Kateri elementi kolobarja $\mathbb{Z}[i]$ so asociirani $m+ni$ 6.10

$$x = u(m+ni) = um + uni$$

kateri so obrnljivi:

$$(x+yi)(a+bi) = 1$$

$$xa - yb + i(ay+xb) = 1$$

$$ay + xb = 0 \qquad xa - yb = 1$$

$$x = \frac{-ay}{b} \qquad -\frac{a^2y}{b} - yb = 1$$

$$-\frac{y}{b}(a^2+b^2) = 1$$

$$y = -\frac{b}{a^2+b^2}$$

$$x = \frac{a}{a^2+b^2} \qquad |a| \leq |a^2| \Rightarrow$$

$$a^2+b^2 \in \{0,1\} \Rightarrow a^2+b^2 = 1$$

$$\Rightarrow a \in \{0,1\}$$
$$b \in \{0,1\} \qquad \text{obrnljivi } \{1, -1, i, -i\}$$

asoc: $\{m+ni, -m-ni, n-mi, -n+mi$

$d \in \mathbb{Z}$

$\mathbb{Z}[\sqrt{d}] = \{m + n\sqrt{d};\ m, n \in \mathbb{Z}\}$

1) Pokaži $\mathbb{Z}[\sqrt{d}]$ je podkolobar $\mathbb{C}$

2) Množica $\mathbb{Q}[\sqrt{d}] = \{q + r\sqrt{d};\ q, r \in \mathbb{Q}\}$ je podpolje $\mathbb{C}$ generirano z $\mathbb{Z}[\sqrt{d}]$

---

1) $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{C}$

zaprtost za seštevanje, množenje, enota

za množenje:

$$(m+n\sqrt{d})(x+x\sqrt{d}) = mx + nyd + \sqrt{d}(nx + ym)$$

enota $= 1 = 1 + 0 \cdot \sqrt{d}$

seštevanje: $(m+n\sqrt{d}) - (x+y\sqrt{d}) = (m-x) + (n-y)\sqrt{d}$

2) inverziranje

je tu

$$\frac{m+n\sqrt{d}}{x+x\sqrt{d}} = \frac{(m+n\sqrt{d})(x-y\sqrt{d})}{x^2-y^2d} = \frac{c}{x^2-y^2d} + \frac{e\sqrt{d}}{x^2-y^2d} \checkmark$$

Avtomorfizmi $\mathbb{Z}[\sqrt{d}] = \{id;\ \underbrace{\sqrt{d} \mapsto -\sqrt{d}}_{\sigma}\}$

Norma $N(x) = x\sigma(x)$

$$N(q+r\sqrt{d}) = (q+r\sqrt{d})(q-r\sqrt{d}) = q^2 - r^2 d$$

3) $\forall x,y \in \mathbb{Z}[\sqrt{d}]: N(xy) = N(x)N(y)$

$$N(x,y) = xy\sigma(x,y) = xy\sigma(x)\sigma(y) = x\sigma(x)y\sigma(y)$$
$$= N(x)\cdot N(y)$$

4) $x \in \mathbb{Z}[\sqrt{d}]$ je obrnljiv $\Leftrightarrow$ $N(x) = \pm 1$

$$xy = 1 \Rightarrow N(xy) = N(1)$$
$$N(x)N(y) = 1$$
$$N(x) = \frac{1}{N(y)} \quad N(y) \in \mathbb{Z} \wedge N(x) \in \mathbb{Z}$$
$$\Rightarrow N(y) \in \{-1, 1\}$$
$$\Rightarrow N(x) \in \{-1, 1\}$$

6)

$p \in \mathbb{P}$; $N(x) = \pm p \Rightarrow x$ nerazcepen

in $N(x) = \pm p$

recimo da x razcepen a,b neobrnljiva

$$x = ab$$
$$N(x) = N(a)N(b) = \pm p \Rightarrow N(a) = 1 \vee N(b) =$$

4) $\Leftarrow$ $N(x) = 1$, $x\sigma(x) = \pm 1 \Rightarrow x^{-1} = \pm\sigma(x)$

5) [illegible]

Mora da je $N(a) \pm 1 \Rightarrow a$ obrnljiv.

$d < -1 \Rightarrow 1, -1$ sta edina obrnljiva v $\mathbb{Z}[\sqrt{d}]$

x obrnljiv $\Rightarrow N(x)\, N(y) = \pm 1$

$N(a+b\sqrt{d}) = (a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - b^2 d = a^2 + b^2 |d|$ $\underset{1}{\vee}$

če je $b^2 > 1 \Rightarrow N > 1$

$\Rightarrow b = 0$

a^2 mora biti $1 \Rightarrow a = \pm 1$ sta edini

~~mamesti~~

in vemo da sta različna to rej. ke je rezultat

Pokaži da so $1+i$, $7+8i$, 3 nerazcepni v $\mathbb{Z}[i]$

Recimo da so razcepni

$1+i = x\,y \quad N(xy) = x\sigma(x)\, y\sigma(y) = 2$

praštevilo

$p \rightarrow$ nerazcepno

$N(7+8i) = 49 + 64 =$ velik $= 113$ praštevilo

~~nje praštevilo~~ Norma je 9

$3 = xy \quad N(xy) = x\sigma(x)\, y\sigma(y) = 9$

vsaj 2 morata biti 1

$N(x) = N(\sigma(x)) = \sigma(x)\sigma\sigma(x) = x\sigma(x)$

$N(x) = n \Rightarrow n$ razcepna

$N(x) = n = x\sigma(x)$

6) Poišči vse delitelje elementa 2 v $\mathbb{Z}[i]$

$x | 2 \Leftrightarrow 2 = xy$ za nek y

$N(x \cdot y) = 4 \quad N(x)N(y) = 4$

$x\sigma(x)y\sigma(y) = 4 = 2 \cdot 2 = xy\sigma(xy) = 2\sigma(xy)$

$\Rightarrow \sigma(xy) = 2 = \sigma(x)\sigma(y) \Rightarrow$

$N(x) = 2 \quad x = \pm 1 \pm i$

$x = \pm 2$ so vse možnosti

$x = \pm 2i$

1)

$a = 3+4i$ $\quad \delta = m^2+n^2$

$b = 1-3i$

$|a| = 9+16 = 25$

$|b| = 1+9 = 10$ $\quad$ [illegible]

$a = kb + r$

$\frac{a}{b} = k + \frac{r}{b}$

$k \in \mathbb{Z}[i]$, $\frac{r}{b} \in \mathbb{Q}[i]$

$a = 3-4i$

$b = 1-3i$

$$\frac{a}{b} = \frac{(3-4i)(1+3i)}{10} = \frac{3+12+i(9-4)}{10} = 1+\frac{1}{2}+\frac{1}{2}i$$

$$\frac{1}{2}(1+i)(1-3i) = \frac{1}{2}(1+3+i(1-3)) = 2-i$$

$a = b + 2 - i \qquad / : (2-i)$

$$\frac{a}{2-i} = \frac{b}{2-i} + 1 \quad \not\Rightarrow$$

$$\frac{a}{2-i} - \frac{b}{2-i} = 1 \quad \Longrightarrow \quad a-b = 2-i$$

2)

$a=6$ $\qquad\qquad \mathbb{Z}[\sqrt{-5}]$

$b=2+2\sqrt{-5}$

$c=2 \xRightarrow{N(a)} = \frac{36}{2} = 3 \quad N(b) = 4+4\cdot 5 = 4\cdot 6 \quad \frac{b}{2} = 1+\sqrt{5}$

$c \mid a \quad \cancel{\ldots} \Rightarrow \cancel{\ldots}\ d \mid \frac{a}{c},\ d \mid b \Rightarrow \ldots \mid 6\cdot 2\cdot 3 \qquad 6\cdot 2\cdot 3$

$\delta(c) = 1+5=6 \qquad 6^2 = \qquad 6\cdot 4 =$

$N(\gcd(a,b)) = \gcd(N(a), N(b)) = \gcd(36, 24) = 12 = x^2+5y^2$

$y^2 \in \{1, 2\}$

$\Downarrow \qquad \Downarrow$

$x^2 = 2\cdot 3 \qquad x^2 = \times$

Tak nedotegn

3) UFD $\Rightarrow$ gcd obstaja

$$a = \prod_{i \in I} p_i^{k_i} \qquad d = \gcd(a,b) = \prod_{i \in I} p_i^{\min\{k_i, l_i\}}$$

$$b = \prod_{i \in I} p_i^{l_i}$$

$$d \mid a \wedge d \mid b$$

Naj bo $c = \prod_{i \in I} p_i^{m_i}$ $\quad c \mid a \wedge c \mid b$

$$a = c\alpha \qquad \alpha = \prod p_i^{n_i}$$

$$a = \prod p_i^{n_i + m_i} = \prod p_i^{k_i} \Rightarrow m_i \le k_i$$

podobno $m_i < l_i \Rightarrow \forall i$

$$c \mid d$$

$\mathbb{Z}[\sqrt{-2}]$ euklidsko

a) $\forall a,b \; b\neq 0 \; \exists q,r \in \mathbb{K}. \; a = qb + r \; \wedge \; r = 0 \vee \delta(r) < \delta(b)$

b) $\delta(a) \leq \delta(a,b) \; \forall a,b \in \mathbb{K} - \{0\}$

$\delta: m + \sqrt{-2}n = m^2 + 2n^2$

Vene da b velja

$a,b \in \mathbb{Z}[\sqrt{-2}]$ poljubna $a = x + \sqrt{-2}y \quad b = m + \sqrt{-2}n$

$$\frac{a}{b} = q_1 + q_2\sqrt{-2}$$

$$a = ([q_1] + [q_2]\sqrt{-2})b + (q_1 - [q_1] + (q_2 - [q_2])\sqrt{-2})b$$

DN:

Naj bo $K[X]$ UFD

Potem ima polinom $n^{\in K}$ enolično faktorizacijo

Tako je tud $K[X]$ UFD

Naj bo K UFD

Naj bo $p(x)$ poljuben polinom

Razcepimo $p(x) = \prod_{k=1}^{r}\left(\sum_{i=1}^{n} a_{ik} X^n\right) =$

$= \prod_{k=1}^{r}\left(a_{nk}X^n + \dots + a_{0k}\right) = X^{nr}\underbrace{\left(\prod_{k=1}^{r} a_{nk}\right)} + \dots$

$\mathrm{cont}(fg) = \mathrm{cont}(f)\,\mathrm{cont}(g)$

$p \mid \mathrm{cont}(f,g) \Rightarrow \gcd(a_0, \dots, a_n) = p$

$fg = 0 \vee K/p[X]$

$\Rightarrow f = 0 \vee g = 0$

$(ax+b)(cx+d) = acx^2 + x(cb+ad) + bd$

$\Rightarrow p \mid \mathrm{cont} f \vee p \mid \mathrm{cont} g$

Recimo da f razcepen $F[x]$

potem lahko zapišemo $af = p(x)q(x) \in K[x]$

Najprej dokažemo $cont(af)$ ker $af = pq$ / cont

$$cont(af) = cont(q)\, cont(p) = a\, cont f$$
$$= a \cdot \text{enolično}$$

enoličnost skupnega delitelja

$$a \mid cont(pq) \Rightarrow \frac{p}{x} \cdot \frac{q}{y} = \frac{fa}{a} = f$$

$$\underbrace{x}_{\mid p}\, ; \underbrace{y}_{\mid q}$$

$f \mid pq \Rightarrow f \mid p \ \vee\ f \mid q$

$kf = pq \qquad cont f\, cont k = cont\, p\, cont q$

pq imata razcep $\Rightarrow f \mid p \vee f \mid q$

$\Rightarrow \forall p_i(x) \ldots$ prafaktor

2) $K[X]$ glavni kolobar $\Leftrightarrow$ K je polje

$a \in K - \{0\}$

$\exists b \quad ab = 1$

(a) je ideal $\quad (a) = \{af \, ; f \in K[X]\}$

20.10

1) $1+\sqrt{-3}$ nerazcepen v $\mathbb{Z}[\sqrt{-3}]$, ampak ni praelement

$1+\sqrt{3}$ nerazcepen

Recimo, da je $(1+\sqrt{3}) = \alpha \cdot \beta$

$$N(\alpha \cdot \beta) = N(\alpha)\cdot N(\beta) = 1+3 = 4$$

$$N(\alpha) = N(\beta) = \pm 2$$

če bi bila 1 bi bila obrnljiva

$$N(a+\sqrt{-3}b) = a^2+3b^2 = \pm 2$$

$$b = 0$$

$$a^2 = \pm 2 \quad \text{ne obstaja}$$

Ni praelement

$$\cancel{(a+b\sqrt{-3})}(1+\sqrt{-3}) = \cancel{a+b\sqrt{-3}+a\sqrt{-3}-b} =$$

Vsak element deli svojo normo

$$(1+\sqrt{-3})(1-\sqrt{-3}) = 4 = 2\cdot 2$$

$$(1+\sqrt{-3}) \nmid 2$$

Recimo, da $k(1+\sqrt{-3}) = 2$ $\quad$ $N(2) = 4 \Rightarrow$

$N(k) = \pm 1$

$\Rightarrow k = \pm 1$

Domača naloga

1)

$$\omega = \frac{(1+\sqrt{-19})}{2} \qquad \mathbb{Z}[\omega] = \{a+b\omega : a,b \in \mathbb{Z}\}$$

$$N(a+b\omega) = |a+b\omega|^2 = a^2+ab+5b^2$$

2 in 3 nerazcepna

$N(2) = 4$

$2 = ab$; a,b neobr. $\Rightarrow N(a) = N(b) = 2$

$a =: a+b\omega$

$N(a) = a^2+ab+5b^2 = 2$

$b \neq 0 \Rightarrow 5b^2 > 2 \Rightarrow N(a) > 2$ ⨯

$N(3) = 9 \qquad 9 = x \cdot y \qquad 9 = 3 \cdot 3 \Rightarrow N(x) = N(y) = 3$

$N(x) = 3 \qquad x = a+b\omega$

$a^2+ab+5b^2 = 3$

$b \neq 0 \Rightarrow 5b^2 > 3 \Rightarrow N(x) > 3$ ⨯

$N(x) = 1 \Rightarrow x \in \{1,-1\}$

$a^2+ab+5b^2 = 1$

$b \neq 0 \Rightarrow 5b^2 > 1$ ⨯ $\Rightarrow b = 0$

$a^2 = 1 \Rightarrow a \in \{1,-1\} \Rightarrow x \in \{1,-1\}$

Moduli

$K = \{$zgornje trikotne matrike nad $F\}$

$M = \begin{bmatrix} F \\ F \end{bmatrix}$ je levi K-modul

iščemo podmodule od M

$KM \subseteq M$

$N = \{\lambda \begin{bmatrix} u \\ v \end{bmatrix} ; \lambda \in F\}$ za $u, v \in F$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} au+bv \\ cv \end{bmatrix} \in N$$

$$\lambda au + \lambda bv = \mu u$$
$$\lambda cv = \mu v \Rightarrow \lambda c = \mu$$

$$\lambda au + \lambda bv = \lambda cu$$
$$au - cu + bv = 0$$
$$bv = (c-a)u$$

DN naprej

$\mathbb{Z}[\omega]$ ni evklidski

$(\exists\ \delta : \mathbb{Z}[\omega] - \{0\} \to \mathbb{N} \cup \{0\})$

$\omega = g x + r \quad \Rightarrow \quad r \in \{1, -1, 0\}$

$r = 0 \Rightarrow \omega = gx \Rightarrow x$ asociiren z ω

$N(x) = N(\omega) = 5$

$2 = g'x \pm 1 \Rightarrow x \in \{\pm 2, \pm 3\} \Rightarrow N(x) \in \{4, 9\}$ $\times$

$r = 1 \Rightarrow \omega = gx + 1$

$-1 + \omega = gx \Rightarrow 1 - 1 + 5 = N(g)N(x) = 5$

$N(x) = 5$ $\times$

$r = 1 \Rightarrow 1 + \omega = gx \Rightarrow 7 = N(g)N(x) \Rightarrow N(x) = 7$ $\times$

$\mathbb{Z}[\omega]$ je glavni

$$\left(\forall \alpha \in \mathbb{C}-\mathbb{Z}[\omega].\ \exists p, g \in \mathbb{Z}[\omega].\ 0<|p\alpha - g|<1\right)$$
$$\Longrightarrow \mathbb{Z}[\omega] \text{ glavni kolobar}$$

Naj bo I poljuben ideal. Naj bo $b \in I$ taki da

$\forall x \in I.\ N(x) \geq N(b)$

Naj bo $a \in I$ poljuben $\quad \alpha = \frac{a}{b} \quad a = \alpha b$

$\alpha \in \mathbb{Z}[\omega] \Rightarrow b | a$ za $\forall a \in I \Longrightarrow I = (b)$

$\alpha \notin \mathbb{Z}[\omega]$ in {pogoj} $\Longrightarrow 0 < |p\frac{a}{b} - g|^2 < 1 \quad / \cdot |b|$

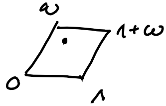
$0 < |\underbrace{pa}_{\in I} - \underbrace{gb}_{\in I}|^2 < |b|^2 \quad \text{—}\!\!\times\!\!\text{—}$

$\forall \alpha \in \mathbb{C}-\mathbb{Z}[\omega] \ \exists \rho, g \in \mathbb{Z}[\omega] : 0 < |\rho\alpha - g| < 1$

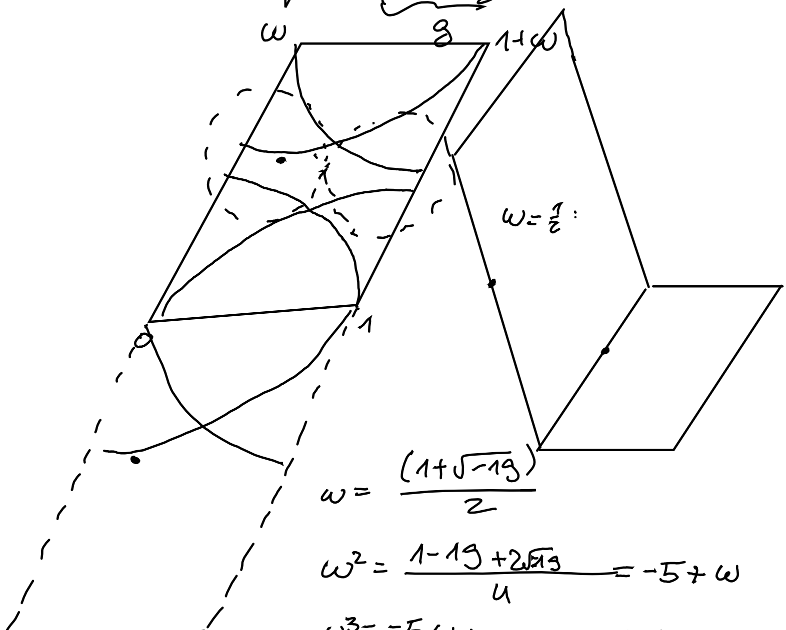
$\alpha \in \mathbb{C}-\mathbb{Z}[\omega]$ poljuben

$\alpha = a + b\omega \quad a, b \in \mathbb{R}$

$\beta = \alpha - (m + \omega n) \quad \beta \in$



$\rho\alpha - g = \rho\beta + \underbrace{\beta(\ldots) - g}$



$$\omega = \frac{(1+\sqrt{-19})}{2}$$

$$\omega^2 = \frac{1-19+2\sqrt{-19}}{4} = -5 + \omega$$

$$\omega^3 = -5\omega + -5 + \omega = -5 - 4\omega$$

DM ze teorie:

$d = 4k+1$

$(a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - db^2 = 4(2-k)$

$b^2 = 1$:

$a^2 - 4k - 1$

$a = 3$

$8 - 4k = 4(2-k)$

$b^2 = 1$

$a^2 = 9$

$$\{(x-1)p(x)\} = \sum a_n = 0$$

Tiste je ideal

$$\sum_{n=0}^{k} a_n x^n$$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 : (x-1) = a_n x^{n-1}$$

$$\underline{-a_n x^n + a_n x^{n-1} \qquad \cancel{+a_n}}$$

$$(a_n + a_{n-1}) x^{n-1} + \dots + a_0$$

ko dobiv pol zavod ostanek

ind. predp.

$$ax - a = a(x-1)$$

Bess ... $\exists$ B $\bar{=}$ $\{f_1 \dots f_g\}$

... element ... expect...

$g_n f_n + \dots + g_n f_n$

$\sqrt{f_n + \dots + f_i^2}$

$\sqrt{\dots + \dots + \dots} \le \sqrt{\dots (g_n \dots)} \sqrt{f_n^2 + \dots + f_n^2}$

$= h_n f_n + \dots + h_n f_n$

$M = {}_{\mathbb{Z}}\mathbb{Z}_{12}$

$\mathbb{Z}_2 = \{0, 6\}$ enostave

$\mathbb{Z}_3 = \{0, 4, 8\}$ enostave

$\mathbb{Z}_4 = \{0, 3, 6, 9\} \supseteq \mathbb{Z}_2$

$\mathbb{Z}_6 = \{0, 2, 4, 6, 8, 10\} \supseteq \mathbb{Z}_2, \mathbb{Z}_3$

3) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{12})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_4)$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{12}, \mathbb{Z}_{12})$

$\mathbb{Z} \longrightarrow \mathbb{Z}_{12}$

$0 \longmapsto 0$ 1 natanko določi

$1 \longmapsto$ karkoli $\Rightarrow$ imamo 12 homomorfizmov

$$\boxed{\begin{array}{l} K \text{ kolobar, } {}_K M \text{ modul} \Rightarrow \\ \mathrm{Hom}(K, M) = \{ x \longmapsto xm : m \in M \} \cong {}_K M \end{array}}$$

$\mathbb{Z}_3 \to \mathbb{Z}_4$

$0 \mapsto 0$

1.	2.	3.
$1 \mapsto 1$	$1 \mapsto 2$	$n \mapsto 0$
$2 \mapsto 2$	$2 \mapsto 1$	
$3 \mapsto 0$	$3 \mapsto 0$	
$4 \mapsto 1$	$4 \mapsto 2$	

$\mathbb{Z}_{12} \mapsto \mathbb{Z}_{12}$

$0 \mapsto 0$

$1 \mapsto a$

$n \mapsto na$ mod 12

K kolobar. K enostaven $\Leftrightarrow$ K obseg

$\Rightarrow$ Recimo da je enostaven, $a \in K$ poljuben

a obrnljiv

$Ka = K \Rightarrow \exists k \in K.\ ka = 1$

$\Leftarrow$ M podlevi modul ima

Vaje za nazaj

1 Naloga M K-modul $M \neq \{0\}$ enostaven $\Leftrightarrow M = Km$ $\forall m \in M$

Naj bo M enostaven. Edina podmodula sta $\{0\}$ in M

$\Rightarrow$ Naj bo $m \in M$ poljuben. $\underline{Km \text{ je podmodul!}}$

$\alpha, \beta \in K \quad \alpha k m - \beta h m = (\alpha k - \beta h) m \in Km$

$\Leftarrow$ $M = Km$ za $\forall m \in M$.

Naj bo N podmodul.

$\forall n \in N$. $Kn = M$ torej $N = M$

2. naloga

levi: K modul ${}_K M$ enostaven $\Leftrightarrow M \cong K/{}_K I$, kjer je ${}_K I \leq K$ maksimalen levi ideal

$\Rightarrow$ Naj bo M enostaven, $a \in M$

$\varphi: K \longrightarrow M$ $\qquad$ φ surjektiven po prejšnji nalogi

$k \longmapsto ka$

$\ker\varphi = \underline{\{k;\ ka = 0\} = {}_K I}$ maksimalen

$I \subsetneq J \subsetneq K$ $\qquad$ $\varphi_*(J) = M \Rightarrow m = ja$ $\forall m \in M$

ker $J \neq \ker\varphi$ $\qquad \Rightarrow j = ja \Rightarrow j(1-a) = 0$

$K \longrightarrow M$ $\qquad 0 \subsetneq \varphi_*(J) \underline{\subsetneq} M$ $\quad \times$ $\qquad 1-a \in \ker\varphi$

$\Rightarrow$

$J \xrightarrow{sur} \varphi_*(J)$ $\qquad 1-a \in J$

$(1-a)+a = 1 \in J$

$I \longrightarrow 0$ $\qquad \Longrightarrow J = I \vee J = K$ $\qquad \Rightarrow J = K$

$\times$

$\Leftarrow$ Naj bo ${}_K M \cong K/{}_K I$; I maksimalen levi ideal

$0 < N < M$ $\qquad \varphi$ surj

$\uparrow \quad \uparrow \quad \uparrow$

$I < J < K$ $\qquad J = \varphi^*(N)$

$N \subseteq \varphi_*(J) = \varphi_*(\varphi^*(N)) \Longrightarrow \varphi_*(J) = N$

$K/I \longrightarrow M$

$1+I \longmapsto e \in M - \{0\}$ $\qquad \Longrightarrow$ $\qquad \varphi: K \longrightarrow M$

$k \longmapsto ke$

$x+I = x(1+I) \longmapsto xe$

$\underline{M \text{ enostaven}}$ $\qquad \underline{I = J} \vee \underline{J = K}$

$I < J_2 < J_1 < K$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$0 < \varphi_*(J_2) = \varphi_*(J_1) < M$ $\qquad /:\varphi(J_2)$

$0 = \frac{\varphi_*(J_1)}{\varphi_*(J_2)} < M/\varphi(J_2)$

??

Vene nelo ge

esparta

Od zadnjič:

$$\operatorname{End}(M\oplus N) \cong \begin{bmatrix} \operatorname{End}(M) & \operatorname{Hom}(N,M) \\ \operatorname{Hom}(M,N) & \operatorname{End}(N) \end{bmatrix}$$

1) določi kolobarje endomorfizmov $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_2\oplus\mathbb{Z}_3)$, $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_6\oplus\mathbb{Z}_3)$

$$\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_2\oplus\mathbb{Z}_3) = \begin{bmatrix} \operatorname{End}\mathbb{Z}_2 & \operatorname{Hom}(\mathbb{Z}_3,\mathbb{Z}_2) \\ \operatorname{Hom}(\mathbb{Z}_2,\mathbb{Z}_3) & \operatorname{End}\mathbb{Z}_3 \end{bmatrix}$$

$\operatorname{End}\mathbb{Z}_2$: $\begin{matrix} 0\mapsto 0 \\ 1\mapsto 1\end{matrix}$ $\quad \begin{matrix} 1\mapsto 0 \\ 0\mapsto 1\end{matrix}$ $\quad \cong \mathbb{Z}_2$

$\boxed{\text{Splošno: } \operatorname{End}\mathbb{Z}_p \cong \mathbb{Z}_p}$ $\Rightarrow \operatorname{End}\mathbb{Z}_3 \cong \mathbb{Z}_3$

$1\mapsto$ red elementa 3 ne obstaja in

$$\operatorname{End}(\mathbb{Z}_2\oplus\mathbb{Z}_3) \cong \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \cong \mathbb{Z}_2\times\mathbb{Z}_3 \cong \mathbb{Z}_6$$

$\operatorname{End}(\mathbb{Z}_6\oplus\mathbb{Z}_3)$: $\operatorname{End}(\operatorname{End}(\mathbb{Z}_2\oplus\mathbb{Z}_3)\oplus\mathbb{Z}_3) =$

$$= \begin{bmatrix} \operatorname{End}(\mathbb{Z}_2\oplus\mathbb{Z}_3) & \operatorname{Hom} \\ \operatorname{Hom} & \operatorname{End}\mathbb{Z}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_6 & \operatorname{Hom} \\ \operatorname{Hom} & \mathbb{Z}_3 \end{bmatrix}$$

$$\operatorname{Hom}(\mathbb{Z}_6,\mathbb{Z}_3) \underset{\mathbb{Z}}{\cong} \operatorname{Hom}(\mathbb{Z}_2\oplus\mathbb{Z}_3,\mathbb{Z}_3) \cong$$

$$\cong \underbrace{\operatorname{Hom}(\mathbb{Z}_2,\mathbb{Z}_3)}_{0} \oplus \operatorname{Hom}(\mathbb{Z}_3,\mathbb{Z}_3) \cong \mathbb{Z}_3$$

$$\operatorname{End}(\mathbb{Z}_6\oplus\mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_6 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}$$

$1\mapsto 2$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \text{ se ne izide}$$

$x \mapsto (2, x \bmod \mathbb{Z}_3)$

$$\begin{bmatrix} 0 & \begin{matrix}\mathbb{Z}_3\to\mathbb{Z}_2\oplus\mathbb{Z}_3\to\mathbb{Z}_6 \\ 1\mapsto(1,0)\mapsto 4\end{matrix} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \begin{matrix}\mathbb{Z}_6\to\mathbb{Z}_2+\mathbb{Z}_3\to\mathbb{Z}_3 \\ (1,0)\mapsto 2 \\ x\mapsto 2x \bmod 3\end{matrix} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x\mapsto 2x \mapsto (2x,0) = 4x \ (\bmod 3) \\ \cancel{2\mapsto 8 \bmod \ldots = 2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\updownarrow \mathbb{Z}_6\to\mathbb{Z}_3 \qquad\qquad \updownarrow$

$$\begin{bmatrix} 0 & x\mapsto 4x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x\mapsto 2x & 0 \end{bmatrix}\begin{bmatrix} x\mapsto 8x & 0 \\ 0 & 0 \end{bmatrix}$$

$\mathbb{Z}_3\to\mathbb{Z}_6$

k $\mathbb{Z}_3 \longrightarrow \mathbb{Z}_6$

1: $1\mapsto(1,0)\mapsto 4$

2: $1\mapsto(2,0)\mapsto 2$

k: $1\mapsto(k,0)\mapsto \ldots$

$\mathbb{Z}_6 \longrightarrow \mathbb{Z}_3$

$k = 1\mapsto k \bmod 3$

Hitreje:

$$\mathbb{Z}_{18} = \mathbb{Z}_6 \oplus \mathbb{Z}_3 = \mathbb{Z}_2 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3)$$

$$\operatorname{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_2 & \operatorname{Hom}(\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_2) \\ \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3) & \operatorname{End}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \end{bmatrix} =$$

$$\operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3) = \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \oplus \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_3)$$

$$= \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \cong$$

$$\underline{\underline{\cong \mathbb{Z}_2 \oplus M_2(\mathbb{Z}_3)}}$$

Bimoduli

$K^{op} = \{ x^{op} \mid x \in K \}$ $\quad x^{op} \cdot y^{op} = (yx)^{op}$

Grupe: $G \cong G^{op}$
$x^{op} = x^{-1}$

${}_K M \cong M_{K^{op}}$ vsak levi modul lahko predstavimo kot desni K modul

Bimodul je ${}_K M_S$, povezuje pogoj asociativnosti zakon
K, S kolobarja $\quad$ ${}_K M$ je levi K-modul
M_S je desni S-modul

$x \in K$, $y \in S$

$m \in M$ $\quad \underbrace{(x \cdot m)}_{\in M} y = x(m \cdot y)$

Primeri

1) ${}_K K_K$ je bimodul

2) ${}_K M_{K^{op}}$ ni bimodul (razen če K komut.

$m \cdot x^{op} := xm$ $\quad$ $x(m y^{op}) = xym$
$\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad (xm)y^{op} = yxm$

3) ${}_K M_{End(M)^{op}}$ je bimodul

$(xm)\varphi^{op} = \varphi(xm) = x\varphi(m) = x(m\varphi^{op})$

2) Naj bosta ${}_K M_S$ in ${}_K N_R$ bimodula (K, S, R kolobarji)

Premisli, da imajo homomorfizmi: $M \to N$ naravno strukturo (S, R)-bimodule

homomorfizmi kot levih K modulov

Zgubi strukturo K modula

$${}_S \mathrm{Hom}(M,N)_R$$

$s \in S \quad s\varphi := m \mapsto \varphi(ms)$

$r \in R \quad \varphi r := m \mapsto \varphi(m)r$

Hom(M,N) je levi S modul

$$(s+t)\varphi = (m \mapsto \varphi(m(s+t))) = (m \mapsto \varphi(ms) + m \mapsto \varphi(mt)) = s\varphi + t\varphi$$

$$\varphi(m(s+t)) = \varphi(ms) + \varphi(mt)$$

$1 \cdot \varphi = (m \mapsto \varphi(m)) = \varphi$

$s(\varphi + \psi) = \checkmark$

Hm. $(st)\varphi\,(m) = \varphi(mst) = \varphi((ms)t) =$

$= (t\varphi)(ms) = s(t\varphi(m)) = s(t\varphi)\,(m)$

ali je to vredu $(t\varphi)(xm) = x\,((t\varphi)(m))$

$(t\varphi)(xm) = \varphi((xm)t) = \varphi(x(mt)) = x\,\varphi(mt)$

Podobno za desni R modul

$s(\varphi r)(m) = (\varphi r)(ms) = \varphi(ms)r = (s\varphi(m))r$

Lev K-modul ${}_K M$. Dual je $M^*_K = \mathrm{Hom}(\cancel{{}_K} M, \cancel{{}_K} K_K)$

Ker je domena bimodul, je tudi M^*_K desni K modul

Funkcional

$$M = \begin{bmatrix} F \\ F \end{bmatrix}$$

3) Določi $\left({}_{M_2(F)} M\right)^*$

$$M^*_{M_2(F)} = \mathrm{Hom}\left({}_{M_2(F)} M, M_2(F)_{M_2(F)}\right) \cong$$

$$\begin{bmatrix} F & F \\ F & F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \cong M \oplus M$$

$$\cong \mathrm{Hom}(M, M \oplus M) = \mathrm{End}(M) \oplus \mathrm{End}(M) \cong F^2$$

To sta obsega (Shurova lema)

$M=\begin{bmatrix} F \\ F \end{bmatrix}$ $M \overset{?}{=} [F\ F]$

$K=\begin{bmatrix} F & F \\ F & F \end{bmatrix}$ $K \cong M \oplus M$ vrstica + vrstica

$\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \oplus \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$

struktura desnega K-modula

$M^* = \mathrm{Hom}(M,K)_K \cong \mathrm{Hom}(M,M) \oplus \mathrm{Hom}(M,M) \cong F \oplus F$

ni desni K modul.

Izgubimo strukturo modula

(*) $F \cong \mathrm{Hom}(M,M) = \mathrm{End}\,M$

End(M) obseg, če M enostaven

$F \cong \mathrm{End}(M)$

2-dim vek prostor

$\supseteq$ V posebnem: $\varphi \in \mathrm{Hom}_K(M,M) \leq \mathrm{Hom}_F(M,M) \cong M_2(F)$

$\varphi(Ax) = A\varphi(x)$ $\varphi = B \Rightarrow$

$BAx = ABx$ za $\forall A \in K$ $\forall x \in M \Rightarrow BA = AB$ $\forall A \in K$.

$B \in Z(M_2(F)) = F \cdot I \cong_F$ diagonalne matrike

$\subseteq$ očitno

(**) M^*_K tej $(F \oplus F)_K$

$\varphi \in \mathrm{Hom}(M,K)$

$\varphi(m) = (\varphi_1(m), \varphi_2(m)) = (\varphi(m)\cdot\begin{bmatrix}1\\0\end{bmatrix}, \varphi(m)\cdot\begin{bmatrix}0\\1\end{bmatrix})$

stolpci

$\varphi(m) = \left[\varphi_1(m) \,\vdots\, \varphi_2(m)\right]$

$\varphi(Am) = \left[\varphi_1(Am) \,\vdots\, \varphi_2(Am)\right]$

$A\varphi(m) = \left[A\varphi_1(m) \,\vdots\, A\varphi_2(m)\right]$ $\varphi_{1,2}$ skalarne večkratnika identitete

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\varphi \cdot A \overset{\in M^*}{=} (m \longmapsto \varphi(m)\cdot A)$ dano po definiciji

$[\varphi_1, \varphi_2]\cdot A = \varphi_1 a + \varphi_2 b + \varphi_1 c + \varphi_2 d = \varphi_1(a+c) + \varphi_2(b+d)$

kako sklepamo da je

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\varphi(m) = [\varphi_1(m), \varphi_2(m)]$ * $m = \begin{bmatrix} x \\ y \end{bmatrix}$

skalarje

$= [\lambda_1 m \ \ \lambda_2 m] = \begin{bmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y & \lambda_2 y \end{bmatrix}$

$\varphi(m)\cdot A = \begin{bmatrix} a\lambda_1 x + b\lambda_2 x & c\lambda_1 x + d\lambda_2 x \\ a\lambda_1 x + b\lambda_2 y & c\lambda_1 y + d\lambda_2 y \end{bmatrix} =$

$= \begin{bmatrix} x(a\lambda_1 + b\lambda_2), & x(c\lambda_1 + d\lambda_2) \\ y(a\lambda_1 + b\lambda_2) & y(c\lambda_1 + d\lambda_2) \end{bmatrix}$

$\varphi = [\lambda_1\ \lambda_2]$ $\varphi \cdot A = [\lambda_1 a + \lambda_2 b, \ c\lambda_1 + d\lambda_2]$

Lažje: $F \oplus F \to \mathrm{Hom}(M,K)$

$[\lambda, \mu] \longmapsto (\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}[\lambda, \mu])$

$(\varphi \cdot A)\begin{bmatrix} x \\ y \end{bmatrix} = (\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}[\lambda, \mu]}_{\varphi})A = \begin{bmatrix} x \\ y \end{bmatrix}([\lambda, \mu]A)$

1) Določi dual ${}_{\mathbb{Z}}\mathbb{Z}_n^*$

$= \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0$

$1 \mapsto x \qquad \mathrm{red}\,1 \mid \mathrm{red}\,x \Rightarrow x = 0$

2) K cel kolobar (brez del. niča, komut)

za ${}_K M$ K-modul definiramo

$\mathrm{tor}(M) = \{m \in M;\ xm = 0,\ x \in K - \{0\}\}$

$\mathrm{tor}\left(\bigoplus_{n=2}^{\infty} \mathbb{Z}_n\right) = \bigoplus_{n=2}^{\infty} \mathbb{Z}_n$; ampak ne obstaja element, ki bi uničil vse

Dokaži da je $\mathrm{tor}(M) \leq M$ podmodul in

$\varphi: M \to N$

$\varphi_*(\mathrm{tor}(M)) \leq \mathrm{tor}(N)$

a) $xm = 0 \quad yn = 0 \quad m, n \in M \quad x, y \in K$

$\underbrace{xy}_{\neq 0}(m+n) = xym + x\underbrace{yn}_{0} = yxm = 0$

$\alpha \in K \quad xm = 0$

$x\alpha m = \alpha x m = 0$

b) φ homomorfizem

$x\varphi(m) = \varphi(xm) = 0 \rightarrow m \in \mathrm{tor}(M) \Rightarrow \varphi(m) \in \mathrm{tor}(N)$

$\Rightarrow \varphi_*(\mathrm{tor}(M)) \subseteq \mathrm{tor}(N)$

je podmodul: $xy(\alpha\varphi(m) + \beta\varphi(n)) =$

$\alpha y \underbrace{\varphi(xm)}_{0} + \beta x \underbrace{\varphi(yn)}_{0} = 0$

$\operatorname{tor}\left(M/\operatorname{tor}(M)\right)=0$ DN

$$x(m+\operatorname{tor}M)=\underbrace{xm}\,+\operatorname{tor}M$$

$=0 \Rightarrow m\in\operatorname{tor}M \rightarrow m+\operatorname{tor}M=0$

$\neq 0 \Rightarrow m\notin\operatorname{tor}M$

$$3.\ M \text{ je prost} \overset{K \text{ cel}}{\Rightarrow} tor(M)=0$$

$$\varphi: M \xrightarrow{\cong} \bigoplus_{i=I} K$$

$$\varphi(tor M) \leq tor(\bigoplus_{i=I} K)$$

φ inj $\Rightarrow$ dovolj je dokazat $tor(\bigoplus_{i=I} K)=0$

(Dovolj je inj vlozitev, ne izomorfizem)

$$\bigoplus_{i\in I} K = \{(x_i)_{i\in I};\ x_i \in K \quad x_i = 0 \text{ za vse razen končno mnogo}\}$$

$$(x_1, x_2, \dots \dots x_i, \dots)$$

$$(x_i)_{i\in I} \in tor(\bigoplus K)$$

ne velja za noben element razen 0

$$y(x_i)_{i\in I} = (yx_i)_{i\in I} \Leftrightarrow \underset{\neq 0}{y} x_i = 0 \ \forall i$$

$$\Rightarrow x_i = 0 \ \forall i$$

4) Ali je $\mathbb{Z}_n$ prost $\mathbb{Z}$-modul?

NE, ker ima torzijo

$$\boxed{\begin{array}{l}\text{Izrek: } M \text{ končno generiran } \mathbb{Z}\text{-modul.} \\ \qquad M \text{ prost} \Leftrightarrow M \text{ torzijska prost}\end{array}}$$

5) Pokaži

${}_{\mathbb{Z}}\mathbb{Q}$ ni prost $\mathbb{Z}$-modul

$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$

$(cb)\frac{a}{b} - (ad)\frac{c}{d} = 0$ netrivialna linearna kombinacija

Ali je lahko baza?

$\mathbb{Q} = \left(\frac{a}{b}\right)$ $\quad \frac{1}{b^2}$ ni oblike $\alpha \frac{a}{b}$

Ali je $(\mathbb{Q}^*, \cdot)$ prost $\mathbb{Z}$ modul

$-1 \in \mathbb{Q}^*$ $\quad (-1)(-1) = 1 \Rightarrow (-1) \in T(\mathbb{Q}^*)$

$\Rightarrow$ ima torzijo, torej ni prost

Kaj pa $(\mathbb{Q}_+, \cdot)$

$\sum \left(\frac{a_i}{b_i}\right)^{n_i} > 0 \neq 0$

Eksaktna zaporedja

kratka ≡ kvocienti

$$0 \xrightarrow{0} L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{0} 0$$

$\operatorname{im}\varphi = \ker\psi \quad \operatorname{im}\psi = \ker 0 = N$

$0 = \operatorname{im} 0 = \ker\varphi$

ψ surjektivna

φ injektivna

$$M/\varphi(L) \cong N$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

$x \longmapsto 2x \qquad x \longmapsto x \bmod 2$

$-2x$

inj

odina možna preslikava

$\ker\psi = 2\mathbb{Z} = \operatorname{im}\varphi$

enotska krožnica

$$0 \xrightarrow{\text{inj} \checkmark} \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{T} \xrightarrow{\text{surj} \checkmark} 0$$

$n \longmapsto \boxed{\begin{matrix} n \\ x \end{matrix}}$ $\begin{matrix} \longmapsto 1 \\ \longmapsto e^{2\pi i x} \end{matrix}$

P je projektiven če $\forall$ eksaktno zaporedje $0 \to L \to M \to P \to 0$ razpade

kratko eksaktno zaporedje razpade če je ekvivalentno $0 \to L \to L \oplus N \to N \to 0$

P je projektiven $\Leftrightarrow$ 1) $\Leftrightarrow$ 2)

$\Leftrightarrow$ 1)

$$\begin{tikzcd} & P \arrow[d, "\forall f"] \arrow[dl, dashed, "\exists g"'] & \\ A \arrow[r, "sur"'] & B \arrow[r] & 0 \end{tikzcd}$$

$\Leftrightarrow$ 2) P je direktni sumand prostega modula $K^{(I)}$

$$\exists N .\ P \oplus N \cong K^{(I)} = \bigoplus_{i \in I} K$$

kolobar

1. naloga 7. 12

$\begin{bmatrix} F \\ F \end{bmatrix}$ je projektiven in ne prost nad $M_2(F)$

2) $M_2(F) \cong \begin{bmatrix} F \\ F \end{bmatrix} \underset{\text{zunanja}}{\oplus} \begin{bmatrix} F \\ F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \underset{\text{notranja}}{\oplus} \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$

$\begin{bmatrix} x & z \\ y & w \end{bmatrix} \longmapsto \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right) \longmapsto \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & z \\ 0 & w \end{bmatrix}$

je očiten izomorfizem

$\Rightarrow$ je projektiven

Recimo da je prost, potem

$\begin{bmatrix} F \\ F \end{bmatrix} = \begin{cases} M_2(F) & = \dim 4 \\ M_2(F) \oplus M_2(F) & = \dim 8 \\ \vdots \end{cases}$ vse dimenzije so prevelike

$\Rightarrow \begin{bmatrix} F \\ F \end{bmatrix}$ ni prost

2. način:

P proj $\Rightarrow P$ orz prost

$\varphi : P \to M$ homomorfizem

$\varphi(\mathrm{tor}(P)) \subseteq \mathrm{tor}(M)$

2) P proj $\underset{(K \text{ cel})}{\Rightarrow}$ P torz. prost

Velja: $\varphi: P \to M$ homo.

$\varphi(\mathrm{tor}(P)) \subseteq \mathrm{tor}(M)$

$P \oplus N \cong K^{(I)}$

$P \longrightarrow K^{(I)}$

$\varphi(\mathrm{tor}) \subseteq \mathrm{tor}(K^{(I)}) = 0$ ker je $K^{(I)}$ prost

Zaradi injektivnosti φ je je tudi $\mathrm{tor}(P) = 0$

prost $\Rightarrow$ proj. $\underset{}{\overset{K \text{ cel}}{\Rightarrow}}$ torz. prost

K glavni $\vee$

končnogeneriran modul

Kategorije

Top, Grp, Ab, Vect_F, K-Mod, ...

K mod:

objekti: K - moduli

morfizmi: homomorfizmi K-modulov

$$\mathrm{Hom}_K(P, \_) : K\,\mathrm{mod} \longrightarrow Ab$$

$$_K M \longmapsto \mathrm{Hom}(P, M)$$

$$(M \xrightarrow{f} N) \longmapsto \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, N)$$

$$\varphi \longmapsto f \circ \varphi$$

3. naloga

$\mathrm{Hom}(P, \_)$ je funktor

- ohranja kompozitum
- ohranja identiteto

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$F(g \circ f) \underset{(x)}{=} (\varphi \mapsto \varphi \circ g \circ f)$$

$$F(g) \circ F(f)(x) = (\varphi \mapsto \varphi \circ g)(\varphi \mapsto \varphi \circ f)(x) =$$

$$= (\varphi \mapsto \varphi \circ g)\, x \circ f = x \circ f \circ g = (\varphi \mapsto \varphi \circ (f \circ g)) = F(f \circ g)$$

dps, zamenjan vrstni red

$$F(id) = (\varphi \mapsto id\,\varphi) = id$$

P proj $\iff \underbrace{\mathrm{Hom}(P,-)}$ eksakten

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

$$0 \to \mathrm{Hom}(P,L) \xrightarrow{F(\varphi)} \mathrm{Hom}(P,M) \xrightarrow{F(\psi)} \mathrm{Hom}(P,N) \to 0$$

$\mathrm{Hom}(P,-)$ je eksakten če slika eksaktna zaporedja v eksaktna

$\Longrightarrow$ 1) $P = K$ $\quad \mathrm{Hom}(K, \underline{{}_K M}) \cong {}_K M$

$f \mapsto \varphi \circ f \qquad f \mapsto f(1)$

$$\begin{tikzcd} \mathrm{Hom}(P,L) \arrow[r, "F(\varphi)"] \arrow[d, phantom, "\wr\wr"] & \mathrm{Hom}(P,M) \arrow[d, phantom, "\wr\wr"] & f \arrow[d] \\ L \arrow[r, "\varphi"'] & M & f(1) \end{tikzcd}$$

2) P je prost $\quad P \cong K^{(I)}$

$$(\mathrm{Hom}, -) = \mathrm{Hom}(\bigoplus_I K, -) = \bigoplus_I \mathrm{Hom}(K,-)$$

$$0 \to \bigoplus \mathrm{Hom}(K,L) \to \bigoplus \mathrm{Hom}(K,M) \to \dots$$

$$(l_1 \dots l_n) \longmapsto (\varphi(l_1), \varphi(l_2) \dots)$$

8. 12

F 1. ura

Tenzorski produkt

ni nujno, ampak je bolj enostavno če je

$_KM$, $_KN$ modula, K komutativen

$$M\otimes_K N = \left\{\sum_{i=1}^{l} m_i\otimes n_i \;\middle|\; m_i\in M \;\; n_i\in N \;\; l\in\mathbb{N}_0\right\}$$

1) bilinearnost

$$(m+m')\otimes n = m\otimes n + m'\otimes n$$

$$m\otimes(n+n') = m\otimes n + m\otimes n'$$

2) biaditivnost $x\in K$

$$x\sum m_i\otimes n_i = \sum x(m_i\otimes n_i) = \sum (xm_i)\otimes n_i = \sum m_i\otimes(xn_i)$$

$$\begin{tikzcd}
\varphi: {}_KM\times {}_KN \arrow[r, "\text{bilin}"] \arrow[d] & {}_KL \\
M\otimes N \arrow[ur, dotted, "\exists!\,\bar\varphi"'] &
\end{tikzcd}$$

$$(m,n) \longmapsto \varphi(m,n), \qquad (m,n)\longmapsto m\otimes n$$

bilinearne

$$\varphi(xm,n) = x\varphi(m,n) = \varphi(m,xn)$$

baza

U,V vek. prostora $\quad U=\operatorname{span}\{e_i\}$ $\quad V=\operatorname{span}\{f_j\}$

$$U\otimes V = \operatorname{span}\{e_i\otimes f_j\}$$

1. Zapiši

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(1,1)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(0,1)+(-1,0)\otimes\left(\tfrac{1}{2},1\right)\in\mathbb{R}^2\otimes\mathbb{R}^2$$

kot enostavni tenzor $(u\otimes v)$

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(1,0)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(0,1)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes(0,1)+\tfrac{1}{2}(-1,0)\otimes(1,0)+(-1,0)\otimes(0,1)=$$

$$=\left(0,\tfrac{1}{2}\right)\otimes(1,0)+(0,1)\otimes(0,1)=$$

$$=\underline{\underline{(0,1)\otimes\left(\tfrac{1}{2},1\right)}}$$

2.

Poišči primer vek. prost. U, V in tenzorja $x \in U \otimes V$, ki ni enostaven

$U, V = \mathbb{R}^2$

$U', V' = \mathbb{R}$

U', V' $\quad \sum (x_i \otimes y_i) = \sum x_i y_i (1 \otimes 1) = \cancel{\ }$ $(1 \otimes \sum x_i y_i \, 1 \otimes 1)$

~~enostavna~~

$$\boxed{M \otimes_k k \cong M}$$

$$(1,0) \otimes (1,0) + (0,1) \otimes (0,1) = x \otimes y$$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$$x \otimes y = (x_1(1,0) + x_2(0,1)) \otimes (y_1(1,0) + y_2(0,1)) =$$

$$= x_1 y_1 (1,0) \otimes (1,0) + x_2 y_1 (0,1) \otimes (1,0) + x_1 y_2 (1,0) \otimes (0,1)$$

$$+ x_2 y_2 (0,1) \otimes (0,1)$$

$x_1 y_1 \neq 0 \quad \Rightarrow x_1 y_1 \neq 0$

$x_2 y_1 = 0 \quad \Rightarrow x_2 = 0$

$x_1 y_2 = 0$

$x_2 y_2 \neq 0 \quad \Rightarrow \cancel{x_2 \neq 0}$

$L^N = \mathrm{Hom}(N, L)$

Ali velja $(L^N)^M \cong L^{N\times M}$

3. Paketi ${}_k\mathrm{Hom}(M\otimes_k N, L) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, L))$

$$\Phi : {}_k\mathrm{Hom}(M\otimes N, L) \longrightarrow \mathrm{Hom}(M, \mathrm{Hom}(N, L))$$
$$(\varphi : M\otimes N, L) \longmapsto (\psi : m \longmapsto (n \longmapsto \varphi(m\otimes n)))$$

$$\Psi : \mathrm{Hom}(M, \mathrm{Hom}(N, L)) \longrightarrow \mathrm{Hom}(M\otimes N, L)$$
$$\psi \longmapsto \varphi$$

$$\begin{tikzcd}
((m, n) \arrow[d] \arrow[r, mapsto] & \psi(m)(n)) \\
m\otimes n \arrow[ur, dashed, "\varphi"']
\end{tikzcd}$$

Dokazati moramo še da je bilinearen

$\psi(m+m')(n) = (\psi(m)+\psi(m'))(n) = \psi(m)(n)+\psi(m')(n)$

ψ homomorfizem

$\psi(m)(n+n') = \psi(m)(n) + \psi(m)(n')$

$\psi(m)$ je homomorfizem

$\Phi\circ\Psi = id$

$(\Phi\circ\Psi)(\psi)(m)(n)$

$\Phi(\psi(m)(n)) = \Phi((m\otimes n) \mapsto \psi(m)(n)) = m \mapsto (n \mapsto \psi(m,n))(m\otimes n$

$\psi(m, n)$

$\forall$ paneme ... razlaga

zakaj iz te naloge sledi

$$\left(\bigoplus M_i\right) \otimes N \cong \bigoplus \left(M_i \otimes N\right)$$

V parabola sem público

2025 1. izpit

k cel M, N tor prosta

$M\otimes_k N$ neničeln

$$\boxed{k\otimes_k M = M}$$

Razširitev skalarjev

to je produkt dveh vek. p-ij. zato je produkt neničeln

$$(M\otimes_k N)\otimes_k F = (M\otimes_k F)\otimes_F (N\otimes_k F) \cong (M\otimes_k F)\otimes_F (F\otimes_k N)$$

$$\cong (M\otimes_k \underset{\substack{\cong \\ F}}{(F\otimes_F F)})\otimes_k N = (M\otimes_k N)\otimes_k F$$

$$\varphi : M \longrightarrow M\otimes_k F$$
$$m \longmapsto m\otimes 1$$
$$\ker\varphi = \operatorname{tor} M$$

ker je M torzijsko prost ga lahko vložimo v $M\otimes_k F$

$$\left(M/\operatorname{tor} M \hookrightarrow M\otimes_k F \right)$$

$$\Rightarrow M\otimes_k F = \{0\} \Rightarrow \ker\varphi = M = \{0\}$$