Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x\,dx$$

$FV(f)$ konvergira k f, če je f zvezna v x, če pa ni, pa konvergira k $\frac{f(x^-) + f(x^+)}{2}$

$$\int x \cos(nx)\,dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int x \sin(nx)\,dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

f soda $\Rightarrow b_n = 0$

f liha $\Rightarrow a_n = 0$

$$\int x^2 \cos(nx)\,dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx)\,dx = \frac{-x^2}{n} \cos x + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

(2)

$f(x) = |x|$ razvij v Fourierovo vrsto in seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx \underset{\text{sodost}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Big|_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(nx) \overset{\text{lihost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2}\cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi}\cdot\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{1}\frac{1}{(2k+1)^2} = -\frac{\pi}{2}\cdot\frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad =? \quad S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

S ... lih.

S' ... ostalo

$$= S + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) =$$

$$= S + \frac{1}{4}\Sigma = \Sigma$$

$$S = \frac{3}{4}\Sigma$$

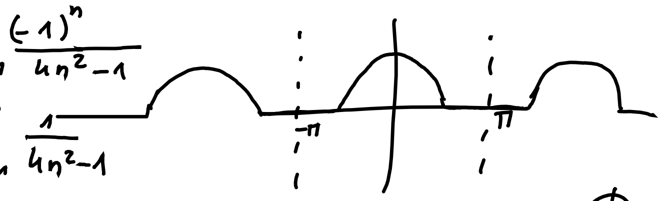
$$\Sigma = \frac{4}{3}S = \frac{\pi^2}{6}$$

③

$f(x) = \max(\cos x, 0)$

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(x) = \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

ničelnost $f(x)$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos(nx) + \cancel{2\int_{\frac{\pi}{2}} \cos(nx)\,dx}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right) =$$

$n \neq 1$

$$\frac{1}{\pi}\left[\frac{1}{n+1}\sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1}\sin\left((n-1)\frac{\pi}{2}\right)\right]$$

$$n = 4k: \frac{1}{\pi}\frac{1}{n+1}\cdot 1 + \frac{1}{n-1}(-1)\frac{1}{\pi} = \frac{-2}{n^2-1}\frac{1}{\pi} \quad \frac{-2}{16k^2-1}\cdot\frac{1}{\pi}$$

n lih: 0

$$n = 4k+2: \frac{1}{\pi}\frac{1}{4k+3}\sin\frac{3\pi}{2} + \frac{1}{\pi}\frac{1}{4k+1}\sin\frac{\pi}{2} =$$

$$= \frac{1}{\pi}\left(-\frac{1}{4k+3} + \frac{1}{4k+1}\right)$$

$$= \frac{1}{\pi}\begin{cases} 0 & ; n \text{ liho} \\ \frac{(-1)^{m+1}\cdot 2}{(2n)^2-1} & ; n = 2m \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{1}{\pi} + \frac{\sin(2x)}{2}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}$$

$b_n = 0$ ker f soda

$$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\cdot 2}{(2m)^2-1}\cos(2mx)$$

$$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}S_1 \Rightarrow$$

$$S_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)\cdot\frac{\pi}{2}(-1) = \frac{1}{2} - \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2-1}\cdot\underbrace{\cos(m\pi)}_{= (-1)^m} =$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\underbrace{\sum_{m=1}^{\infty} \frac{1}{4m^2-1}}_{= S_2}$$

$$S_2 = -\frac{1}{\pi}\cdot\frac{\pi}{2} = -\frac{1}{2}$$

(4)

$f(x) = x^2 \qquad f: [0,\pi] \longrightarrow \mathbb{R}$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v sinusno FV in skiciraj njen graf

1) če f razširimo do sode funkcije

$f_s : [-\pi,\pi] \longrightarrow \mathbb{R}$

$x<0 \Rightarrow f_s(x) = f(-x)$

$FV_{\cos}(f)(x) = FV(f)(x)$

2) $f_l : [-\pi,\pi] \longrightarrow \mathbb{R} \qquad f_l(x) = -f(-x)$

$FV_{\sin}(f)(x) = FV(f_l)(x)$

1) $f_s(x) = x^2$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$b_n = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$

Poglejmo:

$\int x^2 e^{inx} =$

$u = x^2 \qquad dv = e^{inx}dx$

$du = 2x\,dx \qquad v = \frac{1}{in}e^{inx}$

$\boxed{\int x e^{inx} = -\frac{i\,x\,e^{ix}}{n} + \frac{e^{inx}}{n^2}}$

$= x^2\cdot\frac{e^{inx}}{in} - \int 2x\frac{e^{inx}}{in}dx \overset{\text{od prej}}{=}$

$= x^2\cdot\frac{e^{inx}}{in} - \frac{2}{in}\left(\frac{ixe^{inx}}{n} + \frac{e^{inx}}{n^2}\right) + C$

$= \underset{\substack{\shortparallel \\ \cos nx + i\sin nx}}{e^{inx}}\left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3}\right)$

$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$

$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$

$a_n = \frac{1}{\pi}\left(\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) + \frac{2}{n^3}\sin(nx)\right)\Big|_{-\pi}^{\pi} =$

$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right) = \frac{4(-1)^n}{n^2}$

$FV\cos(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(x$

b) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\text{soda}}{f_p(x)\sin(x)} = \frac{2}{\pi} \int_0^{\pi} \underbrace{f_p(x)\sin(nx)}$

$= f(x) \text{ na } [0,\pi] = x^2$

$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2}{n^2}\sin nx + \frac{2}{n^3}(\cos(nx))\right]\Big|_0^{\pi}$

$= \frac{2}{\pi}\left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right)$

$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left((-1)^{n+1}\frac{\pi^2}{2} + \frac{2(1-(-1)^n)}{n^3}\right)\sin(nx)$

$f(x) = x(\pi - x)$ [illegible] ۲

⑥

$f(x)=\sin^3 x$ razvij v FV

Pred premislek:

$f(x)=\sin 2x$ je že FV

$b_2=1$, ostali so 0

$$f(x)=\sin^2 x=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)=$$

$$=\frac{1}{2}\left(\cos(0)-\cos(2x)\right)=\frac{1}{2}-\frac{1}{2}\cos 2x$$

$$f(x)=\sin^2 x\cdot\sin x=\frac{1}{2}\sin x-\frac{1}{2}\cos 2x\,\sin x=$$

$$=\frac{1}{2}\sin x-\frac{1}{2}\cdot\frac{1}{2}\left(\sin(3x)-\sin(x)\right)=$$

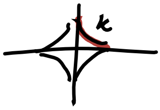
$$=\frac{3}{4}\sin x-\frac{1}{4}\sin(3x)$$

polinom

$\forall p(\sin x,\cos x)$ ima končno furierove vrsto

$p\in\mathbb{R}[x,y]$

$a>0$ Izračunaj težišče homogenega loka astroide



$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

$$x_T=\frac{\int_K x\,dm}{m(K)}=\frac{\int_K x\rho\,ds}{\int_K \rho\,ds} \qquad = l(K)$$

$\vec{r}(t)$ ... parametrizacija

$$ds=|\dot{\vec{r}}(t)|dt=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt$$

u ... skalarno polje

$$\int_K u\,ds=\int_a^b u(\vec{r}(t))|\dot{\vec{r}}(t)|dt$$

$x=a\cos^3 t$

$y=a\sin^3 t$

$t\in[0,\frac{\pi}{2}]$ ← ker je vse pozitivno (gledamo pravi lok)

$$\dot{\vec{r}}(t)=(-3a\cos^2 t\cdot\sin t,\ 3a\sin^2 t\cos t)$$

$$|\dot{\vec{r}}|=3a\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}=$$

$$=3a\cos t\sin t$$

$$l(K)=\int_0^{\frac{\pi}{2}}|\dot{\vec{r}}(t)|dt=\int_0^{\frac{\pi}{2}}3a\cos t\sin t\,dt=$$

$u=\sin t\,dt$

$du=\cos t\,dt$

$$=3a\int_0^1 u\,du=\frac{3}{2}a$$

$$\int_K x\,ds=\int_0^{\frac{\pi}{2}}a\cos^3 t\,3a\cos t\sin t\,dt=$$

$\cos t=u \qquad du=-\sin t$

$$=3a^2\int_0^1 u^4\,du=\frac{3}{5}a^2$$

$$y_T=\frac{3}{5}a^2$$

$a > 0 \quad \alpha \in [0, 2\pi]$

$K = S(0, a) \cap \Pi$

$\Pi : y = x \tan\alpha$

$I : \int_K (y-z) + (z-x)\,dy + (x-y)\,dz$

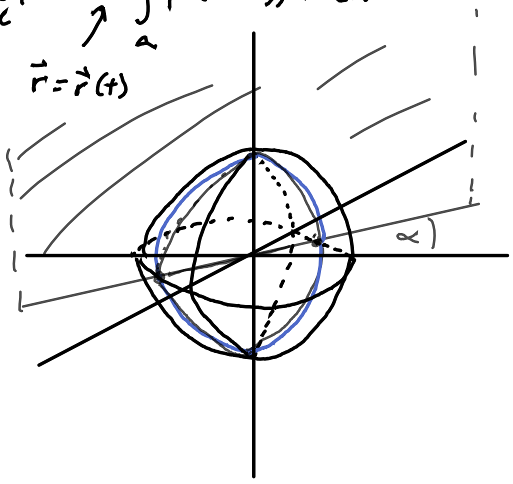
$\vec{f}(x,y,z) = (y-z,\ z-x,\ x-y)$ vektorsko polje

$I = \int_K \vec{f}\, d\vec{r}$

$\vec{r} = (x,y,z) \Rightarrow d\vec{r}$

$\int_K \vec{f}\, d\vec{r} = \int_a^b \vec{f}(\vec{r}(t))\, \dot{\vec{r}}(t)\, dt$

$\vec{r} = \vec{r}(t)$



Parametrizacija K

$x = a\cos\alpha \cdot \cos\vartheta$

$y = a\sin\alpha \cos\vartheta$

$z = a\sin\vartheta$

$r(\vartheta) = (a\cos\vartheta\cos\alpha,\ a\cos\vartheta\sin\alpha,\ a\sin\vartheta)$

$\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\dot{\vec{r}}(\vartheta) = (-a\sin\vartheta\cos\alpha,\ -a\sin\vartheta\sin\alpha,\ a\cos\vartheta)$

$\vec{f}(\vec{r}(\vartheta)) = a(\cos\vartheta\sin\alpha - \sin\vartheta,\ \sin\vartheta - \cos\vartheta\cos\alpha,$
$\cos\vartheta\sin\alpha - \cos\vartheta\cos\alpha)$

$\vec{f} \cdot \dot{\vec{r}} = a^2(-\cos\alpha\cos\cancel{\vartheta\sin\alpha}\sin\vartheta + \sin^2\vartheta\cos\alpha\, \cancel{\cos\alpha}$

$- \sin^2\vartheta\sin\alpha + \cos\vartheta\cancel{\cos\alpha}\sin\vartheta\sin\alpha +$

$+ \cos^2\vartheta\sin\alpha - \cos^2\vartheta\cos\alpha) =$

$= a^2(\cos 2\vartheta \sin\alpha - \cos 2\vartheta \cos\alpha)$ ✗

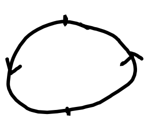
Ne

$= a^2(\cos\alpha - \sin\alpha$

$\int_K \vec{f}\, d\vec{r} = \int_0^{2\pi} a^2(\cos\alpha - \sin\alpha)\, d\vartheta =$

$2\pi a^2(\cos\alpha - \sin\alpha)$

↑ pri orientaciji



$a, b, c \in \mathbb{R}$

$$I = \int_S a\,dydz + b\,dxdz + c\,dxdy$$

$S = \{(x,y,z) ;\ x^2+y^2+z^2=1,\ x,y,z \geq 0\}$

$$I = \int \vec{f}\, d\vec{S}$$

$\vec{f} = (a,b,c)$

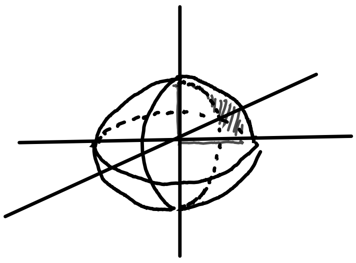
enotske normale

$\int_S (\vec{f} \cdot \vec{N})\, dS$ orientacija ploskve je usmerjena s smerjo normale

$$\int_S \vec{f}\, d\vec{S} = \int_D \vec{f}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\,dv$$

$r = \vec{r}(u,v)$

smer se ujema s predpisano orientacijo



Sferične koordinate

$x = \cos\vartheta \cos\varphi$ $\quad \vartheta \in [0, \frac{\pi}{2}]$

$y = \cos\vartheta \sin\varphi$ $\quad \varphi \in [0, \frac{\pi}{2}]$

$z = \sin\vartheta$

$\vec{r}(\vartheta, \varphi) = (\cos\vartheta \cos\varphi, \cos\vartheta \sin\varphi, \sin\vartheta)$

$\vec{r}_\vartheta = (-\sin\vartheta \cos\varphi, -\sin\vartheta \sin\varphi, \cos\vartheta)$

$\vec{r}_\varphi = (-\cos\vartheta \sin\varphi, \cos\vartheta \cos\varphi, 0)$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2\vartheta \cos\varphi,\ \cos^2\vartheta \sin\varphi,\ -\sin\vartheta\cos\vartheta\cos^2\varphi - \cos\vartheta\sin\vartheta\sin^2\varphi) =$$

$$= (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta) =$$

$$= -\cos\vartheta\,(\underbrace{\cos\vartheta\cos\varphi, -\cos\vartheta\sin\varphi, \sin\vartheta})$$

$$\int_S \vec{f}\,d\vec{S} = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (-a\cos^2\vartheta\cos\varphi + b\cos^2\vartheta\sin\varphi - c\sin\vartheta\cos\vartheta)\, d\vartheta$$

$$= \iint_0^{\frac{\pi}{2}} \left((-a\cos\varphi + b\sin\varphi)\cos^2\vartheta - \frac{c}{2}\sin 2\vartheta\right) d\vartheta =$$

Prevedemo na B funkcijo

$$= \int_0^{\frac{\pi}{2}} \left((-a\cos\varphi + b\sin\varphi)\frac{\pi}{4} - \frac{c}{2}\right) =$$

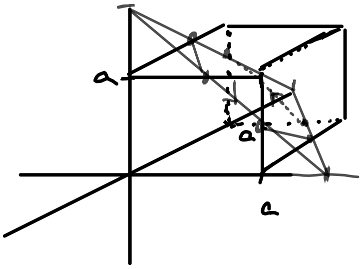
$$= \left(-\frac{\pi a}{4}\sin\varphi + \frac{\pi}{4} b\cos\varphi - \frac{c}{2}\varphi\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4}a - \frac{\pi}{4}b - \frac{\pi}{4}c = -\frac{\pi}{4}(a+b+c)$$

$a > 0$

$\vec{f}(x,y,z) = (y^2 - z^2,\ z^2 - x^2,\ x^2 - y^2)$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z = \frac{3a}{2}$



krivulja 6 kotnik

cirkulacija.... integral vektorskega polja vzdolž sklenjene krivulje

K_1:

$x = x$

$y = 0$

$z = \frac{3a}{2} - x$

$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$

$$\int_{K_1} \vec{f}\,d\vec{r}$$

$\dot{\vec{r}}(x) = (1, 0, -1)$

$\vec{f}(r(x)) = (0 - (\frac{3}{2} - x)^2,\ (\frac{3}{2}a - x)^2 - x^2,\ x^2 - 0)$

$\vec{f} \cdot \dot{\vec{r}} = -(\frac{3}{2}a - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$

$$\int_{a/2}^{a} \left(-(\tfrac{3}{2}a - x)^2 - x^2\right) dx =$$

$$-\frac{1}{3}(\tfrac{3}{2}a - x)^3 \Big|_{a/2}^{a} - \frac{1}{3}x^3 \Big|_{a/2}^{a} =$$

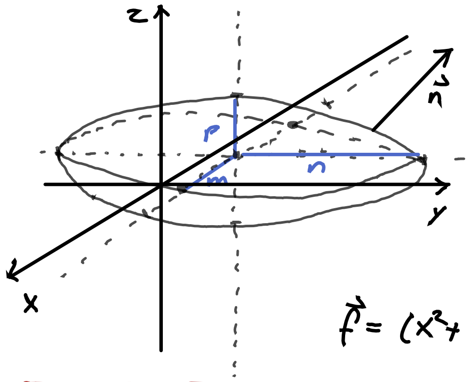
$$= -\cancel{\frac{1}{3}\frac{a^3}{8}} + \cancel{\frac{1}{3}a^3} - \cancel{\frac{1}{3}a^3} + \cancel{\frac{1}{3}\frac{a^3}{8}} = 0$$

$m, n, p > 0$
$a, b, c \in \mathbb{R}$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanja stran te ploskve



$$\vec{f} = (x^2 + y^2 + z^2)$$

$$\boxed{I = \int_{S = \partial D} \vec{f}\, d\vec{S} = \int_D \operatorname{div} \vec{f}\, dV \quad \text{gaussov izrek}}$$

Normala mora biti zunanja

$$\int_D (2x + 2y + 2z)\, dV = 2x_T + 2 \cdot y_T + 2 z_T$$

$$x_T = \int_D x\, dV = x_T \cdot V(D) = a V(D)$$

enotska krogla B: $V(B) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

$$\Downarrow$$

$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3}\pi$$

D dobimo če B raztegnemo za faktor m v smer: x, n, p v drugih dveh

B opišemo:

$x = r\cos\vartheta\cos\varphi$
$y = r\cos\vartheta\sin\varphi$ $\quad r \in [0, 1]$
$z = r\sin\vartheta$

D opišemo:

$x = m r \cos\varphi \cos\vartheta$
$y = n r \cos\vartheta \sin\varphi$
$z = p r \sin\vartheta$

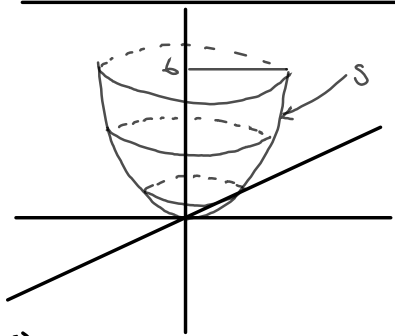
$$I = 2 \frac{4}{3}\pi m \cdot n \cdot p\, (a + b + c)$$

$\vec f(\vec r) = |\vec r|^2 \cdot \vec r \qquad b>0$

izračunaj pretok $\vec f$ skozi

a) rob območja $D = \{x,y,z;\ 2z \geq x^2+y^2;\ z \leq b\}$

b) ploskev $2z = x^2+y^2;\ z \leq b$



Pretok skozi S:
$\int_S \vec f\, d\vec S$

a)
$$\int_{\partial D} \vec f\, d\vec S = \int_D \operatorname{div} \vec f\, dV$$

$$\vec f(x,y,z) = (x^2+y^2+z^2)(x,y,z) =$$
$$(x(x^2+y^2+z^2),\ y(x^2+y^2+z^2),\ z(x^2+y^2+z^2)) =$$
$$(x^3+y^2x+z^2x,\ x^2y+y^3+z^2y,\ x^2z+y^2z+z^3)$$

$$\operatorname{div} \vec f = 3x^2+y^2+z^2 + x^2+3y^2+z^2+y^2+x^2+3z^2$$
$$= 5x^2+5y^2+5z^2$$

$$\int_D \operatorname{div} \vec f\, dV = 5\int_D (x^2+y^2+z^2)\, dV$$

$x = r\cos\varphi \qquad r^2 = x^2+y^2 = 2z$
$y = r\sin\varphi \qquad z = \frac{r^2}{2}$
$z = z$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^{b} (r^2+z^2)\, r\, dz$$
$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \tfrac{1}{3} z^3\right)\Big|_{\frac{r^2}{2}}^{b} dr$$
$$= 10\pi \int_0^{\sqrt{2b}} \left(b r^3 + \tfrac{1}{3} b^3 - \tfrac{r^5}{2} - \tfrac{1}{6} r^6\right) dr =$$
$$= 10\pi \left(\tfrac{1}{4} b r^4 + \tfrac{1}{3} b^3 r - \tfrac{1}{12} r^6 - \tfrac{1}{6\cdot 7} r^7\right)\Big|_0^{\sqrt{2b}} =$$
$$= 10\pi \left(\tfrac{1}{4} b^3 \cdot 4 + \tfrac{1}{3} b^3 \sqrt{2b} - \tfrac{1}{12} 8 b^3 - \tfrac{1}{6\cdot 7} 8 b^3 \sqrt{2b}\right)$$

narobe

$$= 10\pi \left(b^3 + \frac{b^4}{3} - \frac{2}{3} b^3 - \frac{b^4}{12}\right) =$$
$$10\pi \left(\frac{b^3}{3} + \frac{b^4}{4}\right)$$

b) $\partial D = S \cup S_0$

$$\int_S \vec f\, d\vec S = \int_{\partial D} \vec f\, d\vec S - \int_{S_0} \vec f\, d\vec S$$

$$\int_{S_0} \vec f\, d\vec S = \int_{S_0} (\vec f \cdot \vec n)\, dS = \qquad \vec n = (0,0,1)$$
$$= \int_{S_0} (zx^2+zy^2+z^3)\, dS = \int_{S_0} (bx^2+by^2+b^3)\, dS =$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$= b\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r(r^2+b^2)\, dr = b\left(\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr + \underbrace{\iint_{S_0} b^2\, dS}\right)$$

$P(S_0) = 2\pi b$

$$= b\left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr\right) =$$
$$= b\left(2\pi b^3 + 2\pi\, \tfrac{1}{4} \cdot 4 b^2\right) =$$
$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \int_S \vec f\, d\vec S = \frac{10}{3}\pi b^3 + \frac{10}{3}\pi b^4 - 2\pi b^4 - 2\pi b^3$$

Izračunaj $I=\int_K \frac{x\,dy - y\,dx}{x^2+y^2}$

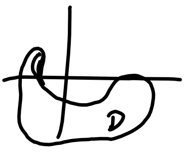
če je $K \subseteq \mathbb{R}^2$ sklenjena krivulja;

a) ki ne obkroži izhodišča

b) ki obkroži izhodišče

Omejimo se na primer: $K=\partial D$ za D z odsekoma gladkim robom

a)



$$\boxed{\int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx\,dy}$$

pozitivna orientacija

greenova formula

$P=\frac{-y}{x^2+y^2}$ $\qquad Q=\frac{x}{x^2+y^2}$

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

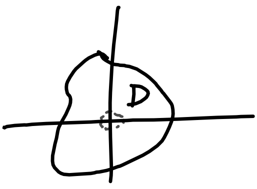
$$P_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\int_D (Q_x - P_y)\,dx\,dy = \int_D 0\,dx\,dy = 0$$

∂D ne obkroži $(0,0) \Longleftrightarrow (0,0) \notin D$

$(x^2+y^2) \neq 0$

b)



Iz D izrežemo dovolj majhen krog $K(0,\varepsilon)$

$D' = D - K(0,\varepsilon)$

Uporabimo greenovo formulo na D'

$$\int_{D'} \cancel{(Q_x + P_y \ldots)} = \int_{\partial D'} (Q_x - P_y)\,dx\,dy = 0$$

$\partial D' = \partial D^+ + \partial K(0,\varepsilon)^-$

$$0 = \int_{\partial D^+} P\,dx + Q\,dy + \int_{\partial K^-} P\,dx + Q\,dy$$

Pazi orientacija obratna kot ponavadi ☹

$$\Longrightarrow \int_{\partial D^+} \ldots = \int_{\partial K^+} \ldots$$

$x = \varepsilon\cos\varphi$ $\qquad y = \varepsilon\sin\varphi$ $\qquad$ ~~$r=\varepsilon$~~

$dx = -\varepsilon\sin\varphi$

$dy = \varepsilon\cos\varphi$

$$\int_{\partial K} \ldots = \frac{1}{\varepsilon}\int_0^{2\pi} \cos^2\varphi + \sin^2\varphi = 2\pi$$

$r_1 \ldots r_n \in \mathbb{R}^3$ različni

$e_1 \ldots e_n \in \mathbb{R}$

$f(\vec r) = \sum_{i=1}^n \operatorname{grad}\left(\frac{-e_i}{4\pi |r-r_i|}\right)$

izračunaj pretok $\vec f$ skozi zaključeno ploskev S, k objema točke $r_1 \ldots r_n$

$r_1 \ldots r_n \in \operatorname{int} D$ $\quad S = \partial D$

$$\int_{\partial D} \vec f \, d\vec S = \int_D \operatorname{div} \vec f \, dV$$

gaussov izrek verjetno ne velja, ker ima f singularne točke v D

$$\operatorname{div}(\vec f(r)) = \sum_{i=1}^n \operatorname{div}\left(\operatorname{grad}\left(\frac{-e_i}{4\pi |r-r_i|}\right)\right)$$

Spomnimo se: $\operatorname{grad}\frac{1}{|\vec r-\vec a|} = -\frac{\vec r-\vec a}{|\vec r-\vec a|^3}$

in $\operatorname{div}\left(\operatorname{grad}\frac{1}{|\vec r-\vec a|}\right) = 0$

Iz D izrežemo krogle $K_i = K(r_i, \varepsilon)$

Tako da $\bar K_i \subset \operatorname{int}(D)$ in $K_i \cap K_j = \emptyset$ $\forall$ za $i \neq j$

$D' = D - \left(\bigcup_{i=1}^n K_i\right)$

Na D' uporabimo gaussov izrek

$$\int_{\partial D'} \vec f(\vec r)\, dS = \int_{D'} \operatorname{div} \vec f \, dV = 0$$

$$\partial D' = \partial D^+ \cup \left(\bigcup_{j=1}^n \partial K_j^-\right) \Rightarrow \int_{\partial D'} = \int_{\partial D^+} \ldots + \sum_{i=1}^n \int_{\partial K_i^-} \ldots = 0$$

$$\Rightarrow \int_{\partial D^+} \ldots = \sum_{i=1}^n \int_{\partial K_i^+} \ldots$$

$$\int_{\partial K_i} \vec f \, d\vec S = \int_{K_i} \sum_{j=1}^n \operatorname{grad}\left(\frac{-e_j}{4\pi |r-r_j|}\right) \cancel{dS}$$

$$\sum_{j=1}^n \int_{\partial K_j} \left(\frac{+e_j}{4\pi} \frac{r-r_j}{|\vec r - \vec r_j|^3}\right) dS$$

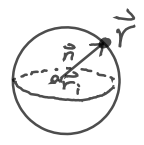
$$\int_{\partial K_j} \frac{\vec r - r_j}{|\vec r - \vec r_j|^3} dS \overset{i \neq j}{\Longrightarrow} r_j \in \bar K_i \Rightarrow$$

lahko uporabimo gaussov izrek

$\Longrightarrow \quad = 0$

$$\sum_{j=1}^n \int_{\partial K_i} \frac{e_j}{4\pi} \frac{\vec r - \vec r_j}{|\vec r - \vec r_j|^3} d\vec S = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{\vec r - \vec r_i}{|\vec r - \vec r_i|^3} d\vec S$$

$d\vec S = \vec n \, dS$



$\vec n = \frac{\vec r - \vec r_i}{|\vec r - \vec r_i|}$

$$= \frac{e_i}{4\pi} \int_{\partial K_i} \frac{|\vec r - \vec r_i|^2}{|\vec r - \vec r_i|^4} dS = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{dS}{|r - r_i|^2} =$$

na ∂K_i je $|r - r_i| = \varepsilon$

$$= \frac{e_i}{4\pi} \int \frac{dS}{\varepsilon^2} =$$

$$\frac{e_i}{4\pi\varepsilon^2} P(\partial K_i) = \frac{e_i}{4\pi\varepsilon^2} 4\pi\varepsilon^2 = e_i$$

$$\Rightarrow \int f \, d\vec S = \sum_{i=1}^n e_i$$

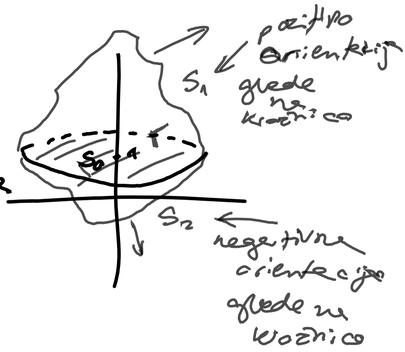
$$I = \int_S (1+x^2) f(x)\,dy\,dz - 2xy\, f(x)\,dz\,dx - 3z\,dx\,dy$$

Določi: $f \in C^1$ da I enak ~~za~~ za vse ploskve S, katerih rob je krožnica $\{(\cos t, \sin t, 1)\}$

Določi to vrednost za I

$$\vec{f} = ((1+x^2)f(x),\ 2xy\, f(x),\ -3z)$$

$$\Longrightarrow I = \int_S \vec{f}\, d\vec{S}$$



$S_1 \cup S_2$ je sklenjena ploskev

$S_1^+ \cup S_2^- = \partial D^+$

Gauss na D:

$$\underbrace{\int_{\partial D^+} \vec{f}\, d\vec{S}}_{\partial D = \int_{S_1} - \int_{S_2} = 0} = \underbrace{\int_D \operatorname{div} \vec{f}\, dV}_{\operatorname{div}\vec{f}\, dV = 0}$$

$$\partial D = \int_{S_1} - \int_{S_2} = 0 \Longleftarrow \operatorname{div}\vec{f}\, dV = 0$$

$\vec{f}$ je solenoidalno $(\operatorname{div}\vec{f} = 0) \Longleftrightarrow f(x)$ je oblike $3\arctan x + C$

$$\int_S \vec{f}\, d\vec{S} = \int_S ((1+x^2)(3\arctan x + c),\ -2xy(3\arctan x + c),\ -3z)\, d\vec{S}$$

$$= \int_{S_0} \vec{f}\, dS =$$

samo krog: $d\vec{S} = \vec{n} \cdot dS \quad \vec{n} = (0,0,1)$

$$= \int_{S_0} -3z\, dS = -3 \int_{S_0} z\, dS = -3 z_T\, P(S_0) =$$

$$= -3 \cdot 1 \cdot \pi$$

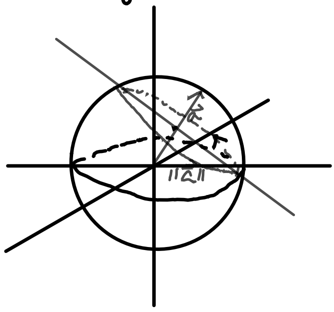
Opomba: Lahko bi dokazali da je

tudi $\operatorname{div}\vec{F}$ potreben pogoj

$\vec{a} \neq \vec{0}$,
$\vec{b} \in \mathbb{R}$

$\vec{F}(\vec{r}) = (\vec{r}-\vec{a}) \times (\vec{r}-\vec{b})$

izračunaj cirkulacijo $\vec{F}$ vzdolž krivulje K: $|\vec{r}| = |\vec{a}| \wedge \vec{r}\cdot\vec{a} = \frac{|\vec{a}|^2}{2}$



enačba ravnine:

$\vec{r}\cdot\vec{n} = d$

normala

ravnina z normalo $\vec{a}$ in točka $\vec{r} = \frac{\vec{a}}{2}$

K je krožnica

B naj bo krog z robom $K = \partial B$

$$\int_{\partial B} \vec{f}\, d\vec{r} = \int_{B} \operatorname{rot} \vec{f}\, d\vec{S}$$

stokesov izrek

normala: B: $\vec{n} = \frac{\vec{a}}{\|\vec{a}\|}$

$$\operatorname{rot} \vec{f} = \operatorname{rot}(\underbrace{\vec{r}\times\vec{r}}_{0} - a \times r - \vec{r}\times\vec{b} + \vec{a}\times\vec{b}) =$$

$$\boxed{\text{Vemo: } \operatorname{rot}(\vec{r}\times\vec{c}) = -2\vec{c}}$$

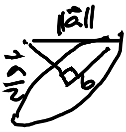
$$= -2\vec{a} + 2\vec{b} + 0 = 2(\vec{b}-\vec{a})$$

$$I = \int_B 2(\vec{b}-\vec{a})\, d\vec{S} = \frac{2}{\|\vec{a}\|}\int_B (\vec{b}-\vec{a})\cdot\vec{a}\, dS =$$

$$= \frac{2(\vec{b}-\vec{a})\cdot\vec{b}}{\|\vec{a}\|} \cdot P(B)$$

$P(B) = ?$

polmer:



$$b = \sqrt{\|a\|^2 - \frac{\|a\|^2}{4}} = \frac{\sqrt{3}}{2}\|a\|$$

$$P(B) = \pi b^2 = \pi \cdot \frac{3}{4}\|a\|$$

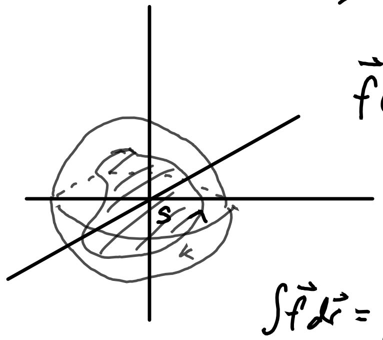
$$I = 3\pi(\vec{b}-\vec{a})\vec{b}\|a\|$$

$\vec{b}$ kot b

K zaključena krivulja na sferi $x^2+y^2+z^2=1$

Izračunaj $I = \int\limits_K \frac{dx+dy+dz}{(x^2+y^2+z^2)^2} = \int\limits_K \vec{f}\, d\vec{r}$

$$\vec{f} = \left(\frac{1}{(x^2+y^2+z^2)^2}, \frac{1}{(x^2+y^2+z^2)^2}, \frac{1}{(x^2+y^2+z^2)^2}\right)$$



$\vec{f}(\vec{r}) = (1,1,1)$ a res $\vec{S}$

$K = \partial S$

$$\int\limits_K \vec{f}\, d\vec{r} = \int\limits_S \operatorname{rot} \vec{f}\, d\vec{S} = \ldots$$

obstaja boljši način

$$\int\limits_K \vec{f}\, d\vec{r} = \int\limits_K \underbrace{(1,1,1)}_{g(\vec{r})} d\vec{r} = \int\limits_S \operatorname{rot}(g(\vec{r}))\, d\vec{S} =$$

$$= \int\limits_S 0\, d\vec{S} = 0$$

Kaj če $K \neq \partial S$

Če je polje $\vec{f}$ potencialno ($\vec{f} = \operatorname{grad} u$) in K krivulja z začetkom v a in koncem v b $\Rightarrow \int\limits_K \vec{f}\, d\vec{r} = u(b) - u(a)$

Posledica: če je K sklenjena je integral vzdolž K od $\int\limits_K \operatorname{grad} u\, d\vec{r} = \vec{0}$

$\vec{f}$ ni potencialno, ampak $\vec{g}$ je potencialno (ker $\operatorname{rot} \vec{g} = 0$)

$\vec{g} = \operatorname{grad}(x+y+z)$ $\qquad I = \int\limits_K \vec{g}\, d\vec{r} = 0$

Opomba: $\int\limits_K h(x^2+y^2+z^2)(dx+dy+dz)$

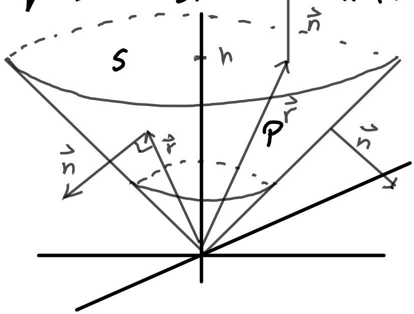
$\Big\downarrow$ na S,

konstantno polje

$h > 0$

$\vec{f}(\vec{r}) = \vec{r}$

Izračunaj pretok $\vec{f}$ skozi plašč in osnovno ploskev stožca: $x^2+y^2 \leq z^2 \quad z \in [0, h]$



$\vec{r} \cdot \vec{n} = 0$

$$\int_P \vec{f}\, d\vec{S} = \int_P \vec{f}\,\vec{n}\, dS = \int_P \vec{r}\,\vec{n}\, dS = \int 0\, dS = 0$$

$$\int_S \vec{f}\, d\vec{S} = \int_S \vec{r}\,\vec{n}\, dS = \int_S z\, dS = z_T \cdot P(S) =$$

$$\vec{n} = (0,0,1) \qquad = h \cdot \pi h^2 = \pi h^3$$

2. način (z gaussom)

$$\int \vec{f}\, dS = \int_D \operatorname{div} \vec{f}\, dV = \int_D (1+1+1)\, dV = 3P(V) =$$

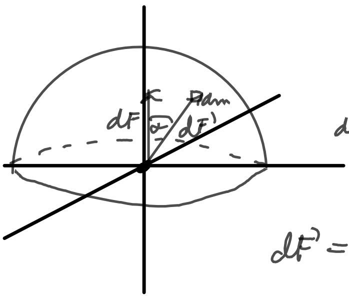
$$= 3 \cdot \frac{\pi h^2 \cdot h}{3} = \pi h^3$$

$$\int_{\partial D} = \int_S + \int_P$$

$a > 0$

Določi privlačno silo med točkastim telesom z maso m_0, ki je v izhodišču in ploskvijo $x^2+y^2+z^2=a^2$ $z \geq 0$ z maso M

homogena



$$F = \frac{m_1 m_2}{r^2} G$$

$$dm = \rho \, dS$$

$$dF' = \frac{m_0 \, dm}{r^2} G = \frac{m_0 \rho \, dS}{r^2} G$$

Rezultanta bo kazala gor torej nas zanima samo z komponenta

$$dF = dF' \cos\alpha$$

$$x = a \cos\vartheta \cos\varphi \qquad \alpha = \frac{\pi}{2} - \vartheta$$

$$y = a \cos\vartheta \sin\varphi$$

$$z = a \sin\vartheta$$

$$dF = \frac{G m_0 \rho}{r^2} \cos\left(\frac{\pi}{2} - \vartheta\right) dS$$

$$dS = a^2 \cos\vartheta \, d\varphi \, d\vartheta$$

od prej z $\sqrt{EG-F^2} = a^2 \cos\vartheta$

$$F = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \frac{G m_0 \rho}{a^2} \sin\vartheta \, a^2 \cos\vartheta \, d\vartheta =$$

$$= G m_0 \rho \, 2\pi \int_0^{\frac{\pi}{2}} \sin\vartheta \cos\vartheta \, d\vartheta =$$

$$G m_0 \rho \, \pi \int_0^{\frac{\pi}{2}} \sin 2\vartheta \, d\vartheta = \frac{\pi}{2} G m_0 \rho \cos 2\vartheta \Big|_0^{\frac{\pi}{2}} =$$

$$= \pi G m_0 \rho = \frac{G m_0 \cdot M}{2a^2}$$

$$\rho = \frac{M}{P(S)} = \frac{M}{2\pi a^2}$$

$D \subseteq \mathbb{R}^3$ gaussasto območje (območje z gladkim robom) in prostornino V

$a \in \mathbb{R}^3$

Izračunaj:

$$\int_{\partial D} (\vec{r} \times \vec{a}) \times d\vec{S} = I$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{y} \cdot \vec{z})\vec{x}$$

$$I = \int_{\partial D} (\vec{r} \cdot d\vec{S})\vec{a} - (\vec{a} \cdot d\vec{S})\vec{r} =$$

$$d\vec{S} = \vec{n} \cdot dS$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n})\vec{a} - (\vec{a} \cdot \vec{n})\vec{r})\, dS =$$

$a = (a_1, a_2, a_3)$

$r = (x, y, z)$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n}) dS)(a_1, a_2, a_3) - (\vec{a}\vec{n}\, x, \vec{a}\vec{n}\, y, \vec{a}\vec{n}\, z) dS$$

$$= \left(\int_{\partial D} (\vec{r} \cdot \vec{n})\, dS\right)\vec{a} - \int (\underbrace{\vec{a}\vec{n}\, x\, dS}_{x\vec{a} \cdot \vec{n}\, dS}, \vec{a}\vec{n}\, y\, dS, \vec{a}\vec{n}\, z\, dS)$$

$$\left(\int (x\vec{a} \cdot \vec{n}\, dS, \int y\vec{a}\vec{n}\, dS, z\vec{a}\vec{n}\, dS\right)$$

$$\int_{\partial D} (\vec{r}\vec{n})\, dS = \int_{\partial D} \vec{r}\, d\vec{S} = \int_D \operatorname{div} \vec{r}\, dV =$$

$$= 3\int_D dV = 3V$$

$$\int_{\partial D} x\vec{a}\vec{n}\, dS = \int_{\partial D} x\vec{a}\, d\vec{S} = \int_D \operatorname{div}(x\vec{a})\, dV = a_1 V$$

$$\operatorname{div} x\vec{a} = \nabla(xa_1, xa_2, xa_3) = a_1$$

Podobno y in

$$I = 3V\vec{a} - (a_1 V, a_2 V, a_3 V) = 2\vec{a}V$$

Dokaži da je za

$$\vec{f} = (2x\cos y - y^2\sin x,\ 2y\cos x - x^2\sin y,\ 4)$$

$\int_K \vec{f}\,d\vec{r}$ enak za vse krivulje med $(0,0,0)$ in $(5,3,\pi)$

in izračunaj

$\int_{K_0} \vec{f}\,d\vec{r}$ za $K_0: \vec{r}(t) = (\cos t, \sin t, t)$; $t \in [-2\pi, 2\pi]$

Dokazujemo: $\vec{f}$ je potencialno

(smo že naredili na začetku)

$u = x^2\cos y + y^2\cos x + 4z$

potencial za $\vec{f}$

$$\int_{K_0} \vec{f}\,d\vec{r} = u(b) - u(a) = u(1, 0, 2\pi) - u(1, 0, -2\pi)$$

$$= 4\cdot 2\pi - (-4\cdot 2\pi) = 16\pi$$

$$I = \int_K u(x,y)(y\,dx + x\,dy)$$

$$\vec{f}(x,y) = y, x)$$

$$\vec{g} = u \cdot \vec{f} = u(x,y) \cdot y,\ u(x,y)\,x)$$

Radi bi da je $\vec{g}$ potencialno polje

$$\vec{g} = \operatorname{grad} v = (v_x, v_y$$

$$v_x = u(x,y) \cdot y \qquad v_y = u(x,y) \cdot x$$

če odvajamo
eno po x enkrat pa po y mora priti enako

$$v_{xy} = u(x,y) + u_y y \qquad v_{yx} = u_x x + u(x,y)$$

$$\rightsquigarrow \quad y u_y = x u_x$$

- Ta pogoj lahko dobimo z greenom
- vložimo v ravnine v $\mathbb{R}^3$

$$t = x \cdot y$$
$$s = \frac{x}{y}$$

$$I = \int_K yu\,dx + xu\,dy + 0\,dz$$

$\vec{h} = (yu, xu, 0)$ je potencialno $\Leftrightarrow \operatorname{rot} \vec{h} = 0$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} = y\frac{\partial}{\partial t} + \frac{1}{y}\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial y} = x\frac{\partial}{\partial t} - \frac{x}{y^2}\frac{\partial}{\partial s}$$

$$x\left(y u_t + \frac{1}{y} u_s\right) - y\left(x u_t - \frac{x}{y^2} u_s\right) = 0$$

$$t u_t + s u_s - t u_t + s u_s = 2 s u_s \qquad s > 0$$

$$u_s = 0 \Rightarrow u = u(t) \quad \ldots \text{samo od } t \text{ odvisen}$$

$$u = h(t)$$

$$u = h(x \cdot y) \qquad \text{kjer } h \in C^1$$

$u, l \in C^2$ $D \subseteq \mathbb{R}^3$ z odsekoma gladkim robom

$\vec{e} = \mathbb{R}^3$ enotski

$$\boxed{\text{Smerni odvod: } \frac{\partial u}{\partial \vec{e}} = \operatorname{grad} u \cdot \vec{e}}$$

Opomba: $\frac{\partial u}{\partial (1,0,0)} = \frac{\partial u}{\partial x} \cdot \vec{e}$

$\vec{n}$ ..... zunanja enotska normala za ∂D

a) Dokaži:

$$\int_{\partial D} u \cdot \frac{\partial v}{\partial \vec{n}} \, dS = \int_D (\operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v) \, dV$$

b) $$\int_{\partial D} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) dS = \int_D (u \Delta v - v \Delta u) \, dV$$

$$\int_{\partial D} u \frac{dv}{d\vec{n}} \, dS = \int_{\partial D} u \operatorname{grad} v \underbrace{\vec{n} \, dS}_{d\vec{S}} = \int_{\partial D} \underbrace{u \operatorname{grad} v}_{\vec{f}} \, d\vec{S}$$

$$\operatorname{div} \vec{f} = \operatorname{div}(u \operatorname{grad} v) = \operatorname{div}(u v_x, u v_y, u v_z) =$$

$$= (u v_x)_x + (u v_y)_y + (u v_z)_z =$$

ko razpišemo

$$= \operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v$$

b)

$$\int_{\partial D} \left(u \frac{dv}{d\vec{n}} - v \frac{du}{\partial \vec{n}}\right) dS = \int_{\partial D} u \frac{dv}{d\vec{n}} \, dS - \int_{\partial D} v \frac{du}{d\vec{n}} \, dS =$$

$$= \int_D \cancel{\operatorname{grad} u \cdot \operatorname{grad} v} + u \Delta v - \cancel{\operatorname{grad} v \cdot \operatorname{grad} u} - v \Delta u$$

$f: \mathbb{R}^3 \to \mathbb{R}$ harmonična ($\Delta f = 0$)

Dokaži da za $a>0$ velja:

$$\frac{1}{4\pi a^2}\int_{S(0,a)} f\,dS = f(0,0,0) = I(a)$$

Nasvet: Parametriziraj $S(0,a)$ in nato s pomočjo odvajanja in gaussovega izreka dokaži trditev.

$x = a\cos\vartheta\cos\varphi \qquad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = a\cos\vartheta\sin\varphi \qquad \varphi \in [0, 2\pi]$

$z = a\sin\vartheta \qquad \rightsquigarrow \vec{r}(\varphi, \vartheta)$

$$dS = |r_\varphi \times r_\vartheta|\, d\varphi d\vartheta = a^2\cos\vartheta\, d\varphi\, d\vartheta$$

$$I(a) = \frac{1}{4\pi \cancel{a^2}}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cancel{a^2}\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))}_{g(a,\varphi,\vartheta)}\, d\vartheta$$

$$I'(a) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g}{\partial a}\, d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, \left(f_x(\vec{r}(\varphi,\vartheta))\cdot x_a + f_y\cdot y_a + f_z \cdot z_a\right) d\vartheta =$$

$$f(x,y,z) = f(a\cos\varphi\cos\vartheta, a\cos\varphi\sin\vartheta, a\sin\vartheta)$$

v $\vec{r}$ se skriva a

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, (f_x \cos\varphi\cos\vartheta + f_y \cos\vartheta\sin\varphi + f_z \sin\vartheta)\, d\vartheta$$

komponente enotske normale $\vec{n}$ za $S(0,1)$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, \operatorname{grad} f\cdot \vec{n}\, d\vartheta = \qquad dS = a^2\cos\vartheta\, d\vartheta\, d\varphi$$

dS za $a=1$

$$= \frac{1}{4\pi}\int_{S(0,1)} \operatorname{grad} f\ \vec{n}\cdot dS = \frac{1}{4\pi}\int_{S(0,1)} \operatorname{grad} f\, d\vec{S} =$$

$$= \frac{1}{4\pi}\int_{K(0,1)} \overbrace{\operatorname{div}(\operatorname{grad} f)}^{\Delta f}\, dV = 0$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

$I'(a)$ je $0 \Rightarrow I(a)$ je konstantna

$$I(a) = \frac{1}{4\pi a^2}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\, d\vartheta$$

$$\lim_{a\to 0} I(a) = \frac{1}{4\pi}\lim_{a\to 0}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\, d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{a\to 0} \cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\, d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, f(0,0,0)\, d\vartheta =$$

$$= \frac{1}{4\pi} f(0,0,0)\cdot \underbrace{4\pi}_{P(S(0,1))} = f(0,0,0)$$

ker je $\cos\vartheta$ determinanta

Holomorfne funkcije

$D \overset{\text{odp}}{\subseteq} \mathbb{C}$ območje $\quad f \in \mathcal{O}(D)$.

Katere od f_1, f_2 in f_3 so holomorfne

$$f_1(z) = \overline{f(z)}$$
$$f_2(z) = f(\bar{z})$$
$$f_3(z) = \overline{f(\bar{z})}$$

$$\boxed{\text{holomorfnost: } \forall a \in D.\ \exists \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} =: f'(a)}$$

$f(x+iy) = u(x,y) + i v(x,y) \quad u = \mathrm{Re} f \quad v = \mathrm{Im} f$

$f \in \mathcal{O}(D) \iff u_x = v_y \ \wedge \ u_y = -v_x$

$u, v \in C^1(D)$

$$f_1(x+yi) = \underbrace{u(x,y)}_{u_1(x,y)} \underbrace{- i v(x,y)}_{v_1(x,y)}$$

$u_{1x} = u_x \qquad u_{1y} = u_y$

$v_{1x} = -v_x \qquad v_{1y} = -v_y$

$u_{1x} = u_x = v_y = -v_{1y}$

$u_{1y} = -v_y$

$\Longrightarrow \quad v_y = -v_y$, $v_x = -v_x$ in ...

$\Downarrow$

$v_x = 0 = v_y$

$u_x = u_y = 0$

u, v konstantni

f_1 ni holomorfna

? ? ?

$$f_2(x+yi) = u_2(x,y) + \cancel{u_2} v_2(x,y); \quad \Big\} f_2 \text{ na } \overline{D}$$
$$= u(x,-y) + v(x,-y)$$

$$\cancel{u_{2x} = u_x} \qquad \cancel{u_{2y} = u_y} \cdot (-1) = -u_y$$
$$\cancel{v_{2x} = v_x} \qquad v_{2y} = -v_y$$

$$u_{2x}(x,y) = u_x(x,-y) \qquad u_{2y}(x,y) = -u_y(x,-y)$$
$$v_{2x}(x,y) = v_x(x,-y) \qquad v_{2y}(x,y) = -v_y(x,-y)$$

$$u_{2x}(x,y) = u_x(x,-y) = v_y(x,-y) = -v_{2y}(x,y)$$

Torej f_2 ni holomorfna, razen če so ~~...~~ u, v konstanti

$$f_3(x+yi) = u_3(x,y) + iv(x,y) \Rightarrow$$

$$u_3(x,y) = u(x,-y)$$
$$v_3(x,y) = -v(x,-y)$$

$$(u_3)_x(x,y) = u_x(x,-y) \qquad v_{3x}(x,y) = v_x(x,y)$$
$$(u_3)_y(x,y) = u_y(x,-x) \qquad v_{3y}(x,y) = +v_y(x,-y)$$

$$(u_3)_x = (v_3)_y \Leftrightarrow$$
$$u_x(x-y) = +v_y(x,-y) \Leftrightarrow u_x = v_y \quad \checkmark$$

$$u_{3y} = -v_{3x} \Leftrightarrow$$
$$-u_y(x,-y) = -v_x(x,-x) \Leftrightarrow v_y = u_x \quad \ldots$$

je holomorfna

Dokaži da obstaj $f \in \mathcal{O}(\mathbb{C})$ za kakro je $u(x,y) = x^3 - 3xy^2$ in jo določi

$$u_x = 3x^2 - 3y^2 = v_y$$
$$u_y = -6xy = -v_x$$

$$v = 3\int (x^2 - y^2)\,dy = 3x^2y - y^3 + C(x)$$
$$v = 6\int xy\,dx = 3x^2y + D(y)$$

$$v = 3x^2y - y^3 + C$$ "unija členov"

$$f(x+yi) = x^3 - 3xy^2 + i(3x y^2 - y^3 + C)$$
$f(z) = z^3 + iC \leftarrow$ z ugibanjem

Brez navdiha:

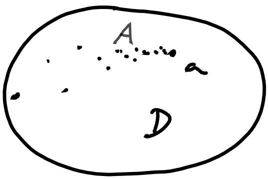
$z = x + yi \Rightarrow x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$

DN: vstavi notri

Imenitna trditev:

$D \subseteq \mathbb{C}$ neko območje, $f, g \in \mathcal{O}(D)$

funkciji ki se ujemata na množici A s stekališčem v D



v A obstaja zaporedje

$(a_n)_n$ $\quad a_n \in A \quad a_n \neq a$

$\lim_{n\to\infty} a_n = a$

$\Longrightarrow f = g$

$u = x^3 - 3xy^2 \qquad v = 3x^3y - y^3 + iC$

če najdemo $g \in \mathcal{O}(\mathbb{C})$ ki se z f ujema na $\mathbb{R}$, po trditvi T sledi $f = g$

$x \in \mathbb{R} \qquad f(x) = f(x + i0) = u(x,0) + v(x,0) = x^3 + iC$

$\Longrightarrow f(z) = z^3 + iC$

Še imenitnejši premislek:

$u(x,y) \rightsquigarrow f = ?$

$x \in \mathbb{R}$: $f(x) = u(x,0) + i\,v(x,0) \overset{x \to z}{\rightsquigarrow} f(z)$:)

$$v(x,0) = \int v_x(x,0)\,dx \underset{\text{CRS}}{=} -\int u_y(x,0)\,dx$$

$u_y = -6xy$

$$v(x,0) = -\int -6x \cdot 0\,dx = +C$$

$$\Longrightarrow f(x) = x^3 + iC \Longrightarrow f(z) = z^3 + iC$$

Podobno

$$u(x,0) = \int u_x(x,0)\,dx = \int v_y(x,0)\,dx$$

$$u(x,y) = e^x(\cos(ky)+\sin(ky))$$

Določi k da je $u = \operatorname{Re} f$ za $f \in \mathcal{O}(\mathbb{C})$

Če je $u = \operatorname{Re} f \Longrightarrow$

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \rightsquigarrow \begin{aligned} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \end{aligned} \Longrightarrow u_{xx} = -u_{yy}$$

Podobno

$$v_{xx} = -v_{yy}$$

$$\Longrightarrow \Delta u = 0$$

$$u_{xx} = e^x(\cos(ky)+\sin(ky)) = u$$
$$u_y = e^x(-k\sin(ky)+k\cos(ky))$$
$$u_{yy} = k^2 e^x(-\cos(ky)-\sin(ky)) = -k^2 v$$

$$\Delta u = e^x(1-k^2)(\cos(ky)+\sin(ky)) = 0 \text{ za } \forall x,y$$

$$\Rightarrow k = \pm 1$$

$$v(x,0) = -\int u_y(x,0)\,dx = -\int e^x k\,dx = -ke^x + C$$

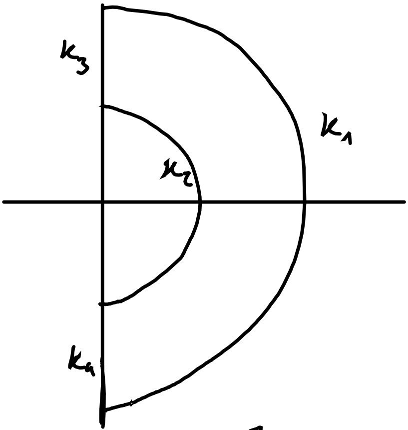
$$f(x) = u(x,0)+iv(x,0) = e^x - ike^x + Ci = e^x(1-ik)+iC$$

$$f(z) = e^z(1 \pm i) + iC$$

12. 4

Integrali holomorfne funkcije

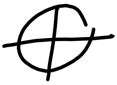
$f(z) = \dfrac{z}{\bar z} \qquad D = \{ z ;\ |z| \in [1,2]\ \ Re(z) > 0 \}$



$$\int_{K_1} f(z)\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2e^{i\varphi}}{2e^{-i\varphi}} 2ie^{i\varphi}\,d\varphi =$$

$z = 2e^{i\varphi}$

$dz = 2ie^{\varphi}$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2ie^{3i\varphi}\,d\varphi = \frac{2}{3}i\ e^{3i\varphi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$\frac{2}{3}(-i - i) = -\frac{2}{3}i$ (kam je šel -)

$$\int_{K_2} f(z)\,dz = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} i\frac{e^{i\varphi}}{e^{-i\varphi}} e^{i\varphi}\,d\varphi = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{1}{3}i\ e^{i3\varphi} =$$

$= \frac{1}{3}(i+i) = \frac{2i}{3}$

$$\int_{K_3 \cup K_4} f(z)\,dz = \qquad z = iy, \quad dz = i\,dy$$

$$= \int_{-1}^{-2} \frac{iy}{-iy} i\,dy + \int_{2}^{1} \frac{iy}{-iy} i\,dy = -i(-2+1) - i(1-2) =$$

$= i(2-1-1+2) \ldots = 2i$

$f(z) = \bar{z}$

$D \overset{\text{dp}}{\subseteq} \mathbb{C}$ s koso na gladkim robom

$\int_{\partial D} f(z)\,dz = ?$

$z = x + iy$

$dz = dx + i\,dy$

$I = \int_{\partial D} (x - iy)(dx + i\,dy) =$

$= \int_{\partial D} (x\,dx + y\,dy) + i(x\,dy - y\,dx) =$

$\overset{\text{green}}{=} \int_{D} (0 - 0) + i \int_{D} (1 - (-1))\,dx\,dy =$

$= 2i \cdot P(D)$

$D^{odp} \subseteq \mathbb{C}$ kosana gładka

$f \in \mathcal{O}(D) \cap C^1(\bar{D})$

$$\int_{\partial D} f(z)\,dz = \int_{\partial D} (u+iv)(dx+idy) =$$

$$= \int_{\partial D} (u\,dx - v\,dy) + i(v\,dx + u\,dy) =$$

$$= \int_D (\underbrace{-v_x - u_y}_{0})\,dx\,dy + i\int (\underbrace{u_x - v_y}_{0})\,dx\,dy = 0$$

$u_x = v_y$

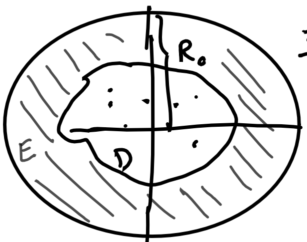
$u_y = -v_x$

$n>1$

$z_1, \dots, z_n \in \mathbb{C}$ poljubne medsebojno različne

$D^{odp} \subseteq \mathbb{C}$ omejeno, ki vsebuje $z_1 \dots z_n$

Dokaži $\int_{\partial D} \frac{dz}{(z-z_1)\cdots(z-z_n)} = 0$



$\exists R_0 > 0.\ \overline{D} \subseteq \Delta(0,R)$ za $\forall R > R_0$

$$\boxed{\text{za } R>R_0 \quad \int_{\partial\Delta(0,R)} f(z)dz = \int_{\partial D} f(z)dz}$$

$E = \Delta(0,R) - \overline{D}$

f je holomorfna na E

$\Rightarrow \int_{\partial E} f(z)dz = 0$

$$\int_{\partial E} f(z)dz = \int_{\partial\Delta(0,R)} f(z)dz - \int_{\partial D} f(z)dz = 0$$

$$\int_{K_R} \frac{dz}{(z-z_1)\cdots(z-z_n)} = \int_{\partial D} \frac{dz}{(z-z_1)\cdots(z-z_n)}$$

$$\int_{\partial E} \dots = 0$$

$$\boxed{\int_{K_1} f(z)dz = \int_{K_2} f(z)dz \quad \text{če krivulji skupaj tvorita } E \text{ in je } f \text{ holomorfna na območju } E}$$

$\int_{K_R}: \quad |z| = Re^{i\varphi}$

$dz = Ri e^{i\varphi}$

$$I = \int_0^{2\pi} \frac{iRe^{i\varphi} d\varphi}{(Re^{i\varphi}-z_1)\cdots(Re^{i\varphi}-z_n)}$$

$$|I| \leq \int_0^{2\pi} \left| \frac{iRe^{i\varphi}}{(Re^{i\varphi}-z_1)\cdots(Re^{i\varphi}-z_n)} \right| d\varphi$$

$$= \left| \frac{R}{(Re^{i\varphi}-z_1)\cdots(Re^{i\varphi}-z_n)} \right| \leq \frac{R}{(R-|z_1|)\cdots(R-|z_n|)}$$

$$|Re^{i\varphi}-z_k| \geq \big||Re^{i\varphi}| - |z_k|\big| = |R-|z_k||$$
$$= R-|z_k|$$

$$|I| \leq \int_0^{2\pi} \frac{R}{(R-|z_1|)\cdots(R-|z_n|)} d\varphi = \frac{2\pi R}{(R-|z_1|)\cdots(R-|z_n|)}$$

$$\xrightarrow{R\to\infty} 0 \Rightarrow I = 0$$

če bi bil $n=1$

$$\int_{\partial D} \frac{dz}{z-z_1} \Rightarrow I = 2\pi i \text{ kr}$$

$$\boxed{\int_{\partial D} \frac{f(z)dz}{z-z_0} \overset{f\in\mathcal{O}(D)}{=} 2\pi i f(z_0)}$$

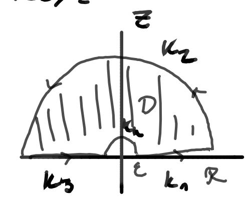
S pomočjo kompleksne integracije izračunaj zu.4

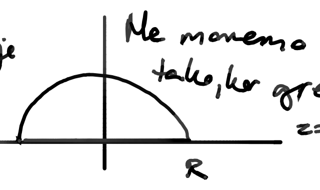
$$\int_0^\infty \frac{\sin x}{x}\,dx$$

Če mamo $\sin x$ ali $\cos x$ v integralu, potem ju zamenjamo z e^x

Drugače pa samo zamenjamo $x \to z$

$f(z) = \frac{e^{iz}}{z}$ območje





Ne moremo tako, ker gre čez $z=0$

R bomo pošlali v ∞ in ε proti 0

$$\int_{\partial D} f(z)\,dz = 0$$

$\uparrow$ $f \in \mathcal{O}(\bar{D})$

∂D razdelimo na gladke kose:

$$\int_{\partial D} f(z)\,dz = \int_{K_1} + \int_{K_2} + \int_{K_3} + \int_{K}$$

K_1: $z = x$ $dz = dx$ $x \in [\varepsilon, R]$

$$\int_{K_1} f(z)\,dz = \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx = \int_{\varepsilon}^{R} \frac{\cos x + i\sin x}{x} \xrightarrow[\varepsilon \to 0^+]{R\to\infty} =$$

$$= \underbrace{\int_0^\infty \frac{\cos x}{x}\,dx}_{\text{ne obstaja } \ddot{\frown}} + i \overbrace{\int_0^\infty \frac{\sin x}{x}\,dx}^{I}$$

"Rešitev": združimo K_1 in K_3

$$\underbrace{\int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\cos x}{dx}\,dx}_{\text{liha funkcija}} + i \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\sin x}{x}\,dx \xrightarrow[\varepsilon\to 0]{R\to\infty} 2iI$$

soda funkcija

K_2: $z = Re^{i\varphi}$

$dz = Rie^{i\varphi}\,d\varphi$

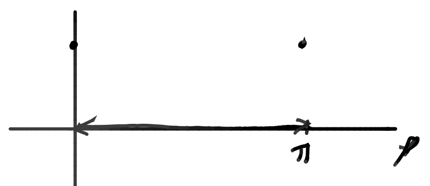
$$\int_{K_2} \ldots = \int_0^\pi \frac{e^{Re^{i\varphi}}}{Re^{i\varphi}} Rie^{i\varphi}\,d\varphi =$$

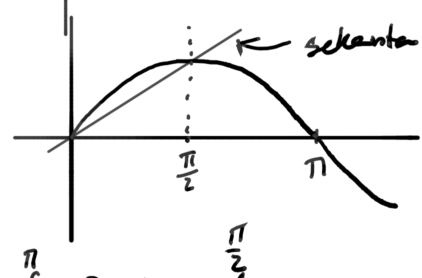
$$= i\int_0^\pi e^{iRe^{i\varphi}}\,d\varphi = \int_0^\pi e^{iR(\cos\varphi + i\sin\varphi)}\,d\varphi$$

$$\left|\int_{K_2} \ldots\right| \leq \int_0^\pi \left|e^{iR(\cos\varphi + i\sin\varphi)}\right| d\varphi =$$

$$= \left|e^{-R\sin\varphi} \cdot e^{i\ldots}\right| = e^{-R\sin\varphi}$$

$$= \int_0^\pi e^{-R\sin\varphi} \xrightarrow{R\to\infty}$$





$$\int_0^\pi e^{-R\sin\varphi}\,d\varphi = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\varphi}\,d\varphi$$

$e^{-R\sin\varphi} \leq$ $\sin\varphi \in [0,1]$ za $\varphi \in$

$-R\sin\varphi \leq 0$

$e^{-R\sin\varphi} \leq 1$ $\ddot{\frown}$

$\sin\varphi \geq \frac{\varphi}{\pi}$

$$\int_0^\pi e^{-R\sin\varphi}\,d\varphi \leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{R2\varphi}{\pi}}\,d\varphi =$$

$$= -2\frac{\pi}{2R}\left(e^{-R} - 1\right) = \frac{\pi}{R} - \frac{\pi}{R}e^{-R} =$$

$$\frac{\pi}{R}\left(1 - e^{-R}\right) \xrightarrow{R\to\infty} 0$$

$\int_{K_4}$: $z = \varepsilon e^{i\varphi}$

...

isto kot pri K_2

$$\int_{K_4} = i\int_\pi^0 e^{i\varepsilon(\cos\varphi + i\sin\varphi)}\,d\varphi \rightsquigarrow i\int_\pi^0 1\,d\varphi = -i\pi$$

$\downarrow$ $\varepsilon \to 0$ $\rightsquigarrow 1$

Iz ana 2: $\underbrace{\int_0^\pi g(\varepsilon,\varphi)\,d\varphi}_{\text{zvezna na } \mathbb{R}\times[0,\pi]} \xrightarrow{\varepsilon\to 0} \int_0^\pi g(0,\varphi)\,d\varphi$ vstavimo $\varepsilon = 0$

$$0 = \int_{\partial D} f(z)\,dz = \int_{K_1} + \ldots + \int_{K_4} \xrightarrow[\varepsilon\to 0]{R\to\infty} 2iI + 0 - i\pi$$

$$\rightsquigarrow \quad I = \frac{i\pi}{2i} = \frac{\pi}{2}$$

$a > 0 \quad p \in (0,1) \quad 0 < \varepsilon < R$

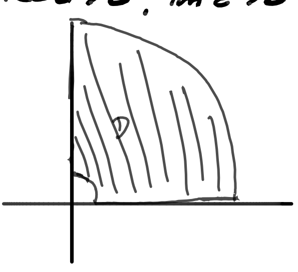
S pomočjo integracije po ∂D izračunaj

$I_1 = \int_0^\infty x^{p-1} \cos(ax)\, dx$

$I_2 = \int_0^\infty x^{p-1} \cos(bx)\, dx$

$D = \{ z;\ |z| \in (\varepsilon, R) \wedge \operatorname{Re} z > 0,\ \operatorname{Im} z > 0$

$f(z) = z^{p-1} e^{iaz}$



holomorfnost:

$e^{iaz} \in \mathcal{O}(\mathbb{C})$

$p = \frac{1}{2}: \quad z^{p-1} = z^{-\frac{1}{2}} = \frac{1}{\sqrt{z}}$

$\sqrt{-1} = \pm i$

$\boxed{\begin{array}{l} \alpha \in \mathbb{C},\ z \neq 0 \\ z^\alpha = e^{\alpha \ln z} \end{array}}$ Definicija kompleksne potence

$\ln z$ je nedoločen do konstante $2k\pi i$

$z^\alpha = e^{\alpha(\overset{\text{neka veja}}{\ln_0(z)} + 2k\pi i)} = (z^\alpha)_0\, e^{2\alpha k\pi i}$

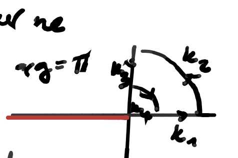
če $\alpha \in \mathbb{Z}$ je z^α enolično definirana

z^α je holomorfno definirana, ko se odločimo za vejo $\ln(z)$

$\arg z \in (\varphi_0, \varphi_0 + 2\pi)$

$\mathbb{C}$ prerezemo v smeri φ_0

Naš primer: $\arg z \in (-\pi, \pi)$ — π: ugodno, ker ne φ_i leže zunaj $\arg = \pi$



Izberemo si rez v smeri π, da ne preseka našega območja

Produkt holomorfnih je holomorfna

z^{p-1} holomorfna

$\mathbb{C} - (-\infty, 0]$

$\int_{k_1} f(z)\, dz = \int_\varepsilon^R x^{p-1} e^{aix}\, dx = \int_\varepsilon^R x^{p-1}(\cos ax + i \sin ax)\, dx$

$z = x$

$\xrightarrow[\varepsilon \to 0]{R \to \infty} I_1 + i I_2$

$k_3: \quad z = ix \qquad dz = i\, dx \qquad \ln(ix) = \ln y + i\frac{\pi}{2}$

$z^{p-1} = e^{(p-1)\ln(z)} = e^{(p-1)(\ln x + \frac{\pi}{2} i)}$

$= e^{(p-1)\ln x} \cdot e^{(p-1)i\frac{\pi}{2}} = x^{p-1} \cdot e^{(p-1)i\frac{\pi}{2}}$

$\int_{R}^{\varepsilon} x^{p-1} e^{(p-1)i\frac{\pi}{2}} \cdot e^{iaix} = \int_{R}^{\varepsilon} x^{\cancel{p-1}\ p-1}\, e^{i(p-1)\frac{\pi}{2}} = \dots$

$= -e^{i(p-1)\frac{\pi}{2}} \int_\varepsilon^R x^{p-1} e^{-ax}\, dx \xrightarrow[\varepsilon \to 0]{R \to \infty}$

$= -e^{i(p-1)\frac{\pi}{2}} \int_0^\infty x^{p-1} e^{-ax} = \dots \frac{1}{a} \int_0^\infty \frac{t^{p-1}}{a^{p-1}} e^{-t}\, dt$

$t = ax$

$dt = a\, dx$

$= -\frac{e^{i(p-1)\frac{\pi}{2}}}{a^p} \Gamma(p)$

$\int_{k_2}: \quad z = Re^{i\varphi} \qquad \cancel{(z_1 \cdot z_2)^\alpha = z_1^\alpha \cdot z_2^\alpha}$ (velja, če si izbereš ustrezne veje)

$z^{p-1} = (Re^{i\varphi})^{p-1} =$

$e^{(p-1)\ln(Re^{i\varphi})} \qquad dz = iRe^{i\varphi} d\varphi$

$\ln(Re^{i\varphi}) = \ln R + i\varphi$

$= e^{\ln R(p-1) + (p-1)i\varphi} = R^{p-1} \cdot e^{i(p-1)\varphi}$

$\int_{k_2} \dots \int_0^{\frac{\pi}{2}} R^{p-1} e^{i(p-1)\varphi} e^{ia R(\cos\varphi + i \sin\varphi)}\, iRe^{i\varphi}\, d\varphi$

$\left|\int_{k_2}\right| \le \int_0^{\frac{\pi}{2}} |\dots|\, d\varphi =$ — abs vrednost $= 1$

$R^p \int_0^{\frac{\pi}{2}} e^{-aR\sin\varphi}\, d\varphi \le \int_0^{\frac{\pi}{2}} R^p e^{-\frac{2\varphi}{\pi} aR}\, d\varphi =$

$\sin\varphi \ge \frac{2\varphi}{\pi}$

$= -R^p \frac{\pi}{2aR}(\underbrace{e^{-aR}}_{} - 1) \to 0$

$\underset{0 \;\; \text{ker } p<1}{\underbrace{}}$ $\quad -1$

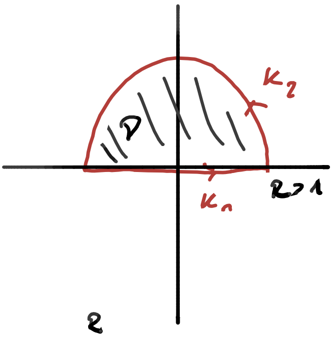
8.5

Tomucda

ene nelage

$$I = \int_0^\infty \frac{\cos x \, dx}{1+x^2}$$

$$f(z) = \frac{e^{iz}}{1+z^2}$$



$$\int_{K_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx = \int_{-R}^{R} \frac{\cos x + i\sin x}{1+x^2}\,dx =$$

$$= \int_{-R}^{R} \frac{\cos x}{1+x^2} + i \underbrace{\int_{-R}^{R} \frac{\sin x}{1+x^2}\,dx}_{\text{0 (lihe)}} = 2\int_0^R \frac{\cos x}{1+x^2} \xrightarrow{R\to\infty} 2I$$

$$\int_{K_2} f(z)\,dz = \int_0^\pi \frac{e^{r(\sin\varphi i + \cos\varphi)}}{1+r^2 e^{2i\varphi}}\, i r e^{i\varphi}\,d\varphi =$$

$z = r e^{i\varphi}$

$$\left| ir \int_0^\pi \frac{e^{-r\sin\varphi} \cdot e^{ir\cos\varphi}}{1+r^2 e^{2i\varphi}} e^{i\varphi}\,d\varphi \right| \leq \quad = 1$$

$$r\int_0^\pi |\ldots|\,d\varphi$$

$r>1$

$$|1+r^2 e^{2i\varphi}| \geq |1 - |r^2 e^{2i\varphi}|| = |1-r^2| \geq r^2-1$$

$$\leq r\int_0^\pi e^{-R\sin\varphi} \frac{1}{r^2-1}\,d\varphi = \frac{r}{r^2-1}\int_0^\pi \underbrace{e^{-r\sin\varphi}}_{\leq 1}\,d\varphi \leq \frac{r}{r^2-1}\,\pi \xrightarrow{r\to\infty} 0$$

f ima singularnost v $i \in D$

$$f(z) = \frac{e^{iz}}{(z-i)(z+i)} = \frac{\frac{e^{iz}}{z+i}}{z-i}$$

$$\boxed{\int_{\partial D} \frac{g(z)}{z-a}\,dz = 2\pi i\, g(a)}$$

$$\int_{\partial D} \frac{\frac{e^{iz}}{z+i}}{z-i} = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

$$\frac{\pi}{e} = \int_{\partial D} f(z)\,dz = \int_{K_1} \ldots + \int_{K_2} \ldots \xrightarrow[R\to\infty]{} 2I + 0 \Rightarrow$$

$$I = \frac{\pi}{2e}$$

$0 < \varepsilon < 1 < R$

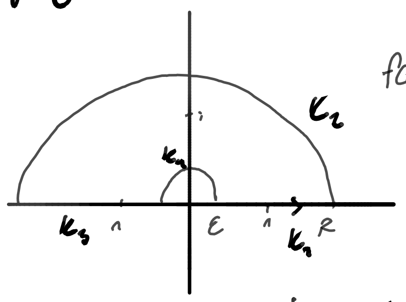
$D := \{ z;\ \varepsilon < |z| < R,\ \operatorname{Im} z > 0 \}$

S pomočjo integracije po ∂D izračunaj

$$\int_0^\infty \frac{x^\alpha\,dx}{1+x^2} = I \qquad \alpha \in (-1,1)$$

in izpelji: $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin \pi p}$

$f(z) = \dfrac{z^\alpha}{1+z^2}$



$z^\alpha = e^{\alpha \ln z}$ ⟵ veja: odstranimo po $\varphi = -\frac{\pi}{2}$

$$\int_{K_1} \ldots = \int_\varepsilon^R \frac{x^\alpha}{1+x^2} \xrightarrow[R\to\infty]{\varepsilon\to 0} I$$

$$\left| \int_{K_2} \ldots \right| =$$

$z = Re^{i\varphi}$ $\qquad z^\alpha = e^{\alpha \ln z} = e^{\alpha(\ln|z| + i \arg z)} =$

$dz = iRe^{i\varphi}d\varphi$

$= e^{\alpha(\ln R + i\varphi)} =$

$= R^\alpha \cdot e^{i\alpha\varphi}$

$$= \left| \int_0^\pi \frac{R^\alpha e^{\alpha i \varphi}}{1+r^2e^{i\varphi}} iRe^{i\varphi}\,d\varphi \right| \leq \int_0^\pi |\ldots|\,d\varphi$$

1

$|1 + r^2 e^{i\varphi}| \geq R^2 - 1$

$$= \int_0^\pi \frac{R^\alpha}{R^2-1} R\,d\varphi = \pi \frac{R^{\alpha+1}}{R^2-1} \xrightarrow[R\to\infty]{\alpha<1} 0$$

$$\int_{K_3} \ldots = \int_{-R}^{-\varepsilon} \frac{x^\alpha}{1+x^2}dx = \int_{-R}^{-\varepsilon} \frac{|x|^\alpha e^{\alpha i \pi}}{1+x^2}d\varphi =$$

$z = x \qquad x^\alpha = z^\alpha = e^{\alpha(\ln|x| + \pi i)} = |x|^\alpha + e^{\alpha\pi i}$

$x = -t \qquad |x| = t$

$dx = -dt$

$$-\int_R^\varepsilon \frac{x^\alpha e^{\alpha i \pi}}{1+x^2}dx = \xrightarrow[R\to\infty]{\varepsilon\to 0} e^{\alpha i \pi} I$$

$$\int_{K_4} \ldots = \int_\pi^0 \frac{\varepsilon^\alpha e^{\alpha i \varphi}}{1+(\varepsilon e^{i\varphi})^2} i\varepsilon e^{i\varphi}\,d\varphi \xrightarrow{\varepsilon\to 0} \int_\pi^0 g(0,\varphi)\,d\varphi = 0$$

$\varepsilon^{\alpha+1} \qquad \alpha + 1 > 0$

$g(\varepsilon, \varphi)$ zvezna za $(-1,1) \times [0,\pi]$

$$\int_{\partial D} f(z)\,dz =$$

$h(z)$

$f(z) = \dfrac{z^\alpha}{(z-i)(z+i)}$ $\qquad \displaystyle\int \frac{h(z)}{z-i}dz = 2\pi i \frac{i^\alpha}{2i} =$

πi^α

$\pi i^\alpha = e^{\alpha i \pi} I + I \qquad i^\alpha = e^{\alpha \ln i} = e^{i\alpha\frac{\pi}{2}}$

$\ln i = 0 + i\frac{\pi}{2}$

$\pi e^{i\alpha\frac{\pi}{2}} = e^{i\alpha\pi} I + I$

$$I = \frac{\pi e^{i\alpha\frac{\pi}{2}}}{1 + e^{i\alpha\pi}} = \frac{\pi}{e^{-i\alpha\frac{\pi}{2}} + e^{i\alpha\frac{\pi}{2}}} = \frac{\pi}{2\operatorname{Re}(e^{i\alpha\frac{\pi}{2}})}$$

$$= \frac{\pi}{2\cos(\alpha\frac{\pi}{2})}$$

Spomnimo se:

$0 < n+1 < m$

$$\int_0^\infty \frac{x^n}{1+x^m}dx = \frac{1}{m} B\left(\frac{n+1}{m}, 1 - \frac{n+1}{m}\right)$$

$n = \alpha$

$m = 2$

$$I = \frac{1}{2} B\left(\frac{\alpha+1}{2}, 1 - \frac{\alpha+1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(1-\frac{\alpha+1}{2})}{\Gamma(\frac{\alpha+1}{2} + 1 - \frac{\alpha+1}{2})}$$

$$= \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(1 - \frac{\alpha+1}{2}\right)$$

$p = \dfrac{\alpha+1}{2} \in (0,1)$

$\alpha = 2p - 1$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\cos(2\frac{\pi}{2}p - \frac{\pi}{2})} = \frac{\pi}{\sin p\pi}$$

Postopkovno koristne naloge !!

Ali obstaja funkcija f holomorfna v okolici 0, da za velike naravne števila $n \in \mathbb{N}$ velja:

a) $f(\frac{1}{n}) = \sin(\frac{n\pi}{2})$

b) $f(\frac{1}{n}) = \frac{1}{2n+1}$

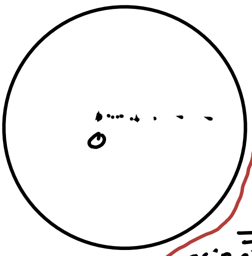
c) $n^{-\frac{5}{2}} < |f(\frac{1}{n})| < 3n^{-\frac{5}{2}}$

a) $\sin\frac{n\pi}{2} = \begin{cases} 0 & n \text{ sodo} \\ 1 & 4k+1 \\ -1 & 4k+3 \end{cases}$

V vsaki okolici 0 ima f vrednosti 0 ali 1

$\Rightarrow$ f ni zvezna v 0 $\Rightarrow$ f ni holomorfna

b)



$A = \{\frac{1}{n}, n \in \mathbb{N}\}$ je množica s stekališčem v D

$\xRightarrow{\text{princip identičnosti}}$ če $\exists f$ je natanko določena

$f(\frac{1}{n}) = \frac{1}{2n+1} \dashrightarrow f(z) = \frac{1}{\frac{2}{z}+1} = \frac{z}{2+z}$

holomorfna v okolici 0

c) če f obstaja ima razvoj v Taylorjevo vrsto v okolici 0

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

$$n^{-\frac{5}{2}} < |c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots| < 3n^{-\frac{5}{2}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \leq |c_0| \leq 0$$

$$|c_0| = 0$$

$/ \cdot n$

$$n^{-\frac{3}{2}} < |c_1 + \frac{c_2}{n} + \dots| < 3n^{-\frac{3}{2}}$$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$/ \cdot n$

$$n^{\frac{1}{2}} \leq |c_3 + \frac{c_4}{n} + \dots| \leq n^{\frac{1}{2}}$$

$$\downarrow \qquad \downarrow n\to\infty \qquad \downarrow$$

$$\infty \qquad c_3 \qquad \infty$$

$\times$

Kaj pa $z^{\frac{5}{2}}$?

je na vsakem prerezu, ampak ne moremo pa je posplošiti na celo okolico 0

Razvij $f(z) = \sin z$ okoli točke i

$$f(z) = c_0 + c_1 (z-i) + c_2 (z-i)^2 + \dots$$

$$\boxed{c_k = \frac{f^{(k)}(i)}{k!} \quad \text{v splošnem}}$$

$$w = z - i$$

$$\sin(z) = \sin(w+i) = \sin(w)\cos(i) + \sin(i)\cos(w)$$

$$= \cos i \cdot \sum_{k=0}^{\infty} \frac{w^{2k+1}}{(2k+1)!}(-1)^k + \sin i \sum_{k=0}^{\infty} \frac{(-1)^k w^{2k}}{(2k)!}$$

$$= \cos i \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z-i)^{2k+1} + \sin i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(z-i)^{2k}$$

$$\sin(x+yi) = \sin x \cos yi + \sin yi \cos x$$

$$\sin iy = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(iy)^{2k+1} =$$

$$= \sum_{k=0}^{\infty} \frac{i^{2k+1} y^{2k+1} (-1)^k}{(2k+1)!} = \sum \frac{(-1)^k \cdot i \, y^{2k+1} (-1)^k}{(2k+1)} =$$

$$= i \sum \frac{y^{2k+1}}{(2k+1)!} = i \operatorname{sh}(y)$$

$$\cos(iy) = \sum_{k=0}^{\infty} \frac{(-1)^k (iy)^{2k}}{(2k)!} = \sum \frac{y^{2k}}{(2k)!} = \operatorname{ch}(y)$$

$$\boxed{\sin(x+yi) = \sin x \operatorname{ch} y + i \operatorname{sh} y \cos x}$$

$$\sin i = 0 + i \operatorname{sh} 1 = i \operatorname{sh} 1$$

$y=1$
$x=0$

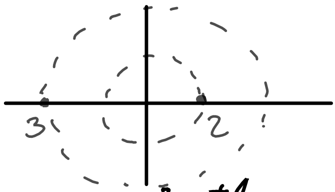
$$\boxed{\begin{aligned}\cos(x+yi) &= \cos x \cos yi - \sin x \sin yi = \\ &\underline{\cos x \operatorname{ch} y - i \sin x \operatorname{sh} y}\end{aligned}}$$

$$\cos i = \operatorname{ch} 1$$

$x=0$
$y=1$

Laurentova vrsta

$$f \in \mathcal{O}(D) \Rightarrow f(z) = \underbrace{\sum_{n=-\infty}^{-1} a_n (z-a)^n}_{\text{glavni del}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{regularni del}}$$



$$f(z) = \frac{2z+1}{z^2+z-6} = \frac{A}{(z+3)} + \frac{B}{(z-2)}$$

$$\Rightarrow B = 1 \quad A = 1$$

$|q| < 1$

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n \qquad \left|\frac{z}{3}\right| < 1 \Rightarrow |z| < 1$$

$$\frac{1}{3+z} = \frac{1}{3} \frac{1}{1-\left(-\frac{z}{3}\right)} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{z}{3}\right)^n$$

$$|z| < 3 \Rightarrow \frac{1}{z+3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{z}{3}\right)^n$$

$$|z| < 2 \Rightarrow -\frac{1}{2} \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$z \in A(0; 0, 2) \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} =$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{3^{n+1}} - \frac{1}{2^{n+1}} \right) z^n$$

$z \in A(0; 2, 3)$

razvoj navzven

$$\frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\frac{2}{z}}$$

$$q = \frac{2}{z} \qquad \left|\frac{2}{z}\right| < 1 \quad 2 < |z|$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n = \sum_{-\infty}^{-1} \frac{z^n}{2^{n+1}}$$

$$m = -(n+1) \Rightarrow n = -m-1$$

$$z \in A(0; 2, 3) \Rightarrow f = \sum_{n=-\infty}^{-1} \frac{1}{2^{n+1}} z^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z$$

$z \in A(0; 3, \infty)$

$$\frac{1}{z+3} = \frac{1}{z} \frac{1}{1+\frac{3}{z}} \qquad q = \left(-\frac{3}{z}\right)$$

$$|z| > 3$$

$$= \frac{1}{z} \sum (-1) \frac{3^n}{z^n} =$$

$$\sum (-1)^n \frac{3^n}{z^{n+1}}$$

$$f(z) = \sum_{n=-\infty}^{-1} \left((-3)^{-n+1} + 2^{-(n+1)} \right) z^n$$

Postopek:

$$f(z) = \frac{p(z)}{q(z)} \qquad p, q \text{ polinoma}$$

$\rightsquigarrow$ razcepimo ulomke

$$\frac{1}{(z-a)^k} = (z-a)^{-k} =$$

binomska vrsta $(1+q)^{\alpha}$

2. Določi glavni del LV

$$f(z) = \frac{\cos z}{z - \sin z} \quad ; a=0$$

$$f(z) = \frac{1}{(z+1)(e^{i\pi z}+1)} \quad ; a=-1$$

izolirani singularnosti v a

$f \in \mathcal{O}(D - \{a\})$

izolirana singularnost

1) Bistvena singularnost
glavni del ima neskončno neničelnih členov

2) Pol stopnje $N \in \mathbb{N}$
glavni del je do N

3) a je odpravljiva singularnost
$\Rightarrow$ glavni del $= 0$

$$f(z) = \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots}{\cancel{z} - (\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots)} =$$

$$= \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} + \dots}{\frac{z^3}{3!} - \frac{z^5}{5!} + \dots} = \frac{1}{z^3}\Bigg(\overbrace{\frac{1 - \frac{z^2}{2} + \dots}{\frac{1}{3!} - \frac{z^2}{5!} + \dots}}^{g(z)} \Bigg)$$

$g(z)$ holomorfen v okolici 0, nima ničle v 0

$\Longrightarrow f(z)$ ima pol 3-stopnje

GD: $\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z}$

$$f(z) = \frac{1 - \frac{z^2}{2} + \dots}{z^3(\frac{1}{3!} - \frac{z^2}{5!} \dots)} = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \underbrace{\dots}_{\text{reg. del}}$$

$$1 - \frac{z^2}{2} + \dots = (a_{-3} + a_{-2} z + a_{-1} z^2 + \dots)(\frac{1}{3!} - \frac{z^2}{5!} + \dots)$$

z^0: $\frac{a_{-3}}{3!} = 1$ z^2: $\frac{a_{-3}}{5!} + \frac{a_{-1}}{3!} = -\frac{1}{2}$

z: $\frac{a_{-2}}{3!} = 0$

...

a_{-1} $\frac{-9 \cdot 6}{20}$

$a_{-3} = 3! = 6$

$a_{-2} = 0$

$a_{-1} =$

GD: $\frac{6}{z^3} - \frac{27}{10 z}$

b) $f(z) = \frac{1}{(z+1)(e^{i\pi z}+1)}$

$e^{i\pi z}$ rozwijamy dla $a=-1$

$w = z - a = z+1$

$e^{i\pi(w-1)} = e^{i\pi w}\, \overset{-1}{\overset{''}{e^{-i\pi}}}$

$(-1)\sum_{n=0}^{\infty} \frac{(i\pi w)^n}{n!} = -1 - \pi w - \dots$

$e^{i\pi z}+1 = -\pi w - \dots = -i\pi(z+1)$

$f(z) = \frac{1}{(z+1)(-i\pi(z+1) - \dots)} =$

$\frac{1}{(z+1)^2(-i\pi - \dots)} = \frac{a_{-2}}{(z+1)^2} + \frac{a_{-1}}{z+1} + \dots$

od drugiej potęgi

$1 = a_2(-i\pi - \dots) + a_{-1}(z+1)(i\pi - \frac{i\pi^2}{2}(z+1) \dots)$

$(z+1)^0:\ a_{-2}\, i\pi$

$(z+1):\ a_{-2}\frac{i\pi^2}{2!} + a_1 i\pi$??

GD: $\frac{1}{\pi(z+1)^2} + \frac{1}{2(z+1)}$

$\operatorname{Res}(f,-1) = a_{-1} = \frac{1}{2}$

$$f(z) = \frac{z}{z^3 - 3z + 2}$$

Rozwiń około $a = -1$

Izrachunaj

$$\int_{|z|=4} f(z)\,dz$$

$z^3 - 3z + 2 =$

$w = z - 1 \quad z = w + 1$

$= w^3 + 3w^2 + \cancel{3w} + 1 - 3w \cancel{-3} + 2 =$

$= w^3 + 3w^2 \cancel{- 2w} = w(\cancel{w^2 + 3w} - 2)$

$= \cancel{w(\ldots \ldots)}$

$= w^2(w+3)$

$w+1 : w+3 = 1 \quad \frac{-2}{w+3}$

$-w-3$

$$f(z) = \frac{w+1}{w^2(w+3)} = \frac{1}{w^2} \frac{w+1}{w+3} =$$

$$\frac{1}{w^2}\left(1 - \frac{2}{w+3}\right)$$

$$1 - \frac{2}{w+3} = 1 - \frac{2}{3} \frac{1}{\frac{w}{3}+1} = -\frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \frac{w^n}{3^n}$$

. . . .

Smun

Formule:

$$\int_{\partial S} f(s)\, d\vec{s}$$

... f holomorfna na okolici D razen v končno točkah nahiše znotraj D

$$\int_{\partial S} f(s)\, d\vec{s} = \sum_{a \in D} \mathrm{Res}(f, a)$$

singularnost

f pol stopnje N v a $\Rightarrow$

$$\mathrm{Res}(f,a) = \frac{1}{(N-1)!} \lim_{z \to a} \left(f(z)(z-a)^N \right)^{(N-1)}$$

$$\int\limits_{|z-2i|=2} \frac{dz}{z^2+9}$$

$a_1 = 3i$

$a_2 = -3i$

singular bod

pola st 1

$$f(z) = \frac{1}{(z+3i)(z-3i)}$$

$$I = 2\pi i \, \mathrm{Res}\,(f, a_1)$$

$$\mathrm{Res}\,(f, 3i) = \lim_{n \to 3i} \frac{1}{z+3i} = \frac{1}{6i}$$

$f(z)(z-3i)$

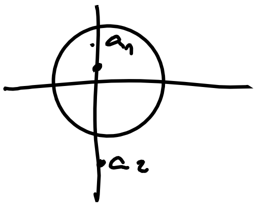
$$\left(\mathrm{Res}\,(f, -3i) = -\frac{1}{6i} \right)$$

$$I = \frac{\pi}{3}$$

b) $\oint\limits_{|z-\frac{i}{2}|=1} \frac{dz}{(z^2+1)^3}$

$f(z) := \frac{1}{(z-i)^3(z+i)^3}$ $\quad a_1 = i$

$a_2 = -i$



$I = 2\pi i \, Res(f, a_1)$

$(f\, a_1) =$

$N = 3$

$Res(f, a_1) = \frac{1}{2!} \lim_{n \to i} \left(\frac{1}{(z+i)^3} \right)'' =$

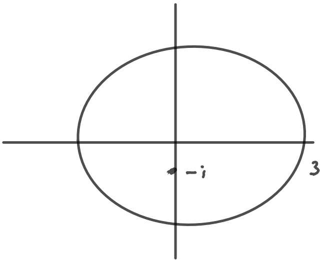
$\left((z+i)^{-3}\right)'' = -3\left((z+i)^{-4}\right)' = 12(z+i)^{-5}$

$Res(f, a_1) = 6 \cdot (2i)^{-5} = \frac{6}{2^5 i} = \frac{3}{2^4 i} = \frac{3}{16i}$

$I = \frac{\pi}{8}$

c) $\displaystyle\int_{|z|=3} \sin\left(\frac{z}{z+i}\right) dz$

$f(z) = \sin\left(\frac{z}{z+i}\right)$ $\qquad a = -i$



$w = z+i$

$z = w-i$

$$f(z) = \sin\left(\frac{w-i}{w}\right) = \sin\left(1 - \frac{i}{w}\right) =$$

$$= \sin 1 \cos\left(\frac{i}{w}\right) - \sin\left(\frac{i}{w}\right)\cos(1) =$$

$$= \sin 1 \left(1 - \frac{\left(\frac{1}{w}\right)^2}{2} + \dots\right) - \cos(1)\left(\frac{i}{w} - \frac{\left(\frac{i}{w}\right)^3}{6} + \dots\right)$$

$\text{Res}(f,i) = C_{-1} \leftarrow$ pri w^{-1}

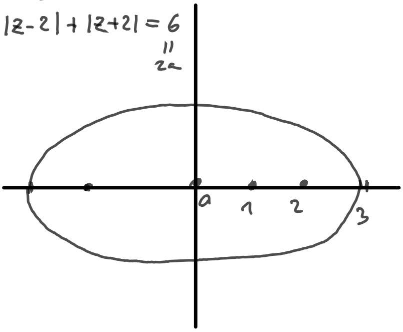
$\Longrightarrow$ $\text{Res}(f,i) = -\cos(1)\, i$

$$\int \frac{1}{z} e^{\frac{1}{1-z}}\, dz$$

$a_1 = 0$ ← pol

$a_2 = 1$

$|z-2| + |z+2| = 6$
$\quad \shortparallel$
$\quad 2a$



$$\underline{I} = \big(\operatorname{Res}(f,1) + \operatorname{Res}(f,0)\big) 2\pi i$$

$$\operatorname{Res}(f,0) = \lim_{z\to 0} (f(z)\cdot z) = \lim_{z\to 0} e^{\frac{1}{1-z}} = e$$

$\operatorname{Res}(f,1) = ?$ bistvena singularnost

z LV v $z=1$

$\frac{1}{z}$ v okolici 1

$$\frac{1}{z} = \frac{1}{z-1+1} \overset{|z-1|<1}{=} \sum_{n=0}^{\infty} (z-1)^n (-1)^n$$

$$e^{\frac{1}{1-z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{1-z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(z-1)^n}$$

$$\frac{1}{z}\cdot e^{\frac{1}{1-z}} = \left(1-(z-1)+(z-1)^2 - \dots\right)\left(1 - \frac{1}{z-1} + \frac{1}{2(z-1)^2} - \frac{1}{3!\,(z-1)^3}\right.$$

$$= \dots + \frac{1}{z-1}\left(-1 - \frac{1}{2} - \frac{1}{3!} - \frac{1}{4!} - \dots\right) + \dots$$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$= -(e-1) = 1-e$$

$$\underline{I} = 2\pi i (e + 1 - e) = 2\pi i$$