Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x\,dx$$

$FV(f)$ konvergira k f, če je f zvezna v x,
če pa ni, pa konvergira k $\frac{f(x^-) + f(x^+)}{2}$

$$\int x \cos(nx)\,dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int x \sin(nx)\,dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

f soda $\Rightarrow b_n = 0$

f liha $\Rightarrow a_n = 0$

$$\int x^2 \cos(nx)\,dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx)\,dx = \frac{-x^2}{n} \cos x + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

(2)

$f(x) = |x|$ razvij v Fourierovo vrsto in seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{1}{\pi} \int_0^{\pi} x \, dx = \frac{1}{2\pi} \pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) \, dx \underset{\text{sodost}}{=} \frac{2}{\pi} \int_{-\pi}^{\pi} x \cos(nx) \, dx =$$

$$= \frac{2}{\pi} \left(\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right) \Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi} \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) \overset{\text{lihost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{(-1)^n - 1}{n^2} \right) \cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2} \cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi} \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{1} \frac{1}{(2k+1)^2} = -\frac{\pi}{2} \cdot \frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad =? \quad s + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

$s \dots$ lih.

$s' \dots$ ostalo

$$= s + \frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) =$$

$$= s + \frac{1}{4} \Sigma = \Sigma$$

$$s = \frac{3}{4} \Sigma$$

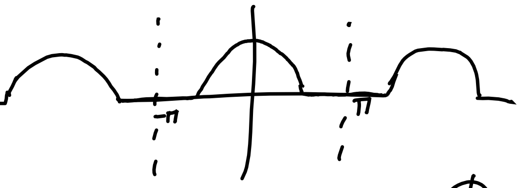
$$\Sigma = \frac{4}{3} s = \frac{\pi^2}{6}$$

③

$f(x) = \max(\cos x, 0)$

$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$

$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$



$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(x) = \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

ničelnost $f(x)$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos(nx) \cancel{+ 2\int_{\frac{\pi}{2}}^{\pi} \cos(nx)\,dx}$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right) =$ $n \neq 1$

$\frac{1}{\pi}\left[\frac{1}{n+1}\sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1}\sin\left((n-1)\frac{\pi}{2}\right)\right]$

$n = 4k: \frac{1}{\pi}\frac{1}{n+1}\cdot 1 + \frac{1}{n-1}(-1)\frac{1}{\pi} = \frac{-2}{n^2-1}\cdot\frac{1}{\pi} \quad \frac{-2}{16k^2-1}\cdot\frac{1}{\pi}$

n lih: 0

$n = 4k+2: \frac{1}{\pi}\frac{1}{4k+3}\sin\frac{3\pi}{2} + \frac{1}{\pi}\frac{1}{4k+1}\sin\frac{\pi}{2} =$

$= \frac{1}{\pi}\left(-\frac{1}{4k+3} + \frac{1}{4k+1}\right)$

$= \frac{1}{\pi}\begin{cases} 0 & ; n \text{ liho} \\ \frac{(-1)^{m+1}\cdot 2}{(2n)^2-1} & ; n = 2m \end{cases}$

$a_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{1}{\pi} + \frac{\sin(2x)}{2}\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}$

$b_n = 0$ ker f soda

$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\cdot 2}{(2m)^2-1}\cos(2mx)$

$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}S_1 \Rightarrow$

$S_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)\cdot\frac{\pi}{2}(-1) = \frac{1}{2} - \frac{\pi}{4}$

$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2-1}\cdot\underbrace{\cos(m\pi)}_{(-1)^m} =$

$= \frac{1}{\pi} - \frac{2}{\pi}\underbrace{\sum_{m=1}^{\infty} \frac{1}{4m^2-1}}_{S_2}$

$S_2 = -\frac{1}{\pi}\cdot\frac{\pi}{2} = -\frac{1}{2}$

④

$f(x) = x^2 \qquad f: [0,\pi] \longrightarrow \mathbb{R}$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v sinusno FV in skiciraj njen graf

1) Če f razširimo do sode funkcije

$f_s : [-\pi,\pi] \longrightarrow \mathbb{R}$

$x<0 \Rightarrow f_s(x) = f(-x)$

$FV_{\cos}(f)(x) = FV(f)(x)$

2) $f_l : [-\pi,\pi] \longrightarrow \mathbb{R} \qquad f_l(x) = -f(-x)$

$FV_{\sin}(f)(x) = FV(f_l)(x)$

1) $f_s(x) = x^2$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

Poglejmo:

$$\int x^2 e^{inx} =$$

$$\boxed{\int x e^{inx} = -\frac{i\cdot x\cdot e^{ix}}{n} + \frac{e^{inx}}{n^2}}$$

$u = x^2 \qquad dv = e^{inx}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{in} e^{inx}$

$$= x^2\cdot\frac{e^{inx}}{in} - \int 2x\,\frac{e^{inx}}{in}\,dx \overset{\text{od prej}}{=}$$

$$= x^2\cdot\frac{e^{inx}}{in} - \frac{2}{in}\left(\frac{ixe^{inx}}{n} + \frac{e^{inx}}{n^2}\right) + C$$

$$= \underbrace{e^{inx}}_{\cos nx + i\sin nx}\left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3}\right)$$

$$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$$

$$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$$

$$a_n = \frac{1}{\pi}\left(\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) + \frac{2}{n^3}\sin(nx)\right)\Big|_{-\pi}^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right) = \frac{4(-1)^n}{n^2}$$

$$FV\cos(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(x$$

b) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\text{soda}}{f_p(x)\sin(x)} = \frac{2}{\pi} \int_0^{\pi} \underbrace{f_p(x) \sin(nx)}_{= f(x) \text{ na } [0,\pi] = x^2}$

$= \frac{2}{\pi} \left[-\frac{x^2}{n} \cos(nx) + \frac{2}{n^2} \sin nx + \frac{2}{n^3} (\cos(nx)) \right] \Big|_0^{\pi}$

$= \frac{2}{\pi} \left(-\frac{\pi^2}{n} (-1)^n + \frac{2}{n^3} (-1)^n - \frac{2}{n^3} \right)$

$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left((-1)^{n+1} \frac{\pi^2}{2} + \frac{2(1-(-1)^n)}{n^3} \right) \sin(nx)$

$f(x) = x(\pi - x)$ [illegible]

⑥

$f(x) = \sin^3 x$ razvij v FV

Pred premislek:

$f(x) = \sin 2x$ je že FV

$b_2 = 1$, ostali so 0

$f(x) = \sin^2 x = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha+\beta)) =$

$= \frac{1}{2}(\cos(0) - \cos(2x)) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2}\sin x - \frac{1}{2}\cos 2x \sin x =$

$= \frac{1}{2}\sin x - \frac{1}{2}\cdot\frac{1}{2}(\sin(3x) - \sin(x)) =$

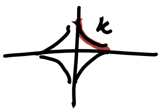
$= \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$

polinom

$\forall p(\sin x, \cos x)$ ima končno furierovo vrsto

$p \in \mathbb{R}[x,y]$

$a > 0$ Izračunaj težišče homogenega

loka astroide 

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

$$x_T = \frac{\int_K x\,dm}{m(K)} = \frac{\int_K x \rho\, ds}{\int_K \rho\, ds} \qquad \int_K \rho\, ds = l(K)$$

$\vec{r}(t)$ ... parametrizacija

$$ds = |\dot{\vec{r}}(t)|\,dt = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt$$

u ... skalarno polje

$$\int_K u\,ds = \int_a^b u(\vec{r}(t))\,|\dot{\vec{r}}(t)|\,dt$$

$x = a\cos^3 t$ $\qquad t \in [0, \frac{\pi}{2}] \leftarrow$ ker je vse pozitivno (gledamo pravilok)

$y = a\sin^3 t$

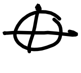
$$\dot{\vec{r}}(t) = (-3a\cos^2 t\cdot \sin t,\ 3a\sin^2 t\cos t)$$

$$|\dot{\vec{r}}| = 3a\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t} =$$
$$= 3a\cos t\sin t$$

$$l(K) = \int_0^{\frac{\pi}{2}} |\dot{\vec{r}}(t)|\,dt = \int_0^{\frac{\pi}{2}} 3a\cos t\sin t\,dt =$$

$u = \sin t\,dt$

$du = \cos t\,dt$



$$= 3a\int_0^1 u\,du = \frac{3}{2}a$$

$$\int_K x\,ds = \int_0^{\frac{\pi}{2}} a\cos^3 t\; 3a\cos t\sin t\,dt =$$

$\cos t = u \qquad du = -\sin t$

$$= 3a^2\int_0^1 u^4\,du = \frac{3}{5}a^2$$

$$y_T = \frac{3}{5}a^2$$

$a>0 \quad \alpha\in[0,2\pi]$

$K = S(0,a)\cap\Pi$

$\Pi: y = x\tan\alpha$

$I: \int_K (y-z)+(z-x)dy+(x-y)dz$

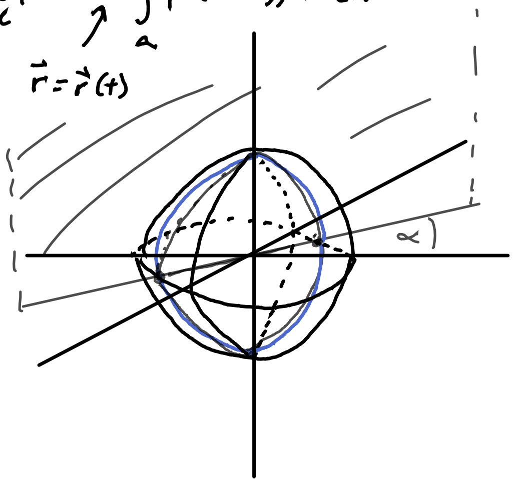
$\vec{f}(x,y,z) = (y-z,\ z-x,\ x-y)$ vektorsko polje

$I = \int_K \vec{f}\, d\vec{r}$

$\vec{r} = (x,y,z) \Rightarrow d\vec{r}$

$\int_K \vec{f}\, d\vec{r} = \int_a^b \vec{f}(\vec{r}(t))\, \dot{\vec{r}}(t)\, dt$

$\vec{r} = \vec{r}(t)$



Parametrizacija K

$x = a\cos\alpha\cdot\cos\vartheta$

$y = a\sin\alpha\cos\vartheta$

$z = a\sin\vartheta$

$r(\vartheta) = (a\cos\vartheta\cos\alpha,\ a\cos\vartheta\sin\alpha,\ a\sin\vartheta)$

$\vartheta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\dot{\vec{r}}(\vartheta) = (-a\sin\vartheta\cos\alpha,\ -a\sin\vartheta\sin\alpha,\ a\cos\vartheta)$

$\vec{f}(\vec{r}(\vartheta)) = a(\cos\vartheta\sin\alpha - \sin\vartheta,\ \sin\vartheta - \cos\vartheta\cos\alpha,$
$\cos\vartheta\sin\alpha - \cos\vartheta\cos\alpha)$

$\vec{f}\cdot\dot{\vec{r}} = a^2(-\cos\alpha\cos\cancel{\vartheta\sin\alpha\sin\vartheta} + \sin^2\vartheta\cos\alpha\ \cancel{\cos\alpha}$

$-\sin^2\vartheta\sin\alpha + \cos\vartheta\cancel{\cos\alpha\sin\vartheta}\sin\alpha +$

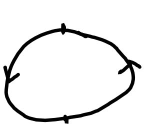
$+\cos^2\vartheta\sin\alpha - \cos^2\vartheta\cos\alpha) =$

$= a^2(\cos 2\vartheta\sin\alpha - \cos 2\vartheta\cos\alpha)$ ✗
Ne

$= a^2(\cos\alpha - \sin\alpha$

$\int_K \vec{f}\, d\vec{r} = \int_0^{2\pi} a^2(\cos\alpha - \sin\alpha)\, d\vartheta =$

$2\pi a^2(\cos\alpha - \sin\alpha)$

↑ pri orientaciji 

$a, b, c \in \mathbb{R}$

$$I = \int_S a\,dydz + b\,dxdz + c\,dxdy$$

$S = \{(x,y,z)\,;\ x^2+y^2+z^2=1\,,\ x,y,z \geq 0\}$

$$I = \int \vec{f}\,d\vec{S}$$

$\vec{f} = (a,b,c)$

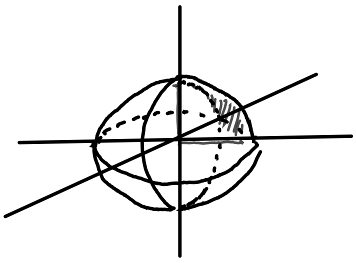
enotske normale

$\int_S (\vec{f}\cdot\vec{N})\,dS$ orientacija ploskve je usmerjena s smerjo normale

$$\int_S \vec{f}\,d\vec{S} = \int_D \vec{f}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dudv$$

$r = \vec{r}(u,v)$

smer se ujema s predpisano orientacijo



Sferične koordinate

$x = \cos\vartheta\cos\varphi \qquad \vartheta \in [0, \frac{\pi}{2}]$

$y = \cos\vartheta\sin\varphi \qquad \varphi \in [0, \frac{\pi}{2}]$

$z = \sin\vartheta$

$\vec{r}(\vartheta,\varphi) = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, \sin\vartheta)$

$\vec{r}_\vartheta = (-\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\vartheta)$

$\vec{r}_\varphi = (-\cos\vartheta\sin\varphi, \cos\vartheta\cos\varphi, 0)$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta\cos^2\varphi - \cos\vartheta\sin\vartheta\sin^2\varphi) =$$

$$= (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta) =$$

$$= -\cos\vartheta\,\underbrace{(\cos\vartheta\cos\varphi, -\cos\vartheta\sin\varphi, \sin\vartheta)}$$

$$\int_S \vec{f}\,d\vec{S} = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (-a\cos^2\vartheta\cos\varphi + b\cos^2\vartheta\sin\varphi - c\sin\vartheta\cos\vartheta)\,d\vartheta$$

$$= \int_0^{\frac{\pi}{2}} \left((-a\cos\varphi + b\sin\varphi)\cos^2\vartheta - \frac{c}{2}\sin 2\vartheta\right)d\vartheta =$$

Prevedemo na B funkcijo

$$= \int_0^{\frac{\pi}{2}} \left((-a\cos\varphi + b\sin\varphi)\frac{\pi}{4} - \frac{c}{2}\right) =$$

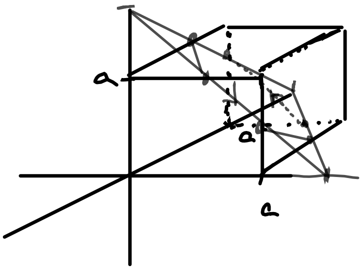
$$= \left(-\frac{\pi a}{4}\sin\varphi + \frac{\pi}{4}b\cos\varphi - \frac{c}{2}\varphi\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4}a - \frac{\pi}{4}b - \frac{\pi}{4}c = -\frac{\pi}{4}(a+b+c)$$

$a > 0$

$$\vec{f}(x,y,z) = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z = \frac{3a}{2}$



krivulja 6 kotnik

cirkulacija.... integral vektorskega polja vzdolž sklenjene krivulje

K_1:

$x = x$

$y = 0$

$z = \frac{3a}{2} - x$

$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$

$$\int_{K_1} \vec{f} dr$$

$\dot{\vec{r}}(x) = (1, 0, -1)$

$\vec{f}(r(x)) = (0 - (\frac{3}{2} - x)^2, (\frac{3}{2}a - x)^2 - x^2, x^2 - 0)$

$\vec{f} \cdot \dot{\vec{r}} = -(\frac{3}{2}a - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$

$$\int_{\frac{a}{2}}^{a} \left(-(\frac{3}{2}a - x)^2 - x^2\right) dx =$$

$$-\frac{1}{3}(\frac{3}{2}a - x)^3 \Big|_{\frac{a}{2}}^{a} - \frac{1}{3}x^3 \Big|_{\frac{a}{2}}^{a} =$$

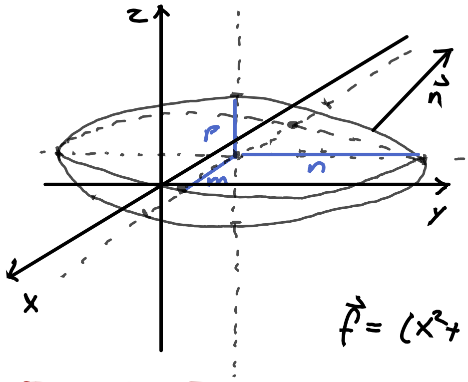
$$= -\cancel{\frac{1}{3}\frac{a^3}{8}} + \cancel{\frac{1}{3}a^3} - \cancel{\frac{1}{3}a^3} + \cancel{\frac{1}{3}\frac{a^3}{8}} = 0$$

$m, n, p > 0$
$a, b, c \in \mathbb{R}$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanja stran te ploskve



$$\vec{f} = (x^2 + y^2 + z^2)$$

$$\boxed{I = \int_{\substack{S \\ \parallel \\ \partial D}} \vec{f} d\vec{S} = \int_D \operatorname{div} \vec{f} dV}$$

gaussov izrek

Normala mora biti zunanja

$$\int_D (2x + 2y + 2z) dV = 2X_T + 2 \cdot y_T + 2 z_T$$

$$X_T = \int_D x dV = X_T \cdot V(D) = a V(D)$$

enotska krogla B: $V(B) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

$$\Downarrow$$

$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3}\pi$$

D dobimo če B raztegnemo za faktor m v smer: x, n, p v drugih dveh

B opišemo:

$$x = r\cos\vartheta\cos\varphi$$
$$y = r\cos\vartheta\sin\varphi$$
$$z = r\sin\vartheta$$

$r \in [0, 1]$

D opišemo:

$$x = m r \cos\varphi\cos\vartheta$$
$$y = n r \cos\vartheta\sin\varphi$$
$$z = p r \sin\vartheta$$

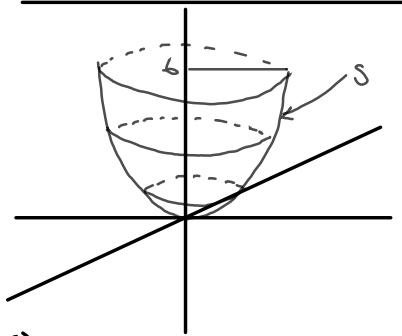
$$I = 2 \tfrac{4}{3}\pi m \cdot n \cdot p (a + b + c)$$

$\vec{f}(\vec{r}) = |\vec{r}|^2 \vec{r} \qquad b>0$

izračunaj pretok $\vec{f}$ skozi

a) rob območja $D=\{x,y,z;\ 2z \geq x^2+y^2;\ z \leq b\}$

b) ploskev $2z = x^2+y^2;\ z \leq b$



Pretok skozi S: $\int_S \vec{f}\, d\vec{S}$

a)
$$\int_{\partial D} \vec{f}\, d\vec{S} = \int_D \operatorname{div} \vec{f}\, dV$$

$$\vec{f}(x,y,z) = (x^2+y^2+z^2)(x,y,z) =$$
$$(x(x^2+y^2+z^2),\ y(x^2+y^2+z^2),\ z(x^2+y^2+z^2)) =$$
$$(x^3+y^2x+z^2x,\ x^2y+y^3+z^2y,\ x^2z+y^2z+z^3)$$

$$\operatorname{div}\vec{f} = 3x^2+y^2+z^2 + x^2+3y^2+z^2 + y^2+x^2+3z^2$$
$$= 5x^2+5y^2+5z^2$$

$$\int_D \operatorname{div}\vec{f}\, dV = 5\int_D (x^2+y^2+z^2)\, dV$$

$x = r\cos\varphi \qquad r^2 = x^2+y^2 = 2z$

$y = r\sin\varphi \qquad z = \frac{r^2}{2}$

$z = z$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^{b} (r^2+z^2)\, r\, dz$$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \frac{1}{3}z^3\right)\Big|_{\frac{r^2}{2}}^{b} dr$$

$$= 10\pi \int_0^{\sqrt{2b}} \left(br^3 + \frac{1}{3}b^3 - \frac{r^5}{2} - \frac{1}{6}r^6\right) dr =$$

$$= 10\pi \left(\frac{1}{4}br^4 + \frac{1}{3}b^3 r - \frac{1}{12}r^6 - \frac{1}{6\cdot 7}r^7\right)\Big|_0^{\sqrt{2b}} =$$

$$= 10\pi \left(\frac{1}{4}b^3\cdot 4 + \frac{1}{3}b^3\sqrt{2b} - \frac{1}{12}8b^3 - \frac{1}{6\cdot 7}8b^3\sqrt{2b}\right)$$

nerabe

$$= 10\pi\left(b^3 + \frac{b^4}{3} - \frac{2}{3}b^3 - \frac{b^4}{12}\right) =$$
$$10\pi\left(\frac{b^3}{3} + \frac{b^4}{4}\right)$$

b) $\partial D = S \cup S_0$

$$\int_S \vec{f}\, d\vec{S} = \int_{\partial D} \vec{f}\, d\vec{S} - \int_{S_0} \vec{f}\, d\vec{S}$$

$$\int_{S_0} \vec{f}\, d\vec{S} = \int_{S_0} (\vec{f}\cdot\vec{n})\, dS = \qquad \vec{n} = (0,0,1)$$

$$= \int_{S_0} (zx^2+zy^2+z^3)\, dS = \int_{S_0} (bx^2+by^2+b^3)\, dS =$$

$x = r\cos\varphi$

$y = r\sin\varphi$

$$= b\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r(r^2+b^2)\, dr = b\left(\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr + \int_{S_0} b^2\, dS\right)$$

$P(S_0) = 2\pi b$

$$= b\left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr\right) =$$

$$= b\left(2\pi b^3 + 2\pi\, \frac{1}{4}\cdot 4b^2\right) =$$

$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \int_S \vec{f}\, d\vec{S} = \frac{10}{3}\pi b^3 + \frac{10}{3}\pi b^4 - 2\pi b^4 - 2\pi b^3$$

Izračunaj $I=\int_k \frac{x\,dy - y\,dx}{x^2+y^2}$

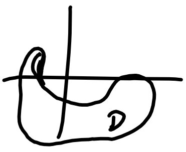
če je $k \subseteq \mathbb{R}^2$ sklenjena krivulja;

a) ki ne obkroži izhodišča

b) ki obhodi izhodišče

Omejimo se na primer: $k=\partial D$ za D z odsekoma gladkim robom

a)



$$\boxed{\int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx\,dy}$$

pozitivna orientacija

greenova formula

$P=\frac{-y}{x^2+y^2}$ $\quad Q=\frac{x}{x^2+y^2}$

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

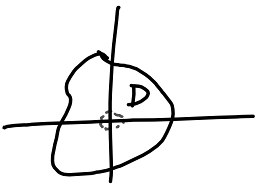
$$P_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\int_D (Q_x - P_y)\,dx\,dy = \int_D 0\,dx\,dy = 0$$

∂D ne obkroži $(0,0) \Leftrightarrow (0,0) \notin D$

$(x^2+y^2) \neq 0$

b)



Iz D izrežemo dovolj majhen krog $K(0,\varepsilon)$

$D' = D - K(0,\varepsilon)$

Uporabimo greenovo formulo na D'

$$\iint_{D'} \cancel{(Q_x + P\,dy)} = \int_{\partial D'} (Q_x - P_y)\,dx\,dy = 0$$

$\partial D' = \partial D^+ + \partial K(0,\varepsilon)^-$

$$0 = \int_{\partial D^+} P\,dx + Q\,dy + \int_{\partial K^-} P\,dx + Q\,dy$$

Pozor orientacija obratna kot ponavadi ☺

$$\Longrightarrow \int_{\partial D^+} \ldots = \int_{\partial K^+} \ldots$$

$x=\varepsilon\cos\varphi$ $\quad y=\varepsilon\sin\varphi$ $\quad$ ~~$\varepsilon=\varepsilon$~~

$dx = -\varepsilon\sin\varphi$

$dy = \varepsilon\cos\varphi$

$$\int_{\partial K} \ldots = \frac{1}{\varepsilon}\int_0^{2\pi} \cos^2\varphi + \sin^2\varphi = 2\pi$$

$r_1 \dots r_n \in \mathbb{R}^3$ različni

$e_1 \dots e_n \in \mathbb{R}$

$$f(\vec r) = \sum_{i=1}^n \text{grad}\left(\frac{-e_i}{4\pi|r-r_i|}\right)$$

izračunaj pretok $\vec f$ skozi zaključeno ploskev S, ki objema točke $r_1 \dots r_n$

$r_1 \dots r_n \in \text{Int} D$ $\quad S = \partial D$

$$\int_{\partial D} \vec f \, d\vec S = \int_D \text{div} \vec f \, dV$$

gaussov izrek vrjetno ne velja,

ker ker ima f singularne točke v D

$$\text{div}(\vec f(r)) = \sum_{i=1}^n \text{div}\left(\text{grad}\left(\frac{-e_i}{4\pi|r-r_i|}\right)\right)$$

Spomnimo se: $\text{grad} \frac{1}{|\vec r - \vec a|} = -\frac{\vec r - \vec a}{|\vec r - \vec a|^3}$

in $\text{div}\left(\text{grad} \frac{1}{|\vec r - \vec a|}\right) = 0$

Iz D izrežemo krogle $K_i = K(r_i, \varepsilon)$

Tako da $\bar K_i \subset \text{Int}(D)$ in $K_i \cap K_j = \emptyset$ za $i \neq j$

$D' = D - \left(\bigcup_{i=1}^n K_i\right)$

Na D' uporabimo gaussov izrek

$$\int_{\partial D'} \vec f(\vec r)\, dS = \int_{D'} \text{div} \vec f \, dV = 0$$

$$\partial D' = \partial D^+ \cup \left(\bigcup_{j=1}^n \partial K_i^-\right) \Rightarrow \int_{\partial D'} = \int_{\partial D^+} \dots + \sum_{i=1}^n \int_{\partial K_i^-} \dots = 0$$

$$\Rightarrow \int_{\partial D^+} \dots = \sum_{i=1}^n \int_{\partial K_i^+} \dots$$

$$\int_{\partial K_i} \vec f \, d\vec S = \int_{K_i} \sum_{j=1}^n \text{grad}\left(\frac{-e_j}{4\pi|r-r_j|}\right) \cancel{dS}$$

$$\sum_{j=1}^n \int_{\partial K_j} \left(\frac{+e_j}{4\pi} \frac{r-r_j}{|\vec r - \vec r_j|^3}\right) dS$$

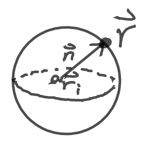
$$\int_{\partial K_j} \frac{\vec r - r_j}{|\vec r - \vec r_j|^3} dS \xRightarrow{i \neq j} r_j \in \bar K_i \Rightarrow$$

lahko uporabimo gaussov izrek

$\Longrightarrow \quad = 0$

$$\sum_{j=1}^n \int_{\partial K_i} \frac{e_j}{4\pi} \frac{\vec r - \vec r_j}{|\vec r - \vec r_j|^3} d\vec S = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{\vec r - \vec r_i}{|\vec r - \vec r_i|^3} d\vec S$$

$d\vec S = \vec n \, dS$



$$\vec n = \frac{\vec r - \vec r_i}{|\vec r - \vec r_i|}$$

$$= \frac{e_i}{4\pi} \int_{\partial K_i} \frac{|\vec r - \vec r_i|^2}{|\vec r - \vec r_i|^4} dS = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{dS}{|r - r_i|^2} =$$

na ∂K_i je $|r - r_i| = \varepsilon$

$$= \frac{e_i}{4\pi} \int \frac{dS}{\varepsilon^2} =$$

$$\frac{e_i}{4\pi\varepsilon^2} P(\partial K_i) = \frac{e_i}{4\pi\varepsilon^2} 4\pi\varepsilon^2 = e_i$$

$$\Rightarrow \int f \, d\vec S = \sum_{i=1}^n e_i$$

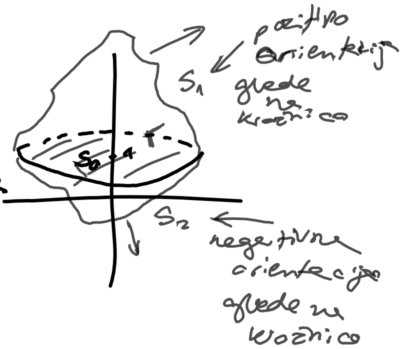
$$I = \int_S (1+x^2) f(x)\, dy\,dz - 2xy f(x)\, dz\,dx - 3z\, dx\,dy$$

Dokaži: $f \in C^1$ da I enak za vse ploskve S, katerih rob je krožnica $\{(\cos t, \sin t, 1)\}$

Določi to vrednost za I

$$\vec{f} = ((1+x^2)f(x),\ 2xy f(x),\ -3z)$$

$$\Longrightarrow I = \int_S \vec{f}\, d\vec{S}$$



$S_1 \cup S_2$ je sklenjena ploskev

$S_1^+ \cup S_2^- = \partial D^+$

Gauss na D:

$$\int_{\partial D^+} \vec{f}\, d\vec{S} = \int_D \operatorname{div} \vec{f}\, dV$$

$$\partial D = \int_{S_1} - \int_{S_2} = 0 \Longleftarrow \operatorname{div} \vec{f}\, dV = 0$$

$\vec{f}$ je solenoidalno ($\operatorname{div} \vec{f} = 0$) $\Longleftrightarrow f(x)$ je oblike $3\arctan x + C$

$$\int_S \vec{f}\, d\vec{S} = \int_S ((1+x^2)(3\arctan x + C), -2xy(3\arctan x + C), -3z)\, d\vec{S}$$

$$= \int_{S_0} \vec{f}\, dS =$$

samo krog: $d\vec{S} = \vec{n} \cdot dS \quad \vec{n} = (0,0,1)$

$$= \int_{S_0} -3z\, dS = -3 \int_{S_0} z\, dS = -3 z_T\, P(S_0) =$$

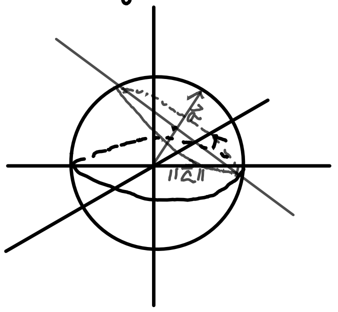
$$= -3 \cdot 1 \cdot \pi$$

Opomba: Lahko bi dokazali da je tudi div $\vec{F}$ potreben pogoj

$\vec a \neq \vec 0$,
$\vec b \in \mathbb{R}$

$\vec F(\vec r) = (\vec r - \vec a) \times (\vec r - \vec b)$

izračunaj cirkulacijo $\vec F$ vzdolž krivulje K: $|\vec r| = |\vec a| \wedge \vec r \cdot \vec a = \frac{|\vec a|^2}{2}$



enačba ravnine:

$\vec r \cdot \vec n = d$
← normala

ravnina z normalo $\vec a$
in točka $\vec r = \frac{\vec a}{2}$

K je krožnica

B naj bo krog z robom ki $K = \partial B$

$$\int_{\partial B} \vec f \, d\vec r = \int_B \operatorname{rot} \vec f \, d\vec S$$

↑ Stokesov izrek

normala: B: $\vec n = \frac{\vec a}{\|\vec a\|}$

$$\operatorname{rot} \vec f = \operatorname{rot}(\underbrace{\vec r \times \vec r}_{0} - a \times r - \vec r \times \vec b + \vec a \times \vec b) =$$

$$\boxed{\text{Vemo: } \operatorname{rot}(\vec r \times \vec c) = -2\vec c}$$

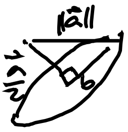
$$= -2\vec a + 2\vec b + 0 = 2(\vec b - \vec a)$$

$$I = \int_B 2(\vec b - \vec a) \, d\vec S = \frac{2}{\|\vec a\|} \int_B (\vec b - \vec a) \cdot \vec a \, dS =$$

$$= \frac{2 (\vec b - \vec a) \cdot \vec b}{\|\vec a\|} \cdot P(B)$$

$P(B) = ?$

polmer:



$$b = \sqrt{\|a\|^2 - \frac{\|a\|^2}{4}} = \frac{\sqrt 3}{2} \|a\|$$

$$P(B) = \pi b^2 = \pi \cdot \frac{3}{4} \|a\|$$

$$I = 3\pi (\vec b - \vec a) \vec b \|a\|$$

← ni isti $\vec b$ kot b