

# Hilbertov prostor

X vektorish. prostor nad R (nad D)  
skalarni produkt  $\langle \cdot, \cdot \rangle$

$$\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$$

$$1) \forall x \in X : \langle x, x \rangle \geq 0$$

$$2) \langle x, x \rangle = 0 \iff x = 0$$

$$3) \forall x, y \in X.$$

$$\langle x, y \rangle = \langle y, x \rangle \text{ nad } \mathbb{R}$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \text{ nad } \mathbb{C}$$

simetričnost oz antisimetričnost

$$4) \forall x, y, z \in X, \forall \lambda, \mu \in \mathbb{R}$$

$$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$$

linearnost upravn faktorju

Za skleni: praktično  
Cavatij - Schwarzova neenakost

$\forall x, y \in X$

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Dokaz nad  $\mathbb{R}$

$$t \mapsto \langle x + ty, x + ty \rangle = s(t) \geq 0$$

$$= \langle x, x \rangle + 2t \langle x, y \rangle + t^2 \langle y, y \rangle$$

$$= \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0$$

$$D \leq 0$$

$$D^2 = 4t^2 \langle x, y \rangle - 4\|x\|^2 \|y\|^2$$

$$\Rightarrow \langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$$

$$\langle x, y \rangle \leq \|x\|^2 \|y\|^2$$

Enako veljače za  $x$  in  $y$   
natančno težaj

linearne aditivne

Nad C:

$$x, y$$

$$\exists \alpha \quad |\alpha| = 1$$

$$\langle x, y \rangle = \alpha |\langle x, y \rangle|$$

$$\Rightarrow \langle x, \alpha y \rangle = |\langle x, y \rangle|$$

$$f(t) = \langle x + t\alpha y, x + t\alpha y \rangle =$$

$$= \|x\|^2 + t \langle \alpha y, x \rangle + \langle x, \alpha y \rangle + t^2 \langle \alpha y, \alpha y \rangle \\ = \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2$$

$$= \|x\|^2 + 2t |\langle x, y \rangle| + t^2 \|y\|^2 \geq 0$$

$$D = 4|\langle x, y \rangle|^2 - 4\|x\|^2 \|y\|^2 \leq 0$$

Norm nedan är det enligt definition

$$\| \cdot \| : X \rightarrow \mathbb{R}$$

$$1) \forall x \in X. \|x\| \geq 0$$

$$2) \|x\| = 0 \Leftrightarrow x = 0$$

$$3) \forall \lambda \in \mathbb{R} \forall x \in X,$$

$$\| \lambda x \| = |\lambda| \|x\| \quad \text{homogenitet}$$

$$4) \text{triangelnska nevenlast}$$

$$\forall x, y \in X. \|x+y\| \leq \|x\| + \|y\|$$

är je  $(X, \langle \cdot, \cdot \rangle)$  vektorslu proster  
s skalarn:m produkten je

$$\|x\| = \sqrt{\langle x, x \rangle} \quad X \text{ vektorslu proster}$$

$\simeq$  normo

$$\|x\| = \sqrt{\langle x, x \rangle}$$

1), 2) Skeli iz 1) 2) Skalarne proizv.

$$3) \|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{|\lambda|^2 \langle x, x \rangle} = |\lambda| \|x\|$$

4)

$$\|x+y\|^2 = \langle x+y, x+y \rangle =$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle =$$

$$= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \leq \|x\|^2 + 2|\langle x, y \rangle|$$

$$+ \|y\|^2$$

$$\leq \lambda^2 \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = \|x\|^2 + \|y\|^2$$

$$(X, <\cdot, \cdot>) \leadsto (X, \| \cdot \|)$$

leadsto metrični prostor  $(X, d)$

$$\forall x, y \in X. \quad d(x, y) = \|x - y\|$$

$$1) d(x, y) \geq 0$$

$$2) d(x, y) = 0 \iff x = y$$

$$3) d(x, y) = d(y, x)$$

$$4) \forall x, y, z \in X$$

$$d(x, y) + d(y, z) \geq d(x, z)$$

Def: Hilbertov prostor je vektorski prostor s skalarnim produktom ki je v metriki posredovan iz skalarnega produkta poln metričnega prostora

Opoomba: Banachov prostor je vektorski prostor  $X$  z normo  $\|\cdot\|$  ki je v metriki posredovan iz norme poln metričnega prostora

Zufällig:

$$1) \mathbb{R}^n, \quad x = (x_1 \dots x_n)$$

$$y = (y_1 \dots y_n)$$

$$x \cdot y = x_1 y_1 + \dots + x_n y_n$$

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \quad d_2 \text{ metrik}$$

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \quad \text{je}$$

poln metrik: präzise

$$(\mathbb{R}^n, \cdot) \text{ je Hilberträum präzise}$$

$(\mathbb{R}^n, d_\infty)$  max  $\{|x_1|, \dots, |x_n|\}$

norm

$(\mathbb{R}^n, d_1)$   $|x_1| + \dots + |x_n|$

~~Banachova postava  
ke size topolosko mesta  $(\mathbb{R}^n, d_1)$~~

~~$\mathbb{R}^n$  je Banachova ke size  
topoloski mesto  $(\mathbb{R}^2, d_2)$~~

amgle ne pride il ekstrarze  
predete

$$2) \quad \mathbb{C}^n \quad z = z_1, \dots, z_n \\ w = w_1, \dots, w_n$$

$$z \cdot w = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

$$\|z\| = \sqrt{z_1^2 + \dots + z_n^2}$$

$$d_z(z, w) = \sqrt{|z_1 - w_1|^2 + \dots + |z_n - w_n|^2}$$

$(\mathbb{C}^n, \cdot)$  je Hilberträum Prostov

Zugel

$$[a, b] \subset \mathbb{R} \quad a < b$$

$$X = C([a, b])$$

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx \quad \text{mit } \mathbb{R}$$

$$3) \langle f, g \rangle = \langle g, f \rangle$$

$$4) \langle f + \mu g, h \rangle = \\ \xrightarrow{\quad} \langle f, h \rangle + \mu \langle g, h \rangle$$

$$1) \cancel{\langle f, f \rangle} = \int_a^b f^2(x) dx \geq 0$$

$$2) f = 0 \Rightarrow \langle f, f \rangle = 0$$

$$\xrightarrow{\quad} \int_a^b f^2(x) dx = 0 \Rightarrow f^2 = 0 \text{ stetig auf } [a, b]$$

$\Rightarrow f = 0$  parab. Kurve zuerst

$\Rightarrow f = 0$  (Niedrigstrahl)

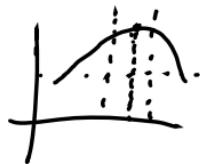
Punkt?

$$\int_a^b f^2(x) dx = 0$$

$\Rightarrow \forall x \in [a, b] f(x) = 0 \Rightarrow$

$$\exists \delta > 0. \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

$$|f(x)| \geq \frac{|f(x_0)|}{2}$$



$$\int_a^b f^2(x) dx \geq \int_{(x_0-\delta, x_0+\delta) \cap [a, b]} f^2(x) dx \geq \frac{f(x_0)^2}{2} \delta$$

$$(x_0 - \delta, x_0 + \delta) \cap [a, b]$$

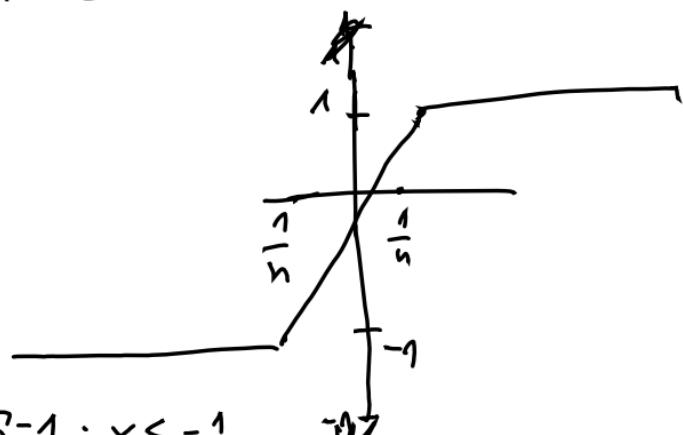
$\Rightarrow f \text{ zero} \Rightarrow f = 0 \text{ passed}$

Trditev:  $(C([-1, 1]), <>)$

n: Hilbertov

Dokaz

$f_n(x)$



$$f_n(x) = \begin{cases} -1 & : x \leq -\frac{1}{n} \\ 1 & : x \geq \frac{1}{n} \\ nx & : -\frac{1}{n} \leq x \leq \frac{1}{n} \end{cases}$$

$(f_n)_n$  je Cauchyjeva zaporedje v  $(C([-1, 1]), d)$   
d-metrice je skalarne produkti

$$d(f, g) = \int_{-1}^1 |f(x) - g(x)|^2 dx$$

$m > n$

$$d(f_n, f_m)^2 =$$

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} (f_n(x) - f_m(x))^2 dx =$$
$$\int_{-\frac{1}{n}}^{\frac{1}{n}} (f_m(x) - f_n(x))^2 dx \leq \int_{-\frac{1}{n}}^{\frac{1}{n}} 1 dx = \frac{2}{n}$$

$$d(f_n, f_m) \leq \sqrt{\frac{2}{n}} < \epsilon$$

$(f_n)_n$  je Cauchyjeva v  $(C([-1, 1]), d)$

$\lim f_n$   
če obstaja  $\lim f_n$  na  $C([-1, 1])$   
meri vejeti, da je  $f(x) = \begin{cases} 1 & : x \leq 1 \\ -1 & : -1 \leq x < 0 \end{cases}$

$\Rightarrow f_n$ : vira v 0

X

Zadaci

$$M = (0, 1)$$

$$d_2(x, y) = |x - y|$$

$\forall M \in \mathbb{N}$ : poln

$M \cup \{0, 1\}$  je pdn

Dodatak: smo mogli imati

Nepotpunite metričke prostore

(Md) nepotpuno  
lakška napodnemo

( $\bar{M}, d$ ) pdn

1)  $M \subseteq \bar{M}$

2)  $\overline{d}_{\bar{M} \times \bar{M}} = d$

3)  $M$  je gost v  $\bar{M}$

$$L^1(A) = \left\{ f: A \rightarrow \mathbb{R} ; \int_A |f(x)| dx < \infty \right\}$$

$\hookrightarrow$  kvadratnointegrable funkecije:

$$L^2([a,b]) = \left\{ f: [a,b] \rightarrow \mathbb{R} ; \int_a^b |f(x)|^2 dx < \infty \right\}$$

Zadaci:

$$1) C([a,b]) \subseteq L^2([a,b])$$

$$2) \text{odsek k na zemlji} \subseteq L^2(\mathbb{Q})$$

$$3) f(x) = \frac{1}{\sqrt[4]{x-a}} \in L^2$$

$$4) g(x) = \frac{1}{\sqrt[2]{x-a}} \notin L^2([a,b])$$

Vrijam:  $f \cdot g$

$$|f \cdot g| \leq \frac{|f|^2 + |g|^2}{2} \Rightarrow f \cdot g \in L^1([a,b])$$

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx$$

$L^2$  je vektorski prostor

$$(f+g)^2 = f^2 + 2fg + g^2$$
$$\in L^2 \quad C_1 \quad \in L^2$$

$$\begin{aligned} \int_a^b (f+g)^2 dx &= \int_a^b f^2 dx + 2 \int_a^b fg dx + \int_a^b g^2 dx \\ &+ \int_a^b g^2 dx < \infty \end{aligned}$$

$$\Rightarrow (f+g) \in L^2$$

$C[a,b] \subseteq L^2([a,b])$   
Hilberträume

Opomba

$\forall f \in L^2([a,b]). \exists f_n \in C[a,b].$

$$\lim_{n \rightarrow \infty} f_n = f \iff \lim_{n \rightarrow \infty} \|f_n - f\| = 0$$

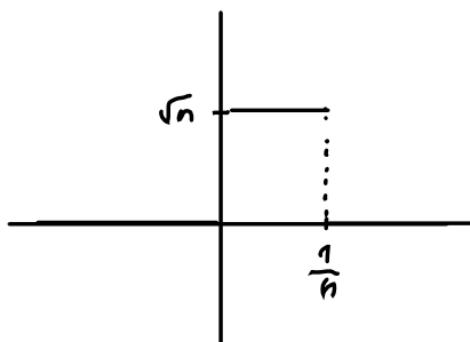
$$\lim_{n \rightarrow \infty} \int_a^b (f_n(x) - f(x))^2 dx = 0$$

Opomba: nach C:  $f = u + v$   $u, v: [a,b] \rightarrow \mathbb{R}$

$$\int_a^b f(x) dx = \int_a^b u(x) dx + \int_a^b v(x) dx$$

Zuged:  $x \in [0, 1]$

$$f_n: \begin{cases} \sqrt{n} & ; 0 < x \leq \frac{1}{n} \\ 0 & ; \text{sonst} \end{cases}$$

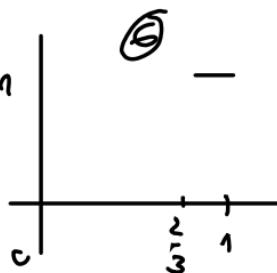
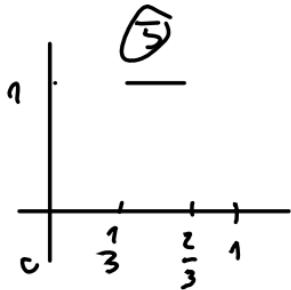
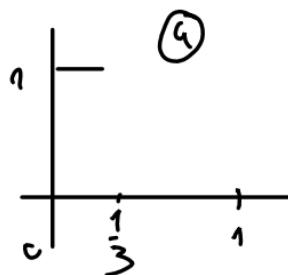
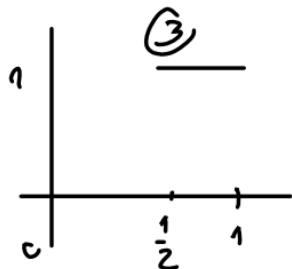
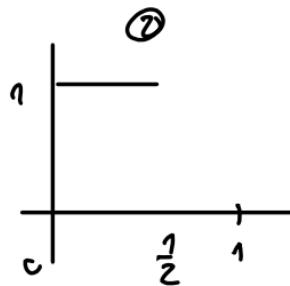
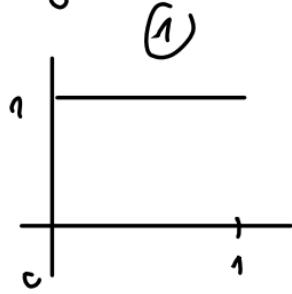


$$\lim f_n(x) = 0 \quad \forall x \in [0, 1]$$

Akt: konvergiert  $\nu L^2$ ?

$$\|f_n - 0\| = \sqrt{\int_0^1 f_n^2(x) dx} = \sqrt{\int_0^{1/n} \sqrt{n}^2 dx} = 1$$

Zgued:



:td

$\lim_{n \rightarrow \infty} f_n(x)$  ne obstege

$\lim f_n(x) \rightarrow L^2 ?$

$$\int_0^1 f_n^2(x) dx \leq \frac{1}{n^2} \text{ tangi limite je } 0$$

Naj bo  $(X, \langle \cdot, \cdot \rangle)$  vek. prostor s  
skalarnim produkтом.



$$A \subseteq X \quad (A \neq \emptyset)$$

$$x, y \in X. \quad x \perp y \Leftrightarrow \langle x, y \rangle = 0$$

(symetria relacji)

$$A \subseteq X \quad A^\perp = \{x \in X. \quad x \perp a \quad \forall a \in A\}$$

Ortogonalny komplement

Trditev:  $A^\perp$  je vektorski podprostor  $X$

Dokaz:

$$x \in A^\perp \rightarrow \lambda \in \mathbb{R} \text{ (ali } 0)$$

$$\forall a \in A. \langle \lambda x, a \rangle \Rightarrow \langle x, a \rangle = \lambda \cdot 0 = 0$$

$$x, y \in A^\perp$$

$$\langle x+y, a \rangle = \langle x, a \rangle + \langle y, a \rangle \stackrel{\text{def}}{=} 0 + 0 = 0$$

Velja:  $A \subseteq (\mathbb{A}^\perp)^\perp$

Trećitev:  $v \in X$

$$f(x)v = \langle x, v \rangle \quad f: X \rightarrow \mathbb{R}$$

$f$  je zvezna na  $X$

Dokaz:

$$x_1, x_2 \in X$$

$$|f(x_1) - f(x_2)| = |\langle x_1 - x_2, v \rangle| \leq \|x_1 - x_2\| \|v\|$$

$f$  je enakomerna zvezna  
(celo Lipschitova zvezna)

Posledica:  $A^\perp$  je zaprt vektorški podprostor

$$C[a,b] \subseteq L^2[a,b] \text{ ni zaprt podprostor}$$

Dokaz:

izpaređje  $x_n$   
 $\lim x_n = x_0 \in X \xrightarrow{?} x_0 \in A^\perp$

$$\forall a \in A. \langle x_n, a \rangle = 0 \quad \forall n$$

$$\lim_{\substack{\longrightarrow \\ n}} \langle x_n, a \rangle = \langle \lim_{\substack{\longrightarrow \\ n}} x_n, a \rangle = \langle x_0, a \rangle = 0$$

Opozme:  $(X, \langle \cdot \rangle)$  hilbertov

$$A \subseteq X \text{ zaprt podprostor} \Rightarrow A = (A^\perp)^\perp$$

Trditev: (Prstagonalni izrek)

$(X, \langle \cdot, \cdot \rangle)$  vektorski prostor s skalarnim produkтом

$$x_1, \dots, x_n \in X \quad \forall i, j \in [n], \quad x_j \perp x_i \quad (\langle x_j, x_i \rangle = 0)$$

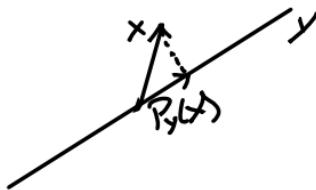
$$\text{Tedaj} \quad \|x_1\|^2 + \dots + \|x_n\|^2 = \|x_1 + \dots + x_n\|^2$$

Dokaz:

$$\|x_1 + \dots + x_n\| = \langle x_1 + x_2 + \dots + x_n, x_1 + \dots + x_n \rangle$$

$$= \left\langle \sum_i x_i; \sum_j x_j \right\rangle = \sum_i \sum_j \langle x_i, x_j \rangle =$$

$$= \|x_1\|^2 + \dots + \|x_n\|^2$$

$(X, < \cdot >)$  $y \leq x$  $x \in X$ 

Definicija: Pravokotna projekcija vektora  $x$  na podprostor  $Y$  (če obstaja) je tak vektor  $P_Y(x) \in Y$  da je

$$x - P_Y(x) \in Y^\perp$$

Trditve: če pravokotna projekcija

$x \in Y$  obstaja, je enolio dobera in  $P_y(x)$  je najboljša aproksimacija vektorskega  $X$  z veletorji iz  $Y$   
(razdejla jo nepravilno)  
 $\|x - P_y(x)\| = \min_{w \in Y} \|x - w\|$

Dokaz:

$$Y \leq X \quad x \in X$$

Dovimo da  $y_1, y_2$  sta pravokotni projektorji:  
 $x \in Y$

$$x - y_1, x - y_2 \in Y^\perp$$

$$(x - y_1) - (x - y_2) \in Y^\perp \quad \text{kor. vek. prostora}$$

$$y_2 - y_1 \in Y^\perp$$

$$y_2 - y_1 \in Y$$

$$\langle y_2 - y_1, y_2 - y_1 \rangle = 0 \iff$$

$$y_2 - y_1 = 0 \quad y_2 = y_1$$

$$w \in Y$$

$$x - w = \underbrace{x - P_y(x)}_{\in Y^\perp} + \underbrace{P_y(x) - w}_{\in Y}$$

pitagon : zek:

$$\|x - w\|^2 = \|x - P_y(x)\|^2 + \|P_y(x) - w\|^2 \geq$$

$$\|x - P_y(x)\|^2$$

Zagled:

$$y = C[a, b] \quad X = L^2[a, b]$$

$f \in X$  - y nime pravokotne projekcije

Tek f nima najbolje aproksimacije  
z veznim funkcijami

Komentar: hilbertov & zaprsten  
ime projekcije

Opozme:

$$1) P_y^2 = P_y$$

$$2) x = \underbrace{x - P_y(x)}_{\|x\|^2} + \underbrace{P_y(x)}_{\|x\| \geq \|P_y(x)\|}$$

$$\|x\|^2 = \|x - P_y(x)\|^2 + \|P_y(x)\|^2$$

$$\|x\| \geq \|P_y(x)\|$$

3)  $P_y$  definisane na celom  $X$ ,  
potonje  $P_y$  linearan in even

$$\|P_{y_1} - P_{y_2}\| = \|P_y(x_1 - x_2)\| \leq \|x_1 - x_2\|$$

$P_y$  je even  
linearnost:

$$\lambda x - P_y(\lambda x)$$

Preglejmo:

$$\lambda x - \lambda P_y(x) = \lambda \underbrace{(x - P_y(x))}_{\in y^\perp}$$

$$\Rightarrow \lambda P_y(x) \in y^\perp$$

$$\lambda x - \lambda P_y(x) \in y^\perp$$

to je pravotna projekcija

$$x_1, x_2 \in X$$

$$P_y(x_1 + x_2) = \underbrace{P_y(x_1)}_{\in y^\perp} + \underbrace{P_y(x_2)}_{\in y^\perp}$$

$$(x_1 + x_2) - \underbrace{(P_y(x_1) + P_y(x_2))}_{\in y^\perp} = x_1 - P_y(x_1) + x_2 - P_y(x_2) \in y^\perp$$

$$\Rightarrow P_y(x_1 + x_2) = P_y(x_1) + P_y(x_2)$$

zad: enačba projekcije

■

~~je  $P_y$  definisan~~

Ce je  $P_Y$  definovan na  $X$  je  
 $Y$  zaprt prostor

Dokaz:

$$\{y_j\} \subseteq Y \quad \lim_{j \rightarrow \infty} y_j = y_0$$

$$P_Y(y_j) = y_j$$

~~$$\lim y_j = \lim P_Y(y_j) = P_Y(y_0)$$~~

||

$y_0$

Če ima  $x$  pravokotna projekcija na  $y$   
torej tudi pravokotna projekcija na  $y^\perp$

$$x; P_y(x)$$

$$P_{y^\perp}(x) = x - P_y(x) \in y^\perp$$

$$x - (x - P_y(x)) \in (y^\perp)^\perp$$

$$\overset{\text{"}}{P_y(x)} \in y \subseteq (y^\perp)^\perp$$

Trditev: Nej bo  $Y \leq X$  končno dimenzionalni podprostor z ortonormirano bazo  $e_1, \dots, e_n$ .  $\langle e_i, e_j \rangle = \delta_{ij}$

Nej bo  $x \in X$ . Teda je

$$P_Y(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i$$



Opomba: Vsak končno dimenzionalni podprostor ima pravokotno projekcijo definirano na  $X$ . In tudi vsi tisti konane kodimenzije

Dokaz:  $P_Y(x) = \sum_1^n \langle x, e_i \rangle e_i \in Y$

Poštejmo  $\sum_1^n \langle x, e_j \rangle e_j$

$$x - \sum_1^n \langle x, e_j \rangle e_j \in Y^\perp$$

$$\langle x - \sum_1^n \langle x, e_j \rangle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \sum_1^n \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$(X, \langle \cdot, \cdot \rangle)$

Sistem vektorer

$$(e_j)_{j=1}^{\infty}$$

je ortogonalen sistem (OS), oe

$$\forall i \neq j \quad \langle e_i, e_j \rangle = 0$$

Tak sistem je ortonormiran (ONS)

$$\forall i, j \quad \langle e_i, e_j \rangle = \delta_{ij}$$

Trditev:  $(X, \langle \cdot, \cdot \rangle)$  Maj bo  $(e_j)$ ; ONS  
 Maj bo  $x \in X$ . Teden je  $\sum |\langle x, e_j \rangle|^2 \leq \|x\|^2$

$\nearrow$   
 $(\text{Besselova neenakost})$

Opomba:  $\langle x, e_j \rangle$  so Fourierovi koeficienti  
 $x$  po ONS  $(e_j)$ ;

Posledica:

$$\lim_{j \rightarrow \infty} \langle x, e_j \rangle = 0$$

Dokaz:  $y_n = \mathcal{L}(\{e_1, \dots, e_n\})$

$$\begin{aligned} & x \in X \\ & \exists P_y(x) = \sum_1^n \langle x, e_j \rangle e_j \quad \begin{array}{l} \text{je projekcija} \\ \text{je velenje krajša od } x \end{array} \\ & \|P_y(x)\|^2 = \sum_1^n \|\langle x, e_j \rangle e_j\|^2 = \sum_1^n |\langle x, e_j \rangle|^2 \leq \|x\|^2 \\ \Rightarrow & \lim_{n \rightarrow \infty} \sum_1^n |\langle x, e_j \rangle|^2 = \sum_1^{\infty} |\langle x, e_j \rangle|^2 \leq \|x\|^2 \end{aligned}$$

Trditev: Naj bo  $(c_j)_{j=1}^{\infty}$  zaporedje  
steril (ali: IR ali: C) za katero velja

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty$$

Naj bo  $(\star, <)$  Hilbertov prostor

$$(e_j)_{j=1}^{\infty} \text{ ONS}$$

Tedaj  $\exists x \in X$ , za katerega velja  $c_j = \langle x, e_j \rangle$   
za vse  $j$ .

$$x = \sum_{j=1}^{\infty} c_j e_j = \lim \sum_{j=1}^{N_j} c_j e_j$$

$(X, \langle \cdot, \cdot \rangle)$  h: libertan e\_j ONs

$$x \in X \rightsquigarrow (\langle x, e_j \rangle)_j \quad \sum_j^{\infty} |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

$$\Rightarrow \exists \tilde{x} = \sum_j^{\infty} \langle x, e_j \rangle e_j$$

$$\forall i: j \in \tilde{x} = x$$

H

Zemude

20.2



Zagled (pred prejšnjim delom)  
Modelni hilbertov prostor  $\ell^2$

Prostor zaporedij

$$\ell^2 = \{(\alpha_j)_j; \alpha_j \in \mathbb{R}, \sum_1^{\infty} |\alpha_j|^2 < \infty\}$$

$$\langle (\alpha_j), (\beta_j) \rangle = \sum \alpha_j \beta_j \text{ nad } \mathbb{R}$$

$$\sum \alpha_j \bar{\beta}_j \text{ nad } \mathbb{C}$$

$$\|\alpha_j\| = \sqrt{\sum |\alpha_j|^2}$$

$(\langle \cdot, \cdot \rangle)$  hilbertov (g) kans

$$x \mapsto \underbrace{\langle x, e_j \rangle}_j \in \ell^2$$

$$e_j = (0 \dots 0, 1, 0 \dots 0)$$

Dokaz:  
 $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$   
 $1 \Rightarrow 6 \Rightarrow 5$

1)  $\Rightarrow 2)$

$$x, y \in X \quad x = \sum_1^{\infty} \langle x, e_j \rangle e_j$$

$$\langle x, y \rangle = \left\langle \sum_1^{\infty} \langle x, e_j \rangle e_j, y \right\rangle =$$

$$= \sum_1^{\infty} \langle x, e_j \rangle \langle e_j, y \rangle$$

2)  $\Rightarrow 3)$

$$x = y \quad \langle x, y \rangle = \|x\|^2 = \sum_1^{\infty} |\langle x, e_j \rangle|^2$$

3)  $\Rightarrow 4)$  ons

čebel:  $\|e_i\| \neq 1$  vsebovan v strogo večjem osnovnem podmnožišču  $e_0 \perp e_j \forall j \quad \|e_0\|=1$

Vstavimo v parcerovalo enakost

$$\|e_0\|^2 = 1^2 = \sum_1^{\infty} |\langle e_0, e_j \rangle|^2 = \sum_1^{\infty} 0 = 0$$

4)  $\Rightarrow 5)$

$$x \perp e_j \forall j$$

$$\text{če } x \neq 0 \text{ trdimo } e_0 = \frac{x}{\|x\|}$$

$(e_0, e_j)$  je strogo večji; konš, ker ne more biti  $\rightarrow$

5)  $\Rightarrow 1)$

$$x \in X \quad \langle x, e_j \rangle_j$$

$$\tilde{x} = \sum_1^{\infty} \langle x, e_j \rangle e_j \quad \text{ali skr enako?}$$

$$\tilde{v} = x - \tilde{x} = x - \sum_1^{\infty} \langle x, e_j \rangle e_j$$

$$\langle v, e_i \rangle = \langle x, e_i \rangle - \sum_1^{\infty} \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$$\tilde{v} = 0 \Rightarrow x = \tilde{x}$$

$$1) \Rightarrow 6) \quad \forall x = \sum_1^{\infty} \langle x, e_j \rangle e_j = \lim \sum_1^N \langle x, e_j \rangle e_j$$

$$\forall \exists > 0 \cdot \exists N \in \mathbb{N} \cdot \|x - \sum_1^N \langle x, e_j \rangle e_j\| < \epsilon$$

6)  $\Rightarrow 5)$

$$\text{Nej bo } x \perp e_j \forall j$$

$$x = 0$$

$$3>0$$

Exkončna linearne kombinacija  $\sum_1^N \lambda_j e_j$

$$\text{daje } \|x - \sum_1^N \lambda_j e_j\| < \epsilon$$

$$\|x\|^2 = \langle x, x \rangle = \langle x - \sum_1^N \lambda_j e_j, x \rangle \leq$$

$$\leq \|x - \sum_1^N \lambda_j e_j\| \cdot \|x\| < \epsilon \|x\|$$

$$1) x = 0 \Rightarrow \checkmark$$

$$2) x \neq 0 \Rightarrow \|x\| < \epsilon \quad \text{a} \vee \epsilon > 0 \Rightarrow x = 0 \quad *$$

Osnadatooima se na  $L^2(-\pi, \pi)$

ONS:  $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \dots$

$\dots, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx), \dots$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx$$

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} f^2(x) dx}$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \quad \|f_0\|_2 = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi} dx} = 1$$

$$\sqrt{\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin^2(nx) dx} = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(2nx)}{2} dx} > 1$$

Opomba: nad  $\Phi: \frac{1}{2\pi} e^{inx} \quad n \in \mathbb{Z}$  je vsemero  
na ONS

$n \neq m$

$$\langle f_n, f_m \rangle = \int_{-\pi}^{\pi} \frac{1}{\pi} e^{inx} e^{imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

$$= \frac{1}{2\pi(n-m)} \left[ e^{i(n-m)x} \right]_{-\pi}^{\pi} = 0$$

$f: (-\pi, \pi) \rightarrow \mathbb{R}$  periodična funkcija  
 $\Leftrightarrow$  perioda  $2\pi$

Koeficijenti Fourierovi:

$$f \in L^2(-\pi, \pi) \quad (\text{Rečimo da imamo kavz})$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n \in \{0, 1, \dots\}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) = \langle f, \frac{1}{\pi} \cos(nx) \rangle =$$

$$= \frac{1}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \rangle = \sqrt{\pi} a_n$$

$$\langle f, \frac{1}{\pi} \sin(nx) \rangle = \sqrt{\pi} b_n$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot 1 dx = \langle f, \frac{1}{\pi} \rangle = \frac{\sqrt{2}}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle =$$

$$\langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{\sqrt{\pi}}{\sqrt{2}} a_0$$

Če je  $\left\{ \frac{1}{\sqrt{2n}}, \frac{1}{\sqrt{n}} \cos x, \dots \right\}$  KNS  
 potem velja parcijskova enakost

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{\pi}{2} a_n + \sum_{n=1}^{\infty} \pi (|a_n|^2 + |b_n|^2)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$

Postavica:

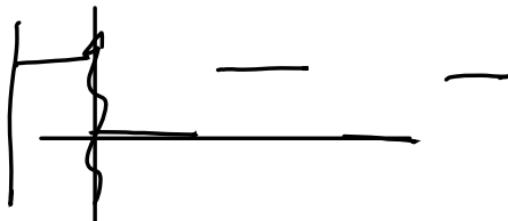
Ker je  $\left\{ \frac{1}{\sqrt{2n}}, \dots \right\}$  ONS velja

$$\lim_{n \rightarrow \infty} a_n = 0 \quad \lim_{n \rightarrow \infty} b_n = 0$$

(Riemann-Lebegueova lema)

Zglicd

$$f(x) = \begin{cases} 1; & 0 \leq x \leq \pi \\ 0; & -\pi < x < 0 \end{cases}$$



$$\begin{aligned}a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \\&= \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx\end{aligned}$$

$$a_0 = 1 \quad n > 0$$

$$a_n = \frac{1}{\pi} \cdot \frac{1}{n} \left. \sin(nx) \right|_0^{\pi}$$

$$\begin{aligned}b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx = -\frac{1}{\pi n} \cos(nx) \Big|_0^{\pi} \\&= \frac{1}{n\pi} (1 - (-1)^n)\end{aligned}$$

$$b_{2k} = 0$$

$$b_{2k+1} = \frac{2}{\pi(2k+1)}$$

KONS:

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x)$$

$\rightarrow 0$

Parsevalova enekost

$$\frac{1}{\pi} \pi = 1 = \frac{1}{2} + \sum_0^{\infty} \frac{4}{\pi^2} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\frac{3}{4} S = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}$$

Izrek:

Naj bo  $f$  odsekoma zvezna in odsekoma odvedljiva periodična funkcija s periodo  $2\pi$

1) Na vsakem intervalu dolžine  $2\pi$  ima največ končno mnogo točk neveznosti in v vseh točki obstajajo levi in desni limiti

$\overbrace{\qquad\qquad\qquad}^{\text{oznaki}}$

$$x_0 \in \mathbb{R}. \exists \lim_{\substack{x \rightarrow x_0}} f(x) = f(x_0 - 0) = f(x_0^-)$$

$$\exists. \lim_{x \downarrow x_0} f(x) = f(x_0 + 0) = f(x_0^+)$$

2) V vseh točki obstajajo levi in desni odvod

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0^-)}{h}$$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0^-)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) - f(x_0^-)}{(-h)}$$

Tedaj  $\forall x \in \mathbb{R}$ . velja  $\frac{f(x+) + f(x-)}{2} =$

$$= \frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

## Panazne trditeve

1) Naj bo  $f: \mathbb{R} \rightarrow \mathbb{R}$  periodična s periodom  $p$   
odsekoma weza  
Teda je  $\int_a^{a+p} f(x) dx = \int_a^p f(x) dx$

### Dokoz:

$$\int_{a}^{a+p} f(x) dx = \int_{a}^r f(x) dx + \int_r^{a+p} f(x) dx$$

$$x = t + \rho$$

$$= \int_a^p f(x) dx + \int_0^a f(t+p) dt =$$

||  
 $f(t)$

$$= \int_a^b p(x) dx + \int_a^b f(x) dx = \int_a^b p(x) dx$$

$$z) \frac{1}{2} + \sum_1^n \cos(kx) = \frac{1}{2} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = D_n(x)$$

(Dirichleova sreda)

Dоказ:

$$\begin{aligned}
 \frac{1}{2} + \sum_1^n \cos(kx) &= \frac{1}{2} + \sum_1^n \frac{e^{ikx} + e^{-ikx}}{2} = \\
 &= \frac{1}{2} \sum_{j=-n}^n e^{ijx} = \frac{1}{2} e^{inx} (1 + e^{ix} + e^{2ix} + \dots + e^{2nx}) \\
 &= \frac{1}{2} e^{-inx} \frac{1 - e^{(2n+1)ix}}{1 - e^{ix}} = \\
 &= \frac{1}{2} \frac{e^{(n+1)ix} - e^{-inx}}{e^{ix} - 1} = \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x} - e^{i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} \\
 &\Rightarrow \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin \frac{x}{2}}
 \end{aligned}$$

3)

$$\int_{-\pi}^{\pi} D_n(x) dx = \pi$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x) dx = 1$$

4)  $D_n(x)$  je sada funkcija, 2π periodična,  
gladka.

$$\begin{aligned}
 5) \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} &= \\
 = \frac{1}{2} \frac{\sin(nx)\cos\frac{x}{2} + \cos(nx)\sin\frac{x}{2}}{\sin\frac{x}{2}} &= \\
 = \frac{1}{2} \left( \sin(nx) \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \cos(nx) \right)
 \end{aligned}$$

Opombe:

f je lihe na  $(-\pi, \pi)$

$$a_n = 0$$

f razvijamo samo po  $\{\sin(nx)\}$

f sode na  $(-\pi, \pi) \Rightarrow b_n = 0$

f razvijemo le po  $\{1, \cos(nx)\}$

f iz  $[0, \pi]$  lako liko rezirimo na  $(-\pi, \pi)$

$$x \mapsto \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

Tako funkcijo razvijamo le po  $\{\sin(nx)\}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{novi } b_n$$

$$\int_0^\pi f(x) \sin(nx) dx = b_n \int_0^\pi \frac{1 - \cos(2nx)}{2} dx = \frac{\pi}{2} b_n$$

če frazirimo le sode funkcijo na  $(-\pi, \pi)$

$$x \mapsto \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

Tako funkcijo razvijamo po  $\{1, \cos(nx)\}$

Opambe 2:

$f \in C^k(\mathbb{R})$  nperiode  $k \in \mathbb{N}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\cos(nx)}_{u} dx =$$

$$= \frac{1}{\pi} \left( \underbrace{\frac{1}{n} \sin(nx) f(x)}_{=0} \Big|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx \right)$$

= ... nekadimo to k kat... =

$$a_n = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx$$

$$a_n = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f''(x) \cos(nx) dx$$

:

$$a_n = \frac{\pm 1}{\pi} \int_{-\pi}^{\pi} f^{(n)}(x) \overset{\text{omejena}}{\underset{\text{nekad to je}}{\overset{\uparrow}{\cos(nx)}}} dx$$

$$a_n, b_n = O\left(\frac{1}{n^2}\right)$$

K velikosti red

Vektor k atje odredjiva f (kat periodična funkcija) h. treće grede Fourierovi koeficijenti pribl. 0

Doktor

$f \in L^2(-\pi, \pi)$

$$\sqrt{\int_{-\pi}^{\pi} |f(x)|^2 dx} = \|f\|_2$$

Klasse von Fourierreihen koeffizienten:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n = 1, 2, \dots$$

$$f_{\text{Four}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Parsevalova enakost

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

f periodična s periodom  $2\pi$  in odsekoma

merjene

•  $\forall x_0 \in \mathbb{R}, \exists f(x_0+), f(x_0-)$  limiti in

• odsekoma odvečljiva

• dostopljata lev: in desni: odvod

$$\text{Potem: } \forall x_0: \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \\ = \frac{f(x_0+) + f(x_0-)}{2}$$

Vemo:

• f periodična s periodo  $p$

•  $\forall x \in \mathbb{R}, \int f(x) dx = \int f(x+p) dx$

$$\cdot \frac{1}{2} + \sum_{n=1}^N \cos(nx) = D_N(x) = \frac{1}{2} \frac{\sin(N + \frac{1}{2})x}{\sin(\frac{1}{2}x)}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_N(x) dx = 1 \quad \text{soda} \\ \frac{2}{\pi} \int_0^{\pi} D_N(x) dx$$

Doktor >>

Dokaz:

$$\begin{aligned} a_n \cos(nx_0) + b_n \sin(nx_0) &= \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \cdot \cos(nx_0) + \\ &\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \cdot \sin(nx_0) = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) (\cos(nt) \cos(nx_0) + \sin(nt) \sin(nx_0)) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(t-x_0)) dt \\ \boxed{\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(x_0-t) dt \dots \text{konvolucija}} \end{aligned}$$

$$t-x_0=y \quad t=x_0+y$$

$$\begin{aligned} &= \frac{1}{\pi} \int_{-\pi-x_0}^{\pi-x_0} f(x_0+y) \cos(ny) dy = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) \cos(ny) dy \\ \frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0+y) dy = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) \cdot \frac{1}{2} dy \end{aligned}$$

$$\frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kx_0) + b_k \sin(kx_0) = S_n(x_0)$$

$$S_n(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) \left( \frac{1}{2} + \dots + \cos(ny) \right) dy =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) D_n(y) dy =$$

$$= \frac{1}{\pi} \left( \int_0^{\pi} f(x_0+y) D_n(y) dy + \int_{-\pi}^0 f(x_0+y) D_n(y) dy \right)$$

$$= \frac{1}{\pi} \left( \int_0^{\pi} f(x_0+y) D_n(y) dy + \int_0^{\pi} f(x_0-y) D_n(y) dy \right) \xrightarrow{\text{sada funkcia}}$$

$$\xrightarrow{n \rightarrow \infty} \frac{f(x_0+) + f(x_0-)}{2}$$

$$\frac{1}{\pi} \int_0^{\pi} (f(x_0+y) + f(x_0-y)) D_n(y) dy - \frac{f(x_0+) + f(x_0-)}{2}$$

$$\frac{2}{\pi} \cdot \int_0^{\pi} D_n(y) dy$$

$$= \frac{1}{\pi} \int_0^{\pi} (f(x_0+y) - f(x_0+)) D_n(y) dy + (f(x_0-y) - f(x_0-)) D_n(y) dy$$

po jednom sumando

$$\frac{1}{2} \int_0^{\pi} (f(x_0+y) - f(x_0+)) \frac{\sin(n+\frac{1}{2})y}{\sin(\frac{1}{2})} dy =$$

$$- \frac{1}{2} \int_0^{\pi} (f(x_0+y) - f(x_0+)) \frac{\cos(\frac{y}{2})}{\sin(\frac{y}{2})} \sin(ny) + \cos(ny) dy$$

$$\text{po jednom sumando}$$

$$\int_0^{\pi} f(x_0+y) \cos(ny) dy \xrightarrow{n \rightarrow \infty} 0$$

$$\lim_{y \rightarrow 0} G(y) = G$$

$$\lim_{y \rightarrow 0} G(y) = \lim_{y \rightarrow 0} \frac{f(x_0+y) - f(x_0+)}{y}$$

Riemann-Lebegova lema:

čistaja

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} G(y) \sin(ny) dy = 0$$

po pretpostavki

za drugi sumand uporabimo podobne argumente



Zadání:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi < x < 0 \end{cases}$$

Furierova vztah:  $\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin((2k+1)x)$



$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2}$$

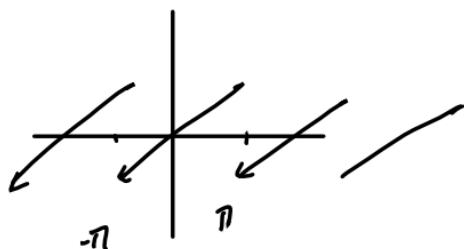
$$x = \frac{\pi}{2}: \quad 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin((2k+1)\frac{\pi}{2})$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

↑  
+1 at:  $p=1$

zgled

$$f(x) = x$$



$$a_n = 0 \quad \forall n$$

ker je na  $(-\pi, \pi)$  funkcijsa l'ke

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx$$

$$u = x \quad dv = \sin(nx) \quad n \geq 1$$

$$du = dx \quad v = -\frac{1}{n} \cos(nx)$$

$$b_n = \frac{1}{\pi} \left( -\frac{x}{n} \cos(nx) \right) \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos(nx) dx \\ \underbrace{= 0}_{= 0}$$

$$= \frac{1}{\pi} \left( -\frac{\pi}{n} \cos(n\pi) + \frac{(-\pi)}{n} \cos(n\pi) \right) =$$

$$= -\frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$

$$x = \sum_{n=1}^{L^2(-\pi, \pi)} (-1)^{n+1} \frac{2}{n} \sin(nx)$$

Po točkoh:

$$x = 0 : 0 = 0$$

$$x = \pi : \frac{\pi + (-\pi)}{2} = 0 \quad \checkmark$$

$$x = \frac{\pi}{2} : \frac{\pi}{2} = 2 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right)$$

Parsevalova enakost

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

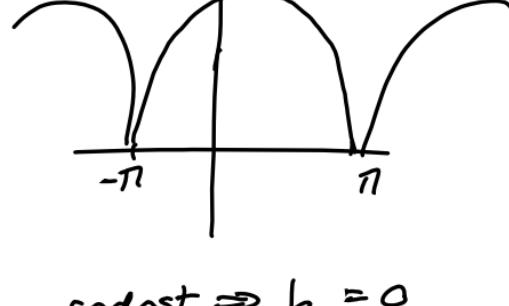
$\uparrow$   
naukadrat

$$\frac{1}{\pi} \left( \frac{1}{3} x^3 \right) \Big|_{-\pi}^{\pi} = \frac{2}{\pi} \pi^3 \cdot \frac{1}{3} = \frac{2}{3} \pi^2 = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Zefek.

$$f(x) = \pi^2 - x^2$$



soda, unegativ  
wenn, v viele  
tak' im levi:  
dann adad

$$\text{sodast} \Rightarrow b_n = 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) dx = \frac{2}{\pi} (\pi^3 - \frac{1}{3} \pi^3) =$$

$$= \frac{4}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{(\pi^2 - x^2)}_u \underbrace{\cos(nx) dx}_v =$$

$$= \frac{1}{\pi} (\pi^2 - x^2) \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} 2x \sin(nx) dx =$$

$$= \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin(nx) dx =$$

$$= \frac{2}{\pi n^2} \left( x \left( -\frac{1}{n} \cos(nx) \right) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos(nx) dx \right) =$$

$$= \frac{2}{\pi n^2} \left( \pi (-1)^{n+1} + (-1)^{n+1} \pi \right) = \frac{4}{n^2} (-1)^{n+1}$$

$$f(x) = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1} \cos(nx) \quad \forall x \in \mathbb{R}$$

$$x=0 \Rightarrow$$

$$f(0) = \pi^2 = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1}$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$x=\pi \Rightarrow$$

$$0 = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \underbrace{(-1)^{n+1}}_{-1} \underbrace{(-1)^n}_{1}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{6}$$

Kons:  $\pi$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2)^2 dx = \frac{1}{2} \frac{16}{3} \pi^4 + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (6\pi^4 - 2x^2 \pi^2 + x^4) dx = \frac{8\pi^4}{3} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{2}{\pi} \left( \pi^5 - \frac{2\pi^5}{3} + \frac{1}{5} x^5 \right) - \frac{8\pi^4}{3} + 16 S$$

$$\frac{15 - 10 + 3}{15} \pi^4 = \frac{4\pi^4}{8} + 8S$$

$$\frac{11}{8\pi^4}$$

$$\frac{1}{15}$$

$$8S = \frac{8\pi^4}{15} - \frac{4\pi^4}{3}$$

$$2S = \frac{2\pi^4}{15} - \frac{\pi^4}{3} = \frac{6\pi^4}{15} - \frac{5\pi^4}{15} = \frac{\pi^4}{15}$$

$$S = \frac{\pi^4}{30} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$$a_1, a_2, \dots \quad \lim_{n \rightarrow \infty} a_n = a$$

$$1, -1, 1, -1$$

$$\frac{1}{1}, \frac{1-1}{2}, \frac{1-1+1}{3} \rightarrow 0$$

$$a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}$$

konvergenz  $\Rightarrow$  konvergenz v parregu

$$S_n \stackrel{(x_0)}{=} \int_{-\pi}^{\pi} f(x_0 + ty) D_n(y) dy$$

$$\frac{S_0(x) + S_1(x) + \dots + S_n(x)}{n} = \sigma_n(x) \xrightarrow{n \rightarrow \infty} f$$

če sárove delne vsak

če je f nena je kavzgenez  
enakomerné

$$\underline{\text{Fejérjevo jedro}}: F_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(x)$$

Trditev:

$$1) F_N(x) = \frac{1}{2N} \left( \frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})} \right)^2$$

2)  $F_N$  je sedež

3)  $F_N(x) \geq 0 \quad \forall x$

$$4) \frac{1}{\pi} \int_{-\pi}^{\pi} F_N(x) dx = 1$$

$$5) \forall a. \quad 0 < a < \pi. \quad \lim_{N \rightarrow \infty} F_N(a) = 0$$

enakomerno na  $a \leq |x| \leq \pi$

Dokaz:

1)  $\Rightarrow$  2) očitno

1)  $\Rightarrow$  3) očitno

Iz definicije  $\Rightarrow$  4)  $\frac{1}{\pi} \int_{-\pi}^{\pi} F_N(x) dx = 1$

1)  $\Rightarrow$  5)



$$\forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ velja } \frac{2}{\pi} |x| \leq |\sin y|$$

$$0 < a \leq |x| \leq \pi$$

$$0 < \frac{a}{2} \leq \left| \frac{x}{2} \right| \leq \frac{\pi}{2}$$

$$\frac{2}{\pi} \cdot \frac{|x|}{2} \leq |\sin(\frac{x}{2})|$$

$$|\sin(\frac{x}{2})| \leq \frac{\pi}{|x|} \leq \frac{\pi}{a}$$

$$F_N(x) = \frac{1}{2N} \frac{|\sin(\frac{Nx}{2})|^2}{|\sin(\frac{x}{2})|^2} \leq \frac{1}{2N} \cdot \frac{\pi^2}{a^2}$$

$$\lim_{n \rightarrow \infty} F_n(x) = 0 \text{ enakomerno na } a \leq |x| \leq \pi$$

$$\begin{aligned} 1) F_N(x) &= \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{2} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = \\ &= \frac{1}{2N} \frac{1}{\sin^2(\frac{x}{2})} \sum_{n=0}^{N-1} \underbrace{\sin(n+\frac{1}{2})x}_{\cos(n+1)x - \cos(n+1)x} \cdot \underbrace{\sin(\frac{x}{2})}_{(1-\cos x) + (\cos x - \cos 2x) + (\cos 2x - \cos 3x) + \dots} = \\ &= \frac{1}{2N \sin^2(\frac{x}{2})} \sum_{n=0}^{N-1} (\cos(n+1)x - \cos(n+1)x) \\ &= 1 - \cos(Nx) \\ &= \frac{1 - \cos(Nx)}{4N \sin^2(\frac{x}{2})} = \frac{1}{2N} \frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})} \end{aligned}$$

brek:

Naj bo  $f$  zvezna in periodična funkcija

Potem cerčrave delne vsote

$$S_N(x) = \frac{1}{N} (S_0(x) + \dots + S_{N-1}(x))$$

Konvergirajo k  $f$  enakomerno na

$[-\pi, \pi]$  ozirome na  $\mathbb{R}$

$$S_n(x_0) = a_0 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$$

Trigonometrični polinom... konvergira  
kombinacije sinusov in kosinusov

Dokaz:

$$S_N(x) = \frac{1}{N} (S_0(x) + \dots + S_{N-1}(x))$$

$$S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+y) F_N(y) dy$$

$$S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+y) \cdot F_N(y) dy \xrightarrow{\text{enakomerno}} f(x)$$

Naj bo  $\epsilon > 0$ . ker je  $f$  zvezna in periodična  
je enakomerna zvezna

$$\exists \delta > 0. \forall y \in \delta. |f(x+y) - f(x)| < \frac{\epsilon}{2}$$

$\forall x \in \mathbb{R}$

$$|S_N(x) - f(x)| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+y) F_N(y) dy - f(x) \frac{1}{\pi} \int_{-\pi}^{\pi} F_N(y) dy \right|$$

$$= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x+y) - f(x)) F_N(y) dy \right| \leq$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+y) - f(x)| |F_N(y)| dy =$$

$$= \frac{1}{\pi} \underbrace{\int_{-\delta}^{\delta} |f(x+y) - f(x)| F_N(y) dy}_{\delta < |y| < \pi} + \underbrace{\int_{-\pi}^{-\delta} |f(x+y) - f(x)| |F_N(y)| dy}_{\delta < |y| < \pi}$$

$$< \frac{1}{\pi} \int_{-\pi}^{\pi} F_N(y) dy \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}$$

$\Rightarrow f$  je zvezna in periodična ter je omogočena

$$\exists M \in \mathbb{R} \quad |f(z)| \leq M \quad \forall z \in \mathbb{R}$$

$$\Rightarrow |f(x+y) - f(x)| \leq 2M \quad \forall x \in \mathbb{R} \quad \forall y \in \mathbb{R}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+y) - f(x)| F_N(y) dy \leq 2M \frac{1}{\pi} \int_{-\pi}^{\pi} F_N(y) dy$$

$$\delta < |y| < \pi \quad \delta < |y| \leq \pi$$

Vemo da je  $\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} F_N(y) dy = 0$

enakomerno na  $\delta \leq |y| \leq \pi$

$$\exists N_0. \forall N \geq N_0. |F_N(y)| \leq \frac{\epsilon}{2M \cdot 4} = \frac{\epsilon}{8M}$$

$$\frac{2M}{\pi} \int_{-\pi}^{\pi} F_N(y) dy \leq \frac{2M}{\pi} \cdot \frac{\epsilon}{8M} \cdot 2\pi = \frac{\epsilon}{2}$$

$\therefore N \geq N_0$

$$|S_N(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R}$$

Izrek:

$$\left\{ \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \cos(nx), \frac{1}{\sqrt{n}} \sin(nx); n \in \mathbb{N} \right\}$$

je kons v  $L^2(-\pi, \pi)$

Dokaz:

$L^2(-\pi, \pi)$  je nepolničev  $C[-\pi, \pi]$  v

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

(ONS je kons  $\Leftrightarrow$  končne lineare kombinacije vektorjev iz ONS ga ste v prostoru

OPOMBI: KONČNE LINEARE KOMBINACIJE

$\sin(nx) \cos(nx) n \in \mathbb{N}_0$  so trigonometrični polinom:

Ali so trigonometrični polinom: gosti v  $L^2(-\pi, \pi)$ ?

1)  $C(-\pi, \pi)$  so goste v  $L^2(-\pi, \pi)$

$f \in L^2(-\pi, \pi), \epsilon > 0, \exists \tilde{f} \in C[-\pi, \pi], \|f - \tilde{f}\|_2 < \frac{\epsilon}{2}$

2) Ali: so trigonometrični polinom goste v  $(C[-\pi, \pi], d_2)$ ?  
če da:

$\exists T(x)$  trigonometrični polinom

$$T(x) = x_0 + \sum_{n=1}^N \lambda_n \cos(nx) + \mu_n \sin(nx)$$

$$\|f - T\|_2 < \frac{\epsilon}{2} \Rightarrow \|f - T\|_2 < \epsilon$$

OPOMBI:

$$g_n \in L^2[-\pi, \pi] \xrightarrow[n \rightarrow \infty]{\text{enakomoč}} f$$

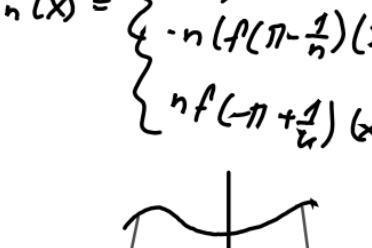
$$\xrightarrow[2]{\lim} \lim_{n \rightarrow \infty} g_n = f \quad d_2$$

$$d_2(f, g_n) = \sqrt{\int_{-\pi}^{\pi} (f - g_n)^2 dx} \xrightarrow{n \rightarrow \infty} 0$$

Vemo:  $f$  je ena 2 $\pi$  periodična  $\Rightarrow$

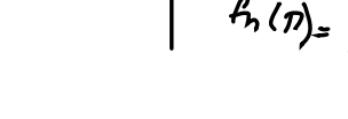
jo lahko poljubro dobra enekomerna približujemo s trigonometričnim polinom:

$$\|f - T_N\|_\infty < \epsilon$$



N: ena 2 $\pi$  periodična funkcija

$$f_n(x) = \begin{cases} f(x) & x \in [-\pi, \pi] \\ -n(f(\pi - \frac{1}{n})(x - \pi)), & \pi - \frac{1}{n} < x \leq \pi \\ n(f(-\pi + \frac{1}{n})(x + \pi)) & -\pi \leq x < -\pi + \frac{1}{n} \end{cases}$$



$$f_n(\pi) = f_n(-\pi) = 0$$

je ena 2 $\pi$  periodična funkcija

$$\|f - f_n\|^2 = \int_{-\pi}^{\pi} (f_n(x) - f(x))^2 dx \leq \frac{2}{n} (2\pi)^2 \xrightarrow{n \rightarrow \infty} 0$$

Teck: (Weierstrassov)

Naj bo  $f$  verne na  $[a,b]$

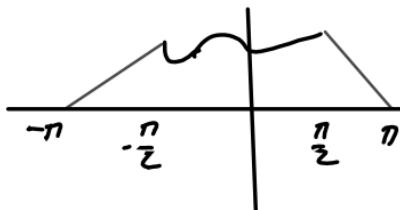
Naj bo  $\epsilon > 0$ . Potem  $\exists$  polinom  $p$  da je

$$\|f - p\|_{\infty} < \epsilon$$

Dokaz: Davolj je operavat  $a = -\frac{\pi}{2}$   $b = \frac{\pi}{2}$

$f$  verne na  $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$f$  verne maksima na  $[0, \pi]$



$$\tilde{f} = \begin{cases} f(x) & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -\frac{\pi}{2} f(\frac{\pi}{2}) (x - \pi) & \frac{\pi}{2} < x \leq \pi \\ \frac{\pi}{2} f(-\frac{\pi}{2})(x + \pi) & -\pi \leq x < -\frac{\pi}{2} \end{cases}$$

$\tilde{f}$  je verne  $2\pi$  periodič.

za  $\epsilon > 0$   $\exists T$  trigonometrični polinom, da je  $\|f - T\|_{\infty} \leq \frac{\epsilon}{2}$

$$\cos(nx) \sin(nx) \quad \text{jih je konan}$$

Aproximacija s Taylorjevimi polinami: enakomerne.

# Vektorska analiza

(u  $\mathbb{R}^2$  ali  $\mathbb{R}^3$ )

$D^{odp} \subseteq \mathbb{R}^3$   $u: D \rightarrow \mathbb{R}$  verz funkcija

jo menujemo skalarne polje

vseki točki v  $D$  privedi skalar

$D \subseteq \mathbb{R}^3$   $\vec{R}: D \rightarrow \mathbb{R}^3$  verz

vektorsko polje vseki točki privedi vektor  $\vec{R}(t)$

$\mathbb{R}^3$   
 $u(x, y, z) = x + b \alpha \cdot \vec{e}_1, \vec{e}_2, \vec{e}_3 (\vec{i}, \vec{j}, \vec{k})$

Baza:  $\vec{p} = \frac{1}{\sqrt{2}} \vec{e}_1 + \frac{1}{\sqrt{2}} \vec{e}_3$

$$\vec{g} = -\frac{1}{\sqrt{3}} \vec{e}_1 + \frac{1}{\sqrt{3}} \vec{e}_2 + \frac{1}{\sqrt{3}} \vec{e}_3$$

$$\vec{r} = \frac{1}{\sqrt{6}} \vec{e}_1 + \frac{2}{\sqrt{6}} \vec{e}_2 - \frac{1}{\sqrt{6}} \vec{e}_3$$

Točka  $(x, y, z)$  imen v Bazi  $(\vec{p}, \vec{g}, \vec{r})$

Koordinate  $(\alpha, \beta, \gamma)$

KOND

$(\alpha, \beta, \gamma) \mapsto \alpha$  n: ista funkcija

ONB je pozitivno orientirane, ce je  
 $[\vec{p}, \vec{g}, \vec{r}] > 0 \quad (\Leftrightarrow \vec{r} = \vec{p} \times \vec{g})$

Ozirno negativno orientiranje kada

$$[\vec{p}, \vec{g}, \vec{r}] < 0 \quad (\Leftrightarrow \vec{r} = -\vec{p} \times \vec{g})$$

(zato da od zdej naprej za množenje  
produkta bo  $(\vec{p}, \vec{g}, \vec{r})$  najbrž

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = U \quad (x, y, z) \quad (e_1, e_2, e_3)$$

$$(\alpha, \beta, \gamma) \quad (\vec{p}, \vec{g}, \vec{r})$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$U^T U = I$$

$$U^{-1} = U^T$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$u(x, y, z) \quad v(e_1, e_2, e_3)$  koordinateh

$$u(\alpha, \beta, \gamma) = u(U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix})$$

Vektordreieck poly

$$\vec{R}(x,y,z) = (X(x,y,z), Y(x,y,z), Z(x,y,z))$$

$\nu(\alpha, \beta, \gamma)$  Winkelwerte

$$\tilde{R}(\alpha, \beta, \gamma) = (U^T \vec{R} \circ U) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\tilde{u} = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$\vec{R}(x,y,z) = (x + 2y + 3z, x + 2y + 3, x + 2y + 5z)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$x = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$y = \frac{\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$

$$z = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{3}} - \frac{\gamma}{\sqrt{6}}$$

$$x + 2y + 3z = \frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma = c$$

$$\text{für } (\vec{R} \circ U) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = (c, c, c)$$

$$U \circ U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U \left( \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}} \right) c$$



$$D^{\partial \partial \rho} \subset \mathbb{R}^3 \quad \text{panzerieren}$$

$$u: D \rightarrow \mathbb{R} \quad C^1$$

$$u \mapsto \vec{\nabla} u = \text{grad } u = (u_x, u_y, u_z)$$

$$\vec{R}: D \rightarrow \mathbb{R}^3 \quad \vec{R} = (X, Y, Z)$$

$$\vec{R}: D \rightarrow \mathbb{R}^3 \quad \vec{R} \mapsto \text{div } \vec{R} = \vec{\nabla} \cdot \vec{R} = X_x + Y_y + Z_z$$

$$\vec{R} \mapsto \text{rot } \vec{R} = \vec{\nabla} \times \vec{R} = (Z_x - Y_z, X_z - Z_x, Y_x - X_z)$$

$$u \in C^2(D), \vec{R}: D \rightarrow \mathbb{R}^3 \quad C^2$$

$$\text{rot grad } u = \vec{0}$$

$$\vec{\nabla} \times \vec{\nabla} u = \vec{0}$$

$$\text{div rot } \vec{R} = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{R}) = 0$$

$$\text{div grad } u = \vec{\nabla} \cdot \vec{\nabla} u = u_{xx} + u_{yy} + u_{zz} = \Delta u$$

zu sch:

Divergenz (3 Punkte, bei vidi-use ostale  
Folge 

$$\vec{R}: D \rightarrow \mathbb{R}^3 \quad \text{rek polje}$$

$$1) \text{ rot } \vec{R} = \vec{0} \Rightarrow \exists u \in C^2(D). \vec{R} = \vec{\nabla} u$$

$$2) \text{ div } \vec{R} = 0 \Rightarrow \exists \vec{G}: D \rightarrow \mathbb{R}^3 \quad C^2 \text{ rek polje}$$

d.h.  $\vec{R} = \text{rot } \vec{G}$

Zgled

da 1) ne velja obratno

$$\mathbb{R}^3 - z=0 \quad D \ni z: \text{wec deko}$$

$$\vec{R} = \left( \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\arctan \frac{y}{x} = u \quad \text{nicht definiert an der Kante}$$

$\Rightarrow$  es ist  $\vec{R}$  potenzial  $u$ , poten so  
vom potenzial; oblike  $u + c$   
(P poveren)

• 

$\vec{R} : D \rightarrow \mathbb{R}^3$  vec. pdje

$$\operatorname{div} \vec{R} = f \quad f \in C^1(D)$$

$$\exists u \text{ na } D, \Delta u = f$$

$$\operatorname{div} \operatorname{grad} u - \Delta u = f = \operatorname{div} \vec{R}$$

$$\operatorname{div} (\vec{R} - \operatorname{grad} u) = 0$$

$$\stackrel{\parallel}{\operatorname{rot}}(\vec{G})$$

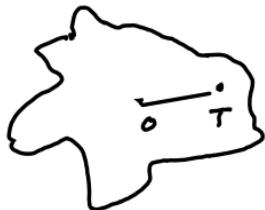
$$\vec{R} = \operatorname{grad} u + \stackrel{\parallel}{\operatorname{rot}} \vec{G}$$

Dokaz (izwleka dve strani nezaj)

$$\text{1) } \vec{r} : D \rightarrow \mathbb{R}^3 \quad C^1(D)$$

$$\text{rot } \vec{\mathcal{R}} = \vec{\sigma} = (z_y - y_z, x_z - z_x, y_x - x_y) = \vec{0}$$

B SZS. je D zweckmäßigste ne



$\exists e \ (x,y,z) \in D.$

$$\forall t \in [0,1], (t \cdot x, t \cdot y, t \cdot z) \in D$$

Dekin;ramo;

$$u(x, y, z) = \int_0^1 [X(tx, ty, tz) \cdot x + Y(tx, ty, tz) \cdot y + Z(tx, ty, tz) \cdot z] dt$$

Uge potental  $\approx$   $\frac{1}{R}$

$$u_x = X \quad u_y = Y \quad u_z = Z$$

$$u_x = \int_0^t (X(t_x, t_y + z) + X_x \cdot t \cdot x + Y_x \cdot t y + Z_x \cdot t z) dz$$

|| || dt  
 X\_y X\_z

$$= \int_0^1 (X(tx, ty, tz) + tx \cdot X_x + ty \cdot X_y + tz \cdot X_z) dt$$

$\overbrace{\hspace{400pt}}$

$$= \frac{d}{dt} (t \cdot X(tx, ty, tz))$$

$$= +x(tx, ty, tz) \Big|_1 = x(x, y, z)$$

Podobno  $U_Y = Y$  iż  $U_2 = Z$

Dher

i)

$$\operatorname{div} \vec{R} = \partial = X_x + Y_y + Z_z$$

(D je merdesta afledte na  $\vec{R}$  o BESZS)

$$\alpha(x, y, z) = \int_0^1 X(tx, ty, tz) dt$$

$$\beta(x, y, z) = \int_0^1 Y(tx, ty, tz) dt$$

$$\gamma(x, y, z) = \int_0^1 Z(tx, ty, tz) dt$$

$$\operatorname{div}(\alpha, \beta, \gamma) = 0 \text{ kør:}$$

$$\alpha_x = \int_0^1 t^2 X_x(tx, ty, tz) dt$$

$$\beta_y = \int_0^1 t^2 Y_y(tx, ty, tz) dt$$

$$\gamma_z = \int_0^1 t^2 Z_z(tx, ty, tz) dt$$

$$= 0$$

$$\vec{G} = (\begin{matrix} \alpha & \beta & \gamma \\ x & y & z \end{matrix}) = (\alpha, \beta, \gamma) \times (x, y, z)$$

$$\vec{G} = (z\beta - y\gamma, x\gamma - z\alpha, y\alpha - x\beta)$$

$$\vec{R} = \text{rot } \vec{G}?$$

1 komponente ad Rot  $\vec{G}$ :

$$(y\alpha - x\beta)_y - (x\gamma - z\alpha)_z =$$

$$= \alpha + y\alpha_y - x\beta_y - x\gamma_z + \alpha_x + z\alpha_z$$

$$= 2\alpha + \alpha_x + y\alpha_y + z\alpha_z$$

Kuparabim prøva eracbn  $\alpha_x + \beta_y + \gamma_z = 0$

$$= \int_0^1 \underbrace{(2 + X(tx, ty, tz) + x t^2 X_x + y t^2 X_y + z t^2 X_z)}_{\frac{d}{dt} (t^2 X(tx, ty, tz))} dt$$

$$t^2 X(tx, ty, tz) \Big|_0^1 = X(x, y, z)$$

Padobno se drug. due komponenter.

Zgled:

$$\vec{R} = (y^2 z^3 + 2, 2xyz^3 + 1, 3xy^2 z^2)$$

$$\text{rot } \vec{R} (6xyz^2 - 6xy^2, 3x^2 z^2 - 3y^2 z^2, 2yz^3 - 2yz^3) = \vec{0}$$

$$u(x,y,z) = \int X(tx, ty, tz) x + \dots dt$$

$$u \text{ potencjal abstaja: } u_x = y^2 z^3 + 2$$

$$u_y = 2xyz^3 + 1$$

$$u_z = 3xy^2 z^2$$

$$u(x,y,z) = xy^2 z^3 + 2x + C(y, z)$$

$$\downarrow \\ u_y = 2xyz^3 + C_y = 2xyz^3 + 1$$

$$C_y = 1$$

$$C(y, z) = y + D(z)$$

$$u(x,y,z) = xy^2 z^3 + 2x + y + D(z)$$

↓

$$u_z = 3xy^2 z^2 + D'(z) = 3xy^2 z^2$$

$$D' = 0$$

$$D = D_0 \text{ konstante}$$

$$u(x,y,z) = xy^2 z^3 + 2x + y + D_0$$

( $\text{Rot } \vec{R} \neq \vec{0}$  bei ej ni potencial(a))

$$u = xy^2 z^3 + 2x + A(yz)$$

$$\downarrow \\ 2xyz^3 + Ay = 2yz^3 + x$$

$$Ay = x \rightarrow \text{kejil } (\text{A fungsi})$$

y in z ja neva

Zufried

$$\vec{R} = (2y - 1, -1, 4x - 2xy)$$

$$\operatorname{div} \vec{R} = (0, 0, 0) = 0$$

suchen  $\vec{G} = ?$  rot  $\vec{G} = \vec{R}$

$$\alpha(x, y, z) = \int_0^y + (2ty - 1) dt = \frac{2}{3}y^3 - \frac{1}{2}$$

$$\beta(x, y, z) = \int_0^1 + (-1) dt = -1$$

$$\gamma(x, y, z) = \int_0^x + (4tx - 2xy + t^2) dt = \frac{4x}{3} - \frac{1}{2}xy$$

$$\vec{G} = \begin{pmatrix} \frac{2}{3}y^3 - \frac{1}{2} & -1 & \frac{4}{3}x - \frac{1}{2}xy \\ x & y & z \end{pmatrix} =$$

$$= \left( -\frac{1}{2}z, -\frac{4}{3}xy + \frac{1}{2}x^2, \frac{4}{3}x^2 - \frac{1}{2}xy - \frac{2}{3}zy + \frac{1}{2}z, \right. \\ \left. \frac{2}{3}y^2 - \frac{1}{2}y + \frac{1}{2}x \right)$$

$$\operatorname{rot} \vec{G} = \left( \frac{4}{3}y^3 - \frac{1}{2} + \frac{2}{3}y - \frac{1}{2}, -\frac{1}{2} - \frac{1}{2}, \frac{8}{3}x - xy + \right. \\ \left. 2y - 1 \right) \\ = 4x - 2xy$$

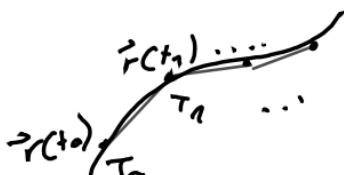
# Krивулјни интеграл

Dolzina krivulje:

$\Gamma$  nej bo  $C^1$  krivulja u  $\mathbb{R}^3$

izg. Parametrizacija  $\vec{r}(t) : \vec{r}[\alpha, \beta] \rightarrow \Gamma$

$$\dot{\vec{r}}(t) \neq 0$$



D delitev  $[\alpha, \beta]$

$$\sum_1^n d(T_{j-1}, T_j) = l(D)$$

$$l(D) = \sum_1^n \sqrt{(x(t_j) - x(t_{j-1}))^2 + \dots + (z(t_j) - z(t_{j-1}))^2}$$

$$= \sum_1^n \sqrt{\dot{x}^2(\vec{t}_j) + \dot{y}^2(\vec{t}_j) + \dot{z}^2(\vec{t}_j)}$$

$$= \sum_1^n \sqrt{\dot{x}^2(\vec{t}_j) + \dot{y}^2(\vec{t}_j) + \dot{z}^2(\vec{t}_j)} =$$

$$= R(\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}, D, \tau) \xrightarrow{\max \Delta t_j \rightarrow \infty}$$

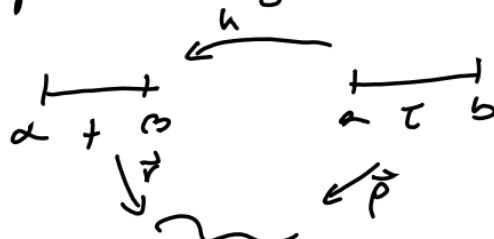
$$\rightarrow \int_{\alpha}^{\beta} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

Naj bo  $\Gamma$   $C^1$  krivulja v  $\mathbb{R}^3$  z regelarno parametrizacijo:  $\vec{r}: [\alpha, \beta] \rightarrow \Gamma$

Dolin: samo delnice krivulje

$$l(\Gamma) = \int_{\alpha}^{\beta} |\vec{r}'| dt = \int_{\alpha}^{\beta} \sqrt{x^2 + y^2 + z^2} dt$$

Ali je ta definicija neodvisna od regelarne parametrizacije



$$\begin{array}{ccc} \vec{h} : [\alpha, \beta] & \xrightarrow{DF} & [\alpha, \beta] \\ \vec{p} = \vec{r} \circ \vec{h} \end{array}$$

$$\vec{p}' = \vec{r}'(h) h'$$

$$\int_a^b |\vec{p}'| dt = \int_{\alpha}^{\beta} |\vec{r}'| dt$$

$$\int_{\alpha}^{\beta} |\vec{r}(t)| dt = \int_{[\alpha, \beta]}^{[\alpha, \beta]} |\vec{r}(t)| dt = \int_{[a, b]} |\vec{r}(h(t))| |h'(t)| dt$$

$$= \int_{[a, b]} |\vec{p}| dt$$

$[a, b]$

$\Gamma$  nej bo c<sup>1</sup> krivulja  $\vec{r}(t)$  reg. param.

$$S(t) = \int_{\alpha}^t |\dot{\vec{r}}(t)| dt \quad s: [\alpha, \beta] \rightarrow [0, l(\Gamma)]$$

= dolžina  $\Gamma$  med  $\vec{r}(s)$  in  $\vec{r}(t)$

$$\text{odvod: } \dot{S} = |\dot{\vec{r}}(t)| > 0$$

$s$  je strogo naraščajna od  $[\alpha, \beta] \times [0, l(\Gamma)]$

tački imajo inverz:

$$T: [0, l(\Gamma)] \rightarrow [\alpha, \beta]$$

$$s \mapsto T(s) \text{ inverz od } S$$

Pogojno parametrizacija  $\Gamma$ :

$$s \mapsto \vec{r}(T(s))$$

$$\text{Vemo } S \circ T = id_{[0, l(\Gamma)]} \text{ in } T \circ S = id_{[\alpha, \beta]}$$

odnosno:

$$\dot{S}(T(s)) \cdot T'(s) = 1$$

izračunajmo velikost parametrizacije

$$s \mapsto \vec{r}(T(s))$$

$$\frac{d}{ds} (\vec{r}(T(s))) = \dot{\vec{r}}(T(s)) \cdot T'(s) =$$

$$= \frac{\dot{\vec{r}}(T(s))}{\dot{S}(T(s))} = \frac{\dot{\vec{r}}(T(s))}{|\dot{\vec{r}}(T(s))|}$$

velikost hitosti je 1

Tako parametrizirana krivulja je  
naročno parametrizirana  $\Leftrightarrow$  in  $s$  je  
naravní parameter

Zagled:

$v_{ij} = \text{on; ca:}$

$$+ \longmapsto (a \cos t, a \sin t, bt) = \vec{r}(t)$$

$$|\vec{r}| = \sqrt{a^2 + b^2}$$

$$S(t) = \int_0^t \sqrt{a^2 + b^2} dt = \sqrt{a^2 + b^2} \cdot t = s$$

$$T(s) = \frac{s}{\sqrt{a^2 + b^2}}$$

naravno parametrizirane vijeknice

$$s \longmapsto \left( a \cos \left( \frac{s}{\sqrt{a^2 + b^2}} \right), a \sin \left( \frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} \right)$$

$$\stackrel{\parallel}{\vec{n}}(s)$$

$$|\vec{n}(s)| = 1$$

# Orientacija krivulje



$\Gamma$  gladka  $C^1$  krivulja

Orientacija  $\Gamma$  je zvezni izobar enotskega tangentnega vektorja vzdolž  $\Gamma$ :  $\vec{T}$

$$|\vec{T}| = 1$$

$\Gamma$  paverzne  $\Rightarrow \Gamma$  ima dve orientaciji:  
 $\vec{T} : n - \vec{T}$

$\vec{r} : [\alpha, \beta] \rightarrow \Gamma$  reg. parametrizacija

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} \text{ en močni izobar}$$

V tem primerni sta orientacija in parametrizacija usklajeni:

$$(\Gamma, \vec{T}) \text{ ali } \vec{r}$$

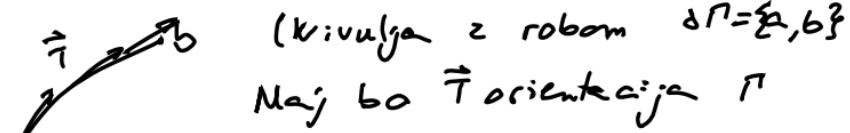
Vsake krivulje je orientabilna

Odeščemo gladka krivulje



$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$$

$\Gamma$  je robni me točkama a, b



Orientacija  $\Gamma$  parodi orientacijo  $\delta\Gamma$

Ena od teh dveh točk je prva oz zadetna  $\Gamma$  ( $z_\Gamma$ ) in druga je končna točka  $\Gamma$  ( $k_\Gamma$ )

Gleda na sliko je  $a = z_\Gamma$   $b = k_\Gamma$

S tem je rob  $\Gamma$  orientiran skladno z orientacijo  $\Gamma$

Naj bo  $\Gamma$  odsekoma gladke krivulje.

$$\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$$

1)  $\Gamma_j$  je gladka krivulja z robom

2)  $\Gamma_j \cap \Gamma_l = \emptyset \Leftrightarrow |j-l| \text{ mod } n \geq$

3)  $\Gamma_j \cap \Gamma_{j+1} = \sum_{i=1}^m p_{ij}^i \quad j \in \{1, \dots, n-1\}$   $p_{ij}^i$  robe  
vsi samotakozi s  $\Gamma_j$  in  $\Gamma_{j+1}$   
 $\Gamma_n$  in  $\Gamma_1$  se lahko selete ali pa ne

Orientacija  $\Gamma$  je tak izber orientacij

$\Gamma_1 \dots \Gamma_n$ , da je  $K_{\Gamma_j} = P_j = \sum_{i=1}^{m_{j+1}}$

če se  $\Gamma_n$  in  $\Gamma_1$  selete je  $K_{\Gamma_n} = \sum_{i=1}^{m_1}$

# Krивулјни интеграл

Dve vrsti krivuljnih integralov:

1) Integral skalarnega polja po krivulji  $\Gamma$   
(orientacije tenu potrebujemo)

$\Gamma$  gladka krivulja omejena

$\underline{u}: \Gamma \rightarrow \mathbb{R}$  zvezna skalarno polje  
 $\hat{r}: [\alpha, \beta] \rightarrow \Gamma$  regulerne parametrizacije

Definisramo  $\int_{\Gamma} u ds := \int_{\alpha}^{\beta} u(\hat{r}(t)) \hat{r}'(t) dt$

Opozabe:

1)  $u=1 \Rightarrow$  je to dolzina  $\Gamma$

2) Ta rednost je neodvisna od  
reg. parametrizacije

3)  $u =$  dolzinske gostote  $\Rightarrow$   
dobimo maso  $\Gamma$

če je  $u =$  po konstante, je  $\Gamma$  homogena

če je  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  odeljene gladke

$$\text{je } \int_{\Gamma} u ds = \sum_{j=1}^n \int_{\Gamma_j} u ds$$

2) krivuljni integral vektorskega polja  
po orientirani krivulji

>>

Zufed:  $\Gamma$  homogene polkugel

$$\Gamma = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = a^2, \quad y \geq 0 \}$$

Legt  $m(\Gamma)$  fest: ?

$$m(\Gamma) = \int_{\Gamma} p_0 \, ds \quad x_{\Gamma}(\Gamma) = \frac{1}{m(\Gamma)} \int_{\Gamma} x p_0 \, ds$$
$$y_{\Gamma}(\Gamma) = \frac{1}{m(\Gamma)} \int_{\Gamma} y p_0 \, ds$$

$$m(\Gamma) = \int_{\Gamma} p_0 \, ds = \pi a p_0$$

Parametrisierung:

$$x(t) = a \cos t \quad y(t) = a \sin t$$

$$ds = \sqrt{1 + \dot{x}^2} dt$$

$$\dot{x} = -a \sin t \quad x^2 + y^2 = a^2$$
$$\dot{y} = a \cos t$$

$$m(\Gamma) = \int_0^\pi p_0 a dt = p_0 a \pi$$

$$x_{\Gamma}(\Gamma) = \frac{1}{m(\Gamma)} \int_0^\pi a \cos t + p_0 a dt = 0$$

$$\frac{p_0 a \pi}{\pi} \int_0^\pi \sin t \cos t dt = \frac{p_0 a^2}{2} (-\cos^2 t) \Big|_0^\pi =$$
$$\frac{p_0 a^2}{2}$$

Vztrajnost: moment umoci:  $\frac{1}{2} m(\Gamma) \cdot a^2$

$$J_y = \int_{\Gamma} (a^2 z^2) p \, ds = \int_0^\pi a^2 \cdot a^2 \cdot a \cdot dt = \int_0^\pi a^3 \int \frac{1 + \cos^2(2t)}{2} dt$$

$$= \frac{\pi}{2} p_0 a^3 = \frac{1}{2} m(\Gamma) \cdot a^2$$

2) Krivuljni integral vektorskega polja po orientirani krivulji;  
 (Delo sile vzdolž  $\vec{r}$ )

$\vec{R}: \vec{r} \rightarrow \mathbb{R}^3$  zeno vek. polje  
 $\vec{r} = (x, y, z)$



če je  $\Gamma$  gladka in

$\vec{r}: [\alpha, \beta] \rightarrow \Gamma$  param. k. je  
 vsakojene z orientacijo  $\Gamma$

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

Doliniramo  $\int_{\Gamma} \vec{R} d\vec{r} = \int_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \cdot \vec{r}'(t) dt$  s lekarni produkt

$$= \int_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \cdot \dot{\vec{r}} dt = \int_{\Gamma} (\vec{R} \cdot \vec{T}) ds$$

Rezultat je neodvisen od parametrizacije  $\Gamma$ , k. je vsakojene z orientacijo

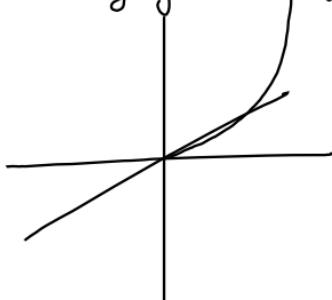
$$\int_{-\Gamma} \vec{R} d\vec{r} = - \int_{\Gamma} \vec{R} d\vec{r}$$

Zadaci:

$$\vec{R}(x, y, z) = (xy, z, x-2)$$

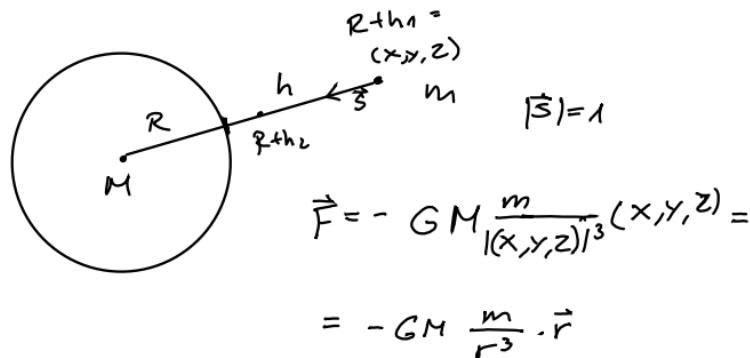
$$r(t) = (t, t, \frac{1}{2}t^2) \quad t \in [0, 1]$$

orientacija je ista kada je usklojene  $z = \vec{r}$



$$\begin{aligned}\int_{\vec{r}}^{\vec{R}} d\vec{r} &= \int_0^1 (t^2, \frac{1}{2}t^2, t - \frac{1}{2}t^2) \cdot (1, 1, t) dt = \\ &= \int_0^1 (\frac{3}{2}t^2 + t^2 - \frac{1}{2}t^2) dt = \frac{5}{6} - \frac{1}{8} = \frac{17}{24}\end{aligned}$$

Zefed:



$\vec{r} \rightarrow \vec{s} \cdot \vec{r}$  + force due to  $R+h_1$  do  $R+h_2$

$$\int_{R+h_1}^{R+h_2} \vec{F} dr = \int_{R+h_1}^{R+h_2} -GM \frac{m}{r^3} (\vec{r} \cdot \vec{s}) \cdot \vec{s} dt =$$

$$= -GMm \int_{R+h_1}^{R+h_2} \frac{dt}{r^2} = -GMm \left[ \frac{1}{r} \right]_{R+h_1}^{R+h_2} = -GMm \left( \frac{1}{R+h_2} - \frac{1}{R+h_1} \right)$$

$$= \frac{GM (h_1 - h_2)m}{(R+h_1)(R+h_2)} \doteq \underbrace{\frac{GM}{R^2}}_g m \cdot \Delta h = mg \Delta h$$

Oponka

$$\text{.) } \int_{\Gamma} \vec{R} dr = \int_{\Gamma} X dx + Y dy + Z dz$$

diferencijalna forma

..) če je  $\Gamma$  sklenjena ( $Z_\Gamma = k_\Gamma$ )

$$\int_{\Gamma} \vec{R} dr$$

Naj bo  $\vec{R}$  potencialno vektorsko polje

Naj bo  $\Gamma$   $C^1$  krviljë s parametrizacijo  $\vec{r}(t)$

Orientacija vsklajjena z  $\vec{r}$   $\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$

$$\begin{aligned}\int_{\vec{\alpha}}^{\vec{\beta}} \vec{R} d\vec{r} &= \int_{\vec{\alpha}}^{\vec{\beta}} \text{grad } u d\vec{r} = \int_{\alpha}^{\beta} (u_x, u_y, u_z) (\vec{r}(t)) \cdot \dot{\vec{r}}(t) dt = \\ &= \int_{\alpha}^{\beta} \vec{r}(t) = (x(t), y(t), z(t)) = \int_{\alpha}^{\beta} \underbrace{(u_x \cdot \dot{x} + u_y \cdot \dot{y} + u_z \cdot \dot{z})}_{\frac{d}{dt} (u(x(t), y(t), z(t)))} dt = \\ &= u(\vec{r}(\beta)) - u(\vec{r}(\alpha)) = u(k_{\vec{\tau}}) - u(z_{\vec{\tau}})\end{aligned}$$

**Trditev:** Če je vek. polje  $\vec{R}$  potencialno, je vrednost integrala  $\vec{R}$  po orientirani kružnici  $\Gamma$  enaka razlike potenciala polja  $\vec{R}$  med končno točko  $\vec{r}_1$  in začetno točko  $\vec{r}_2$

**Postedica:** Če je  $\vec{R}$  potencialno polje in je  $\Gamma$  sklenjena jo integral  $\int_{\Gamma} \vec{R} dr = 0$

Zgled:

$$\vec{R} = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\vec{F} = \{ (x, y, 0); x^2+y^2=R^2; R>0 \}$$



$$x = R \cos t$$

$$t \in [0, 2\pi)$$

$$y = R \sin t$$

$$z = 0$$

$$\dot{x} = -R \sin t$$

$$\dot{y} = R \cos t$$

$$\dot{z} = 0$$

$$\int_{\Gamma} \vec{F} d\vec{r} = \int \left( \frac{1}{R^2} R \sin t, \frac{1}{R^2} R \cos t, 0 \right) \cdot (-R \sin t, R \cos t, 0) dt =$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi \neq 0$$

$$\Rightarrow \text{Ta polje n} ; \text{potencialno v } \mathbb{R}^3 - \Sigma z \cdot \cos^2 t$$
$$(Vemo pa da je \operatorname{rot} \vec{R} = \vec{0})$$

Izrek (karakterizacija potencialnih vek. polj)

Naj bo  $D \subseteq \mathbb{R}^3$  odpr in  $\vec{R}: D \rightarrow \mathbb{R}^3$  zvezna vektorška polje:

Nasledje tr: izjave so ekvivalentne:

- 1)  $\vec{R}$  je potencialno na  $D$   
( $\exists U \in C^1(D), \vec{R} = \text{grad } u$ )
- 2) Integral  $\vec{R}$  po orientiranih krivuljah je neodvisen od krivulj z isto začetno in isto končno točko
- 3) Integral  $\vec{R}$  po sklenjenih krivuljah je enak 0

Necrtež zvez: členi vektorsk

Rabam nacrt barvargb → path  
Ustvarjajtekrivulj vseh vrst  
če da se vedno pravilno!

Trajet

Dokaz:

$$1) \Rightarrow 3)$$

$\vec{r}$  silenjene:  $t \rightarrow \vec{r}(t)$  parametrizacija  $C$   
 $t \in [\alpha, \beta]$ ;  $\vec{r}(\alpha) = \vec{r}(0)$

$$\int_{\vec{\Gamma}} R dr = \int_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \vec{r}'(t) dt = \vec{R}(\vec{r}(\beta)) - \vec{R}(\vec{r}(\alpha)) = 0$$

$$\int_{\vec{\Gamma}} \frac{\partial}{\partial t} (\vec{R}(\vec{r}(t))) dt$$

$$3) \Rightarrow 2) \quad \vec{r}_1$$



$$\int_{\vec{\Gamma}} \vec{R} dr = 0$$

$$\int_{\vec{\Gamma}_1} \vec{R} dr - \int_{\vec{\Gamma}_2} \vec{R} dr = 0 \rightarrow \int_{\vec{\Gamma}_1} R dr = \int_{\vec{\Gamma}_2} \vec{R} dr$$

$$2) \Rightarrow 1)$$

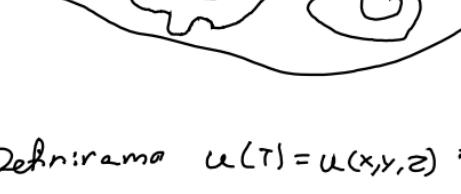
Potrebuješmo potencial  $u$ .  $\vec{\nabla} u = \vec{R}$

Naj bo  $D$  povezane ( $\exists n$ : delenje po vseh komponentih posebej)

$D$  je s potmi povezane

za vsko dve točki obstaja odsekoma gladka pot od  $A$  do  $B$

Naj bo  $A_0 \in D$  neka točka



Naj bo  $T \in D$ .

Naj bo  $\vec{r} \subseteq D$

odsekoma gladka pot od  $A_0$  do  $T$

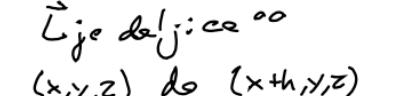
$$\text{Definiramo } u(T) = u(x, y, z) = \int_{\vec{\Gamma}} \vec{R} dr$$

Le debro definirano, ker n: odkrišen od poti od  $A_0$  do  $T$

$$\text{Tovimo: } \vec{\nabla} u = (u_x, u_y, u_z) = \vec{R} = (x, y, z)$$

$$\lim_{h \rightarrow 0} \frac{u(x+h, y, z) - u(x, y, z)}{h} =$$

$$u(x, y, z) = \int_{\vec{\Gamma}} \vec{R} dr$$



$$A_0 \quad \vec{r} = \vec{\Gamma} \cup \vec{L}$$

$\vec{L}$  je delj: ce so

$(x, y, z)$  do  $(x+h, y, z)$

$$\frac{u(x+h, y, z) - u(x, y, z)}{h} = \frac{1}{h} \left( \int_{\vec{\Gamma} \cup \vec{L}} \vec{R} dr - \int_{\vec{\Gamma}} \vec{R} dr \right) =$$

$$= \frac{1}{h} \int_{\vec{L}} \vec{R} dr$$

parametrizacija  $\vec{L}: \vec{r}: t \rightarrow (x+h, y, z)$

$$t \in [0, 1]$$

$$\vec{r}(t)(h, 0, 0) = h(1, 0, 0)$$

$$\frac{1}{h} \int_0^1 \vec{R}(x+h, y, z) h(1, 0, 0) dt =$$

$$= \int_0^1 X(x+h, y, z) dt \xrightarrow{h \rightarrow 0} X(x, y, z)$$

Res  $u_x = X$  podobno se  $u_y$  in  $u_z$

Opombe:

$$1) \int_{\overline{\Omega}} \vec{\nabla} u \cdot d\vec{r} = u(k_{\overline{\Omega}}) - u(z_{\overline{\Omega}})$$

če je  $v$  nek drug potencial  $\vec{R} = \vec{\nabla} v = \vec{\nabla} u$

$$\text{D povezne} \Rightarrow v = u + C$$

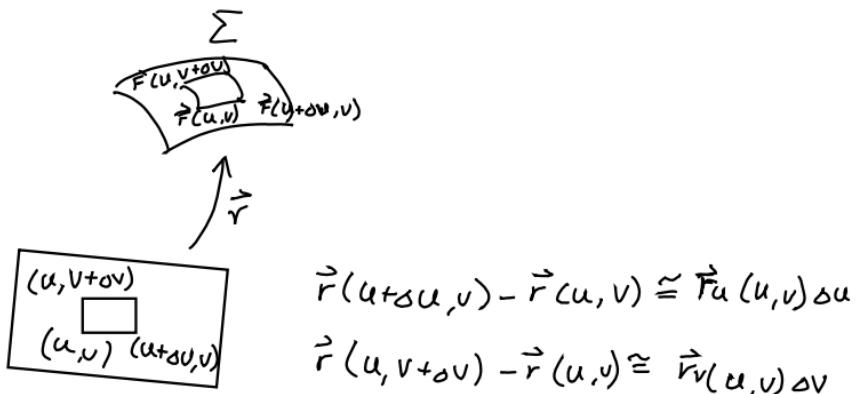
2) D zvezdesto glede na nelo teko,  $\text{rot } \vec{R} = \vec{0}$

$$u(T) = u(x, y, z) \int_{[A_0, T]} R \, d\vec{r} = \int_0^1 (x X(t_x, t_y, t_z) +$$

$$y Y(x, y, z) + z Z(x, y, z)) \, dt$$

# Površina ploške

$\Sigma \subseteq \mathbb{R}^3$  gladka ploška



$$\Delta P = |\vec{r}_u \times \vec{r}_v| dudv$$

Definiramo:

$$P(\Sigma) = \iint_D |\vec{r}_u \times \vec{r}_v| dudv$$

$$\text{Opremba: } (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \vec{c})(\vec{b} \vec{d}) - (\vec{a} \vec{d})(\vec{b} \vec{c})$$

$$(\vec{r}_u \times \vec{r}_v) \cdot (\vec{r}_u \times \vec{r}_v) = |\vec{r}_u|^2 |\vec{r}_v|^2 - (\vec{r}_u \cdot \vec{r}_v)^2 \geq 0$$

$$\begin{aligned} E &= |\vec{r}_u|^2 \\ F &= \vec{r}_u \cdot \vec{r}_v \\ G &= |\vec{r}_v|^2 \end{aligned}$$

$$\frac{1}{EG - F^2} = \begin{vmatrix} E & F \\ F & G \end{vmatrix}$$

$$P(\Sigma) = \iint_D \sqrt{EG - F^2} dudv$$

Parabolische Grade

$$\sum \{x, y, f(x,y)) \mid (x,y) \in D \subseteq \mathbb{R}^2\}$$

$$\vec{r}(x,y) = (x, y, \tilde{f}(x,y))$$

$$\vec{r}_x = (1, 0, f_x(x,y))$$

$$r_y = (0, 1, f_y(x,y))$$

$$E = 1 + f_x^2$$

$$F = f_x f_y$$

$$G = 1 + f_y^2$$

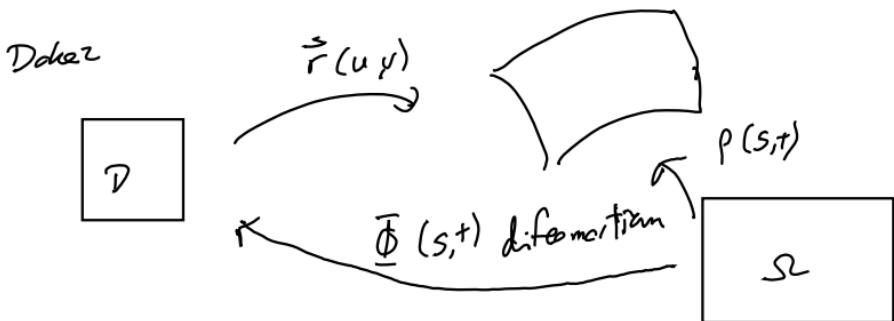
$$EG - F^2 = (1 + f_x^2) + (1 + f_y^2) - f_x^2 f_y^2 =$$

$$= 1 + f_x^2 + f_y^2$$

$$P(Q(f)) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx, dy$$

Opomba: EF, G so koeficienti I fundamentalne  
fame ploske  $E u^2 + 2FuU + Gu^2$

Trditev: Definicija površine ploskve je neodvisna od regularne parametrizacije ploskve



$$\tilde{P}(s,t) = (\vec{r} \circ \underline{\Phi})(s,t)$$

$$\iint_{\mathcal{R}} |\tilde{P}_s \times \tilde{P}_t| ds dt = \iint_{\mathcal{R}} (\vec{r}_u \times \vec{r}_v) |ds dt|$$

$$\underline{\Phi}: \mathcal{R}_{(s,t)} \rightarrow P_{u,v}$$

$$\underline{\Phi}(s,t) = (U(s,t), V(s,t))$$

$$P_s = \vec{r}_u \cdot U_s + \vec{r}_v \cdot V_s$$

$$P_t = r_u V_t + \vec{r}_v \cdot V_t$$

$$P_s \cdot P_t = (\vec{r}_u \times \vec{r}_v) U_s V_t - (\vec{r}_u \times \vec{r}_v) U_t V_s = \\ = (\vec{r}_u \times \vec{r}_v) (U_s V_t - U_t V_s)$$

$$|P_s \times P_t| = |\vec{r}_u \times \vec{r}_v| \left| \begin{matrix} U_s & V_t \\ V_s & V_t \end{matrix} \right|$$

$$\iint_D |\vec{r}_u \times \vec{r}_v| du dv = \iint_{\mathcal{R}} |(\vec{r}_u \times \vec{r}_v) \underline{\Phi}(s,t)| |J \underline{\Phi}(s,t)| ds dt =$$

$$(u,v) \in \underline{\Phi}(s,t)$$

$$= \iint_{\mathcal{R}} |P_s \times P_t| ds dt$$

zgfele:

$$\rightarrow S(0, \alpha) = \{ (x, y, z) \mid x^2 + y^2 + z^2 = \alpha^2 \}$$

$$x(\vartheta, \rho) = \alpha \cos \vartheta \sin \rho$$

$$y(\vartheta, \rho) = \alpha \cos \vartheta \sin \rho$$

$$z(\vartheta, \rho) = \alpha \cos \vartheta$$

$$\vec{r}(\vartheta, \rho) = x(\vartheta, \rho), y(\vartheta, \rho), z(\vartheta, \rho)$$

$$\vec{r}_\vartheta = (\alpha \cos \vartheta \cos \rho, \alpha \sin \vartheta \cos \rho, -\alpha \sin \vartheta)$$

$$\vec{r}_\rho = (-\alpha \sin \vartheta \sin \rho, \alpha \cos \vartheta \sin \rho, 0)$$

$$E = \alpha^2$$

$$F = 0$$

$$G = \alpha^2 \sin^2 \vartheta$$

$$P(S) = \int_0^{2\pi} \int_0^\pi \alpha^2 \sin^2 \vartheta \, d\vartheta \, d\rho =$$

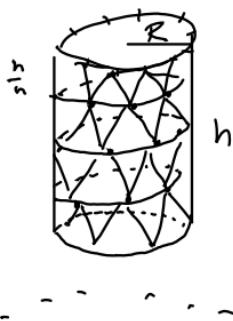
$$= \alpha^2 2\pi \left[ -\cos \vartheta \right]_0^\pi = 4\pi \alpha^2$$

$$2) f(x, y) = \frac{1}{2} (x^2 + y^2) \quad f_x = x \quad x = r \cos \varphi \\ D = \{ (x, y) ; x^2 + y^2 \leq 1 \} \quad f_y = y \quad y = r \sin \varphi$$

$$P(G(f)) = \iint \sqrt{1 + x^2 + y^2} \, dx \, dy =$$

$$\int_0^{\pi} \int_0^1 r \sqrt{1 + r^2} \, dr \, d\varphi = 2\pi \frac{2}{3} \left( 1 + r^2 \right)^{\frac{3}{2}} \Big|_0^1 = \frac{2\pi}{3} (2^{\frac{3}{2}} - 1)$$

Zgled (Schwarzova lánčovna at. Schwarzov  
skorň)

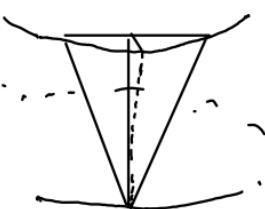


Příčleněna povrchu  $2\pi Rh$

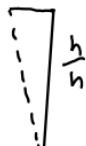
m lze využít dožív.

n může být různou

2nm trapezů (ne leží na pláštvi)



$$2 - R \cos \frac{\pi}{m} = \\ 2(1 - \cos \frac{\pi}{m}) = \\ = R 2 \cos^2 \frac{\pi}{2m} \\ 2R \cdot \sin \frac{\pi}{m} \text{ délka řetěze}$$



Plocha jednoho trojúhelníku:

$$\frac{1}{2} R \sin \frac{\pi}{m} \sqrt{\frac{h^2}{n^2} + R^2 \sin^2 \frac{\pi}{2m}}$$

Povrchu všech těch trojúhelníků je

$$P(n, m) = 2nm R \sin \frac{\pi}{m} \sqrt{\frac{h^2}{n^2} + 4R^2 \sin^2 \frac{\pi}{2m}}$$

$$1) \lim_{m \rightarrow \infty} P(m, m) = \lim_{m \rightarrow \infty} 2R m^2 \sin \frac{\pi}{m} \sqrt{\frac{h^2}{m^2} + 4R^2 \sin^2 \frac{\pi}{2m}} =$$

$$= 2R \lim_{m \rightarrow \infty} m^2 \frac{\pi}{m} \sqrt{\frac{h^2}{m^2} + 4R^2} \cancel{\frac{\pi^4}{16m^4}} =$$

$$= 2R \lim_{m \rightarrow \infty} \pi \sqrt{h^2 + 4R^2 \frac{\pi^4}{4m^2}} = 2\pi R h$$

$$\lim_{m \rightarrow \infty} P(m^2, m) = 2R \lim_{m \rightarrow \infty} m^3 \frac{\pi}{m} \sqrt{\frac{h^2}{m^4} + 4R^2 \frac{\pi^4}{16m^4}} =$$

$$= 2R \lim_{m \rightarrow \infty} \sqrt{h^2 + \frac{\pi^4 R^2}{4}} > 2\pi R h$$

$$3) \lim_{m \rightarrow \infty} P(m^3, m) = 2R \lim_{m \rightarrow \infty} m^4 \frac{\pi}{m} \sqrt{\frac{h^2}{m^8} + 4R^2 \frac{\pi^4}{16m^4}} =$$

$$= 2R \lim_{m \rightarrow \infty} \sqrt{h^2 + \frac{\pi^4 R^2}{4}} m^2 = \infty$$

## Orientacija ploskev

(za hivnje:  
 $|\vec{T}|=1$  wegi tangentni vektorji vzdolž  $\Gamma$ )

$\Sigma \subseteq \mathbb{R}^3$  gladke ploskev

orientacija  $\Sigma$  je vezan izhod enotne normale  
na  $\Sigma$

$$\vec{N} : \Sigma \rightarrow \mathbb{R}^3$$

$$|\vec{N}|=1 \quad \vec{N}(x,y,z) \perp T_{(x,y,z)}\Sigma$$

če  $\Sigma$  lahko orientiramo je  $\Sigma$  orientabilna  
(sicer je neorientabilna)

$(\Sigma, \vec{N})$  če je  $\Sigma$  površina in orientabilna ima  
netanko dve različni orientaciji:

$$(\Sigma, \vec{N}), (-\Sigma, \vec{N})$$

Zadani:

1)  $\Sigma = G(f) = \{(x, y, f(x, y)) ; (x, y) \in D \subseteq \mathbb{R}^n\}$

$$f \in C_1(D)$$
$$\vec{N} = \frac{(-f_x - f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

2)  $S^2 = \{(x, y, z) \in \mathbb{R}^3 ; x^2 + y^2 + z^2 = R^2\}$

$$\vec{N} = \frac{(x, y, z)}{R}$$

zunanja normala:

$$x^2 + y^2 + z^2 = R^2 \quad R > 0$$



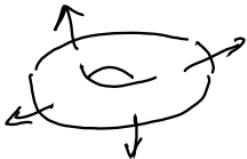
3)



$$\Sigma = \{(x, y, z) ; x^2 + y^2 = R^2, z \in \mathbb{R}\}$$

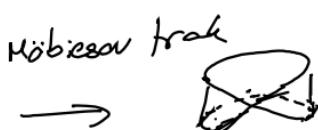
$$\vec{N} = \frac{(x, y, 0)}{R}$$

④



v  $\mathbb{R}^3$  so vse sklenjene (komplexe) plastične orientabilne

5)



N: orientabilne

$\Sigma \subseteq \mathbb{R}^3$  je lako orientabilna ali pa n:  
orientabilna

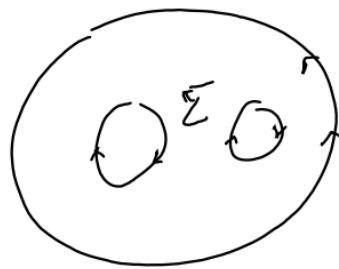
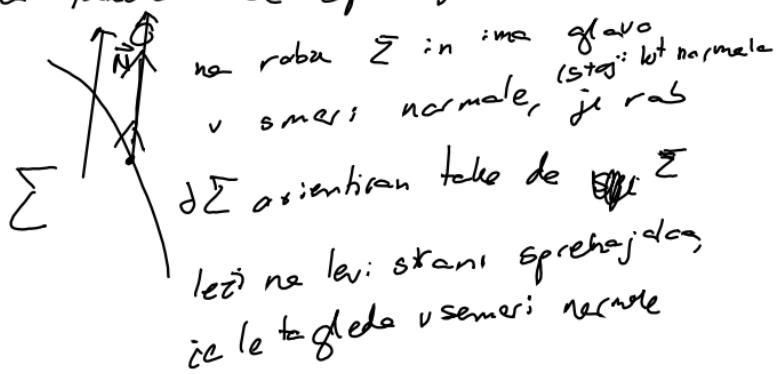
↓  
2 možni orientaciji  
za površinu  $\Sigma$

6)  $\Sigma \subseteq \mathbb{R}^2 \times \mathbb{S}^1$

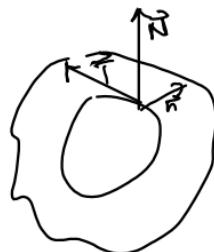
če n: rečno drugačje jo orientiramo z  
normalo  $(0,0,1) = \vec{n}$   
(pozitivna orientacija)

Naj bo  $\Sigma$  gladka plošča z robom (konica  
unija odsekoma gladkih sklenjenih krvulj)

Naj bo  $(\Sigma, \vec{n})$  plošča z orientacijo  
skladne oz. kohesivne orientacije  $\delta\Sigma$  je  
definirana takole: če sprehajalec stoji:



če  $\bar{\Sigma} \subseteq \tilde{\Sigma}$



$\vec{T} = \vec{n} \times \vec{n}$

$\vec{n}$  ... evračna normala  
čez krvuljo

$(\Sigma, \vec{n}) \rightsquigarrow (\delta\Sigma, \vec{T})$   
skladna orientacija

$(\Sigma, -\vec{n}) \rightsquigarrow (\delta\Sigma, -\vec{T})$

Opoomba:  $\mathcal{M}^2$

če:  $\vec{r}: D \rightarrow \Sigma$  regularna parametrizacija

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

Orientacija  $\vec{N}$  je vektor s parametrizacijo  $\vec{r}$

### Odselkovna gladka plaskov

$$\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n \subset \mathbb{R}^n$$

1)  $\Sigma_1 \cup \dots \cup \Sigma_n$  so gladke plaskove z robom

2) če  $\Sigma_i \cap \Sigma_j \neq \emptyset$  je presek del raba  $\Sigma_i$  in  $\Sigma_j$   
 $i \neq j$

3)  $\Sigma_i \cap \Sigma_j \cap \Sigma_k \in \{\emptyset, \Sigma\}$

$i \neq j \neq k$  (presek je ali: prazen ali tacik)

Orientacija  $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$  če obstaja,  
je tak neobar orientacija, da sta ne vseh  
preselkih

$\Pi_{ij} = \Sigma_i \cap \Sigma_j$  inducirani orientaciji:  
nespratni

# Ploskovni integral

1) Ploskovni integral skalarnega polja

$$u: \Sigma \rightarrow \mathbb{R} \quad \text{verno skalarne polje}$$
$$\vec{r}: D \xrightarrow{\text{regularne parametrične}} \Sigma$$

$$\iint_{\Sigma} u dS := \iint_D u(\vec{r}(u,v)) \sqrt{E-G-F^2} du dv =$$

$$\iint_D u(\vec{r}(u,v)) \| \vec{r}_u \times \vec{r}_v \| du dv$$

$$E = |\vec{r}_u|^2 \quad F = \vec{r}_u \cdot \vec{r}_v \quad G = |\vec{r}_v|^2$$

Opoomba:  $u=1 \Rightarrow$  to je površine ploskve

Opoomba: Ta vrednost je neodvisna od regularne parametrične zacepite  $\vec{r}$

Opoomba: Ta ne potrebujeamo orientacije

Ta integral obstaja tudi na Möbiusovem trakcu

Opoomba:  $u=\rho \dots$  površinega ogostota, potem je

$$\iint_{\Sigma} \rho dS = m(\Sigma)$$

$\sum \rho$  konstanten  $\Rightarrow$  ploskev je homogene

Zgfd

$$S = \{ (x, y, z) ; x^2 + y^2 + z^2 = R^2 \} \quad R > 0$$

homogene Stoff

$$J_z = \iint_S \rho (x^2 + y^2) dS$$

$$(\vartheta, \rho) \mapsto (R \cos \varphi \sin \vartheta, R \sin \varphi \cos \vartheta, R \cos \vartheta)$$

$$\sqrt{E\mathbf{G} - F^2} = R^2 \sin \vartheta$$

$$J_z = \iint_0^{2\pi} \rho_0 R^2 \sin^2 \vartheta R^2 \sin \vartheta d\vartheta d\varphi =$$

$$= \rho_0 2\pi R^4 \int_0^\pi \sin^3 \vartheta d\vartheta = 2\pi \rho_0 R^4 \int_0^1 (1-t^2) dt$$
$$2\pi \rho_0 R^4 \cdot 2 \left(1 - \frac{1}{3}\right) = \frac{8}{3} \rho_0 R^4 =$$

$$m(S) = 4\pi R^2 \rho_0$$

$$= \frac{2}{3} m R^2$$

2) Ploščovni integral vektorstva polja po orientirani plošči

$(\Sigma, \vec{N})$   $\vec{R}: \Sigma \rightarrow \mathbb{R}^3$  vemo vek. polje

$$\int_{\Gamma} \vec{R} d\vec{s} = \int (\vec{R} \cdot \vec{T}) ds$$



Pretok  $\vec{R}$  skozi  $\Sigma$

$$\iint_{\Sigma} \vec{R} d\vec{S} := \iint_{\Sigma} (\vec{R} \cdot \vec{N}) dS \quad \text{Neodvisno reg.}$$

$\Sigma$  parametrizacija, usklajene z orientacijo  $\Sigma$   
parametrizacija, usklajene z orientacijo  $\Sigma$

$$\text{Tudi } \iint_{\Sigma} \vec{R} (-\vec{N}) dS = - \iint_{\Sigma} (\vec{R} \vec{N}) dS$$

$\vec{r}$  je vektorska funkcija  $\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

Potem:

$$\iint_{\Sigma} \vec{R} dS = \iint_{\Sigma} \vec{R} \vec{N} dS = \iint_D \vec{R} (\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

$$= \iint_D (\vec{r}_u, \vec{r}_v, \vec{R}) du dv$$

$\mathbb{R}^D$   $\wedge$  mehanični produkt

če je  $\Sigma = \sum_1^n \dots \sum_1^n$

$$\iint_{\Sigma} adS = \sum_{j=1}^n \iint_{\Sigma_j} adS$$

$$\iint_{\Sigma} \vec{R} d\vec{S} = \sum_j \iint_{\Sigma_j} \vec{R} d\vec{S}$$

Zgled

$$S = f(x, y, z); x^2 + y^2 + z^2 = R^2$$

$$\begin{aligned}\vec{R} &= \frac{\vec{r}}{|r|^3} = \left( \frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \\ &= -\vec{\nabla} \frac{1}{|r|}, \quad \operatorname{div} \vec{R} = 0 \quad \delta \frac{1}{r} = 0 \text{ na } \mathbb{R}^3 - \{0\}\end{aligned}$$

$$\vec{N} = (0, 0, 1) = (0, 0, 1)$$

$$\begin{aligned}\iint_S \vec{R} d\vec{s} &= \iint_S \frac{\vec{r}}{|r|^3} \cdot \frac{\vec{r}}{|r|} \cdot d\vec{s} = \iint_S \frac{R^2}{R^4} d\vec{s} = \\ &= \iint_S \frac{1}{R^2} d\vec{s} = \frac{1}{R^2} 4\pi R^2 = 4\pi\end{aligned}$$

Oponha Diferenciável 1-forma

$$\int_{\Gamma} \vec{R} \cdot d\vec{r} = \int_{\tilde{\Gamma}} X dx + Y dy + Z dz$$

$$\iint_{\tilde{\Gamma}} \vec{R} \cdot d\vec{S} = \int X dx \wedge dz + Y dz \wedge dx + Z dx \wedge dy$$

## Integralni izreki

$$\int_a^b f(x) dx = f(b) - f(a)$$

$$I = [a, b]$$

$$\vec{T} = \vec{1}$$

Integral po orientiranem intervalu  $f$  po  
 $\delta[a, b]$  arsentriranem skupaj s ogstirjo  $\delta[a, b]$

Že vemo:

$$\int_{\vec{\Pi}} \vec{v} u d\vec{x} = u(k_{T_r}) - u(z_{T_r})$$

oblique integration by parts

$M$  orientierungsmässig z. Raum  $\partial M$

$$M = [a, b] \quad M = \Gamma, M = D \subset \mathbb{R}^2, M = \Sigma \subset \mathbb{R}^3, M = \mathcal{R} \subset \mathbb{R}^3$$

Idee: integriere oblique:

$$\int_M \omega = \int_{\partial M} dw \quad (\text{Stokes' Law})$$

rek objekt

# Gaussov izrek (Gaussov - ostrogradski izrek)

Naj bo  $S \subseteq \mathbb{R}^3$  omejena odprtta množica z robom sestavljenim iz končne števile odsekoma gladkih sklenjenih plaskov, orientiranih z zunajjo normalo  $\vec{n}$  glede na  $S$



Naj bo  $\vec{R}: S \rightarrow \mathbb{R}^3$  vekt. pošte

Tedaj je

$$\iint_S \vec{R} d\vec{S} = \iint_S \operatorname{div} \vec{R} dV$$

?? preveriti, če je ta enačba pravilna.

$$\iiint_D \operatorname{div} \vec{R} dV = \iint_D \vec{R} d\vec{S} = \iint_D \vec{R} \cdot \vec{n} dS$$

$$n=2: D \subseteq \mathbb{R}^2$$

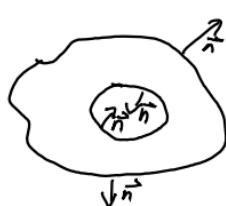
omejena odpta množica v  $\mathbb{R}^2$  z odsekoma gladkim robom sestavljenim iz končne števile odsekoma gladkih sklenjenih krivulj

Naj bo  $\vec{R}: D \rightarrow \mathbb{R}^2$  vekt. pošte

Tedaj je

$$\iint_D \operatorname{div} \vec{R} dS = \iint_D \vec{R} \cdot \vec{n} dS$$

$\leftarrow$  zunanj e endiske  
 $\vec{n}$  normale



Greenov izrek (greenova formula) :

Naj bo  $D \subseteq \mathbb{R}^2$  mejana, odp z odsekoma gladkim robom sestavljenim iz konznega stekla odsekoma gladkih sklenjenih krivulj, orientiranih pozitivno glede na  $D$

Naj bosta  $x, y \in C^1(D)$

Tedaj

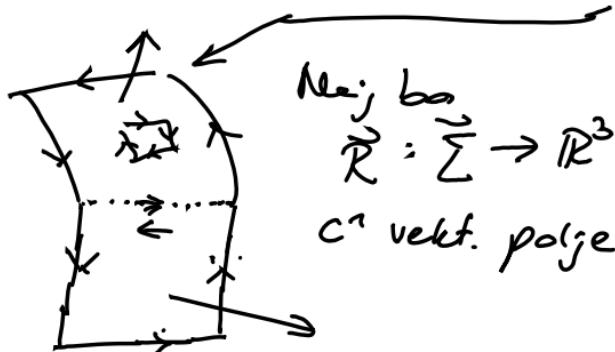
$$\int\limits_{\partial D} X dx + y dy = \iint\limits_D (y_x - x_y) dx dy$$

"

$$\int\limits_{\partial D} \vec{R} dr \quad \vec{R} = (x, y)$$

Stokesov izrek:

Naj bo  $\sum \subset \mathbb{R}^3$  ravnjava, orientirana, odsekana gladka ploskev z odsekano gladkim robom, sestavljenim iz končnega števila odsekana gladkih sklenjenih kurvij orientiranih skladno z orientacijo  $\sum$



Tedaj je

$$\int_{\sum} \vec{R} d\vec{r} = \iint_{\Sigma} \text{rot} \vec{R} d\vec{s}$$

Zgled

1)  $D = k(0, R_0)$   $R_0 > 0$

$$\vec{R} = (x, y, z)$$

$$X(x, y, z) = x$$

$$Y(x, y, z) = y$$

$$Z(x, y, z) = z$$

$$\iiint_D \operatorname{div} \vec{R} dV =$$

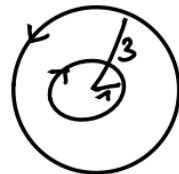
$$= \iiint_D 3 dV = 3 V(D) = 3 \frac{4\pi R_0^3}{3} = 4\pi R_0^3$$

$$\iint_D \vec{R} d\Omega = \iint_{\{(0, R_0)\}} \frac{(x, y, z)}{R_0} d\Omega =$$

$$= R_0 \iint_{S(0, R_0)} 1 d\Omega = R_0 (4\pi R_0^2)$$

$$2) D = \{(x, y) ; 1 < x^2 + y^2 < 9\}$$

$$X = -y \quad Y = x$$



$$\iint_D (x_y - y_x) dx dy =$$

$$= \iint_D 2 dx dy = 2 P(D) =$$

$$2(\pi \cdot 9 - \pi \cdot 1) = 16\pi$$

$$\iint_D x dx + y dy = \int_{x^2+y^2=3^2} -y dx + x dy + \int_{x^2+y^2=1} -y dx + x dy$$

(E)      .)      (Q)      .)

$$\bullet) x = 3 \cos t \quad dx = -3 \sin t dt$$

$$y = 3 \sin t \quad t \in [0, 2\pi] \quad dy = 3 \cos t dt$$

$$\int_{x^2+y^2=3^2} -y dx + x dy = \int_0^{2\pi} (-3 \sin t) (-3 \sin t / dt) +$$

$$\bullet) \quad + 3 \cos t \cdot 3 \cos t dt =$$

$$= 9 \int_0^{2\pi} 1 dt = 18\pi$$

$$\bullet) X = \cos t \quad t \in [0, 2\pi] \quad \text{do } 0$$

$$\int_0^{2\pi} (-\sin t) (-\sin t) dt + \cos t \cdot \cos t dt =$$

$$= \int_0^{2\pi} 1 dt = -2\pi$$

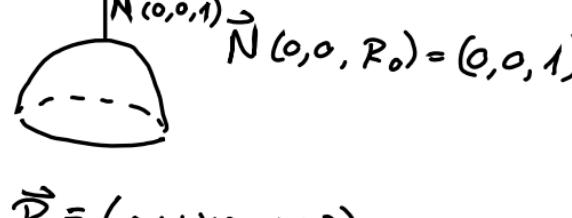
$$\bullet) + \bullet) = 16\pi$$

$D \subseteq \mathbb{R}^2$  lat v greenov; formelle.

$$\int_{\partial D} -y dx + x dy = 2 \cdot Pl(D)$$

$$Pl = \frac{1}{2} \int_{\partial D} -y dx + x dy$$

$$3) R_o > 0 \quad \Sigma : \{x, y, z \in \mathbb{R}^3 ; x^2 + y^2 + z^2 = R_o^2, z > 0\}$$



$$\vec{R} = (xy, x, x-y+z)$$

$$\int_{\Sigma} \vec{R} d\sigma$$

parametriziamo  $\Sigma$

$$x = R_o \cos t$$

$$y = R_o \sin t$$

$$z = 0 \quad t \in [0, 2\pi]$$

$$R(R_o \cos t, R_o \sin t, 0) =$$

$$= (0, R_o \cos t, R_o (\cos t - \sin t))$$

$$\dot{x} = -R_o \sin t$$

$$\dot{y} = R_o \cos t$$

$$\dot{z} = 0$$

$$\vec{R} \cdot \vec{\tau} = R_o^2 \cos^2 t$$

$$\begin{aligned} \int_{\Sigma} \vec{R} d\sigma &= \int_{\Sigma} x dx + y dy + z dz = \int_0^{2\pi} -R_o^2 \cos^2 t dt = \\ &= R_o^2 \int_0^{2\pi} \frac{1 + \cos(2t)}{2} dt = \pi R_o^2 \end{aligned}$$

$$\text{rot } \vec{R} = (-1, y-1, 1-z)$$

$$\iint_{\Sigma} \text{rot } \vec{R} dS = \iint_{\Sigma} (-1, y-1, 1-z) \cdot \frac{(x, y, z)}{R_o} dS$$

$$= \frac{1}{R_o} \iint_{\Sigma} (-x + y^2 - y + z - z^2) dS$$

Parametrizzazione polsfere

$$x = R_o \cos \varphi \sin \vartheta$$

$$y = R_o \sin \varphi \sin \vartheta$$

$$z = R_o \cos \vartheta \quad \varphi \in [0, 2\pi] \quad \vartheta \in [0, \frac{\pi}{2}]$$

$$\sqrt{EG - F^2} = R_o^2 \sin \vartheta$$

$$J = \frac{1}{R_o} \iiint_0^{\frac{\pi}{2}} R_o^2 \sin \vartheta (-R_o \cos \varphi \sin \vartheta \sin \varphi \sin \vartheta + R_o \cos \varphi \sin \vartheta \sin \vartheta \sin \vartheta + R_o \sin \varphi \sin \vartheta + R_o \cos \varphi \sin \vartheta - R_o^2 \cos^2 \vartheta) d\vartheta d\varphi =$$

$$= R_o \int_0^{\frac{\pi}{2}} (\pi R_o \sin^3 \vartheta + \pi R_o \sin(2\vartheta) - 2\pi R_o^2 \cos^2 \vartheta) d\vartheta =$$

$$= R_o \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 - \cos^2 \vartheta \sin \vartheta - 2R_o \cos^2 \vartheta \sin \vartheta + \sin 2\vartheta) d\vartheta =$$

$$= \pi R_o^2 \left( R_o \int_0^1 (1 - 3\cos^2 t) dt + 1 \right) = \pi R_o^2$$

$$\underbrace{R_o(1-1)+1}_{R_o(1-1)+1}$$

$$\rightarrow = \pi R_o^2$$

$$\vec{R} = (zy, x, x-y+z)$$

$$\operatorname{rot} \vec{R} = (-1, y-1, 1-z)$$



$$\sum U \{ (x, y, 0) | x^2 + y^2 \leq R^2 \}$$

Guassov izrek

$$\iint_D \operatorname{rot} \vec{R} d\vec{s} = \iiint_D \operatorname{div} (\operatorname{rot} \vec{R}) dv$$

$$\iint_D \operatorname{rot} \vec{R} d\vec{s} + \iint_{\overset{\circ}{D}} \operatorname{rot} \vec{R} d\vec{s}$$

$$N = (0, 0, -1)$$

$$\iint_D \operatorname{rot} \vec{R} d\vec{s} = \iint_D (-1, y-1, 1) (0, 0, -1) dS$$

$$= - \iint_D 1 dS = -\pi R_0^2$$

Gaußsche Integrale über  $\mathbb{R}^2$

$\Rightarrow$  Greenova Formel

Daher v Greenovas Formel:

v Gaußschem Integ. für  $n=2$

$$\int\limits_{\partial D} X dx + Y dy = \int\limits_{\partial D} \vec{F} \cdot d\vec{s} = \int\limits_{\partial D} \vec{R} \cdot \vec{T} \cdot ds =$$

$\partial D$  je positiv orientiert

$$= \int\limits_{\partial D} (X \cdot T_1 + Y T_2) ds \quad \vec{T} = (T_1, T_2)$$



$$\int\limits_{\partial D} \vec{R} \cdot \vec{T} ds = \int\limits_{\partial D} (-i) \vec{R} \cdot (-; \vec{T}) ds$$

$$\vec{R} = (x, y) \quad " - ; R " = (y, -x)$$

$$\vec{n} = (T_2, -T_1) = " \vec{T}_1 + i \vec{T}_2 "$$

$$= \int\limits_{\partial D} \underbrace{(y \cdot T_2 + (-x) \cdot (-T_1))}_{(y-x) \cdot \vec{n}} ds$$

gauss

$$= \iint_D (y_x - x_y) dx dy$$

Stokesov izrek  $\Rightarrow$  greenova formula

$$\int_{\partial\Sigma} \vec{P} d\vec{\sigma} = \iint_{\Sigma} \text{rot } \vec{P} d\vec{s}$$

$$\Sigma = D, \quad \vec{N} = (0, 0, 1)$$

$$\vec{P} = (x_{(x,y)}, y_{(x,y)}, 0)$$

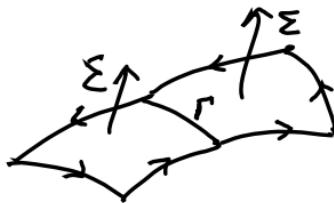
$$\text{rot } \vec{P} = (0, 0, y_x - x_y)$$

$$\Rightarrow \int_{\partial D} R d\vec{\sigma} = \iint_D (y_x - x_y) dx dy$$

$$\Sigma = \Sigma_1 \cup \Sigma_2 \subset \mathbb{R}^3$$

Ploščev v  $\text{P}_t^3$ , orientirane, omogene  
z odsekoma gladkim robom

$$\overline{\sum}_n \cap \overline{\sum}_z = \text{P kriwage}$$



$$\vec{R}: \overline{\Sigma} \rightarrow \mathbb{R}^3 \quad c^1 \text{ vekt. pdje}$$

$$\iint \text{rot } \vec{R} d\vec{S} = \iint_{\Sigma_1} \text{rot } \vec{R} dS + \iint_{\Sigma_2} \text{rot } R^2 dS =$$

Frosch: me plasdiro o

$$= \iint_{\Sigma_1} \vec{R} dr + \int_{\Sigma_2} \vec{R} dr$$

ce vige stabilisere  $\Sigma_1$  m  $\Sigma_2$

Patem velja trudi za {

$$= \iint_{\sum_1} R dr^2 + \overbrace{\int_{\Gamma} \vec{R} dr}^{=0} + \int_{\Gamma} \vec{R} dr + \int_{\delta \sum_2 - \Gamma} R d\vec{R} = \int_{\delta \sum} R dr$$

orientirane  
 grænse på  $\sum_1$       orientirte  
 grænse  
 $\sum_2$

Greenova formula  $\Rightarrow$  Stokesova iock

Omejimo se lahko na primer, ko je

$$\sum \text{graf nad } D \subseteq \mathbb{R}^2$$

$$\sum = \{(x, y, f(x, y)) ; x, y \in D \subseteq \mathbb{R}^2\}$$

$$f \in C^2(\bar{D})$$

$\sum$  naj bo orientirane z normalo, ki keže navzgor  $\vec{N} = \frac{(-f_x, f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$

$\partial\sum$  je graf f nad  $\partial D$

Pozitivna orientacija  $\partial D$  nam da pozitivno (oz skledno) orientacijo  $\partial\sum$

$$\vec{R} = (x, y, z)$$

$$\int_{\partial\sum} \vec{R} d\vec{s} = \int_{\partial\sum} X dx + Y dy + Z dz =$$

Naz  $\partial\sum$  je  $z = f(x, y)$  torej

$$dz = f_x dx + f_y dy$$

$$= \int_D X(x, y, f(x, y)) dx + Y(x, y, f(x, y)) dy + Z(f_x dx + f_y dy) =$$

$$= \int_D (X + 2f_x) dx + (Y + 2f_y) dy =$$

$$G.F = \iint_D (Y + 2f_y)_x - (X + 2f_x)_y dx dy =$$

$$= \iint_D (Y_x + Y_z f_x + Z_x f_y + Z_y f_x \cdot f_y + Z f_{yx}) - (X_y + X_z f_y + Z_x f_x + Z_z \cdot f_y \cdot f_x + Z f_{xy}) dx dy$$

$$\text{rot } \vec{R} = (z_y - y_z, x_z - z_x, y_x - x_y)$$

$$= \iint_D (-f_x)(z_y - y_z) + (-f_y)(x_z - z_x) +$$

$$+ (y_x - x_y) dx dy =$$

$$= \iint_D \frac{(-f_x - f_y, 1)}{\sqrt{1+f_x^2+f_y^2}} \cdot (\text{rot } \vec{R})(x, y) \underbrace{\sqrt{1+f_x^2+f_y^2}}_{ds} ds$$

$$= \iint_{\sum} \text{rot } \vec{R} \vec{N} ds$$

Zufall:

$$\vec{R} = \frac{\vec{r}}{\|\vec{r}\|^3} = \vec{r} \left( -\frac{1}{r} \right)$$

✓ 19.3  
primer  
(oständig)

Gaußsche Formel ( $n=3$ )

Die gaußsche Formel für  $D_1 \cup D_2 \Rightarrow$   
vert. von  $D$

$$\iiint_D \operatorname{div} \vec{R} dV = \iiint_{D_1} \operatorname{div} \vec{R} dV + \iiint_{D_2} \operatorname{div} \vec{R} dV$$

# Dokaz (Gaussova rečenica)

Ko je  $\mathbf{R}$  tečje oblike, da ned uveliko  
od treh koordinatnih ravni teči  
med dva druga tečena  $C^1$  funkcij  
ned enjeno adaptivno množico s ploskev

$$\iint_{\Sigma} \mathbf{R} = (x_1 z)$$

$$N = (N_x, N_y, N_z)$$

$$\iint_{\Sigma} \mathbf{R} N dS = \iiint_{V} \mathbf{R} dV$$

Oz. ravn

$$\begin{aligned} & \iint_{\partial D} x (y_x + z) dS = \\ & \quad \iint_{\partial D} (x_1 + y_1 + z_1) dV \end{aligned}$$

Oblike  $\Sigma$  implicira

$$\iint_{\partial D} x N_x dS = \iiint_{V} x_1 dV$$

$$\iint_{\partial D} y N_y dS = \iiint_{V} y_1 dV$$

$$\iint_{\partial D} z N_z dS = \iiint_{V} z_1 dV$$



$$\iiint_{V} z_1 dV = \iint_D \left( \int_{z=f(x,y)}^{z=g(x,y)} dz \right) dx dy =$$

$$= \iint_D (z(x,y, f(x,y)) - z(x,y, g(x,y))) dx dy$$

$$\iint_{\partial D} z N_z dS = \iint_{\text{graf } f} z N_z dS + \iint_{\text{navpiši}} z N_z dS + \iint_{\text{graf } g} z N_z dS$$

$$N = \frac{(-f_x, f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$$

$$N_z = 0$$

$$N_z = \frac{(g_x, g_y, -1)}{\sqrt{1+g_x^2+g_y^2}}$$

$$= \iint_D z(x,y, f(x,y)) \frac{1}{\sqrt{1+f_x^2+f_y^2}} \underbrace{\sqrt{1+f_x^2+f_y^2} dx dy}_{dS} +$$

$$+ 0 +$$

$$+ \iint_D z(x,y, g(x,y)) \frac{(-1)}{\sqrt{1+f_x^2+f_y^2}} \underbrace{\sqrt{1+f_x^2+f_y^2} dx dy}_{dS}$$

## Greenovi identiteti

Naj bo  $\mathcal{S} \subset \mathbb{R}^3$  kot v Gaussovem izreku

Naj bo  $u, v \in C^2(\bar{\mathcal{S}})$

Tedaj velja:

$$1) \iint_{\partial \mathcal{S}} u \frac{du}{d\vec{n}} d\mathcal{S} = \iiint_{\mathcal{S}} (\vec{\nabla} u \cdot \vec{\nabla} v + u \Delta v) dv \quad \xrightarrow{\text{Laplace}}$$

$$2) \iint_{\partial \mathcal{S}} (u \frac{dv}{d\vec{n}} - v \frac{du}{d\vec{n}}) d\mathcal{S} = \iiint_{\mathcal{S}} (u \Delta v - v \Delta u) dv$$

Opozme: i) Enost ka zunanjega normala na  $\partial \mathcal{S}$

$$\text{ii}) \frac{\partial u}{\partial \vec{n}} = \text{smerni odvod } u \text{ v smeri } \vec{n} \\ = \vec{\nabla} u \cdot \vec{n}$$

$$\text{iii}) \frac{\partial u}{\partial \vec{n}} d\mathcal{S} = \vec{\nabla} u \cdot \vec{n} \cdot d\mathcal{S} = \vec{\nabla} u d\mathcal{S}$$

....) "integracija po delih"

(....)  $u$ ;  $\Delta u = 0$  harmonična

$$\Rightarrow \iiint_{\mathcal{S}} v \Delta u dv = 0$$

$u$ ;  $\forall v$   prirabuje 0

$$\iiint_{\mathcal{S}} u \Delta v dv = 0 \quad \xrightarrow{?} \text{u harmonična}$$

Dоказ:

$$\vec{R} = u \vec{\nabla} v = (uv_x, uv_y, uv_z) \Leftrightarrow$$

$$\operatorname{div} \vec{R} = (uv_x)_x + (uv_y)_y + (uv_z)_z =$$

$$= u_x v_x + u_y v_y + u_z v_z + u \Delta v =$$

$$= \vec{\nabla} u \cdot \vec{\nabla} v + u \Delta v$$

1)

$$\iint_{\Sigma} u \frac{dv}{d\vec{n}} d\vec{s} = \iint_{\Sigma} u \vec{\nabla} v \cdot \vec{n} \cdot d\vec{s} = \iint_{\Sigma} u \vec{\nabla} v \cdot d\vec{s}$$

также

$$= \iiint_{\Omega} \operatorname{div}(u \vec{\nabla} v) dV = \iiint_{\Omega} (\vec{\nabla} u \cdot \vec{\nabla} v + u \Delta v) dV$$

2) Всегда туди:

$$\iint_{\Sigma} v \frac{du}{d\vec{n}} d\vec{s} = \iiint_{\Omega} (\vec{\nabla} v \cdot \vec{\nabla} u + v \Delta u) dV$$

о доказано

Brež koordinatna definicija divergencije:

$$\vec{R} : D^{\text{od}} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad C^1$$



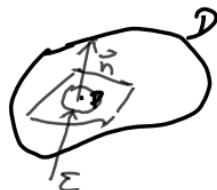
$$\text{Velja: } \iint_{S(P, \epsilon)} \vec{R} d\vec{s} = \iiint_{K(P, \epsilon)} \operatorname{div} \vec{R} dV$$

$$\frac{1}{V(K(P, \epsilon))} \iint_{S(P, \epsilon)} \vec{R} d\vec{s} \xrightarrow[\epsilon \rightarrow 0]{} \frac{1}{V(K(P, \epsilon))} \iint_{K(P, \epsilon)} \operatorname{div} \vec{R} dV$$

$$\lim_{\epsilon \rightarrow 0} = \frac{3}{4\pi \epsilon^3} \iint_{S(P, \epsilon)} \vec{R} d\vec{s} = \operatorname{div} \vec{R}(P)$$

Pretok  $\vec{R}$  skozi sfero

# Brekoordinatne definicije rotacije



$$p \in D$$

$$\vec{R} : D \rightarrow \mathbb{R}^3$$

$$|\vec{n}| = 1$$

$\sum$  krog v ravni in sredina v r  
in polmerom  $\epsilon$

$$\int\limits_{\partial\Sigma} \vec{R} d\vec{r} = \iint\limits_{\Sigma} \text{rot } \vec{R} \vec{n} dS$$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon^2} \int\limits_{\partial\Sigma} \vec{R} d\vec{r} = (\text{rot } \vec{R})(p) \cdot \vec{n}$$

Diferenciálne forme  $\Omega \overset{\text{odp}}{\subseteq} \mathbb{R}^3$

0-forme: funkcia na  $\Omega$  (vzaj  $C^2$ )

$u: \Omega \rightarrow \mathbb{R}$  odvod - diferencial

$$du = u_x dx + u_y dy + u_z dz$$

$dx$  = diferencial funkcia  $(x, y, z) \mapsto x$

$$dy = -1$$

$$dz = -1$$

$$du(\vec{v}) = u_x v_1 + u_y v_2 + u_z v_3$$

## Ekvivalentne forme

$w$  je ekvivalentna ce je odred  $0$ -forma  
 $w = du$

Forme množimo:

Proizvod dveh  $1$ -form je  $2$ -forma

1 vnanji produkt

- Na  $1$ -formama je antikomutativan

$$dx \wedge dy = -dy \wedge dx$$

$$dy \wedge dz = -dz \wedge dy$$

$$dz \wedge dx = -dx \wedge dz$$

$$\Rightarrow dx \wedge dx = -dx \wedge dx = 0 = dy \wedge dy = dz \wedge dz$$

- Standardna baza prostora  $2$ -forma v

vseki takoči je  $(dy \wedge dz, dz \wedge dx, dx \wedge dy)$

Vseki  $2$ -forma je oblike

$$\nu = A(x,y,z)dy \wedge dz + B(x,y,z)dz \wedge dx + C(x,y,z)dx \wedge dy$$

odwołanie 1-forme

$$\omega = X dx + Y dy + Z dz$$

$$d\omega = dX \wedge dx + dY \wedge dy + dZ \wedge dz =$$

$$= (X_x dx \wedge dx + X_y dx \wedge dy + X_z dx \wedge dz) \wedge dx + \\ + (Y_x dy \wedge dx + Y_y dy \wedge dy + Y_z dy \wedge dz) \wedge dy +$$

$$\# \dots =$$

$$= -X_y dx \wedge dy + X_z dz \wedge dx +$$

$$+ \cancel{Y_z dz \wedge dx} + Y_x dx \wedge dy$$

$$- Y_z dy \wedge dz - Z_x dx \wedge dx + Z_y dy \wedge dz =$$

$$d\omega = (Z_y - Y_z) dy \wedge dz + (X_z - Z_x) dz \wedge dx +$$

$$(Y_x - X_y) dx \wedge dy$$

1-forma  $\omega$  je sklenjena  $\bar{c}c \cdot \omega = 0$

• Vsake eksaktna forma je sklenjena

$$d^2 = 0$$

$d_0: 0\text{-forma} \rightarrow 1\text{-forma}$

$d_1: 1\text{-forma} \rightarrow 2\text{-forma}$

$$d_1 d_0 = d^2 \quad \text{oznaka}$$

---

$$\int_{\tilde{\Pi}} du = \int_{\partial \tilde{\Pi}} u = \int_{\tilde{\Pi}} du$$

$$\int_{\partial \Sigma} \omega = \iint_{\Sigma} d\omega \quad \int_{\partial \Sigma} \hat{R} d\tilde{r} = \iint_{\Sigma} \text{rot} \hat{R} d\tilde{s}$$

3-forme

$$\begin{aligned} V &= u(x,y,z) dx \wedge dy \wedge dz = \\ &= -u dy \wedge dx \wedge dz = u dy \wedge dz \wedge dx \end{aligned}$$

2-forme

2 forma  $V$  je ekvivalentna, če  $\exists$  w 1-forma,

$$\text{dejte } V = dw$$

2-forma  $V$  je ekvivalentna če je  $dV = 0$

$$V = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$$

$$dV = dA dy \wedge dz + dB dz \wedge dx + dC dx \wedge dy$$

$$= (A_x dx + A_y dy + A_z dz) dy \wedge dz + \dots + \dots$$

$$= (A_x + B_y + C_z) dx \wedge dy \wedge dz$$

Vseka ekvivalentna 2-forma je ekvivalentna

odred 3-forme je enak 0 (npr. v 3D)

Vseke 3 forme je sklenjene

$$0 \xrightarrow{d_1} 0\text{-forme} \xrightarrow{d_2} 1\text{-forme} \xrightarrow{d_3} 2\text{-forme} \xrightarrow{d_4} 0$$
$$3\text{-forme} \xrightarrow{d_5} 0$$

im  $d_j$  se kerdi<sub>j+1</sub>

Sklenjene 0-forme

$$du = 0$$

ne vsek parcen: komponenti soje u konst.

52 parcene:  $\mathbb{R}$        $\mathbb{R}^{52(r)}$  za parcene<sup>r</sup>  
 $\bigcup \bigcup : \mathbb{R}^2$

Gaußsche Formel:

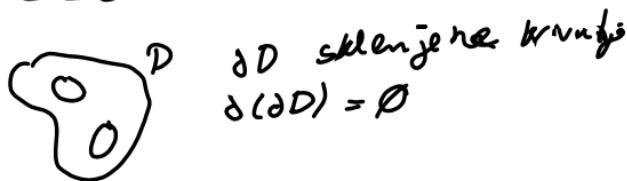
$$\iint_{\partial \Omega} \vec{R} \cdot d\vec{s} = \iiint_{\Omega} \text{div} \vec{R} \, dv$$

$$\iint_{\partial \Omega} V = \iiint_{\Omega} dV$$

$$\Omega \subseteq \mathbb{R}^2$$

$\leftarrow$  kompakt  $D \times \Omega$

$$D \subseteq \subseteq \Omega$$



$\Gamma$  Kurve  
 $\delta \Gamma = \emptyset$  (je sklängliche Kurve)

$$\exists D, \partial D = \Gamma ?$$