

Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

\uparrow
 $\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$FV(f)$ konvergira k f , če je f zvezna v x ,
če pa n: , pa konvergira k $\frac{f(x^-) + f(x^+)}{2}$

$$\int x \cos(nx) dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int x \sin(nx) dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

$$f \text{ sade} \Rightarrow b_n = 0$$

$$f \text{ l:h} \Rightarrow a_n = 0$$

$$\int x^2 \cos(nx) dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx) dx = -\frac{x^2}{n} \cos(x) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

(2)

$f(x) = |x|$ razvij v Fourierovo vrsto in
sestej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \pi^2 = \frac{\pi^2}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx =$$

sodost

$$= \frac{2}{\pi} \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right) \Big|_0^\pi = \quad \bigoplus$$

$$= \frac{2}{\pi} \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx \stackrel{\text{lhost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{(-1)^n - 1}{n^2} \right) \cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2} \cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi} \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = -\frac{\pi}{2} \cdot \frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatekno

$$\sum = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = ? \quad S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

S ... l.h.
S' ... ostalo

$$= S + \frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) =$$

$$= S + \frac{1}{4} \sum = \sum$$

$$S = \frac{3}{4} \sum$$

$$\sum = \frac{4}{3} S = \frac{\pi^2}{6}$$

③

$$f(x) = \max(\cos x, 0)$$

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos(x) dx = \frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) \cdot \cos(nx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (\cos x \cdot \cos nx) + 2 \int_{\frac{\pi}{2}}^{\pi} \cos(n x) dx \\ &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{2} (\cos((n+1)x) + \cos((n-1)x)) dx \\ &\quad \text{... even } n \\ &= \frac{1}{\pi} \left[\frac{1}{n+1} \sin((n+1)\frac{\pi}{2}) + \frac{1}{n-1} \sin((n-1)\frac{\pi}{2}) \right] \end{aligned}$$

$$n=4k: \frac{1}{\pi} \frac{1}{n+1} \cdot 1 + \frac{1}{n-1} (-1)^{\frac{n}{2}} = \frac{-2}{n^2-1} \frac{1}{\pi} \frac{-2}{16k^2-1} \cdot \frac{1}{\pi}$$

n l.h.: 0

$$\begin{aligned} n &= 4k+2: \frac{1}{\pi} \frac{1}{4k+3} \sin \frac{3\pi}{2} + \frac{1}{4k+1} \sin \frac{\pi}{2} = \\ &= \frac{1}{\pi} \left(-\frac{1}{4k+3} + \frac{1}{4k+1} \right) \end{aligned}$$

$$= \frac{1}{\pi} \left\{ \begin{array}{l} 0 ; n \text{ l.h.o} \\ \frac{(-1)^{m+1} \cdot 2}{(2n)^2 - 1} ; n = 2m \end{array} \right.$$

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx = \frac{1}{\pi} + \frac{\sin(2x)}{2} \Big|_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \end{aligned}$$

$$b_n = 0 \text{ ker } f \text{ sode}$$

$$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2} \cos(x) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cdot 2}{(2m)^2 - 1} \cos(2mx)$$

$$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} S_1 \Rightarrow$$

$$S_1 = \left(\frac{1}{2} - \frac{1}{\pi} \right) \cdot \frac{\pi}{2} (-1) = \frac{1}{2} - \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2 - 1} \cdot \cos(m\pi) =$$

$$= \frac{1}{\pi} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \stackrel{\text{"}}{S_2}$$

$$S_2 = -\frac{1}{\pi} \cdot \frac{\pi}{2} = -\frac{1}{2}$$

(4)

$$f(x) = x^2 \quad f: [0, \pi] \rightarrow \mathbb{R}$$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v sinusno FV in skiciraj njen graf

1) ce f razbirimo do sade funkcije

$$f_s: [-\pi, \pi] \rightarrow \mathbb{R}$$

$$x < 0 \Rightarrow f_s(x) = f(-x)$$

$$FV_{\cos}(f)(x) = FV(f)(x)$$

$$2) f_s: [-\pi, \pi] \rightarrow \mathbb{R} \quad f_s(x) = -f(-x)$$

$$FV_{\sin}(f)(x) = FV(f_s)(x)$$

$$\Rightarrow f_s(x) = x^2$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \int_0^\pi x^2 dx = \frac{\pi^2}{3}$$

$$b_n = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

Poglejmo:

$$\int x^2 e^{inx} =$$

$$u = x^2 \quad dv = e^{inx} dx$$

$$du = 2x dx \quad v = \frac{1}{in} e^{inx}$$

$$\boxed{\begin{aligned} \int x e^{inx} dx \\ -\frac{i \cdot x \cdot e^{inx}}{n} + \frac{e^{inx}}{n^2} \end{aligned}}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \int 2x \frac{e^{inx}}{in} dx = \text{vad proj}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \frac{2}{in} \left(\frac{ix e^{inx}}{n} + \frac{e^{inx}}{n^2} \right) + C$$

$$= e^{inx} \left(\frac{ix^2}{n} + \frac{2x}{n^2} + \frac{2i}{n^3} \right)$$

$\cos nx + i \sin nx$

$$\int x^2 \cos(nx) dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx) dx = -\frac{x^2}{n} \cos(nx) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

$$a_n = \frac{1}{\pi} \left(\frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) + \frac{2}{n^3} \sin(nx) \right) \Big|_{-\pi}^{\pi} =$$

$$= \frac{2}{\pi} \left(\frac{2\pi}{n^2} (-1)^n \right) = \frac{4(-1)^n}{n^2}$$

$$FV_{\cos}(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_0(x) \sin(nx) dx = \frac{2}{\pi} \underbrace{\int_0^{\pi} f_0(x) \sin(nx) dx}_{= f(x) \text{ on } [0, \pi]} = x^2 \\
 &= \frac{2}{\pi} \left[-\frac{x^2}{n} \cos(nx) + \frac{2}{n^2} \sin(nx) + \frac{2}{n^3} (\cos(nx)) \right] \Big|_0^{\pi} \\
 &= \frac{2}{\pi} \left(-\frac{\pi^2}{n} (-1)^n + \frac{2}{n^3} \left(-1^n - \frac{2}{n^3} \right) \right)
 \end{aligned}$$

$$FV_{\sin n}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left((-1)^{n+1} \frac{\pi^2}{n^2} + \frac{2(1 - (-1)^n)}{n^3} \right) \sin(nx)$$

$f(x) = x(n-x)$ \rightarrow $ax^2 + bx + c$

6

$$f(x) = \sin^3 x \quad \text{reduz } v \quad FV$$

1. cd premissek:

$$f(x) = \sin 3x \quad j \in \mathbb{Z} \in FV$$

$$b_2 = 1, \text{ odd: } \leq 0 \quad 0$$

$$f(x) = \sin^2 x = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) =$$

$$= \frac{1}{2} (\cos(0) - \cos(2x)) = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$f(x) = \sin^3 x \cdot \sin x = \frac{1}{2} \sin x - \frac{1}{2} \cos 2x \sin x =$$

$$= \frac{1}{2} \sin x - \frac{1}{2} \cdot \frac{1}{2} (\sin(3x) - \sin(x)) =$$

$$= \frac{3}{4} \sin x - \frac{1}{4} \sin(3x)$$

polynom
f_p(sin x, cos x) \rightarrow končna Fourierova vrsto

$$p \in \mathbb{R}[x, y]$$

$a > 0$ kvadratnej tečnosti homogenega

like astroid



$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

$$x_T = \frac{\int x dm}{m(K)} = \frac{\int x \rho ds}{\int \rho ds} = l(K)$$

$\vec{r}(t) \dots$ parametrizacija

$$ds = |\dot{\vec{r}}(t)| dt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

$u \dots$ skalarne koje

$$\int_K u ds = \int_a^b u(\vec{r}(t)) |\vec{r}'(t)| dt$$

$$x = a \cos^3 t$$

$$y = a \sin^3 t$$

$$t \in [0, \frac{\pi}{2}] \leftarrow \begin{array}{l} \text{ker jo vse pospino} \\ (\text{gledomo pravilne}) \end{array}$$

$$\vec{r}(t) = (-3a \cos^2 t \cdot \sin t, 3a \sin^2 t \cos t)$$

$$|\vec{r}'| = 3a \sqrt{\cos^4 t + \sin^4 t + \sin^4 t \cos^2 t} = \\ = 3a \cos t \sin t$$

$$l(K) = \int_0^{\frac{\pi}{2}} |\vec{r}'(t)| dt = \int_0^{\frac{\pi}{2}} 3a \cos t \sin t dt =$$

$$u = \sin t dt$$

$$du = \cos t dt$$



$$= 3a \int_0^1 u du = \frac{3}{2} a$$

$$\int_K x ds = \int_0^{\frac{\pi}{2}} a \cos^3 t \cdot 3a \cos t \sin t dt =$$

$$\cos t = u \quad du = -\sin t$$

$$= 3a^2 \int_0^1 u^4 du = \frac{3}{5} a^2$$

$$y_T = \frac{3}{5} a^2$$

$$a > 0 \quad \alpha \in [0, 2\pi]$$

$$\kappa = S(0, a) \cap \Pi$$

$$\Pi: y = x \tan \alpha$$

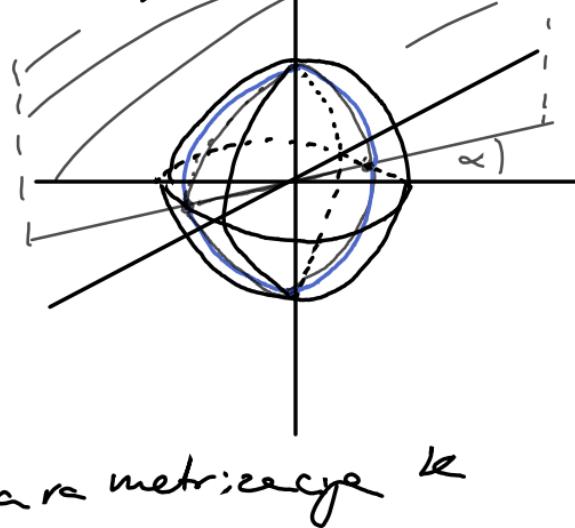
$$I: \int_{\kappa} (y-z) + (z-x) dy + (x-y) dz$$

$$\vec{f}(x, y, z) = (y-z, z-x, x-y) \quad \begin{matrix} \text{vektor} \\ \text{pa\ddot{g}e} \end{matrix}$$

$$I = \int_{\kappa} \vec{f} d\vec{r}$$

$$\vec{r} = (x, y, z) \Rightarrow d\vec{r}$$

$$\int_{\kappa} \vec{f} d\vec{r} = \int_a^b \vec{f}(\vec{r}(t)) \dot{\vec{r}}(t) dt$$



Parametrizace κ

$$x = a \cos \alpha \cdot \cos \vartheta$$

$$y = a \sin \alpha \cdot \cos \vartheta$$

$$z = a \sin \vartheta$$

$$\vec{r}(\vartheta) = (a \cos \alpha \cos \vartheta, a \cos \alpha \sin \vartheta, a \sin \alpha)$$

$$\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\dot{\vec{r}}(\vartheta) = (-a \sin \alpha \cos \vartheta, -a \sin \alpha \sin \vartheta, a \cos \alpha)$$

$$\vec{f}(\vec{r}(\vartheta)) = (\cos \vartheta \sin \alpha - \sin \vartheta, \sin \vartheta - \cos \vartheta \cos \alpha, \cos \vartheta \sin \alpha - \cos \vartheta \cos \alpha)$$

$$\vec{f} \cdot \dot{\vec{r}} = a^2 (-\cos \alpha \cos \vartheta \sin \alpha \sin \vartheta + \sin^2 \vartheta \cos \alpha \cos \alpha)$$

$$- \sin^2 \vartheta \sin \alpha + \cos \vartheta \cos \alpha \sin \vartheta \sin \alpha + \cos^2 \vartheta \sin \alpha - \cos^2 \vartheta \cos \alpha) =$$

$$= a^2 (\cos 2\vartheta \sin \alpha - \cos 2\vartheta \cos \alpha) \cancel{x}$$

$$= a^2 (\cos \alpha - \sin \alpha)$$

ϑ

$$\int_{\kappa} \vec{f} d\vec{r} = \int_0^{2\pi} a^2 (\cos \alpha - \sin \alpha) d\vartheta =$$

$$2\pi a^2 (\cos \alpha - \sin \alpha)$$

↑ pri. orientaci:



$$a, b, c \in \mathbb{R}$$

$$I = \int_S a dy dz + b dx dz + c dx dy$$

$$S = \{(x, y, z) ; x^2 + y^2 + z^2 = 1, x, y, z \geq 0\}$$

$$I = \int_S \vec{f} d\vec{s}$$

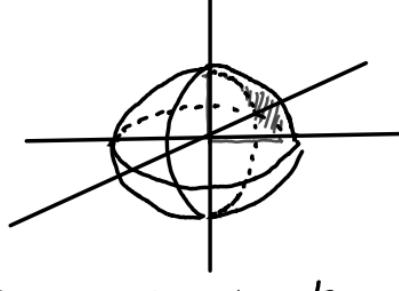
$$\vec{f} = (a, b, c)$$

$$\int_S (\vec{f} \cdot \vec{N}) dS \quad \begin{matrix} \text{orientacija} \\ \text{usmerjena s smrjo normale} \end{matrix} \quad \begin{matrix} \text{plaskve je} \\ \text{smer se ujem} \end{matrix}$$

$$\int_S \vec{f} d\vec{s} = \int_0^\pi \vec{f}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

$$\vec{r} = \vec{r}(u, v)$$

\uparrow smer se ujema
s predpisano orientacijo



Sferične koordinate

$$x = \cos \vartheta \cos \varphi \quad \vartheta \in [0, \frac{\pi}{2}]$$

$$y = \cos \vartheta \sin \varphi \quad \varphi \in [0, \pi]$$

$$z = \sin \vartheta$$

$$\vec{r}(\vartheta, \varphi) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta)$$

$$\vec{r}_\vartheta = (-\sin \vartheta \cos \varphi, -\sin \vartheta \sin \varphi, \cos \vartheta)$$

$$\vec{r}_\varphi = (-\cos \vartheta \sin \varphi, \cos \vartheta \cos \varphi, 0)$$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2 \vartheta \cos \varphi, \cos^2 \vartheta \sin \varphi,$$

$$-\sin \vartheta \cos \vartheta \cos \varphi - \cos \vartheta \sin \vartheta \sin \varphi) =$$

$$= (-\cos^2 \vartheta \cos \varphi, \cos^2 \vartheta \sin \varphi, -\sin \vartheta \cos \vartheta) =$$

$$= -\cos \vartheta \underbrace{(\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta)}$$

$$\stackrel{\frac{\pi}{2}}{\stackrel{\frac{\pi}{2}}{\stackrel{\pi}{\curvearrowright}}}$$

$$\int_S \vec{f} d\vec{s} = \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\frac{\pi}{2}} \vec{f}(-\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta) d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (-a \cos \vartheta \cos \varphi + b \cos \vartheta \sin \varphi - c \sin \vartheta) d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} (-a \cos \vartheta \cos \varphi + b \sin \vartheta \sin \varphi) \frac{\pi}{4} - \frac{c}{2} \vartheta =$$

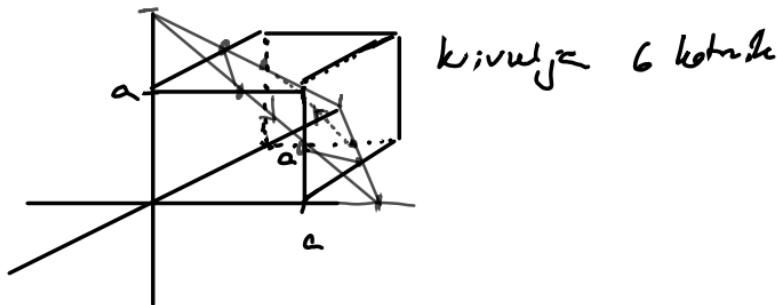
$$= \left(-\frac{\pi}{4} a \sin \vartheta \cos \varphi + \frac{\pi}{4} b \sin \vartheta \sin \varphi - \frac{c}{2} \vartheta \right) \Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4} a - \frac{\pi}{4} b - \frac{\pi}{4} c = -\frac{\pi}{4} (a + b + c)$$

070

$$\vec{f}(x,y,z) = (y^2-z^2, z^2-x^2, x^2-y^2)$$

Izračunaj cirkulacijo f vzdolž
preseka rba kocke $[0,a]^3$ in
ravnine $x+y+z = \frac{3a}{2}$



cirkulacija... integral vektorskega polja
vzdolž sklenjene krivulje

K_1 :

$$x=x$$

$$y=0$$

$$z = \frac{3a}{2} - x$$

$$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$$

$$\int_{K_1} \vec{f} d\vec{r}$$

$$\vec{r}(x) = (1, 0, -1)$$

$$\vec{f}(r(x)) = (0 - (\frac{3}{2}a - x)^2; (\frac{3}{2}a - x)^2 - x^2, x^2 - 0)$$

$$\vec{f} \cdot \vec{r} = -(\frac{3}{2}a - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$$

$$\int_{\frac{3a}{2}}^a (-(\frac{3}{2}a - x)^2 - x^2) dx =$$

$$-\frac{1}{3} (\frac{3}{2}a - x)^3 \Big|_{\frac{3a}{2}}^a - \frac{1}{3} x^3 \Big|_{\frac{3a}{2}}^a =$$

$$= -\frac{1}{3} \cancel{\frac{a^3}{8}} + \cancel{\frac{1}{3} a^3} - \cancel{\frac{1}{3} a^3} + \cancel{\frac{1}{3} \frac{a^3}{8}} = 0$$

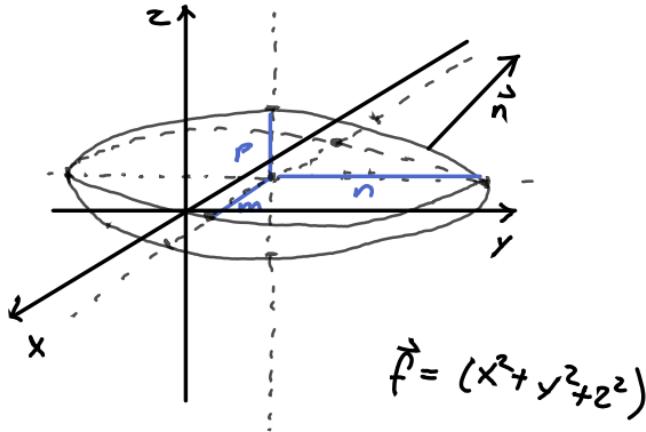
$$m, n, p > 0$$

$$a, b, c \in \mathbb{R}$$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanje stran te ploskve



$$I = \int_S \vec{f} d\sigma = \int_D \operatorname{div} \vec{f} dV \quad \text{gaussov zakon}$$

Normale mera biti
zunanje

$$\int_D (2x + 2y + 2z) dV = 2x_T + 2y_T + 2z_T$$

$$x_T = \int_D x dV = x_T \cdot V(D) = a V(D)$$

$$\text{enotake kugle } B: V(B) = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi$$



$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3} \pi$$

D dobimo ce B raztegnemo za faktor
m v smere: x, n, p v drugih duvh

B opisemo:

$$\begin{aligned} x &= r \cos \varphi \cos \rho \\ y &= r \cos \varphi \sin \rho \\ z &= r \sin \varphi \end{aligned} \quad r \in [0, 1]$$

D opisemo:

$$\begin{aligned} x &= m r \cos \varphi \cos \rho \\ y &= n r \cos \varphi \sin \rho \\ z &= p r \sin \varphi \end{aligned}$$

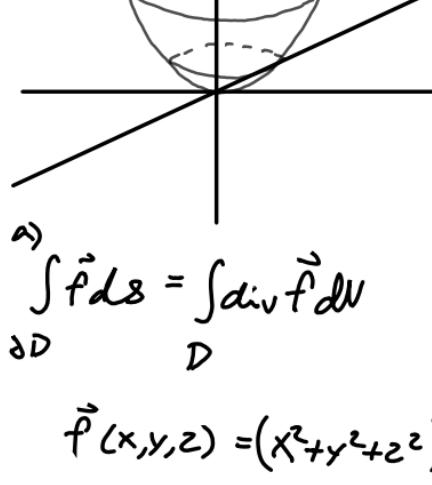
$$I = 2 \frac{4}{3} \pi m \cdot n \cdot p (a+b+c)$$

$$\vec{f}(\vec{r}) = |\vec{r}|^2 \vec{r} \quad b > 0$$

izracunaj pretok \vec{f} skazi

a) rob območja $D = \{(x,y,z); 2z \geq x^2 + y^2, z \leq b\}$

b) plasko $2z = x^2 + y^2, z \leq b$



Pretok skazi $S:$

$$\int \vec{f} dS$$

a) $\int \vec{f} dS = \int_{\partial D} \operatorname{div} \vec{f} dV$

$$\begin{aligned} \vec{f}(x,y,z) &= (x^2 + y^2 + z^2)(x, y, z) = \\ &= (x(x^2 + y^2 + z^2), y(x^2 + y^2 + z^2), z(x^2 + y^2 + z^2)) = \\ &= (x^3 + x^2y^2 + x^2z^2, y^3 + y^2x^2 + y^2z^2, z^3 + z^2x^2 + z^2y^2) \\ \operatorname{div} \vec{f} &= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + y^2 + x^2 + 3z^2 \\ &= 5x^2 + 5y^2 + 5z^2 \end{aligned}$$

$$\int_D \operatorname{div} \vec{f} dV = 5 \int_D (x^2 + y^2 + z^2) dV$$

$$\begin{aligned} x &= r \cos \varphi & r^2 &= x^2 + y^2 = 2z \\ y &= r \sin \varphi & z &= \frac{r^2}{2} \\ z &= z = \end{aligned}$$

$$\begin{aligned} &= 5 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^b (r^2 + z^2) r dz \\ &= 5 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \frac{1}{3} z^3 \right) \Big|_{\frac{r^2}{2}}^b dr \\ &= 10\pi \int_0^{\sqrt{2b}} \left(b r^3 + \frac{1}{3} b^3 - \frac{r^5}{2} - \frac{1}{6} r^6 \right) dr = \\ &= 10\pi \left(\frac{1}{4} b^4 r^4 + \frac{1}{3} b^3 r - \frac{1}{12} r^6 - \frac{1}{6} r^7 \right) \Big|_0^{\sqrt{2b}} = \\ &= 10\pi \left(\frac{1}{4} b^3 \cdot 4 + \frac{1}{3} b^3 \sqrt{2b} - \frac{1}{12} \cdot 8 b^3 - \frac{1}{6} \cdot 8 b^3 \sqrt{2b} \right) \\ &\quad \cancel{-} \text{ reverse} \\ &= 10\pi \left(b^3 + \frac{b^4}{3} - \frac{2}{3} b^3 - \frac{b^4}{12} \right) = \\ &= 10\pi \left(\frac{b^3}{3} + \frac{b^4}{4} \right) \end{aligned}$$

b) $\partial D = S \cup S_0$

$$\int_S \vec{f} dS = \int_{\partial D} \vec{f} dS - \int_{S_0} \vec{f} dS$$

$$\int_S \vec{f} dS = \int_{S_0} (\vec{f} \cdot \vec{n}) dS = \quad \vec{n} = (0, 0, 1)$$

$$= \int_{S_0} (2x^2 + 2y^2 + z^2) dS = \int_{S_0} (b x^2 + b y^2 + b^3) dS =$$

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$= b \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r (r^2 + b^2) dr = b \left(\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr + b^2 \int_{S_0} dS \right)$$

$$= b \left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr \right) =$$

$$= b \left(2\pi b^3 + \int_0^{2\pi} \frac{1}{4} \cdot 4 b^4 \right) =$$

$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \int_S \vec{f} dS = \frac{10}{3}\pi b^3 + \frac{10}{3}\pi b^4 - 2\pi b^4 - 2\pi b^3$$

$$P(S_0) = 2\pi b$$

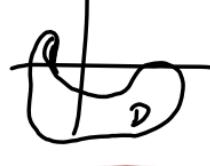
$$\text{to solve} \quad I = \int \frac{x dx - y dy}{x^2 + y^2}$$

$\bar{c} \in \mathbb{R}^2$ skiljerne kvaliteter;

- a) ki ne abkozi izhadisə
 - b) ki dəhədi izhadisə

Omejimo se na prímes: $K = dD \in D^{\mathbb{C}}$ odsekmou
grafikom ročním

5



$$\int\limits_{\partial D} P dx + Q dy = \int\limits_D (Q_x - P_y) dx dy$$

↑
 positive orientation
 green's formula

$$y = \frac{1}{x^2+y^2} \quad Q = x^2+y^2$$

$$Q_x = \frac{x+y}{(x^2+y^2)^{1/2}} = \frac{\sqrt{x^2+y^2}}{(x^2+y^2)^{1/2}}$$

$$r_y = \sqrt{x^2 + y^2}$$

$$(0,0) \in D$$

$$x^2 + y^2 \neq 0$$

devalj mejhon krcog
 $K(0, \varepsilon)$

-a lime green

$$\int (Q_x + P_y) dx dy = \int (Q_x - P_y) dx dy$$

$\Delta D'$ $\Delta D'$

$\int =$ \int_B

$$\delta D^+ \quad \delta K^- \quad \text{Polar} \\ \text{ket}^+$$

k^+ ponavadi

$$\delta D^+ = \delta K^+$$

$$c = -\frac{1}{2} \pi \sin y$$

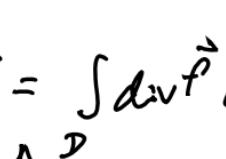
$$dx = E \cos y$$

$$\int_{-\pi}^{\pi} \frac{1}{2} (\cos^2 p + \sin^2 p) = 2\pi$$

$r_1, \dots, r_n \in \mathbb{R}^3$ vektori

$e_1, \dots, e_n \in \mathbb{R}$

$f(\vec{r}) = \sum_{i=1}^n \text{grad} \left(\frac{-e_i}{4\pi |r - r_i|} \right)$
izračunaj potok \vec{f} kroz zložen
planinski s. k. s površinom r_1, \dots, r_n



$r_1, \dots, r_n \in \text{Int}(D) \quad S = \partial D$

$$\int_D \vec{f} \cdot d\vec{s} = \int_D \text{div} \vec{f} dV$$

gaussov učinkovitost je veljača,
ker ker ima f singulirne točke v^D

$$\text{div}(\vec{f}(r)) = \sum_{i=1}^n \text{div}(\text{grad} \left(\frac{-e_i}{4\pi |r - r_i|} \right))$$

$$\boxed{\text{Spomnimo se: } \text{grad} \frac{1}{|\vec{r} - \vec{a}|} = -\frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3}}$$

$$\text{in } \text{div}(\text{grad} \frac{1}{|\vec{r} - \vec{a}|}) = 0$$

Iz D izrežemo kugle $K_i = K(r_i, \varepsilon)$

Tako da $\bar{K}_i \subset \text{Int}(D)$ in $K_i \cap K_j = \emptyset \quad \forall i \neq j$

$$D' = D - \left(\bigcup_{i=1}^n K_i \right)$$

Na D' uporabimo gaussov učink

$$\int_D f(\vec{r}) d\vec{s} = \int_{D'} \text{div} \vec{f} dV = 0$$

$$\partial D' = \partial D^+ \cup \left(\bigcup_{j=1}^n \partial K_i^- \right) \Rightarrow \int_{\partial D'} = \int_{\partial D^+} + \sum_{i=1}^n \int_{\partial K_i^-} = 0$$

$$\Rightarrow \int_{\partial D^+} = \sum_{i=1}^n \int_{\partial K_i^-} \dots$$

$$\int_{\partial K_i^-} \vec{f} d\vec{s} = \int_{K_i^-} \sum_{j=1}^n \text{grad} \left(\frac{e_j}{4\pi |r - r_j|} \right) dV$$

$$\sum_{j=1}^n \int_{K_i^-} \left(\frac{e_j}{4\pi} \frac{r - r_j}{|r - r_j|^3} \right) dV$$

$$d\vec{s} = \vec{n} d\vec{s}$$

$$\vec{n} = \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}$$

$$= \frac{e_i}{4\pi} \int_{K_i^-} \frac{|\vec{r} - \vec{r}_i|^2}{|\vec{r} - \vec{r}_i|^4} dV = \frac{e_i}{4\pi} \int_{K_i^-} \frac{dV}{|\vec{r} - \vec{r}_i|^2} =$$

$$\approx \frac{e_i}{4\pi \varepsilon^2} V(K_i^-) = \frac{e_i}{4\pi \varepsilon^2} 4\pi \varepsilon^2 = e_i$$

$$\Rightarrow \int_D f(\vec{r}) d\vec{s} = \sum_{i=1}^n e_i$$

$$I = \int_S (1+x^2) f(x,y,z) dy dz - 2xy f(x) dx dx - 3z \frac{\partial f}{\partial x} dy$$

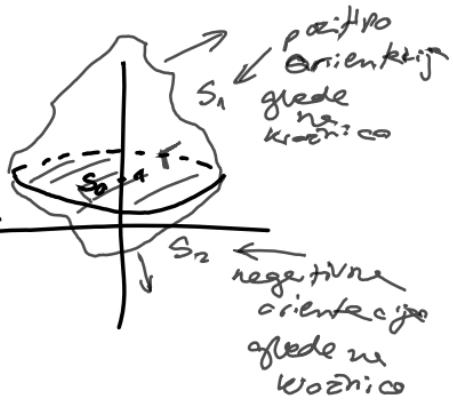
Dobře: $f \in C^1$ de I erak ~~a~~ vše pláškav
 S , kterého rob je kožnice $\{(\cos t, \sin t, 1)\}$
 Dobře to vzhled $\Rightarrow I$

$$\vec{f} = ((1+x^2)f(x), 2xy f(x), -3z)$$

$$\Rightarrow I = \int_S \vec{f} d\vec{s}$$

$S_1 \cup S_2$ je salenjene
 pláškav

$$S_1^+ \cup S_2^- = \partial D^+$$



Gauss na D:

$$\int_D \vec{f} d\vec{s} = \int_D \operatorname{div} \vec{f} dV$$

$$\int_D = \int_{S_1} - \int_{S_2} = 0 \Leftrightarrow \operatorname{div} \vec{f} dV = 0$$

\vec{f} je salo idlano $\Leftrightarrow \operatorname{div} \vec{f} = 0$ \Leftrightarrow $P(x)$ je
 obliké 3arctan $x + C$

$$\int_S \vec{f} d\vec{s} = \int_S ((1+x^2)(3 \arctan x + C), -2xy(3 \arctan x + C), -3z) d\vec{s}$$

$$= \int_{S_0} \vec{f} d\vec{s} =$$

$$\text{salog: } d\vec{s} = \vec{n} \cdot dS \quad \vec{n} = (0, 0, 1)$$

$$= \int_{S_0} -3z dS = -3 \int_{S_0} z dS = -3z_0 P(S_0) =$$

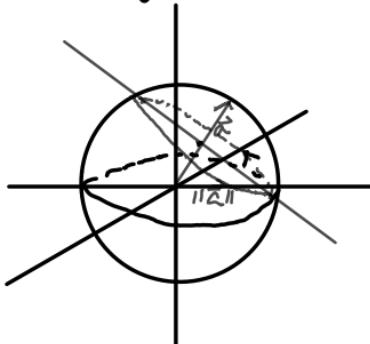
$$= -3 \cdot 1 \cdot \pi$$

Opanša: Lasko bi dokazal: da je
tadi div[†] potresen pa ga

$$\vec{\alpha} \neq \vec{0}, \\ \vec{b} \in \mathbb{R}$$

$$\vec{F}(\vec{r}) = (\vec{r} - \vec{\alpha}) \times (\vec{r} - \vec{b})$$

izmērītajā cirkulācijā \vec{F} veidojās
krūvīgā kārtība: $|\vec{r}| = |\vec{\alpha}|$ un $\vec{r} \cdot \vec{\alpha} = \frac{1}{2} |\vec{\alpha}|^2$



enešķķērīgā ravnīne:

$$\vec{r} \cdot \vec{n} = d \\ \text{normale}$$

ravnīne \vec{n} normālo $\vec{\alpha}$
in točkā $\vec{r} = \frac{\vec{\alpha}}{2}$

K je krustnīca

B nej bo krustīgā α \in radošā mī K = ∂B

$$\int\limits_{\partial B} f \, d\vec{r} = \int\limits_B \text{rot } \vec{f} \, d\vec{s}$$

$$\text{stāvoklis: } \vec{n} \\ \text{normāla: } B \ni \vec{n} = \frac{\vec{\alpha}}{|\vec{\alpha}|}$$

$$\text{rot } \vec{f} = \text{rot} \left(\underbrace{\vec{r} \times \vec{r}}_0 - \vec{\alpha} \times \vec{r} - \vec{r} \times \vec{b} + \vec{\alpha} \times \vec{b} \right) =$$

$$\boxed{\text{Vērtība: } \text{rot} (\vec{r} \times \vec{r}) = -2\vec{c}}$$

$$= -2\vec{\alpha} + 2\vec{b} + 0 = 2(\vec{b} - \vec{\alpha})$$

$$I = \int\limits_B 2(\vec{b} - \vec{\alpha}) \, d\vec{s} = 2 \int\limits_B (\vec{b} - \vec{\alpha}) \cdot \vec{n} \, d\vec{s} =$$

$$= \frac{2(\vec{b} - \vec{\alpha}) \cdot \vec{b}}{|\vec{\alpha}|} \cdot P(B)$$

$$P(B) = ?$$

polumeris:



$$b = \sqrt{|\vec{\alpha}|^2 - \frac{|\vec{\alpha}|^2}{4}} = \frac{\sqrt{3}}{2} |\vec{\alpha}|$$

$$P(B) = \pi b^2 = \cancel{\pi} \cdot \pi \cdot \frac{3}{4} |\vec{\alpha}|^2$$

$$I = 3\pi (\vec{b} - \vec{\alpha}) \vec{b} |\vec{\alpha}|$$

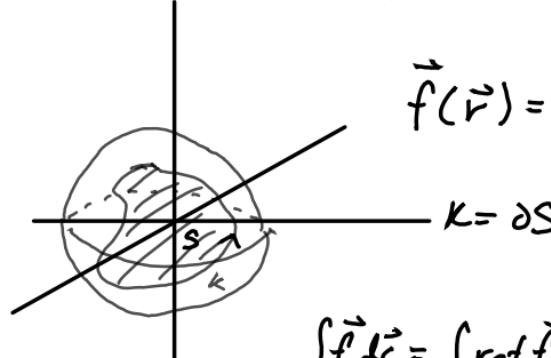
K ir \vec{b} katētēs \vec{b} katētēs b

K zelenjčena kugla

$$\text{na sferi } x^2 + y^2 + z^2 = 1$$

$$\text{izracunaj } I = \int_K \frac{dx + dy + dz}{(x^2 + y^2 + z^2)^2} = \int_K \vec{f} d\vec{r}$$

$$\vec{f} = \left(\frac{1}{(x^2 + y^2 + z^2)}, \frac{1}{(x^2 + y^2 + z^2)}, \frac{1}{(x^2 + y^2 + z^2)} \right)$$



$$\vec{f}(r) = (1, 1, 1) \propto r \vec{e}_S$$

$$\int_K \vec{f} d\vec{r} = \int_S \text{rot } \vec{f} d\vec{s} = \dots$$

obstaja boljši način

$$\int_K \vec{f} d\vec{r} = \int_K \underbrace{(1, 1, 1)}_{g(\vec{r})} d\vec{r} = \int_S \text{rot}(g(\vec{r})) d\vec{s} =$$

$$= \int_S 0 d\vec{s} = 0$$

Kaj je $K \neq S$

če je polje \vec{f} potencialno ($\vec{f} = \text{grad } u$)
in K kružuje \Rightarrow razlikom v a in koncem
 $v - b \Rightarrow \int_K \vec{f} d\vec{r} = u(b) - u(a)$

Poštedica: če je K sklopjena po integralu
vedeli $\vec{r} \in K$ od $\int_K \text{grad } u d\vec{r} = \vec{0}$

\vec{f} ni potencialno, ampak \vec{g} je potencialno
(ker $\text{rot } \vec{g} = 0$)

$$\vec{g} = \text{grad}(x + y + z) \quad I = \int_K \vec{g} d\vec{r} = 0$$

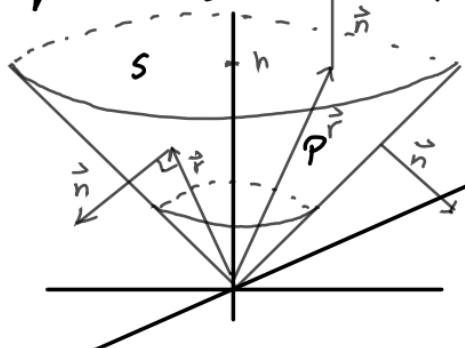
Opomba: $\int_K (x^2 + y^2 + z^2) (dx + dy + dz)$
na S,

konstantno
polje

$$h > 0$$

$$\vec{f}(\vec{r}) = \vec{r}$$

terčuňaj pretek \vec{P} skri plášť in
 ploškov súčasť: $x^2 + y^2 \leq z^2$ $z \in [0, h]$



$$\vec{r} \cdot \vec{n} = 0$$

$$\int_P \vec{f} d\Omega = \int_P \vec{f} \cdot \vec{n} d\Omega = \int_P \vec{r} \cdot \vec{n} ds = \int_P 0 ds = 0$$

$$\int_S \vec{f} d\Omega = \int_S \vec{r} \cdot \vec{n} ds = \int_S z ds = z_T \cdot P(S) = \\ \vec{n} = (0, 0, 1) = h \cdot \pi h^2 = \pi h^3$$

2. násob (z gaussova)

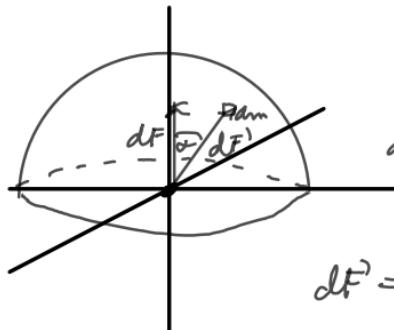
$$\int_D \vec{f} d\Omega = \int_D \operatorname{div} \vec{f} dV = \int_D (1+1+1) dV = 3 P(V) = \\ = 3 \cdot \frac{\pi h^2}{3} \cdot h = \pi h^3$$

$$\int_D = \int_S + \int_P$$

$a > 0$

Določi privlačno silo med telesom z maso m_1 , ki je v izhodišču in plaskujejo $x^2 + y^2 + z^2 = a^2 \geq 0$ z maso M

homogene



$$F = \frac{m_1 m_2}{r^2} G$$

$$dm = \rho d\Omega$$

$$dF = \frac{m_1 dm}{r^2} G = \frac{m_1 \rho d\Omega}{r^2} G$$

Rezultante bo ker ka gr. terj nas
čenime samo z komponento

$$dF = dF \cos \alpha$$

$$x = a \cos \vartheta \cos \varphi$$

$$\alpha = \frac{\pi}{2} - \vartheta$$

$$y = a \cos \vartheta \sin \varphi$$

$$z = a \sin \vartheta$$

$$dF = \frac{G m_1 \rho}{r^2} \cos(\frac{\pi}{2} - \vartheta) d\Omega$$

$$d\Omega = a^2 \cos \vartheta d\varphi d\vartheta$$

$$\text{ot proj z } \sqrt{EG - F^2} = a^2 \cos \vartheta$$

$$F = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \frac{G m_1 \rho}{a^2} \sin \vartheta \cos \vartheta d\vartheta =$$

$$= G m_1 \rho 2\pi \int_0^{\frac{\pi}{2}} \sin \vartheta \cos^2 \vartheta d\vartheta =$$

$$G m_1 \rho \pi \int_0^{\frac{\pi}{2}} \sin \vartheta \cos^2 \vartheta d\vartheta = \frac{\pi}{2} G m_1 \rho \cos^2 \vartheta \Big|_0^{\frac{\pi}{2}} =$$

$$= \pi G m_1 \rho = \frac{G m_1 \cdot \pi}{2 a^2}$$

$$\rho = \frac{M}{P(S)} = \frac{M}{2\pi a^2}$$

$D \subseteq \mathbb{R}^3$ gaussasto območje (območje z gladkimi robami) in prosternino V

$$a \in \mathbb{R}^3$$

Izračunej:

$$\int_D (\vec{r} \times \vec{a}) \times d\vec{s} = I$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z}) \vec{y} - (\vec{y} \cdot \vec{z}) \vec{x}$$

$$I = \int_D (\vec{r} \cdot d\vec{s}) \vec{a} - (\vec{a} \cdot d\vec{s}) \vec{r} =$$

$$d\vec{s} = \vec{n} \cdot dS$$

$$= \int_D ((\vec{r} \cdot \vec{n}) \vec{a} - (\vec{a} \cdot \vec{n}) \vec{r}) dS = \quad a = (a_1, a_2, a_3)$$

$$= \int_D ((\vec{r} \cdot \vec{n}) dS) (a_1, a_2, a_3) - (\vec{a} \cdot \vec{n} x, \vec{a} \cdot \vec{n} y, \vec{a} \cdot \vec{n} z) dS \quad r = (x, y, z)$$

$$= \left(\int_D (\vec{r} \cdot \vec{n}) dS \right) \vec{a} - \underbrace{\int_D (\vec{a} \cdot \vec{n} x, \vec{a} \cdot \vec{n} y, \vec{a} \cdot \vec{n} z) dS}_{x \vec{a} \cdot \vec{n} dS}$$

$$\hookrightarrow \left(\int_D (x \vec{a} \cdot \vec{n} dS, y \vec{a} \cdot \vec{n} dS, z \vec{a} \cdot \vec{n} dS) \right)$$

$$\int_D (\vec{r} \cdot \vec{n}) dS = \int_D \vec{r} d\vec{s} = \int_D \operatorname{div} \vec{r} dV =$$

$$= 3 \int_D dV = 3V$$

$$\int_D x \vec{a} \cdot \vec{n} dS = \int_D x \vec{a} d\vec{s} = \int_D \operatorname{div}(x \vec{a}) dV = a_1 V$$

$$\operatorname{div} x \vec{a} = \nabla(x a_1, x a_2, x a_3) = a_1$$

Podelno y in

$$I = 3V \vec{a} - (a_1 V, a_2 V, a_3 V) = 2 \vec{a} V$$

Dokazati da je \mathbf{z}

$$\vec{f} = (2x \cos y - y^2 \sin x, 2y \cos x - x^2 \sin y, u)$$

$\int_{\kappa} \vec{f} d\vec{r}$ enak za vse krvulje med
(0,0,0) in (5,3,7)
in izračunaj

$$\int_{K_0} \vec{f} d\vec{r} \simeq K_0: \vec{r}(t) = (\cos t, \sin t, t); \quad t \in [-2\pi, 2\pi]$$

Dokazujmo: \vec{f} je potencial no
(smo že naredili na enakov)

$$u = x^2 \cos y + y^2 \cos x + u z$$

\vec{U} potencial za \vec{f}

$$\begin{aligned} \int_{L_0} \vec{f} d\vec{r} &= u(b) - u(a) = u(1, 0, 2\pi) - u(1, 0, -2\pi) \\ &= 4 \cdot 2\pi - (-4 \cdot 2\pi) = 16\pi \end{aligned}$$

$$I = \int_K u(x, y)(y dx + x dy)$$

$$\vec{f}(x, y) = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\vec{g} = u \cdot \vec{f} = u(x, y) \cdot y, u(x, y) \cdot x$$

Radi bude je \vec{g} potencijalne projekcije

$$\vec{g} = \text{grad } v = (v_x, v_y)$$

$$v_x = u(x, y) \cdot y \quad v_y = u(x, y) \cdot x$$

če odvajamo

eno po x eno pa po y mora postati celo

$$v_{xy} = u_{xx} + u_{yy} \quad v_{yx} = u_{xy} + u_{yy}$$

$$\Rightarrow y u_y = x u_x$$

- Ta projekcija lahko dobimo z greenom

- vložimo v ravnine \mathbb{R}^3

$$t = x \cdot y$$

$$s = \frac{x}{y}$$

$$I = \int_K y u dx + x u dy + 0 dz$$

$$\vec{h} = (y u, x u, 0) \text{ je potencijalno} \Leftrightarrow \text{rot } \vec{h} = 0$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x} = y \frac{\partial}{\partial t} + \frac{1}{y} \frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t} \frac{\partial t}{\partial y} + \frac{\partial}{\partial s} \frac{\partial s}{\partial y} = x \frac{\partial}{\partial t} - \frac{x}{y^2} \frac{\partial}{\partial s}$$

$$x(y u_t + \frac{1}{y} u_s) - y(x u_t - \frac{x}{y^2} u_s) = 0$$

$$tu_t + su_s - tu_t + su_s = 2su_s \quad s > 0$$

$$u_s = 0 \Rightarrow u = u(t) \dots \text{same od } t$$

$$u = h(t) \quad \text{odvisno}$$

$$u = h(x \cdot y) \quad \text{kjer } h \in C^1$$

$u, v \in C^2$ $D \subseteq \mathbb{R}^3$ z odsekama
glatkim robom

$\vec{e} = \mathbb{R}^3$ enotski:

$$\text{Smerni odvod: } \frac{\partial u}{\partial \vec{e}} = \text{grad } u \cdot \vec{e}$$

$$\text{Opamka: } \frac{\partial u}{\partial (1,0,0)} = \frac{\partial u}{\partial x} \cdot \vec{e}$$

\vec{n} zunanje enotske normale za ∂D

a) Dokaz:

$$\int_D u \cdot \frac{\partial v}{\partial \vec{n}} dS = \int_D \text{grad } u \cdot \text{grad } v + u \Delta v dv$$

$$b) \int_D \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \right) dS = \int_D (u \Delta v - v \Delta u) dv$$

$$\int_D u \frac{\partial v}{\partial \vec{n}} dS = \int_D u \text{grad } v \cdot \vec{n} dS = \int_D u \text{grad } v d\tilde{S}$$

$$\text{div } \vec{f} = \text{div}(u \text{grad } v) = \text{div}(uv_x, uv_y, uv_z) =$$

$$= (uv_x)_x + (uv_y)_y + (uv_z)_z =$$

ko regrisišemo

$$= \text{grad } u \cdot \text{grad } v + u \Delta v$$

b)

$$\int_D \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \right) dS = \int_D u \frac{\partial v}{\partial \vec{n}} dS - \int_D v \frac{\partial u}{\partial \vec{n}} dS =$$

$$= \int_D \cancel{\text{grad } u \text{grad } v} + u \Delta v - \cancel{\text{grad } v \text{grad } u} - v \Delta u$$

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ harmonična ($\Delta f = 0$)

Dokaz da je $a > 0$ velja:

$$\frac{1}{4\pi a^2} \int f dS = f(0,0,0) = I(a)$$

Nasvet: Parametriziraj $S(0,a)$ i nato s pomočjo odvejanja in agresivnega izreka dokazi trditv.

$$x = a \cos \vartheta \cos \varphi \quad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$y = a \cos \vartheta \sin \varphi \quad \varphi \in [0, 2\pi]$$

$$z = a \sin \vartheta \quad \mapsto \vec{r}(\rho, \vartheta)$$

$$dS = |\vec{r}_\rho \times \vec{r}_\vartheta| d\rho d\vartheta = a^2 \cos \vartheta d\rho d\vartheta$$

$$I(a) = \frac{1}{4\pi a^2} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos \vartheta \underbrace{f(\vec{r}(\rho, \vartheta))}_{g(a, \rho, \vartheta)} d\rho$$

$$I'(a) = \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g}{\partial a} d\rho =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta (f_x(\vec{r}(\rho, \vartheta)) \cdot x_a + f_y \cdot y_a + f_z \cdot z_a) d\rho =$$

$$f(x, y, z) = f(a \cos \vartheta \cos \vartheta, a \cos \vartheta \sin \vartheta, a \sin \vartheta)$$

V \vec{r} se shriva a

$$= \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta (f_x \cos \vartheta \cos \vartheta + f_y \cos \vartheta \sin \vartheta + f_z \sin \vartheta) d\rho$$

komponente enotske
normalne $\vec{n} \in S(0, 1)$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta \underbrace{\text{grad } f \cdot \vec{n}}_{dS} d\rho = \quad dS = a^2 \cos \vartheta d\rho d\vartheta$$

$$= \frac{1}{4\pi} \int_{S(0,1)} \text{grad } f \cdot \vec{n} dS = \frac{1}{4\pi} \int_{S(0,1)} \text{grad } f d\vec{S} =$$

$$= \frac{1}{4\pi} \int_{K(0,1)} \overbrace{\text{div}(\text{grad } f)}^{0} dV = 0$$

$$\text{of} = f_{xx} + f_{yy} + f_{zz}$$

$I'(a) \neq 0 \Rightarrow I(a) \neq \text{konstantne}$

$$I(a) = \frac{1}{4\pi a^2} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos \vartheta f(\vec{r}(\rho, \vartheta)) d\rho$$

$$\lim_{a \rightarrow 0} I(a) = \frac{1}{4\pi} \lim_{a \rightarrow 0} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta f(\vec{r}(\rho, \vartheta)) d\rho =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{a \rightarrow 0} \cos \vartheta f(\vec{r}(\rho, \vartheta)) d\rho =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta f(0,0,0) d\rho =$$

$$= \frac{1}{4\pi} f(0,0,0) \cdot 4\pi = f(0,0,0) \quad \leftarrow P(S(0,1))$$

ker je cosine determinante

Holomorphe Funktionen

$D \subseteq \mathbb{C}$ offene Menge $f \in C(D)$.
Ketene auf f_1, f_2, f_3 in f_3 so holomorphe

$$f_1(z) = \overline{f(z)}$$

$$f_2(z) = \overline{f(\bar{z})}$$

$$f_3(z) = \overline{\overline{f(z)}}$$

holomorphen, $\forall a \in D, \exists \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} =: f'(a)$

$$f(x+i y) = u(x,y) + i v(x,y) \quad u = \operatorname{Re} f \quad v = \operatorname{Im} f$$

$$f \in C^1(D) \iff u_x = v_y \quad \wedge \quad u_y = -v_x \quad u, v \in C^1(D)$$

$$f_1(x+iy) = \underbrace{u(x,y)}_{u_1(x,y)} - i \underbrace{v(x,y)}_{v_1(x,y)}$$

$$\cancel{u_{1x} = u_x} \quad \cancel{u_{1y} = u_y}$$

$$\cancel{v_{1x} = v_x} \quad \cancel{v_{1y} = -v_y}$$

$$\cancel{u_{1x} = u_x = v_y = -v_{1y}}$$

$$u_{1y} = -v_y$$

$$\Rightarrow v_y = -v_y \quad \text{in } \dots \\ v_x = -v_x$$

JK

? ? ?

$$v_x = 0 = v_y$$

$$u_x = u_y = 0$$

u, v konstant;

f_1, n : holomorphe

$$f_2(x+y) = u(x,y) + i v(x,y); \quad \int \overline{D} f_2 \text{ na}$$

||

$$u(x,-y) + v(x,-y)$$

$$\begin{array}{ll} u_{2x} = u_x & u_{2y} = u_y \cdot (-1) = -u_y \\ \cancel{v_{2x} = v_x} & \cancel{v_{2y} = -v_y} \end{array}$$

$$\begin{array}{ll} u_{2x}(x,y) = u_x(x,-y) & u_{2y}(x,y) = -u_y(x,-y) \\ v_{2x}(x,y) = v_x(x,-y) & v_{2y}(x,y) = -v_y(x,-y) \end{array}$$

$$u_{2x}(x,y) = u_x(x,-y) = v_y(x,-y) = -v_{2y}(x,y)$$

To zeigen f_2 ist holomorphe, reichen die ~~u, v~~ konstant:

$$f_3(x+y) = u_3(x,y) + v(x,y) \Rightarrow$$

$$u_3(x,y) = u(x,-y)$$

$$v_3(x,y) = -v(x,-y)$$

$$(u_3)_x(x,y) = u_x(x,-y) \quad v_{3x}(x,y) = v_x(x,y)$$

$$(u_3)_y(x,y) = u_y(x,-x) \quad v_{3y}(x,y) = +v_y(x,-y)$$

$$(u_3)_x = (v_3)_y \Leftrightarrow$$

$$u_x(x,-y) = +v_y(x,-y) \Leftrightarrow u_x = v_y \quad \checkmark$$

$$u_{3y} = -v_{3x} \Leftrightarrow$$

$$-u_y(x,-y) = -v_x(x,-x) \Leftrightarrow v_y = u_x \quad \dots$$

je holomorphe

Dokaži da obstoji $f \in O(\mathbb{C})$ za katero je $u(x,y) = x^3 - 3xy^2$ in jo določi

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy = -v_x$$

$$v = 3 \int (x^2 - y^2) dy = 3x^2y - y^3 + C(x)$$

$$v = 6 \int xy dx = 3x^2y + D(y)$$

$$v = 3x^2y - y^3 + C \quad \text{"unija členov"}$$

$$f(x+y; i) = x^3 - 3xy^2 + i(3x^2y - y^3 + C)$$

$$f(z) = z^3 + i \underbrace{C}_{z \text{ ugibanjem}}$$

Brez navodila:

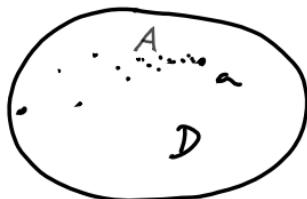
$$z = x + yi \Rightarrow x = \frac{z + \bar{z}}{2} \quad y = \frac{z - \bar{z}}{2i}$$

DN: ustvari natri;

Imenitne trditve:

$D \subseteq \mathbb{C}$ neko območje, $f, g \in O(D)$

funkciji ki se ujemata na množici A s stekatistcem ν_D



ν_A obstaja zaporedje $(a_n)_n$, $a_n \in A$, $a_n \neq a$
 $\lim_{n \rightarrow \infty} a_n = a$

$$\Rightarrow f = g$$

$$u = x^3 - 3xy^2 \quad v = 3x^3y - y^3 + iC$$

če najdemo $g \in O(C)$ ki se $z = f$ ujema na \mathbb{R} , po trditvi T sledi $f = g$

$$x \in \mathbb{R} \quad f(x) = f(x+iy) = u(x, 0) + v(x, 0) = x^3 + iC$$

$$\Rightarrow f(z) = z^3 + iC$$

Se imenitejší premíslit:

$$u(x, y) \rightsquigarrow f = ?$$

$$x \in \mathbb{R} : f(x) = u(x, 0) + v(x, 0) \xrightarrow[x \rightarrow z]{?} f(z) = ?$$

$$v(x, 0) = \int_{CRS} v_x(x, 0) dx = - \int u_y(x, 0) dx$$

$$u_y = -6xy$$

$$v(x, 0) = - \int -6x \cdot 0 dx = +C$$

$$\Rightarrow f(x) = x^3 + C \Rightarrow f(z) = z^3 + C$$

Podobno

$$u(x, 0) = \int u_x(x, 0) dx = \int v_y(x, 0) dx$$

$$u(x,y) = e^x (\cos(ky) + \sin(ky))$$

Dado si k deje $u = \text{Ref}$ $\Rightarrow f \in O(\alpha)$

De je $u = \text{Ref} \Rightarrow$

$$\begin{aligned} u_x &= v_y & u_{xx} &= v_{yx} \\ u_y &= -v_x & u_{yy} &= -v_{xy} \end{aligned} \Rightarrow u_{xx} = -u_{yy}$$

Podobno

$$v_{xx} = -v_{yy}$$

$$\Rightarrow \Delta u = 0$$

$$u_{xx} = e^x (\cos(ky) + \sin(ky)) = u$$

$$u_y = e^x (-k\sin(ky) + k\cos(ky))$$

$$u_{yy} = k^2 e^x (-\cos(ky) - \sin(ky)) = -k^2 v$$

$$\Delta u = e^x (1 - k^2) (\cos(ky) + \sin(ky)) = 0 \Rightarrow k = \pm 1$$

$$\Rightarrow k = \pm 1$$

$$v(x,0) = - \int u_y(x,0) dx = - \int e^x k dx = -k e^x + C$$

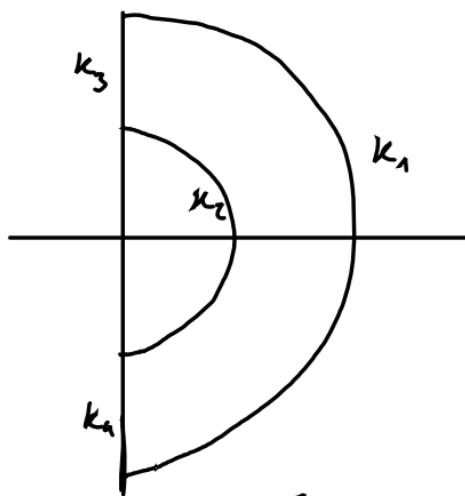
$$f(x) = u(x,0) + i v(x,0) = e^x - i k e^x + C = e^x (1 - i k) + C$$

$$f(z) = e^z (1 \pm i) + C$$

✓ 12.4

Integral: holomorphe Funktion

$$f(z) = \frac{z}{\bar{z}} \quad D = \{ z; |z| \in [1, 2] \quad \operatorname{Re}(z) > 0 \}$$



$$\int_{k_1} f(z) dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2e^{i\varphi}}{2e^{-i\varphi}} 2 \cdot e^{\varphi} d\varphi =$$

$$z = 2e^{i\varphi}$$

$$dz = 2i e^{i\varphi} d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2i e^{3i\varphi} d\varphi = \left. \frac{2}{3} i e^{3i\varphi} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$



$$\frac{2}{3} i (-i - i) = -\frac{2}{3}; \quad (\text{Kern je Sch } -)$$

$$\int_{k_2} f(z) dz = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{e^{i\varphi}}{\bar{e}^{i\varphi}} e^{i\varphi} d\varphi = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{1}{3} i e^{3i\varphi} d\varphi =$$

$$= \frac{2}{3} (i + i) = \underline{\underline{2i}}$$

$$\int_{k_3 \cup k_u} f(z) dz = \text{wegen } z = iy \quad dz = i dy$$

$k_3 \cup k_u$

$$= \int_{-1}^2 \frac{iy}{-iy} i dy + \int_{2+iy}^1 \frac{iy}{z-iy} i dy = -i(2+1) + i(1-2) =$$

$$= i(z-1-1+z) \dots = 2i$$

$$f(z) = \bar{z}$$

$D \subseteq \mathbb{C}$ s koso ma gladkim robom

$$\int_D f(z) dz = ?$$

$$z = x + iy$$

$$dz = dx + idy$$

$$I = \int_D (x - iy)(dx + idy) =$$

$$= \int_D (x dx + y dy) + i(x dy - y dx) =$$

green

$$= \int_D (0 + 0) + i \int_D (1 - (-1)) dx dy =$$

$$= 2i \cdot P(D)$$

$D \subset \mathbb{C}$ kompakte glatte

$$f \in C(D) \cap C^1(D)$$

$$\begin{aligned} \int_D f(z) dz &= \int_D (u + iv)(dx + idy) = \\ &= \int_D (u dx - v dy) + i(v dx + u dy) = \\ &= \int_D \underbrace{(v_x - u_y)}_0 dx dy + i \int_D \underbrace{(u_x - v_y)}_0 dx dy = 0 \end{aligned}$$

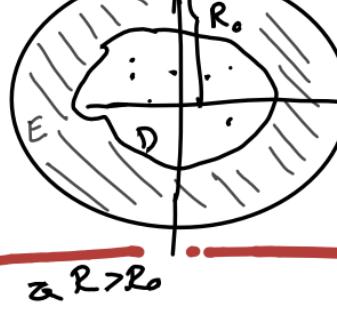
\uparrow \nearrow
 $u_x = v_y$
 $u_y = v_x$

$n > 1$

$z_1, \dots, z_n \in \mathbb{C}$ poljubne nejedno realne

$D^{odg} \subseteq \mathbb{C}$ onjeno k i vsebuje z_1, \dots, z_n

Dokazi $\int_{\partial D} \frac{dz}{(z-z_1) \dots (z-z_n)} = 0$



$\exists R_0 > 0. \bar{D} \subseteq \Delta(0, R) \Rightarrow R > R_0$

$\exists R > R_0$

$$\int_{\partial \Delta(0, R)} f(z) dz = \int_D f(z) dz$$

$$E = \Delta(0, R) - \bar{D}$$

f je holomorfna na E

$$\Rightarrow \int_E f(z) dz = 0$$

$$\int_{\partial C} f(z) dz = \int_{\partial \Delta(0, R)} f(z) dz - \int_D f(z) dz = 0$$

$$\int_{K_R} \frac{dz}{(z-z_1) \dots (z-z_n)} = \int_{\partial D} \frac{dz}{(z-z_1) \dots (z-z_n)}$$

\uparrow

$$\int_{\partial G} \dots = 0$$

$$\int_{K_1} f(z) dz = \int_{K_2} f(z) dz \quad \text{je kvadrat skupaj}$$

trditev E je f holomorfna na delovje

$$\int_{K_R} : |z| = R e^{i\varphi} \\ dz = R i e^{i\varphi}$$

$$I = \int_0^{2\pi} \frac{i R e^{i\varphi} dy}{(R e^{i\varphi} - z_1) \dots (R e^{i\varphi} - z_n)}$$

$$|I| \leq \int_0^{2\pi} \left| \frac{i R e^{i\varphi}}{(R e^{i\varphi} - z_1) \dots (R e^{i\varphi} - z_n)} \right| dy \\ = \left| \frac{R}{(R - |z_1|) \dots (R - |z_n|)} \right| \leq \frac{R}{(R - |z_1|) \dots (R - |z_n|)}$$

$$|R e^{i\varphi} - z_n| \geq |R e^{i\varphi}| - |z_n| = |R - |z_n|| \\ = R - |z_n|$$

$$|I| \leq \int_0^{2\pi} \frac{R}{(R - |z_1|) \dots (R - |z_n|)} dy = \frac{2\pi R}{(R - |z_1|) \dots (R - |z_n|)}$$

$$\xrightarrow[R \rightarrow \infty]{} 0 \Rightarrow I = 0$$

ce bilo; $n=1$

$$\int_{\partial D} \frac{dz}{z-z_1} \Rightarrow I = 2\pi \text{ ker}$$

$$\int_{\partial D} \frac{f(z) dz}{z-z_0} \stackrel{f(z_0) = 0}{=} 2\pi \cdot f(z_0)$$

S pomočjo kompleksne integracije 2u.4

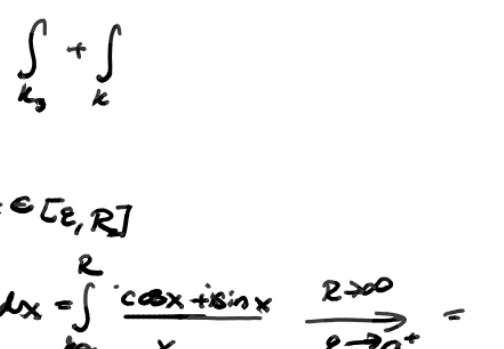
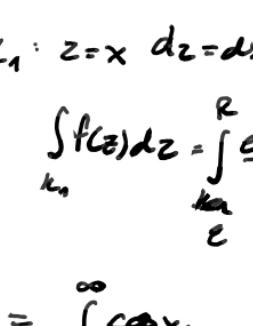
izracunaj:

$$\int_0^\infty \frac{\sin x}{x} dx$$

če mamo $\sin x$ ali $\cos x$ v integralu, potem ju zamenjamo z e^{ix}

Druži če pa smo zamenjamo $x = z$

$$f(z) = \frac{e^{iz}}{z} \quad \text{območje}$$



R bomo posledi v ∞ in z proti 0

$$\int_D f(z) dz = 0$$

\uparrow
 $f(z)$ je v \bar{D}

∂D razdelimo na gladke kose:

$$\int_D f(z) dz = \int_{K_1} + \int_{K_2} + \int_{K_3} + \int_K$$

$$K_1: z=x \quad dz=dx \quad x \in [\epsilon, R]$$

$$\int_{K_1} f(z) dz = \int_{\epsilon}^R \frac{e^{ix}}{x} dx = \int_{\epsilon}^R \frac{\cos x + i \sin x}{x} dx \xrightarrow[\epsilon \rightarrow 0^+]{R \rightarrow \infty} =$$

$$= \int_0^\infty \frac{\cos x}{x} dx + i \int_0^\pi \frac{\sin x}{x} dx$$

$\underbrace{\hspace{10em}}$

$$K_2: z = \rho e^{i\varphi}$$

$$dz = \rho i e^{i\varphi} d\varphi$$

$$\int_{K_2} \dots = \int_0^\pi \frac{e^{i\varphi}}{\rho} \cdot \rho i e^{i\varphi} d\varphi =$$

$$= i \int_0^\pi e^{2i\varphi} d\varphi = \int_0^\pi e^{-2R\sin\varphi} d\varphi$$

$$| \int_{K_2} \dots | \leq \int_0^\pi | e^{-2R\sin\varphi} | d\varphi = \int_0^\pi e^{-2R\sin\varphi} d\varphi$$

$$= \int_0^\pi e^{-2R\sin\varphi} d\varphi \xrightarrow[R \rightarrow \infty]{\varphi \rightarrow \infty} 0$$

$$\int_{K_3} : z = \rho e^{i\varphi}$$

...
zato kot $\rho \rightarrow \infty$

$$\int_{K_3} = i \int_{\pi}^0 e^{i\varphi} (\cos\varphi + i \sin\varphi) d\varphi \xrightarrow[\rho \rightarrow \infty]{\varphi \rightarrow 0} i \int_{\pi}^0 1 d\varphi = -i\pi$$

$$\text{Kazalo: } \int_0^\pi g(\rho, \varphi) d\varphi \xrightarrow[\rho \rightarrow \infty]{\varphi \rightarrow 0} \int_0^\pi g(0, \varphi) d\varphi$$

\uparrow
vira na $\mathbb{R} \times [0, \pi]$

$$0 = \int_D f(z) dz = \int_{K_1} + \dots + \int_{K_3} \xrightarrow[\rho \rightarrow \infty]{\varphi \rightarrow 0} 2iI + C - i\pi$$

$$\Rightarrow I = \frac{i\pi}{2i} = \frac{\pi}{2}$$

$$\int_0^\pi g(\rho, \varphi) d\varphi = \int_0^\pi g(0, \varphi) d\varphi$$

$a > 0 \quad p \in (0, 1) \quad 0 < c < R$

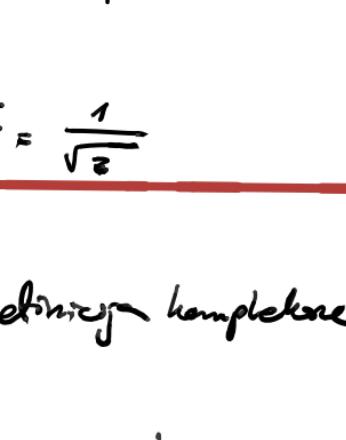
S pomočjo integracije po ∂D izračunaj

$$I_1 = \int_{-\infty}^{\infty} x^{p-1} \cos(ax) dx$$

$$I_2 = \int_{-\infty}^{\infty} x^{p-1} \cos(bx) dx$$

$$D = \{z; |z| \in (\epsilon, R) \wedge \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$$

$$f(z) = z^{p-1} e^{iaz}$$



holomorfost:

$$e^{iaz} \in \mathcal{O}(C)$$

$$p = \frac{1}{z}: z^{p-1} = z^{-\frac{1}{z}} = \frac{1}{\sqrt[z]{z}}$$

$$\sqrt{-1} = \pm i$$

$$\alpha \in \mathbb{C}, z \neq 0 \\ z^\alpha = e^{\alpha \ln z}$$

Definicija kompleksne potence

$\ln z$ je nedoločen do konstante $2k\pi i$:

$$z^\alpha = e^{\alpha(\ln z + 2k\pi i)} = (z^\alpha)_0 e^{2ak\pi i}$$

če $\alpha \in \mathbb{Z}$ je z^α endična definicija

z^α je holomorfnost finančnosti, ko se odločimo

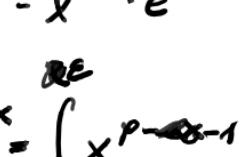
za vlogo $\ln(z)$

$$\arg z \in (0, 0+2\pi)$$

C pravčnem omeri: f_0

Nekj prime: $\begin{cases} n: \text{vključno, ker ne} \\ b: \text{tisto} \\ z \text{emo} \Rightarrow \arg = \pi \end{cases}$

$$\arg z \in (-\pi, \pi)$$



Produkt holomorfnosti je holomorfnost rez vmes: π , da ne prečka nizega dormoja

z^{p-1} holomorfnost

$$\mathbb{C} - (-\infty, 0]$$

$$\int_{k_1}^R f(z) dz = \int_{k_1}^R x^{p-1} e^{iaz} dx = \int_{k_1}^R x^{p-1} (\cos ax + i \sin ax) dx =$$

$$\xrightarrow[k_1 \rightarrow \infty]{R \rightarrow \infty} I_1 + i I_2$$

$$k_3: z = ix \quad dz = i dx \quad \ln(ix) = \ln y + i \frac{\pi}{2}$$

$$z^{p-1} = e^{(p-1)\ln(z)} = e^{(p-1)(\ln x + \frac{\pi}{2}i)}$$

$$= e^{(p-1)\ln x} \cdot e^{(p-1)i\frac{\pi}{2}} = x^{p-1} \cdot e^{(p-1)i\frac{\pi}{2}}$$

$$\int_{k_3}^R x^{p-1} e^{(p-1)i\frac{\pi}{2}} \cdot e^{iaz} dx = \int_{k_3}^R x^{p-1} e^{(p-1)i\frac{\pi}{2}} e^{iax} dx$$

$$= - e^{(p-1)i\frac{\pi}{2}} \int_{k_3}^R x^{p-1} e^{-ax} dx \xrightarrow[k_3 \rightarrow \infty]{R \rightarrow \infty}$$

$$= - e^{(p-1)i\frac{\pi}{2}} \int_0^\infty x^{p-1} e^{-ax} dx = \dots \int_0^\infty \frac{1}{a^{p-1}} e^{-t} dt$$

$$t = ax \quad dt = adx$$

$$= - \frac{e^{(p-1)i\frac{\pi}{2}}}{a^p} \Gamma(p)$$

$$\int_{k_2}^R: z = Re^{iy}$$

$$(z_1 - z_2)^\alpha = z_1^\alpha \cdot z_2^\alpha \quad \text{ni}$$

$$z^{p-1} = (Re^{iy})^{p-1} =$$

(Velja če s: izbereti ustrezeni vege)

$$e^{(p-1)\ln(Re^{iy})}$$

$$dz = Re^{iy} dy$$

$$\ln(Re^{iy}) = \ln R + iy$$

$$= e^{\ln R(p-1) + (p-1)iy} = R^{p-1} \cdot e^{i(p-1)y}$$

$$\int_{k_2}^R: \int_0^\pi R^{p-1} e^{i(p-1)y} e^{ia(ay + iy)} iRe^{iy} dy$$

$$|S| \leq \int_0^\pi | \dots | dy = \text{abs mednost} = 1$$

$$R^p \int_0^\pi e^{-aR \sin y} dy \leq \int_0^\pi R^p e^{-\frac{aR}{2} \sin y} dy =$$

$$\sin y \geq \frac{2y}{\pi}$$

$$= -R^p \frac{\pi}{2aR} (e^{-aR} - 1) \rightarrow 0$$

$$0 < a < 1$$

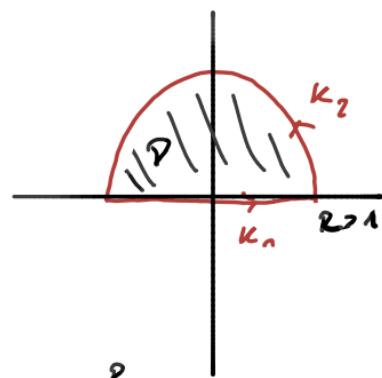
8.5

Tanuca

one village

$$I = \int_0^\infty \frac{\cos x \, dx}{1+x^2}$$

$$f(x) = \frac{e^{iz}}{1+z^2}$$



$$\begin{aligned} \int_{K_1} f(z) dz &= \int_{-R}^R \frac{e^{ix}}{1+x^2} dx = \int_{-R}^R \frac{\cos x + i \sin x}{1+x^2} dx = \\ &= \int_{-R}^R \frac{\cos x}{1+x^2} dx + i \int_{-R}^R \frac{\sin x}{1+x^2} dx = 2 \int_0^R \frac{\cos x}{1+x^2} dx \xrightarrow{R \rightarrow \infty} 2I \end{aligned}$$

"Ortliche"

$$\int_{K_2} f(z) dz = \int_0^\pi \frac{e^{i(\theta+iy) + iz}}{1+r^2 e^{2i\theta}} i r e^{i\theta} d\theta =$$

$$\left| \int_0^\pi \frac{e^{-rsin\theta} \cdot e^{ircos\theta}}{1+r^2 e^{2i\theta}} e^{iz} d\theta \right| \leq$$

$$r \int_0^\pi |...| d\theta$$

$$|1+r^2 e^{2i\theta}| \geq |1-r^2 e^{2i\theta}| = |1-r^2| \geq r^2-1$$

$$\leq r \int_0^\pi e^{-rsin\theta} \frac{1}{r^2-1} d\theta = \frac{r}{r^2-1} \int_0^\pi e^{-rsin\theta} d\theta \xrightarrow{r \rightarrow \infty} 0$$

f ist eine Singularität v. i ∈ D

$$f(z) = \frac{e^{iz}}{(z-i)(z+i)} = \frac{\frac{e^{iz}}{z+i}}{z-i}$$

$$\boxed{\int_D \frac{g(z)}{z-a} dz = 2\pi i g(a)}$$

$$\int_D \frac{\frac{e^{iz}}{z+i}}{z-i} dz = 2\pi i \cdot \frac{e^{-i}}{2i} = \frac{\pi}{e}$$

$$\frac{\pi}{e} = \int_D f(z) dz = \int_{K_1} + \int_{K_2} \xrightarrow{R \rightarrow \infty} 2I + 0 \Rightarrow I = \frac{\pi}{2e}$$

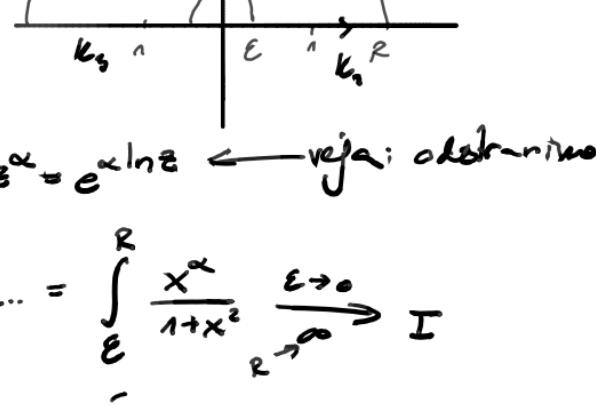
$$0 < \varepsilon < 1 < R$$

$$D = \{ z; \varepsilon < |z| < R, \operatorname{Im} z > 0 \}$$

Sposób całkowania po drodze resztynej

$$\int_0^\infty \frac{x^\alpha dx}{1+x^2} = I \quad \alpha \in (-1, 1)$$

$$\text{wtedy: } \Gamma(p) \Gamma(1-p) = \frac{\pi}{\sin \pi p}$$



$$z^\alpha = e^{\alpha \ln z} \leftarrow \text{rej. odstraniono po } p = -\frac{\pi}{2}$$

$$\int_{K_1} \dots = \int_{-\infty}^{\infty} \frac{x^\alpha}{1+x^2} \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}]{} I$$

$$\left| \int_{K_2} \dots \right| =$$

$$z = re^{i\varphi} \quad z^\alpha = e^{\alpha \ln z} = e^{\alpha(\ln r + i\arg z)} =$$

$$dz = re^{i\varphi} d\varphi \quad = e^{\alpha(\ln r + i\varphi)}$$

$$= \left| \int_0^\pi \frac{r^\alpha e^{i\alpha\varphi}}{1+r^2 e^{i2\varphi}} \cdot r e^{i\varphi} d\varphi \right| \leq \int_0^\pi 1 \dots d\varphi$$

\downarrow

$$|1+r^2 e^{i2\varphi}| \geq R^2 - 1$$

$$= \int_0^\pi \frac{r^\alpha}{R^2 - 1} R d\varphi = \pi \frac{R^{\alpha+1}}{R^2 - 1} \xrightarrow[R \rightarrow \infty]{\alpha < 1} 0$$

$$\int_{K_3} \dots = \int_{-R}^{-\varepsilon} \frac{x^\alpha}{1+x^2} dx = \int_{-R}^{-\varepsilon} \frac{|x|^\alpha e^{\alpha i\pi}}{1+x^2} dx =$$

$$z = x \quad x^\alpha = z^\alpha = e^{\alpha(\ln|x| + i\pi)} = |x|^\alpha + e^{\alpha i\pi}$$

$$x = -t \quad |x| = t$$

$$dx = -dt$$

$$- \int_{-R}^{-\varepsilon} \frac{x^\alpha e^{\alpha i\pi}}{1+x^2} dx = \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}]{} e^{\alpha i\pi} I$$

$$\int_{K_4} \dots = \int_0^\pi \frac{e^\alpha e^{\alpha i\varphi}}{1+(e^{i\varphi})^2} i e^{i\varphi} d\varphi \xrightarrow[e \rightarrow 0]{\substack{\alpha < 1 \\ \varphi \in [0, \pi]}} \int_0^\pi g(0, \varphi) d\varphi = 0$$

$$g(\varepsilon, \varphi) \text{ average na } (-1, 1) \times (0, \pi)$$

$$\int_D P(z) dz =$$

$$f(z) = \frac{z^\alpha}{(z-i)(z+i)}$$

$$\int \frac{h(z)}{z-i} dz = 2\pi i \frac{i^\alpha}{\pi i^\alpha} =$$

$$\pi i^\alpha = e^{\alpha i\pi} I + I$$

$$i^\alpha = e^{\alpha \ln i} = e^{i\alpha \frac{\pi}{2}}$$

$$\pi e^{i\alpha \frac{\pi}{2}} = e^{i\alpha \pi} I + I$$

$$\ln i = 0 + i \frac{\pi}{2}$$

$$I = \frac{\pi e^{i\alpha \frac{\pi}{2}}}{1+e^{i\alpha \pi}} = \frac{\pi}{e^{i\alpha \frac{\pi}{2}} + e^{i\alpha \pi}} = \frac{\pi}{2 \operatorname{Re}(e^{i\alpha \frac{\pi}{2}})}$$

$$= \frac{\pi}{2 \cos(\alpha \frac{\pi}{2})}$$

Sposób narysu se:

$0 < n < m$

$$\int_0^n \frac{x^n}{1+x^m} dx = \frac{1}{m} B\left(\frac{n+1}{m}, 1 - \frac{n+1}{m}\right)$$

$$\frac{n=\alpha}{m=2} \quad I = \frac{1}{2} B\left(\frac{\alpha+1}{2}, 1 - \frac{\alpha+1}{2}\right) = \frac{1}{2} \frac{\pi \left(\frac{\alpha+1}{2}\right) \Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2} + \frac{1}{2}\right)}$$

$$= \frac{1}{2} \pi \left(\frac{\alpha+1}{2}\right) \Gamma\left(1 - \frac{\alpha+1}{2}\right)$$

$$\rho = \frac{\alpha+1}{2} \in (0, 1)$$

$$\alpha = 2\rho - 1$$

$$\Gamma(p) \Gamma(1-p) = \frac{\pi}{\cos(\frac{\pi}{2}p - \frac{\pi}{2})} = \frac{\pi}{\sin p\pi}$$

Ali obsteja funkcija f holomorfna v diskici O , da za velike naravne števile $n \in \mathbb{N}$

velja:

$$a) f\left(\frac{1}{n}\right) = \sin\left(\frac{n\pi}{2}\right)$$

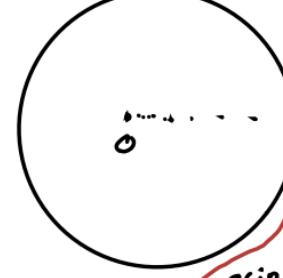
$$b) f\left(\frac{1}{n}\right) = \frac{1}{2n+1}$$

$$c) n^{-\frac{5}{2}} < |f\left(\frac{1}{n}\right)| < 3n^{-\frac{5}{2}}$$

$$\Rightarrow \sin\frac{n\pi}{2} = \begin{cases} 0 & n \text{ sodo} \\ \frac{1}{2k+1} & n \text{ nesodo} \end{cases}$$

V vsaki diskici O je f vrednosti 0 ali 1
 $\Rightarrow f$ ni avansa $\Rightarrow f$ n: holomorfna

b)



$A = \{\frac{1}{n}, n \in \mathbb{N}\}$ je množica
 = stekeliscem v D

$\xrightarrow{\text{princip identičnosti}}$ Če f je natančno
 deljarene

$$f\left(\frac{1}{n}\right) = \frac{1}{2n+1} \quad \Rightarrow \quad f(z) = \frac{1}{\frac{2}{z}+1} = \frac{z}{2+z}$$

holomorfna v
 okolici 0

c) če f obstaja ima razoj v
 Taylorjevo vrsto v diskici O

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

$$n^{-\frac{5}{2}} < |c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots| < 3n^{-\frac{5}{2}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \leq |c_0| \leq 0$$

$$n^{-\frac{3}{2}} < |c_1 + \frac{c_2}{n} + \dots| < 3n^{-\frac{3}{2}}$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = 0$$

$$n^{\frac{1}{2}} \leq |c_3 + \frac{c_4}{n} + \dots| \leq n^{\frac{1}{2}}$$

$$\downarrow \qquad \downarrow_{n \rightarrow \infty} \qquad \downarrow$$

$$\infty \qquad c_3 \qquad \infty$$

*

Kaj pa $z \geq \frac{5}{2}$?

je nevsekem prezen, ampak ne
 moremo pa je posplošiti na cele
 diskici O

Rewj: $f(z) = \sin z$ dali töche:

$$f(z) = c_0 + c_1(z-i) + c_2(z-i)^2 + \dots$$

$$c_n = \frac{f^{(n)}(i)}{n!} \quad \text{v erlegen}$$

$$w = z-i;$$

$$\sin(z) = \sin(w+i) = \sin(w)\cos(i) + \sin(i)\cos(w)$$

$$= \cos(i) \cdot \sum_{k=0}^{\infty} \frac{w^{2k+1}}{(2k+1)!} (-1)^k + \sin(i) \cdot \sum_{k=0}^{\infty} \frac{(-1)^k w^{2k}}{(2k)!}$$

$$= \cos(i) \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (z-i)^{2k+1} + \sin(i) \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (z-i)^{2k}$$

$$\sin(x+iy) = \sin x \cos y i + \sin x \cos x$$

$$\sin iy = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (iy)^{2k+1} =$$

$$= \sum_{k=0}^{\infty} i \frac{y^{2k+1} (-1)^k}{(2k+1)!} = \sum \frac{(-1)^k \cdot i \cdot y^{2k+1} (-1)^k}{(2k+1)!} =$$
$$= i \sum \frac{y^{2k+1}}{(2k+1)!} = i \operatorname{sh}(y)$$

$$\cos(iy) = \sum_{k=0}^{\infty} \frac{(-1)^k (iy)^{2k}}{(2k)!} = \sum \frac{y^{2k}}{(2k)!} = \operatorname{ch}(y)$$

$$\sin(x+iy) = \sin x \operatorname{ch} y i + i \operatorname{sh} y \cos x$$

$$\sin i = 0 + i \operatorname{sh} 1 = i \operatorname{sh} 1$$

$$\begin{aligned} y &= 1 \\ x &= 0 \end{aligned}$$

$$\cos(x+iy) = \cos x \cos y i - \sin x \sin y i =$$
$$\underline{\cos x \operatorname{ch} y - i \sin x \operatorname{sh} y}$$

$$\cos i = \begin{aligned} x &= 0 & \operatorname{ch} 1 \\ y &= 1 \end{aligned}$$

Laurentova vrsta

$$f \in \mathcal{O}(D) \Rightarrow f(z) = \underbrace{\sum_{n=-\infty}^{-1} a_n (z-a)^n}_{\text{glavní del}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{regularní del}}$$



glavní del

regularní del

$$f(z) = \frac{z^2 + 1}{z^2 + z - 2} = \frac{4}{z+3} + \frac{B}{z-2}$$

$$|z| < 1 \Rightarrow B = 1 \quad A = 1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |\frac{z}{3}| < 1 \Rightarrow |z| < 1$$

$$\frac{1}{z+3} = \frac{1}{3} \frac{1}{1-(\frac{z}{3})} = \frac{1}{3} \sum_{n=0}^{\infty} (\frac{z}{3})^n$$

$$|z| < 3 \Rightarrow \frac{1}{z+3} = \frac{1}{3} \sum_{n=0}^{\infty} (-\frac{z}{3})^n$$

$$|z| < 2 \Rightarrow -\frac{1}{2} \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} (\frac{z}{2})^n$$

$$z \in A(0; 0, 2) \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \\ = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{3^{n+1}} - \frac{1}{2^{n+1}} \right) z^n$$

$$z \in A(0; 2, 3)$$

$$\frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\frac{2}{z}} \quad \text{rozvoj na závěr} \\ g = \frac{2}{z} \quad |\frac{2}{z}| < 1 \quad z < |z|$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{z} \right)^n = \sum_{n=-\infty}^{-1} \frac{z^{n+1}}{2^{n+1}}$$

$$m = -(n+1) \Rightarrow n = -m-1$$

$$z \in A(0; 2, 3) \Rightarrow f = \sum_{n=-\infty}^{-1} \frac{1}{z^{n+1}} z^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n$$

$$z \in A(0; 3, \infty)$$

$$\frac{1}{z+3} = \frac{1}{z} \frac{1}{1+\frac{3}{z}} \quad g = \left(-\frac{3}{z} \right)$$

$$\frac{1}{z} \sum_{n=0}^{\infty} (-1) \frac{3^n}{z^n} = \quad |z| > 3$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^{n+1}}$$

$$f(z) = \sum_{n=-\infty}^{-1} \left((-3)^{-n+1} + 2^{-n-n} \right) z^n$$

Pozitiviek:

$$f(z) = \frac{p(z)}{q(z)} \quad p, q \text{ polinome}$$

\rightsquigarrow rozčleněno uložíme

$$\frac{1}{(z-a)^k} = (z-a)^{-k} =$$

\nwarrow binomická vztah
 $(1+g)^{\alpha}$

z: Dolaci glavní del LV

$$f(z) = \frac{\cos z}{z - \sin z} ; z=0$$

izolirani:

$$f(z) = \frac{1}{(z+1)(e^{i\pi z} + 1)} ; z=-1$$

singularnost
v a

$f \in C^0(D - \text{zaj})$

• izolirane singularnosti

1) Bistvena singularnost

glavní del ima neskončno
nemělenných členů

2) Pol stopnje $N \in \mathbb{N}$

glavní del je do N

3) z je odpovídající singularnost

\Rightarrow glavní del = 0

$$f(z) = \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots}{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots)} =$$

$$= \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} + \dots}{\frac{z^2}{3!} - \frac{z^5}{5!} + \dots} = \frac{1}{z^3} \left(\frac{1 - \frac{z^2}{2} + \dots}{\frac{1}{3!} - \frac{z^2}{5!} + \dots} \right)$$

$g(z)$ hänomorf i $\mathbb{C} \setminus \{0\}$, nima
nidev 0

$\Rightarrow f(z)$ imma pol 3-styrige

$$\text{GD: } \frac{a_{-3} + a_{-2}}{z^3} + \frac{a_{-1}}{z}$$

$$f(z) = \frac{1 - \frac{z^2}{2} + \dots}{z^3 \left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right)} = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \dots$$

reg. del

$$1 - \frac{z^2}{2} + \dots = (a_{-3} + a_{-2} z + a_{-1} z^2 + \dots) \left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right)$$

$$z^0: \quad \frac{a_{-3}}{3!} = 1 \quad z^2: \quad \frac{a_{-3}}{5!} + \frac{a_{-1}}{3!} = -\frac{1}{2}$$

$$z: \quad \frac{a_{-2}}{3!} = 0$$

$$\dots \quad a_{-1} \quad \frac{-3 \cdot 6}{20}$$

$$a_{-3} = 3! = 6$$

$$a_{-2} = 0$$

$$a_{-1} = \boxed{18}$$

$$\text{GD: } \frac{6}{z^3} - \frac{\cancel{348}}{\cancel{2012}} \frac{27}{10z}$$

$$b) f(z) = \frac{1}{(z+1)} \left(e^{i\pi z} + 1 \right)$$

$e^{i\pi z}$ razvijemo okol $a = -1$

$$w = z - a = z + 1$$

$$e^{i\pi(w+1)} = e^{i\pi w} e^{i\pi}$$

$$(-1) \sum_{n=0}^{\infty} \frac{(i\pi w)^n}{n!} = -1 - i\pi w - \dots$$

$$e^{i\pi z+1} = -i\pi w - \dots = -i\pi(z+1)$$

$$f(z) = \frac{1}{(z+1) \left(-i\pi(z+1) - \dots \right)} = \frac{1}{(z+1)^2 (-i\pi - \dots)} = \frac{a_2}{(z+1)^2} + \frac{a_{-1}}{z+1} + \dots$$

pol druge sljednje

$$1 = a_2(-i\pi - \dots) + a_{-1}(z+1)(i\pi - \dots) + \frac{c_i \pi^2}{z+1}$$

$$(z+1)^0: a_{-2} i\pi$$

$$(z+1)^1: a_2 \frac{i\pi^2}{2!} + a_1 i\pi \quad ??$$

$$GD: \frac{1}{\pi(z+1)^2} + \frac{1}{z(z+1)}$$

$$\text{res}(f, -1) = a_{-1} = \frac{1}{2}$$

$$f(z) = \frac{z}{z^3 - 3z + 2}$$

Rozwiąż okl:
a = -1
izrachunaj

$$\int f(z) dz$$

$|z|=6$

$$z^3 - 3z + 2 =$$

$$w = z - 1 \quad z = w + 1$$

$$= w^3 + 3w^2 + 3w + 1 - 3w - 3 + 2 =$$

$$= w^3 + 3w^2 - 2w = w(w^2 + 3w - 2)$$

$$= w(w - \dots)$$

$$= w^2(w+3)$$

$$\begin{matrix} w+1; w+3=1 \\ w-3 \end{matrix} \quad \frac{1}{w+3}$$

$$f(z) = \frac{w+1}{w^2(w+3)} = \frac{1}{w^2} \frac{w+1}{w+3} =$$

$$\frac{1}{w^2} \left(1 - \frac{2}{w+3}\right)$$

$$1 - \frac{2}{w+3} = 1 - \frac{2}{3} \frac{1}{\frac{w}{3} + 1} = -\frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \frac{w^n}{3^n}$$

- - - -

symm

Formula:

$$\int \oint_C f(z) ds \quad \dots \quad f \text{ holomorphic in } \\ \text{abreie } D \text{ mit } \\ \text{runden unregelmäßigen} \\ \text{Winkelwinkeln in } D$$

$$\int_C f(z) ds = \sum_{a \in D} \operatorname{Res}(f, a)$$

\uparrow
singular point

f pol stopt auf N v $a \Rightarrow$

$$\operatorname{Res}(f, a) = \frac{1}{(N-1)!} \lim_{z \rightarrow a} (f(z)(z-a)^N)^{(N-1)}$$

$$\int \frac{dz}{z^2 - 9}$$

$|z - 2| = 2$

$$f(z) = \frac{1}{(z+3)}(z-3)$$

$$\alpha_1 = 3;$$

$$\alpha_2 = -3;$$

singular point
pole st 1

$$I = 2\pi; \operatorname{Res}(f, \alpha_n)$$

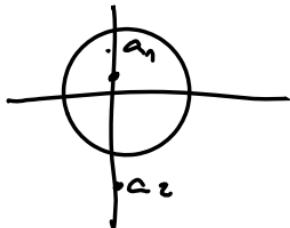
$$\operatorname{Res}(f, 3) = \lim_{n \rightarrow \infty} \frac{1}{z+3} = \frac{1}{6}$$

\nearrow
 $f(z)(z-3)$

$$\left(\operatorname{Res}(f, -3) = -\frac{1}{6} \right) \quad I = \frac{\pi}{3}$$

$$5) \oint \frac{dz}{(z^2+i)^3} \\ |z - \frac{i}{2}| = 1$$

$$f(z) = \frac{1}{(z-i^3)(z+i)^3} \quad a_1 = ; \\ a_i = -i$$



$$I = 2\pi; \operatorname{Res}(f, a_1)$$

$$\operatorname{Res}(a_1) =$$

$$N = 3$$

$$\operatorname{Res}(f, a_1) = \frac{1}{2!} \lim_{n \rightarrow 1} \left(\frac{1}{(z+i)^3} \right)^{(1)} =$$

$$\left((z+i)^{-3} \right)^{(1)} = -3 \left((z+i)^{-4} \right)^{(1)} = 12 (z+i)^{-5}$$

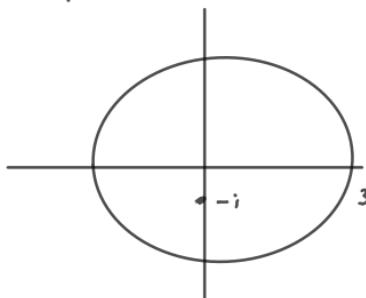
$$\operatorname{Res}(f, a_1) = 6 \cdot (2i)^{-5} = \frac{6}{2^5 i} = \frac{3}{16 i}$$

$$I = \frac{\pi}{8}$$

$$c) \int \sin\left(\frac{z}{z+i}\right) dz$$

$$|z|=3$$

$$f(z) = \sin\left(\frac{z}{z+i}\right) \quad a = \frac{-i}{z+i}$$



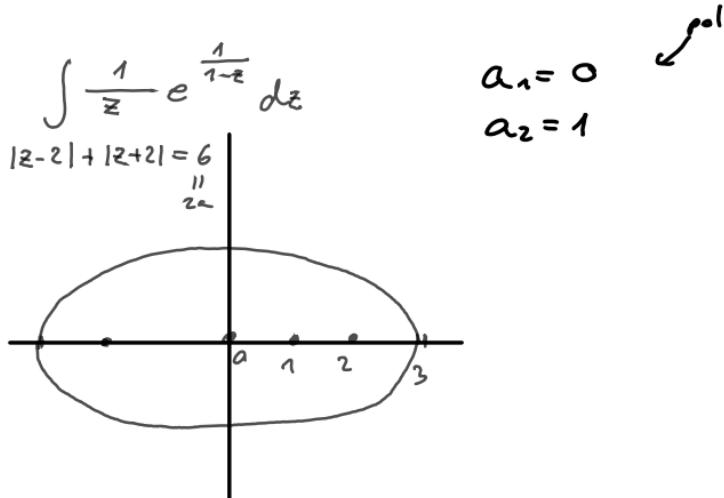
$$\omega = z+i$$

$$z = \omega - i$$

$$\begin{aligned}
 f(z) &= \sin\left(\frac{\omega-i}{\omega}\right) = \sin\left(1 - \frac{i}{\omega}\right) = \\
 &= \sin 1 \cos\left(\frac{i}{\omega}\right) - \sin\left(\frac{i}{\omega}\right) \cos(1) = \\
 &= \sin_1\left(1 - \frac{\left(\frac{1}{\omega}\right)^2}{2} + \dots\right) - \cos(1)\left(\frac{i}{\omega} - \frac{i\left(\frac{1}{\omega}\right)^3}{6} + \dots\right)
 \end{aligned}$$

$$\operatorname{Res}(f, i) = C_{-1} \leftarrow \dots \omega^{-n}$$

$$\Rightarrow \operatorname{Res}(f, i) = -\cos(1);$$



$$I = (\operatorname{Res}(f, 1) + \operatorname{Res}(f, 0))2\pi i;$$

$$\operatorname{Res}(f, 0) = \lim_{z \rightarrow 0} (f(z) \cdot z) = \lim_{z \rightarrow 0} e^{\frac{1}{1-z}} = e$$

$\operatorname{Res}(f, 1) = ?$ bivene singularitet

$z \in LV \vee z=1$

$$\frac{1}{z} \text{ v okolici } 1$$

$$\frac{1}{z} = \frac{1}{z-1+1} = \sum_{n=0}^{\infty} (z-1)^n (-1)^n$$

$$e^{\frac{1}{1-z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{1-z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(z-1)^n}$$

$$\begin{aligned} \frac{1}{z} \cdot e^{\frac{1}{1-z}} &= (1 - (z-1) + (z-1)^2 - \dots) \left(1 - \frac{1}{z-1} + \frac{1}{2(z-1)^2} - \frac{1}{3!} \frac{1}{(z-1)^3} \right. \\ &= \dots + \frac{1}{z-1} \left(-1 - \frac{1}{2} - \frac{1}{3!} - \frac{1}{4!} - \dots \right) + \dots \end{aligned}$$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \quad)$$

$$= -(e-1) = 1-e$$

$$I = 2\pi i (e + 1 - e) = 2\pi i;$$

$$\text{St niceł } z \quad p(z) = z^5 + 7z^4 - z^2 + 2$$

$$\text{Na a) } A(0,1) \quad b) \overline{A(0,1)} \quad c) A(0, 2, 10)$$

Rouchejeva sreća:

$D \subseteq \mathbb{C}$ omejeno lepn območje (glede na δD)

$f, F \in \sigma$ (okoliš D)

$$\forall z \in \partial D \mid f(z) \mid < |F(z)|$$

\Rightarrow St niceł vedravost: z je enako številu niceł z $f+F$

$$a) p(z) = z^5 + 7z^4 - z^2 + 2 = f(z) + F(z)$$

$$z \in \partial D \Leftrightarrow |z| = 1$$

$$F(z) = 7z^5 \Rightarrow |F(z)| = 7$$

$$f(z) = z^5 - z^2 + 2 \Rightarrow |f(z)| \leq 1 + 1 + 2 = 4$$

$$\xrightarrow{\text{R.I.}} \text{St niceł } p(z) \text{ na } A(0,1) \text{ je } 4$$

b)

Opomba

$$z \in \partial D. \mid F(z) \mid > |f(z)|$$

$$|F(z) + f(z)| \geq |F(z)| - |f(z)| > 0$$

$$\Rightarrow \underbrace{F(z) + f(z)}_{g(z)} \neq 0 \quad \forall z \in \partial D$$

$g(z)$ nima niciole na ∂D

$$c) A(0, 2, 10)$$

$$p(z) = z^5 + 7z^4 - z^2 + 2$$

$$z \in \partial D \Leftrightarrow |z| = 2 \vee |z| = 10$$

$$D = \underbrace{A(0, 10)}_{D_1} - \underbrace{\overline{A(0, 2)}}_{D_2}$$

Prestojemo niciole na D_1 in potem na D_2

$$D_1: |z| = 10$$

$$F = z^5 \Rightarrow |F(z)| = 10^5$$

$$f(z) = 7z^4 - z^2 + 2 \Rightarrow |f(z)| \leq 7012$$

$$D_2: |z| = 2$$

$$P(z) = z^5 \quad P(z) = 2^4$$

$$|P(z)| \leq z^5 + 4 + 2$$

\Rightarrow ~~oblike:~~

$$5 - 4 = 1$$

Dokz) da ist $z \cdot \sin z = 1$ same reale reziproze

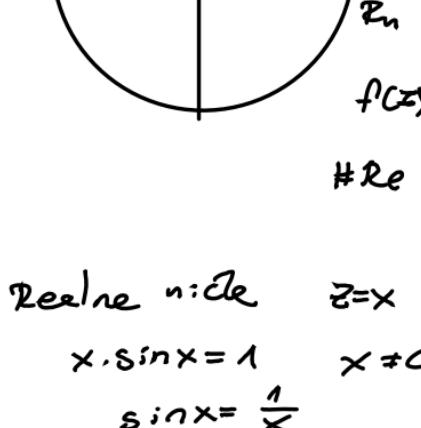
$\sin z$ ist eine reelle Welle

$$\sin z = 0$$

$$z = \pi k, k \in \mathbb{Z}$$

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) =$$

$$= \sin x \cosh y + i \sinh(y) \cos(x)$$



$$D = \Delta(0, R_n)$$

$$R_n = 2n\pi + \frac{\pi}{2}$$

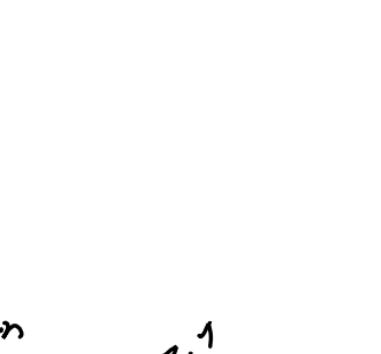
$$f(z) = z \sin z - 1$$

Reelles Bild von f auf D
= st. v. sen. wellen

Reelle Welle $z = x$

$$x \cdot \sin x = 1 \quad x \neq 0$$

$$\sin x = \frac{1}{x}$$



↪ rotation in $(0, R_n)$:

$$z_n + 1$$

Reelle Welle in D :

$$z \cdot \sin z - 1 = 0$$

$$f(z) = z \cdot \sin(z)$$

$$\lg z = -1$$

$$\Rightarrow |z| = R_n$$

$$|z \cdot \sin(z)| \leq R_n \cdot |\dots|$$

$$g(z) = 1$$

Nachrechnung $z = (x+iy)$

$$w(x, y) = |\sin(x+iy)|^2 =$$

$$\sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

$$\min \text{ min in } \Delta(0, R_n) \quad x^2 + y^2 = R_n^2$$

Vierer: elektrom.:

$$L = \sin^2 x \cosh^2 y + \sin^2 y \cosh^2 x - \alpha (x^2 + y^2) R_n^2$$

$$L_x = 2 \sin x \cos x \cosh^2 y - 2 \sin^2 x \cosh^2 y \dots$$

3

$$\Rightarrow \sin 2x = 2\alpha x$$

$$\sin 2y = 2\alpha y$$

$$\sin t = \alpha t$$

$$\sin u = \alpha u$$



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sehr viele Wellen

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$$\alpha >$$

$$F(z) = z + \sin z \quad f(z) = -1$$

$$|F(z)| > f(z)$$

1. ~~g~~ niederst der

4n niederst prege 1

4n+2 nicht \leftarrow sarpde \rightarrow stevilem
real rich reicher
so so vug reiche

Konformne preslikave

ohranja kote in orientacijo
med krivuljami:

= holomorfne z neničelnim odvodom

$$f: D_1 \longrightarrow D_2 \quad D_1, D_2 \subseteq \mathbb{C}$$

biholomorfna ekvivalenca

f bijektiuna, holomorfna, f^{-1} holomorfna

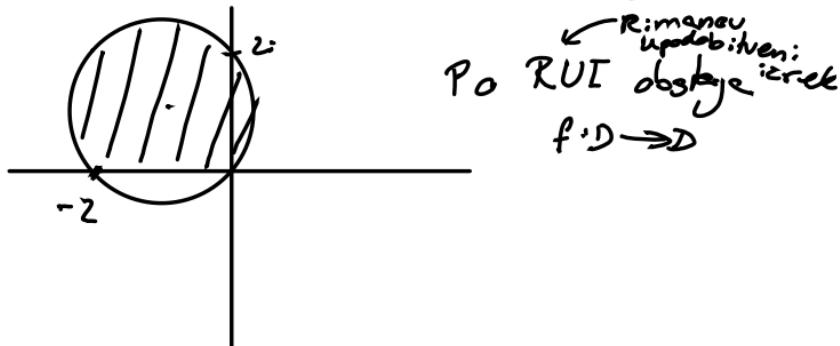
$D \subseteq \mathbb{C}$ je biholomorfno ekvivalenten Δ



$D \neq \mathbb{C} \wedge D$ enostavna povezan
(nima luknj)

Ugatovi da je D bih. obr Δ , oč je,
dakci je biholomorficna

$$a) D = \{z; |z-i+1| < \sqrt{2}, \operatorname{im} z > 0\}$$



$$g(z) = \frac{az+b}{cz+d}$$

lomljena linearne
preslikave

$\{\text{kražnice, premice}\} \longrightarrow \{\text{kražnice, premice}\}$

$$\mathbb{CP}^1 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Premice so kražnice
kroz $z_i = \infty$

$g: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ biholomorfna

Poisadi LUP

$$g: g(-2) = 0 \quad g(-1) = 1 \quad g(0) = \infty$$

$$\frac{az+b}{cz+d}$$

$$\frac{a(z+2)}{cz}$$

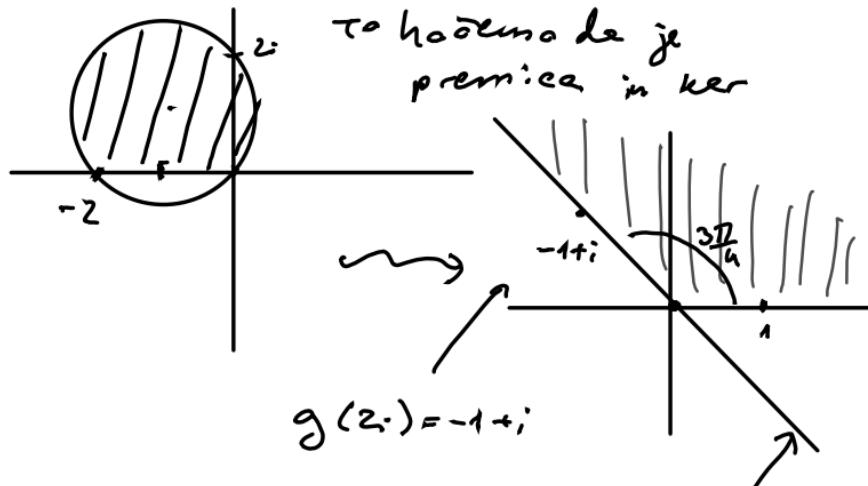
$$a=1$$

$$\frac{-1+z}{c(z-1)} = 1$$

$$1 = -c$$

$$g = \frac{z+2}{-z}$$

To smo gledali ker



$$g = f_n$$

$$g(D) = D_n$$

Naprej

$$f_n: D_n \rightarrow H$$

$$\frac{3\pi}{4}, \dots, \pi$$

$$y, \dots, x \quad x = \frac{\varphi \pi u}{3\pi} = \frac{u}{3} \varphi$$

$$z \mapsto |z| e^{\frac{u}{3} \arg z}$$

$$\text{lahko tudi } z \mapsto z^{\frac{u}{3}}$$

$$z^\alpha = e^{\alpha \ln z}$$

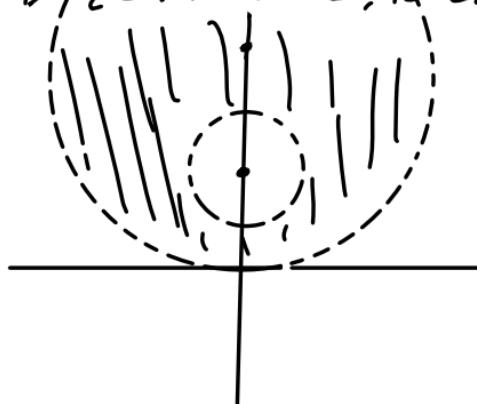
$$f_0(z): H \longrightarrow \Delta$$

$$f_0(z) = \frac{iz+1}{iz-1}$$

$$f = f_0 \circ f_2 \circ f_1$$

b)

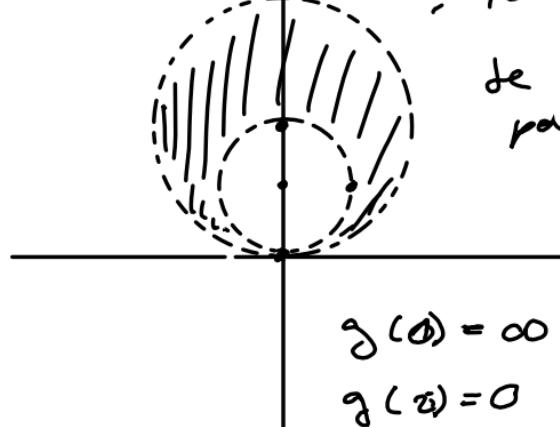
$$D = \{z \mid |z-i| > \frac{1}{2}, |z-2i| < 2\}$$



D ni enastava
paverano \Rightarrow
 $n: b:h$ olev \leq

c)

$$D = \{z \mid |z-i| > 1, |z-2i| < 2\}$$



de enastava
paverano alue

$$g(0) = \infty$$

$$g(\infty) = 0$$

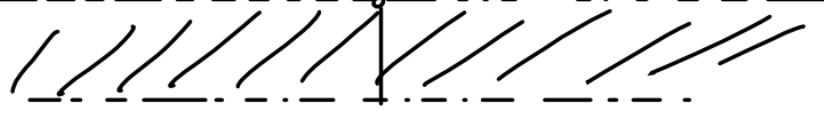
$$g(i+1) = 1$$

$$\frac{a(z-2i)}{cz} \quad \frac{a}{c} \frac{(1-i)}{1+i} = \frac{a}{c} \frac{1-2i-1}{2} = -\frac{ai}{c}$$

$$\frac{a}{c} = -i$$

$$g = \frac{2-z_i}{z}$$

$$g(u_i) = \frac{2+4}{ai} = -\frac{3}{2i}$$



$$f_2: z \mapsto e^z$$

$$e^{x+iy} = e^x \cdot e^{iy} \quad \text{y ja ket}$$

$$y \in (0, \pi) \quad e^x \in (0, \infty)$$

$$x \in (-\infty, \infty)$$

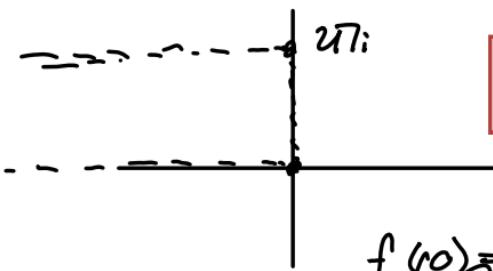
$$e^{iy} \in (\varrho, \infty)$$

s1, kev H

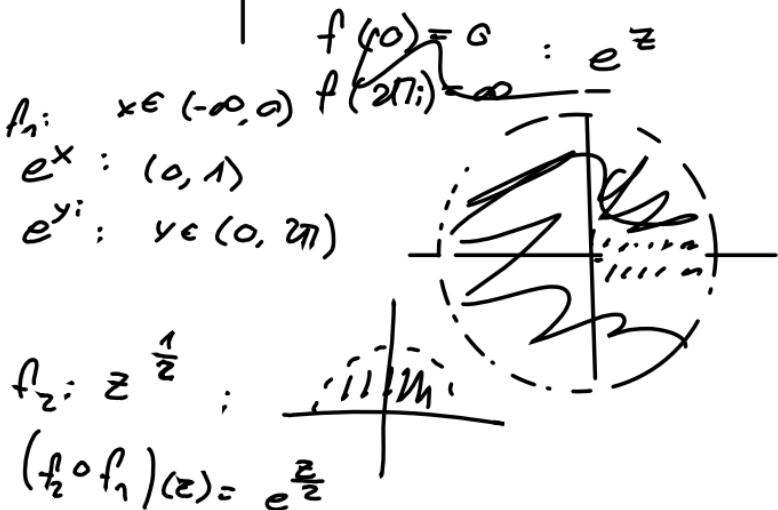
$$f_0: D_0 \rightarrow \Delta$$

$$f = f_0 \circ f_3 \circ f_2 \circ f_1$$

$$D = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0 \text{ and } \operatorname{Im}(z) \in (0, 2\pi)\}$$



e^z reziproque

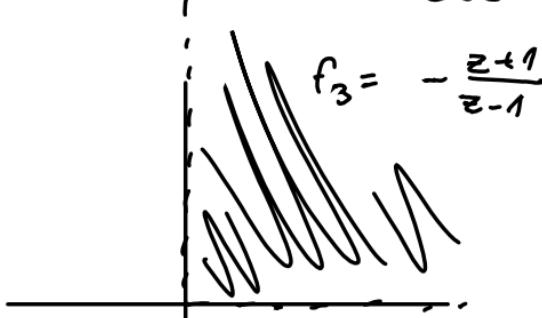


$f_3:$

$g(-1) = 0 \quad \frac{az+b}{cz+d}$

$g(0) = 1$

$g(\infty) = \infty \quad \frac{a(z+1)}{c(z-1)} - \frac{a}{c} = 1$



$$f_4: z^2$$

$$f_5: H \rightarrow \Delta$$

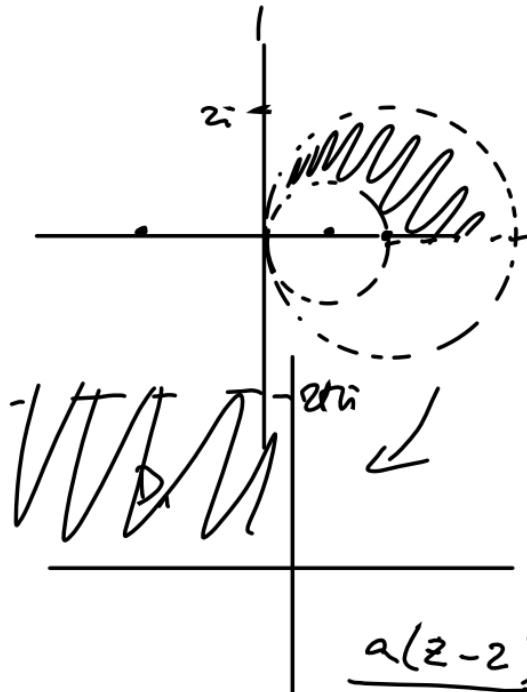


$$g = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$$



$$D = \{z \in \mathbb{C} \mid |z-1| > 1 \text{ and } |z-2| < 2\}$$

$\operatorname{Im}(z) \in (0, 2\pi)$



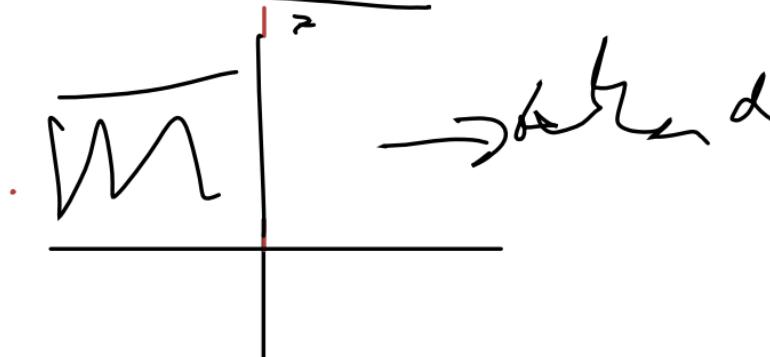
$$\frac{\alpha(z-2)}{cz}$$

$$\frac{\alpha}{c} \frac{4-2}{4} = \frac{\alpha}{c} \frac{1}{2} = 2\pi i;$$

$$\frac{\alpha}{c} =$$

$$\frac{\alpha}{c} = 4\pi i$$

$$\frac{4\pi i(z-2)}{z}$$



Dolacij vse pove celih funkcij
za katero je $|f(z)|^2 + |g(z)|^2 = 1$

cela funkcija - holomorfna na \mathbb{C}
kandidata:

$$f(z) = \sin(h(z)) \quad g(z) = \cos(h(z))$$

$$(f(z))^2 + (g(z))^2 = (\underbrace{f(z) + ig(z)}_{k(z)}) (\underbrace{f(z) - ig(z)}_{\ell(z)}) = 1$$

k nima nide

Izrek: Če je $k \in O(D)$ in je D enostavno
povezano, \exists veja $\ln(k(z)) \in O(D)$
 $(k(z) = e^{\ell(z)})$ $\overbrace{\ell(z)}$

$D = \mathbb{C}$ enostavno povezano

$$k(z) = e^{\ell(z)}$$

$$2f(z) = e^{i\ell(z)} - e^{-i\ell(z)}$$

$$f(z) = \cos(\ell(z))$$

$$g(z) = \sin(\ell(z))$$

f je holomorfna na povezanij od prati
okolici $\infty \in \mathbb{CP}^1$

Dolozit f , $\exists c \geq n \in \mathbb{N}$ $n \geq n_0$ velj.

$$f(z) = \frac{n^2 \sqrt[n]{c}}{z+n^2}$$

f holomorfna u $\infty \Leftrightarrow f\left(\frac{1}{z}\right)$ holomorfna u 0

Princip identitete