{x_r^2 - x_{r-1}^2} =$$

$$= \frac{x_{r-1}x_r(x_r - x_{r-1}) + a(x_r - x_{r-1})}{x_r^2 - x_{r-1}^2} = \frac{x_{r-1}x_r + a}{x_r + x_{r-1}}$$

$\underline{x_{r+1} < x_r}$

$$x_{r+1} < \frac{x_{r-1}x_r + a}{x_r + x_{r-1}} \rightsquigarrow x_{r+1}x_r + x_{r+1}x_{r-1} < x_{r-1}x_r + a$$

$<$

$$2x_{r+1}\cancel{x_{r-1}} < \cancel{x_{r-1}}x_r \cancel{+a}_{x_{r+1}}$$

$a < x_r$

$$x_{r+1} < \frac{x_r + \cancel{a}}{2} < x_r$$

~..

3.3 naloga

$N_\infty(A) = \max |a_{ij}|$

1) $\underline{N}$ $\underline{ni}$ matrična norma

$N(A)N(B) = N(AB)$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ $\times$

2) $\|A\| = n N_\infty(A)$ je matrična norma

pozitivna definitnost

- $\|A\| \geq 0$: sledi iz def. in $n>0$

$\|A\| = 0 \Leftrightarrow n=0 \lor a_{ij}=0 \ \forall a_{ij} \Rightarrow A=0$

- homogenost

$\|\alpha A\| = n N(\alpha A) = n|\alpha| N(A) = |\alpha| \|A\|$ ✓

- trik. neenakost

$\|A+B\| = n N_\infty(A+B) = n \max_{ij} |a_{ij}+b_{ij}| \leq n \max |a_{ij}| + n \max \ldots$

$\leq \|A\| + \|B\|)$

- multiplikativnost

$\|A \cdot B\| \leq \|A\| \|B\|$

$\| n$

$n \max \left| \sum_{c=1}^{n} a_{ic} b_{cj} \right| \leq n \sum |a_{ic} b_{cj}|$

$n \max |a_{ij}| \left| \sum_{c=1}^{n} b_{cj} \right| \leq \max a_{ij}$ naprej

$\leq \|A\| \ldots$

3.8 naloga

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

$$\|A\|_2 = \max \sqrt{\underbrace{\lambda_i(A^H A)}_{\text{lastne vrednost od } A^H A}}$$

1) $\frac{1}{\sqrt{n}} \|A_F\| \leq \|A\|_2 \leq \|A\|_F$

$$\det(A^H A - \lambda I) = 0$$

3,10

$$\|A\|_2 \leq \|A\|_2$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^H = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$A^H A = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

V esperienza P

LU razcep

- brez pivotiranja (zadnjič ☺)
- delno pivotiranje $PA = LU$

$$Ax = b$$
$$L\underbrace{Ux}_{y} = Pb$$

1) $Ly = Pb$
2) $Ux = y$

- kompletno pivotiranje $Ly = Pb$ 2) $Uz = y$ 3) $Q^{-1}x = z$

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 2 & 1 & -4 & 2 \\ 3 & -2 & 3 & -1 \\ -1 & 3 & -1 & 1 \end{bmatrix}$$

izračunaj LU razcep in tako $\det(A)$

$1-(2)\left(\frac{2}{3}\right)$

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ 2 & 1 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & \boxed{-2} & 3 & -1 \\ \boxed{\frac{2}{3}} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & \frac{7}{3} & -4 & \frac{5}{3} \\ -\frac{1}{3} & \frac{7}{3} & 0 & \frac{2}{3} \end{bmatrix} \sim$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

$1-(-2)\left(\frac{2}{3}\right)$

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & 1 & 2 & -1 \\ -\frac{1}{3} & 1 & 6 & -2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$-1-\left(-\frac{2}{3}\right)$

$$U = \begin{bmatrix} 3 & -2 & 3 & -1 \\ & \frac{7}{3} & -6 & \frac{8}{3} \\ & & 6 & -2 \\ & & & -\frac{1}{3} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & & & \\ \frac{2}{3} & 1 & & \\ -\frac{1}{3} & 1 & 1 & \\ \frac{2}{3} & 1 & \frac{1}{3} & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det(A) = \frac{\det(U)\det(L)}{\det(P)} = \det(U) = (7)(-2) = -14$$

$$\det(P) = \pm 1 = (-1)^{\text{št. zam. vrstic}}, \quad \text{št. zam. vrstic} = 2$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix}$$

LU s kompletnim pivotiranjem + sistem $Ax=b$

$$A = \begin{bmatrix} 4 & 2 & -6 \\ 1 & 2 & -3 \\ -3 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ -3 & 2 & 1 \\ 3 & 3 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 4 & -1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

zamenjane vrstice

$$L = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -6 & 2 & 4 \\ & 4 & -1 \\ & & -\frac{3}{4} \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

zamenjava stolpcev

$Ly = Pb$

$$\begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$y_1 = 14$

$\frac{14}{2} + y_2 = 0 \rightsquigarrow y_2 = +7$

$y_3 = -\frac{3}{4}$

$Uz = y$

$x = Qz$

a) $A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 5 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Izračunajte LU razcep brez pivotiranja

Kaj opazite

b) Naj bo $\begin{bmatrix} a_1 & b_1 & & \\ c_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix}$ splošna 3-diagonalna matrika

Zapišite algoritem za razcep te tridiagonalne matrike in preštejte št. operacij

koliko operacij potrebujemo za reševanje sistema

$Ax = z$

a) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & -\frac{2}{7} & \frac{11}{7} & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & -\frac{2}{7} & \frac{11}{7} & 6 \\ 0 & 0 & \frac{7}{11} & -\frac{20}{11} \end{bmatrix}$

$6 \cdot 7 = 42$

$2 - \frac{42}{11} = \frac{-20}{11}$

$L = I \quad U = A_{zg}$

b) for i in range(1, n):

$= L[i][i-1] = A[i][i-1] \,/\, A[i-1][i-1]$

$U[i][i] = A[i][i] - L[i][i-1] \cdot A[i-1][i]$

$u_1 = a_1$

for $i = (1:n)$

$l_i = \frac{c_i}{u_i}$

$u_{i+1} = a_{i+1} - l_i \cdot b_i$

end

$n-1$ deljenja l

$n-1$ množenj

$n-1$ odštevanj

$3 \cdot (n-1)$ operacij

$Ax = z$

$Ly = z$: $y_1 = z_1$ $\quad y_2 = l_1 \cdot y_1 + z_2 \quad y_i = z_i - l_{i-1} y_{i-1}$

$2 \cdot (n-1)$ operacij

$Ux = z$: $x_n = u_n^{-1} z_n$

$b_{n-1} x_n + u_{n-1} x_{n-1} = z_n$

$x_{n-1} = \frac{z_n - b_{n-1} x_n}{u_{n-1}} \qquad x_i = \frac{z_{i+1} - b_i x_{i+1}}{u_i}$

$= 3 \cdot (n-1) + 1 = 3n-2$ operacij

skupaj $5n-4$ operacij

Opišite postopek za reševanje sistema linearnih enačb

$$\begin{bmatrix} U & -I \\ B & L \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$ kjer je $B = LU$ nesingularna

Preštejte število operacij

$$\begin{bmatrix} Ux - y \\ Bx + Ly \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$Bx + Ly = LUx + Ly = L(Ux + Iy)$$

$$y = Ux - a \qquad Ux = y + a$$

$$Bx + Ly = L(Ux + Ux - a) = L(2Ux - a) =$$

$$= L(\underbrace{2y + a}_{z}) = b$$

1. $Lz = b$ $\quad n^2$ operacij
2. $y = \frac{z - a}{2}$ $\quad 2n$ operacij
3. $Ux = a + y$ $\quad n + \underbrace{n^2}_{\text{obrat. sub.}} + n =$

skupaj: $2n + 4n$

15.11

V menjkole
ene uro

Razcep Choleskega

$A \in \mathbb{R}^{n\times n}$ simetrična, pozitivno definitna

$(\forall x \in \mathbb{R} - \{0\}.\ x^T A x > 0$

$A = V \cdot V^T$ $\quad V =$ nesingularna s pozitivnimi diag. elementi

for $i = 1:n$

$$v_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} v_{ik}^2}$$

for $j = i+1:n$

$$v_{ji} = \frac{1}{v_{ii}}\left(a_{ji} - \sum_{k=1}^{j-1} v_{jk} v_{ik}\right)$$

a) Določi faktor Choleskega (V)

$$A=\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 8 & -2 & 8 \\ -2 & -2 & 14 & -11 \\ 3 & 8 & -11 & 15 \end{bmatrix}$$

b) Naj bo

$$A=\begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

i) za katere α je to poz. def.

ii) $\alpha = 23$ Reši sistem

$$Ax=\begin{bmatrix} 6 \\ -5 \\ 2 \\ 1 \end{bmatrix}$$

$$V=\begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ -2 & 1 & 3 & \\ 3 & 1 & -2 & 1 \end{bmatrix}$$

$$v_{ji} = \frac{1}{v_{ii}}\left(a_{ji} - \sum_{k=1}^{j-1} v_{jk} v_{ik}\right)$$

$$v_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} v_{ik}^2}$$

$$A=\begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

$$v_{21} = \frac{1}{1}(2-0)$$

$$v_{22} = \sqrt{8 - 2^2} = 2$$

$$v_{32} = \frac{1}{2}(-2-(-4)) = 1$$

$$v_{42} = \frac{1}{2}(8-(3\cdot 2)) = 1$$

$$V=\begin{bmatrix} 2 & & & \\ 3 & 3 & & \\ 1 & -1 & 1 & \\ -2 & 3 & 1 & \sqrt{\alpha-14} \end{bmatrix}$$

$\alpha > 14$

26.11

Dani sta matriki A in $B \in \mathbb{R}^{n \times n}$. B je pozitivno definitna. Sestavite učinkovit postopek za izračun sledi od $A^T B^{-1} A$ in preštejte št. operacij

Na tabli:

Reševanje sistemov nelinearnih enačb

$$f_1(x_1 \dots x_n) = 0$$
$$f_2(x_1 \dots x_n) = 0$$
$$\vdots$$
$$f_n(x_1 \dots x_n) = 0$$

$$\rightsquigarrow F(x_1 \dots x_n) = \begin{bmatrix} f_1(x_1 \dots x_n) \\ \vdots \\ f_n(x_1 \dots x_n) \end{bmatrix}$$

$$F(\underline{x}) = 0$$

Jakobijeva iteracija

$$G(x) = \begin{bmatrix} g_1(\underline{x}) \\ \vdots \\ g_n(\underline{x}) \end{bmatrix} \qquad x^{(r+1)} = G(x^{(r)})$$

Seidlova iteracija

$$x_i^{(r+1)} = g_i(x_1^{(r+1)}, \dots, x_{i-1}^{(r+1)}, x_i^{(r)} \dots x_n^{(r)})$$

- $\underline{\alpha}$ je privlačna negibna točka, če obstaja taka matrična norma, da velja $\|J_G(\alpha)\| < 1$
  (G mora biti odvedljiva v α)

Potem $\exists$ okolica $\Omega \ni \alpha$. $\forall \underline{x}^{(0)} \in \Omega$ zaporedje konvergira k α

Dan je sistem enačb $x = \sin\left(\frac{2x-y}{4}\right)$

$y = \cos\left(\frac{x+2y}{4}\right)$

a) pri začetnem približku $x^0 = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ naredite dva koraka Jakobijeve in seidlove iteracije

b) Pokažite, da iteraciji konvergirata za $\forall$ začetni približek

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{4}\right) \\ \cos\left(\frac{x+2y}{4}\right) \end{bmatrix}$$

$$x^1 = G\begin{pmatrix} 2\pi \\ 0 \end{pmatrix} = \begin{bmatrix} \sin\left(\frac{4\pi}{4}\right) \\ \cos\left(\frac{2\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x^2 = G\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Seidlova:

$$x^1 = G\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x^2 = G\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\frac{1}{4} \\ \cos\left(\frac{-\sin\frac{1}{4} + 2}{4}\right) \end{bmatrix}$$

$$G(x,y)=\begin{bmatrix}\sin\left(\frac{2x-y}{4}\right)\\ \cos\left(\frac{x+2y}{4}\right)\end{bmatrix}$$

$$J_G(x,y)=\begin{bmatrix}\cos\left(\frac{2x-y}{4}\right)\cdot\frac{1}{2} & -\cos\left(\frac{2x-y}{4}\right)\frac{1}{4}\\ -\sin\left(\frac{x+2y}{4}\right)\cdot\frac{1}{4} & -\sin\left(\frac{x+2y}{4}\right)\cdot\frac{1}{2}\end{bmatrix}$$

$$\underbrace{}_{||<1}$$

Vzememo ∞ normo

vsota po abs vrednost vrstic:

$$\|J_G(x,y)\|_\infty=\max\left\{\left|\cos\left(\frac{2x-y}{4}\right)\frac{1}{2}\right|+\left|\frac{\cos\left(\frac{2x-y}{4}\right)}{4}\right|, \dots\right\}\le$$

$$\le\max\left\{\frac{1}{2}+\frac{1}{4},\ \frac{1}{2}+\frac{1}{4}\right\}<1$$

To je za $\forall x,y$ kar obstaja pomeni da krogle ki ga ne pr... ~~?~~ kam se vse približuje

Za seidle:

$$G(x,y)=\begin{bmatrix}\sin\left(\frac{2x-y}{4}\right)\\ \cos\left(\frac{\sin\frac{(2x-y)}{4}+2y}{4}\right)\end{bmatrix}$$

$$JG=\begin{bmatrix}\frac{1}{2}\cos\left(\frac{2x-y}{4}\right) & -\frac{1}{4}\cos\left(\frac{2x-y}{4}\right)\\ -\frac{1}{2}\cos\left(\frac{2x-y}{4}\right)\sin\left(\frac{\sin\left(\frac{2x-y}{4}\right)+2y}{4}\right), & \dots\end{bmatrix}$$

spet preverimo $||_\infty$ normo in vidimo da je manjše od 1

Newtnova metoda

$$x^{(r+1)} = x^{(r)} - J_F(x^{(r)})^{-1} F(x^{(r)})$$

$$\rightsquigarrow$$

$$J_F(x^{(r)})\left(x^{(r+1)} - x^{(r)}\right) = -F(x^{(r)})$$

1) $J_F(\underline{x}^{(r)})\,\underline{\Delta x}^{(r)} = -F(x^{(r)})$

2) $x^{(r+1)} = \Delta x^{(r)} + x^{(r)}$

Dan je sistem $x^2+y^2=4$

$x^2-y^2=1$

a) Naredite 2. koraka newtnove metode pri $x^{(0)}=(2,1)$

$$\underline{x}=\begin{bmatrix} x \\ y \end{bmatrix} \quad F=\begin{bmatrix} x^2+y^2-4 \\ x^2-y^2-1 \end{bmatrix}$$

$$JF=\begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$J_F(x^0)\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}=\begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}=\begin{bmatrix} 4\tilde{x}+2\tilde{y} \\ 4\tilde{x}-2\tilde{y} \end{bmatrix}=-\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 2 & -1 \\ 4 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 2 & -1 \\ 0 & -4 & -1 \end{array}\right] \rightsquigarrow y=\frac{1}{4}$$

$4x+2y=-1$

$4x=-\frac{3}{2}$

$x=-\frac{3}{8}$

3.12

¥ 1. nelogo

Predoločeni sistemi

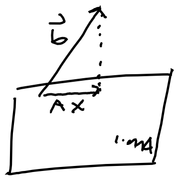
$Ax = b \quad A \in \mathbb{R}^{m \times n}, \; m \geq n \quad b \in \mathbb{R}^m$

Iščemo x, ki minimizira $\|Ax - b\|_2$

A polnega ranga $\Rightarrow$ enolična rešitev

Normalni sistem

$Ax \in \operatorname{Im}(A)$



$\Rightarrow Ax - b \perp \operatorname{im}(A)$

$\Leftrightarrow$

$(Ax - b) \perp A^T$ postolpcih

$\Leftrightarrow$

$A^T(Ax - b) = 0$

$\Leftrightarrow$

$A^T A x = A^T b$

$A^T A$ je simetrična, pozitivno definitnega

$\Rightarrow$ lahko uporabimo razcep choleskega

1) $A^T A = V V^T$

2) $A^T A x = A^T b$

$$V \underbrace{V^T x}_{y} = A^T b$$

$$V y = A^T b$$

f podana v točkah

$f(-1)=\frac{11}{4}$ Poiščite parabolo, ki se

$f(0)=\frac{7}{4}$ po metodi najmanjših

$f(1)=\frac{1}{4}$ kvadratov najbolje prilega

$f(2)=\frac{13}{4}$ funkciji f. Izpeljite normalen

in ga rešite z razcepom Choleskega

$p(x)=ax^2+bx+c$

$p(-1)=a-b+c=\frac{11}{4}$

$p(0)=c=\frac{7}{4}$

$p(1)=a+b+c=\frac{1}{4}$

$p(2)=4a+2b+c=\frac{13}{4}$

$A\begin{bmatrix} c \\ b \\ a \end{bmatrix}=\underline{b}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \qquad A^TA=\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

Cholesky

$\frac{2}{2}$, $\frac{6}{2}$

$$V=\begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \frac{8-3}{\sqrt{5}} & \sqrt{18-9-5} \end{bmatrix}=$$

$$=\begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix} \qquad Vy=A^Tb$$

$$A^Tb=\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 11 \\ 7 \\ 1 \\ 13 \end{bmatrix}=\begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} \rightsquigarrow$$

$y_1=4$

$4+\sqrt{5}y_2=4 \rightsquigarrow y_2=0$

$12+2y_3=16 \rightsquigarrow y_3=2$

$V^Tx=y$

$$\begin{bmatrix} 2 & 1 & 3 \\ & \sqrt{5} & \sqrt{5} \\ & & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \rightsquigarrow$$

$x_3=1$

$\sqrt{5}(x_2+1)=0 \rightsquigarrow x_2=-1$

$2x_1-1+3=4$

$x_1=1$

$$x=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{array}{l} a=1 \\ b=-1 \\ c=1 \end{array}$$

$$\underline{\underline{p(x)=x^2-x+1}}$$

QR razcep

$m > n$

$A = QR \quad A \in \mathbb{R}^{m \times n} \quad Q \in \mathbb{R}^{m \times n} \quad R \in \mathbb{R}^{n \times n}$

ortonormirani stolpci

zgornje trikotna

$Q^T Q = I$

$Q Q^T \neq I$ ker vrstice niso ortonormirane

- **Modificiran Gram-Schmidtov postopek**

$k = 1, \dots, n$

$\quad q_k = a_k$

$\quad i = 1, \dots, k-1$

$\quad\quad r_{ik} = q_i^T q_k$

$\quad\quad q_k = q_k - r_{ik} q_i$

$\quad r_{kk} = \|q_k\|_2$

$\quad q_k = \dfrac{q_k}{r_{kk}}$

S pomočjo MGS (modificiran gram Schmidt) izračunajte QR razcep za matriko

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix} \qquad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

$$g_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \|g_1\| = \sqrt{2}$$

$$g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \rightsquigarrow r_{12} = 0 \quad g_2 = g_2 - 0 \rightsquigarrow r_{22} = \sqrt{2} \quad q_2 = \frac{g_2}{\sqrt{2}}$$

$$g_3 = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow r_{13} = -\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} = -2\sqrt{2}$$

$$g_3 = -\begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow$$

$$r_{23} = 0 \,. \quad r_{33} = \cancel{\sqrt{12}} = 2\sqrt{3} \quad 2$$

$$g_3 = g_3 \rightsquigarrow$$

$$q_3 = \frac{1}{2\sqrt{3}} g_3$$

Modificiran Gramm-Schmidt

$k = 1 : n$

$\quad g_k = a_k$

$\quad j = 1, \dots, k-1$

$\qquad r_{jk} = g_j^T g_k$

$\qquad g_k = g_k - r_{jk} g_j$

$\quad r_{kk} = \|g_k\|_2$

$\quad g_k = \frac{1}{r_{kk}} g_k$

Pri reševanju sistema z MGS naredimo QR razcep na razširjeni matriki

$$[A \,\vdots\, b] = [Q \,\vdots\, g_{n+1}] \begin{bmatrix} R & z \\ 0 & \rho \end{bmatrix} \qquad \begin{pmatrix} Ax = b \\ \min\limits_x \|Ax - b\|_2 \end{pmatrix}$$

Rešujemo $Rx = z$ $\qquad \min\limits_x \|Ax - b\| = \rho$

Od zadnjic

$$A=\begin{bmatrix}1&0&-1\\1&0&-3\\0&-1&1\\0&1&1\end{bmatrix}=\quad Q=\begin{bmatrix}\frac{1}{\sqrt2}&0&\frac12\\\frac{1}{\sqrt2}&0&-\frac12\\0&\frac{1}{\sqrt2}&\frac12\\0&-\frac{1}{\sqrt2}&\frac12\end{bmatrix}\quad R=\begin{bmatrix}\sqrt2&0&-2\sqrt2\\&\sqrt2&0\\&&2\end{bmatrix}$$

b) Preko QR razcepa rešite predoločen sistem

$$Ax=\begin{bmatrix}4\\6\\-1\\2\end{bmatrix}$$ Po metodi najmanjših kvadratov.

Koliko je $\min\|Ax-b\|_2$

$$[A;b]=\begin{bmatrix}1&0&-1&4\\1&0&-3&6\\0&-1&-1&-1\\0&1&1&2\end{bmatrix}\qquad q_4=\begin{bmatrix}6\\6\\-1\\2\end{bmatrix}$$

$j=1$

$$r_{14}=\frac{4}{\sqrt2}+\frac{6}{\sqrt2}=\frac{10}{\sqrt2}=5\sqrt2$$

$$q_4=q_4-5\sqrt2 q_1$$

$$q_4=\begin{bmatrix}-1\\1\\-1\\2\end{bmatrix}$$

$j=2$:

$$r_{24}=-\frac{1}{\sqrt2}-\frac{2}{\sqrt2}=-\frac{3}{\sqrt2}=-\frac{\sqrt2\cdot3}{2}$$

$$q_4=q_4+\frac{3}{\sqrt2}q_2=\begin{bmatrix}-1\\1\\-\frac12\\\frac12\end{bmatrix}$$

$j=3$

$$r_{34}=-\frac12$$

$$q_3=q_4+\frac12 q_3=\begin{bmatrix}\frac14\\-\frac14\\\frac14\\\frac14\end{bmatrix}=\frac34\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

$$r_{44}=\sqrt{\frac{9}{16}\cdot2}=\sqrt{\frac98}=\frac{3}{2\sqrt2}=$$

$$q_4=\frac{1}{r_{44}}q_4=\frac12\cdot\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

$Rx=z$

$$R=\begin{bmatrix}\sqrt2&0&-2\sqrt2\\&\sqrt2&0\\&&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5\sqrt2\\-\frac32\sqrt2\\-\frac12\end{bmatrix}$$

$$x_3=-\frac14$$

$$x_2=-\frac32$$

$$\sqrt2 x_1-\frac{2\sqrt2}{4}=5\sqrt2$$

$$x_1=\frac12+5=\frac92$$

$$x=\begin{bmatrix}\frac92\\-\frac32\\-\frac14\end{bmatrix}$$

$$\min_x\|Ax-b\|_2=2\text{ "?}$$

V 10.12

1.Wa

Razširjeni QR razcep

$A = \tilde{Q}\tilde{R}$

$\tilde{Q}$: $m \times m$ ortogonalna $\tilde{Q}^T\tilde{Q} = \tilde{Q}\tilde{Q}^T = I$

$\tilde{R}$: $m \times n$ zgornja trapezna

$$A = \tilde{Q}\tilde{R} = \begin{bmatrix} Q & Q_1 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Reševanje sistema:

množenje z $\tilde{Q}$ ne spremeni 2-norme

$$\|Ax - b\|_2 = \|\tilde{Q}\tilde{R}x - b\|_2 = \|\tilde{Q}^T(Q\tilde{R}x - b)\|_2 =$$

$$= \|\tilde{R}x - Q^T b\| = \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} x - \begin{bmatrix} Q^T \\ Q_1^T \end{bmatrix} b \right\|$$

x, ki to minimizira dobimo iz $Rx = Q^T b$

minimum $= \|Q_1^T b\|_2$

Givensove rotacije

vhodni podatki: A, b

izhod: $\tilde{R}$ $Q^T b$

for $i = 1:n$

  for $k = i+1:m$

    if $a_{ki} \neq 0$:

$$R^T_{ik} = I$$

$$r = \sqrt{a_{ii}^2 + a_{ki}^2}$$

$$c = \frac{a_{ii}}{r}$$

$$s = \frac{a_{ki}}{r}$$

$$R^T_{ik}([i\ k],[i\ k]) = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$A = R^T_{ik} A$$

$$b = R^T_{ik} b$$

$(R^T_{ik})_{ii} = c$

$(R^T_{ik})_{ik} = s$

izhod: $A (= \tilde{R})$, $b (= \tilde{Q}^T b)$

Naloga: s pomočjo givensovih rotacij reši predoločen sistem. Koliko je $\min_x \|Ax-b\|$?

$$\begin{bmatrix} 3 \\ -4 \\ 12 \\ -84 \end{bmatrix} x = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 85 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 5 \\ -\frac{7\cdot 5}{13} \\ \frac{84}{85} \end{bmatrix}$$

$i=1 \quad k=2$

$r = 5$

$c = \frac{3}{5}$

$s = -\frac{4}{5}$

$$R = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & & \\ \frac{4}{5} & \frac{3}{5} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad RA \qquad Rb$$

$$\begin{array}{r} 144 \\ 25 \\ \hline 169 \end{array}$$

$i=1 \quad k=3$

$r = \sqrt{5^2 + 12^2} = 13$

$c = \frac{5}{13} \quad s = \frac{12}{13}$

$$R = \begin{bmatrix} \frac{5}{13} & 0 & \frac{12}{13} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad RA \qquad Rb$$

$i=1 \quad k=4$

$r = \sqrt{13^2 + 84^2} = 85$

$c = \frac{13}{85} \quad s = -\frac{84}{85}$

$$R = \begin{bmatrix} \frac{13}{85} & & & -\frac{84}{85} \\ & 1 & & \\ & & 1 & \\ \frac{84}{85} & & & \frac{13}{85} \end{bmatrix}$$

$$A = \begin{bmatrix} 85 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix} \qquad Q = 1 \qquad \rightsquigarrow R = 85$$

$$Q_1^T b = \begin{bmatrix} 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix} \qquad \text{minimum:} \sqrt{25 + \frac{35^2}{13^2} + \frac{84^2}{13^2}}$$

$$= \sqrt{74}$$

V manjkanje 1,5 krat

7 dni prej
in
24.12

Interpolacija

Iščemo polinom, ki bo potekal skozi dane točke. Za n+1 točk obstaja enoličen polinom stopnje n, ki jih interpolira

Lagrangeeva oblika interpolacijskega polinoma

Lagrangevi bazni polinomi:

$$l_{i,n}(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

interpolacijski polinom:

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_{i,n}(x)$$

Dobcite L.I.P.

$$f(x)=\frac{40}{x+1} \quad \text{v tockah } \{\overset{x_0}{0}, \overset{x_1}{\tfrac{1}{3}}, \overset{x_2}{\tfrac{2}{3}}, \overset{x_3}{1}\}$$

$n=3$

$$l_0=\frac{x-\frac{1}{3}}{-\frac{1}{3}}\cdot\frac{x-\frac{2}{3}}{-\frac{2}{3}}\cdot\frac{x-1}{-1}=-(3x-1)(\tfrac{3}{2}x-1)(x-1)$$

$+40=f(x_i)$

$$l_1=\frac{x}{\frac{1}{3}}\cdot\frac{x-\frac{2}{3}}{-\frac{1}{3}}\cdot\frac{x-1}{-\frac{2}{3}}$$

$f(x_1)=30$

$$l_2=\frac{x}{\frac{2}{3}}\cdot\frac{x-\frac{1}{3}}{\frac{1}{3}}\cdot\frac{x-1}{-\frac{1}{3}}$$

$f(x_2)=24$

$$l_3=\frac{x}{1}\cdot\frac{x-\frac{1}{3}}{\frac{2}{3}}\cdot\frac{x-\frac{2}{3}}{\frac{1}{3}}$$

$f(x_3)=20$

$$p(x)=\frac{9}{2}(x-\tfrac{2}{3})(x-1)((3x-1)(-40)+30x)+$$

$$+\frac{9}{2}x(x-\tfrac{1}{3})(-24(x-1)+60(x-\tfrac{2}{3}))=$$

$$\begin{array}{r} 60 \\ 24 \\ \hline 36 \end{array} \qquad \begin{array}{r} 40 \\ -24 \\ \hline 16 \end{array}$$

$$=\frac{6}{2}(3x-2)(x-1)(-90x+40)+$$

$$\frac{6}{2}(3x^2-1)(36x-16)=$$

$$=\frac{6}{2}(9x-4)(-10(3x^2-5x+2)+$$

$$+\cancel{4}\,12x^2-4)=$$

$$=\frac{6}{2}(9x-4)(-18x^2-50x-24)=$$

$$=6(9x-4)(-9x^2-25x-12)=$$

$$=6(x^3\cdot(-81)+x^2(36)+x(-9\cdot12+4\cdot25)+12\cdot4$$

Newtnova oblika interpolacijskega polinoma

Baza iz premaknjenih potenc

$1,\ x-x_0,\ (x-x_0)(x-x_1),\ \dots,\ (x-x_0)\dots(x-x_{n-1})$

$$p(x)=\sum_{i=0}^{n}\underbrace{[x_0,x_1\dots x_i]\cdot f}_{\text{deljene diference}}(x-x_0)\dots(x-x_{i-1})$$

$$[x_0\dots x_k]f=\frac{[x_1\dots x_k]f-[x_0\dots x_{k-1}]f}{x_k-x_0}$$

ni nula

x_i	$[x_i]f$	$[\ ,\ ]f$	$[\ ,\ ,\ ]f$	
0	40	$\frac{30-40}{\frac{1}{3}} = -30$		
$\frac{1}{3}$	30	$\frac{24-30}{\frac{1}{3}} = -18$	$\frac{-18+30}{\frac{2}{3}} = 18$	
$\frac{2}{3}$	24	$\frac{20-24}{\frac{1}{3}} = -12$	$\frac{-12+18}{\frac{2}{3}} = 9$	$\frac{9-18}{1} = -9$
1	20			

$$p(x) = 40 - 30(x-0) + 18x\left(x-\frac{1}{3}\right) - 9x\left(x-\frac{1}{3}\right)\left(x-\frac{2}{3}\right)$$

$$40 - 30x + 18x^2 - 6x - 9x^3 + 9x^2 - 2 =$$

$$= 40 - 38x + 27x - 9x^3$$

Vemo s predavanj

$$f(x) - p(x) = \omega(x)\frac{f^{(n+1)}(\xi)}{(n+1)!} \qquad \omega(x) = \prod_{i=1}^{n}(x - x_i)$$

$$\omega(x) = x\left(x-\frac{1}{3}\right)\left(x-\frac{2}{3}\right)(x-1)$$

Nejhorší om

menjkala

Deljene diference

$$[x_i, \ldots x_{i+k}]f = \begin{cases} \dfrac{f^{(k)}(x_i)}{k!} & ; x_i = \ldots = x_{i+k} \\ \dfrac{[x_{i+1}, \ldots x_{i+k}]f - [x_i, \ldots x_{i+k-1}]f}{x_{i+k} - x_i} & ; \text{sicer} \end{cases}$$

$$f(x) - p(x) = \omega(x) \frac{f^{(n+1)}(\xi)}{(n+1)!} \qquad \omega(x) = \prod_{i=0}^{n} (x - x_i)$$

$$f(x) - p(x) = \omega(x) [x_0, x_1, \ldots x_n, x]$$

Naj bo $f(x) = \frac{40}{x+1}$

a) Določite polinom p, ki interpolira vrednosti in odvode funkcije f v točkah 0 in 1 in nato oceniti napako $\|f-p\|_{\infty,[0,1]}$

$$f' = -\frac{40}{(x+1)^2}$$

$p(0) = 40$

$p(1) = 20$

$p'(0) = -40$

$p'(1) = -10$

$f(0)$ $=$ $[\bullet]f$

		$[\bullet]f$	$[\bullet,\bullet]f$	$[\bullet,\bullet,\bullet]f$
x_0	0	40		
x_1	0	40	$\frac{-40}{1!} = \frac{f'(0)}{1!}$	$\frac{-20-(-40)}{1-0} = 20$
x_2	1	20	$\frac{20-40}{1-0} = -20$	$\frac{-10-(-20)}{1-0} = 10$
x_3	1	20	-10	

$[\bullet,\bullet,\bullet,\bullet]f$

$$\frac{10-20}{1-0} = -10$$

$$p(x) = 40 - 40x + 20x^2 - 10x^2(x-1)$$

$n = 3$ .... stopnja polinoma

Torej rabimo 4. odvod

$$\|f-p\|_{\infty,[0,1]} \leq \|\omega\|_{\infty,[0,\infty]} \frac{\|f^{(n+1)}\|_{\infty,[0,1]}}{(n+1)!}$$

$$f''(x) = \frac{80}{(x+1)^3} \qquad 80 = 2\cdot 40$$

$$f'''(x) = -\frac{2\cdot3\cdot40}{(x+1)^4}$$

$$f''''(x) = \frac{2\cdot3\cdot4\cdot40}{(x+1)^5}$$

največji bo pri $x=0$

$$\Rightarrow \|f^{(4)}(x)\|_\infty = 2\cdot3\cdot4\cdot40$$

$$\|f-p\|_\infty \leq \|\omega\| \frac{4!\,40}{4!}$$

$\|\omega\| = ?$

$$\omega(x) = x^2(x-1)^2$$

$$\omega'(x) = 2x(x-1)^2 + 2x^2(x-1) =$$

$$= 2x(x-1)(2x-1)$$

ničle: $0, 1, \frac{1}{2}$

$$\longrightarrow \|\omega\| = \omega\left(\tfrac{1}{2}\right) = \frac{1}{16}$$

$$\|f-p\|_\infty \leq 40 \cdot \frac{1}{16} = \frac{5}{2}$$

$$
\begin{array}{r}
01100001 \\
01100010 \\
\hline
11000011
\end{array}
$$

F

nisem

poslusala