

Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

\nearrow
 $\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$FV(f)$ konvergira k f , če je f zvezna v x ,
če pa ni, pa konvergira k $\frac{f(x^-) + f(x^+)}{2}$

$$\int x \cos(nx) dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int x \sin(nx) dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

$$f \text{ sode} \Rightarrow b_n = 0$$

$$f \text{ liha} \Rightarrow a_n = 0$$

$$\int x^2 \cos(nx) dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx) dx = -\frac{x^2}{n} \cos(nx) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

②

$f(x) = |x|$ razvij v Fourierovo vrsto in
seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx =$$

sodast

$$= \frac{2}{\pi} \left(\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right) \Big|_0^{\pi} =$$

$$= \frac{2}{\pi} \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx = 0$$

↙ lihost

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{(-1)^n - 1}{n^2} \right) \cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2} \cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = -\frac{\pi}{2} \cdot \frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = ? \quad S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

S... li.

S'... ostalo

$$= S + \frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) =$$

$$= S + \frac{1}{4} \Sigma = \Sigma$$

$$S = \frac{3}{4} \Sigma$$

$$\Sigma = \frac{4}{3} S = \frac{\pi^2}{6}$$

③

$$f(x) = \max(\cos x, 0)$$

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos(x) dx = \frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos(x-nx) + \cos(x+nx)}{2} dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{2} (\cos((n+1)x) + \cos((n-1)x)) dx$$

$$\frac{1}{\pi} \left[\frac{1}{n+1} \sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1} \sin\left((n-1)\frac{\pi}{2}\right) \right]$$

$$n=4k: \frac{1}{\pi} \frac{1}{4k+1} \cdot 1 + \frac{1}{4k-1} \cdot (-1) = \frac{-2}{4k^2-1} \cdot \frac{1}{\pi}$$

$$n=4k+2: 0$$

$$n=4k+2: \frac{1}{\pi} \frac{1}{4k+3} \sin\left(\frac{3\pi}{2}\right) + \frac{1}{4k+1} \sin\left(\frac{\pi}{2}\right) =$$

$$= \frac{1}{\pi} \left(-\frac{1}{4k+3} + \frac{1}{4k+1} \right)$$

$$= \frac{1}{\pi} \begin{cases} 0 & ; n \text{ l.h.o} \\ \frac{(-1)^{m+1} \cdot 2}{(2m)^2 - 1} & ; n=2m \end{cases}$$

$$a_1 = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx = \frac{1}{\pi} + \frac{\sin(2x)}{2} \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$b_n = 0 \text{ ker } f \text{ sodd}$$

$$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2} \cos(x) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cdot 2}{(2m)^2 - 1} \cos(2mx)$$

$$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} S_1 \Rightarrow$$

$$S_1 = \left(\frac{1}{2} - \frac{1}{\pi} \right) \cdot \frac{\pi}{2} (-1) = \frac{1}{2} - \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2 - 1} \cdot \cos(2m\pi) =$$

$$= \frac{1}{\pi} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2 - 1}$$

$$S_2 = -\frac{1}{\pi} \cdot \frac{\pi}{2} = -\frac{1}{2}$$

④

$$f(x) = x^2 \quad f: [0, \pi] \rightarrow \mathbb{R}$$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij v sinusno FV in skiciraj njen graf

1) če f razberimo do sode funkcije

$$f_s: [-\pi, \pi] \rightarrow \mathbb{R}$$

$$x < 0 \Rightarrow f_s(x) = f(-x)$$

$$FV_{\cos}(f)(x) = FV(f)(x)$$

$$2) f_f: [-\pi, \pi] \rightarrow \mathbb{R} \quad f_f(x) = -f(-x)$$

$$FV_{\sin}(f)(x) = FV(f_f)(x)$$

$$1) f_s(x) = x^2$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$$

$$b_n = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

Reglejmo:

$$\int x^2 \cdot e^{inx} =$$

$$u = x^2 \quad dv = e^{inx} dx$$

$$du = 2x dx \quad v = \frac{1}{in} e^{inx}$$

$$\boxed{\int x e^{inx} = \frac{-i \cdot x \cdot e^{inx}}{n} + \frac{e^{inx}}{n^2}}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \int 2x \frac{e^{inx}}{in} dx = \text{od prej}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \frac{2}{in} \left(\frac{ix e^{inx}}{n} + \frac{e^{inx}}{n^2} \right) + C$$

$$= e^{inx} \left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3} \right)$$

$$\cos nx + i \sin nx$$

$$\int x^2 \cos(nx) dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx) dx = -\frac{x^2}{n} \cos nx + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

$$a_n = \frac{1}{\pi} \left(\frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) + \frac{2}{n^3} \sin(nx) \right) \Big|_{-\pi}^{\pi} =$$

$$= \frac{2}{\pi} \left(\frac{2\pi}{n^2} (-1)^n \right) = \frac{4(-1)^n}{n^2}$$

$$FV_{\cos}(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{n^2} \cos(x)$$

$$b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\text{Soda}}(x) \sin(nx) = \frac{2}{\pi} \underbrace{\int_0^{\pi} f(x) \sin(nx)}_{= f(x) \text{ on } [0, \pi] = x^2}$$

$$= \frac{2}{\pi} \left[-\frac{x^2}{n} \cos(nx) + \frac{2}{n^2} \sin(nx) + \frac{2}{3n} (\cos(nx)) \right] \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left(-\frac{\pi^2}{n} (-1)^n + \frac{2}{n^2} (-1)^n - \frac{2}{n^3} \right)$$

$$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left((-1)^{n+1} \frac{\pi^2}{2} + \frac{2(1 - (-1)^n)}{n^3} \right) \sin(nx)$$

$$f(x) = x(\pi - x) \quad \text{rezi. j'}$$

6)

$$f(x) = \sin^3 x \quad \text{rozvij v FV}$$

Pred premislek:

$$f(x) = \sin^2 x \quad \text{je že FV}$$

$$b_2 = 1, \text{ ostalo: } 0 \quad 0$$

$$f(x) = \sin^2 x = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) =$$

$$= \frac{1}{2} (\cos(0) - \cos(2x)) = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2} \sin x - \frac{1}{2} \cos 2x \sin x =$$

$$= \frac{1}{2} \sin x - \frac{1}{2} \cdot \frac{1}{2} (\sin(3x) - \sin(x)) =$$

$$= \frac{3}{4} \sin x - \frac{1}{4} \sin(3x)$$

polinom
 $\forall p(\sin x, \cos x)$ ima končno furierovo vrsto
 $p \in \mathbb{R}[x, y]$

$a > 0$ kvadrantnej težišče homogénega

loke astroide



$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

$$x_T = \frac{\int_K x dm}{m(K)} = \frac{\int_K x \rho ds}{\int_K \rho ds} = l(K)$$

$\vec{r}(t)$... parametrizácia

$$ds = |\dot{\vec{r}}(t)| dt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

u, \dots skalárna funkcia

$$\int_K u ds = \int_a^b u(\vec{r}(t)) |\dot{\vec{r}}(t)| dt$$

$$x = a \cos^3 t$$

$$y = a \sin^3 t$$

$t \in [0, \frac{\pi}{2}]$ ← keď je vše pozitívne (vyššie pravidlo)

$$\vec{r}(t) = (-3a \cos^2 t \cdot \sin t, 3a \sin^2 t \cos t)$$

$$|\dot{\vec{r}}| = 3a \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} =$$
$$= 3a \cos t \sin t$$

$$l(K) = \int_0^{\frac{\pi}{2}} |\dot{\vec{r}}(t)| dt = \int_0^{\frac{\pi}{2}} 3a \cos t \sin t dt =$$

$$u = \sin t dt$$

$$du = \cos t dt$$

$$= 3a \int_0^1 u du = \frac{3}{2} a$$



$$\int_K x ds = \int_0^{\frac{\pi}{2}} a \cos^3 t + 3a \cos t \sin t dt =$$

$$\cos t = u \quad du = -\sin t$$

$$= 3a^2 \int_0^1 u^4 du = \frac{3}{5} a^2$$

$$y_T = \frac{3}{5} a^2$$

$$a > 0 \quad \alpha \in [0, 2\pi]$$

$$K = S(0, a) \cap \Pi$$

$$\Pi: y = x \tan \alpha$$

$$I = \int_K (y-z) + (z-x) dy + (x-y) dz$$

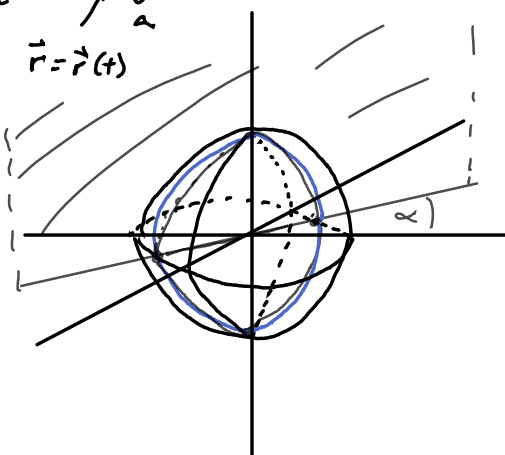
$$\vec{f}(x, y, z) = (y-z, z-x, x-y) \quad \text{vektor sho po'ye}$$

$$I = \int_K \vec{f} d\vec{r}$$

$$\vec{r} = (x, y, z) \Rightarrow d\vec{r}$$

$$\int_K \vec{f} d\vec{r} = \int_a^b \vec{f}(\vec{r}(t)) \vec{r}'(t) dt$$

$$\vec{r} = \vec{r}(t)$$



Para metrizecyje K

$$x = a \cos \alpha \cdot \cos \vartheta$$

$$y = a \sin \alpha \cos \vartheta$$

$$z = a \sin \vartheta$$

$$\vec{r}(\vartheta) = (a \cos \vartheta \cos \alpha, a \cos \vartheta \sin \alpha, a \sin \vartheta)$$

$$\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\vec{r}'(\vartheta) = (-a \sin \vartheta \cos \alpha, -a \sin \vartheta \sin \alpha, a \cos \vartheta)$$

$$\vec{f}(\vec{r}(\vartheta)) = a (\cos \vartheta \sin \alpha - \sin \vartheta, \sin \vartheta - \cos \vartheta \cos \alpha, \cos \vartheta \sin \alpha - \cos \vartheta \cos \alpha)$$

$$\vec{f} \cdot \vec{r}' = a^2 (-\cos \alpha \cos \vartheta \sin \alpha \sin \vartheta + \sin^2 \vartheta \cos \alpha \cos \alpha -$$

$$- \sin^2 \vartheta \sin \alpha + \cos \vartheta \cos \alpha \sin \vartheta \sin \alpha +$$

$$+ \cos^2 \vartheta \sin \alpha - \cos^2 \vartheta \cos \alpha) =$$

$$= a^2 (\cos 2\vartheta \sin \alpha - \cos 2\vartheta \cos \alpha) \quad \text{Me}$$

$$= a^2 (\cos \alpha - \sin \alpha$$

$$2\pi$$

$$\int_K \vec{f} d\vec{r} = \int_0^{2\pi} a^2 (\cos \alpha - \sin \alpha) d\vartheta =$$

$$2\pi a^2 (\cos \alpha - \sin \alpha)$$

\uparrow pri. orientatsiya



$$a, b, c \in \mathbb{R}$$

$$I = \int_S a \, dy \, dz + b \, dx \, dz + c \, dx \, dy$$

$$S = \{ (x, y, z) ; x^2 + y^2 + z^2 = 1, x, y, z \geq 0 \}$$

$$I = \int \vec{f} \, d\vec{S}$$

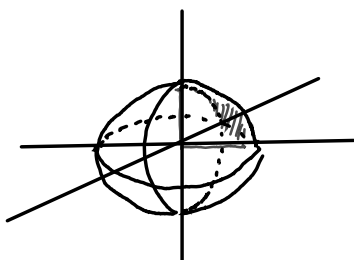
$$\vec{f} = (a, b, c)$$

$$\int (\vec{f} \cdot \vec{N}) \, dS$$

\nwarrow *enotska normala*
orijentacija ploške je usmerjena s smerom normale

$$\int_S \vec{f} \, d\vec{S} = \int_S \vec{f}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv$$

\nearrow $\vec{r} = \vec{r}(u, v)$
 \nwarrow *smer se ujema s predpisano orijentacijom*



Sferične koordinate

$$\begin{aligned} x &= \cos \vartheta \cos \varphi & \vartheta &\in [0, \frac{\pi}{2}] \\ y &= \cos \vartheta \sin \varphi & \varphi &\in [0, \frac{\pi}{2}] \\ z &= \sin \vartheta \end{aligned}$$

$$\vec{r}(\vartheta, \varphi) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta)$$

$$\vec{r}_\vartheta = (-\sin \vartheta \cos \varphi, -\sin \vartheta \sin \varphi, \cos \vartheta)$$

$$\vec{r}_\varphi = (-\cos \vartheta \sin \varphi, \cos \vartheta \cos \varphi, 0)$$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2 \vartheta \cos \varphi, \cos^2 \vartheta \sin \varphi,$$

$$-\sin \vartheta \cos \vartheta \cos^2 \varphi - \cos \vartheta \sin \vartheta \sin^2 \varphi) =$$

$$= (-\cos^2 \vartheta \cos \varphi, \cos^2 \vartheta \sin \varphi, -\sin \vartheta \cos \vartheta) =$$

$$= -\cos \vartheta (\underbrace{\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta}_{\vec{r}})$$

$$\int_S \vec{f} \, d\vec{S} = \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\frac{\pi}{2}} d\varphi (a \cos^2 \vartheta \cos \varphi + b \cos^2 \vartheta \sin \varphi - c \sin \vartheta \cos \vartheta)$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (-a \cos \varphi + b \sin \varphi) \cos^2 \vartheta + \frac{c}{2} \sin 2\vartheta \, d\vartheta \, d\varphi =$$

Pravedimo na Bernouliju

$$= \int_0^{\frac{\pi}{2}} (-a \cos \varphi + b \sin \varphi) \left(\frac{\pi}{4} - \frac{c}{2} \right) d\varphi =$$

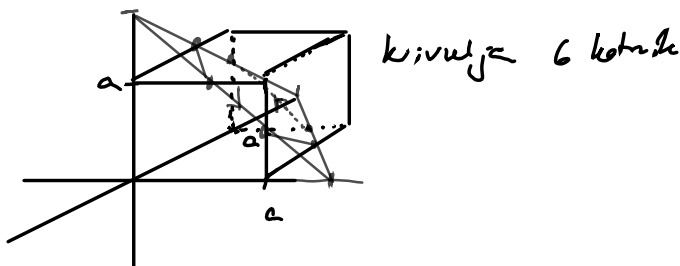
$$= \left(-\frac{\pi a}{4} \sin \varphi + \frac{\pi b}{4} \cos \varphi - \frac{c}{2} \varphi \right) \Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4} a - \frac{\pi}{4} b - \frac{\pi}{4} c = -\frac{\pi}{4} (a + b + c)$$

200

$$\vec{f}(x,y,z) = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z = \frac{3a}{2}$



cirkulacija... integral vektorskega polja vzdolž sklenjene krivulje

K_1

$$x = x$$

$$y = 0$$

$$z = \frac{3a}{2} - x$$

$$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$$

$$\int_{K_1} \vec{f} d\vec{r}$$

$$\dot{\vec{r}}(x) = (1, 0, -1)$$

$$\vec{f}(\vec{r}(x)) = (0 - (\frac{3a}{2} - x)^2, (\frac{3a}{2} - x)^2 - x^2, x^2 - 0)$$

$$\vec{f} \cdot \dot{\vec{r}} = -(\frac{3a}{2} - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$$

$$\int_{\frac{a}{2}}^a (-2x^2 + 3ax - \frac{9}{4}a^2) dx =$$

$$= -\frac{1}{3} \left(\frac{3a}{2} - x \right)^3 \Big|_{\frac{a}{2}}^a - \frac{1}{3} x^3 \Big|_{\frac{a}{2}}^a =$$

$$= -\frac{1}{3} \frac{a^3}{8} + \frac{1}{3} a^3 - \frac{1}{3} a^3 + \frac{1}{3} \frac{a^3}{8} = 0$$

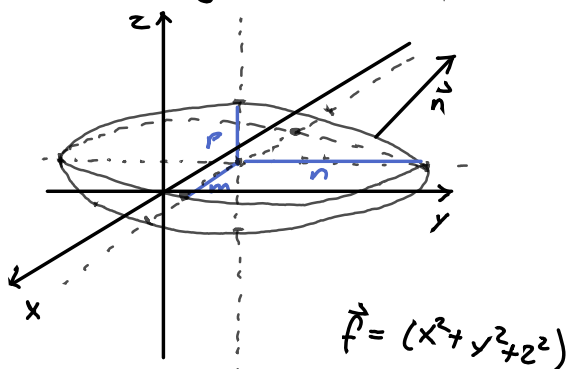
$$m, n, p > 0$$

$$a, b, c \in \mathbb{R}$$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanje stran te ploskve



$$I = \int_S \vec{f} d\vec{s} = \int_D \text{div} \vec{f} dV \quad \text{gaussov : zrek}$$

Normale mere biti
zunanje

$$\int_D (2x + 2y + 2z) dV = 2x_T + 2y_T + 2z_T$$

$$x_T = \int_D x dV = x_T \cdot V(D) = x_T V(D)$$

$$\text{enotska krogla } B: V(B) = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi$$



$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3} \pi$$

D dobimo če B raztegemo o ca faktor
m v smeri x, n, p v drugih dveh

B opisemo:

$$\begin{aligned} x &= r \cos \varphi \cos \psi \\ y &= r \cos \varphi \sin \psi \\ z &= r \sin \varphi \end{aligned} \quad r \in (0, 1]$$

D opisemo:

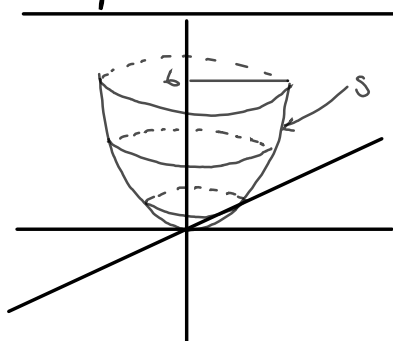
$$\begin{aligned} x &= m r \cos \varphi \cos \psi \\ y &= n r \cos \varphi \sin \psi \\ z &= p r \sin \varphi \end{aligned}$$

$$I = 2 \frac{4}{3} \pi m \cdot n \cdot p (a + b + c)$$

$\vec{f}(\vec{r}) = |\vec{r}|^2 \cdot \vec{r} \quad b > 0$ ~~irracionalnej pretok \vec{f} skozi~~

a) rob obmedzena $D = \{x, y, z; 2z \geq x^2 + y^2; z \leq b\}$

b) ploška $2z = x^2 + y^2; z \leq b$



Pretok skozi S:

$$\int_S \vec{f} d\vec{s}$$

$$\int_{\partial D} \vec{f} d\vec{s} = \int_D \operatorname{div} \vec{f} dV$$

$$\vec{f}(x, y, z) = (x^2 + y^2 + z^2)(x, y, z) =$$

$$(x(x^2 + y^2 + z^2), y(x^2 + y^2 + z^2), z(x^2 + y^2 + z^2)) =$$

$$(x^3 + yx^2 + z^2x, xy^2 + y^3 + z^2y, x^2z + y^2z + z^3)$$

$$\operatorname{div} \vec{f} = 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + y^2 + x^2 + 3z^2 = 5x^2 + 5y^2 + 5z^2$$

$$\int_D \operatorname{div} \vec{f} dV = 5 \int_D (x^2 + y^2 + z^2) dV$$

$$x = r \cos \varphi \quad r^2 = x^2 + y^2 = 2z$$

$$y = r \sin \varphi$$

$$z = \frac{r^2}{2}$$

$$z = z =$$

$$= 5 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^b (r^2 + z^2) r dz$$

$$= 5 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \frac{1}{3} z^3 \right) \Big|_{\frac{r^2}{2}}^b dr$$

$$= 10\pi \int_0^{\sqrt{2b}} \left(br^3 + \frac{1}{3} b^3 - \frac{r^5}{2} - \frac{1}{6} r^6 \right) dr =$$

$$= 10\pi \left(\frac{1}{4} br^4 + \frac{1}{3} b^3 r - \frac{1}{12} r^6 - \frac{1}{6 \cdot 7} r^7 \right) \Big|_0^{\sqrt{2b}} =$$

$$= 10\pi \left(\frac{1}{4} b^3 \cdot 4 + \frac{1}{3} b^3 \sqrt{2b} - \frac{1}{12} \cdot 8 b^3 - \frac{1}{6 \cdot 7} \cdot 8 b^3 \sqrt{2b} \right)$$

~~nerobí~~

$$= 10\pi \left(b^3 + \frac{b^4}{3} - \frac{2}{3} b^3 - \frac{b^4}{12} \right) =$$

$$10\pi \left(\frac{b^3}{3} + \frac{b^4}{4} \right)$$

$$b) \partial D = S \cup S_0$$

$$\int_S \vec{f} d\vec{s} = \int_{\partial D} \vec{f} d\vec{s} - \int_{S_0} \vec{f} d\vec{s}$$

$$\int_S \vec{f} d\vec{s} = \int_{S_0} (\vec{f} \cdot \vec{n}) d\vec{s} = \quad \vec{n} = (0, 0, 1)$$

$$= \int_{S_0} (2x^2 + 2y^2 + z^3) d\vec{s} = \int_{S_0} (bx^2 + by^2 + b^3) d\vec{s} =$$

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$= b \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r(r^2 + b^3) dr = b \left(\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr + \int_{S_0} b^3 d\vec{s} \right)$$

$$= b \left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr \right) = \quad P(S_0) = 2\pi b$$

$$= b \left(2\pi b^3 + 2\pi \cdot \frac{1}{4} \cdot 4b^2 \right) =$$

$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \int_S \vec{f} d\vec{s} = \frac{10}{3} \pi b^3 + \frac{10}{3} \pi b^4 - 2\pi b^4 - 2\pi b^3$$

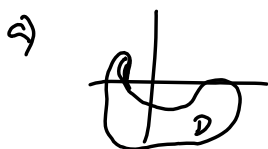
traznaja $I = \int_K \frac{x dy - y dx}{x^2 + y^2}$

če je $K \subseteq \mathbb{R}^2$ sklenjena krivulja;

a) ki ne obkroži izhodišča

b) ki dohodi izhodišče

Omejimo se na primer: $K = \partial D$ za D^2 odsekoma gladkim robom



$$\int_{\partial D} P dx + Q dy = \int_D (Q_x - P_y) dx dy$$

↑
pozitivna orientacija

↑
greenova formula

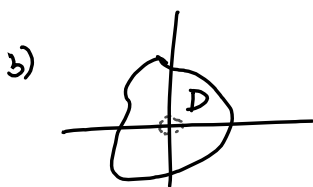
$$P = \frac{-y}{x^2 + y^2} \quad Q = \frac{x}{x^2 + y^2}$$

$$Q_x = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^4} = \frac{y^2 - x^2}{(x^2 + y^2)^4}$$

$$P_y = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$$\int_D (Q_x - P_y) dx dy = \int_D 0 dx dy = 0$$

∂D ne obkroži $(0,0) \Leftrightarrow (0,0) \notin D$
 $(x^2 + y^2) \neq 0$



K je rečano dovolj majhen krog $K(0, \epsilon)$

$$D' = D - K(0, \epsilon)$$

Uporabimo Greenovo formulo na D'

$$\int_{\partial D'} (Q dx + P dy) = \int_{\partial D'} (Q_x - P_y) dx dy = 0$$

$$\partial D' = \partial D^+ + \partial K(0, \epsilon)^-$$

$$0 = \int_{\partial D^+} P dx + Q dy + \int_{\partial K^-} P dx + Q dy$$

Pozor orientacije obratna
 K^+ ponaradi

$$\Rightarrow \int_{\partial D^+} \dots = \int_{\partial K^+} \dots$$

$$x = \epsilon \cos \varphi \quad y = \epsilon \sin \varphi$$

$$\epsilon = \epsilon$$

$$dx = -\epsilon \sin \varphi$$

$$dy = \epsilon \cos \varphi$$

$$\int_{\partial K} \dots = \frac{1}{\epsilon} \int_0^{2\pi} \cos^2 \varphi + \sin^2 \varphi = 2\pi$$

$r_1, \dots, r_n \in \mathbb{R}^3$ rationali

$e_1, \dots, e_n \in \mathbb{R}$

$$f(\vec{r}) = \sum_{i=1}^n \text{grad} \left(\frac{e_i}{4\pi |\vec{r} - \vec{r}_i|} \right)$$

irracionalnej pretok \vec{f} skozi elipsoid

plošče s, k odprene take r_1, \dots, r_n



$r_1, \dots, r_n \in \text{int}(D)$ $S = \partial D$

$$\int_{\partial D} \vec{f} d\vec{s} = \int_D \text{div} \vec{f} dV$$

gaussov izrek vjetno ne velja,
ker ker ima f singularne točke v D

$$\text{div}(\vec{f}(\vec{r})) = \sum_{i=1}^n \text{div} \left(\text{grad} \left(\frac{e_i}{4\pi |\vec{r} - \vec{r}_i|} \right) \right)$$

Spomimo se: $\text{grad} \frac{1}{|\vec{r} - \vec{a}|} = -\frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3}$
in $\text{div} \left(\text{grad} \frac{1}{|\vec{r} - \vec{a}|} \right) = 0$

Iz D izrežemo kugle $K_i = K(\vec{r}_i, \epsilon)$

Tako da $\bar{K}_i \subset \text{int}(D)$ in $K_i \cap K_j = \emptyset$ za $i \neq j$

$$D' = D - \left(\bigcup_{i=1}^n K_i \right)$$

Na D' uporabimo gaussov izrek

$$\int_{\partial D'} \vec{f}(\vec{r}) d\vec{s} = \int_{D'} \text{div} \vec{f} dV = 0$$

$$\partial D' = \partial D^+ \cup \left(\bigcup_{j=1}^n \partial K_j^- \right) \Rightarrow \int_{\partial D'} = \int_{\partial D^+} + \sum_{i=1}^n \int_{\partial K_i^-} = 0$$

$$\Rightarrow \int_{\partial D^+} = - \sum_{i=1}^n \int_{\partial K_i^-}$$

$$\int_{\partial K_i^-} \vec{f} d\vec{s} = \int_{K_i} \sum_{j=1}^n \text{grad} \left(\frac{e_j}{4\pi |\vec{r} - \vec{r}_j|} \right) dV$$

$$\sum_{j=1}^n \int_{\partial K_i^-} \left(\frac{e_j}{4\pi} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3} \right) d\vec{s}$$

$$\int_{\partial K_i^-} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3} d\vec{s} \xrightarrow{i \neq j} \vec{r}_j \in \bar{K}_i \Rightarrow$$

lahko uporabimo
gaussov izrek

$$\Rightarrow = 0$$

$$\sum_{j=1}^n \int_{\partial K_i^-} \frac{e_j}{4\pi} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3} d\vec{s} = \frac{e_i}{4\pi} \int_{\partial K_i^-} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} d\vec{s}$$

$$d\vec{s} = \vec{n} dS$$



$$\vec{n} = \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}$$

$$= \frac{e_i}{4\pi} \int_{\partial K_i^-} \frac{|\vec{r} - \vec{r}_i|^2}{|\vec{r} - \vec{r}_i|^4} dS = \frac{e_i}{4\pi} \int_{\partial K_i^-} \frac{dS}{|\vec{r} - \vec{r}_i|^2} =$$

na ∂K_i je
 $|\vec{r} - \vec{r}_i| = \epsilon$

$$= \frac{e_i}{4\pi} \int \frac{dS}{\epsilon^2} =$$

$$\frac{e_i}{4\pi \epsilon^2} P(\partial K_i) = \frac{e_i}{4\pi \epsilon^2} 4\pi \epsilon^2 = e_i$$

$$\Rightarrow \int \vec{f} d\vec{s} = \sum_{i=1}^n e_i$$

$$I = \int_S (1+x^2) f(x) dy dz - 2xy f(x) dz dx - 3z \frac{\partial f}{\partial x} dx dy$$

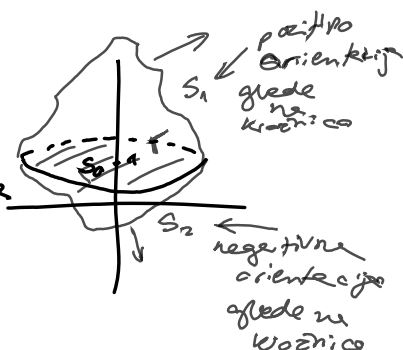
Dalo: $f \in C^1$ da I enak ~~na~~ vse ploskve S , katerih rob je krožnica $\{(\cos t, \sin t, 1) \}$
 Dokaži to ravnostno I

$$\vec{F} = ((1+x^2)f(x), 2xy f(x), -3z)$$

$$\Rightarrow I = \int_S \vec{F} d\vec{S}$$

$S_1 \cup S_2$ je sklenjena ploskev

$$S_1^+ \cup S_2^- = \partial D^+$$



Gauss na D :

$$\int_{\partial D} \vec{F} d\vec{S} = \int_D \operatorname{div} \vec{F} dV$$

$$\partial D = \int_{S_1} - \int_{S_2} = 0 \iff \operatorname{div} \vec{F} dV = 0$$

\vec{F} je solenoidalno ($\operatorname{div} \vec{F} = 0$) $\Rightarrow f(x)$ je oblike $3 \arctan x + c$

$$\int_S \vec{F} d\vec{S} = \int_S ((1+x^2)(3 \arctan x + c), -2xy(3 \arctan x + c), -3z) d\vec{S}$$

$$= \int_{S_0} \vec{F} d\vec{S} =$$

$$\text{skrajšano: } d\vec{S} = \vec{n} \cdot dS \quad \vec{n} = (0, 0, 1)$$

$$= \int_{S_0} -3z dS = -3 \int_{S_0} z dS = -3z_T P(S_0) =$$

$$= -3 \cdot 1 \cdot \pi$$

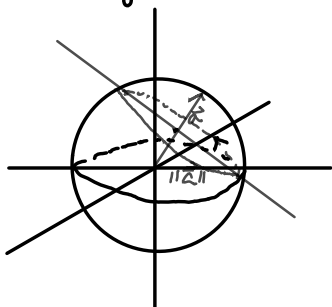
Opomba: Lahko bi dokazali da je
tudi $\operatorname{div} \vec{F}$ polje

$$\vec{a} \neq \vec{0}$$

$$\vec{b} \in \mathbb{R}$$

$$\vec{F}(\vec{r}) = (\vec{r} - \vec{a}) \times (\vec{r} - \vec{b})$$

izračunaj cirkulacijo \vec{F} vzdolž
krivulje K : $|\vec{r}| = |\vec{a}|$ in $\vec{r} \cdot \vec{a} = \frac{|\vec{a}|^2}{2}$



enačba ravnine:

$$\vec{r} \cdot \vec{n} = d$$

\vec{n} normala

ravnina z normalo \vec{a}
in točka $\vec{r} = \frac{\vec{a}}{2}$

K je krožnica

B naj bo krožnica K z roboto $K = \partial B$

$$\int_{\partial B} \vec{F} d\vec{r} = \int_B \text{rot } \vec{F} d\vec{S}$$

Stokesov izrek
normala: $B \ni \vec{n} = \frac{\vec{a}}{|\vec{a}|}$

$$\text{rot } \vec{F} = \text{rot}(\underbrace{\vec{r} \times \vec{r}}_0 - \vec{a} \times \vec{r} - \vec{r} \times \vec{b} + \vec{a} \times \vec{b}) =$$

$$\text{Vemari: } \text{rot}(\vec{r} \times \vec{c}) = -2\vec{c}$$

$$= -2\vec{a} + 2\vec{b} + 0 = 2(\vec{b} - \vec{a})$$

$$I = \int_B 2(\vec{b} - \vec{a}) d\vec{S} = \frac{2}{|\vec{a}|} \int_B (\vec{b} - \vec{a}) \cdot \vec{a} dS =$$

$$= \frac{2(\vec{b} - \vec{a}) \cdot \vec{b}}{|\vec{a}|} \cdot P(B)$$

$$P(B) = ?$$

polmer:



$$b = \sqrt{|\vec{a}|^2 - \frac{|\vec{a}|^2}{4}} = \frac{\sqrt{3}}{2} |\vec{a}|$$

$$P(B) = \pi b^2 = \pi \cdot \frac{3}{4} |\vec{a}|^2$$

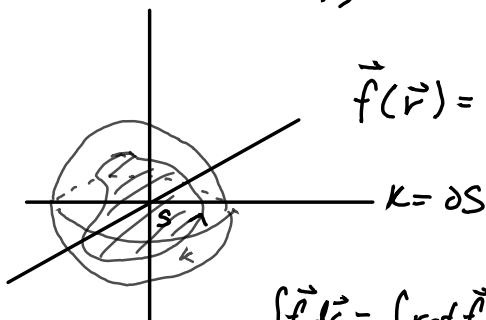
$$I = 3\pi (\vec{b} - \vec{a}) \cdot \vec{b} |\vec{a}|$$

K is izh \vec{b} kot b

K zaključena krogla
na sferi: $x^2 + y^2 + z^2 = 1$

Izračunaj
$$I = \int_K \frac{dx + dy + dz}{(x^2 + y^2 + z^2)^2} = \int_K \vec{f} d\vec{r}$$

$$\vec{f} = \left(\frac{1}{(x^2 + y^2 + z^2)^2}, \frac{1}{(x^2 + y^2 + z^2)^2}, \frac{1}{(x^2 + y^2 + z^2)^2} \right)$$



$$\vec{f}(\vec{r}) = (1, 1, 1) \text{ a } \text{res}$$

$$\int_K \vec{f} d\vec{r} = \int_S \text{rot} \vec{f} d\vec{S} = \dots$$

↑
obstaja boljši način

$$\int_K \vec{f} d\vec{r} = \int_K \underbrace{(1, 1, 1)}_{\vec{g}(\vec{r})} d\vec{r} = \int_S \text{rot}(\vec{g}(\vec{r})) d\vec{S} =$$

$$= \int_S 0 d\vec{S} = 0$$

Kaj če $K \neq \partial S$

če je poje \vec{f} potencialno ($\vec{f} = \text{grad } u$)
in K krivulja z začetkom v a in koncem
v b $\Rightarrow \int_K \vec{f} d\vec{r} = u(b) - u(a)$

Pozvedica: če je K sklenjena je integral
vzdolž K od $\int_K \text{grad } u d\vec{r} = 0$

\vec{f} ni potencialno, ampak \vec{g} je potencialno
(ker $\text{rot} \vec{g} = 0$)

$$\vec{g} = \text{grad}(x + y + z) \quad I = \int_K \vec{g} d\vec{r} = 0$$

Opomba: $\int_K \frac{1}{(x^2 + y^2 + z^2)^2} (dx + dy + dz)$

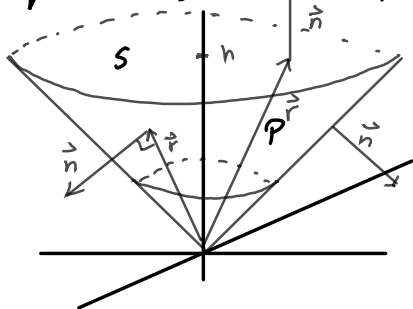
↓ na S_1

konstantno
poje

$$h > 0$$

$$\vec{f}(\vec{r}) = \vec{r}$$

lucuraj pretok \vec{f} skozi plosče in osnovno ploskev sklopa: $x^2 + y^2 \leq z^2$ $z \in [0, h]$



$$\vec{r} \cdot \vec{n} = 0$$

$$\int_P \vec{f} d\vec{s} = \int_P \vec{f} \vec{n} d\vec{s} = \int_P \vec{r} \vec{n} d\vec{s} = \int 0 d\vec{s} = 0$$

$$\int_S \vec{f} d\vec{s} = \int_S \vec{r} \vec{n} d\vec{s} = \int_S z d\vec{s} = z_T \cdot \mathcal{P}(S) =$$

$$\vec{n} = (0, 0, 1) \quad = h \cdot \pi h^2 = \pi h^3$$

2. način (z gaussonom)

$$\int \vec{f} d\vec{s} = \int_D \text{div} \vec{f} dV = \int_D (1+1+1) dV = 3\mathcal{P}(V) =$$

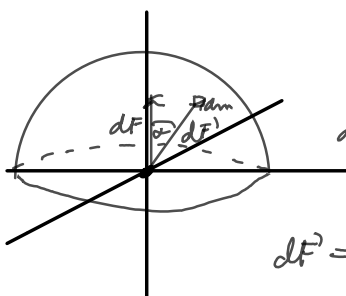
$$= 3 \cdot \frac{\pi h^2 \cdot h}{3} = \pi h^3$$

$$\int_{\partial D} = \int_S + \int_P$$

$$a > 0$$

Dobri privlačno silo med točkastim telesom z maso m_0 , ki je v izhodišču in ploskvijo $x^2 + y^2 + z^2 = a^2 \geq 0$ z maso M

homogena



$$F = \frac{m_1 m_2}{r^2} G$$

$$dm = \rho dS$$

$$dF = \frac{m_0 dm}{r^2} G = \frac{m_0 \rho dS}{r^2} G$$

Rezultante bo kerala gor tvoj nes
zanima samo z komponente
 $dF = dF \cos \alpha$

$$x = a \cos \vartheta \cos \varphi$$

$$\alpha = \frac{\pi}{2} - \vartheta$$

$$y = a \cos \vartheta \sin \varphi$$

$$z = a \sin \vartheta$$

$$dF = \frac{G m_0 \rho}{r^2} \cos(\frac{\pi}{2} - \vartheta) dS$$

$$dS = a^2 \cos \vartheta d\vartheta d\varphi$$

$$\text{or prej } \sqrt{EG - F^2} = a^2 \cos \vartheta$$

$$F = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \frac{G m_0 \rho}{a^2} \sin \vartheta a^2 \cos \vartheta d\vartheta =$$

$$= G m_0 \rho 2\pi \int_0^{\frac{\pi}{2}} \sin \vartheta \cos \vartheta d\vartheta =$$

$$G m_0 \rho 2\pi \int_0^{\frac{\pi}{2}} \sin 2\vartheta d\vartheta = \frac{\pi}{2} G m_0 \rho \cos 2\vartheta \Big|_0^{\frac{\pi}{2}} = \oplus$$

$$= \pi G m_0 \rho = \frac{G m_0 \cdot \pi}{2 a^2}$$

$$\uparrow \rho = \frac{M}{P(S)} = \frac{M}{2\pi a^2}$$

$D \subseteq \mathbb{R}^3$ gaussesto območje (območje z gladiko robom) in prostornino V

$$a \in \mathbb{R}^3$$

Izračunaj:

$$\int_{\partial D} (\vec{r} \times \vec{a}) \times d\vec{s} = I$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{y} \cdot \vec{z})\vec{x}$$

$$I = \int_{\partial D} (\vec{r} \cdot d\vec{s}) \vec{a} - (\vec{a} \cdot d\vec{s}) \vec{r} =$$

$$d\vec{s} = \vec{n} \cdot dS$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n}) \vec{a} - (\vec{a} \cdot \vec{n}) \vec{r}) dS = \quad \begin{array}{l} a = (a_1, a_2, a_3) \\ r = (x, y, z) \end{array}$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n}) dS) (a_1, a_2, a_3) - (\vec{a} \cdot \vec{n} x, \vec{a} \cdot \vec{n} y, \vec{a} \cdot \vec{n} z) dS$$

$$= \left(\int_{\partial D} (\vec{r} \cdot \vec{n}) dS \right) \vec{a} - \underbrace{\int_{\partial D} (\vec{a} \cdot \vec{n} x, \vec{a} \cdot \vec{n} y, \vec{a} \cdot \vec{n} z) dS}_{\int_{\partial D} x \vec{a} \cdot \vec{n} dS}$$

$$\rightarrow \left(\int_{\partial D} (x \vec{a} \cdot \vec{n} dS, y \vec{a} \cdot \vec{n} dS, z \vec{a} \cdot \vec{n} dS) \right)$$

$$\int_{\partial D} (\vec{r} \cdot \vec{n}) dS = \int_{\partial D} \vec{r} \cdot d\vec{s} = \int_D \operatorname{div} \vec{r} dV =$$

$$= 3 \int_D dV = 3V$$

$$\int_{\partial D} x \vec{a} \cdot \vec{n} dS = \int_{\partial D} x \vec{a} \cdot d\vec{s} = \int_D \operatorname{div} (x \vec{a}) dV = a_1 V$$

$$\operatorname{div} x \vec{a} = \nabla(x a_1, x a_2, x a_3) = a_1$$

Podobno y in

$$I = 3V \vec{a} - (a_1 V, a_2 V, a_3 V) = 2 \vec{a} V$$

Dokazati da je za

$$\vec{f} = (2x \cos y - y^2 \sin x, 2y \cos x - x^2 \sin y, u)$$

$\int_K \vec{f} d\vec{r}$ enak za vse krivulje med
 $(0,0,0)$ in $(5,3,\pi)$
in izračunaj

$$\int_{K_0} \vec{f} d\vec{r} \quad \text{za } K_0: \vec{r}(t) = (\cos t, \sin t, t); \quad t \in [-2\pi, 2\pi]$$

Dokazujemo: \vec{f} je potencialno
(smo že našli na računalniku)

$$u = x^2 \cos y + y^2 \cos x + u z$$

↑
potential za \vec{f}

$$\int_{K_0} \vec{f} d\vec{r} = u(b) - u(a) = u(1, 0, 2\pi) - u(1, 0, -2\pi)$$

$$= 4 \cdot 2\pi - (-4 \cdot 2\pi) = 16\pi$$

$$I = \int_K u(x, y) (y dx + x dy)$$

$$\vec{f}(x, y) = (y, x)$$

$$\vec{g} = u \cdot \vec{f} = u(x, y) \cdot (y, x)$$

Radi bi da je \vec{g} potencijalno polje

$$\vec{g} = \text{grad } v = (v_x, v_y)$$

$$v_x = u(x, y) \cdot y \quad v_y = u(x, y) \cdot x$$

če odvajamo

eno po x eno po y mera pitajmo

$$v_{xy} = u(x, y) + u_y y \quad v_{yx} = u_x x + u(x, y)$$

$$\leadsto y u_y = x u_x$$

• To prajj lahko dobimo z greenom

• vložimo v ravnino v \mathbb{R}^3

$$t = x \cdot y$$

$$s = \frac{x}{y}$$

$$I = \int_K y u dx + x u dy + 0 dz$$

$$\vec{h} = (y u, x u, 0) \text{ je potencijalno} \Leftrightarrow \text{rot } \vec{h} = 0$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x} = y \frac{\partial}{\partial t} + \frac{1}{y} \frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t} \frac{\partial t}{\partial y} + \frac{\partial}{\partial s} \frac{\partial s}{\partial y} = x \frac{\partial}{\partial t} - \frac{x}{y^2} \frac{\partial}{\partial s}$$

$$x(y u_t + \frac{1}{y} u_s) - y(x u_t - \frac{x}{y^2} u_s) = 0$$

$$t u_t + s u_s - t u_t + s u_s = 2 s u_s \quad s > 0$$

$$u_s = 0 \Rightarrow u = u(t) \dots \text{ samo od } t$$

$$u = h(t)$$

adizem

$$u = h(x \cdot y) \quad \text{kjer } h \in C^1$$

$u, l \in C^2 \quad D \in \mathbb{R}^3$ z odsekoma gladkim robom

$\vec{e} = \mathbb{R}^3$ enotski:

$$\text{Smerni odvod: } \frac{\partial u}{\partial \vec{e}} = \text{gradu} \cdot \vec{e}$$

Opomba: $\frac{\partial u}{\partial (1,0,0)} = \frac{\partial u}{\partial x} \cdot \vec{e}$

\vec{n} ... zunanja enotska normala za ∂D

a) Dokaži:

$$\int_{\partial D} u \cdot \frac{\partial v}{\partial \vec{n}} dS = \int_D \text{gradu} \cdot \text{grad} v + u \Delta v \, dv$$

$$b) \int_{\partial D} \left(u \frac{dv}{\partial \vec{n}} - v \frac{du}{\partial \vec{n}} \right) dS = \int_D (u \Delta v - v \Delta u) \, dv$$

$$\int_{\partial D} u \frac{dv}{\partial \vec{n}} dS = \int_{\partial D} u \text{grad} v \cdot \underbrace{\vec{n} dS}_{\vec{\rho} dS} = \int_{\partial D} u \underbrace{\text{grad} dS}_{\vec{\rho} dS}$$

$$\text{div } \vec{f} = \text{div}(u \text{grad} v) = \text{div}(u v_x, u v_y, u v_z) =$$

$$= (u v_x)_x + (u v_y)_y + (u v_z)_z =$$

ko razpišemo

$$= \text{gradu} \cdot \text{grad} v + u \Delta v$$

b)

$$\int_{\partial D} \left(u \frac{dv}{\partial \vec{n}} - v \frac{du}{\partial \vec{n}} \right) dS = \int_{\partial D} u \frac{dv}{\partial \vec{n}} dS - \int_{\partial D} v \frac{du}{\partial \vec{n}} dS =$$

$$= \int_D \text{grad} u \cdot \text{grad} v + u \Delta v - \text{grad} v \cdot \text{grad} u - v \Delta u \, dv$$

$$f: \mathbb{R}^3 \rightarrow \mathbb{R} \text{ harmonična } (\Delta f = 0)$$

Dokaži da za $a > 0$ velja:

$$\frac{1}{4\pi a^2} \int_{S(0,a)} f dS = f(0,0,0) = I(a)$$

Nasvet: Parametriziraj $S(0,a)$ in nato s pomočjo odvajanja in gaussovega izreka dokaži trditv.

$$x = a \cos \vartheta \cos \varphi \quad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$y = a \cos \vartheta \sin \varphi \quad \varphi \in [0, 2\pi]$$

$$z = a \sin \vartheta \quad \leadsto \vec{r}(\varphi, \vartheta)$$

$$dS = |\vec{r}_\varphi \times \vec{r}_\vartheta| d\varphi d\vartheta = a^2 \cos \vartheta d\varphi d\vartheta$$

$$I(a) = \frac{1}{4\pi a^2} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{a^2 \cos \vartheta f(\vec{r}(\varphi, \vartheta))}_{g(a, \varphi, \vartheta)} d\vartheta$$

$$I'(a) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g}{\partial a} d\vartheta =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta (f_x(\vec{r}(\varphi, \vartheta)) \cdot x_a + f_y(\vec{r}(\varphi, \vartheta)) \cdot y_a + f_z(\vec{r}(\varphi, \vartheta)) \cdot z_a) d\vartheta =$$

$$f(x, y, z) = f(a \cos \vartheta \cos \varphi, a \cos \vartheta \sin \varphi, a \sin \vartheta)$$

$\forall \vec{r}$ se skriva a

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta (f_x \cos \varphi \cos \vartheta + f_y \sin \varphi \cos \vartheta + f_z \sin \vartheta) d\vartheta$$

komponente enotske
normalne \vec{n} na $S(0,1)$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta \text{grad } f \cdot \vec{n} d\vartheta = \quad dS = a^2 \cos \vartheta d\vartheta d\varphi$$

dS za $a=1$

$$= \frac{1}{4\pi} \int_{S(0,1)} \text{grad } f \cdot \vec{n} dS = \frac{1}{4\pi} \int_{S(0,1)} \text{grad } f d\vec{S} =$$

$$= \frac{1}{4\pi} \int_{K(0,1)} \underbrace{\text{div}(\text{grad } f)}_{\Delta f} dV = 0$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

$I'(a)$ je 0 $\Rightarrow I(a)$ je konstantna

$$I(a) = \frac{1}{4\pi a^2} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos \vartheta f(\vec{r}(\varphi, \vartheta)) d\vartheta$$

$$\lim_{a \rightarrow 0} I(a) = \frac{1}{4\pi} \lim_{a \rightarrow 0} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta f(\vec{r}(\varphi, \vartheta)) d\vartheta =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{a \rightarrow 0} \cos \vartheta f(\vec{r}(\varphi, \vartheta)) d\vartheta =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta f(0,0,0) d\vartheta =$$

$$= \frac{1}{4\pi} f(0,0,0) \cdot 4\pi = f(0,0,0) \quad \leftarrow P(S(0,1))$$

ker je $\cos \vartheta$
determinant

Holomorfne funkcije

$D \subseteq \mathbb{C}$ otvorena oblast, $f \in O(D)$

Kakve su f_1, f_2 i f_3 sa holomorfne

$$f_1(z) = \overline{f(z)}$$

$$f_2(z) = f(\bar{z})$$

$$f_3(z) = \overline{f(\bar{z})}$$

$$\text{holomorfnost: } \forall a \in D, \exists \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} =: f'(a)$$

$$f(x+iy) = u(x,y) + i v(x,y) \quad u = \operatorname{Re} f \quad v = \operatorname{Im} f$$

$$f \in O(D) \iff u_x = v_y \wedge u_y = -v_x$$

$u, v \in C^1(D)$

$$f_1(x+iy) = \underbrace{u(x,y)}_{u_1(x,y)} - i \underbrace{v(x,y)}_{v_1(x,y)}$$

$$\begin{array}{ll} u_{1,x} = u_x & u_{1,y} = u_y \\ v_{1,x} = -v_x & v_{1,y} = -v_y \end{array}$$

$$u_{1,x} = u_x = v_y = -v_{1,y}$$

$$u_{1,y} = -v_y$$

$$\Rightarrow \begin{array}{l} v_y = -v_y \\ v_x = -v_x \end{array} \quad \text{in } \dots$$

\Downarrow

$$v_x = 0 = v_y$$

$$u_x = u_y = 0$$

???

u, v konstantni;

f_1 ni holomorfne

$$f_2(x+yi) = u_2(x,y) + i v_2(x,y); \quad \left. \begin{array}{l} \\ \end{array} \right\} f_2 \text{ na } \overline{D}$$

$$\parallel$$

$$u(x,-y) + i v(x,-y)$$

~~$$u_{2x} = u_x \quad u_{2y} = u_y \cdot (-1) = -u_y$$

$$v_{2x} = v_x \quad v_{2y} = -v_y$$~~

$$u_{2x}(x,y) = u_x(x,-y) \quad u_{2y}(x,y) = -u_y(x,-y)$$

$$v_{2x}(x,y) = v_x(x,-y) \quad v_{2y}(x,y) = -v_y(x,-y)$$

$$u_{2x}(x,y) = u_x(x,-y) = v_y(x,-y) = -v_{2y}(x,y)$$

Torej f_2 ni holomorfe, razen če so ~~u, v~~ ^{u, v} konstanti

$$f_3(x+yi) = u_3(x,y) + i v(x,y) \Rightarrow$$

$$u_3(x,y) = u(x,-y)$$

$$v_3(x,y) = -v(x,-y)$$

$$(u_3)_x(x,y) = u_x(x,-y)$$

$$v_{3x}(x,y) = v_x(x,-y)$$

$$(u_3)_y(x,y) = u_y(x,-y)$$

$$v_{3y}(x,y) = +v_y(x,-y)$$

$$(u_3)_x = (v_3)_y \Leftrightarrow$$

$$u_x(x,-y) = +v_y(x,-y) \Leftrightarrow u_x = v_y \quad \checkmark$$

$$u_{3y} = -v_{3x} \Leftrightarrow$$

$$-u_y(x,-y) = -v_x(x,-y) \Leftrightarrow v_y = u_x \quad \dots$$

je holomorfe

Dokaži da obstaja $f \in O(\mathbb{C})$ za katero
je $u(x,y) = x^3 - 3xy^2$ in jo določi

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy = -v_x$$

$$v = 3 \int (x^2 - y^2) dy = 3x^2y - y^3 + C(x)$$

$$v = 6 \int xy dx = 3x^2y + D(y)$$

$$v = 3x^2y - y^3 + C$$

"unija členov"

$$f(x+yi) = x^3 - 3xy^2 + i(3x^2y - y^3 + C)$$

$$f(z) = z^3 + iC \leftarrow \text{z ugibanjem}$$

Brez navdiha:

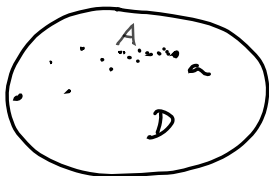
$$z = x + yi \Rightarrow x = \frac{z + \bar{z}}{2} \quad y = \frac{z - \bar{z}}{2i}$$

DN: ugotovi natri

Imenitna trditev:

$D \subseteq \mathbb{C}$ neka območje, $f, g \in \mathcal{O}(D)$

funkciji ki se ujemata na množici A s
sklepiščem v D



v A obstaja zaporedje
 $(a_n)_n$ $a_n \in A$ $a_n \neq a$
 $\lim_{n \rightarrow \infty} a_n = a$

$$\Rightarrow f = g$$

$$u = x^3 - 3xy^2 \quad v = 3x^2y - y^3 + iC$$

če najdemo $g \in \mathcal{O}(C)$ ki se z f ujemata
na \mathbb{R} , po trditvi: T sledi $f = g$

$$x \in \mathbb{R} \quad f(x) = f(x + i0) = u(x, 0) + v(x, 0) = x^3 + iC$$

$$\Rightarrow f(z) = z^3 + iC$$

Še imenitnejši premislek:

$$u(x, y) \rightsquigarrow f = ?$$

$$x \in \mathbb{R} : f(x) = u(x, 0) + i \underset{?}{v(x, 0)} \xrightarrow{x \rightarrow z} f(z) =$$

$$v(x, 0) = \int_{\text{CRS}} v_x(x, 0) dx = - \int u_y(x, 0) dx$$

$$u_y = -6xy$$

$$v(x, 0) = - \int -6x \cdot 0 dx = +C$$

$$\Rightarrow f(x) = x^3 + iC \Rightarrow f(z) = z^3 + iC$$

Podobno

$$u(x, 0) = \int u_x(x, 0) dx = \int v_y(x, 0) dx$$

$$u(x,y) = e^x (\cos(ky) + \sin(ky))$$

Dokaži k da je $u = \operatorname{Re} f$ za $f \in \mathcal{O}(\mathbb{C})$

Če je $u = \operatorname{Re} f \Rightarrow$

$$\begin{aligned} u_x &= v_y & u_{xx} &= v_{yx} \\ u_y &= -v_x & u_{yy} &= -v_{xy} \end{aligned} \Rightarrow u_{xx} = -u_{yy}$$

Podobno

$$v_{xx} = -v_{yy}$$

$$\Rightarrow \Delta u = 0$$

$$u_{xx} = e^x (\cos(ky) + \sin(ky)) = u$$

$$u_y = e^x (-k \sin(ky) + k \cos(ky))$$

$$u_{yy} = k^2 e^x (-\cos(ky) - \sin(ky)) = -k^2 u$$

$$\Delta u = e^x (1 - k^2) (\cos(ky) + \sin(ky)) = 0 \text{ za } k = \pm 1$$

$$\Rightarrow k = \pm 1$$

$$v(x,0) = - \int u_y(x,0) dx = - \int e^x k dx = -k e^x + C$$

$$f(x) = u(x,0) + i v(x,0) = e^x - i k e^x + C = e^x (1 - i k) + C$$

$$f(z) = e^z (1 \pm i) + i C$$