

# Osnove Newtonove mehanike

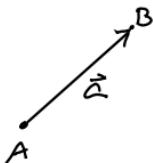
Def: Afini prostor je ned rektorski prostorom  $\mathcal{V}$  je množica od z binarno operacijo  $\mathcal{A} \times \mathcal{V} \rightarrow \mathcal{A}: (A, \vec{a}) \mapsto A + \vec{a}$  z lastnostmi:

i)  $(A + \vec{a}) + \vec{b} = A + (\vec{a} + \vec{b})$

ii)  $\forall A, B \in \mathcal{A}, \exists \vec{a} \in \mathcal{V}: B = A + \vec{a}$

$$\dim \mathcal{A} = \dim \mathcal{V}$$

Primeri:



Def: Definiramo operacijo  $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{V}$ .

s predpisom  $B - A = \vec{a} \Leftrightarrow \vec{B} = A + \vec{a}$

Trditve:

i)  $A - A = \vec{0}$

ii)  $(A - B) + (B - A) = \vec{0}$

iii)  $(A - B) + (B - C) + (C - A) = \vec{0}$

iv)  $(A - B) + \vec{a} = (A + \vec{a}) - B$

v)  $(A + B) - C = B + (A - C)$

Dokaz:

i)  $A - A = \vec{a} \Leftrightarrow A = A + \vec{a} \Leftrightarrow A = (A + \vec{a}) + \vec{a} = A + 2\vec{a}$

$$\Rightarrow \vec{a} = 2\vec{a} \quad (\text{ker je } \vec{a} \text{ neprazna dolacen})$$

$$\Rightarrow \vec{a} = \vec{0}$$

v)  $A - C = \vec{a} \Rightarrow A = C + \vec{a}$

$$B + (A - C) = B + \vec{a} = C + (\vec{b} + \vec{a})$$

$B - C = \vec{b} \Rightarrow B = C + \vec{b}$  vedno, zato lahko zamenjamo oblike

$$(A + B) - C = ((C + \vec{a}) + (C + \vec{b})) - C =$$
$$= C + \vec{a} + (C + \vec{b} - C) = \underbrace{C}_{\text{vector}} + \vec{a} + \vec{b}$$

$(\mathcal{A}, \mathcal{V})$   $(\mathcal{A}', \mathcal{V}')$

$g: \mathcal{A} \rightarrow \mathcal{A}'$

Def: Preslikava  $g: \mathcal{A} \rightarrow \mathcal{A}'$  je **afina** če obstaja

$d_g \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$  tako da velja  $g(A) - g(B) = d_g(A - B)$   
za  $\forall A, B \in \mathcal{A}$

$$g(A) = g(O) + d_g(A - O)$$

$g(A) = g(O) + d_g(A - O)$ ;  $O$  je **pol affine preslikave**

Izbira pola je poljubn

$$\begin{aligned}\tilde{g}(A) &= g(\tilde{o}) + d_g(A - \tilde{o}) = g(O) + d_g(\tilde{o} - o) + d_g(A - \tilde{o}) \\ &= g(O) + \underbrace{d_g((\tilde{o} - o) + (A - \tilde{o}))}_{A - o} = g(A)\end{aligned}$$

Izberimo  $O \in \mathcal{A}$   $\vec{o} = A - O$

$$A \in \mathcal{A}; A = O + (A - O) = O + \vec{o}$$

$\forall A \in \mathcal{A}$  lahko identificiramo  $\vec{o} \in \mathcal{V}$

Definicija: Galilejeva struktura  $G$  je trojica  $(\mathcal{A}, \tau, d)$ , kjer je  $\mathcal{A}$  stiri razsežen afini prostor,  $\tau \in \mathcal{L}(V, \mathbb{R})$  in  $d$  evklidske razdalje nad linearne preslikave

Funkcionalu  $\tau$  pravimo **časovnost**. Dve delki  $A, B \in \mathcal{A}$  sta **istočasna**, če  $A$ -bekert

Definicija: Galilejev strukturi  $G(\mathcal{A}, \tau, d)$  in  $\tilde{G}(\tilde{\mathcal{A}}, \tilde{\tau}, \tilde{d})$  sta **ekvivalentni**, če obstaja afina bijekcija  $g: \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ , ki ohranja časovnost in razdaljo istočasnih delkov

$$\tilde{\tau}(g(A) - g(B)) = \tau(A - B)$$

$A, B$  istočasna  $\Leftrightarrow g(A), g(B)$  istočasna

$$d(A, B) = \tilde{d}(g(A), g(B))$$

Definicija

$\mathbb{R} \times E$  afini prostor, kjer je  $E$  torazsežen: evklidski prostor

Na  $\mathbb{R} \times E$  vpeljena naravna Galilejeva struktura.

$$A \in \mathbb{R} \times E \Rightarrow A = (t, P) \quad t \in \mathbb{R}, P \in E$$

imata normo poravnana s skalarnim produktom

$$\tau(A_2 - A_1) = t_2 - t_1$$

$$d(A_1, A_2) = \|P_1 - P_2\|$$

Ta struktura; pravimo naravna Galilejeva struktura

Definicija: koordinatni sistem na afinem prostoru oz.

je bijektivna preslikava  $\varphi: A \rightarrow \mathbb{R}$

$$A \mapsto \varphi(A) = (\Pi_t \varphi(A), \Pi_p \varphi(A))$$

za heterogeno afino preslikavo  $= \varphi_t(A), \varphi_p(A)$

$$(\mathbb{R} \times E, \tau, d)$$

$$\tau(A - B) = t(\varphi_t(A) - \varphi_t(B))$$

$$d(A, B) = \|\varphi_p(A) - \varphi_p(B)\|$$

Lahko merimo razdaljo tudi med neistotnimi dogodki

$$A \xrightarrow{\varphi} \mathbb{R} \times E$$
$$\downarrow$$
$$A \xrightarrow{\varphi} \mathbb{R} \times E$$

Kdaj sta dve naravnji geometrijski strukturi ekvivalentni:

$$g: \mathbb{R} \times E \rightarrow \mathbb{R} \times E$$

$$A = \begin{bmatrix} t \\ P \end{bmatrix} \mapsto g(A) = \begin{bmatrix} t' \\ P' \end{bmatrix} = g(O) + dg(O)(A - O)$$

$$\begin{aligned} &= \begin{bmatrix} t_o \\ P_o \end{bmatrix} + \begin{bmatrix} \alpha & \vec{\alpha}^T \\ \vec{\alpha} & Q \end{bmatrix} \begin{bmatrix} t - t_o \\ P - P_o \end{bmatrix} \\ O = \begin{bmatrix} t_o \\ P_o \end{bmatrix} \quad &\xrightarrow{\text{matr. } Q \text{ je ortogonalna}} \end{aligned}$$

$$A_1 = \begin{bmatrix} t_1 \\ P_1 \end{bmatrix} \quad A_2 = \begin{bmatrix} t_2 \\ P_2 \end{bmatrix}$$

$$+ (g(A_2) - g(A_1)) = + (A_2 - A_1)$$

$$g(A_2) - g(A_1) = \begin{bmatrix} \alpha & \vec{\alpha}^T \\ \vec{\alpha} & Q \end{bmatrix} \begin{bmatrix} t_2 - t_1 \\ P_2 - P_1 \end{bmatrix} = \begin{bmatrix} \alpha(t_2 - t_1) + \vec{\alpha}(P_2 - P_1) \\ (t_2 - t_1)\vec{\alpha} + Q(P_2 - P_1) \end{bmatrix}$$

$$\Rightarrow \alpha(t_2 - t_1) + \vec{\alpha}(P_2 - P_1) = t_2 - t_1 \Rightarrow \alpha = 1, \vec{\alpha} = \vec{0}$$

To velja da se obranja časovnost

obranjanje razdeljje med istočasnim: dogodki:

$$d(g(A_1), g(A_2)) = d(A_2, A_1) \geq t_1 - t_2$$

$$\|Q(P_2 - P_1)\| \quad \|P_2 - P_1\|$$

$$\Rightarrow Q \in O(3)$$

• ortogonalna matrika  $3 \times 3$

Definicija: Preslikave, ki ohranja Galilejevo strukturo pravimo **Galilejeve preslikave**

Tedatev: Galilejeve preslikave med mernavna Galilejevima strukturama  $\mathbb{R} \times E$  je oblike

$$\begin{bmatrix} \tilde{t} \\ \tilde{P} \end{bmatrix} \mapsto \begin{bmatrix} t' \\ P' \end{bmatrix} = \begin{bmatrix} \tilde{t}_0 + t - t_0 \\ \tilde{P}_0 + \tilde{c}(t - t_0) + Q(P - P_0) \end{bmatrix} =$$

kjer je  $Q \in O(3)$ ,  $\tilde{c}$  poljuben vektor, to poljubno  
stevilo in  $P'_0$  poljubna točka

$$= \begin{bmatrix} \tilde{t}_0 + t \\ \tilde{P}_0 + \tilde{c}t + Q(P - P_0) \end{bmatrix}$$

$$t_0' - t_0 = \tilde{t}_0$$

Dva apsorvalca:  $\varphi(t, p)$ ,  $\varphi'(t, p)$

objavljanje  $t \mapsto P(t)$  teraktorija točke

$$\vec{v} = \frac{dP}{dt} \quad \text{veličina h:trazi } \lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h}$$

$$\ddot{P} = \dot{\vec{v}} = \vec{a} = \frac{d\vec{v}}{dt} \quad \text{veličina pospeška}$$

$$|\vec{v}| = v \quad \text{brzina} \quad \overset{Q(P(t) - P_0)}{,} (P(t) - P_0)$$

$$P'(t) = P_0 + \overset{Q(P(t) - P_0)}{\overbrace{\vec{C}(t - t_0)}} + Q(P(t) - P_0) \quad \text{teraktorija v } \varphi'$$

$$\vec{v}' = \frac{dP'}{dt} = \vec{C} + Q \frac{dP}{dt} (t - t_0) = \vec{C} + Q P(t)$$

$$\ddot{P}'(t) = \vec{C} + Q \ddot{P}(t)$$

$\uparrow$   
najprej ortica, nato odvod

$$\ddot{P}'(t) = \vec{a} = \frac{d\vec{v}'}{dt} = Q \ddot{P}(t)$$

$$\begin{array}{c|c} \varphi & \varphi' \\ \hline P & P = P_0 + \vec{C} t + Q(P - P_0) \\ \vec{a} = P_2 - P_1 & \vec{a}' = P_2 - P_1 = Q(P_2 - P_1) = Q \vec{a} \end{array}$$

Definicije: vektor  $a = P_2 - P_1$  je koordinatno neodvisen

$$A \in \mathcal{L}(\mathcal{V}, \mathcal{V}) \quad A^T = Q^T A Q$$

$$\lambda \quad \lambda' \Rightarrow$$

sistem materialnih točk  $(P_1, \dots, P_n) = P$

$$\underline{P} = \underline{P}_0 + \vec{c} t + Q (\underline{P} - \underline{P}_0)$$

$$\underline{P}' = (P'_1, \dots, P'_n)$$

$$Q(\underline{P} - \underline{P}_0) = (Q(P_1 - P_0), \dots, Q(P_n - P_0))$$

$$\vec{c} = (\vec{c}_1, \dots, \vec{c}_n)$$

$$\Rightarrow \dot{\underline{P}} = \vec{c} + Q \dot{\underline{P}}$$

$$\ddot{\underline{P}} = Q \ddot{\underline{P}}$$

### Princip determiniranosti

V danem KCS (koordinatni sistem) je trektorij sistema materialnih točk netanko določena z začetnim položajem in začetno hitrostjo.

To posebeno pomeni, da obstaja funkcija interakcije  $\vec{f}$  tako da je  $\ddot{\underline{P}} = \vec{f}(\underline{t}, \underline{P}, \dot{\underline{P}})$

$$(P(t) = \vec{f}(t, \underline{P}(t), \dot{\underline{P}}(t)) \text{ nadalje})$$

**Princip relativnosti** obstaja razred koordinatnih sistemov v katerem je funkcija interakcije invariantna za Galilejeve transformacije. Koordinatni sistem : z tege pravimo **mercialn**; koordinatni sistem

$$(t, \underline{P}) \xrightarrow{GT} (\underline{t}, \underline{P}) \text{ Galilejeva transformacija.}$$

$$\text{Potem je } \underline{\ddot{P}} = \underline{\ddot{P}}' \quad (\ddot{P} = \ddot{P}')$$

$$\ddot{\underline{P}} = \ddot{\underline{f}}(t, \underline{P}, \dot{\underline{P}})$$

$$Q \ddot{\underline{P}} = Q \ddot{\underline{f}}(t, \underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(t_0 + t, \underline{P}_0 + \vec{c}t + Q(\underline{P} - \underline{P}_0), \dot{\underline{P}} + Q\dot{\underline{P}})$$

$$\Rightarrow \vec{c} = \vec{\sigma} \quad Q = I \quad \underline{P}_0 = \underline{P}_0$$

$$\ddot{\underline{f}}(t, \underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(t_0 + t, \underline{P}, \dot{\underline{P}}) \propto \forall t_0$$

$\Rightarrow f_n$ ; eksplicitna odvisnost od čase ( $t$ )

(homogenost čase)

$$ii) \vec{c} = 0, Q = I, \dot{\underline{P}} = \underline{P}_0 + \vec{\alpha}$$

$$\ddot{\underline{f}}(\underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(\underline{P} + \vec{\alpha}, \dot{\underline{P}}) \propto \forall \vec{\alpha}$$

||  
 $(\underline{P}_1 + \vec{\alpha}, \dots, \underline{P}_n + \vec{\alpha})$

$\Rightarrow \ddot{\underline{f}}$  je odvisna samo od relativnih položajev

$$\ddot{\underline{f}}(\underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(\underline{P}_i - \underline{P}_j, \dot{\underline{P}})$$

$i \neq j$  & vse kombinacije tege

(homogenost prostora)

$$iii) Q = I$$

$$f(P_i - P_j, \dot{\underline{P}}) = f(P_i - P_j, \vec{c} + \dot{\underline{P}}) \Rightarrow \ddot{\underline{f}}(P_i - P_j, \dot{\underline{P}}_k - \dot{\underline{P}}_l)$$

(homogenost prostora = hirost)

$$iv) Q \text{ poljuben}$$

$$Q \ddot{\underline{f}}(P_i - P_j, \dot{\underline{P}}_k - \dot{\underline{P}}_l) = f(Q(P_i - P_j), Q(\dot{\underline{P}}_k - \dot{\underline{P}}_l))$$

$f$  je izotropična funkcija

$$Q \vec{g}(\vec{\alpha}) = \vec{g}(Q \vec{\alpha}) \quad \forall Q \in O(3)$$

Posebni primer:  $N=1$  izolirana točka

$$\ddot{\vec{P}} = \vec{f}(\cdot) \quad \text{brez argumentov (konstante)}$$

Se vedno velja izotrapacnost  $Q\ddot{\vec{P}} = Q\vec{f} = \vec{0} \Rightarrow \vec{f} = 0$   
 $\Rightarrow \ddot{\vec{P}} = \vec{0}$

v IKS (inercijskih koordinatnih sistem) se izolirane materialne točke gibljiv premoorimo s konstantno brzinom.  $\vec{P} = \vec{v}_0 t + \vec{P}_0(t=0)$

$N=2 \Rightarrow$

$$\ddot{\vec{P}}_1 = \vec{f}_1(\vec{P}_1 - \vec{P}_2, \dot{\vec{P}}_1 - \dot{\vec{P}}_2) \quad \text{če se tekoče gibljejo pravimiči proti drugi. Leta včne osteli na tej premici.}$$
$$\ddot{\vec{P}}_2 = \vec{f}_2(\vec{P}_2 - \vec{P}_1, \dot{\vec{P}}_2 - \dot{\vec{P}}_1) \quad (\text{Podobno tr: tekoče obsegajo ne ravni})$$

$$\ddot{\vec{P}}_1 = f_1(\vec{P}_1 - \vec{P}_2, \dot{\vec{P}}_1 - \dot{\vec{P}}_2, \underline{\underline{\vec{P}_2 - \vec{P}_3}}, \dots)$$

$t$

tudi  
to je pomembno

detraktivje: Interakcija  $\ddot{\vec{P}}_i = f_i(\dots)$  je **parška**  
 če je odvisna samo od relativnih položajev  
 in hitrosti glede na  $\vec{P}_i$  in je delovanje  
 tako autonomno

$$\vec{f}_i(P_i - P_j, \dot{P}_i - \dot{P}_j)$$

$$j \neq i \quad k \neq i$$

$$\vec{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^n \vec{f}_{ij}(P_i - P_j, \dot{P}_i - \dot{P}_j)$$

Def: Interakcija  $\vec{f}_{ij} = \vec{f}_j(P_i - P_j, \dot{P}_i - \dot{P}_j)$  je **lokalna**  
 če velja  $\lim_{|P_i - P_j| \rightarrow \infty} \vec{f}_{ij} = \vec{0}$

## Princip sorazmernosti

V IKS za sistem materialnih točk  $P_1, \dots, P_N$  obstajajo realne dolocene konstante  $\alpha_{ij}$  tako, da ne glede na interakcije  $\vec{f}_i$  velja  $\vec{f}_i = -\sum_{j=1}^N \alpha_{ij} \vec{f}_j$

Lema: Za konstante  $\alpha_{ij}$  velja

- i)  $\alpha_{ij} \alpha_{ji} = 1$
- ii)  $\alpha_{ij} \alpha_{jk} \alpha_{ki} = 1$

Dokaz:

$\vec{f}_i = \vec{f}_i(P_i, -P_n) = \vec{f}_i(P_i, -P_1, P_i, -P_2, \dots, P_i, -P_N)$   
in naj bodo  $P_i$  lokale:

$P_k$  posljedno postoji  $\infty$  razen  $P_i$  in  $P_j$

$$\vec{f}_i = -\sum_{k=1}^N \alpha_{ki} \vec{f}_k = \underbrace{\alpha_{ji}}_{k \neq i} \vec{f}_j$$

$$\vec{f}_j = -\sum_{k=1}^N \alpha_{kj} \vec{f}_k = -\alpha_{ij} \vec{f}_i$$

$$f_i = -\alpha_{ij} f_j \Rightarrow \alpha_{ij} \alpha_{ji} = 1$$

$P_i, P_j, P_k$  obstajajo, ostale grejo prehodno

$$\begin{aligned} \vec{f}_i &= -\alpha_{ij} \vec{f}_j - \alpha_{ki} \vec{f}_k = -\alpha_{ij} (-\alpha_{jj} \vec{f}_i - \alpha_{kj} \vec{f}_k) \alpha_{ik}^{-1} = \\ \vec{f}_i &= \underbrace{-\alpha_{ij} \vec{f}_i}_{\vec{f}_i} - \alpha_{ki} \vec{f}_k \end{aligned}$$

$$\underbrace{\alpha_{ki} \vec{f}_k}_{\vec{f}_k} = \alpha_{ij} \alpha_{kj} \vec{f}_i / \alpha_{ik}$$

$$\vec{f}_k = \alpha_{ij} \alpha_{kj} \alpha_{ik} \vec{f}_i \Rightarrow \alpha_{ik} \alpha_{kj} \alpha_{ji} = 1$$

Lemai: Naj za pozitivna števila  $\alpha_{ij}$  velja

i)  $\alpha_{ij} \cdot \alpha_{ji} = 1$

ii)  $\alpha_{ij} \cdot \alpha_{jk} \cdot \alpha_{ki} = 1$ .

Potem  $\exists$  pozitivna števila  $m_i$ , tako da velja da je

$$\alpha_{ij} = \frac{m_i}{m_j}$$

Števila  $m_i$  so določene to sorazmernostne faktorje načanost. Številom  $m_i$  pravimo **inercijske mase**

Dokaz:  $\underline{\alpha_{ii}} = 1$

$$\alpha_{ij} \alpha_{jj} \alpha_{ji} = 1 \Rightarrow \alpha_{jj} = 1$$

$$l_{ij} = \log \alpha_{ij} \quad l_{ij} + l_{jk} + l_{ki} = 0$$

$$l_{ij} = -l_{ji}$$

$$l_{io} + l_{ji} + l_{ki} = 0$$

$$l_{ij} - l_{io} + l_{oi} - l_{ki} = 0 \Rightarrow$$

$$l_{ij} - l_{io} = -l_{oi} + l_{ki} = l_{ki} - l_{io} \quad \forall j, k$$

$$\Rightarrow l_{ij} - l_{io} = n_{io}$$

$$l_{ij} = n_{io} + l_{io} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow 0 = n_{io} + l_{io} \Rightarrow l_{io} = n_{io}$$

$$l_{ij} = l_{io} - l_{ji}$$

$$l_{io} = \log m_i \text{ delkin: same}$$

$$\Rightarrow l_{ij} = \log m_i - \log m_j = \log \frac{m_i}{m_j} \Rightarrow \alpha_{ij} = \frac{m_i}{m_j}$$

Kreki: V inertialnom koordinatnom sistemu velja

$$m_1 \ddot{\vec{P}}_1 + m_2 \ddot{\vec{P}}_2 + \dots + m_n \ddot{\vec{P}}_n = \vec{0}$$

Dokaz:

$$\ddot{f}_i = - \sum_{\substack{j=1 \\ j \neq i}}^N \alpha_{ij} f_j = - \sum_{j=1}^N \frac{m_j}{m_i} f_j$$

$$\ddot{\vec{P}}_i = - \sum_{j=1}^N \frac{m_j}{m_i} \ddot{\vec{P}}_j \quad /m;$$

$$\sum_{i=1}^N m_i \ddot{\vec{P}}_i = \vec{0}$$

$$\vec{P}_* = \frac{1}{m} \sum_{i=1}^N m_i \vec{P}_i, \quad m = \sum_{i=1}^N m_i \quad \text{masno srediste}$$

sestavanje tako:

$$\sum_{i=1}^N \vec{P}_i = \sum (m_i(\vec{P}_i - \vec{0}) + m_i \vec{0}) = \sum_{i=1}^N m_i(\vec{P}_i - \vec{0}) + \underbrace{\sum_{i=1}^N m_i \vec{0}}_0$$

$\vec{v}_\text{ektor}$

$$\vec{P}_* = \vec{0} + \frac{1}{m} \sum_{i=1}^N m_i(\vec{P}_i - \vec{0})$$

$$m \ddot{\vec{P}}_* = \vec{0}$$

Sile  $n = P_i$   
 Definicija: je produkt interakcije fiz mase  $m$ :  
 $m; \ddot{P}_i = m; f_i(\dots) = \vec{F}_i$

Princip omaza mase materialnih točki je enaka  
 v vseh koordinatnih sistemih

$$\ddot{P}_1 = f_1(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \quad m_1 \dot{P}_1 = \vec{F}_1$$

$$\ddot{P}_2 = f_2(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \quad m_2 \dot{P}_2 = \vec{F}_2$$

$$m_1 \ddot{\vec{P}}_1 + m_2 \ddot{\vec{P}}_2 = \vec{F}_1 + \vec{F}_2 = 0$$

$$\vec{F}_2 = -\vec{F}_1$$

$$\vec{\vec{F}}_1 = \vec{\vec{F}}_{21}$$

$$\vec{\vec{F}}_{12} = -\vec{\vec{F}}_{21}$$

Trditev: Če so vse sile parske in lokalne velja  
zakon akcelir in rezultir (3NZ)

Dokaz:  $f_i = \sum_{\substack{j=1 \\ j \neq i}}^N f_{ij} (p_j - p_i, \dot{p}_j - \dot{p}_i)$

$$\downarrow \quad \left. \begin{array}{l} p_j \rightarrow \infty \quad j \notin \{i, k\} \end{array} \right\}$$

$$\begin{aligned} m_i \ddot{f}_i &= m_i \ddot{p} \\ m_i f_{ji} + m_j f_{ij} &= \vec{\sigma} \\ \vec{F}_{ji} &= -\vec{F}_{ij} \end{aligned}$$

Y

10.10  
omadika prva uro

adrad uzdalī trakcije  $\frac{d}{dt} E(t, P(t), \dot{P}(t)) =$

$$\frac{\partial E}{\partial t} + \left( \frac{\partial E}{\partial P} \right)^T \dot{P} + \left( \frac{\partial E}{\partial \dot{P}} \right)^T \ddot{P}$$

Izrek o množenju: Maj bo sila  $\vec{F}$  povezana s potencialom  $U(t, P, \dot{P})$ . Potem je vsata funkcija in potencialne energije konstanta gibanja (= njen adrad uzdalī trakcije je 0)  
 $\Leftrightarrow$  moč sil je nasprotna enaka adradu potencialne energije uzdalī trakcije

$$D\dot{E}_0 : \Rightarrow E_0 = T + U = \frac{1}{2} m |\dot{P}|^2 + U$$

$$0 = \frac{\partial E_0}{\partial t} - m \ddot{P} \dot{P} + \vec{F} \cdot \dot{P} + \frac{dU}{dt} \Rightarrow \frac{dU}{dt} = - \vec{F} \cdot \dot{P}$$

$$\begin{aligned} & \leftarrow \begin{array}{l} T_1 = T_2 - T_1 \\ U_1 = U_2 - U_1 \end{array} \\ A = & \int_{T_1}^{T_2} \vec{F} \cdot \dot{P} dt = - \int_{T_1}^{T_2} \frac{dU}{dt} dt = U_1 - U_2 \end{aligned}$$

$$\Rightarrow U_1 + T_1 = U_2 + T_2 = E_0$$

Sila je konzervativna, če je potencialna in  
odvisna samo od položaja

Posledica: če je sila konzervativna velja izrek  
o energiji:

$$\text{Dokaz: } \frac{d}{dt} U(P(t)) = \left( \frac{\partial U}{\partial P} \right)^T \dot{P} = -F \dot{P}$$

$$F \dot{P} = - \frac{dU}{dt}$$

$$\begin{aligned} F' \ddot{P}' &= QF(\vec{c} + Q\dot{P}) = QF \cdot \vec{c} + \underbrace{Q\vec{F} \cdot Q\dot{P}}_{\stackrel{||}{F} \cdot \dot{P}} = - \frac{dU}{dt} + m\ddot{P}^2 \vec{c}^2 \\ &= - \frac{d}{dt} (U - m\dot{P}' \vec{c}') \quad \begin{matrix} \uparrow \\ \text{arlegatna} \\ \text{preoblikovana} \\ \text{eksterni produkt} \end{matrix} \end{aligned}$$

$$\tilde{U} = U - m\dot{P}' \vec{c}'$$

$$\tilde{U}(P', \dot{P}') = U'(P') - m\dot{P}' \vec{c}'$$

Kræft: Kræft o. energi; je invarianten za  
Galilejev transformatioj

$$\begin{aligned}T' &= T + \tilde{U} = \frac{1}{2} m |\vec{c}'|^2 + mc \vec{Q} \cdot \vec{P} + T + \tilde{U} - m \vec{P}' \cdot \vec{c}' = \\&= E_0 + \frac{1}{2} m |\vec{c}'|^2 + m \vec{c}' (\vec{Q} \cdot \vec{P} - \vec{P}') = E_0 - \frac{1}{2} m |\vec{c}'|^2 \\T &= \frac{1}{2} m \vec{P}' \cdot \vec{P}' = \frac{1}{2} m (\vec{c} + \vec{Q} \cdot \vec{P}) (\vec{c} + \vec{Q} \cdot \vec{P}) = \frac{1}{2} m |\vec{c}|^2 + m \vec{c} \cdot \vec{Q} \cdot \vec{P} \\&\quad + \frac{1}{2} m |\vec{P}|^2 \\&\vec{P}' = \vec{c}' + \vec{Q} \cdot \vec{P}\end{aligned}$$

# Premočrtno gibanje

= pospešek ima konstantno smer

$$\vec{a} = a(t) \vec{e}$$

$$\vec{v} = \int_{t_0}^t \vec{a}(t) dt + \vec{v}(t_0) = \left( \int_{t_0}^t a(t) dt \right) \vec{e} + \vec{c}(t_0)$$

Obstaja XS v heterem tir ledi na premici

$$\overline{x} = x \quad \dot{\overline{x}} = \dot{x} \quad \ddot{\overline{x}} = \ddot{x}$$

$$m \ddot{x} = f(t, x, \dot{x})$$

$v = \dot{x}$  je lahko zeloj tudi negativen

omejimo se ko bo f konzervativna

$$m \dot{x} = f(x)$$

Kdaj je sila v odvisnosti položaja potencialna?

$$f = -\frac{dU}{dx} \Rightarrow U = \int_{x_0}^x f(\xi) d\xi + U_0 \quad (\text{potencialna} \rightarrow \text{konzervativna})$$

Silu f(x) je potencialna če je f zvezna

Ce je f(x) zvezna  $\Rightarrow$  velik izrok energiji

$$m\ddot{x} = F(x)$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = E_0$$

$$\dot{x}^2 = \frac{2}{m}(E_0 - U(x))$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E_0 - U(x))}$$

$$\pm \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = dt$$

$$\operatorname{sgn} x \int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \int_{t_0}^t dt = t - t_0 \Rightarrow t = t(x) \Rightarrow x = x(t)$$

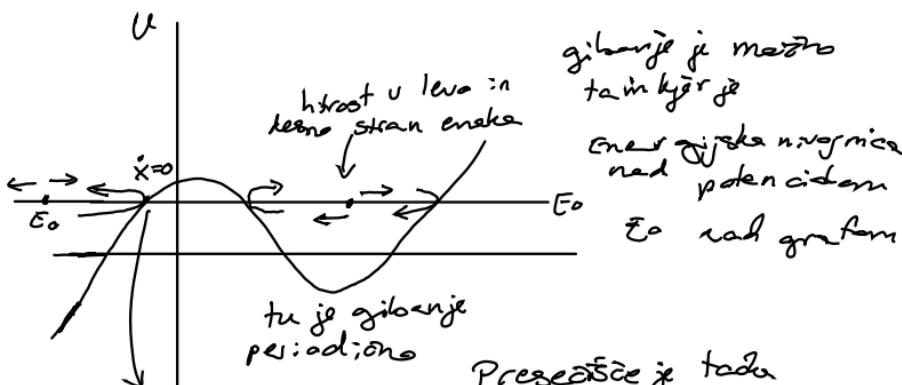
↑

Lahko krenemo  
ko  $x \neq 0$

( $\operatorname{sgn} x$  je  
konstanten)

$$t = t_0 \pm \int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

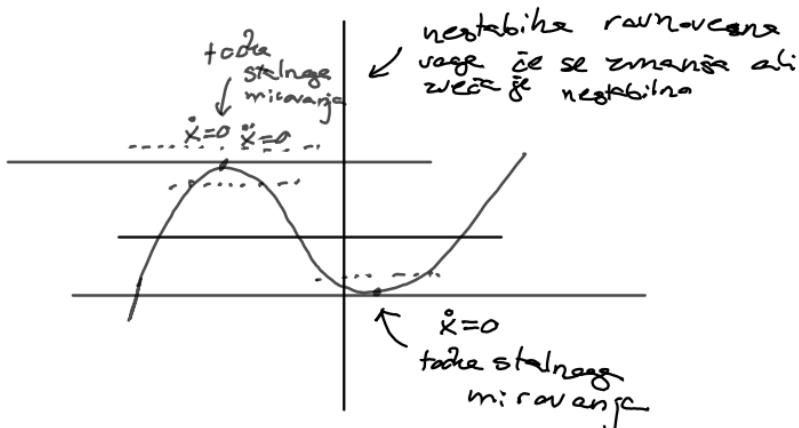
Kvalitativne obravnavi gibanje



$$0 = \dot{x} = \frac{F(x)}{m} = -\frac{dU}{dx}$$

Premeseče je točka

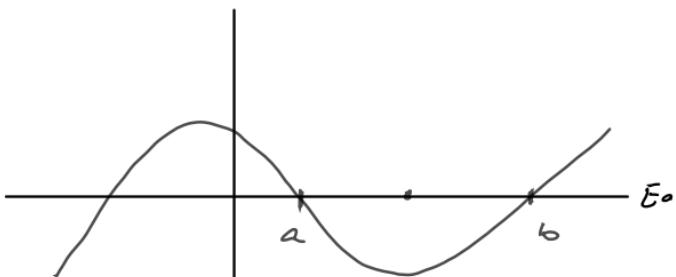
trenutnega mirovanja  
(točke obrate gibanje)



Lokalni minimum je stabilna ravnotezna lega  
(če se energija nato povečuje pride do majhnejših periodičnih gibanj (majhen odmak od ravnotezne vase)

Prevoj je tudi nestabilna lega

Kakšna je periooda periodičnega gibanja



gibanje:  $t_1 = \int_{x_a}^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$      $t_2 = -\int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$

$$t_3 = \int_a^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

$T = t_1 + t_2 + t_3 = 2 \int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$

↑ perioda

Ali je rez periodično in se ne ustavi: prej?

atd

$$\int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

$$U(x) = U(a) \left( \frac{du}{dx} \right)_a^x (x-a)$$

$$\frac{du}{dx}(a) < 0 \quad \text{a je presečišče}$$

$$\exists \delta > 0. \exists c \in (a, a+\delta) \Rightarrow 2 \frac{du}{dx}(z) < \frac{1}{2} \frac{du}{dx}(a)$$

$$E_0 - U(x) = \frac{du}{dx}(z)(x-a) < -2 \frac{du}{dx}(a)(x-a)$$

$\Rightarrow$  integral je konvergenten

$$\text{ker } \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}} \leq \frac{1}{\sqrt{-\frac{1}{2} U'(a)}} \int_a^b \frac{dx}{\sqrt{x-a}} < \infty$$

Če se energijska nivojnica dotika lokalnega maksimuma, potem ne prideemo do konvergencije

v končnem času

$$\text{Lema} \quad \int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi$$

$$\text{Dokaz: } x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a) \cos \vartheta$$

$$b-x = \frac{1}{2}(b-a)(1-\cos \vartheta)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos \vartheta)$$

$$(b-x)(x-a) = \left(\frac{1}{2}(b-a)\right)^2 (1-\cos^2 \vartheta) = \left(\frac{1}{2}(b-a)\right)^2 \sin^2 \vartheta$$

$$dx = -\frac{1}{2}(b-a) \sin \vartheta d\vartheta$$

$$I = - \int_{\pi}^0 \frac{\frac{1}{2}(b-a) \sin \vartheta d\vartheta}{\frac{1}{2}(b-a) \sin \vartheta} = - \int_{\pi}^0 d\vartheta = \pi$$

Primer:

$$\text{Harmonični oscilator} \quad U = \frac{1}{2}kx^2$$

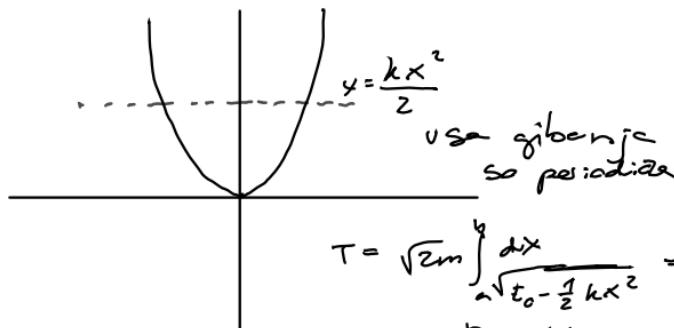
$$f = -kx$$

$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0 \quad \frac{k}{m} = \omega^2$$

$$\ddot{x} + \omega x = 0$$

$$x = A \cos(\omega t - \delta)$$



T periode je neodvisne od energije  
(izobrščene gibanje)

$$\text{Primer: } U = \alpha x^2 + b x^{-2}$$

$$[U] \text{ dimensionen erthalten}$$

$$[U] = [a x^2] = [a] [x^2] = [a] L^2$$

$$[U] = [b x^{-2}] = [b] L^{-2}$$

$$\Rightarrow [a] L^2 = [b] L^{-2}$$

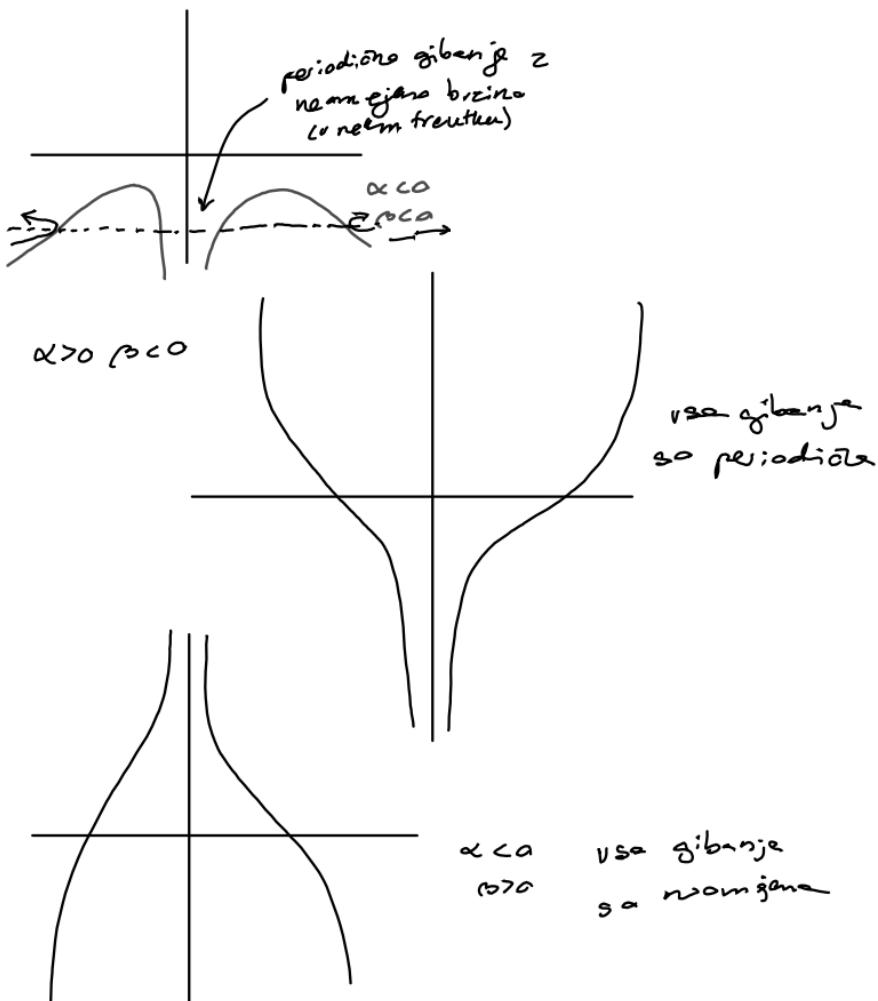
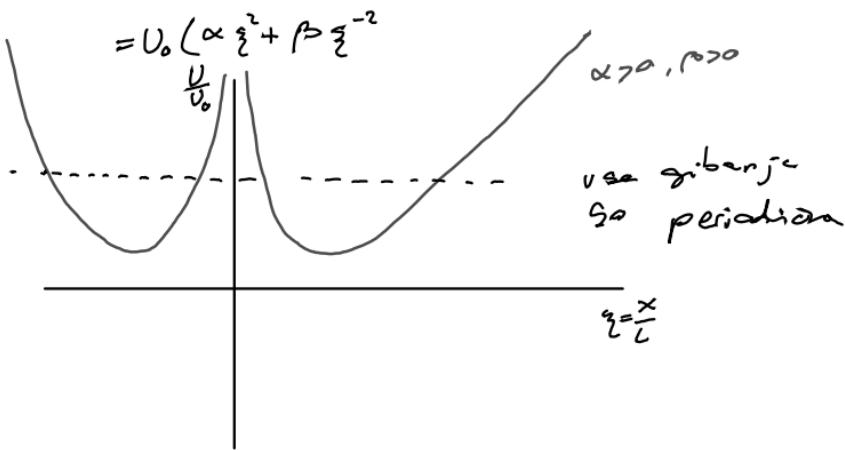
$$L^4 = \frac{[b]}{[a]} \Rightarrow L = \sqrt[4]{\frac{[b]}{[a]}}$$

$$x = L \xi \quad L = \left(\frac{b}{a}\right)^{\frac{1}{4}}$$

$\xi$  zweidimensionale Größe

$$U = a \sqrt{\frac{|b|}{|a|}} \xi^2 + b \sqrt{\frac{|a|}{|b|}} \xi^{-2} = \underbrace{\sqrt{|a||b|}}_{U_0} (\alpha \xi^2 + \beta \xi^{-2}) =$$

$$a = \underbrace{\frac{a}{\sqrt{|a|}}}_{\alpha} |a| \quad b = \underbrace{\frac{b}{\sqrt{|b|}}}_{\beta} |b|$$



Računajmo perioda za  $\alpha, \beta > 0$

$$U = \sqrt{|\alpha||\beta|} \left( \xi^2 + \xi^{-2} \right)$$

$$\begin{matrix} \\ \\ U_0 \end{matrix}$$

stevilo  
 $E_0 = U_0 C_0$

$$T = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - (\alpha x^2 + \beta x^{-2})}} = \sqrt{2m} \int_{\xi_1}^{\xi_2} \frac{L d\xi}{\sqrt{U_0 C_0 - U_0 (\xi^2 - \xi^{-2})}} =$$
$$= \sqrt{2m} \frac{L}{\sqrt{U_0}} \int_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{C_0 - \xi^2 - \xi^{-2}}} d\xi = \frac{\sqrt{2m}}{\sqrt{U_0}} L \int \frac{\xi d\xi}{\sqrt{\xi^2 C_0 - \xi^4 - 1}} d\xi$$

$$u = \xi^2$$
$$= \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1^2}^{\xi_2^2} \frac{\frac{1}{2} du}{\sqrt{-u^2 + u C_0 - 1}} = \frac{\sqrt{2m} L}{\sqrt{U_0}} \frac{1}{2} \pi$$

$\nearrow$   
v kojih jima  
je kvadrat  
koren  
 $\approx u^2 - 1$

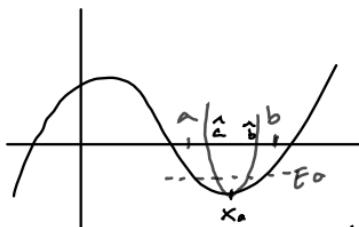
$$U_0 = \sqrt{|\alpha||\beta|}$$

$$\frac{L}{\sqrt{U_0}} = \frac{|\beta|^{\frac{1}{4}}}{|\alpha|^{\frac{1}{4}} |\alpha^{\frac{1}{2}}| |\beta|^{\frac{1}{2}}} = 2 \sqrt{\frac{1}{|\alpha|}}$$

$$T = \sqrt{\frac{m}{2|\alpha|}} \pi \quad \text{zpet je izobranjeno  
(neodvisno od energije)}$$

in natrii  $n = 6$

## Harmonična optimizacija periode



$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

$\Rightarrow$  her je lok. minimum.

$$U(x) = U(x_0) + \frac{du}{dx}(x_0)(x - x_0) + \frac{1}{2} \frac{du}{dx}(x_0)(x - x_0)^2$$

$$T \approx \sqrt{2m} \int_a^b \frac{dx}{\sqrt{\sqrt{E_0 - U_0 - \frac{1}{2} (U''(x_0)) (x - x_0)^2}}} =$$

$$= \sqrt{2m} \cdot \frac{1}{\sqrt{\frac{1}{2} U''(x_0)}} T = 2\pi \sqrt{\frac{m}{U''(x_0)}}$$

$$E_0 = U(x) + \frac{1}{2} m g c x \dot{x}^2$$

$$U(x) = \int_{x_0}^x \sqrt{g(x)} dx \quad \dot{x} = \ddot{x} \sqrt{g(x)} \\ \dot{x}^2 = \dot{x}^2 g(x)$$

$$\frac{1}{2} m \dot{x}^2 + U(x) = E_0$$

$$\hat{U}(u) = U(x(u))$$

$$U(x) = \hat{U}(u) \quad \stackrel{\text{"V(x)"}}{=} \\ U'(x) = \frac{d\hat{U}}{du} \frac{du}{dx} \quad \left. \frac{du}{dx} \right|_{u_0} = 0$$

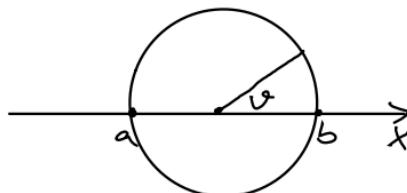
$$U''(x_0) = \frac{d^2 \hat{U}}{du^2} \left( \frac{du}{dx} \right)^2 + C = \frac{d^2 \hat{U}}{du^2}(u_0) \cdot g(x_0)$$

$\Rightarrow$  harmonische Approximation

$$T = 2\pi \sqrt{\dots}$$

# Libnocijska aproksimacija periode

$$t - t_0 = \operatorname{sgn}(\dot{x}) \int_{x_0}^x \frac{dx}{\sqrt{(E_0 - U(x)) \cdot \frac{2}{m}}}$$



$$x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$$

$$E_0 - U(x) = (b-x)(x-a) \chi(x)$$

↗ verneka me nizle

$$b-x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$

$$= \operatorname{sgn}\dot{x} \int_{V_0}^V \frac{\frac{1}{2}(b-a)\sin\vartheta d\vartheta}{\sqrt{\frac{1}{2}(b-a)^2 \sin^2 \chi(v)}}$$

$$\leq -\operatorname{sgn}(\dot{x}) \int_{V_0}^V \frac{\sin\vartheta d\vartheta}{|\sin\vartheta| \sqrt{\chi(v)}}$$

$$\forall \vartheta \in (0, \pi) \Rightarrow \dot{x} < 0, \sin\vartheta > 0$$

$$= \int_{V_0}^V \frac{d\vartheta}{\sqrt{\chi(v)}}$$

$$\forall \vartheta \in (\pi, 2\pi) \Rightarrow \dot{x} > 0, \sin\vartheta < 0$$

$$= \int_{V_0}^V \frac{d\vartheta}{\sqrt{\chi(v)}}$$

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{V_0}^V \frac{d\vartheta}{\sqrt{\chi(v)}} \Rightarrow T = 2\sqrt{\frac{m}{2}} \int_0^\pi \frac{d\vartheta}{\sqrt{\chi(v)}} \stackrel{\text{trapezna formula}}{=} \quad \text{trapezna formula}$$

$$= 2\sqrt{\frac{m}{2}} \frac{1}{2}\pi \left( \frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right) = \pi \sqrt{\frac{m}{2}} \left( \frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right)$$

$$U(a) = E_0$$

$$U(b) = E_0$$

$$\chi(x) = \frac{e_0 - U(x)}{(b-x)(x-a)}$$

$$\chi(a) = \lim_{x \rightarrow a} \chi(x) = -\frac{U'(a)}{b-a}$$

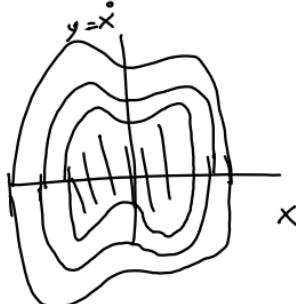
$$\chi(b) = \frac{U'(b)}{b-a}$$



$$T = \pi \sqrt{\frac{m}{2}} \sqrt{b-a} \left( \frac{1}{\sqrt{-U'(a)}} + \frac{1}{\sqrt{U'(b)}} \right) \approx 2\pi \sqrt{\frac{m}{U'(x_0)}}$$

$$U'(a) = U'(x_0) + U''(x_0)(b-a)$$

Fazni portret



$$\frac{1}{2} m \dot{x}^2 + U(x) = E_0 \rightarrow$$

$$\frac{1}{2} m \dot{y}^2 + U(x) = E_0$$

nivojnice  
u form. ravnni

$$y = \pm \sqrt{\frac{2}{m} (E_0 - U(x))}$$

plasōde

$$A(E) = 2 \int_{a(E)}^{b(E)} \sqrt{\frac{2}{m}(E_0 - U(x))}$$

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m}} \cdot \frac{1}{2} \frac{1 - (-U)'(x)}{\sqrt{E_0 - U(x)}}$$

$$\frac{dt}{dE} = 2 b'(E) \cdot 0 - 2 a'(E) \cdot 0 + 2 \sqrt{\frac{2}{m}} \int_a^b \frac{1}{2} \frac{dx}{\sqrt{E - U(x)}} = \frac{1}{m} T$$

$$\Rightarrow T = m \frac{da}{dt}$$

Primer: Harmonic oscillator

$$U = \frac{1}{2} kx^2 \quad (\text{hooker potential})$$

$$\frac{1}{2} my^2 + \frac{1}{2} kx^2 = E_0$$

$$\left( \frac{y^2}{\sqrt{\frac{2E_0}{m}}} \right)^2 + \frac{x^2}{\frac{\sqrt{2E_0}}{k}} = 1$$

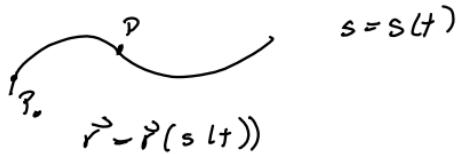
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3 plasenne

plasenne  
↓

$$A = \pi \cdot \frac{2E_0}{\sqrt{km}} \Rightarrow \frac{dA}{dE_0} = \frac{2\pi}{\sqrt{km}} \Rightarrow T = 2\pi \sqrt{\frac{m}{k}}$$

izobrazidna  
periode

# Gibanje po krivulji:



$$\vec{v} = \frac{d\vec{s}}{dt} = \left( \frac{ds}{dt} \right) \vec{s} = \ddot{s} \vec{e}_t, \quad |\vec{v}| = |\ddot{s}|$$

$\vec{e}_r$

$$\vec{\alpha} = \frac{d}{dt} (\dot{s} \vec{e}_t) = \ddot{s} \vec{e}_t + \dot{s} \frac{d\vec{e}_t}{ds} = \ddot{s}$$

$$\frac{d\vec{e}_t}{ds} = \gamma \vec{e}_n$$

$$\frac{d\vec{e}_t}{ds} \perp \vec{e}_t$$

$$\frac{d\vec{e}_t}{ds} = \frac{d\vec{e}_t}{ds} \Big| \frac{d\vec{e}_t}{ds} \Big| \quad \gamma = \frac{1}{R} \quad R \dots \text{krivuljens radius}$$

normalni vektor

$$|\vec{v} \times \vec{\alpha}| = \gamma |\vec{s}|^3 \Rightarrow \gamma = \frac{|\vec{v} \times \vec{\alpha}|}{|\vec{s}|^3}$$

$$\vec{\alpha} = \ddot{s} \vec{e}_t + \gamma s^2 \vec{e}_n \quad \text{tangenti + normalni pravac}$$

$$\vec{e}_b = \vec{e}_x \times \vec{e}_m \quad \dots \text{binormal}$$

$$m\vec{a} = F \quad m\vec{a} = \vec{F} + \vec{S} \quad \begin{matrix} \nearrow \\ \text{resultante vec} \end{matrix}$$

resultante  
aktiv in sil

$$m\vec{s}' = \vec{F}\vec{e}_x + \vec{S}\vec{e}_y$$

$$m\vec{s}^2 = \vec{F}\vec{e}_n + \vec{S}\vec{e}_n \quad \begin{matrix} + \text{ konstruiamo no} \\ \text{zvaca sila veci} \end{matrix}$$

$$0 = \vec{F}\vec{e}_n + \vec{S}\vec{e}_n$$

Gibanje pokrevalj "brez trenja":  $\vec{S} \cdot \vec{e}_x = 0$

Idealna krovija, gladka krovija

$$\vec{S} \cdot \vec{e}_x = -\frac{\vec{v}}{m} k \sqrt{|\vec{S} \cdot \vec{e}_n|^2 + |\vec{S} + \vec{F}_S|^2} \cdot \vec{e}_x$$

Gibanje po gladkoj kružnici

$$\vec{s} - \vec{e}_r = 0$$

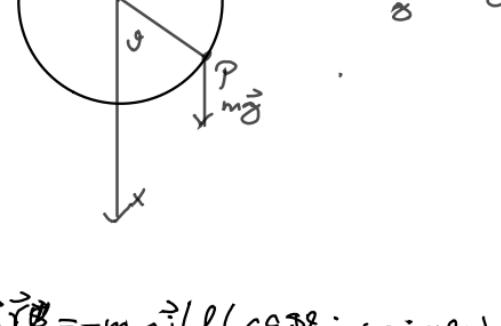
$$m\ddot{\vec{s}} = \vec{F} \cdot \vec{e}_x \quad F = -\text{grad } U$$

$$\frac{dU}{ds} = \text{grad } U \quad \frac{d\vec{s}}{ds} = -\vec{e}_r$$

$$m\ddot{\vec{s}} = \vec{F} \cdot \vec{e}_x \cdot \dot{s} \rightarrow \frac{1}{m} \ddot{s}^2 + U(s) = E_0$$

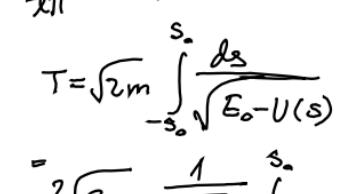
gibanje po gladkoj kružnici je integrabilno

Primer:  
Matematičko nihalo = gibanje po  
gladkoj kružnici pod uticajem sile teže



$$U = -mg\vec{r} \cdot \vec{e}_z = -mg(l(\cos\theta; + \sin\theta)) = -mg\cos\theta$$

$$s = l\vartheta \quad U(s) = mg\cos\left(\frac{s}{l}\right)$$



Rejade matematičke  
nihalje

$$T = \sqrt{2m} \int_{-s_0}^{s_0} \frac{ds}{\sqrt{E_0 - U(s)}} = E_0 = -mg l \cos\left(\frac{s_0}{l}\right)$$

$$= \sqrt{2m} \frac{1}{\sqrt{mg}} \int_0^{s_0} \frac{ds}{\sqrt{1 - \cos\left(\frac{s}{l}\right) - \cos\left(\frac{s_0}{l}\right)}} \quad s = l\vartheta$$

$$= \sqrt{\frac{2\sqrt{2}}{g}} \int_0^{\frac{\pi}{2}} \frac{l d\vartheta}{\sqrt{\cos\vartheta - \cos\frac{s_0}{l}}} = \sqrt{\frac{2}{g}} \int_0^{\frac{\pi}{2}} \frac{d\vartheta}{\sqrt{\cos\vartheta - \cos\frac{s_0}{l}}}$$

$$1 = \cos^2 \frac{\vartheta}{2} + \sin^2 \frac{\vartheta}{2}$$

$$\cos\vartheta = \cos \frac{\vartheta}{2} - \sin \frac{\vartheta}{2}$$

$$1 - \cos\vartheta = \frac{2 \sin^2 \frac{\vartheta}{2}}{2}$$

$$= \sqrt{\frac{2}{g}} \sqrt{\frac{l}{2}} \int_0^{\frac{\pi}{2}} \frac{d\vartheta}{\sqrt{2 \sin^2 \frac{\vartheta}{2} - 2 \sin^2 \frac{s_0}{l}}} =$$

$$u = \frac{\sin \frac{\vartheta}{2}}{\sin \frac{s_0}{l}}$$

$$\sin^2 \frac{\vartheta}{2} = \sin^2 \frac{s_0}{l} u^2$$

$$du = \frac{1}{\sin^2 \frac{\vartheta}{2}} \cos \frac{\vartheta}{2} \cdot \frac{1}{2} d\vartheta$$

$$= \sqrt{\frac{2}{g}} \int_0^1 \frac{2 du \cdot \sin \frac{s_0}{l}}{\cos^2 \frac{\vartheta}{2} \sqrt{\sin^2 \frac{s_0}{l}(1-u^2)}} =$$

$$= \sqrt{\frac{2}{g}} \int_0^1 \frac{du}{\sqrt{1 - \frac{\sin^2 \frac{s_0}{l}}{2} u^2 \sqrt{1-u^2}}} =$$

$$\sqrt{\frac{2}{g}} \int_a^1 \frac{du}{\sqrt{1-k^2 u^2} \sqrt{1-u^2}} \quad \text{to je popolni eliptični integral}$$

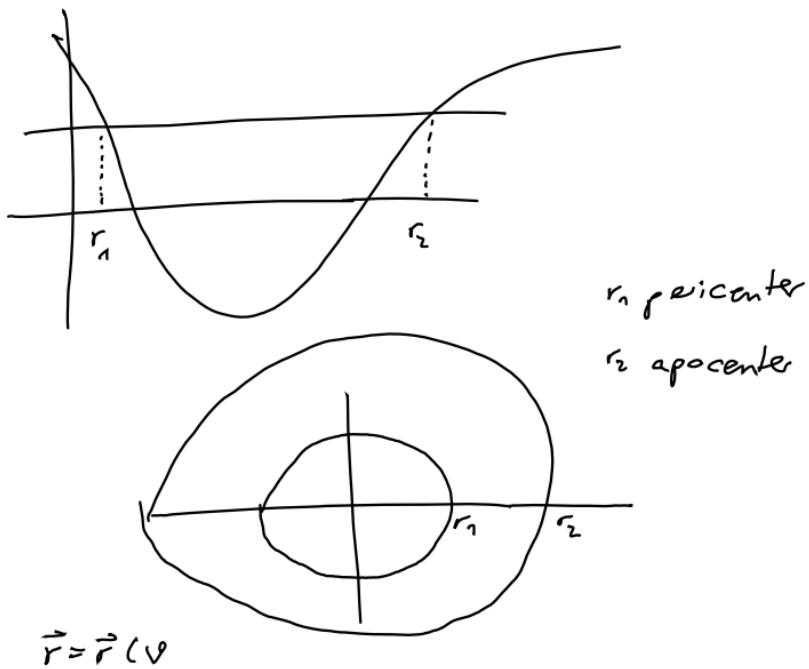
$$= K(k^2) \quad \text{pone vrake}$$

$$T = 4\sqrt{\frac{2}{g}} K(k^2)$$

5.10

✓

ene wa ← ene wa · possegnē



$$\vec{r} = \vec{r}(\varphi)$$

$$\frac{\vec{r}}{dt} = \dot{r} \hat{e}_r + r \dot{\varphi} \hat{e}_\varphi$$

Terej se letíku může být  
konečný

$\dots$   $a$

tři je směřován

Megeő Filipovszkij 2??

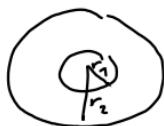
$$\frac{0.9}{\pi} \in \mathbb{Q} \Leftrightarrow \text{tir je zrati}$$

periodično  
gibanje  
koordinate  $r$

12.11.25

Trditev: Nezrati: tir "omejenega" gibanja je geste množica nad absolutnim razdaljim

Dokaz:



Dovolj je dokazati, da so presečine tira s vsako krožnico med absidnima ~~radijema~~<sup>radijama</sup> geste množice te krožnice.

Betravljavo izrek:

Edine potenciatele za katera so vsi tiri v okolic krožnega tira zrati; sta geometriški potencial in Hookov potencial

$$\hookrightarrow U = \frac{1}{2} kr^2$$

??  
Neizven  
reni je po  
besede

# Relativno gibanje

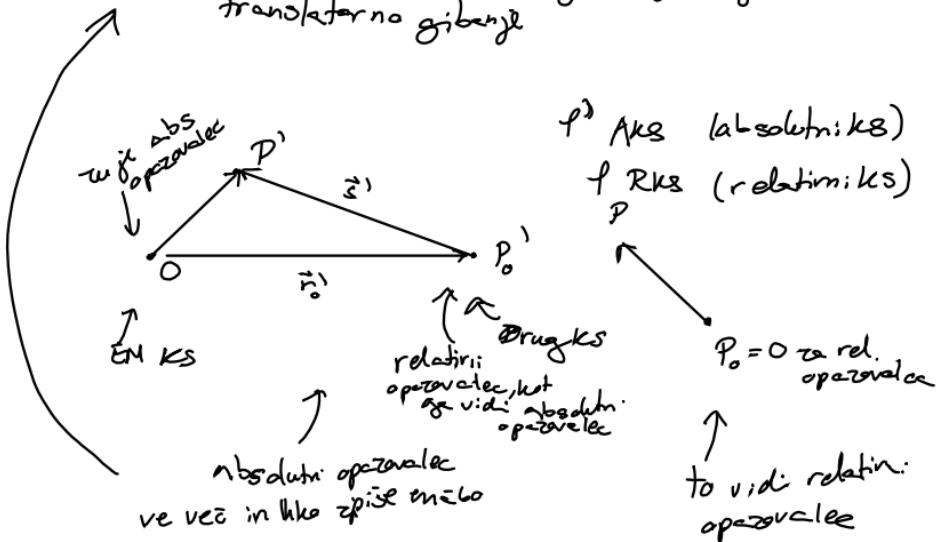
Koordinatni sistem  $\varphi(t, p)$  se giba je glede na koordinatni sistem  $\varphi'(t', p')$ , če je  $\exists$  trajice

$(P_0 \in E, P_0 : \mathbb{R} \rightarrow E, U : \mathbb{R} \rightarrow SO(3))$  take da velja

$t = t'$   $\leftarrow$  pri analizir. je bil  $Q$  konstanten

$$P' = P_0(t) + Q(t)(P - P_0)$$

↑  
translatorno gibanje  $\nwarrow$  rotacijsko gibanje



Trditev: Rotacijšči del gibanja je neodvisen od trajice

Dokaz:  $\tilde{P}_o, \tilde{P}_o^1, \tilde{Q}$

$$\tilde{P}_o + Q(P - \tilde{P}_o) = \tilde{P}_o + \tilde{Q}(P - \tilde{P}_o) \approx AP$$

$$P = P_o \Rightarrow \tilde{P}_o = \tilde{P}_o + Q(P_o - \tilde{P}_o)$$

$$\tilde{P}_o + \tilde{Q}(P - \tilde{P}_o) + Q(P - \tilde{P}_o) = \tilde{P}_o + \tilde{Q}(P - \tilde{P}_o)$$

$$\tilde{P}_o + \tilde{Q}(P_o - \tilde{P}_o) + Q(P - \tilde{P}_o) = \tilde{P}_o + Q(P - \tilde{P}_o) \Rightarrow$$

$$Q(P - \tilde{P}_o) = \tilde{Q}(P - \tilde{P}_o) - \tilde{Q}(P_o - \tilde{P}_o) =$$

$$\tilde{Q}((P - \tilde{P}_o) - (P_o - \tilde{P}_o)) = \tilde{Q}(P - P_o)$$

$$\Rightarrow Q = \tilde{Q}$$

$$P = P(t) \Rightarrow$$

$$\dot{P} = \dot{P}_o + \dot{Q}(P - P_o) + Q(\dot{P})$$

$$\begin{array}{c} \uparrow \\ \dot{V}_{rel} = \dot{P} \text{ relative hitrost} \\ \hline V_o \text{ translatorske hitrost} \end{array} \quad \text{rotacijska hitrost}$$

$$Q^T Q = I = Q Q^T$$

$$(Q^T)^* Q + Q^T \dot{Q} = 0$$

$$\dot{Q}^T Q + Q^T \dot{Q} = 0$$

$$W = Q^T \cdot \dot{Q}$$

$$W^T = \dot{Q}^T Q$$

$$W^T = -W \Rightarrow W \text{ je parnno simetrični tenzor}$$

Linearna preslikave,  $=$  Tensor (drugega reda)

Trotter  $W = Q^T \vec{Q}$  je postaveno simetrický tensor  
 Trotter: Nejbo  $W$  postaveno simetrický tensor (p.s.t)  
 definovan na tranzitivním vektorském prostoru.  
 Potom dostaje vektor  $w$  je  $W\vec{w} = \vec{w} \times \vec{a}$   
 ze  $\vec{a} \neq \vec{0}$

Dokaz:

$$W\vec{p} = \lambda \vec{p}$$

$$\vec{p} \cdot W\vec{p} \Rightarrow |\vec{p}|^2$$

$$W^T \vec{p} \cdot \vec{p}$$

"

$$-W\vec{p} \cdot \vec{p} =$$

"

$$-\lambda|\vec{p}|^2 - |\vec{p}|^2 \Rightarrow \lambda = 0$$

$\{\vec{p}, \vec{g}, \vec{r}\}$  ONB

$$W\vec{p} = \vec{0}$$

$$W\vec{g} = (W\vec{g} \cdot \vec{p})\vec{p} + (\underbrace{W\vec{g} \cdot \vec{g}}_{\vec{0}})\vec{g} + (W\vec{g} \cdot \vec{r})\vec{r}$$

$$\underbrace{\vec{g} \cdot \vec{g}}_{\vec{0}} \quad \underbrace{\vec{g} \cdot \vec{r}}_0$$

$$W\vec{r} = W((\vec{a} \cdot \vec{p})\vec{p} + (\vec{a} \cdot \vec{g})\vec{g} + (\vec{a} \cdot \vec{r})\vec{r}) = \underbrace{(\vec{a} \cdot \vec{p})\vec{p}}_{\cancel{W\vec{p} + W\vec{g}}} + \underbrace{(\vec{a} \cdot \vec{g})\vec{g}}_{\cancel{W\vec{p} + W\vec{g}}}$$

$$\vec{p} \times \vec{g} = \vec{r} \quad \text{in } \vec{p} \times \vec{p} = \vec{0} \text{ kde ONB}$$

$$= (W\vec{g} \cdot \vec{r}) [\vec{g} \cdot (\vec{p} \times \vec{g}) - (\vec{g} \cdot \vec{r})(\vec{r} \times \vec{p})] =$$

$$= (W\vec{g} \cdot \vec{r}) \vec{p} \times (\vec{a} \cdot \vec{g})\vec{g} + (\vec{a} \cdot \vec{r})\vec{r} = (W\vec{g} \cdot \vec{r})(\vec{p} \times \vec{a})$$

$$\underbrace{(\vec{a} \cdot \vec{p})\vec{p}}_{\vec{a}} = \vec{0}$$

$$\vec{a} = (W\vec{g} \cdot \vec{r})$$

Vektorja  $\vec{\omega}$  pravimo osni vektor p.s.t W

$$\vec{\omega} = \vec{\omega}(W)$$

$$e_{ijk} = \begin{cases} 1 & \text{je jedsoda permutacija štev: } 123 \\ -1 & \text{ije je like} \\ 0 & \text{če n: permutacija} \end{cases}$$

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^3 e_{ijk} a_j b_k \vec{e}_i = e_{ijk} a_j b_k e_i$$

$\uparrow$  pogost zapis brez vsake

sumacijeske konjunkcije  
če se indeks ponovi, se sumira

$$\vec{a} = \sum a_i \vec{e}_i = a_i \vec{e}_i$$

$$\vec{a} \cdot \vec{b} = a_i b_i;$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\begin{aligned} \delta_{ii} &= \delta_{11} + \delta_{22} + \delta_{33} = 3 \\ \underline{\delta_{ii}} &= 1 \end{aligned}$$

$$W_{ij} = e_{jik} w_k \quad \begin{matrix} \leftarrow \text{seštevanje} \\ \text{sam po k} \end{matrix}$$

$$W \vec{a} = \underbrace{W_{ij} a_j}_{\vec{a}} e_i$$

$$w \times \vec{a} = \underbrace{e_{ijk} w_j}_{\vec{a}} \underbrace{a_k e_i}_{\vec{a}}$$

$$W_{ij} a_j = e_{jik} w_k a_k = \vec{a}$$

$$W_{ij} a_j = e_{ikj} w_k a_j \quad \begin{matrix} \leftarrow \text{j in k menjajo} \end{matrix}$$

$$\Rightarrow W_{ij} = e_{ikj} w_k = -e_{jik} w_k$$

$$e_{ijk} e_{mnl} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \quad \text{velja (dette ne so)}$$

$$e_{ijk} e_{mnl} = \delta_{im} \delta_{jn} - \delta_{ij} \delta_{lm} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$$

$$e_{ijk} e_{ijk} = 6$$

$$e_{ijp} w_{;j} = -e_{ijp} e_{jkl} w_k = -2\delta_{pk} w_k = -2w_p$$

$$w_p = -\frac{1}{2} e_{ijp} w_{ij}$$

Trägheits  $A \in \mathcal{L}(U, V)$

$$A^*(\vec{a} \times \vec{b}) = (A\vec{a}) \times (A\vec{b})$$

$$\text{A* adjungierte A-Jet} \quad A^{-1} = \frac{1}{\det A} (A^*)^T$$

Daher:  $\det A \neq 0 \quad \vec{a}, \vec{b}, \vec{c}$  lin. unabh.

$$\det A = \frac{[\vec{a}, \vec{b}, \vec{c}]}{[\vec{a}, \vec{b}, \vec{c}]}$$

$$(A\vec{a} \times A\vec{b})\vec{c} = (A\vec{a} \times A\vec{b})(AA^{-1}\vec{c}) = [\vec{a}, \vec{b}, AA^{-1}\vec{c}] =$$

$$= \det A [\vec{a}, \vec{b}, A^{-1}\vec{c}] = \det A (\vec{a} \times \vec{b}) A^{-1}\vec{c} = \\ = (\det A) A^{-T} (\vec{a} \times \vec{b}) \vec{c} \Rightarrow A\vec{a} \times A\vec{b} = \det(A) A^{-T} (\vec{a} \times \vec{b})$$

$$A \cancel{\rightarrow} A \quad A\vec{a} \times A\vec{b} = A^*(\vec{a} \times \vec{b})$$

$$\det A = 0 \Rightarrow \forall \delta > 0. \exists A_\delta. \det A_\delta = 0 . |A - A_\delta| < \delta$$

$$\lim_{\delta \rightarrow 0} A_\delta \vec{a} \times A_\delta \vec{b} = A^*(\vec{a} \times \vec{b})$$

$$A_\delta \rightarrow A \quad A_\delta^* \xrightarrow{?} A^*$$

$$\text{aus } \vec{a} \text{ und } \vec{b} \text{ linear abhängig} \\ [\vec{a}, \vec{b}, \vec{c}] = \det A [\vec{a}, \vec{b}, A\vec{c}] \text{ beliebig}$$

$$A = Q \in O(3)$$

$$Q^* = \det(Q) (Q^{-1})^T = \det Q Q$$

||  
 $(Q^T)^T = Q$

$$Q \in SO(3) \Rightarrow Q(\vec{a} \times \vec{b}) = (Q\vec{a}) \times (Q\vec{b})$$

$$W = Q^T \overset{\circ}{Q}$$

osnemu vektorju p.s.t pravimo  
rotacijski vektor

Vektorju  $\vec{\omega}^1 = Q \vec{u}$  pa pravimo vektor katne hitrosti  
rotacije  $Q$

$$Q^T \overset{\circ}{Q} \vec{a} = \vec{\omega} \times \vec{a}$$

$$\overset{\circ}{Q} \vec{a} = Q(\vec{\omega} \times \vec{a}) = \vec{\omega}^1 \times Q \vec{a}$$

Traktor

Osim vektora p.s.t  $\vec{W} = \dot{\vec{Q}}\vec{Q}^T$  je vektor korene hitrosti rotacije  $\vec{Q}$

$$\text{Dakle: } \dot{\vec{Q}}\vec{Q}^T\vec{a} = \vec{\omega} \times \vec{a}$$

$$\dot{\vec{Q}}\vec{a} = \vec{\omega} \times \vec{Q}\vec{a}$$

$$\dot{\vec{Q}}\vec{Q}^T\vec{a} = \vec{\omega} \times \vec{Q}(\vec{Q}^T\vec{a}) = \vec{\omega} \times \vec{a}$$

$$\left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \hat{\vec{y}} = \vec{\omega}$$

$Q \in SO(3)$

$W_1 W_2$  n: posebno simetrični tensor

$$[W_1, W_2] = W_1 W_2 - W_2 W_1 \text{ je posebno sim. tenz}$$

parametar

$$(W, +, [ ]) \text{ je algebra}$$

množica p.s.t

$$(V, +, \times) \text{ je tudi algebra}$$

$\uparrow$

je tudi algebra  
(L: jeva algebra)

prostir 3D vektora jev

T r dle:  $\omega([w_1, w_2]) = \omega(w_1) \times \omega(w_2)$   
 $\omega \mapsto W(\omega)$  je homomorfizam algebre

$$\begin{aligned} \text{Dokaz } [w_1, w_2] \vec{\alpha} &= (w_1 w_2 - w_2 w_1) \vec{\alpha} = \\ &= \vec{w}_1 \times (\vec{w}_2 \times \vec{\alpha}) - \vec{w}_2 \times (\vec{w}_1 \times \vec{\alpha}) = \\ &= (\vec{w}_1 \vec{\alpha}) \vec{w}_2 - (\cancel{(w_1 w_2)} \vec{\alpha}) - ((\vec{w}_2 \vec{\alpha}) w_1 - (\cancel{(w_2 w_1)} \vec{\alpha})) = \\ &= (w_1 \times w_2) \times \vec{\alpha} \end{aligned}$$

Izrek: Najboj W p.s.t z enoskim osnim vektanjem  $\hat{e}$ .  
Potem je  $e^{vW}$  rotacija okrog osi  $\hat{e}$  za kot  $v$

Izrek: KS  $\rho(t, P)$  in  $\varphi(t, P')$  sta vistem razredu KS

$$\Leftrightarrow P' = P_0 + Q(P - P_0) \text{ presek vseh}$$

premočrtna gibanje s konstantno brzino

v KS  $\rho$  z premočrtna s konstantno brzino

v KS  $\varphi'$

Dokaz

( $\Rightarrow$ ) očitno

$$(\Leftarrow) P = P_0 + \vec{v} + \mapsto P' = P'(t)$$

$$\vec{\alpha} = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{\hat{\omega}} + \vec{\omega} \times (\vec{\omega} \times \vec{\hat{\omega}}) + 2 \vec{\omega} \times \vec{v}_{re}$$

$$\vec{\alpha} = \vec{a}_0 + t \vec{\omega} \times \vec{v} + t \vec{\omega} \times (\vec{\omega} \times \vec{v}) + 2 \vec{\omega} \times \vec{v} \quad / \cdot \vec{v}$$

$$\vec{\alpha} = \vec{a}_0 \vec{v} + t (\vec{\omega} \times (\vec{\omega} \times \vec{v})) \vec{v}$$

$$\vec{\alpha} = \vec{a}_0 \vec{v} + (\vec{v} \times \vec{\omega})(\vec{\omega} \times \vec{v}) = \vec{a}_0 \vec{v} + \|\vec{v} \times \vec{\omega}\|_2$$

$$\vec{v} = \lambda \vec{v}_0$$

$$0 = \lambda \vec{a}_0 \vec{v}_0 + \lambda^2 \|\vec{v}_0 \times \vec{\omega}\|_2 \quad \forall \lambda$$

$$\Rightarrow \vec{a}_0 \vec{v}_0 = 0 \quad \wedge \quad \vec{v}_0 \times \vec{\omega} = 0 \quad \forall \vec{v}_0$$

$$\vec{a}_0 = 0 \quad \Rightarrow \vec{\omega} = 0$$



$\Leftrightarrow$   
Q je konstanten

$$\Rightarrow P_0(t) = P'_1 + \vec{c}t$$

Poštedica (Prvi Newtonov zakon)

KS je IKS  $\Leftrightarrow$  A prosta materijalna točka se giblje prema orbiti s konstantnom brzinom

Dokaz

( $\Rightarrow$ ) ✓

Prosta materijalna točka - izolirana od vpliva ostalih

( $\Leftarrow$ ) ✓

Lahko bi nemesto prosta rešetka masno težišće  
(prosta točka mora obstojati)

RK8

Newton'sche Gesetze v. relativem Koordinatensysteme

$$\nabla \text{IKS} \quad m\ddot{\vec{r}} = \vec{F}$$

inertialen  
rotierende Sile

centrifugale Sile

$$m\ddot{\vec{r}}_{re} = \vec{F} - m\ddot{\vec{r}}_0 - m\vec{\omega}^2 \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) -$$

inertiale  
Sile

fiktive Sile

$$-2m\vec{\omega} \times \vec{v}_{re} = \vec{F}_{rel}$$

Coriolosova  
Sile

$$m\ddot{\vec{r}}_{re} = \vec{F}_{re}$$

$$\vec{F} = Q^T \vec{F} \quad F_{re} = Q^T F_{rel}$$

Trditev:  $\vec{F}$  potencialna  $\Rightarrow$   $F$  potencialna

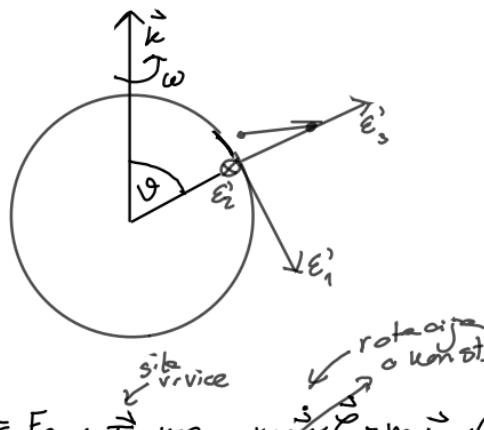
Dokaz:  $\vec{F}' = -\text{grad}_p U' = \left(\frac{\partial U}{\partial p}\right)^T = -\left(\frac{\partial U}{\partial p} \frac{\partial p}{\partial p'}\right)^T =$

$$U(p) = U'(p') = U(p_0) + Q(p - p_0)$$

$$p = Q^T(p' - p_0) + p_0$$

$$\Rightarrow = -Q \left(\frac{\partial U}{\partial p}\right)^T = Q \vec{F} \Rightarrow \vec{F} = -\left(\frac{\partial U}{\partial p}\right)^T = -\text{grad}_p U$$

## Foucaultův nihalo



$$m \ddot{\alpha}_r = F_g + \vec{T} - m \ddot{\alpha}_o - m \vec{\omega} \times \vec{\epsilon} - m \vec{\omega} \times (\vec{\omega} \times \vec{\epsilon}) - 2m \vec{\omega} \times \vec{v}_{rel}$$

→ *síla gravitace*      → *rotacej o konst.*  
 ↓                            ↓

$$\omega^2 = \omega k^2$$

$$\omega = \frac{2\pi}{24 \cdot 3600} \text{ rad/s} < 10^{-4}$$

$\approx 10^{-8}$

$$\ddot{\alpha}_o = \vec{\omega} \times (\vec{\omega} \times \vec{\epsilon})$$

$$\ddot{\epsilon} = \ddot{\alpha}_o + \vec{\omega} \times \vec{\epsilon} + \vec{\omega} \times (\vec{\omega} \times \vec{\epsilon}) + 2 \vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

↑      "      "      "      "      "      "      "      "  
 0      0      0      0      0      0      0      0

přesného kruhu

zde neplatí kvantitativní zákon

$$F_g - m \ddot{\alpha}_o = m \ddot{\epsilon}$$

$$\ddot{\epsilon} = T \cdot \frac{\ddot{T}}{T} = \frac{\vec{\epsilon}_1}{l} \ell_1 + \frac{\ell - l}{l} \vec{\epsilon}_3 - \frac{\vec{\epsilon}_2}{l} \vec{\epsilon}_2$$

1

jd neli

1

## Sistem materialnih točaka

$P_1, \dots, P_n$

$$m_i \ddot{P}_i = \vec{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^N \vec{F}_{ji} \quad \vec{F}_{ji} = -\vec{F}_{ij}$$

$$\sum_{i=1}^n m_i \ddot{P}_i = \sum_{i=1}^n \vec{F}_i + \sum_{i \neq j} \vec{F}_{ji}$$

$$\ddot{P}_x = 0 + \sum_{i=1}^n m_i (\ddot{P}_i - 0)$$

$$m = \sum_{i=1}^n m_i$$

$$\overset{\circ}{P}_x = \frac{1}{m} \sum_{i=1}^n m_i \overset{\circ}{P}_i \quad \ddot{P}_x = \frac{1}{m} \sum_{i=1}^n m_i \ddot{P}_i$$

$$m \ddot{P}_x = \vec{F} - \sum_{i=1}^n m_i \ddot{P}_i \quad \begin{matrix} \text{Enačba gibanje} \\ \text{masnega sredistva} \end{matrix}$$

