$x = 0,1$

Pokaži da velja *

$$x = \sum_{i=1}^{\infty} (2^{-4i} + 2^{-4i-1})$$

b) binarni zapis za x

c) zapiši v IEEE standardu

IEEE754

enojna natančnost $P(2, 24, -125, 128)$

$$(-1)^{\sigma}(1+m)\,2^{\tilde{e}-127}$$

m dolžine 23
$\tilde{e}$ dolžine 8
σ dolžine 1

dvojna natančnost $P(2, 53, -1021, 1024)$

$$(-1)^{\sigma}(1+m)\,2^{\tilde{e}-1023}$$

m 52
$\tilde{e}$ 11
σ 1

a) $x = \sum_{i=1}^{\infty} 2^{-4i} + \dots = \frac{\frac{1}{16}}{1-\frac{1}{16}} + \frac{1}{2}\,\frac{\frac{1}{16}}{1-\frac{1}{16}} = \frac{\cancel{3}}{2\cdot\cancel{16}}\;\frac{\cancel{16}}{\cancel{15}\,5} = \frac{1}{10}$

$= 0{,}1$

b) $0{,}0001100110011 = 0{,}0\overline{0011}_{(2)}$

c)

$1{,}1\overline{0011}\cdot 2^{-4} =$

$1 + 0{,}1\overline{0011}\cdot 2^{-4}$

$0{,}100110\ldots\ldots001101$

$\tilde{e} - 127 = -4 \Rightarrow \tilde{e} = 123 = 1111011$

$123 : 2 = 61$	1
$61 : 2 = 30$	1
$30 : 2 = 15$	0
$15 : 2 = 7$	1
$7 : 2 = 3$	1
$3 : 2 = 1$	1
$1 : 2 = 0$	1

$x = 2^{-1} + 2^{-k} + 2^{-t}$

$y = 2^{-1} + 2^{-k}$

$k = \frac{t}{2} + 1$

$t = 2k - 2$

x^2+y^2 izračunamo z izrazom

$x \cdot x - y \cdot y$

z obravnavo relativne napake pokažite da izračun ni direktno stabilen

$$x^2 = (2^{-1}+2^{-k}+2^{-t})^2 = 2^{-2} + 2^{-2k} + 2^{-2t} + 2\cdot 2^{-1-k} + 2\cdot 2^{-k-t} + 2\cdot 2^{-t-1} =$$

$$\cancel{2^{-2}} + \cancel{2^{-2k}} + \underbrace{2^{-4k+4}}_{=2^{-2t}} + \cancel{2^{-k}} + 2^{-k-t+1} + 2^{-t}$$

$$y^2 = (2^{-1}+2^{-k})^2 = \cancel{2^{-2}} + \cancel{2 \cdot 2^{-k}} + \cancel{2^{-2k}}$$

$$x^2 - y^2 = 2^{-4k+4} + 2^{-k-t+1} + 2^{-t}$$

$$= 2^{-2t} + 2^{-t} + 2^{-k-t+1}$$

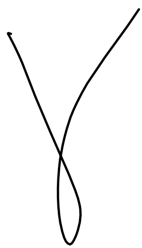
$$fl(x^2) = 0{,}0\overset{-2}{1}\ldots\overset{-k}{1}\ldots\overset{-t}{1}\,0\,|\,\overset{-2k}{1}\ldots\overset{-t-k+1}{1}\ldots =$$

$$= 0{,}01\ldots1\ldots1\overset{-t}{1}$$

$$fl(y^2) = 0{,}01\ldots\overset{-k}{1}\ldots\overset{-t}{0}0\,|\,1 = 0{,}01\ldots\overset{-k}{1}\ldots01\overset{-t}{0}$$

ni več desno zadi

zato izberemo število s sodo zadnjo števko, ker je 1 iz mesta -2^k zadnja cifra v ~~...~~ zapisu



22.10

$g(x) = -x^2 + 8x - 12 \qquad x_{r+1} = g(x_r)$

$\lim\limits_{r\to\infty} x_r = 4$ za $\forall x_0 \in (3,5)$

$\Longleftrightarrow$

$\lim\limits_{r\to\infty} |x_r - 4| = 0$

$= |x_{r+1} - 4|$

$\lim |x_{r+1} - 4| = \lim\limits_{r\to\infty} |-x_r^2 + 8x_r - 16| =$

$= \lim\limits_{r\to\infty} |x_r - 4|^2 = \lim\limits_{r\to\infty} |x_{r-1} - 4|^4 = \dots = \lim\limits_{r\to\infty} |x_0 - 4|^{2^{r+1}}$

konvergira za $|x_0 - 4| < 1 \Rightarrow x_0 \in (3,5)$

proti 0

Red konvergence

$\alpha \ldots$ negibna točka

$$g'(\alpha) = g''(\alpha) = \ldots = g^{(p-1)}(\alpha) = 0 \qquad g^{(p)} \neq 0$$

potem je red konvergence enak p

a) pokažite da lahko $\sqrt{\ }$ pozitivnega števila $a>$ izračunamo z iteracijo $x_{r+1} = \underbrace{x_r \cdot \frac{x_r^2+3a}{3x_r^2+a}}_{g(x_r)}$

b) določi red konvergence

c) dokaži da iteracija konvergira k $\sqrt{a}$ za $\forall$ začetni približek $x_0>0$

1. Preverimo če je $\sqrt{a}$ negibna točka iteracijske funkcije

$$\sqrt{a} \cdot \frac{\sqrt{a}^2+3a}{3\sqrt{a}^2+a} = \sqrt{a} \cdot \frac{4a}{4a} = \sqrt{a}$$

1. negibnost
2. privlačnost

2. preverjamo privlačnost $g'(\sqrt{a}) < 1$

$$g(x) = \frac{x^3+3ax}{3x^2+a}$$

$$g'(x) = \frac{(3x^2+3a)(3x^2+a) - 6x(x^3+3ax)}{(3x^2+a)^2} =$$

$$g'(\sqrt{a}) = \frac{(3a+3a)(3a+a) - 6a^2 - 18a\cdot a}{(3a+a)^2} =$$

$6a\cdot 3a + 6a^2$

$$= 0 < 1$$

b) $$g''(x) = \left(3\frac{\cancel{3}(x^2+a)(3x^2+a) - \cancel{6}^{2}x^2(x^2+3a)}{(3x^2+a)^2}\right)' =$$

$$= 3\frac{\big((2x+a)(3x^2+a) + 6x(x^2+a) - 4x(x^2+3a) - 4x^3\big)(3x^2+a)}{}$$

$$- \frac{2\cancel{(3x^2+a)}\big((x^2+a)(3x^2+a) - 2x^2(x^2+3a)\big)}{(3x^2+a)^{\cancel{4}\,3}}$$

$$= -3\frac{(6x^3+3ax^2+2ax+a+6x^3+6a-4x^3+12xa-4x^3)(3x^2+a)}{}$$

$$- \frac{\qquad}{}$$

$$g'' = \frac{48xa}{(3x^2+a)} \cdot (x^2+a)$$

$h_1 \quad h_2$

$$g''' = h_1'h_2 + h_1h_2' = h_1'(x)(x^2+a) + h_1(x)2x$$

$$g'''(\sqrt{a}) = h_1(\sqrt{a})\cdot 0 + h_1(\sqrt{a})2\sqrt{a}$$

$$h_1\sqrt{a} = \frac{48a\sqrt{a}}{3a+a}$$

$$g'''\sqrt{a} = \frac{48\cdot 2a^2}{(4a)^3} = \frac{3}{2a} \quad > 0 \neq 0$$

$\Rightarrow$ red konvergence $= 3$

c) $g(x) = x\dfrac{x^2+3a}{3x^2+a}$

ločimo primere ko $x_0 < \sqrt{a}$

$x_1 > x_0$?

$x_1 > \sqrt{a}$ ali $x_1 < \sqrt{a}$?

$$x_1 = \cancel{x_0}\frac{x_0^2+3a}{3x_0^2+a} > \cancel{x_0}$$

$$x_0^2+3a > 3x_0^2+a$$

$$2a > 2x_0^2 \quad \checkmark$$

$$\sqrt{a} > x_0 \quad \checkmark$$

$x_1 < \sqrt{a}$

$$x_0(x_0^2+3a) < \sqrt{a}(3x_0^2+a)$$

$$\Leftarrow$$

$$\sqrt{a}(x_0^2+3a) < \sqrt{a}(3x_0^2+a)$$

$$2a < 2x_0$$

sledi: $x_1 < \sqrt{a}$

torej x_r naraščajo in so omejeni
konvergirajo nekam

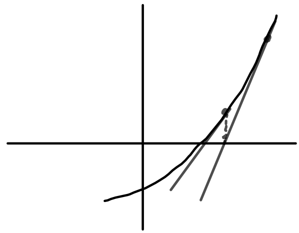
konvergira k $\sqrt{a}$?

$x_0 \in (\sqrt{a}, \infty)$

pokažemo da imamo padajoče
navzdol omejeno zaporedje

Tangentna metoda

$$g(x) = x - \frac{f(x)}{f'(x)}$$



$$x_{r+1} = g(x_r)$$

Babilonska metoda za računanje $\sqrt{a}$, $a>0$ temelji na iteraciji

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{a}{x_r}\right)$$

a) Preverite, da iteracija ustreza tangentni metodi za funkcijo $f(x) = x^2 - a$

b) določite red konvergence

c) Dokažite da iteracija konvergira k $\sqrt{a}$ za $\forall x_0 > 0$

a)

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \stackrel{?}{=} x - \frac{f(x)}{f'(x)} = x - \frac{x^2-a}{2x} =$$

$$= \frac{2x^2 - x^2 + a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right) \checkmark$$

b)

$$g'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) \rightsquigarrow g'(\sqrt{a}) = 0$$

$$g''(x) = \frac{2}{2}\left(\frac{a}{x^3}\right) = \frac{a}{x^3} \quad \frac{1}{\sqrt{a}} > 0$$

red konvergence je 2

c) $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$

g narašča/pojoča za $x \in (\sqrt{a}, \infty)$

$\sqrt{a} < x_{r+1} < x_r$

še za

$\sqrt{a} < \frac{1}{2}\left(x_r + \frac{a}{x_r}\right) < x_r$

$2\sqrt{a}x_r < x_r^2 + a < 2x_r^2$

$a < x_r^2$ ✓

$x_r^2 - 2\sqrt{a}x_r + a > 0$

$(x_r - \sqrt{a})^2 > 0$

padajoče navzdol omejeno zaporedje

$x_0 \in (0, \sqrt{a}) \Rightarrow x_1 \in (\sqrt{a}, \infty)$

$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) \overset{?}{>} \sqrt{a}$

$x_0^2 + a > 2x_0\sqrt{a}$

$x_0^2 - 2x_0\sqrt{a} + a > 0$

$(x_0 - \sqrt{a})^2 > 0$ ✓

Naj bo $f \in \mathcal{C}^2$ α njena enostavna ničla

a) Dokažite, da metoda

$$x_{r+1} = x_r - \frac{2f(x_r)\cdot f'(x_r)}{2f'(x_r)^2 - f(x_r)f''(x_r)}$$ ustreza tangentni metodi za funkcijo $F(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$

b) poenostavite metodo za $f(x) = x^2 - a$

a) $$F'(x) = \frac{f'(x)\sqrt{|f'(x)|} - \sqrt{|f'(x)|}'}{|f'(x)|} =$$

$$\sqrt{|f'(x)|}' = \frac{f''(x)}{2\sqrt{|f'(x)|}}\frac{f'(x)}{|f'(x)|}$$

$$= \frac{\frac{f'(x)}{\sqrt{|f'(x)|}}\left(1 - \frac{f''(x)}{2|f'(x)|}\right)}{|f'(x)|} = \frac{\operatorname{sgn} f\left(1 - \frac{1}{2}\operatorname{sgn} f\right)}{\sqrt{|f'(x)|}} =$$

$$= \frac{\operatorname{sgn} f - \frac{1}{2}}{\sqrt{|f'(x)|}}$$

$$\frac{F(x)}{F'(x)} = \frac{f(x)}{\cancel{\sqrt{|f'(x)|}}}\frac{\cancel{\sqrt{|f'(x)|}}}{(\operatorname{sgn} f - \frac{1}{2})} = \frac{f(x)}{\operatorname{sgn} f - \frac{1}{2}}$$

1.1

$P(2,3,-1,3)$ $\qquad \pm m \cdot b^e \quad L \leq e \leq U$

$P(b,t,L,U)$

baza

0	e: -1	1	2	3
0,100	0,01	1	10	100
0,101	0,0101	1,01	10,1	101
0,110	0,011	1,1	11	110
0,111	0,0111	1,11	11,1	111

1.2

$P(2,9,-10,10)$

$x = 47,712$

$47:2 = 23$ in 1

07

1 ost

$23:2 = 11$ in 1

$11:2 = 5$ in 1

$5:2 = 2$ in 1

$2:2 = 1$ in 0

$1:2 = 1$ in 1

$1+2+2^2+2^3+2^5=$

32
8
4
3

47

10111

$0,712 \cdot 2 = 1,424$ $\quad$ 1

$0,\!101111010\!\!\!/ \cdot 10^6$

$0,424 \cdot 2 = 0,848$ $\quad$ 0

$0,848 \cdot 2 = 1,696 = 1$

$0,696 \cdot 2 = 1,392$ $\quad$ 1

$0,392 \cdot 2 = 0,784$ $\quad$ 0

$$0,1 = \sum_{i=1}^{\infty} \left(2^{-4i} + 2^{-4i-1}\right)$$

$$\sum_{i=1}^{\infty} (2^{-4})^i = \frac{1}{1-2^{-4}} = \frac{2^4}{2^4-1}$$

$$\frac{1}{2} \sum_{i=1}^{\infty} (2^{-4})^i = \frac{2^3}{2^4-1}$$

$$+ \quad = \frac{2^4+2^3}{15} = \frac{24}{15} = \frac{8}{5} = \frac{16}{10} = 1,6$$

$$\frac{3}{2} \quad \frac{16}{15} = \frac{8}{5} =$$

1 2.3

$f(x)=x^5-10x+1$

$f(0)=1$

$f(0,2)=0,2^5-2+1<0$

ima vsaj eno ničlo

če ima več, ima stacionarno točko

$f'(x)=5x^4-10=0$

$x^4=2$

$x=\pm\sqrt[4]{2}$ ni med 0 in 0,2

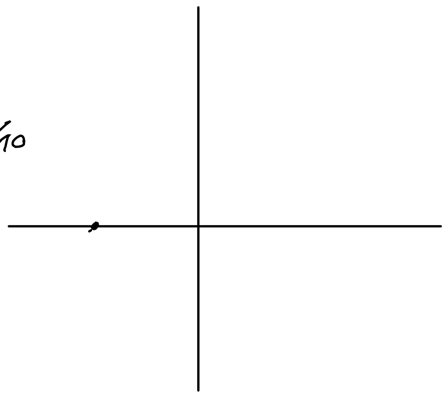
b)

$x_0=0$

$x_{r+1}=g(x_r)$

$g(x)=(x^5+1)/10$

$$x_{r+1}=\frac{(x_r^5+1)}{10}$$



$\lim_{r\to\infty} f(x_r)$

$$g'(x)=\frac{5x^4}{10} \qquad g'(0)=0<$$

$$g(g(0))=g\left(\frac{1}{10}\right)=\frac{\frac{1}{10^5}+1}{10}=\frac{10^5+1}{10^6}$$

$g(x) = -x^2 + 8x - 12$

$g(x) = x$

$-x^2 + 8x - 12 - x = 0$

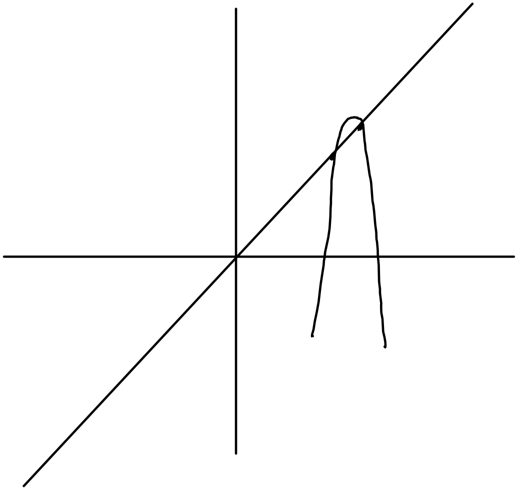
$-x^2 + 7x - 12 = 0$

$-(x-3)(x-4) = 0$

$g'(x) = -2x + 8$ $\qquad$ $|-2x+8| < 1$

$g'(3) = 2$ $\quad$ [illegible] $\qquad$ $\in \left(\frac{7}{2}, \frac{9}{2}\right)$

$g'(4) = 0$ $\quad$ [illegible]



$$b_n = \frac{10}{3} b_{n-1} - b_{n-2}$$

$$b_{n-2} = \frac{10}{3} b_{n-1} - b_n$$

2,7 naloga $\qquad f(x) = (x^2-a)x = x^3 - ax$

$$g(x) = x\,\frac{x^2+3a}{3x^2+a} \quad \text{je}$$

negibne točke x: $\sqrt{a}, -\sqrt{a}, 0$

za $x_0 > 0$ konvergira k $\sqrt{a}$

$$x_r - \sqrt{a} = g(x_{r-1}) - \sqrt{a} = \frac{(x_{r-1}-\sqrt{a})^3}{\underbrace{3x_{r-1}^2+a}_{>0}} < 0$$

$x_0 - \sqrt{a} < 0$

$\Rightarrow x_{r-1} - \sqrt{a} < 0$

Ali je stroga neenakost? $x_r < \sqrt{a}$

2.19 naloga

$f(x) = x^2 - a \quad a > 0$

Hallyjeva metoda

$$\boxed{g(x) = x - \frac{2f(x)f'(x)}{2f'(x)^2 - f(x)f''(x)}}$$

$f'(x) = 2x \qquad f''(x) = 2$

$$g(x) = x - \frac{2(x^2-a)2x}{2(2x)^2 - (x^2-a)2} =$$

$$= x - \frac{4x^3 - 4ax}{8x^2 - 2x^2 + 2a} = \frac{2x^3 + 6ax}{6x^2 + 2a} =$$

$$= \frac{x(x^2+3a)}{3x^2+a}$$

2.15 naloga

$$x_{r+1} = \frac{x_{r-1} x_r + a}{x_{r-1} + x_r}$$

$x_0, x_1 > \sqrt{a}$

$$\boxed{\begin{array}{l}\text{sekantna metoda} \\ x_{r+1} = x_r - \dfrac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}\end{array}}$$

$f(x) = x^2 - a$

$$x_{r+1} = x_r - \frac{(x_r^2 - a)(x_r - x_{r-1})}{x_r^2 - \cancel{a} - x_{r-1}^2 + \cancel{a}} =$$

$$= \frac{\cancel{x_r^3} - x_{r-1}^2 x_r - \cancel{x_r^3} + a x_r + x_{r-1} x_r^2 - a x_{r-1}}{x_r^2 - x_{r-1}^2} =$$

$$= \frac{x_{r-1} x_r (x_r - x_{r-1}) + a(x_r - x_{r-1})}{x_r^2 - x_{r-1}^2} = \frac{x_{r-1} x_r + a}{x_r + x_{r-1}}$$

$x_{r+1} < x_r$

$$x_{r+1} < \frac{x_{r-1} x_r + a}{x_r + x_{r-1}} \rightsquigarrow x_{r+1} x_r + x_{r+1} x_{r-1} < x_{r-1} x_r + a$$

$$<$$

$$2 x_{r+1} \cancel{x_{r-1}} < \cancel{x_{r-1}} x_r + \frac{a}{x_{r+1}}$$

$$a < x_r$$

$$x_{r+1} < \frac{x_r + \cancel{1}}{2} \downarrow < x_r$$

$\rightsquigarrow$ ..

3.3 naloga

$N_\infty(A) = \max |a_{ij}|$

1) N ni matrična norma

$N(A)N(B) = N(AB)$

$\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}2&2\\0&0\end{bmatrix}$ $\times$

2) $\|A\| = n N_\infty(A)$ je matrična norma

pozitivna definitnost

- $\|A\| \geq 0$: sledi iz def. in $n>0$

$\|A\| = 0 \Leftrightarrow n = 0 \vee a_{ij} = 0 \; \forall a_{ij} \Rightarrow A = 0$

- homogenost

$\|\alpha A\| = n N(\alpha A) = n|\alpha| N(A) = |\alpha| \, \|A\|$ ✓

- trik. neenakost

$\|A+B\| = n N_\infty(A+B) = n \max_{ij} |a_{ij} + b_{ij}| \leq n \max |a_{ij}| + n \max ...$

- multiplikativnost $\leq \|A\| + \|B\|)$

$\|A \cdot B\| \leq \|A\| \|B\|$

$\| n \max \left| \sum_{c=1}^{n} a_{ic} b_{cj} \right| \leq n \sum |a_{ic} b_{cj}|$

$n \max |a_{ij}| \left| \sum_{c=1}^{n} b_{cj} \right| \leq \max a_{ij}$ naprej

$\leq \|A\| \cdot ...$

3.8 naloga

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}$$

$$\|A\|_2 = \max \sqrt{\underbrace{\lambda_i(A^H A)}_{\text{lastne vrednost od } A^H A}}$$

1) $\frac{1}{\sqrt{n}} \|A_F\| \leq \|A\|_2 \leq \|A\|_F$

$$\det(A^H A - \lambda I) = 0$$

3,10

$$\|A\|_2 \leq \|A\|_2$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^H = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$A^H A = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

LU razcep

- brez pivotiranja (zadnjič ☺)
- delno pivotiranje $PA = LU$

$$Ax = b$$
$$L\underbrace{Ux}_{y} = Pb \qquad 1)\ Ly = Pb \qquad 2)\ Ux = y$$

- kompletno pivotiranje $\quad Ly = Pb \quad 2)\ Uz = y \quad 3)\ Q^{-1}x = z$

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 2 & 1 & -4 & 2 \\ 3 & -2 & 3 & -1 \\ -1 & 3 & -1 & 1 \end{bmatrix}$$

izračunaj LU razcep in tako $\det(A)$

$$1-(2)\left(\tfrac{2}{3}\right)$$

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ 2 & 1 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & \boxed{-2} & 3 & -1 \\ \boxed{\frac{2}{3}} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & \frac{7}{3} & -4 & \frac{5}{3} \\ -\frac{1}{3} & \frac{7}{3} & 0 & \frac{2}{3} \end{bmatrix} \sim$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

$$1-(-2)\left(\tfrac{2}{3}\right)$$

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & 1 & 2 & -1 \\ -\frac{1}{3} & 1 & 6 & -2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$-1-\left(-\tfrac{2}{3}\right)$$

$$U = \begin{bmatrix} 3 & -2 & 3 & -1 \\ & \frac{7}{3} & -6 & \frac{8}{3} \\ & & 6 & -2 \\ & & & -\frac{1}{3} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & & & \\ \frac{2}{3} & 1 & & \\ -\frac{1}{3} & 1 & 1 & \\ \frac{2}{3} & 1 & \frac{1}{3} & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det(A) = \frac{\det(U)\det(L)}{\det(P)} = \det(U) = (+7)(-2) = -14$$

$$\det(P) = \pm 1 = (-1)^{\text{št zam. vrstic}}, \qquad \text{št zam. vrstic} = 2$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix}$$

LU s kompletnim pivotiranjem + sistem $Ax=b$

$$A = \begin{bmatrix} 4 & 2 & -6 \\ 1 & 2 & -3 \\ -3 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ -3 & 2 & 1 \\ 3 & 3 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 4 & -1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

zamenjane vrstice

$$L = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -6 & 2 & 4 \\ & 4 & -1 \\ & & -\frac{3}{4} \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

zamenjava stolpcev

$Ly = Pb$

$$\begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$y_1 = 14$

$\frac{14}{2} + y_2 = 0 \rightsquigarrow y_2 = +7$

$y_3 = -\frac{3}{4}$

$Uz = y$

$x = Qz$

a) $A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 5 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Izračunajte LU razcep brez pivotiranja

Kaj opazite

b) Naj bo $\begin{bmatrix} a_1 & b_1 & & \\ c_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix}$ splošna 3-diagonalna matrika

Zapišite algoritem za razcep te tridiagonalne matrike in preštejte št. operacij

koliko operacij potrebujemo za reševanje sistema

$Ax = z$

a) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & -\frac{2}{7} & \frac{11}{7} & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & -\frac{2}{7} & \frac{11}{7} & 6 \\ 0 & 0 & \frac{7}{11} & -\frac{20}{11} \end{bmatrix}$

$6 \cdot 7 = 42$

$2 - \frac{42}{11} = \frac{-20}{11}$

$L = I \quad U = A_{zg}$

b) for i in range(1, n):

= L[i][i-1] = A[i][i-1] / A[i-1][i-1]

U[i][i] = A[i][i] - L[i][i-1] · A[i-1][i]

$u_1 = a_1$

for $i = (1:n)$

$l_i = \frac{c_i}{u_i}$

$u_{i+1} = a_{i+1} - l_i \cdot b_i$

end

n-1 deljenja l

n-1 množenj

n-1 odštevanj

3·(n-1) operacij

$Ax = z$

$Ly = z: \quad y_1 = z_1 \quad y_2 = l_1 \cdot y_1 + z_2 \quad y_i = z_i - l_{i-1} y_{i-1}$

2·(n-1) operacij

$Ux = z: \quad x_n = u_n^{-1} z_n$

$b_{n-1} x_n + u_{n-1} x_{n-1} = z_n$

$x_{n-1} = \frac{z_n - b_{n-1} x_n}{u_{n-1}} \qquad x_i = \frac{z_{i+1} - b_i x_{i+1}}{u_i}$

$= 3 \cdot (n-1) + 1 = 3n - 2$ operacij

skupaj $5n-4$ operacij

Opišite postopek za reševanje sistema linearnih enačb

$$\begin{bmatrix} U & -I \\ B & L \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$ kjer je $B = LU$ nesingularna

Preštejte število operacij

$$\begin{bmatrix} Ux - y \\ Bx + Ly \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$Bx + Ly = LUx + Ly = L(Ux + Iy)$$

$$y = Ux - a \qquad Ux = y + a$$

$$Bx + Ly = L(Ux + Ux - a) = L(2Ux - a) =$$

$$= L(\underbrace{2y + a}_{z}) = b$$

1. $Lz = b$ $\quad n^2$ operacij
2. $y = \frac{z-a}{2}$ $\quad 2n$ operacij
3. $Ux = a + y$ $\quad n + \underbrace{n^2}_{\text{obrat. sub}} + n =$

skupaj: $2n + 4n$

15.11

V menjkale

ene uro

Razcep Choleskega

$A \in \mathbb{R}^{n \times n}$ simetrična, pozitivno definitna

$(\forall x \in \mathbb{R}-\{0\}.\ x^T A x > 0)$

$A = V \cdot V^T$ $\quad V =$ nesingularna s pozitivnimi diag. elementi

for $i = 1:n$

$$v_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} v_{ik}^2}$$

for $j = i+1:n$

$$v_{ji} = \frac{1}{v_{ii}}\left(a_{ji} - \sum_{k=1}^{j-1} v_{jk} v_{ik}\right)$$

a) Določi faktor Choleskega (V)

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 8 & -2 & 8 \\ -2 & -2 & 14 & -11 \\ 3 & 8 & -11 & 15 \end{bmatrix}$$

b) Naj bo

$$A = \begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

i) za katere α je to poz. def.

ii) $\alpha = \cancel{\ }23$ Reši sistem

$$Ax = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ -2 & 1 & 3 & \\ 3 & 1 & -2 & 1 \end{bmatrix}$$

$$V_{ji} = \frac{1}{V_{ii}}\left(a_{ji} - \sum_{k=1}^{j-1} V_{jk} V_{ik}\right)$$

$$V_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} V_{ik}^2}$$

$$A = \begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

$$V_{21} = \frac{1}{1}(2-0)$$

$$V_{22} = \sqrt{\cancel{1}8 - \cancel{\ }2^2} = \cancel{4}\cancel{1}\,2$$

$$V_{32} = \frac{1}{2}(-2-(-4)) = 1$$

$$V_{42} = \frac{1}{2}(8 - (3\cdot 2)) = 1$$

$$V = \begin{bmatrix} 2 & & & \\ 3 & 3 & & \\ 1 & -1 & 1 & \\ -2 & 3 & 1 & \sqrt{\alpha - 14} \end{bmatrix}$$

$\alpha > 14$

26.11

Dani sta matriki A in B $\in \mathbb{R}^{n \times n}$. B je pozitivno definitna. Sestavite učinkovit postopek za izračun sledi od $A^T B^{-1} A$ in preštejete št. operacij

Na tabli:

Reševanje sistemov nelinearnih enačb

$$f_1(x_1 \dots x_n) = 0$$
$$f_2(x_1 \dots x_n) = 0$$
$$\vdots$$
$$f_n(x_1 \dots x_n) = 0$$

$$\rightsquigarrow F(x_1 \dots x_n) = \begin{bmatrix} f_1(x_1 \dots x_n) \\ \vdots \\ f_n(x_1 \dots x_n) \end{bmatrix}$$

$$F(\underline{x}) = 0$$

Jakobijeva iteracija

$$G(x) = \begin{bmatrix} g_1(\underline{x}) \\ \vdots \\ g_n(\underline{x}) \end{bmatrix} \qquad x^{(r+1)} = G(x^{(r)})$$

Seidlova iteracija

$$x_i^{(r+1)} = g_i(x_1^{(r+1)}, \dots, x_{i-1}^{(r+1)}, x_i^{(r)}, \dots, x_n^{(r)})$$

- $\underline{\alpha}$ je privlačna negibna točka, če obstaja taka matrična norma, da velja $\|J_G(\alpha)\| < 1$
  (G mora biti odvedljiva v α)

Potem $\exists$ okolica $\Omega \ni \alpha$. $\forall \underline{x}^{(0)} \in \Omega$ zaporedje konvergira k α

Dan je sistem enačb $x = \sin\left(\frac{2x-y}{4}\right)$

$y = \cos\left(\frac{x+2y}{4}\right)$

a) pri začetnem približku $x^0 = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ naredite dva koraka Jakobijeve in seidlove iteracije

b) Pokažite, da iteraciji konvergirata za $\forall$ začetni približek

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{4}\right) \\ \cos\left(\frac{x+2y}{4}\right) \end{bmatrix}$$

$$x^1 = G\begin{pmatrix} 2\pi \\ 0 \end{pmatrix} = \begin{bmatrix} \sin\left(\frac{4\pi}{4}\right) \\ \cos\left(\frac{2\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x^2 = G\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Seidlova:

$$x^1 = G\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x^2 = G\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\frac{1}{4} \\ \cos\left(\frac{-\sin\frac{1}{4}+2}{4}\right) \end{bmatrix}$$

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{4}\right) \\ \cos\left(\frac{x+2y}{4}\right) \end{bmatrix}$$

$$J_G(x,y) = \begin{bmatrix} \cos\left(\frac{2x-y}{4}\right)\cdot\frac{1}{2} & -\cos\left(\frac{2x-y}{4}\right)\frac{1}{4} \\ -\sin\left(\frac{x+2y}{4}\right)\cdot\frac{1}{4} & -\sin\left(\frac{x+2y}{4}\right)\cdot\frac{1}{2} \end{bmatrix}$$

$\underbrace{}_{||<1}$

Vzememo ∞ normo .
vsota po abs vrednost vrstic:

$$\|J_G(x,y)\|_\infty = \max\left\{ \left|\cos\left(\frac{2x-y}{4}\right)\frac{1}{2}\right| + \frac{\left|\cos\left(\frac{2x-y}{4}\right)\right|}{4}, \dots \right\} \le$$

$$\le \max\left\{\frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}\right\} < 1$$

To je za $\forall x,y$ kar obstaja podlaga krogle ker se ujo prilega.

Za seidla:

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{4}\right) \\ \cos\left(\frac{\sin\frac{(2x-y)}{4} + 2y}{4}\right) \end{bmatrix}$$

$$JG = \begin{bmatrix} \frac{1}{2}\cos\left(\frac{2x-y}{4}\right) & -\frac{1}{4}\cos\left(\frac{2x-y}{4}\right) \\ -\frac{1}{2}\cos\left(\frac{2x-y}{4}\right)\sin\left(\frac{\sin\left(\frac{2x-y}{4}\right)+2y}{4}\right), & \dots \end{bmatrix}$$

spet preverimo $||_\infty$ normo in vidimo da je manjše od 1

Newtnova metoda

$$x^{(r+1)} = x^{(r)} - J_F(x^{(r)})^{-1} F(x^{(r)})$$

$$\rightsquigarrow$$

$$J_F(x^{(r)})(x^{(r+1)} - x^{(r)}) = -F(x^{(r)})$$

1) $J_F(\underline{x}^{(r)})\,\underline{\Delta x}^{(r)} = -F(x^{(r)})$

2) $x^{(r+1)} = \Delta x^{(r)} + x^{(r)}$

Dan je sistem $x^2+y^2=4$

$x^2-y^2=1$

a) Naredite 2. korake newtnove metodo pri $x^{(0)} = (2,1)$

$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad F = \begin{bmatrix} x^2+y^2-4 \\ x^2-y^2-1 \end{bmatrix}$

$JF = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$

$J_F(x^0)\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 4\bar{x}+2\bar{y} \\ 4\bar{x}-2\bar{y} \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 4 & 2 & -1 \\ 4 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 2 & -1 \\ 0 & -4 & -1 \end{array}\right] \rightsquigarrow y = \frac{1}{4}$

$4x+2y=-1$

$4x=-\frac{3}{2}$

$x=-\frac{3}{8}$