Hilbertov prostor

X vektorski prostor nad $\mathbb{R}$ (nad D)

skalarni produkt $\langle \ \rangle$

$$\langle \ \rangle : X \times X \longrightarrow \mathbb{R}$$

1) $\forall x \in X : \langle x, x \rangle \geq 0$ Pozitivne definitnost

2) $\langle x, x \rangle = 0 \iff x = 0$

3) $\forall x, y \in X$

$\langle x, y \rangle = \langle y, x \rangle$ nad $\mathbb{R}$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$ nad $\mathbb{C}$

simetričnost oz antisimetričnost

4) $\forall x, y, z \in X, \ \forall \lambda \mu \in \mathbb{R}$

$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

linearnost v prvem faktorju

Za skalarni produkt velja

Cauchy-Schwarzova neenakost

$\forall x, y \in X$

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}$$

$$\|x\| = \sqrt{\langle x,x\rangle}$$

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|$$

Dokaz nad $\mathbb{R}$

$$t \longmapsto \langle x+ty, x+ty\rangle = s(t) \geq 0$$

$$= \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle$$

$$= \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\| \geq 0$$

$$D \leq 0$$

$$D = 4\langle x,y\rangle - 4\|x\|^2\|y\|^2$$

$$\Rightarrow \langle x,y\rangle^2 \leq \|x\|^2\|y\|^2$$

$$\langle x,y\rangle \leq \|x\|^2\|y\|^2$$

enakost velja natanko tedaj če sta x in y linearno odvisna

Nad $\mathbb{C}$:

x, y

$\exists \alpha \quad |\alpha| = 1$

$$\langle x, y \rangle = \alpha |\langle x, y \rangle|$$

$$\Rightarrow \quad \langle x, \alpha y \rangle = |\langle x, y \rangle|$$

$$f(t) = \langle x + t\alpha y, x + t\alpha y \rangle =$$

$$= \|x\|^2 + t \underbrace{\langle \alpha y, x \rangle + \langle x, \alpha y \rangle)}_{= 2 \cdot |\langle x, y \rangle|} + t^2 \langle \alpha y, \alpha y \rangle$$

$$= \|x\|^2 + 2t|\langle x, y \rangle| + t^2 \|y\|^2 \geq 0$$

$$D = 4|\langle x, y \rangle|^2 - 4\|x\|^2\|y\|^2 \leq 0$$

Norma nad vektorskem prostorom X

$$\|\ \| : X \to \mathbb{R}$$

1) $\forall x \in X.\ \|x\| \geq 0$

2) $\|x\| = 0 \Leftrightarrow x = 0$

3) $\forall \lambda \in \mathbb{R} \vee \mathbb{C},\ \forall x \in X$

$$\|\lambda x\| = |\lambda|\,\|x\| \quad \text{homogenost}$$

4) trikotniška neenakost

$$\forall x, y \in X.\ \|x+y\| \leq \|x\| + \|y\|$$

če je $(X, \langle\ \rangle)$ vektorski prostor s skalarnim produktom je

$\|x\| = \sqrt{\langle x, x \rangle}$ X vektorski prostor z normo

$$\|x\| = \sqrt{\langle x, x\rangle}$$

1), 2) sledi iz 1) 2) skalarnega produkta

3) $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{|\lambda|^2 \langle x, x\rangle} =$

$= |\lambda| \, \|x\|$

4)

$\|x+y\|^2 = \langle x+y, x+y\rangle =$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle =$

$= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \leqslant \|x\|^2 + 2|\langle x, y\rangle|$

$+ \|y\|^2$

$\leqslant \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

$(X, \langle\ \rangle) \rightsquigarrow (X, \|\ \|)$

$\rightsquigarrow$ metrični prostor (X, d)

$\forall x, y \in X \,.\, d(x,y) = \|x-y\|$

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \iff x = y$

3) $d(x,y) = d(y,x)$

4) $\forall x, y, z \in X$

$$d(x,y) + d(y,z) \geq d(x,z)$$

Def: Hilbertov prostor je vektorski prostor s skalarnim produktom ki je v metriki porojeni iz skalarnega produkta poln metrični prostor

Opomba: Banachov prostor je vektorski prostor X z normo $\|\ \|$ ki je v metriki porojeni iz norme poln metrični prostor

Zgledi:

1) $\mathbb{R}^n$, $x = (x_1 \dots x_n)$

$y = (y_1 \dots y_n)$

$$x \cdot y = x_1 y_1 + \dots \pm x_n y_n$$

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$$ d_2 metrika

$d_2(x,y) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$ je poln metrični prostor

$(\mathbb{R}^n, \cdot)$ je Hilbertov prostor

$(\mathbb{R}^n, d_\infty)$ $\quad \|\ \|_\infty$ $\quad \max\{|x_1| \dots |x_n|\}$

normi

$(\mathbb{R}^n, d_1)$ $\quad |x_1| + \dots + |x_n|$ $\quad \|\ \|_1$

Banachova prostora

ker sta topološko enaka $(\mathbb{R}^n, d_2)$

$\mathbb{R}^n$ sta Banachova ker sta topološki ekvivalentna $(\mathbb{R}^2, d_2)$

Ampak ne prideta iz skalarnega produkta

2) $\mathbb{C}^n$ $\quad z = z_1, \ldots z_n$

$w = w_1, \ldots w_n$

$$z \cdot w = z_1 \overline{w_1} + \ldots + z_n \overline{w_n}$$

$$\|z\| = \sqrt{z_1^2 + \ldots + z_n^2}$$

$$d_2(z,w) = \sqrt{|z_1 - w_1|^2 + \ldots + |z_n - w_n|^2}$$

$(\mathbb{C}^n, \cdot)$ je Hilbertov prostor

Zgled

$[a,b] \subseteq \mathbb{R}$ $a<b$

$X = C([a,b])$

nad $\mathbb{R}$

$$\langle f,g\rangle = \int_a^b f(x)\,g(x)\,dx$$

3) $\langle f,g\rangle = \langle g,f\rangle$

4) $\langle \lambda f + \mu g, h\rangle =$
$\lambda \langle f,h\rangle + \mu \langle g,h\rangle$

1) $\langle f,f\rangle = \int_a^b f^2(x)\,dx \geq 0$

2) $f=0 \Rightarrow \langle f,f\rangle = 0$

$\int_a^b f^2\,dx = 0 \Rightarrow f^2 = 0$ skoraj povsod

$\Rightarrow f^2 = 0$ povsod ker zvezna

$\Rightarrow f = 0$ (Naslednja stran)

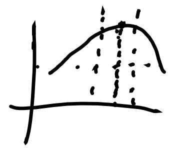
Polnost ?

$$\int_a^b f^2(x)\,dx = 0$$

Ce $\exists\, x_0 \in [a,b]$ $f(x_0) \neq 0 \Longrightarrow$

$\exists\, \delta > 0 : \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a,b].$

$$|f(x)| \geq \frac{|f(x_0)|}{2}$$



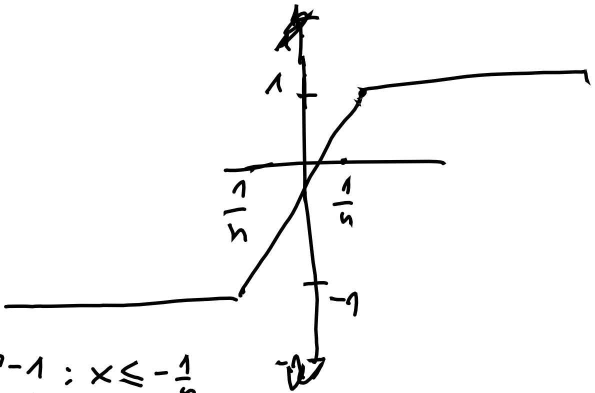
$$\int_a^b f^2(x)\,dx \geq \int_{(x_0-\delta,\, x_0+\delta)\cap[a,b]} f^2(x)\,dx \geq \frac{f(x_0)^2}{2}\,\delta$$

Ce f непр $\Longrightarrow f = 0$ равно

Trditev: $(C[a,b], \langle\ \rangle)$
ni Hilbertov

Dokaz

$f_n(x)$



$$f_n(x) = \begin{cases} -1 &: x \leq -\frac{1}{n} \\ 1 &: x \geq \frac{1}{n} \\ nx &: -\frac{1}{n} \leq x \leq \frac{1}{n} \end{cases}$$

$(f_n)_n$ je Cauchyjevo zaporedje v $(C[-1,1], d)$
d-metrika iz skalarnega produkta

$$d^2(f,g) = \int_{-1}^{1} |f(x) - g(x)|^2 dx$$

$m > n$

$$d(f_n, f_m)^2 =$$

$$\int_{-1}^{1} (f_n(x) - f_m(x))^2 dx =$$

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} ((f_m(x) - f_n(x)))^2 \leq \int_{-\frac{1}{n}}^{\frac{1}{n}} 1 dx = \frac{2}{n}$$

$$d(f_n, f_m) \leq \sqrt{\frac{2}{n}} < \varepsilon$$

$(f_n)_n$ je Cauchyjevo v $(C[-1,1], d)$

$\lim f_n$

Če obstaja $\lim$ za $(f_n)_n$ v $C[-1,1]$
mora veljati, da je $f(x) = \begin{cases} 1 &: 0 < x \leq 1 \\ -1 &: -1 \leq x < 0 \end{cases}$

$\Rightarrow f$ ni zvezna v 0 $\times$

Zgled

$M = (0,1)$

$d_2(x,y) = |x-y|$

~~M~~ M ni poln

$M \cup \{0,1\}$ je poln

Dodali smo možne limite

Napolnitev metričnega prostora

(M,d) napolnimo lahko napolnemo

$(\tilde{M},\tilde{d})$ poln

1) $M \subseteq \tilde{M}$

2) $\tilde{d}|_{M\times M} = d$

3) M je gost v $\tilde{M}$

$$L^1(A) = \{ f : A \to \mathbb{R} \ ; \int_A |f(x)|\, dx < \infty \}$$

s kvadratom integrabilne funkcije:

$$L^2([a,b]) = \{ f : [a,b] \to \mathbb{R} \ \int_a^b |f|^2\, dx < \infty \}$$

Zgled:

1) $C\ell[a,b] \subseteq L^2([a,b])$

2) odsekoma zvezne $\subseteq L^2$

3) $f(x) = \dfrac{1}{\sqrt[4]{x-a}} \in L^2$

4) $g(x) = \dfrac{1}{\sqrt[2]{x-a}} \notin L^2([a,b])$

Velja: f, g

$$|f \cdot g| \leq \frac{|f|^2 + |g|^2}{2} \Rightarrow f \cdot g \in L^1([a,b])$$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

L^2 je vektorski prostor

$$(f+g)^2 = f^2 + 2fg + g^2$$

$\in L^2 \quad \in L^1 \quad \in L^2$

$$\int_a^b (f+g)^2\, dx = \int_a^b f^2(x)\, dx + 2\int f(x) g(x)\, dx + \int_a^b g^2(x) < \infty$$

$\Rightarrow \quad (f+g) \in L^2$

$C[a,b] \subseteq L^2([a,b])$

hilbertov

Opomba

$\forall f \in L^2([a,b]) . \exists f_n \in C[a,b] .$

$$\lim_{n\to\infty} f_n = f \iff \lim_{n\to\infty} \|f_n - f\| = 0$$

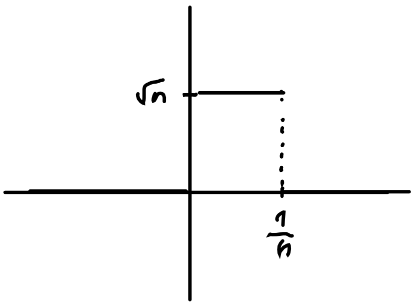
$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x))^2 dx = 0$$

Opomba: nad $\mathbb{C}$: $f = u + iv \quad u,v : [a,b] \to \mathbb{R}$

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx$$

Zgled: na $[0,1]$

$$f_n: \begin{cases} \sqrt{n} & ; \ 0 < x \leq \frac{1}{n} \\ 0 & ; \ \text{sicer} \end{cases}$$

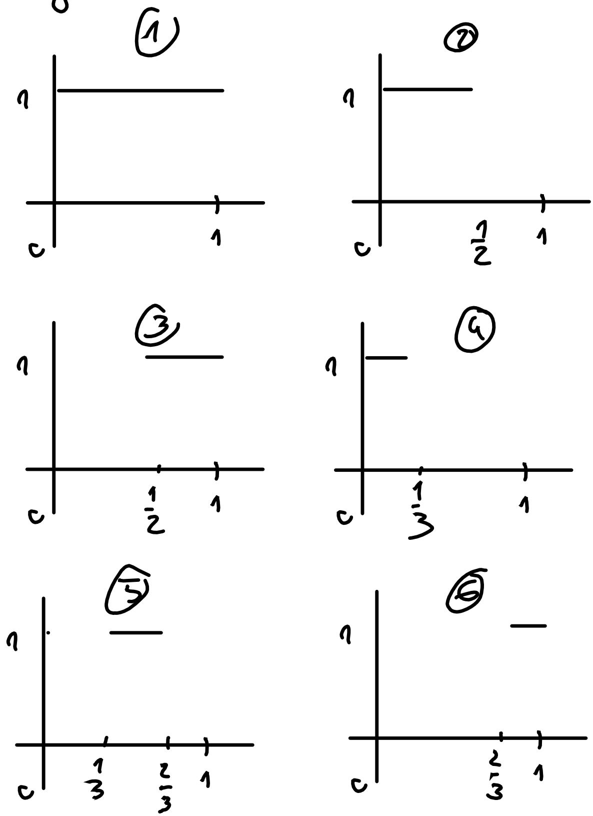


$\lim f_n(x) = 0$ za $x \in [0,1]$

Ali konvergira v L^2 ?

$$\|f_n - 0\| = \sqrt{\int_0^1 f_n^2(x)\,dx} = \sqrt{\int_0^{\frac{1}{n}} \sqrt{n}^2\,dx} = 1$$

Zgled:



itd

$\lim_{n\to\infty} f_n(x)$ ne obstaja

$\lim f_n(x)$ v L^2 ?

$\int_0^1 f_n^2(x)dx \leq \frac{1}{n^2}$ torej limita je 0

Naj bo $(X, \langle\ \rangle)$ vek. prostor s skalarnim produktom.

$A \subseteq X \quad (A \neq \emptyset)$

$x, y \in X. \quad x \perp y \Leftrightarrow \langle x, y \rangle = 0$

(simetricna relacija)

$A \subseteq X \quad A^{\perp} = \{x \in X.\ x \perp a \text{ za } \forall a \in A\}$

Ortogonalni komplement

Trditev: $A^\perp$ je vektorski podprostor v X

Dokaz:

$x \in A^\perp$ $\lambda \in \mathbb{R}$ (ali $\mathbb{C}$)

$a \in A$. $\langle \lambda x, a \rangle = \lambda \langle x, a \rangle = \lambda \cdot 0 = 0$

$x, y \in A^\perp$

$$\langle x+y, a \rangle = \underbrace{\langle x, a \rangle}_{=0} + \underbrace{\langle y, a \rangle}_{=0} = 0$$

Velja: $A \subseteq (A^{\perp})^{\perp}$

Trditev: $v \in X$

$f(x) = \langle x, v \rangle \quad f: X \to \mathbb{R}$

f je zvezna na X

Dokaz:

$x_1, x_2 \in X$

$|f(x_1) - f(x_2)| = |\langle x_1 - x_2, v \rangle| \leq \|x_1 - x_2\| \|v\|$

f je enakomerno zvezna
(celo Lipschitzova zvezna)

Posledica: $A^\perp$ je zaprt vektorski podprostor

$C[a,b] \subseteq L^2[a,b]$ ni zaprt podprostor

Dokaz:

zaporedje x_n

$\lim x_n = x_0 \in X \overset{?}{\Longrightarrow} x_0 \in A^\perp$

$\forall a \in A . \langle x_n, a \rangle = 0$ za $\forall n$

$$\underbrace{\lim \langle x_n, a \rangle}_{=0} = \langle \lim x_n, a \rangle = \langle x_0, a \rangle = 0$$

Opomba: $(X, \langle\,\rangle)$ hilbertov

$A \leq X$ zaprt podprostor $\Rightarrow A = (A^\perp)^\perp$

Trditev: (Pitagorov izrek)

$(X, \langle\ \rangle)$ vektorski prostor s skalarnim produktom

$x_1 \dots x_n \in X \quad \forall i \neq j \in [n] .\ x_j \perp x_i$

$(\langle x_j, x_i \rangle = 0)$

Tedaj $\|x_1\|^2 + \dots + \|x_n\|^2 = \|x_1 + \dots x_n\|^2$

Dokaz:

$\|x_1 + \dots + x_n\| = \langle x_1 + x_2 \dots + x_n, x_n + \dots + x_n \rangle$

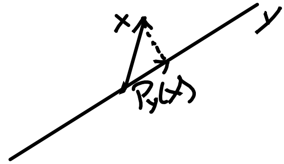
$= \langle \sum_i x_i ; \sum x_j \rangle = \sum_i \sum_j \langle x_i, x_j \rangle =$

$= \|x_1\|^2 + \dots + \|x_n\|^2$

$(X, \langle\ \ \rangle)$

$Y \leq X$

$x \in X$



Definicija: Pravokotna projekcija vektorja x na podprostor Y (če obstaja) je tak vektor $P_Y(x) \in Y$ da je

$$x - P_Y(x) \in Y^{\perp}$$

Trditev: če pravokotna projekcija x na Y obstaja, je enolično določena in $P_Y(x)$ je najboljša aproksimacija vektorja x z vektorji iz Y (razdalja je najmanjša)

$$\left(\|x-P_Y(x)\| = \min_{w\in Y}\|x-w\|\right)$$

Dokaz:

$Y \leq X \quad x\in X$

Denimo da y_1 y_2 sta pravokotni projekciji x na Y

$x-y_1,\ x-y_2 \in Y^\perp$

$(x-y_1)-(x-y_2) \in Y^\perp$ ← ker vek. prostor

$y_2-y_1 \in Y^\perp$

$y_2-y_1 \in Y$

$\langle y_2-y_1, y_2-y_1\rangle = 0 \iff$

$y_2-y_1=0 \quad y_2=y_1$

$w\in Y$

$$x-w = \underbrace{x-P_Y(x)}_{\in Y^\perp} + \underbrace{P_Y(x)-w}_{\in Y}$$

pitagorov izrek:

$$\|x-w\|^2 = \|x-P_Y(x)\|^2 + \|P_Y(x)-w\|^2 \geq$$
$$\|x-P_Y(x)\|^2$$

■

Zgled:

$Y = C[a,b]$ $\qquad X = L^2[a,b]$

$f \in X - Y$ nima pravokotne projekcije

Tak f nima najboljše aproksimacije z zveznimi funkcijami

Komentar: hilbertov + zaprt podprostor ima projekcije

Opombe:

1) $P_Y^2 = P_Y$

2) $x = \underbrace{x - P_Y(x)} + \underline{P_Y(x)}$

$\|x\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x)\|^2$

$\|x\| \geq \|P_Y(x)\|$

3) Če je P_Y definirana na celem X, potem je P_Y linearen in zvezen

$\|P_Y x_1 - P_Y x_2\| = \|P_Y(x_1 - x_2)\| \leq \|x_1 - x_2\|$

P_Y je zvezna

linearnost:

$\lambda x - P_Y(\lambda x)$

Poglejmo:

$\lambda x - \lambda P_Y(x) = \underbrace{\lambda\overbrace{(x - P_Y(x))}^{\in Y^\perp}}_{\in Y^\perp}$

$\Rightarrow \lambda P_Y(x) \in Y$

$\lambda x - \underbrace{\lambda P_Y(x)} \in Y^\perp$

~~zar~~ To je pravokotna projekcija λx

$x_1, x_2 \in X$

$P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$

$(x_1 + x_2) - \underbrace{(P_Y(x_1) + P_Y(x_2))}_{\in Y} = \underbrace{x_1 - P_Y(x_1) + x_2 - P_Y(x_2)}_{\in Y^\perp}$

$\Rightarrow P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$

zaradi: enoličnosti projekcije ∎

~~Če P_Y definiran~~

Če je P_Y definiran na X je

Y zaprt prostor

Dokaz:

$\{y_j\} \subseteq Y \quad \lim_{j\to\infty} y_j = y_0$

$P_Y(y_j) = y_j$

$$\underset{\underset{y_0}{\shortparallel}}{\lim y_j} = \lim P_Y(y_j) \overset{\text{zveznost}}{=} P_Y(y_0)$$

Če ima X pravokotno projekcijo na Y ima tudi pravokotno projekcijo na $Y^\perp$

$x;\ P_Y(x)$

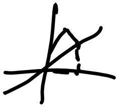
$P_{Y^\perp}(x) = x - P_Y(x) \in Y^\perp$

$x - (x - P_Y) \in (Y^\perp)^\perp$

$$\| $$

$P_Y(x) \in Y \subseteq (Y^\perp)^\perp$

Trditev: Naj bo $Y \leq X$ končno dimen podprostor z ortonormirano bazo $e_1, \ldots, e_n$. $\langle e_i, e_j \rangle = \delta_{ij}$



Naj bo $x \in X$. Tedaj

$$P_Y(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$$

Opomba: Vsak končno dimenzionalni podprostor ima pravokotno projekcijo definirano na X

In tudi vsi tisti končne kodimenzije

Dokaz: $P_Y(x) = \sum_{1}^{n} \langle x, e_i \rangle e_i \in Y$

Poglejmo še

$$x - \sum_{1}^{n} \langle x, e_j \rangle e_j \in Y^{\perp}$$

$$\langle x - \sum_{1}^{n} \langle x, e_j \rangle e_j , e_i \rangle =$$

$$\langle x, e_i \rangle - \sum_{1}^{n} \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$(X, \langle\ \rangle)$

Sistem vektorjev

$(e_j)_{j=1}^{\infty}$

je ortogonalen sistem (OS), če

$\forall i \neq j \ \langle e_i, e_j \rangle = 0$

Tak sistem je ortonormiran (ONS)

$\forall i,j \ \langle e_i, e_j \rangle = \delta_{ij}$

Trditev: $(X, \langle\ \rangle)$ Naj bo $(e_j)_j$ ONS

Naj bo $x \in X$. Tedaj je $\sum |\langle x, e_j \rangle|^2 \leq \|x\|^2$

(Besselova neenakost)

Opomba: $\langle x, e_j \rangle$ so Fourierovi koeficienti x po ONS $(e_j)_j$

Posledica:

$$\lim_{j \to \infty} \langle x, e_j \rangle = 0$$

Dokaz: $Y_n = \mathcal{L}(\{e_1, \dots, e_n\})$

$x \in X$

$$\exists P_Y(x) = \sum_1^n \langle x, e_j \rangle e_j$$

projekcija je vedno krajša od x

$$\|P_Y(x)\|^2 = \sum_1^n \|\langle x, e_j \rangle e_j\|^2 = \sum_1^n |\langle x, e_j \rangle| \leq \|x\|^2$$

$$\Rightarrow \lim_{n \to \infty} \sum_1^n \|\langle x, e_j \rangle\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

Trditev: Naj bo $(c_j)_{j=1}^{\infty}$ zaporedje števil (ali $\mathbb{R}$ ali $\mathbb{C}$) za katero velja

$$\sum_1^{\infty} |c_j|^2 < \infty$$

Naj bo $(X, \langle\,\rangle)$ Hilbertov prostor $(e_j)_1^{\infty}$ ONS

Tedaj $\exists x \in X$, za katerega velja $c_j = \langle x, e_j \rangle$ za $\forall j$.

$$x = \sum_1^{\infty} c_j e_j = \lim \sum_1^{N} c_j e_j$$

$(X, \langle\ \rangle)$ hilbertov $\quad e_j$ ONB

$x \in X \rightsquigarrow (\langle x, e_j \rangle)_j \quad \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$

$\Rightarrow \exists \tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$

Ali je $\tilde{x} = x$

F

Zemuda

20.2

Zgled (pred prejšnjim dokazom)

<u>Modelni hilbertov prostor</u> ℓ^2

Prostor zaporedij

$$\ell^2 = \{(a_j)_j \,;\, a_j \in \mathbb{R} \,,\, \sum_1^\infty |a_j|^2 < \infty\}$$

$$\langle (a_j),(b_j)\rangle = \sum a_j b_j \quad \text{nad } \mathbb{R}$$

$$\sum a_j \bar{b}_j \quad \text{nad } \mathbb{C}$$

$$\|a_j\|^2 = \sum |a_j|^2$$

$(X, \langle\,\rangle)$ hilbertov $\quad (e_j)$ KONS

$$x \longmapsto (\langle x, e_j\rangle)_j \in \ell^2$$

$$e_j = (0, \ldots, 0\, 1\, 0 \ldots 0)$$

Dokaz:
$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Rightarrow 6 \Rightarrow 5$

$1) \Rightarrow 2)$

$x, y \in X \quad x = \sum_1^\infty \langle x, e_j \rangle e_j$

$\langle x, y \rangle = \langle \sum_1^\infty \langle x, e_j \rangle e_j, y \rangle =$

$= \sum_1^\infty \langle x, e_j \rangle \langle e_j, y \rangle$

$2) \Rightarrow 3)$

$x = y \quad \langle x, y \rangle = \|x\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2$

$3) \Rightarrow 4)$

če bi bil ONS vsebovan v strogo večjem ONS

potem $\exists e_0 \perp e_j$ za $\forall j$ $\|e_0\| = 1$

Vstavimo v parcevalovo enakost

$\|e_0\|^2 = 1^2 = \sum_1^\infty |\langle e_0, e_j \rangle|^2 = \sum_1^\infty 0 = 0$

$4) \Rightarrow 5)$

$x \perp e_j$ za $\forall j$

če $x \neq 0$ tvorimo $e_0 = \frac{x}{\|x\|}$

$(e_0, (e_j)_j)$ je strogo večji KONS, ker ne more biti $\times$

$5) \Rightarrow 1)$

$x \in X \qquad \langle x, e_j \rangle_j$

$\tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$ ali sta enaka?

$\vec{v} = x - \tilde{x} = x - \sum_1^\infty \langle x, e_j \rangle e_j$

$\langle v, e_i \rangle = \langle x, e_i \rangle - \sum_1^\infty \langle x, e_j \rangle \langle e_j, e_i \rangle =$

$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$

$\vec{v} = 0 \Rightarrow x = \tilde{x}$

$1) \Rightarrow 6) \quad \forall x = \sum_1^\infty \langle x, e_j \rangle e_j = \lim \sum_1^N \langle x, e_j \rangle e_j$

$\forall \varepsilon > 0 . \exists N_0 \in \mathbb{N} . \|x - \sum_1^{N_0} \langle x, e_j \rangle e_j\| < \varepsilon$

$6) \Rightarrow 5)$

Naj bo $x \perp e_j$ za $\forall j$

$x = 0$

$\varepsilon > 0$

$\exists$ končna linearna kombinacija $\sum_1^N \lambda_j e_j$

daje $\|x - \sum_1^N \lambda_j e_j\| < \varepsilon$

$\|x\|^2 = \langle x, x \rangle = \langle x - \sum_1^N \lambda_j e_j, x \rangle \leq$

ker pravokotnost

$\leq \|x - \sum_1^N \lambda_j e_j\| \cdot \|x\| < \varepsilon \|x\|$

1) $x = 0 \Rightarrow \checkmark$

2) $x \neq 0 \Rightarrow \|x\| < \varepsilon$ za $\forall \varepsilon > 0 \Rightarrow x = 0$ $\times$

Osredotočimo se na $L^2(-\pi,\pi)$

ONS: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \ldots \right.$

$\ldots \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx), \ldots$

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} f^2(x)}$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \qquad \|f_0\|_2 = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi}} = 1$$

$$\sqrt{\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin^2(nx)\, dx} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\, dx} = 1$$

Opomba: nad $\mathbb{C}$: $\frac{1}{2\pi} e^{inx}$ $n \in \mathbb{Z}$ vzamemo za ONS

$n \neq m$

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{inx} e^{imx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\, dx$$

$$= \frac{1}{2\pi(n-m)} e^{i(n-m)x}\Big|_{-\pi}^{\pi} = 0$$

$f: (-\pi, \pi) \Rightarrow \mathbb{R}$ periodična funkcija s periodo 2π

Klasični furierovi koeficienti

$f \in L^2(-\pi, \pi)$ (Recimo da imamo konos)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n \in \{0, 1, \dots\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \langle f, \frac{1}{\pi}\cos(nx)\rangle =$$

$$= \frac{1}{\sqrt{\pi}}\langle f, \frac{1}{\sqrt{\pi}}\cos(nx)\rangle = \sqrt{\pi}\,a_n$$

$$\langle f, \frac{1}{\pi}\sin(nx)\rangle = \sqrt{\pi}\,b_n$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot 1\,dx = \langle f, \frac{1}{\pi}\rangle = \frac{\sqrt{2}}{\sqrt{\pi}}\langle f, \frac{1}{\sqrt{2\pi}}\rangle =$$

$$\langle f, \frac{1}{\sqrt{2\pi}}\rangle = \frac{\sqrt{\pi}}{\sqrt{2}}\,a_0$$

Če je $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x \ldots \right\}$ KONS

potem velja parcevalova enakost

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2(x)\,dx = \frac{\pi}{2} a_n + \sum \pi\left(|a_n|^2 + |b_n|^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$

Posledica:

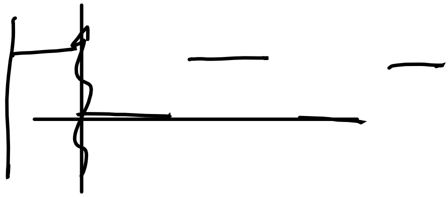
ker je $\left\{ \frac{1}{\sqrt{2\pi}} \ldots \right\}$ ONS velja

$$\lim_{n\to\infty} a_n = 0 \qquad \lim_{n\to\infty} b_n = 0$$

(Riemann-Lebegueova lema)

Zgled

$$f(x) = \begin{cases} 1; & 0 \leq x \leq \pi \\ 0; & -\pi < x < 0 \end{cases}$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx =$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx$$

$a_0 = 1 \quad n > 0$

$$a_n = \frac{1}{\pi} \cdot \frac{1}{n} \sin(nx) \Big|_0^{\pi}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx)\, dx = -\frac{1}{\pi n} \cos(nx) \Big|_0^{\pi}$$

$$= \frac{1}{n\pi} \left(1 - (-1)^n\right)$$

$$b_{2k} = 0$$

$$b_{2k+1} = \frac{2}{\pi(2k+1)}$$

KONS:

$$f(x) \overset{L^2}{=} \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x)$$

$=0$

Parsevalova enakost

$$\frac{1}{\pi} \pi = 1 = \frac{1}{2} + \sum_{0}^{\infty} \frac{4}{\pi^2} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\frac{3}{4} S = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}$$

Izrek:

Naj bo f odsekoma zvezna in odsekoma odvedljiva periodična funkcija s periodo 2π

1) Na vsakem intervalu dolžine 2π ima največ končno mnogo točk nezveznosti in v vsaki točki obstajata leva in desna limita

$$x_0 \in \mathbb{R}. \ \exists \lim_{x \uparrow x_0} f(x) = \underbrace{f(x_0 - 0) = f(x_0^-)}_{\text{oznaki}}$$

$$\exists. \lim_{x \downarrow x_0} f(x) = f(x_0 + 0) = f(x_0 +)$$

2) V vsaki točki obstajata levi in desni odvod

$$\lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0^+)}{h}$$

$$\lim_{h \uparrow 0} \frac{f(x_0 + h) - f(x_0^-)}{h} = \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0^-)}{(-h)}$$

Tedaj za $\forall x \in \mathbb{R}$. velja $\dfrac{f(x+) + f(x-)}{2} =$

$$= \frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

Pamožne trditeve

1) Naj bo $f: \mathbb{R} \to \mathbb{R}$ periodična s periodo p odsekoma zvezna.
Tedaj je za $\forall a \in \mathbb{R}$

$$\int_a^{a+p} f(x)\,dx = \int_a^p f(x)\,dx$$

Dokaz:

$$\int_a^{a+p} f(x)\,dx = \int_a^p f(x)\,dx + \int_p^{a+p} f(x)\,dx$$

$$x = t+p$$

$$= \int_a^p f(x)\,dx + \int_0^a \underbrace{f(t+p)}_{= f(t)}\,dt =$$

$$= \int_a^p f(x)\,dx + \int_0^a f(t)\,dt = \int_a^p f(x)\,dx$$

2)

$$\frac{1}{2} + \sum_{1}^{n} \cos(kx) = \frac{1}{2} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = D_n(x)$$

(Dirichletova jedro)

Dokaz:

$$\frac{1}{2} + \sum_{1}^{n} \cos(kx) = \frac{1}{2} + \sum_{1}^{n} \frac{e^{ikx} + e^{-ikx}}{2} =$$

$$= \frac{1}{2} \sum_{j=-n}^{n} e^{ijx} = \frac{1}{2} e^{-inx}(1+e^{ix}+e^{2ix}+\ldots+e^{2nix})$$

$$= \frac{1}{2} e^{-inx} \frac{1-e^{(2n+1)ix}}{1-e^{ix}} =$$

$$= \frac{1}{2} \frac{e^{(n+1)ix}-e^{-inx}}{e^{ix}-1} = \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x}-e^{-i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}$$

$$= \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}$$

3)

$$\int_{-\pi}^{\pi} D_n(x)\,dx = \pi$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$$

4) $D_n(x)$ je soda funkcija, 2π periodična, gladka.

5) $\frac{1}{2}\frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} =$

$$= \frac{1}{2}\frac{\sin(nx)\cos\frac{x}{2} + \cos(nx)\sin(\frac{x}{2})}{\sin\frac{x}{2}} =$$

$$= \frac{1}{2}\left(\sin(nx)\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \cos(nx)\right)$$

Opombe:

f je lihe na $(-\pi,\pi)$

$a_n = 0$

f razvijamo samo po $\{\sin(nx)\}$

f soda na $(-\pi,\pi) \Rightarrow b_n = 0$

f razvijemo le po $\{1, \cos(nx)\}$

f iz $[0,\pi]$ lahko liho razširimo na $(-\pi,\pi)$

$$x \mapsto \begin{cases} f(x) & x>0 \\ -f(-x); & x<0 \end{cases}$$

Tako funkcijo lahko razvijamo le po $\sin(nx)$

$$f(x) = \sum_1^\infty b_n \sin(nx)$$

novi b_n

$$\int_0^\pi f(x)\sin(nx)dx = b_n\int_0^\pi \frac{1-\cos(2nx)}{2} = \frac{\pi}{2}b_n$$

Če f razširimo kot soda funkcija na $(-\pi,\pi)$

$$x \longmapsto \begin{cases} f(x) & x>0 \\ f(-x) & x<0 \end{cases}$$

Tako funkcijo razvijamo po $\{1, \cos(nx)\}$

Opombe 2:

$f \in C^k(\mathbb{R})$ 2π periodična $k \in \mathbb{N}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{u} \underbrace{\cos(nx)\,dx}_{dv} =$$

$$= \frac{1}{\pi}\left(\underbrace{\frac{1}{n}\sin(nx) f(x)\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx \right)$$

$=$ ... naredimo to k krat ... $=$

$$a_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx$$

$$a_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx$$

$\vdots$

$$a_n = \frac{\pm 1}{\quad}\int_{-\pi}^{} f^{(k)}(x) \underbrace{\begin{matrix}\cos(nx)\\ \sin(nx)\end{matrix}}_{\text{neki od tega}} \overset{\text{omejena}}{} dx$$

$$a_n, b_n = \mathcal{O}\left(\frac{1}{n^2}\right)$$

velikostni red

večkrat kot je odvedljiva f (kot periodična funkcija) hitreje gredo furierovi koeficienti proti 0

Dokaz

$f \in L^2(-\pi, \pi)$

$$\sqrt{\int_{-\pi}^{\pi} |f|^2 dx} = \|f\|_2$$

Klasični: furijejevi koeficienti:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad n = 0, 1, 2 \dots)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \quad n = 1, 2 \dots$$

$$f(x) \overset{L_2}{=} \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Parsevalova enakost

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx = \frac{|a_0|^2}{2} + \sum_{1}^{\infty} (|a_n|^2 + |b_n|^2)$$

f periodična s periodo 2π in odsekoma zvezna

- $\forall x_0 \in \mathbb{R}.\ \exists f(x_0+), f(x_0-)$ limiti in
- odsekoma odvedljiva
- dostajata levi in desni odvod

Potem: $\forall x_0$: $\frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) =$

$$= \frac{f(x_0+) + f(x_0-)}{2}$$

Vemo:

- f periodična s periodo p
- $\forall a \in \mathbb{R}.\ \int_a^{a+p} f(x)\, dx = \int_0^p f(x)\, dx$
- $\frac{1}{2} + \sum_{1}^{N} \cos(kx) = D_N(x) = \frac{1}{2} \frac{\sin(N + \frac{1}{2})x}{\sin(\frac{1}{2}x)}$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x)\, dx = 1 \quad \text{soda}$$

$$\| $$

$$\frac{2}{\pi} \int_0^{\pi} D_n(x)\, dx$$

Dokaz >>

Dokaz:

$$a_n\cos(nx_0)+b_n\sin(nx_0) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt\cdot\cos(nx_0) +$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt\ \sin(nx_0) =$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\big(\cos(nt)\cos(nx_0)+\sin(nt)\sin(nx_0)\big)\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n(t-x_0))\,dt$$

(opomba) $\int_{-\pi}^{\pi} f(t)g(x_0-t)\,dt$ ..... konvolucija

$$t-x_0=y \qquad t=x_0+y$$

$$=\frac{1}{\pi}\int_{-\pi-x_0}^{\pi-x_0} f(x_0+y)\cos(ny)\,dy =$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\cos(ny)\,dy$$

$$\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x_0+y)\,dy=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\cdot\frac{1}{2}\,dy$$

$$\frac{a_0}{2}+\sum_{1}^{n} a_k\cos(kx_0)+b_k\sin(kx_0) = S_n(x_0)$$

$$S_n(x_0)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)\left(\tfrac{1}{2}+\cos y+\dots\cos(ny)\right)dx=$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+y)D_n(y)\,dy =$$

$$=\frac{1}{\pi}\Big(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy+\int_{-\pi}^{0} f(x_0+y)D_n(y)\,dy$$

$y\mapsto -y$

$$=\frac{1}{\pi}\Big(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy+\int_{0}^{\pi} f(x_0-y)D_n(y)\,dy$$

soda funkcija

$$\xrightarrow{n\to\infty} \frac{f(x_0+)+f(x_0-)}{2}$$

$$\frac{1}{\pi}\int_{0}^{\pi} \big(f(x_0+y)+f(x_0-y)\big)D_n(y)\,dy - \frac{f(x_0+)+f(x_0-)}{2}$$

$$\frac{2}{\pi}\cdot\int_{0}^{\pi} D_n(y)\,dy = 1$$

$$=\frac{1}{\pi}\int_{0}^{\pi} \big(f(x_0+y)-f(x_0+)\big)D_n(y)+\big(f(x_0-y)-f(x_0-)\big)D_n(y)\,dy$$

Pogledamo sumando

$$\frac{1}{2}\int_{0}^{\pi} \big(f(x_0+y)-f(x_0^+)\big)\frac{\sin(n+\frac{1}{2})y}{\sin(\frac{y}{2})}\,dy =$$

$$=\frac{1}{2}\int_{0}^{\pi} \big(f(x_0+y)-f(x_0)\big)\frac{\cos(\frac{y}{2})}{\sin(\frac{y}{2})}\sin(ny)+\cos(ny)\,dy$$

Pogledamo sumand

$$\int_{0}^{\pi} f(x_0+y)-f(x_0^+)\cos(ny)\,dy \xrightarrow{n\to\infty} 0$$

$$F(y)=\begin{cases} f(x_0+y)-f(x_0^+); & 0\le y\le\pi\\ 0; & -\pi<y<0\end{cases}$$

odsekoma zvezna

$$=\int_{-\pi}^{\pi} F(y)\cos(ny)\,dy \xrightarrow[n\to\infty]{\text{Rieman-Lebesguova lema}} 0$$

Še

$$\int_{0}^{\pi} f(x_0+y)-f(x_0^+)\big)\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}\sin(ny)\,dy$$

$$=\int_{0}^{\pi} \frac{f(x_0+y)-f(x_0+)}{y}\,\frac{y}{\sin\frac{y}{2}}\cos(\tfrac{y}{2})\sin(ny)\,dy$$

$$G(y)=\begin{cases}\dfrac{f(x_0+y)-f(x_0+)}{y}\,\dfrac{y}{\sin\frac{y}{2}}\cdot\cos\frac{y}{2}; & 0<y\le\pi\\ 0; & -\pi<y<0\end{cases}$$

$\int_{-\pi}^{\pi} G(y)\sin(ny)\,dy$ za $y\neq 0$ je f odsekoma zvezna

za $y=0$ pa obstaja limita

$$\lim_{y\nearrow 0} G(y)=0$$

$$\lim_{y\searrow 0} G(y)=2\lim_{y\searrow 0}\frac{f(x_0+y)-f(x_0+)}{y}$$

obstaja po predpostavki

Riemann-Lebesguova lema: $\lim_{n\to\infty}\int_{-\pi}^{\pi} G(y)\sin(ny)\,dy=0$

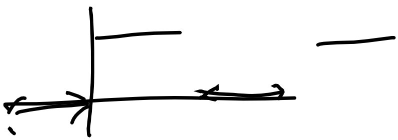
Za drugi sumand uporabimo podobne argumente

■

Zgled:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi < x < 0 \end{cases}$$

Furierova vrsta: $\frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x$



$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2}$$

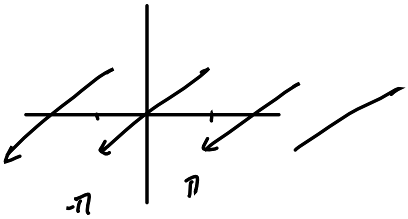
$x = \frac{\pi}{2}$: $\quad 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{1}{2k+1} \sin(2k+1)\frac{\pi}{2}$

+1 ali pa −1

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

zgled

$f(x)=x$



$a_n = 0$ za $\forall n$

ker je na $(-\pi,\pi)$ funkcija liha

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$u=x \quad dv=\sin(nx) \quad n\geq 1$

$du=dx \quad v=-\frac{1}{n}\cos(nx)$

$$b_n = \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx)\right)\Big|_{-\pi}^{\pi} + \underbrace{\frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx}_{=0}$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + \frac{(-\pi)}{n}\cos(n\pi)\right) =$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$x \overset{L^2(-\pi,\pi)}{=} \sum_{1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx)$$

Po točkah:

$x=0: \quad 0=0$

$x=\pi: \quad \frac{\pi+(-\pi)}{2} = 0$ ✓

$x=\frac{\pi}{2}: \quad \frac{\pi}{2} = 2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)$

Parsevalova enakost

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx = \sum_{1}^{\infty}\frac{4}{n^2} = 4\sum_{1}^{\infty}\frac{1}{n^2}$$

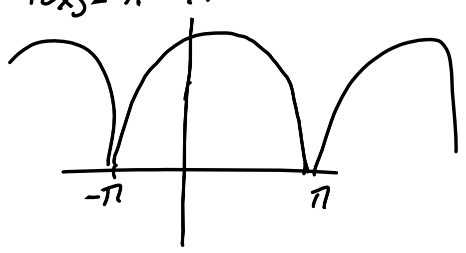
na kvadrat

$$\frac{1}{\pi}\left(\frac{1}{3}x^3\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi}\pi^3\cdot\frac{1}{3} = \frac{2}{3}\pi^2 = 4\sum_{1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{1}^{\infty}\frac{1}{n^2}$$

Zgled.

$f(x) = \pi^2 - x^2$



soda, nenegativna zvezna, v vsaki točki ima levi in desni odvod

sodost $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)dx = \frac{2}{\pi}(\pi^3 - \frac{1}{3}\pi^3) =$$

$$= \frac{4}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{(\pi^2 - x^2)}_{u}\underbrace{\cos(nx)dx}_{dv} =$$

$$= \frac{1}{\pi}(\pi^2 - x^2)\frac{1}{n}\sin(x)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} 2x\sin(nx)dx =$$

$$= \frac{2}{\pi n}\int_{-\pi}^{\pi} \underbrace{x}_{u} \underbrace{\sin(nx)\,dx}_{dv} =$$

$$= \frac{2}{\pi n}\left(x(-\frac{1}{n}\cos(nx))\Big|_{-\pi}^{\pi} - \overbrace{\int_{-\pi}^{\pi}\cos(nx)dx}^{0}\right) =$$

$$= \frac{2}{\pi n^2}\left(\pi(-1)^{n+1} + (-1)^{n+1}\pi\right) = \frac{4}{n^2}(-1)^{n+1}$$

$$f(x) = \frac{2}{3}\pi^2 + \sum_{1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nx) \quad \forall x \in \mathbb{R}$$

$x = 0 \Rightarrow$

$$f(0) = \pi^2 = \frac{2}{3}\pi^2 + \sum_{1}^{\infty}\frac{4}{n^2}(-1)^{n+1}$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$x = \pi \Rightarrow$

$$0 = \frac{2}{3}\pi^2 + \sum_{1}^{\infty}\frac{4}{n^2}\overbrace{(-1)^{n+1}(-1)^n}^{-1}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

kons:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)^2 dx = \frac{1}{2}\frac{16}{9}\pi^4 + \sum_{1}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{\pi}\int_{0}^{\pi}(\pi^4 - 2x^2\pi^2 + x^4)dx = \frac{8\pi^4}{9} + 16\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}(\pi^5 - \frac{2\pi^5}{3} + \frac{1}{5}x^5) = \frac{8\pi^4}{9} + 16S$$

$$\frac{15 - 10 + 3}{15}\pi^4 = \frac{4\pi^4}{9} + 8S$$

$$\| \qquad \frac{8\pi^4}{15}$$

$$8S = \frac{8\pi^4}{45} - \frac{4\pi^4}{9}$$

$$2S = \frac{2\pi^4}{15} - \frac{\pi^4}{9} = \frac{6\pi^4}{45} - \frac{5\pi^4}{45} = \frac{\pi^4}{45}$$

$$S = \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$a_1, a_2, \ldots \qquad \lim_{n\to\infty} a_n = a$

$1, -1, 1, -1$

$\frac{1}{1}, \frac{1-1}{2}, \frac{1-1+1}{3} \to 0$

$a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}$

konvergira $\Rightarrow$ konvergira u poprečju

$$S_n^{(x_0)} = \int_{-\pi}^{\pi} f(x_0+y)\, D_n(y)\, dy$$

$$\frac{S_0(x) + S_1(x) + \dots + S_{n-1}(x)}{n} = \sigma_n(x) \xrightarrow[n\to\infty]{} f$$

cesárove delne vsote

če je f zvezna je konvergenca enakomerna

Fejérjevo jedro: $F_N(x) = \frac{1}{N} \sum_{0}^{N-1} D_n(x)$

Trditev:

1) $F_N(x) = \frac{1}{2N} \left(\frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})} \right)^2$

2) F_N je soda

3) $F_N(x) \geq 0$ za $\forall x$

4) $\frac{1}{\pi} \int_{-\pi}^{\pi} F_N(x)\,dx = 1$

5) $\forall a.\ 0 < a < \pi.\ \lim_{N \to \infty} F_N(x) = 0$

enakomerno na $a \leq |x| \leq \pi$

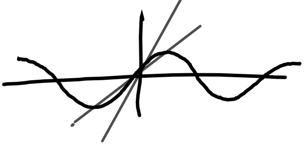
Dokaz:

1) $\Rightarrow$ 2) očitno

1) $\Rightarrow$ 3) očitno

Iz definicije $\Rightarrow$ 4) $\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x)\,dx = 1$

1) $\Rightarrow$ 5)



$\forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ velja $\frac{2}{\pi}|y| \leq |\sin y|$

$0 < a \leq |x| \leq \pi$

$0 < \frac{a}{2} \leq |\frac{x}{2}| \leq \frac{\pi}{2}$

$\frac{2}{\pi} \cdot \frac{|x|}{2} \leq |\sin(\frac{x}{2})|$

$\frac{1}{|\sin\frac{x}{2}|} \leq \frac{\pi}{|x|} \leq \frac{\pi}{a}$

$F_N(x) = \frac{1}{2N} \frac{|\sin(\frac{Nx}{2})|^2}{|\sin(\frac{x}{2})|^2} \leq \frac{1}{2N} \cdot \frac{\pi^2}{a^2}$

$\lim_{n \to \infty} F_n(x) = 0$ enakomerno na $a \leq |x| \leq \pi$

1) $F_N(x) = \frac{1}{N} \sum_{0}^{N-1} D_n(x) = \frac{1}{N} \sum_{0}^{N-1} \frac{1}{2} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} =$

$= \frac{1}{2N} \frac{1}{\sin^2(\frac{x}{2})} \sum_{0}^{N-1} \sin(n+\frac{1}{2})x \cdot \sin(\frac{x}{2}) =$

$= \frac{1}{2N \sin^2(\frac{x}{2})} \frac{1}{2} \underbrace{\sum_{0}^{N-1} (\cos(nx) - \cos(n+1)x)}_{(1-\cos x) + (\cos x - \cos 2x) + (\cos 2x - \cos 3x) + \dots}$

$= 1 - \cos(Nx)$

$= \frac{1 - \cos(Nx)}{4N \sin^2(\frac{x}{2})} = \frac{1}{2N} \frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}$

Izrek:
Naj bo f zvezna 2π periodična funkcija
Potem Cezárove delne vsote

$$\sigma_N(x) = \frac{1}{N}\left(S_0(x)+\ldots+S_{N-1}(x)\right)$$

konvergirajo k f enakomerno na
$[-\pi,\pi]$ oziroma na $\mathbb{R}$

$$S_N(x_0) = a_0 + \sum_{1}^{N}\left(a_k\cos(kx) + b_k\sin(kx)\right)$$

Trigonometrični polinom... končne kombinacije sinusov in kosinusov

Dokaz:

$$\sigma_N(x) = \frac{1}{N}\left(S_0(x)+\ldots+S_{N-1}(x)\right)$$

$$S(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\, D_N(y)\, dy$$

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y)\cdot F_N(y)\, dy \xrightarrow{\text{enakomerno}} f(x)$$

Naj bo $\varepsilon > 0$. ker je f zvezna in periodična je enakomerno zvezna

$$\exists \delta > 0 .\ \forall y < \delta\ |f(x+y) - f(x)| < \frac{\varepsilon}{2}$$
$\forall x \in \mathbb{R}$

$$|\sigma_N(x) - f(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y)F_N(y)dy - f(x)\frac{1}{\pi}\int_{-\pi}^{\pi}F_N(y)dy\right|$$

$$= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x+y)-f(x))F_N(y)dy\right| \leq$$

$F_N(y) \geq 0$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+y)-f(x)|\, F_N(y)dy =$$

$$= \underbrace{\frac{1}{\pi}\int_{-\delta}^{\delta}|f(x+y)-f(x)|F_N(y)dy}_{< \frac{1}{\pi}\int_{-\delta}^{\delta}F_N(y)dy\cdot\frac{\varepsilon}{2} < \frac{\varepsilon}{2}} + \int_{\delta\leq|y|\leq\pi}|f(x+y)-f(x)|F_N(y)dy$$

f je zvezna in periodična torej je omejena

$$\exists M \in \mathbb{R}\quad |f(z)| \leq M\ \forall z \in \mathbb{R}$$
$$\Rightarrow f(x+y) - f(x) \leq 2M \quad \text{za } \forall x \text{ in } \forall y \in \mathbb{R}$$

$$\frac{1}{\pi}\int_{\delta<|y|<\pi}|f(x+y)-f(x)|\, F(y)dy \leq 2M\frac{1}{\pi}\int_{\delta<|y|\leq\pi}F_N(y)dy$$

Vemo da je limita $\lim_{N\to\infty} F_N(y) = 0$
enakomerno na $\delta \leq |y| \leq \pi$

$$\exists N_0 .\ \forall N \geq N_0 .\ |F_N(y)| \leq \frac{\varepsilon}{2M\cdot 4} = \frac{\varepsilon}{8M}$$

$$\frac{2M}{\pi}\int_{\delta<|y|\leq\pi}F_N(y)dy \leq \frac{2M}{\pi}\cdot\frac{\varepsilon}{8M}\cdot 2\pi = \frac{\varepsilon}{2}$$

Za $N \geq N_0$ je

$$|\sigma_N(x) - f(x)| < \varepsilon \quad \text{za } \forall x \in \mathbb{R}$$

Izrek:

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx);\ n\in\mathbb{N}\right\}$$

je KONS v $L^2(-\pi,\pi)$

Dokaz:

$L^2(-\pi,\pi)$ je napolnitev $C([-\pi,\pi])$ v

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

ONS je KONS $\Leftrightarrow$ končne lineare kombinacije vektorjev iz ONS goste v prostoru

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OPOMBA: KONČNE LINEARE KOMBINACIJE $\sin(nx)$ $\cos(nx)$ $n\in\mathbb{N}_0$ so trigonometrični polinomi.

Ali so trigonometrični polinomi gosti v $L^2(-\pi,\pi)$?

1) $C(-\pi,\pi)$ so goste v $L^2(-\pi,\pi)$ (privzamemo)

$f\in L^2(-\pi,\pi)$. $\varepsilon>0$. $\exists\, \tilde f\in C[-\pi,\pi]$. $\|f-\tilde f\|<\frac{\varepsilon}{2}$

2) Ali so trigonometrični polinomi gosti v $(C[-\pi,\pi], d_2)$

če da:

$\exists T(x)$ trigonometrični polinom

$$T(x)=\lambda_0+\sum_{1}^{N}\lambda_n\cos(nx)+\mu_n\sin(nx)$$

$$\|\tilde f-T\|_2<\frac{\varepsilon}{2} \Rightarrow \|f-T\|_2<\varepsilon$$

OPOMBA:

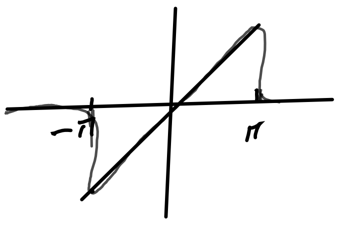
$$g_n\in L^2[-\pi,\pi] \xrightarrow[\text{na } [-\pi,\pi]]{\text{enakomerno}} f$$

$$\overset{?}{\Rightarrow} \lim_{n\to\infty} g_n = f \text{ v } d_2$$

$$d_2(f,g_n)^2=\int_{-\pi}^{\pi}(f-g_n)^2 \xrightarrow{n\to\infty} 0$$

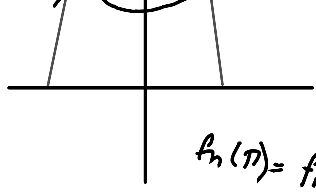
Vemo: f zvezna 2π periodična $\Rightarrow$ jo lahko poljubno dobro enakomerno aproksimiramo s trigonometričnimi polinomi:

$$\|f-\sigma_N\|_\infty<\varepsilon$$



Ni: zvezne 2π periodične funkcije

$$f_n(x)=\begin{cases} f(x) \\ -n\left(f(\pi-\frac{1}{n})\right)(x-\pi); & \pi-\frac{1}{n}<x\le\pi \\ nf(-\pi+\frac{1}{n})(x+\pi) & -\pi\le x<-\pi+\frac{1}{n}\end{cases}$$



$f_n(\pi)=f_n(-\pi)=0$

zvezna 2π periodična funkcija

$$\|f-f_n\|_2^2=\int_{-\pi}^{\pi}(f_n(x)-f(x))^2\,dx\le\frac{2}{n}(2M)^2\xrightarrow{n\to\infty}0$$

Izrek: (Weierstrassov)

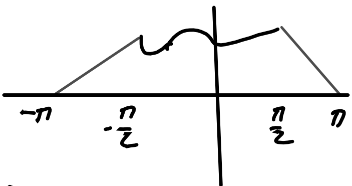
Naj bo f zvezna na $[a,b]$

Naj bo $\varepsilon > 0$. Potem $\exists$ polinom p da je
$\|f-p\|_\infty < \varepsilon$

Dokaz: Dovolj je opazovati $a=-\frac{\pi}{2}$ $b=\frac{\pi}{2}$

f zvezna na $[-\frac{\pi}{2}, \frac{\pi}{2}]$

f zvezno razširimo na $[-\pi, \pi]$



$$\tilde{f} = \begin{cases} f(x) & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -\frac{2}{\pi} f(\frac{\pi}{2})(x-\pi) & \frac{\pi}{2} < x \leq \pi \\ \frac{2}{\pi} f(-\frac{\pi}{2})(x+\pi) & -\pi \leq x < -\frac{\pi}{2} \end{cases}$$

$\tilde{f}$ je zvezna 2π periodična.

Za $\varepsilon > 0$ $\exists$ T trigonometrični polinom, da je $\|\tilde{f} - T\|_\infty < \frac{\varepsilon}{2}$

$\cos(nx), \sin(nx)$ jih je končno

Aproksimiramo s Taylorjevimi polinomi enakomerno.

Vektorska analiza
(v $\mathbb{R}^2$ ali $\mathbb{R}^3$)

$D^{odp} \subseteq \mathbb{R}^3$ $\quad u: D \to \mathbb{R}$ zvezna funkcija

jo imenujemo <u>skalarno polje</u>

vsaki točki v D priredi skalar

$D \subseteq \mathbb{R}^3$ $\quad \vec{R}: D \longrightarrow \mathbb{R}^3$ zvezna

<u>vektorsko polje</u> vsaki točki priredi

vektor $\vec{R}(t)$

$\mathbb{R}^3$

$u(x,y,z) = x$ v bazi $\vec{e}_1, \vec{e}_2, \vec{e}_3$ $(\vec{i}, \vec{j}, \vec{k})$

Baza: $\vec{p} = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_3$

$\vec{q} = -\frac{1}{\sqrt{3}} e_1 + \frac{1}{\sqrt{3}} e_2 + \frac{1}{\sqrt{3}} e_3$

$\vec{r} = \frac{1}{\sqrt{6}} e_1 + \frac{2}{\sqrt{6}} e_2 - \frac{1}{\sqrt{6}} e_3$

Točka (x,y,z) ima v Bazi $(\vec{p}, \vec{q}, \vec{r})$

ONB

koordinate (α, β, γ)

$(\alpha, \beta, \gamma) \longmapsto \alpha$ ni ista funkcija

ONB je pozitivno orientirana, če je
$[\vec{p}, \vec{q}, \vec{r}] > 0 \ (\Leftrightarrow \vec{r} = \vec{p} \times \vec{q})$

oziroma negativno orientirana ko
$[\vec{p}, \vec{q}, \vec{r}] < 0 \ (\Leftrightarrow \vec{r} = -\vec{p} \times \vec{q}$

(oznake od zdej naprej za mešani produkt bo $(\vec{p}, \vec{q}, \vec{r})$ najbrž

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} =: U$$

(x, y, z) $(e_1 \ldots e_n)$

(α, β, γ) $(\vec{p}, \vec{q}, \vec{r})$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$U^T U = I$$

$$U^{-1} = U^T$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$u(x, y, z)$ v (e_1, e_2, e_3) koordinatah

$$\tilde{u}(\alpha, \beta, \gamma) = u\left(U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}\right)$$

Vektorska polja

$$\vec{R}(x,y,z) = \left(X(x,y,z),\ Y(x,y,z),\ Z(x,y,z)\right)$$

v (α,β,γ) koordinatah

$$\tilde{R}(\alpha,\beta,\gamma) = \left(U^T \circ \vec{R} \circ U\right)\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

$$\tilde{u} = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$\vec{R}(x,y,z) = (x+2y+3z,\ x+2y+3,\ x+2y+3z)$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

$$x = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$y = \frac{\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$

$$z = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{3}} - \frac{\gamma}{\sqrt{6}}$$

$$x+2y+3z = \frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma = c$$

To je $(R \circ U)\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = (c, c, c)$

$$U\left(R \circ U\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}\right) = \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}\right)c$$

$D^{odp} \subseteq \mathbb{R}^3$ ponovitev:

$u : D \to \mathbb{R} \quad C^1$

$u \longmapsto \vec{\nabla} u = \operatorname{grad} u = (u_x, u_y, u_z)$

$\vec{R} = (X, Y, Z)$

$\vec{R} : D \to \mathbb{R}^3$

$\vec{R} \longmapsto \operatorname{div} \vec{R} = \vec{\nabla} \cdot \vec{R} = X_x + Y_y + Z_z$

$\vec{R} \longmapsto \operatorname{rot} \vec{R} = \vec{\nabla} \times \vec{R} = (Z_y - Y_z, X_z - Z_x, Y_x - X_y)$

$u \in C^2(D), \vec{R} : D \to \mathbb{R}^3 \quad C^2$

$\operatorname{rot} \operatorname{grad} u = \vec{0}$

$\vec{\nabla} \times \vec{\nabla} u = \vec{0}$

$\operatorname{div} \operatorname{rot} \vec{R} = 0$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{R}) = 0$

$\operatorname{div} \operatorname{grad} u = \vec{\nabla} \cdot \vec{\nabla} u = u_{xx} + u_{yy} + u_{zz} = \Delta u$

Izrek:

D zvezdasto ($\exists$ točka, ki vidi vse ostale točke)

$\vec{R} : D \to \mathbb{R}^3$ vek. polje

1) $\operatorname{rot} \vec{R} = \vec{0} \Rightarrow \exists u \in C^2(D).\ \vec{R} = \vec{\nabla} u$

2) $\operatorname{div} \vec{R} = 0 \Rightarrow \exists \vec{G} : D \to \mathbb{R}^3$ C^2 vek. polje da je $\vec{R} = \operatorname{rot} \vec{G}$

Zgled

ad 1) ne velja obratno

$\mathbb{R}^3 - z\text{-os}$ D ni zvezdasto

$\vec{R} = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

$\arctan \frac{y}{x} = u$ ni dobro definirana na $\mathbb{R}^3 - z\text{-os}$

•) Če ima $\vec{R}$ potencial u, potem so vsi potenciali oblike $u + c$ (D povezan)

~~Zzg~~

$\vec{R} : D \to \mathbb{R}^3$ vec. polje

$\operatorname{div} \vec{R} = f \quad f \in C^1(D)$

$\exists u$ na D. $\Delta u = f$

$\operatorname{div} \operatorname{grad} u = \Delta u = f = \operatorname{div} \vec{R}$

$$\operatorname{div}(\underbrace{\vec{R} - \operatorname{grad} u}_{= \operatorname{rot}(\vec{G})}) = 0$$

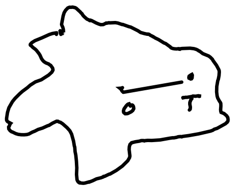
$\vec{R} = \operatorname{grad} u + \operatorname{rot} \vec{G}$

Dokaz (izreka dve strani nazaj)

1) $\vec{R}: D \to \mathbb{R}^3 \quad C^1(D)$

$\operatorname{rot} \vec{R} = \vec{0} = (Z_y - Y_z, X_z - Z_x, Y_x - X_y) = \vec{0}$

BŠZS. je D zvezdasto glede na 0



če $(x,y,z) \in D$.

$\forall t \in [0,1].\ (t\cdot x, t y, t\cdot z) \in D$

Definiramo:

$$u(x,y,z) = \int_0^1 (X(tx,ty,tz)\cdot x + Y(tx,ty,tz) y + Z(tx,ty,tz)\cdot z)\, dt$$

u je potencial za $\vec{R}$

$u_x = X \quad u_y = Y \quad u_z = Z$

$$u_x = \int_0^1 (X(tx,ty,tz) + X_x \cdot t\cdot x + \underbrace{Y_x}_{= X_y} \cdot ty + \underbrace{Z_x}_{= X_z} \cdot tz)\, dt$$

$$= \int_0^1 \underbrace{(X(tx,ty,tz) + tx\cdot X_x + ty X_y + tz \cdot X_z)}_{= \frac{d}{dt}(t\cdot X(tx,ty,tz))} dt$$

$$= tX(tx,ty,tz)\Big|_0^1 = X(x,y,z)$$

Podobno $U_y = Y$ in $U_z = Z$

Dokaz

2)

$div \vec{R} = \vec{0} = X_x + Y_y + Z_z$

(D je zvezdasta glede na 0 BŠZS)

$\alpha(x,y,z) = \int_0^1 t X(tx, ty, tz)\, dt$

$\beta(x,y,z) = \int_0^1 t Y(tx, ty, tz)\, dt$

$\gamma(x,y,z) = \int_0^1 t Z(tx, ty, tz)\, dt$

$div(\alpha, \beta, \gamma) = 0$ ker:

$$\left.\begin{aligned} \alpha_x &= \int_0^1 t^2 X_x(tx, ty, tz)\, dt \\ \beta_y &= \int_0^1 t^2 Y_y(tx, ty, tz)\, dt \\ \gamma_z &= \int_0^1 t^2 Z_z(tx, ty, tz)\, dt \end{aligned}\right\} +$$

$= 0$

$\vec{G} = \begin{pmatrix} \alpha & \beta & \gamma \\ x & y & z \end{pmatrix} = (\alpha, \beta, \gamma) \times (x, y, z)$

$\vec{G} = (z\beta - y\gamma,\ x\gamma - z\alpha,\ y\alpha - x\beta)$

$\vec{R} = rot\, \vec{G}$?

1 komponenta od $Rot\, \vec{G}$:

$(y\alpha - x\beta)_y - (x\gamma - z\alpha)_z =$

$= \alpha + y\alpha_y - x\beta_y - x\gamma_z + \alpha + z\alpha_z$

$= 2\alpha + \alpha_x + y\alpha_y + z\alpha_z$

uporabimo prvo enačbo $\alpha_x + \beta_y + \gamma_z = 0$

$= \int_0^1 \big(\underbrace{2t X(tx, ty, tz) + x t^2 X_x + y t^2 X_y + z t^2 X_z}_{\frac{d}{dt}\left(t^2 X(tx, ty, tz)\right)}\big)\, dt$

$t^2 X(tx, ty, tz)\Big|_0^1 = X(x,y,z)$

Podobno še drugi dve komponenti

zgled:

$$\vec{R} = (y^2 z^3 + 2,\ 2xyz^3 + 1,\ 3xy^2 z^2)$$

$$\operatorname{rot}\vec{R}\ (\underbrace{6xyz^2 - 6xyz^2}_{=0},\ \underbrace{3y^2z^2 - 3y^2z^2}_{=0},\ 2yz^3 - 2yz^3) = \vec{0}$$

$$u(x,y,z) = \int_0^1 X(tx, ty, tz)\, x + \dots)\, dt$$

u potencial obstaja: $u_x = y^2 z^3 + 2$

$u_y = 2xyz^3 + 1$

$u_z = 3xy^2 z^2$

$$u(x,y,z) = xy^2z^3 + 2x + C(y,z)$$

$\downarrow$

$$u_y = 2xyz^3 + C_y = 2xyz^3 + 1$$

$$C_y = 1$$

$$C(y,z) = y + D(z)$$

$$u(x,y,z) = xy^2z^3 + 2x + y + D(z)$$

$\downarrow$

$$u_z = 3xy^2z^2 + D'(z) = 3xy^2z^2$$

$$D' = 0$$

$D = D_0$ konstanta

$$u(x,y,z) = xy^2z^3 + 2x + y + D_0$$

(Rot $\vec{R} \neq \vec{0}$ takoj ni potenciala)

$$u = xy^2z^3 + 2x + A(yz)$$

$\downarrow$

$$2xyz^3 + A_y = 2yz^3 + x$$

$A_y = x$ * ker je A funkcija y in z ja nexa

Zgled

$\vec{R} = (2y-1, -1, 4x-2xy)$

$\operatorname{div}\vec{R} = (0,0,0) = 0$

iščemo $\vec{G} = ?$ $\operatorname{rot}\vec{G} = \vec{R}$

$$\alpha(x,y,z) = \int_0^1 t(2ty-1)\,dt = \frac{2}{3}y - \frac{1}{2}$$

$$\beta(x,y,z) = \int_0^1 t(-1)\,dt = -\frac{1}{2}$$

$$\gamma(x,y,z) = \int_0^1 t(4tx-2xyt^2)\,dt = \frac{4x}{3} - \frac{1}{2}xy$$

$$\vec{G} = \begin{pmatrix} \frac{2}{3}y-\frac{1}{2} & -\frac{1}{2} & \frac{4}{3}x-\frac{1}{2}xy \\ x & y & z \end{pmatrix} =$$

$$= \left(-\frac{1}{2}z - \frac{4}{3}xy + \frac{1}{2}xy^2,\ \frac{4}{3}x^2 - \frac{1}{2}x^2y - \frac{2}{3}zy + \frac{1}{2}z,\ \frac{2}{3}y^2 - \frac{1}{2}y + \frac{1}{2}x\right)$$

$$\operatorname{rot}\vec{G} = \Big(\underbrace{\frac{4}{3}y - \frac{1}{2} + \frac{2}{3}y - \frac{1}{2}}_{=2y-1},\ \underbrace{-\frac{1}{2} - \frac{1}{2}}_{=-1},\ \underbrace{\frac{8}{3}x - xy + \frac{4}{3}x - xy}_{=4x-2xy}\Big)$$

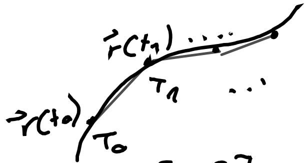
Krivuljni integral

Dolžina krivulje:

Γ naj bo C^1 krivulja v $\mathbb{R}^3$

reg. Parametrizacija $\vec{r}(t)$: $\vec{r}[\alpha, \beta] \longrightarrow \Gamma$

$\dot{\vec{r}}(t) \neq 0$



D delitev $[\alpha, \beta]$

$$\sum_1^n d(T_{j-1}, T_j) = l(D)$$

$$l(D) = \sum_1^n \sqrt{(x(t_j) - x(t_{j-1}))^2 + \ldots + (z(t_j) - z(t_{j-1}))^2}$$

$$\overset{\text{Lagrange}}{=} \sum_1^n \sqrt{\dot{x}^2(\tau_j) + \dot{y}^2(\tau_j') + \dot{z}^2(\tau_j'')}$$

$$\doteq \sum_1^n \sqrt{\dot{x}(\tau_j) + \dot{y}^2(\tau_j) + z^2(\tau_j'')} =$$

$$= R\left(\sqrt{\dot{x} + \dot{y}^2 + \dot{z}^2}, D, \tau\right) \xrightarrow[\max \Delta t_j \to \infty]{}$$

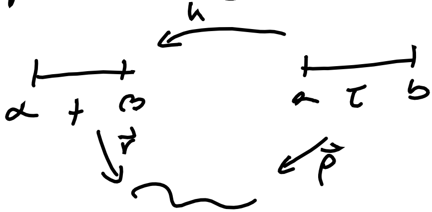
$$\longrightarrow \int_\alpha^\beta \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Naj bo Γ C^1 krivulja v $\mathbb{R}^3$ z regularno parametrizacijo $\vec{r}: [\alpha,\beta) \to \Gamma$

Definiramo dolžino krivulje

$$l(\Gamma) = \int_\alpha^\beta |\dot{\vec{r}}|\,dt = \int_\alpha^\beta \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt$$

Ali je ta definicija neodvisna od regularne parametrizacije



$$\vec{h}: [a,b] \xrightarrow{\text{Dif}} [\alpha,\beta]$$

$$\vec{\rho} = \vec{r} \circ \vec{h}$$

$$\vec{\rho}\,' = \dot{\vec{r}}(h)\,h'$$

$$\int_a^b |\rho'|\,d\tau = \int_\alpha^\beta |\dot{\vec{r}}|\,dt$$

$$\int_\alpha^\beta |\dot{\vec{r}}(t)|\,dt = \int_{[\alpha,\beta]} |\dot{\vec{r}}(t)|\,dt = \int_{[a,b)} |\dot{\vec{r}}(h(\tau))|\,|h'(\tau)|\,d\tau$$

$t = h(\tau)$

$$= \int_{[a,b]} |\vec{\rho}\,'|\,d\tau$$

Γ naj bo C^1 krivulja $\vec{r}(t)$ reg. param.

$$s(t) = \int_\alpha^t |\dot{\vec{r}}(\tau)|\, d\tau \qquad s : [\alpha, \beta] \rightarrow [0, l(\Gamma)]$$

= dolžina Γ med $\vec{r}(\alpha)$ in $\vec{r}(t)$

Odvod: $\dot{s} = |\dot{\vec{r}}(t)| > 0$

s je strogo naraščajoča od $[\alpha, \beta]$ v $[0, l(\Gamma)]$

Torej ima inverz:

$$T : [0, l(\Gamma)] \longrightarrow [\alpha, \beta]$$

$$s \longmapsto T(s) \quad \text{inverz od } S$$

Poglejmo parametrizacijo Γ:

$$s \longmapsto \vec{r}(T(s))$$

Vemo $S \circ T = id_{[0, l(\Gamma)]}$ in $T \circ S = id_{[\alpha, \beta]}$

odvajamo:

$$\dot{s}(T(s)) \cdot T'(s) = 1$$

izračunajmo velikost parametrizacije

$$s \longmapsto \vec{r}(T(s))$$

$$\frac{d}{ds}(\vec{r}(T(s))) = \dot{\vec{r}}(T(s)) \cdot T'(s) =$$

$$= \frac{\dot{\vec{r}}(T(s))}{\dot{s}(T(s))} = \frac{\dot{\vec{r}}(T(s))}{|\dot{\vec{r}}(T(s))|}$$

velikost hitrosti je 1

Tako parametrizirana krivulja je naravno parametrizirana in s je naravni parameter

Zgled:

vijačnica:

$a > 0$

$b \neq 0$

$$t \longmapsto (a\cos t, a \sin t, bt) = \vec{r}(t)$$

$$|\dot{\vec{r}}| = \sqrt{a^2+b^2}$$

$$S(t) = \int_0^t \sqrt{a^2+b^2}\, d\tau = \sqrt{a^2+b^2} \cdot t = s$$

$$T(s) = \frac{s}{\sqrt{a^2+b^2}}$$

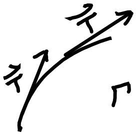
naravno parametrizirana vijačnica

$$s \longmapsto \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right)$$

$$\| \quad \vec{r}_N(s)$$

$$|\vec{r}_N'(s)| = 1$$

Orientacija krivulje



Γ gladka C^1 krivulja

<u>orientacija</u> Γ je zvezen izbor enotskega tangentnega vektorja vzdolž Γ: $\vec{T}$

$|\vec{T}| = 1$

Γ povezana $\Rightarrow$ Γ ima dve orientaciji

$\vec{T}$ in $-\vec{T}$

$\vec{r}:[\alpha,\beta] \to \Gamma$ reg. parametrizacija

$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$ en možni izbor

V tem primeru sta orientacija in parametrizacija usklajeni:

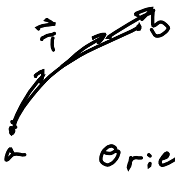
$(\Gamma, \vec{T})$ ali $\vec{\Gamma}$

Vsaka krivulja je orientabilna

Odsekoma gladka krivulja

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$

Γ z robnima točkama a, b



(krivulja z robom $\partial\Gamma = \{a, b\}$)

Naj bo $\vec{T}$ orientacija Γ

Orientacija Γ porodi orientacijo $\partial\Gamma$

Ena od teh dveh točk je prva oz začetna Γ (Z_Γ) in druga je končna točka Γ (K_Γ)

Glede na sliko je $a = Z_\Gamma$ $b = K_\Gamma$

S tem je rob Γ orientiran skladno z orientacijo Γ

Naj bo Γ odsekoma gladka krivulja.

$\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$

1) Γ_j je gladka krivulja z robom

2) $\Gamma_j \cap \Gamma_l = \emptyset \Leftarrow |j-l| \bmod n \geq$

3) $\Gamma_j \cap \Gamma_{j+1} = \{P_j\}$ $j \in \{1, \dots, n-1\}$ P_j robna točka

← ena sama točka za Γ_j in Γ_{j+1}

Γ_n in Γ_1 se lahko sekata ali pa ne

Orientacija Γ je tak izbor orientacij $\Gamma_1 \dots \Gamma_n$, da je $K_{\Gamma_j} = P_j = Z_{\Gamma_{j+1}}$

če se Γ_1 in Γ_n sekata je $K_{\Gamma_n} = Z_{\Gamma_1}$

Krivuljni integral

Dve vrsti krivuljnih integralov:

1) Integral skalarnega polja po krivulji Γ
(orientacije tu ne potrebujemo)

Γ gladka krivulja omejena

$u: \Gamma \longrightarrow \mathbb{R}$ vzemo skalarno polje

$\vec{r}: [\alpha, \beta] \longrightarrow \Gamma$ regularna parametrizacija

Definiramo $\int_\Gamma u\, ds := \int_\alpha^\beta u(\vec{r}(t)) |\dot{\vec{r}}(t)|\, dt$

Opombe:

1) $u = 1 \Rightarrow$ je to dolžina Γ

2) Ta vrednost je neodvisna od reg. parametrizacije

3) $u =$ dolžinska gostota $\Rightarrow$ dobimo maso Γ

Če je $u = \rho_0$ konstanta, je Γ homogena

Če je $\Gamma_1 \cup \dots \cup \Gamma_n$ odsekoma gladka

je $\int_\Gamma u\, ds = \sum_{j=1}^{n} \int_{\Gamma_j} u\, ds$

2) Krivuljni integral vektorskega polja po orientirani krivulji

» »

Zgled: Γ homogena polkrožnica

$$\Gamma = \{(x,y) \in \mathbb{R}^2, \ x^2+y^2=a^2, \ y \geq 0\}$$

Lega težišča: ?

$$m(\Gamma) = \int_\Gamma \rho_0 \, ds \qquad x_T(\Gamma) = \frac{1}{m_\rho} \int_\Gamma x \rho_0 \, ds$$

$$y_T(\Gamma) = \frac{1}{m(\Gamma)} \int_\Gamma y \rho_0 \, dy$$

$$m(\Gamma) = \int_\Gamma \rho_0 \, ds = \pi a \rho_0$$

Parametrično:

$x(t) = a\cos t \qquad y(t) = a \sin t$

$ds = |\dot{\vec{r}}| dt$

$\dot{x} = -a \sin t \qquad x^2+y^2=a^2$

$\dot{y} = a \cos t$

$$m(\Gamma) = \int_0^\pi \rho_0 a \, dt = \rho_0 a \, dt$$

$$x_T(\Gamma) = \frac{1}{m(\Gamma)} \int_0^\pi a\cos t \, \rho_0 \, a dt = 0$$

$$\frac{1 \rho_0 a}{\rho_0 a \pi} \int_0^\pi \sin t \, a \, dt = \frac{a}{\pi}(-\cos t)\Big|_0^\pi =$$

$$\frac{2}{\pi} \cdot a$$

Vztrajnostni moment okoli ~~x~~ y osi

$$J_y = \int_\Gamma \underbrace{(x^2+z^2)}_{\substack{\| \\ 0}} \rho \, ds = \int_0^\pi a^2 \cos^2 t \cdot a \, dt = \rho_0 a^3 \int_0^\pi \frac{1+\cos(2t)}{2} dt$$

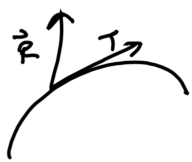
$$= \frac{\pi}{2} \rho_0 a^3 = \frac{1}{2} m(\Gamma) \cdot a^2$$

2) Krivuljni integral vektorskega polja po orientirani krivulji

(Delo sile vzdolž $\vec{\Gamma}$)

$\vec{R} : \vec{\Gamma} \to \mathbb{R}^3$ zvezno vek. polje

$\vec{R} = (X, Y, Z$



če je Γ gladka in

$\vec{r} : [\alpha, \beta] \longrightarrow \Gamma$ param. ki je vsklajena z orientacijo Γ

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

skalarni produkt

Definiramo $\int\limits_{\Gamma} \vec{R}\, d\vec{r} = \int\limits_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\, dt$

$$= \int\limits_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \cdot |\dot{\vec{r}}|\, dt = \int\limits_{\Gamma} (\vec{R} \cdot \vec{T})\, ds$$

Rezultat je neodvisen od parametrizacije Γ, ki je vsklajena z orientacijo

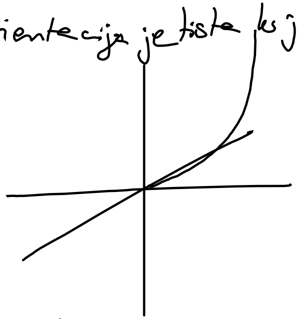
$$\int\limits_{-\vec{\Gamma}} \vec{R}\, dr = -\int\limits_{\vec{\Gamma}} \vec{R}\, dr$$

Zgled:

$$\vec{R}(x,y,z) = (xy, z, x-z)$$

$$r(t) = (t, t, \tfrac{1}{2}t^2) \qquad t\in[0,1]$$

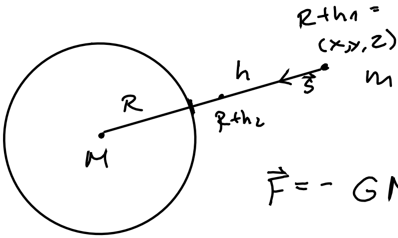
orientacija je tista, ki je usklajena z $\vec{r}$



$$\int_{\vec{r}} \vec{R}\,d\vec{r} = \int_0^1 (t^2, \tfrac{1}{2}t^2, t-\tfrac{1}{2}t^2)\cdot(1,1,t)\,dt =$$

$$= \int_0^1 (\tfrac{3}{2}t^2 + t^2 - \tfrac{1}{2}t^2)\,dt = \frac{5}{6} - \frac{1}{8} = \frac{17}{24}$$

Zgled:



$|\vec{s}|=1$

$$\vec{F} = -GM\frac{m}{|(x,y,z)|^3}(x,y,z) =$$

$$= -GM\frac{m}{r^3}\cdot\vec{r}$$

$t \longmapsto \vec{s}\cdot t \qquad t$ teče od $R+h_1$ do $R+h_2$

$$\int_{\Gamma}\vec{F}\,d\vec{r} = \int_{R+h_1}^{R+h_2} -GM\frac{m}{t^3}(t\cdot\vec{S})\cdot\vec{s}\;dt =$$

$$= -GMm\int_{R+h_1}^{R+h_2}\frac{dt}{t^2} = -GMm\left.\frac{1}{t}\right|_{R+h_1}^{R+h_2} = -GmM\left(\frac{1}{R+h_2}-\frac{1}{R+h_1}\right)$$

$$= \frac{GM(h_1-h_2)m}{(R+h_1)(R+h_2)} \doteq \underbrace{\frac{GM}{R^2}}_{g}m\cdot\Delta h = mg\Delta h$$

Opomba

•) $\int\limits_{\vec{\Gamma}} \vec{R}\, dr = \int\limits_{\vec{\Gamma}} \underbrace{X\,dx + Y\,dy + Z\,dz}$

diferencialna forma

••) če je Γ sklenjena ($Z_\Gamma = K_\Gamma$)

$$\oint\limits_{\vec{\Gamma}} \vec{R}\, d\vec{r}$$

Naj bo $\vec{R}$ potencialno vektorsko polje

Naj bo Γ C^1 krivulja s parametrizacijo $\vec{r}(t)$

Orientacija usklajena z $\vec{r}$ $\quad \vec{T} = \frac{|\dot{\vec{r}}|}{|\dot{\vec{r}}|}$

$$\int_{\vec{\Gamma}} \vec{R}\, dr = \int_{\vec{\Gamma}} \operatorname{grad} u\, d\vec{r} = \int_{\alpha}^{\beta} (u_x, u_y, u_z)(r(t)) \cdot \dot{\vec{r}}(t)\, dt =$$

$$= \int_{\alpha}^{\beta} \quad \vec{r}(t) = (x(t), y(t), z(t)) = \int_{\alpha}^{\beta} \underbrace{(u_x \cdot \dot{x} + u_y \dot{y} + u_z \dot{z})}_{\frac{d}{dt}\left(u(x(t), y(t), z(t))\right)} dt =$$

$$= u(\vec{r}(\beta)) - u(\vec{r}(\alpha)) = u(k_{\vec{\Gamma}}) - u(z_{\vec{\Gamma}})$$

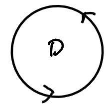
Trditev: Če je vek. polje $\vec{R}$ potencialno, je vrednost integrala $\vec{R}$ po orientirani krivulji $\vec{r}$ enaka razliki potenciala polja $\vec{R}$ med končno točko $\vec{r}$ in začetno točko $\vec{r}$

Posledica: Če je $\vec{R}$ potencialno polje in je r sklenjena je integral $\oint_{r} \vec{R}\,dr = 0$

Zgled:

$$\vec{R} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\vec{\Gamma} = \{ (x,y,0);\ x^2+y^2 = R^2;\ R>0 \}$$



$$x = R\cos t \qquad t \in [0, 2\pi)$$
$$y = R\sin t$$
$$z = 0$$

$$\dot{x} = -R\sin t$$
$$\dot{y} = R\cos t$$
$$\dot{z} = 0$$

$$\int_{\Gamma} \vec{R}\,d\vec{r} = \int \left(-\frac{1}{R^2} R\sin t, \frac{1}{R^2} R\cos t, 0 \right) \cdot (-R\sin t, R\cos t, 0)\,dt =$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi \neq 0$$

$\Rightarrow$ To polje ni potencialno v $\mathbb{R}^3 - \{z\text{-os}\}$

(Vemo pa da je $\operatorname{rot}\vec{R} = \vec{0}$

Izrek (karakterizacija potencialnih vek. polj)

Naj bo $D \subseteq \mathbb{R}^3$ odp in $\vec{R} : D \to \mathbb{R}^3$ zvezno vektorsko polje:

Naslednje tri izjave so ekvivalentne:

1) R je potencialno na D
$(\exists u \in C^1(D).\ \vec{R} = \operatorname{grad} u)$

2) Integral $\vec{R}$ po orientiranih krivuljah je neodvisen od krivulj z isto začetno in isto končno točko

3) Integral po sklenjenih krivuljah je enak 0

~~Neč[illegible]~~ Level: Tiles [illegible]

Rabim [illegible] barva rgb → path

Usak blok [illegible] vrb [illegible]

za dva vrednost [illegible] definirane!

Trart[illegible]

Dokaz:

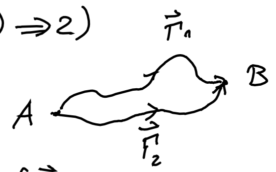
1) $\Rightarrow$ 3)

$\vec{\Gamma}$ sklenjena: $t \longrightarrow \vec{r}(t)$ parametrizacija C^1

$t \in [\alpha, \beta]$; $\vec{r}(\alpha) = \vec{r}(\beta)$

$$\int_{\vec{\Gamma}} \vec{R}\,d\vec{r} = \int_{\alpha}^{\beta} \underbrace{\vec{R}(\vec{r}(t))\dot{\vec{r}}(t)dt}_{\frac{d}{dt}(\vec{R}(\vec{r}(t)))dt} = R(\vec{r}(\beta)) - R(\vec{r}(\alpha)) = 0$$

3) $\Rightarrow$ 2)



$\vec{\Gamma} = \vec{\Gamma}_1 \cup (-\vec{\Gamma}_2)$

$$\int_{\vec{\Gamma}} \vec{R}\,dr = 0$$

$$\int_{\vec{\Gamma}_1} \vec{R}\,dr - \int_{\vec{\Gamma}_2} \vec{R}\,dr = 0 \quad \Longrightarrow \quad \int_{\vec{\Gamma}_1} \vec{R}\,dr = \int_{\vec{\Gamma}_2} \vec{R}\,dr$$

2) $\Rightarrow$ 1)

Potrebujemo potencial u. $\vec{\nabla} u = \vec{R}$

Naj bo D povezana (če ni delamo po vsaki komponenti posebi)

D je s potmi povezana

Za vsaki dve točki obstaja odsekoma gladka pot od A do B

Naj bo $A_0 \in D$ neka točka

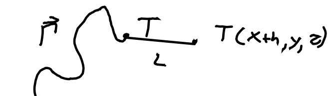
Naj bo $T \in D$.
Naj bo $\vec{\Gamma} \subseteq D$ odsekoma gladka pot od A_0 do T

Definiramo $u(T) = u(x,y,z) = \int_{\vec{\Gamma}} \vec{R}\,dr$

Je dobro definiran, ker ni odvisen od poti od A_0 do T

Trdimo: $\vec{\nabla} u = (u_x, u_y, u_z) = \vec{R} = (X, Y, Z)$

$$\lim_{h \to 0} \frac{u(x+h,y,z) - u(x,y,z)}{h} =$$



$\vec{\tilde{\Gamma}} = \vec{\Gamma} \cup \vec{L}$

$\vec{L}$ je daljica od (x,y,z) do $(x+h,y,z)$

$$u(x,y,z) = \int_{\vec{\Gamma}} \vec{R}\,dr$$

$$\frac{u(x+h,y,z) - u(x,y,z)}{h} = \frac{1}{h}\left(\int_{\vec{\Gamma} \cup \vec{L}} \vec{R}\,dr - \int_{\vec{\Gamma}} \vec{R}\,dr\right) =$$

$$= \frac{1}{h} \int_{\vec{L}} \vec{R}\,dr$$

Parametrizacija $\vec{L}: \vec{r}: t \longmapsto (x+th, y, z)$

$t \in [0,1]$

$\dot{\vec{r}}(t) = (h,0,0) = h(1,0,0)$

$$\frac{1}{h}\int_0^1 \vec{R}(x+th,y,z)\, h\,(1,0,0)\,dt =$$

$$= \int_0^1 X(x+th,y,z)\,dt \xrightarrow{h \to 0} X(x,y,z)$$

Res $u_x = X$ podobno se u_y in u_z

Opombe:

1) $\int_{\vec{\Gamma}} \vec{\nabla} u \, dr = u(k T_{\vec{\Gamma}}) - u(z T_{\vec{\Gamma}})$

Če je v nek drug potencial $\vec{R} = \vec{\nabla} v = \vec{\nabla} u$

D povezana $\Rightarrow v = u + C$

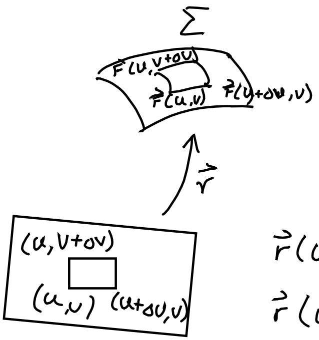
2) D zvezdasto glede na neko točko, $\operatorname{rot} \vec{R} = \vec{0}$

$u(T) = u(x, y, z) \int_{[A_0, T]} R \, d\vec{r} = \int_0^1 \big(x X(tx, ty, tz) +$

$y Y(x, y, z) + z Z(x, y, z) \big) \, dt$

Površina ploskve

$\Sigma \subseteq \mathbb{R}^3$ gladka ploska



$$\vec{r}(u+\Delta u, v) - \vec{r}(u,v) \cong \vec{r}_u(u,v)\Delta u$$

$$\vec{r}(u, v+\Delta v) - \vec{r}(u,v) \cong \vec{r}_v(u,v)\Delta v$$

$$\Delta P = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

Definiramo:

$$P(\Sigma) = \iint_D |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Opomba: $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a}\vec{c})(\vec{b}\vec{d}) - (\vec{a}\vec{d})(\vec{b}\vec{c})$

$$(\vec{r}_u \times r_v) \cdot (\vec{r}_u \times r_v) = |\vec{r}_u|^2 |\vec{r}_v|^2 - (\vec{r}_u \cdot \vec{r}_v)^2 \geq 0$$

$$\parallel$$

$E = |\vec{r}_u|^2$

$F = \vec{r}_u r_v$

$G = |\vec{r}_v|^2$

$$= EG - F^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix}$$

$$P(\Sigma) = \iint \sqrt{EG - F^2} \, du \, dv$$

Površina grafa

$$\Sigma = \{x, y, f(x,y)) \quad (x,y) \subseteq D \subseteq \mathbb{R}^2\}$$

$$\vec{r}(x,y) = (x, y, \vec{f}(x,y))$$

$$\vec{r}_x = (1, 0, f_x(x,y))$$

$$r_y = (0, 1, f_y(x,y))$$

$$E = 1 + f_x^2$$

$$F = f_x f_y$$

$$G = 1 + f_y^2$$

$$EG - F^2 = (1 + f_x^2) + (1 + f_y^2) - f_x^2 f_y^2 =$$

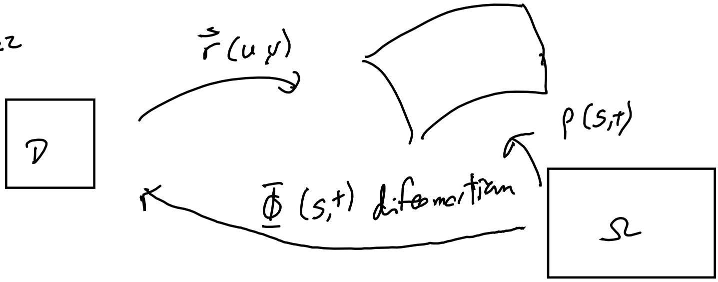
$$= 1 + f_x^2 + f_y^2$$

$$P(Q(f)) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx, dy$$

Opomba: E, F, G so koeficienti I fundamentalne forme ploskve $Eu^2 + 2Fuv + Gv^2$

Trditev: Definicija površine ploskve je neodvisna od regularne parametrizacije ploskve

Dokaz



$$\vec{\rho}(s,t) = (\vec{r} \circ \Phi)(s,t)$$

$$\iint_{\Omega} |\vec{\rho}_s \times \vec{\rho}_t| \, ds\,dt = \iint |\vec{r}_u \times r_v| \, du\,dv$$

$$\Phi : \Omega_{(s,t)} \longrightarrow D_{(u,v)}$$

$$\Phi(s,t) = (U(s,t), V(s,t))$$

$$\rho(s,t) = \vec{r}(U(s,t), V(s,t))$$

$$\rho_s = \vec{r}_u \cdot U_s + \vec{r}_v V_s$$

$$\rho_t = r_u V_t + \vec{r}_v \cdot V_t$$

$$\rho_s \cdot \rho_t = (\vec{r}_u \times \vec{r}_v) U_s V_t - (\vec{r}_u \times \vec{r}_v) U_t V_s =$$

$$= (\vec{r}_u \times \vec{r}_v)(U_s V_t - U_t V_s)$$

$$\begin{vmatrix} U_s & U_t \\ V_s & V_t \end{vmatrix}$$

$$|\rho_s \times \rho_t| = |\vec{r}_u \times \vec{r}_v| \, |J\Phi|$$

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, du\,dv = \iint_{\Omega} |(\vec{r}_u \times \vec{r}_v)\Phi(s,t))| \, |J\Phi(s,t)| \, ds\,dt =$$

$$(u,v) = \Phi(s,t)$$

$$= \iint_{\Omega} |\rho_s \times \rho_t| \, ds\,dt$$

Zgled:

1) $S(0,a) = \{(x,y,z)\ x^2+y^2+z^2 = a^2\}$

$x(\vartheta,\varphi) = a\cos\varphi\sin\vartheta$

$y(\vartheta,\varphi) = a\cos\varphi\sin\vartheta$

$z(\vartheta,\varphi) = a\cos\vartheta$

$\vec{r}(\vartheta,\varphi) = x(\vartheta,\varphi), y(\vartheta,\varphi), z(\vartheta,\varphi)$

$\vec{r}_\vartheta = (a\cos\varphi\cos\vartheta, a\sin\varphi\cos\vartheta\ -a\sin\vartheta)$

$\vec{r}_\varphi = (-a\sin\varphi\sin\vartheta,\ a\cos\varphi\sin\vartheta, 0)$

$E = a^2$

$F = 0$

$G = a^2\sin^2\vartheta$

$P(S) = \int_0^{2\pi}\int_0^{\pi} a^2\sin\vartheta\, d\vartheta\, d\varphi = a^2 2\pi(-\cos\vartheta)\Big|_0^{\pi} = 4\pi a^2$

2) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$D = \{(x,y);\ x^2+y^2 \leq 1\}$

$f_x = x$

$f_y = y$

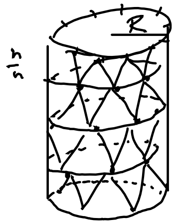
$x = r\cos\varphi$

$y = r\sin\varphi$

$P(G(f)) = \iint_D \sqrt{1+x^2+y^2}\, dx, dy =$

$\int_0^1 dr \int_0^{2\pi} d\varphi\ r\sqrt{1+r^2}\, d\varphi\, dr = 2\pi\frac{2}{3}(1+r^2)^{\frac{3}{2}}\frac{1}{2}\Big|_0^1 = \frac{2\pi}{3}(2^{\frac{3}{2}}-1$

zgled (Schwarzova lanterna al. Schwarzov škorn

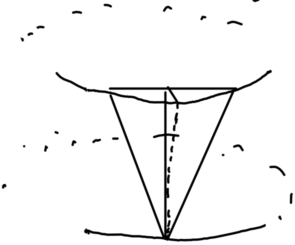


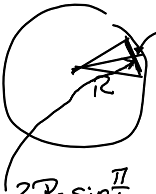
Pričakujemo površino $2\pi r h$

m lokov enako dolgih

n enakih trakov

2mn trikotnikov (ne ležijo na ploskvi





$R - R\cos\frac{\pi}{m} =$

$R\left(1-\cos\left(\frac{\pi}{m}\right)\right) =$

$= R\,2\cos^2\frac{\pi}{2m}$

$2R\cdot\sin\frac{\pi}{m}$ dolžina tetive

Ploščina trikotnika:

$$\frac{1}{2}2R\sin\frac{\pi}{m}\sqrt{\frac{h^2}{n^2}+R^2\sin^4\frac{\pi}{2m}}$$

Površina vseh teh trikotnikov je

$$P(n,m) = 2nmR\sin\frac{\pi}{m}\sqrt{\frac{h^2}{n^2}+4R^2\sin^4\frac{\pi}{2m}}$$

1) $$\lim_{m\to\infty} P(m,m) = \lim_{m\to\infty} 2Rm^2\sin\frac{\pi}{m}\sqrt{\frac{h^2}{m^2}+4R^2\sin^4\frac{\pi}{2m}} =$$

$$= 2R\lim_{m\to\infty} m^2\frac{\pi}{m}\sqrt{\frac{h^2}{m^2}+4R^2\frac{\pi^4}{16m^4}} =$$

$$= 2R\lim_{m\to\infty}\pi\sqrt{h^2+\cancel{4}R^2\frac{\pi^4}{4m^2}} = 2\pi Rh$$

$$\lim_{m\to\infty} P(m^2,m) = 2R\lim_{m\to\infty} m^3\frac{\pi}{m}\sqrt{\frac{h^2}{m^4}+\frac{4R^2\pi^4}{16m^4}} =$$

$$= 2R\pi\lim_{m\to\infty}\sqrt{h^2+\frac{\pi^4R^2}{4}} > 2\pi Rh$$

3) $$\lim_{m\to\infty} P(m^3,m) = 2R\lim_{m\to\infty} m^4\frac{\pi}{m}\sqrt{\frac{h^2}{m^6}+4R^2\frac{\pi^4}{16m^4}} =$$

$$= 2R\lim_{m\to\infty}\sqrt{h^2+\frac{\pi^4R^2}{4}m^2} = \infty$$

Orientacija ploskev

(Za krivulje:

$|\vec{T}| = 1$ wem tangentni vektorji vzdolž Γ)

$\Sigma \subseteq \mathbb{R}^3$ gladka ploskev

orientacija Σ je wezen izbor enotske normale na Σ

$\vec{N} : \Sigma \longrightarrow \mathbb{R}^3$

$|\vec{N}| = 1 \qquad \vec{N}(x,y,z) \perp T_{(x,y,z)}\Sigma$

če Σ lahko orientiramo je Σ orientabilna
(sicer je neorientabilna)

$(\Sigma, \vec{N})$ če je Σ povezana in orientabilna ima natanko dve različni orientaciji

$(\Sigma, \vec{N}), (\Sigma, -\vec{N})$

Zgledi:

1) $\Sigma = G(f) = \{(x,y,f(x,y)); (x,y) \in D \subseteq \mathbb{R}^n$

$f \in C_1(D)$

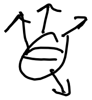
$$\vec{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$$

2) $S^2 = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2=R\}$

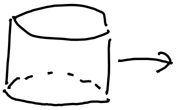
$$\vec{N} = \frac{(x,y,z)}{R}$$

Zunanja normala:

$x^2+y^2+z^2=R^2 \quad R>0$



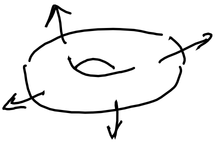
3)



$\Sigma = \{(x,y,z); x^2+y^2=R^2; z \in \mathbb{R}\}$

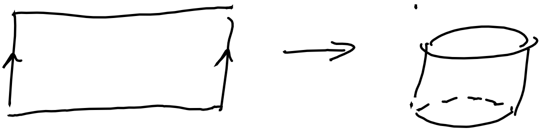
$$\vec{N} = \frac{(x,y,0)}{R}$$

4)

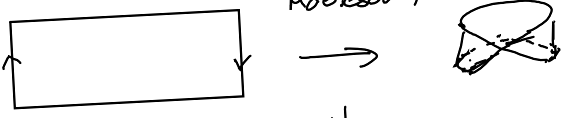


v $\mathbb{R}^3$ so vse sklenjene (kompaktne) ploskve orientabilne

5)



Möbiusov trak



Ni orientabilne

$\Sigma \subseteq \mathbb{R}^3$ je lahko orientabilna ali pa ni orientabilna

2 možni orientaciji za povezan Σ

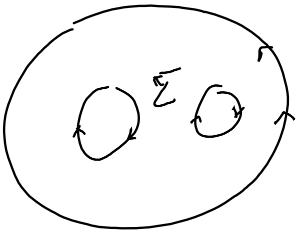
6) $\Sigma \in \mathbb{R}^2 \times \{a\}$

če ni rečeno drugače je orientirana z normalo $(0,0,1) = \vec{N}$ (pozitivna orientacija

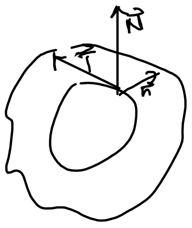
Naj bo Σ gladka ploskev z robom (končna unija odsekoma gladkih sklenjenih krivulj)

Naj bo $(\Sigma, \vec{N})$ ploskev z orientacijo

skladna oz. koherentna orientacija $\partial\Sigma$ je definirana takole: če sprehajalec stoji na robu Σ in ima glavo v smeri normale, (stoji kot normala) je rob $\partial\Sigma$ orientiran tako da Σ leži na levi strani sprehajalca, če le ta gleda v smeri normale



če $\Sigma \subseteq \tilde{\Sigma}$



$\vec{T} = \vec{N} \times \vec{n}$

$\vec{n}$ ... zunanja normala (na krivuljo

$(\Sigma, \vec{N}) \rightsquigarrow (\partial\Sigma, \vec{T})$

skladna orientacija

$(\Sigma, -\vec{N}) \rightsquigarrow (\partial\Sigma, -\vec{T})$

Opomba: $\subseteq \mathbb{R}^2$

če: $\vec{r}: D \to \Sigma$ regularna parametrizacija

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

Orientacija $\vec{N}$ je usklajena s parametrizacijo $\vec{r}$

Odsekoma gladka ploskev

$$\Sigma = \Sigma_1 \cup \ldots \Sigma_n \subset \mathbb{R}^n$$

1) $\Sigma_1 \ldots \Sigma_n$ so gladke ploskve z robom

2) če $\Sigma_i \cap \Sigma_j \neq \emptyset$ je presek del roba $\Sigma_i \cap \Sigma_j$
$i \neq j$

3) $\Sigma_i \cap \Sigma_j \cap \Sigma_k \in \{\emptyset, \{x\}\}$
$i \neq j \neq k$ (presek je ali prazen ali točka)

Orientacija $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$ če obstaja, je tak nabor orientacij, da sta na vseh presekih
$\Gamma_{ij} = \Sigma_i \cap \Sigma_j$ inducirani orientaciji:
nasprotni

Ploskovni integral

1) Ploskovni integral skalarnega polja

$u : \Sigma \to \mathbb{R}$ zvezno skalarno polje

$\vec{r} : D \xrightarrow{\text{regularna parametrizacija}} \Sigma$

$$\iint_\Sigma u\, dS := \iint_D u(\vec{r}(u,v)) \sqrt{EG - F^2}\, du\, dv =$$

$$\iint_D u(\vec{r}(u,v)) \|r_u \times r_v\|\, du\, dv$$

$E = |\vec{r_u}|^2 \qquad F = \vec{r_u} \cdot \vec{r_v} \qquad G = |\vec{r_v}|$

Opomba $u = 1 \Rightarrow$ to je površina ploskve

Opomba: Ta vrednost je neodvisna od regularne parametrizacije $\vec{r}$

Opomba: Tu ne potrebujemo orientacije
Ta integral obstaja tudi na Möbiusovem traku

Opomba: $u = \rho$ ... površinska gostota, potem je

$$\iint_\Sigma \rho\, dS = m(\Sigma)$$

ρ konstanten $\Rightarrow$ ploskev je homogena

Zad

$S = \{ (x,y,z) ; x^2 + y^2 + z^2 = R^2 \}$ $R > 0$

homogena sfera

$J_z = \iint_S \rho \, (x^2+y^2) \, dS$

$(\vartheta, \varphi) \longmapsto (R\cos\varphi \sin\vartheta, R\sin\varphi \cos\vartheta, R\cos\vartheta)$

$\sqrt{EG - F^2} = R^2 \sin\vartheta$

$J_z = \int_0^{2\pi}\int_0^{\pi} \rho_0 R^2 \sin^2\vartheta \, R^2 \sin\vartheta \, d\vartheta \, d\varphi =$

$= \rho_0 \, 2\pi R^4 \int_0^{\pi} \sin^3\vartheta \, d\vartheta = 2\pi \rho_0 R^4 \int_{-1}^{1} (1-t^2) \, dt$

$2\pi \rho_0 R^4 \cdot 2 \left(1 - \frac{1}{3}\right) = \frac{8}{3} \rho_0 R^4 =$

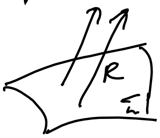
$m(S) = 4\pi R^2 \rho_0$

$= \frac{2}{3} m R^2$

2) Ploskovni integral vektorskega polja po orientirani ploskvi

$(\Sigma, \vec{N})$ $\quad \vec{R} : \Sigma \to \mathbb{R}^3$ zvezno vek. polje

$$\int_{\vec{\Gamma}} \vec{R} d\vec{r} = \int_{\Gamma} (\vec{R} \vec{T}) ds$$



Pretok $\vec{R}$ skozi Σ

$$\iint_{\Sigma} \vec{R} d\vec{S} := \iint_{\Sigma} (\vec{R} \cdot \vec{N}) dS$$

Neodvisno reg. parametrizacije, usklajene z orientacijo Σ

Tudi $\iint_{\Sigma} \vec{R} (-\vec{N}) dS = -\iint_{\Sigma} (\vec{R} \vec{N}) dS$

$\vec{r}$ je usklajena ko $\vec{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

Potem:

$$\iint_{\Sigma} \vec{R} dS = \iint_{\Sigma} \vec{R} \vec{N} dS = \iint_{D} \vec{R} (\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

$$= \iint_{D} (\vec{r}_u, \vec{r}_v, \vec{R}) du dv$$

mešani produkt

Če je $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$

$$\iint_{\Sigma} u dS = \sum_{j=1}^{n} \iint_{\Sigma_j} u dS$$

$$\iint_{\Sigma} \vec{R} d\vec{S} = \sum_{j} \iint_{\Sigma_j} \vec{R} d\vec{S}$$

Zgled:

$S = \{(x,y,z) ; x^2+y^2+z^2 = R^2\}$

$$\vec{R} = \frac{\vec{r}}{|\vec{r}|^3} = \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right)$$

$$= -\vec{\nabla} \frac{1}{|\vec{r}|}, \quad \operatorname{div} R = 0 \qquad \Delta \frac{1}{r} = 0 \text{ na } \mathbb{R}^3 - \{0\}$$

$\vec{N} = (0,0,1) = (0,0,1)$

$$\iint_S \vec{R}\, d\vec{S} = \iint_S \frac{\vec{r}}{|\vec{r}|^3} \cdot \underbrace{\frac{\vec{r}}{|\vec{r}|}}_{\vec{N}} \cdot dS = \iint_S \frac{R^2}{R^4} dS =$$

$$= \iint_S \frac{1}{R^2} dS = \frac{1}{R^2} 4\pi R^2 = 4\pi$$

Opombe

Diferencialna 1-forma

$$\int_{\vec{r}} \vec{R}\, d\vec{r} = \int_{\vec{r}} X dx + Y dy + Z dz$$

$$\iint \vec{R}\, d\vec{S} = \int X dx \wedge dz + Y dz \wedge dx + dz \wedge dy$$

Integralski izreki

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

$\Gamma = [a,b]$

$\vec{T} = \vec{1}$

Integral po orientiranem intervalu f' po

$\partial[a,b]$ orientiranem skupaj s orientacijo $[a,b]$

Že vemo:

$$\int_{\vec{\Gamma}} \vec{\nabla} u \, d\vec{r} = u(KT_{\Gamma}) - u(ZT_{\Gamma})$$

oblike integralnih izrekov

M orientirana mnogoterost z robom ∂M

$M=[a,b]$, $M=\Gamma$, $M=D\subset\mathbb{R}^2$, $M=\Sigma\subset\mathbb{R}^3$, $M=\Omega\subseteq\mathbb{R}^3$

Izreki imajo obliko:

$$\int_{\partial M}\omega = \int_M d\omega \qquad \text{(Stokesov izrek)}$$

nek objekt

Gaussov izrek (Gaussov-ostrogradski izrek)

Naj bo $\Omega \subseteq \mathbb{R}^3$ omejena odprta množica z robom sestavljenim iz končnega števila odsekoma gladkih sklenjenih ploskev, orientiranih z zunanjo normalo glede na Ω

Naj bo $\vec{R}: \Omega \to \mathbb{R}^3$ C^1 vek. polje

Tedaj je

$$\iint_{\partial\Omega} \vec{R}\, d\vec{S} = \iint_{\Omega} \operatorname{div} \vec{R}\, dV$$

??? preveri pravilnost tega

$$\iiint_{D} \operatorname{div} \vec{R}\, dV = \iint_{\partial D} R\, d\vec{S} = \iint_{\partial D} \vec{R} \cdot \vec{N}\, dS$$

$n=2$: $D^{odp} \subseteq \mathbb{R}^2$

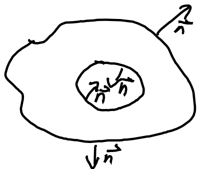
omejena odprta množica v $\mathbb{R}^2$ z odsekoma gladkim robom sestavljenim iz končnega števila odsekoma gladkih sklenjenih krivulj

Naj bo $\vec{R}: \vec{D} \to \mathbb{R}^2$ vekt. polje

Tedaj je

$$\iint_{D} \operatorname{div} \vec{R}\, dS = \int_{\partial D} \vec{R} \vec{n}\, ds$$

zunanje enotske normale



Greenov izrek (greenova formula):

Naj bo $D \subseteq \mathbb{R}^2$ omejena, odp z odsekoma gladkim robom sestavljenim iz končnega števila odsekoma gladkih sklenjenih krivulj, orientiranih pozitivno glede na D

Naj bosta $X, Y \in C^1(\bar{D})$

Tedaj

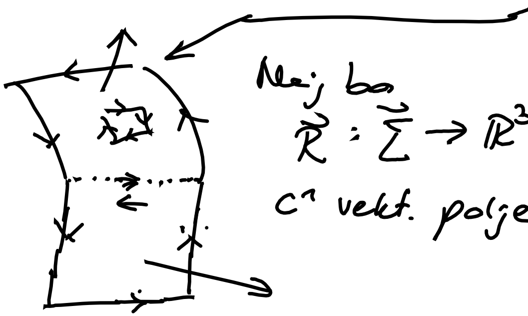
$$\int_{\partial D} X\,dx + Y\,dy = \iint_D (Y_x - X_y)\,dx\,dy$$

$$\parallel$$

$$\int_{\partial D} \vec{R}\,d\vec{r} \qquad \vec{R} = (X, Y)$$

Stokesov izrek:

Naj bo $\Sigma \subseteq \mathbb{R}^3$ omejena, orientirana, odsekoma gladka ploskev z odsekoma gladkim robom, sestavljenim iz končnega števila odsekoma gladkih sklenjenih krivulj orientiranih skladno z orientacijo Σ



Naj bo
$\vec{R} : \vec{\Sigma} \to \mathbb{R}^3$
C^1 vekt. polje

Tedaj je

$$\int_{\partial \Sigma} \vec{R}\, d\vec{r} = \iint_{\Sigma} \operatorname{rot} \vec{R}\, d\vec{S}$$

Zgled

1) $D = K(0, R_0) \quad R_0 > 0$

$\vec{R} = (x, y, z)$

$X(x,y,z) = x$

$Y(x,y,z) = y$

$Z(x,y,z) = z$

$$\iiint_D \operatorname{div} \vec{R}\, dV =$$

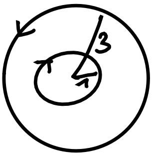
$$= \iiint_D 3\, dV = 3V(D) = 3\frac{4\pi R_0^3}{3} = 4\pi R_0^3$$

$$\iint_{\partial D} \vec{R}\, dS = \iint_{S(0,R_0)} (x,y,z) \overbrace{\frac{(x,y,z)}{R_0}}^{\vec{N}} dS =$$

$$= R_0 \iint_{S(0,R_0)} 1\, dS = R_0 (4\pi R_0^2)$$

2) $D = \{(x,y)\ ;\ 1 < x^2+y^2 < 9\}$

$X = -y \qquad Y = x$



$$\iint_D (X_y - Y_x)\,dx\,dy =$$

$$= \iint_D 2\,dx\,dy = 2\,Pl(D) =$$

$$2(\pi \cdot 9 - \pi \cdot 1) = 16\pi$$

$$\int_{\partial D} X\,dx + Y\,dy = \underbrace{\int_{x^2+y^2=3^2} -y\,dx + x\,dy}_{\bullet)} + \underbrace{\int_{x^2+y^2=1} -y\,dx + x\,dy}_{\bullet\bullet)}$$

•) $x = 3\cos t$ $\qquad dx = -3\sin t\,dt$

$y = 3\cos t \quad t \in [0, 2\pi]$ $\qquad dy = 3\cos t\,dt$

$$\int_{x^2+y^2=3^2} -y\,dx + x\,dy = \int_0^{2\pi} (-3\sin t)(-3\sin t\,dt) +$$

$$+ 3\cos t \cdot 3\cos t\,dt =$$

$$= 9\int_0^{2\pi} 1\,dt = 18\pi$$

••) $X = \cos t$ $\quad t \in$ od 2π do 0

$y = \sin t$

$$\int_{2\pi}^{0} (-\sin t)(-\sin t)\,dt + \cos t \cdot \cos t\,dt =$$

$$= \int_{2\pi}^{0} 1\,dt = -2\pi$$

•) + ••) $= 16\pi$

$D \subseteq \mathbb{R}^2$ kot v greenovi formuli:

$$\int_{\partial D} -y\,dx + x\,dy = 2 \cdot Pl(D)$$

$$Pl = \frac{1}{2} \int_{\partial D} -y\,dx + x\,dy$$

3) $R_0 > 0 \quad \Sigma : \{x,y,z \in \mathbb{R}^3 : x^2+y^2+z^2 = R_0^2, z>0\}$

$\vec{N}(0,0,R_0) = (0,0,1)$

$\vec{R} = (xy, x, x-y+z)$

$\int_{\partial\Sigma} \vec{R}\, d\vec{r}$

parametriziramo $\partial\Sigma$

$x = R_0 \cos t$

$y = R_0 \sin t$

$z = 0 \qquad t \in [0, 2\pi)$

$R(R_0\cos t, R_0 \sin t, 0) =$

$= (0, R_0 \cos t, R_0(\cos t - \sin t))$

$\dot{x} = -R_0 \sin t$

$\dot{y} = R_0 \cos t$

$\dot{z} = 0$

$\vec{R}\cdot\dot{\vec{r}} = R_0^2 \cos^2 t$

$$\int_{\partial\Sigma} \vec{R}\, d\vec{r} = \int_{\partial\Sigma} X dx + Y dy + Z dz = \int_0^{2\pi} -R^2\cos^2 t\, dt =$$

$$= R^2 \int_0^{2\pi} \frac{1+\cos(2t)}{2}\, dt = \pi R_0^2$$

$\text{rot}\,\vec{R} = (-1, y-1, 1-z)$

$$\iint_\Sigma \text{rot}\,\vec{R}\, dS = \iint_\Sigma (1, y-1, 1-z)\cdot \frac{(x,y,z)}{R_0}\, dS$$

$$= \frac{1}{R_0}\iint_\Sigma (-x + y^2 - y + z - z^2)\, dS$$

Parametrizacija polsfere

$x = R_0 \cos\varphi \sin\vartheta \qquad \varphi \in [0, 2\pi)$

$y = R_0 \sin\varphi \sin\vartheta \qquad \vartheta \in [0, \frac{\pi}{2}]$

$z = R_0 \cos\vartheta$

$\sqrt{EG - F^2} = R_0^2 \sin\vartheta$

$$I = \frac{1}{R_0}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} R_0^2 \sin\vartheta \Big(-R_0 \overset{0}{\cos\varphi} \sin\vartheta + R_0^2 \sin^2\varphi \sin^2\vartheta$$
$$- R_0 \overset{0}{\sin\varphi} \sin\vartheta + R_0 \cos\vartheta$$
$$- R_0^2 \cos^2\vartheta\Big)\, d\vartheta\, d\varphi$$

$$= R_0 \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R_0^2 \frac{1-\cos(2\varphi)}{2} \sin^3\vartheta + R_0 \frac{1}{2}\sin(2\vartheta) -$$
$$- R_0^2 \cos^2\vartheta \sin\vartheta\,)\, d\vartheta)\, d\varphi =$$

$$= R_0 \int_0^{\frac{\pi}{2}} \left(\pi R_0 \sin^3\vartheta + \pi R_0 \sin(2\vartheta) - 2\pi R_0^2 \cos^2\vartheta \sin\vartheta\right) d\vartheta$$

$$= R_0^2 \pi \int_0^{\frac{\pi}{2}} R_0 (1-\cos^2\vartheta \sin\vartheta - 2R_0 \cos^2\vartheta \sin\vartheta$$
$$+ \sin 2\vartheta)\, d\vartheta$$

$$= \pi R_0^2 \left(R_0 \int_0^{\frac{\pi}{2}} (1 - 3\cos^2\vartheta)\sin\vartheta + \overbrace{\frac{1}{2}\cos(2\vartheta)\Big|_0^{\frac{\pi}{2}}}^{1}\right)$$

$$= \pi R_0^2 \Big(\underbrace{R_0 \int_0^1 (1-3t^2)\,dt + 1}_{R_0(1-1)+1}\Big)$$

$$\rightsquigarrow = \pi R_0^2$$

$\vec{R} = (zy, x, x-y+z)$

$\text{rot}\,\vec{R} = (-1, y-1, 1-z$

$\Sigma \cup \{(x,y,0)\ x^2+y^2 \le R^2\}$

Gussov izrek

$$\iint_{\partial D} \text{rot}\,R\, d\vec{S} = \iiint_D \text{div}(\text{rot}\,\vec{R})\, dV$$

$$\iint_\Sigma \text{rot}\,\vec{R}\, d\vec{S} + \iint_\Delta \text{rot}\,\vec{R}\, d\vec{S}$$

$N = (0,0,-1)$

$$\iint_\Delta \text{rot}\,\vec{R}\, d\vec{S} = \iint_\Delta (-1, y-1, 1)\ (0,0,-1)\, dS$$

$$= -\iint_\Delta 1\, dS = -\pi R_0^2$$

Gaussov izrek v $\mathbb{R}^2$

$\Rightarrow$ Greenova formula

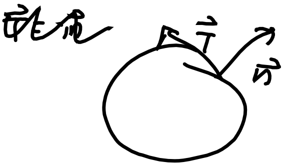
Dokaz v Greenovi formuli:

V Gaussovem izrek za $n=2$

$$\int_{\partial D} X\,dx + Y\,dy = \int_{\partial D} \vec{R}\,d\vec{r} = \int_{\partial D} \vec{R}\cdot\vec{T}\cdot ds =$$

∂D je pozitivno orientiran

$$= \int_{\partial D} (X\cdot T_1 + Y T_2)\,ds \qquad \vec{T} = (T_1, T_2)$$



$\vec{T} = i\vec{n}$

$\vec{n} = -i\,\vec{T}$

$$\int_{\partial D} \vec{R}\vec{T}\,ds = \int_{\partial D} (-i)\vec{R}\cdot(-i\vec{T})\,ds$$

$\vec{R} = (X, Y)$ "$-iR$" $= (Y, -X)$

$\vec{n} = (T_2, -T_1)$ = "$T_1 + i\,T_2$"

$$= \int_{\partial D} \big(\underbrace{(Y\cdot T_2) + (-X)(-T_1)}_{(Y,-X)\cdot\vec{n}\,ds}\big)\,ds$$

gauss

$$= \iint_D (Y_x - X_y)\,dx\,dy$$

stoaksov izrek $\Rightarrow$ greenova formula

$$\int_{\partial\Sigma} \vec{R}\, d\vec{r} = \iint_{\Sigma} \operatorname{rot} \vec{R}\, d\vec{S}$$

$\Sigma = D$, $\vec{N} = (0,0,1)$

$$\vec{R} = (X_{(x,y)}, Y_{(x,y)}, 0)$$

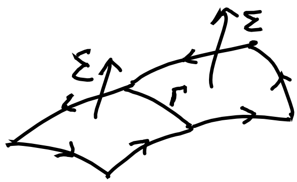
$$\operatorname{rot} \vec{R} = (0, 0, Y_x - X_y)$$

$$\Rightarrow \int_{\partial D} R\, d\vec{r} = \iint_D (Y_x - X_x)\, dx\, dy$$

$$\Sigma = \Sigma_1 \cup \Sigma_2 \subseteq \mathbb{R}^3$$

Ploskev v $\mathbb{R}^3$, orientirane, omejena z odsekoma gladkim robom

$$\overline{\Sigma}_1 \cap \overline{\Sigma}_2 = \Gamma \text{ krivulje}$$



$\vec{R} : \overline{\Sigma} \to \mathbb{R}^3$ C^1 vekt. polje

$$\iint_{\Sigma} \operatorname{rot} \vec{R}\, d\vec{S} = \iint_{\Sigma_1} \operatorname{rot} \vec{R}\, dS + \iint_{\Sigma_2} \operatorname{rot} R^2\, dS =$$

presek ima ploščino 0

$$= \iint_{\partial \Sigma_1} \vec{R}\, dr + \int_{\partial \Sigma_2} \vec{R}\, d\vec{r}$$

če velja Stokesov izrek za Σ_1 in Σ_2 potem velja tudi za Σ

$$= \iint_{\partial \Sigma_1 - \Gamma} R\, d\vec{r} + \underbrace{\int_{\Gamma} \vec{R}\, dr + \int_{\Gamma} \vec{R}\, dr}_{=0} + \int_{\partial \Sigma_2 - \Gamma} R\, d\vec{r} = \int_{\partial \Sigma} R\, d\vec{r}$$

orientiran glede na Σ_1

orientira glede na Σ_2

Greenova formula $\Rightarrow$ Stokesov izrek

Omejimo se lahko na primer, ko je Σ graf nad $D \subseteq \mathbb{R}^2$

$$\Sigma = \{(x, y, f(x,y)); x, y \in D \subseteq \mathbb{R}^2$$

$$f \in C^2(\bar{D})$$

Σ naj bo orientirana z normalo, ki kaže navzgor $\vec{N} = \dfrac{(-f_x, f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$

$\partial\Sigma$ je graf f nad ∂D

Pozitivna orientacija ∂D nam da pozitivno (oz skladno) orientacijo $\partial\Sigma$

$$\vec{R} = (X, Y, Z)$$

$$\int_{\partial\Sigma} \vec{R}\, d\vec{r} = \int_{\partial\Sigma} X dx + Y dy + Z dz =$$

Na $\partial\Sigma$ je $z = f(x,y)$ torej

$$dz = f_x dx + f_y dy$$

$$= \int_{\partial D} X(x, y, f(x,y)) dx + Y dy + Z(f_x dx + f_y dy) =$$

$$= \int_{\partial D} (X + Z f_x) dx + (Y + Z f_y) dy =$$

$$\overset{G.F}{=} \iint_D \left((Y + Z f_y)_x - (X + Z f_x)_y\right) dx dy =$$

$$= \iint_D Y_x + Y_z f_x + Z_x f_y + Z_z f_x \cdot f_y + Z f_{yx}$$
$$- (X_y + X_z f_y + Z_y f_x + Z_z \cdot f_y \cdot f_x + Z f_{xy}) dx dy$$

$$\operatorname{rot}\vec{R} = (Z_y - Y_z, X_z - Z_x, Y_x - X_y)$$

$$= \iint_D (-f_x)(Z_y - Y_z) + (-f_y)(X_z - Z_x) +$$
$$+ (Y_x - X_y) dx dy =$$

$$= \iint_D \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}} \cdot (\operatorname{rot}R)(x,y) \underbrace{\sqrt{1+f_x^2+f_y^2}}_{dS}$$

$$= \iint_\Sigma \operatorname{rot}\vec{R}\, \vec{N}\, dS$$

Zgledi:

$$\vec{R} = \frac{\vec{r}}{\|\vec{r}\|^3} = \vec{\nabla}\left(-\frac{1}{r}\right.$$

19.3

primer

(ostalo je)

Gaussov izrek $(n=3)$

Če gaussov izrek velja za D_1 in D_2 $\Rightarrow$ velja za D

$$\iiint_D \operatorname{div}\vec{R}\,dV = \iiint_{D_1} \operatorname{div}\vec{R}\,dV + \iiint_{D_2} \operatorname{div}\vec{R}\,dV$$

Dokaz (Gaussovega izreka)

ko je Ω take oblike, da med vsako od treh koordinatnih ravnin leži med dvema grafoma C^1 funkcij nad omejeno odprto množico s ploščino

Ω $\vec{R} = (x,y,z)$

$\vec{N} = (N_x, N_y, N_z)$

$$\iint_{\Omega} \vec{R}\vec{N}\,dS = \iiint_{\Omega} \operatorname{div}\vec{R}\,dV$$

oziroma

$$\iint_{\partial\Omega} x(y_x + y_z + zN_z)\,dS =$$

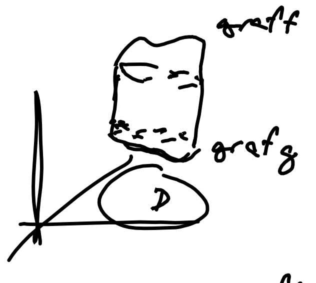
$$\overset{?}{=} \iiint_{\Omega} (x_x + y_y + z_z)\,dV$$

Oblika Ω implicira

$$\iint_{\partial\Omega} xN_x\,dS = \iiint_{\Omega} x_x\,dV$$

$$\iint_{\partial\Omega} yN_y\,dS = \iiint_{\Omega} y_y\,dV$$

$$\iint_{\partial\Omega} zN_z\,dS = \iiint_{\Omega} z_z\,dV$$



$$\iiint_{\Omega} z_z\,dV = \iint_D \left(\int_{g(x,y)}^{f(x,y)} z_z\,dz\right)dx\,dy =$$

$$= \iint_D \big(z(x,y,f(x,y)) - z(x,y,g(x,y))\big)\,dx\,dy$$

$$\iint_{\partial\Omega} zN_z\,dS) = \iint_{\text{graf}(f)} zN_z\,dS + \iint_{\text{navpični del}} zN_z\,dS + \iint_{\text{graf } g} z\cdot N_z\,dS$$

$$\vec{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}} \qquad N_z = 0 \qquad N_z = \frac{(g_x, g_y, -1)}{\sqrt{1+g_x^2+g_y^2}}$$

$$= \iint z(x,y,f(x,y))\frac{1}{\cancel{\sqrt{1+f_x^2+f_y^2}}} \cdot \overbrace{\cancel{\sqrt{1+f_x^2+f_y^2}}\,dx\,dy}^{dS} +$$

$$+ 0 +$$

$$+ \iint z(x,y,g(x,y))\frac{(-1)}{\cancel{\sqrt{1+f_x^2+f_y^2}}} \cdot \overbrace{\cancel{\sqrt{1+f_x^2+f_y^2}}\,dx\,dy}^{dS}$$

■

Greenovi identiteti

Naj bo Ω v $\mathbb{R}^3$ kot v Gaussovem izreku

Naj bo $u, v \in C^2(\bar{\Omega})$

Tedaj velja:

Laplace

1) $$\iint_{\partial\Omega} u \frac{dv}{d\vec{N}}\, dS = \iiint_{\Omega} (\vec{\nabla} u\, \vec{\nabla} v + u \Delta v)\, dV$$

2) $$\iint_{\partial\Omega} \left(u \frac{dv}{d\vec{N}} - v \frac{du}{d\vec{N}}\right) dS = \iiint_{\Omega} (u \Delta v - v \Delta u)\, dV$$

Opombe: ·) Enotska zunanja normala na $\partial\Omega$

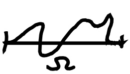
··) $\frac{\partial u}{\partial \vec{N}}$ = smerni odvod u v smeri $\vec{N}$
$= \vec{\nabla} u \cdot \vec{N}$

···) $\frac{du}{d\vec{n}} dS = \nabla \vec{u}\, \vec{n} \cdot dS = \vec{\nabla} u\, d\vec{S}$

····) "integracija po delih"

(·····) u ; $\Delta u = 0$ harmonična

$$\Rightarrow \iiint_{\Omega} v \Delta u\, dV = 0$$

u : $\forall v$  na robu je 0

$$\iiint_{\Omega} u \Delta v\, dV = 0 \overset{?}{\Longrightarrow} u \text{ harmonična}$$)

Dokaz:

$$\vec{R} = u\vec{\nabla}v = (uv_x, uv_y, uv_z)$$

$$\operatorname{div}\vec{R} = (uv_x)_x + (uv_y)_y + (uv_z)_z =$$

$$= u_x v_x + u_y v_y + u_z v_z + u\Delta v =$$

$$= \vec{\nabla}u \cdot \vec{\nabla}v + u\Delta v$$

1)

$$\iint_{\partial\Omega} u\frac{dv}{d\vec{n}}\,dS = \iint_{\partial\Omega} u\vec{\nabla}v\cdot\vec{n}\cdot dS = \iint_{\partial\Omega} \overbrace{u\vec{\nabla}v}^{\vec{R}}\cdot d\vec{S}$$

Gaussov

$$= \iiint_{\Omega} \operatorname{div}(u\vec{\nabla}v)\,dS = \iiint_{\Omega} (\vec{\nabla}u\vec{\nabla}v + u\Delta v)\,dV$$

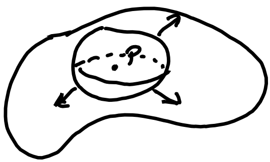
2) Velja tudi

$$\iint_{\partial\Omega} v\frac{du}{d\vec{n}}\,dS = \iiint_{\Omega} (\vec{\nabla}v\cdot\vec{\nabla}u + v\Delta u)\,dV$$

o dštejemo

Brezkoordinatna definicija divergence:

$$\vec{R} : D \overset{\text{odp}}{\subseteq} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad C^1$$



$P \in D \; ; \; \varepsilon > 0$

Velja: $$\iint\limits_{S(P,\varepsilon)} \vec{R}\, d\vec{S} = \iiint\limits_{K(P,\varepsilon)} \operatorname{div} R \, dV$$

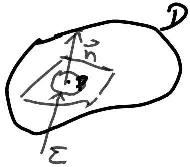
$$\frac{1}{V(K(P,\varepsilon))} \iint\limits_{S(P,\varepsilon)} \vec{R}\, d\vec{S} = \frac{1}{V(K(P,\varepsilon))} \iiint\limits_{K(P,\varepsilon)} \operatorname{div} \vec{R}\, dV$$

$$\Big\downarrow \; \varepsilon \to 0$$

$$\lim_{\varepsilon \to 0} = \frac{3}{4\pi\varepsilon^3} \iint\limits_{S(P,\varepsilon)} \vec{R}\, d\vec{S} = \operatorname{div} \vec{R}\,(P)$$

pretok $\vec{R}$ skozi sfero

Brezkoordinatna definicija rotacije



$p \in D$

$\vec{R} : D \to \mathbb{R}^3$

$|\vec{n}| = 1$

Σ krog v ravnini s središčem v p in polmerom ε

$$\int_{\partial\Sigma} \vec{R}\, d\vec{r} = \iint_{\Sigma} \operatorname{rot} \vec{R}\, \vec{n}\, dS$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi \varepsilon^2} \int_{\partial\Sigma} \vec{R}\, d\vec{r} = (\operatorname{rot} \vec{R})(p) \cdot \vec{n}$$

Diferencialne forme $\Omega \overset{odp}{\subseteq} \mathbb{R}^3$

0-forme: funkcije na Ω (vsaj C^2

$u: \Omega \to \mathbb{R}$ odvod - diferencial

$$du = u_x dx + u_y dy + u_z dz$$

dx = diferencial funkcije $(x,y,z) \mapsto x$

dy = –||–

dz = –||–

$$du(\vec{v}) = u_x v_1 + u_y v_2 + u_z v_3$$