

Osnove Newtonove mehanike

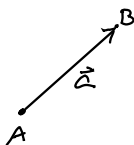
Def: **Afini prostor** \mathcal{A} nad vektorskim prostorom \mathcal{V} je množica \mathcal{A} z binarno operacijo $\mathcal{A} \times \mathcal{V} \rightarrow \mathcal{A}, (A, \vec{a}) \mapsto A + \vec{a}$ z lastnostmi:

$$i) (A + \vec{a}) + \vec{b} = A + (\vec{a} + \vec{b})$$

$$ii) \forall A, B \in \mathcal{A}, \exists \vec{a} \in \mathcal{V}: B = A + \vec{a}$$

$$\dim \mathcal{A} = \dim \mathcal{V}$$

Primeri:



Def: Definiramo operacijo $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{V}$.

$$\text{s predpisom } B - A = \vec{a} \Leftrightarrow \vec{B} = A + \vec{a}$$

Trditve:

$$i) A - A = \vec{0}$$

$$ii) (A - B) + (B - A) = \vec{0}$$

$$iii) (A - B) + (B - C) + (C - A) = \vec{0}$$

$$iv) (A - B) + \vec{a} = (A + \vec{a}) - B$$

$$v) (A + B) - C = B + (A - C)$$

Dokaz:

$$i) A - A = \vec{a} \Leftrightarrow A = A + \vec{a} \Leftrightarrow A = (A + \vec{a}) + \vec{c} = A + 2\vec{a}$$

$$\Rightarrow \vec{a} = 2\vec{a} \text{ (ker je } \vec{a} \text{ natančno določen)}$$

$$\Rightarrow \vec{a} = \vec{0}$$

$$v) A - C = \vec{a} \Rightarrow \overset{A = C + \vec{a}}{B + (A - C) = B + \vec{a} = C + (\vec{b} + \vec{a})}$$

$$B - C = \vec{b} \Rightarrow B = C + \vec{b}$$

$$(A + B) - C = \underbrace{(C + \vec{a}) + (C + \vec{b})}_{\text{vektor}}, \text{ zato lahko zamenjamo skupaj}$$

$$= C + \vec{a} + (C + \vec{b} - C) = \vec{C} + \vec{a} + \vec{b}$$

vektor

$$(A, V) \quad (A', V')$$

$$g: A \rightarrow A'$$

Def: Preslikava $g: A \rightarrow A'$ je **afina** če obstaja

$$d_g \in L(V, V') \text{ tako da velja } g(A) - g(B) = d_g(A - B) \\ \text{za } \forall A, B \in A$$

$$g(A) = g(B) + d_g(A - B)$$

$$g(A) = g(0) + d_g(A - 0); \quad 0 \text{ je } \text{pol afine preslikave}$$

Izbira 0 da je poljubna

$$\begin{aligned} \tilde{g}(A) &= g(\tilde{0}) + d_g(A - \tilde{0}) = g(0) + d_g(\tilde{0} - 0) + d_g(A - \tilde{0}) \\ &= g(0) + d_g(\underbrace{(\tilde{0} - 0) + (A - \tilde{0})}_{A - 0}) = g(A) \end{aligned}$$

$$\text{Izberimo } 0 \in A \quad \vec{a} = A - 0$$

$$A \in A; \quad A = 0 + (A - 0) = 0 + \vec{a}$$

$\forall A \in A$ lahko identificiramo z vektorjem $\vec{a} \in V$

Definicija: **Galilejeva struktura** G je trojica (\mathcal{A}, τ, d) , kjer je \mathcal{A} štirirazsežen afini prostor, $\tau \in \mathcal{L}(V, \mathbb{R})$ in d evklidska razdelje nad \mathcal{A} .
 \uparrow
 linearna presliheva prstocam istocasnih dogedkov.

Funkcionalu τ pravimo **časovnost**. Daggolke $A, B \in \mathcal{A}$ sta **istocasna**, oe $A-B \in \tau$

Definicija: Galilejevi strukturi $G(\mathcal{A}, \tau, d)$ in $\tilde{G}(\tilde{\mathcal{A}}, \tilde{\tau}, \tilde{d})$ sta **ekvivalentni**, oe obstaja afina bijekcija $g: \mathcal{A} \rightarrow \tilde{\mathcal{A}}$, ki ohranja časovnost in razdeljo istocasnih dogedkov

$$\tilde{\tau}(g(A) - g(B)) = \tau(A - B)$$

$$A, B \text{ istocasna} \Leftrightarrow g(A), g(B) \text{ istocasna}$$

$$d(A, B) = \tilde{d}(g(A), g(B))$$

~~Definicija~~
 $\mathbb{R} \times E$ afini prostor, kjer je E tovarazsežni: evklidski prostor

Na $\mathbb{R} \times E$ vpeljemo naravno Galilejevo strukturo.

$$A \in \mathbb{R} \times E \Rightarrow A = (t, P) \quad t \in \mathbb{R}, P \in E$$

τ ima normo porojeno s skalarnim produktom

$$\tau(A_2 - A_1) = t_2 - t_1$$

$$d(A_1, A_2) = \|P_1 - P_2\|$$

Taj struktur; pravimo **naravna Galilejeva struktura**

Definicija: **Koordinatni sistem** na afinem prostoru \mathcal{A} je bijektivna preslikava $\varphi: \mathcal{A} \rightarrow \mathbb{R}$

$$A \mapsto \varphi(A) = (\pi_t \varphi(A), \pi_P \varphi(A))$$

za katero φ φ_t afina preslikave $= \varphi_t(A), \varphi_P(A)$

$$(\mathbb{R} \times E, \tau, \|\cdot\|)$$

$$\tau(A - B) = t(\varphi_t(A) - \varphi_t(B))$$

$$d(A, B) = \|\varphi_P(A) - \varphi_P(B)\|$$

Lahko merimo razdaljo tudi med neistovaznimi dogodki.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathbb{R} \times E \\ & \searrow \varphi & \updownarrow \\ & & \mathbb{R} \times E \end{array}$$

Kdaj sta dve naravni galilejski strukturi ekvivalentni

$$g: \mathbb{R} \times E \rightarrow \mathbb{R} \times E$$

$$A = \begin{bmatrix} t \\ p \end{bmatrix} \mapsto g(A) = \begin{bmatrix} t' \\ p' \end{bmatrix} = g(0) + d_g(A-0)$$

$$= \begin{bmatrix} t_0 \\ p_0 \end{bmatrix} + \begin{bmatrix} \alpha \vec{a}^T \\ \vec{c} \quad Q \end{bmatrix} \begin{bmatrix} t - t_0 \\ p - p_0 \end{bmatrix}$$

$$0 = \begin{bmatrix} t_0 \\ p_0 \end{bmatrix} \quad \begin{matrix} 3 \times 3 \\ \text{matrica} \end{matrix} \curvearrowright$$

$$A_1 = \begin{bmatrix} t_1 \\ p_1 \end{bmatrix} \quad A_2 = \begin{bmatrix} t_2 \\ p_2 \end{bmatrix}$$

$$t(g(A_2) - g(A_1)) = t(A_2 - A_1)$$

$$g(A_2) - g(A_1) = \begin{bmatrix} \alpha \vec{a}^T \\ \vec{c} \quad Q \end{bmatrix} \begin{bmatrix} t_2 - t_1 \\ p_2 - p_1 \end{bmatrix} = \begin{bmatrix} \alpha(t_2 - t_1) + \vec{a}(p_2 - p_1) \\ (t_2 - t_1)\vec{c} + Q(p_2 - p_1) \end{bmatrix}$$

$$\Rightarrow \alpha(t_2 - t_1) + \vec{a}(p_2 - p_1) = t_2 - t_1 \Rightarrow \alpha = 1, \vec{a} = \vec{0}$$

Ta velja, da se ohranjanje časovnosti

ohranjanje razdelje med istočasnost; dogodki:

$$d(g(A_1), g(A_2)) = d(A_2, A_1) \quad \text{za } t_1 = t_2$$

$$\begin{matrix} \parallel \leftarrow t_2 - t_1 = 0 & \parallel \\ \parallel Q(p_2 - p_1) \parallel & \parallel p_2 - p_1 \parallel \end{matrix}$$

$$\Rightarrow Q \in O(3)$$

\leftarrow ortogonalna matrika 3×3

Definicija: Preslikava, ki ohranja Galilejevo strukturo pravimo **Galilejeva preslikava**

Trditev: Galilejeve preslikave med marnima Galilejevima strukturama $\mathbb{R} \times E$ je oblike

$$[\tilde{t}] \mapsto [\tilde{t}', \tilde{p}'] = \begin{bmatrix} t_0' + t - t_0 \\ p_0' + \vec{c}(t - t_0) + Q(P - P_0) \end{bmatrix} =$$

kjer je $Q \in O(3)$, \vec{c} poljubni vektor, t_0 poljubno število in P_0 poljubna točka

$$= \begin{bmatrix} \tilde{t}_0 + t \\ \tilde{p}_0 + \vec{c}t + Q(P - P_0) \end{bmatrix}$$

\uparrow
 $t_0' - t_0 = \tilde{t}_0$

Dva apazovalca: $p(t, P), p'(t', P)$

gibanje $t \mapsto P(t)$ *trajektorija točke*

$$\vec{v} = \frac{dP}{dt} \quad \text{vektor hitrosti} \quad \lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h}$$

$$\ddot{p} = \dot{\vec{v}} = \ddot{a} = \frac{d\vec{v}}{dt} \quad \text{vektor pospeška}$$

$$|\vec{v}| = v \quad \text{brzina} \quad \underbrace{Q(P(t') - t_0') - P_0}$$

$$P'(t') = P_0' + \vec{c}(t' - t_0') + \underbrace{Q(P(t) - P_0)}_{\text{trajektorija v } \varphi'}$$

$$\vec{v}' = \frac{dP'}{dt'} = \vec{c} + Q \frac{dP}{dt}(t' - t_0') = \vec{c} + QP(t)$$

$$\vec{p}'(t') = \vec{c} + Q\dot{P}(t)$$

↑ najprej otika, nato odvod

$$\dot{\vec{p}}'(t') = \ddot{a}' = \frac{d\vec{v}'}{dt'} = Q\ddot{P}(t)$$

φ	φ'
P	$P = P_0' + \vec{c}t + Q(P - P_0)$
$\vec{a} = P_2 - P_1$	$\vec{a}' = P_2' - P_1' = Q(P_2 - P_1) = Q\vec{a}$

Definicija: vektor $a = P_2 - P_1$ je *koordinatno neodvisen*

$$A \in \mathcal{L}(V, V) \quad A' = Q^T A Q$$

$$\lambda \quad \lambda' = \lambda$$

Sistem materialnih točk $(p_1, \dots, p_n) = P$

$$\underline{P}' = \underline{P}_0' + \underline{\dot{C}}t + Q(\underline{P} - \underline{P}_0)$$

$$\underline{P}_0' = (p_0', \dots, p_n')$$

$$Q(\underline{P} - \underline{P}_0) = (Q(p_1 - p_0), \dots, Q(p_n - p_0))$$

$$\underline{\dot{C}} = (\dot{c}_1, \dots, \dot{c}_n)$$

$$\Rightarrow \underline{\dot{P}}' = \underline{\dot{C}} + Q \underline{\dot{P}}$$

$$\underline{\ddot{P}}' = Q \underline{\ddot{P}}$$

Princip determiniranosti

V danem KS (koordinatni sistem) je trajektorija sistema materialnih točk natanko določena z začetnim položajem in začetno hitrostjo.

To specialno pomeni, da obstaja funkcija interakcije \vec{f} tako da je $\underline{\ddot{P}} = \vec{f}(t, \underline{P}, \underline{\dot{P}})$

$$\left(P(t) = \vec{f}(t, \underline{P}(t), \underline{\dot{P}}(t)) \text{ nedolgo} \right)$$

Princip relativnosti ^{8.10} Obstaja razred

koordinatnih sistemov v katerem je funkcija interakcije invariantna za Galilejeve transformacije. Koordinatnim sistemom iz tega pravimo **ercialni koordinatni sistem**

$$(t, \underline{P}) \xrightarrow{GT} (t', \underline{P}') \text{ Galilejeva transformacija.}$$

$$\text{Potem je } \underline{t} = \underline{t'} \quad (\underline{\ddot{P}} = \underline{\ddot{P}'})$$

$$\underline{\ddot{P}'} = \underline{\ddot{f}}(t', \underline{P}', \underline{\dot{P}'})$$

$$Q \underline{\ddot{P}} = Q \underline{\ddot{f}}(t, \underline{P}, \underline{\dot{P}}) = \underline{\ddot{f}}(t_0' + t, \underline{P}_0' + \underline{\dot{C}}t + Q(\underline{P} - \underline{P}_0), \underline{\dot{C}}t + Q \underline{\dot{P}})$$

$$i) \quad \underline{\dot{C}} = \underline{\dot{0}} \quad Q = I \quad \underline{P}_0' = \underline{P}_0$$

$$\underline{\ddot{f}}(t, \underline{P}, \underline{\dot{P}}) = \underline{\ddot{f}}(t_0' + t, \underline{P}, \underline{\dot{P}}) \text{ za } \forall t_0'$$

$\Rightarrow f$ ni eksplicitno odvisna od čase (t)

(homogenost časa)

$$ii) \quad \underline{\dot{C}} = \underline{\dot{0}}, Q = I, \underline{\dot{P}'} = \underline{\dot{P}} + \underline{\dot{a}}$$

$$\underline{\ddot{f}}(\underline{P}, \underline{\dot{P}}) = \underline{\ddot{f}}(\underline{P} + \underline{\dot{a}}, \underline{\dot{P}}) \text{ za } \forall \underline{\dot{a}}$$

$$\parallel$$

$$(\underline{P}_1 + \underline{\dot{a}}, \dots, \underline{P}_n + \underline{\dot{a}})$$

$\Rightarrow \underline{\ddot{f}}$ je odvisna samo od relativnih položajev

$$\underline{\ddot{f}}(\underline{P}, \underline{\dot{P}}) = \underline{\ddot{f}}(\underline{P}_i - \underline{P}_j, \underline{\dot{P}})$$

$i \neq j$ in

vse kombinacije tega

(homogenost prostora)

$$iii) \quad Q = I$$

$$f(\underline{P}_i - \underline{P}_j, \underline{\dot{P}}) = f(\underline{P}_i - \underline{P}_j, \underline{\dot{C}} + \underline{\dot{P}}) \Rightarrow \underline{\ddot{f}}(\underline{P}_i - \underline{P}_j, \underline{\dot{P}}_k - \underline{\dot{P}}_l)$$

(homogenost prostora hitrost)

iv) Q poljuben

$$Q \underline{\ddot{f}}(\underline{P}_i - \underline{P}_j, \underline{\dot{P}}_k - \underline{\dot{P}}_l) = \underline{\ddot{f}}(Q(\underline{P}_i - \underline{P}_j), Q(\underline{\dot{P}}_k - \underline{\dot{P}}_l))$$

f je izotropna funkcija

$$Q \underline{\ddot{g}}(\underline{\ddot{a}}) = \underline{\ddot{g}}(Q \underline{\ddot{a}}) \quad \forall Q \in O(3)$$

Posebni primer: $N=1$ (izdelirana točka)

$$\ddot{\vec{p}} = \vec{f}(\underline{\quad}) \quad \text{brez argumentov (konstanta)}$$

se vedno velja izotropno $\& \quad Q\ddot{\vec{p}} = Q\vec{f} = \vec{f} \Rightarrow \vec{f} = 0$
 $\Rightarrow \ddot{\vec{p}} = \vec{0}$

v IKS (inercialni koordinatni sistem) se izdelirane materialna telesa giblje premočrtno s konstantno
brzino. $\vec{p} = \vec{v}_0 t + \vec{p}_0 (t=0)$

$N=2 \Rightarrow$

$$\ddot{\vec{p}}_1 = \vec{f}_1(\vec{p}_1 - \vec{p}_2, \dot{\vec{p}}_1 - \dot{\vec{p}}_2)$$

$$\ddot{\vec{p}}_2 = \vec{f}_2(\vec{p}_2 - \vec{p}_1, \dot{\vec{p}}_2 - \dot{\vec{p}}_1)$$

če se telesa gibata po
potnici proti drugi telesi
vedno osteli na tej premici:
(Podobno to: telesa ostanejo
na ravni!)

$$\ddot{\vec{p}}_1 = \vec{f}_1(\vec{p}_1 - \vec{p}_2, \vec{p}_1 - \vec{p}_3, \vec{p}_2 - \vec{p}_3, \dots)$$

\nwarrow

tudi
to je pomembno

Definicija: Interakcija $\vec{P}_i = f(\dots)$ je **parska** če je odvisna samo od relativnih položajev in hitrosti glede na P_i in je delovanje tudi avtonomno

$$\vec{P}_i(P_i - P_j, \dot{P}_i - \dot{P}_j)$$

$j \neq i$ $k \neq i$

$$\vec{P}_i = \sum_{\substack{j=1 \\ j \neq i}}^n \vec{P}_{ji}(P_i - P_j, \dot{P}_i - \dot{P}_j)$$

Def: Interakcija $\vec{P}_{ji} = \vec{P}_{ji}(P_i - P_j, \dot{P}_i - \dot{P}_j)$ je **lokalna** če velja $\lim_{|P_i - P_j| \rightarrow \infty} \vec{P}_{ji} = \vec{0}$

Princip sorazmernosti

V IKS za sistem materialnih točk P_1, \dots, P_N obstajajo natanko določene konstante α_{ij} tako, da ne glede na interakcije \vec{F}_i velja $\vec{F}_i = - \sum_{j=1}^N \alpha_{ij} \vec{F}_j$

Lema: Za konstante α_{ij} velja

i) $\alpha_{ij} \alpha_{ji} = 1$

ii) $\alpha_{ij} \alpha_{jk} \alpha_{ki} = 1$

Dokaz:

$$\vec{F}_i = \vec{F}_i; (P_i, -P_k) = \vec{F}_i; (P_i, -P_1, P_i, -P_2, \dots, P_i, -P_N)$$

in naj bode P_i lokalni

P_k poslejmo proti ∞ razen P_i in P_j

$$\vec{F}_i = - \sum_{\substack{k=1 \\ k \neq i}}^N \alpha_{ki} \vec{F}_k = -\alpha_{ji} \vec{F}_j$$

$$\vec{F}_j = - \sum_{\substack{k=1 \\ k \neq j}}^N \alpha_{kj} \vec{F}_k = -\alpha_{ij} \vec{F}_i$$

$$F = -\alpha_{ji} \vec{F}_j = \alpha_{ji} \alpha_{ij} \vec{F}_i \Rightarrow \alpha_{ji} \alpha_{ij} = 1$$

P_i, P_j, P_k obstajajo, ostale gredo proti ∞

$$\vec{F}_i = -\alpha_{ji} \vec{F}_j - \alpha_{ki} \vec{F}_k = -\alpha_{ji} (-\alpha_{ij} \vec{F}_i - \alpha_{kj} \vec{F}_k) - \alpha_{ki} \vec{F}_k =$$

$$\vec{F}_i = -\alpha_{ji} \vec{F}_j - \alpha_{ki} \vec{F}_k$$

$$F_i = \underbrace{\alpha_{ji} \alpha_{ij}}_1 F_i + \alpha_{ji} \alpha_{kj} F_k - \alpha_{ki} F_k$$

$$\alpha_{ki} F_k = \alpha_{ji} \alpha_{kj} F_k / \alpha_{ik}$$

$$F_k = \alpha_{ji} \alpha_{kj} \alpha_{ik} F_k \Rightarrow \alpha_{ik} \alpha_{kj} \alpha_{ji} = 1$$

Lema: Naj za pozitivna števila α_{ij} velja

$$i) \alpha_{ij} \alpha_{ji} = 1$$

$$ii) \alpha_{ij} \alpha_{jk} \alpha_{ki} = 1.$$

Potem \exists pozitivna števila m_i , tako da velja da je

$$\alpha_{ij} = \frac{m_i}{m_j}$$

Števila m_i so določena do sorazmernostnega faktorja natančnost. Številom m_i pravimo **inersijske mase**

Dokaz: $\alpha_{ii} = 1$

$$\alpha_{ij} \alpha_{jj} \alpha_{ji} = 1 \Rightarrow \alpha_{jj} = 1$$

$$l_{ij} = \log \alpha_{ij} \quad l_{ij} + l_{jk} + l_{ki} = 0$$

$$l_{ij} = -l_{ji}$$

$$l_{i_0 j} + l_{jk} + l_{ki_0} = 0$$

$$l_{ij} - l_{i_0 j} + l_{ki} - l_{ki_0} = 0 \Rightarrow$$

$$l_{ij} - l_{i_0 j} = -l_{ki} + l_{ki_0} = l_{ki_0} - l_{ki} \quad \forall j, k$$

$$\Rightarrow l_{ij} - l_{i_0 j} = n_{ii_0}$$

$$\left. \begin{array}{l} l_{ij} = n_{ii_0} + l_{i_0 j} \\ j=1 \end{array} \right\} \Rightarrow 0 = n_{ii_0} + l_{i_0 i} \Rightarrow l_{i_0 i} = -n_{ii_0}$$

$$l_{ij} = l_{i_0 i} - l_{j i_0}$$

$$l_{i_0 i} = \log m_i \text{ definisano}$$

$$\Rightarrow l_{ij} = \log m_i - \log m_j = \log \frac{m_i}{m_j} \Rightarrow \alpha_{ij} = \frac{m_i}{m_j}$$

Izreki: V inercialnem koordinatnem sistemu velja

$$m_1 \ddot{\mathbf{p}}_1 + m_2 \ddot{\mathbf{p}}_2 + \dots + m_n \ddot{\mathbf{p}}_n = \vec{0}$$

Dokaz:

$$f_i = - \sum_{\substack{j=1 \\ j \neq i}}^N \alpha_j \cdot f_j = - \sum_{j=1}^N \frac{m_j}{m_i} f_j$$

$$\ddot{\mathbf{p}}_i = - \sum_{j=1}^N \frac{m_j}{m_i} \ddot{\mathbf{p}}_j \quad / m_i$$

$$\sum_{i=1}^N m_i \ddot{\mathbf{p}}_i = \vec{0}$$

$$\mathbf{P}_* = \frac{1}{m} \sum_{i=1}^N m_i \mathbf{p}_i, \quad m = \sum_{i=1}^N m_i$$

masno središče

seštevanje točk:

$$\sum_{i=1}^N \mathbf{p}_i = \sum (m_i (\underbrace{\mathbf{p}_i - \mathbf{0}}_{\text{vektor}}) + m_i \mathbf{0}) = \sum_{i=1}^N m_i (\mathbf{p}_i - \mathbf{0}) + \underbrace{\sum_{i=1}^N m_i \mathbf{0}}_{\mathbf{0}}$$

$$\mathbf{P}_* = \mathbf{0} + \frac{1}{m} \sum_{i=1}^N m_i (\mathbf{p}_i - \mathbf{0})$$

$$m \ddot{\mathbf{P}}_* = \vec{0}$$

Sila $n = P_i$
 Definicija: je produkt interakcije A z maso m_i :

$$m_i \ddot{\vec{P}}_i = m_i A(\dots) =: \vec{F}_i$$

Princip omasi masa materialnih tade je enaka
 v vseh koordinatnih sistemih

$$\ddot{\vec{P}}_1 = f_1(\vec{P}_1 - \vec{P}_2, \dot{\vec{P}}_1 - \dot{\vec{P}}_2) \quad m_1 \ddot{\vec{P}}_1 = \vec{F}_1$$

$$\ddot{\vec{P}}_2 = f_2(\vec{P}_1 - \vec{P}_2, \dot{\vec{P}}_1 - \dot{\vec{P}}_2) \quad m_2 \ddot{\vec{P}}_2 = \vec{F}_2$$

$$\overline{m_1 \ddot{\vec{P}}_1 + m_2 \ddot{\vec{P}}_2} = \vec{F}_1 + \vec{F}_2 = 0$$

$$\vec{F}_2 = -\vec{F}_1$$

$$\vec{F}_1 = \vec{F}_{21}$$

$$\vec{F}_2 = \vec{F}_{12}$$

$$\vec{F}_{12} = -\vec{F}_{21}$$

Trditav: Če so vse sile parske in lokalne velja zakon akcije in reakcije (3. NZ)

$$Dokaz: F_i = \sum_{\substack{j=1 \\ j \neq i}}^N f_{ij}(\vec{r}_{ij} - \vec{r}_i, \dot{\vec{r}}_j - \dot{\vec{r}}_i)$$

$$\downarrow$$

$$\{ \vec{r}_j \rightarrow \infty \quad j \in \{i, k\} \}$$

$$m_i \ddot{\vec{r}}_i = m_i \ddot{\vec{r}}$$

$$m_i f_{ji} + m_j f_{ij} = \vec{0}$$

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

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odvod vzdolž trakecije $\frac{d}{dt} \in (1, p(t), \dot{p}(t)) =$

$$\frac{\partial E}{\partial \dot{p}} + \left(\frac{\partial E}{\partial p} \right)^T \dot{p} + \left(\frac{\partial E}{\partial p} \right)^T \ddot{p}$$

Izrek o množenju: Naj bo sila \vec{F} potencialna s
potencialom $U(t, \vec{p}, \dot{\vec{p}})$. Potem je vsota

kinetične in potencialne energij konstanta
gibanja (= njen odvod vzdolž trakecije je 0)

\Leftrightarrow moč sil je nasprotno enaka odvodu
potencialne energije vzdolž trakecije

$$\text{Ded: } \Rightarrow E_0 = T + U = \frac{1}{2} m |\dot{\vec{p}}|^2 + U$$

$$0 = \frac{\partial E_0}{\partial t} = m \ddot{\vec{p}} \dot{\vec{p}} + \dot{\vec{F}} \dot{\vec{p}} + \frac{dU}{dt} \Rightarrow \frac{dU}{dt} = -\vec{F} \dot{\vec{p}}$$

\Leftarrow

$$\begin{aligned} A &= T_2 - T_1 \\ A &= \int_{t_1}^{t_2} \vec{F} \cdot \dot{\vec{p}} dt = - \int_{t_1}^{t_2} \frac{dU}{dt} dt = U(t_1) - U(t_2) \end{aligned}$$

$$\Rightarrow U_1 + T_1 = U_2 + T_2 = E_0$$

Sila je **konzervativna**, če je potencialna in odvisna samo od položaja

Posledica: če je sila konzervativna velja izrek o energiji

$$\text{Dokaz: } \frac{d}{dt} U(\mathbf{p}(t)) = \left(\frac{\partial U}{\partial \mathbf{p}} \right)^T \dot{\mathbf{p}} = -\mathbf{F} \dot{\mathbf{p}}$$

$$\mathbf{F} \dot{\mathbf{p}} = -\frac{dU}{dt}$$

$$\mathbf{F} \dot{\mathbf{p}} = Q \mathbf{F}(\dot{\mathbf{c}} + Q \dot{\mathbf{p}}) = \underbrace{Q \mathbf{F} \cdot \dot{\mathbf{c}}}_{\mathbf{F} \cdot \dot{\mathbf{p}}} + \underbrace{Q \mathbf{F} \cdot Q \dot{\mathbf{p}}}_{\mathbf{F} \cdot \dot{\mathbf{p}}} = -\frac{dU}{dt} + m \ddot{\mathbf{c}} \cdot \dot{\mathbf{c}} =$$

$$= -\frac{d}{dt} (U - m \dot{\mathbf{p}} \cdot \dot{\mathbf{c}})$$

↑ ortogonalna preslikava določa skalarni produkt

$$\tilde{U} = U - m \dot{\mathbf{p}} \cdot \dot{\mathbf{c}}$$

$$\tilde{U}(\mathbf{p}, \dot{\mathbf{p}}) = U(\mathbf{p}) - m \dot{\mathbf{p}} \cdot \dot{\mathbf{c}}$$

Izrek: Izrek o energiji je invarianten za Galilejeve transformacije

$$\begin{aligned}
 t_0' &= T' + \tilde{U}' = \frac{1}{2} m |\vec{c}|^2 + m c Q \dot{P} + T + \tilde{U} - m \dot{P}' \cdot \vec{c} = \\
 &= E_0 + \frac{1}{2} m |\vec{c}|^2 + m \vec{c} (Q \dot{P} - \dot{P}') = E_0 - \frac{1}{2} m |\vec{c}|^2 \\
 T' &= \frac{1}{2} m \dot{P}' \cdot \dot{P}' = \frac{1}{2} m (\vec{c} + Q \dot{P}) \cdot (\vec{c} + Q \dot{P}) = \frac{1}{2} m |\vec{c}|^2 + m \vec{c} \cdot Q \dot{P} \\
 &\quad + \frac{1}{2} m |\dot{P}|^2
 \end{aligned}$$

$$\dot{P}' = \vec{c} + Q \dot{P}$$

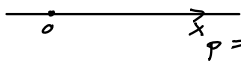
Premočno gibanje

= pospešek ima konstantno smer

$$\vec{a} = a(t) \vec{e}$$

$$\vec{v} = \int_{t_0}^t \vec{a}(t) dt + \vec{v}(t_0) = \left(\int_{t_0}^t a(t) dt \right) \vec{e} + \vec{v}(t_0)$$

Obstaja KS v katerem tir leži na premici



$$p = x \quad \dot{p} = \dot{x} \quad \ddot{p} = \ddot{x}$$

$$m\ddot{x} = f(t, x, \dot{x})$$

$v = \dot{x}$ je lahko celo tudi negativen

omejimo se ko bo f konzervativna

$$m\ddot{x} = f(x)$$

kdaj je sila v odvisnosti položaja potencialna?

(potencialna \Rightarrow konzervativna)

$$f = -\frac{dU}{dx} \Rightarrow U = \int_{x_0}^x f(\xi) d\xi + U_0$$

sila $f(x)$ je potencialna če je f zvezna

če je $f(x)$ zvezna \Rightarrow velja izrek o energiji

$$m\ddot{x} = F(x)$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = E_0$$

$$\dot{x}^2 = \frac{2}{m}(E_0 - U(x))$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E_0 - U(x))}$$

$$\pm \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = dt$$

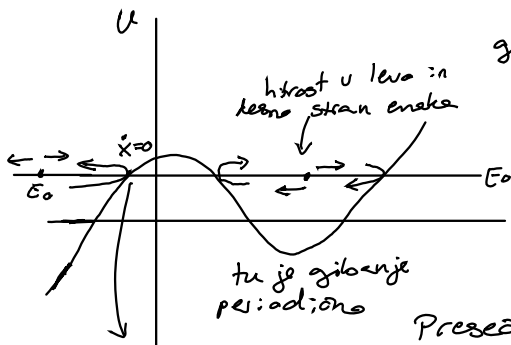
$$\text{sgn} x \int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \int_{t_0}^t dt = t - t_0 \Rightarrow t = T(x) \Rightarrow x = x(t)$$

↑

lahko bramo
ko $x \neq 0$
(sgn x je
konstanten)

$$t = t_0 \pm \int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

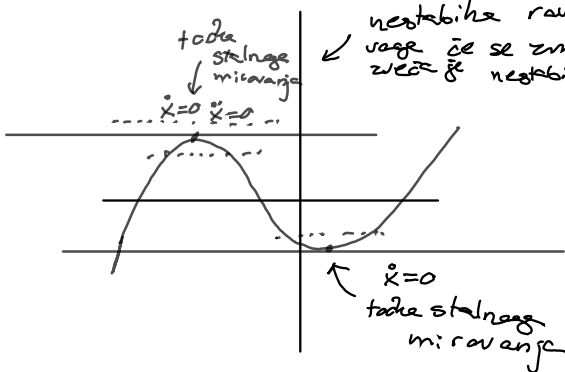
Kvalitativna obravnava gibanja



gibanje je možno
tam kjer je
energija nižja od
potencialne
 E_0 nad grafom

$$0 = \dot{x} = \frac{F(x)}{m} = -\frac{dU}{dx}$$

Presečišče je tuda
trenutnega miravanja
(točka obrate gibanja)

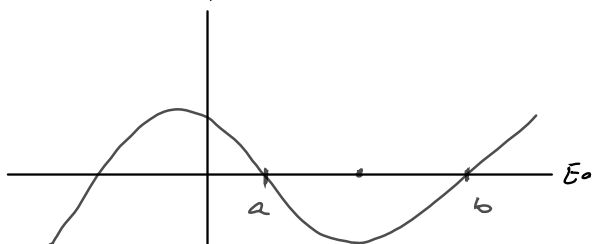


nestabilna ravnovesna
vaga če se zmanjša ali
zveča je nestabilna

Lokalni minimum je stabilna ravnovesna lega
(če se energija malo spremeni pride do majhnega
periodičnega gibanja (majhen odklon od ravnovesne
vage))

Prevoj je tudi nestabilna lega

kakšna je perioda periodičnega gibanja



gibanje: $t_1 = \int_{x_a}^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$ $t_2 = -\int_a^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$

$$t_3 = \int_a^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

$T = t_1 + t_2 + t_3 = 2 \int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$

↑
perioda

Ali je res periodično in se ne ustavi prej?

atd

$$\int_a^{\infty} \frac{dx}{\sqrt{E_0 - U(x)}}$$

$$U(x) = U(a) + \left(\frac{dU}{dx}(a)\right)(x-a)$$

$$\frac{dU}{dx}(a) < 0 \quad a \text{ je presečišče}$$

$$\exists \delta > 0, \forall x \in (a, a+\delta) \Rightarrow 2 \frac{dU}{dx}(a) < \frac{1}{2} \frac{dU}{dx}(a)$$

$$E_0 - U(x) = -\frac{dU}{dx}(a)(x-a) < -2 \frac{dU}{dx}(a)(x-a)$$

\Rightarrow integral je konvergenten

$$\text{ker } \int_a^{\infty} \frac{dx}{\sqrt{E_0 - U(x)}} \leq \frac{1}{\sqrt{-\frac{1}{2}U'(a)}} \int_a^{\infty} \frac{dx}{\sqrt{x-a}} < \infty$$

Če se energijske nivojnica dotika lokalnega maksimuma, potem ne pridemo do dotikališča

v končnem času

lema $\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi$

Dokaz: $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$

$$b-x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$

$$(b-x)(x-a) = \left(\frac{1}{2}(b-a)\right)^2 (1-\cos^2\vartheta) = \overset{\sin^2\vartheta}{\sqrt{\left(\frac{1}{2}(b-a)\right)^2}}$$

$$dx = -\frac{1}{2}(b-a)\sin\vartheta d\vartheta$$

$$I = - \int_{\pi}^0 \frac{\frac{1}{2}(b-a)\sin\vartheta d\vartheta}{\frac{1}{2}(b-a)\sin\vartheta} = - \int_{\pi}^0 d\vartheta = \pi$$

Primer:

Harmonični oscilator $U = \frac{1}{2}kx^2$

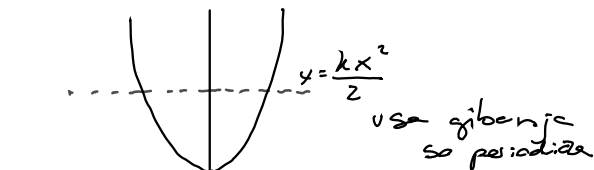
$$F = -kx$$

$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0 \quad \frac{k}{m} = \omega^2$$

$$\ddot{x} + \omega^2 x = 0$$

$$x = A \cos(\omega t - \delta)$$



$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - \frac{1}{2}kx^2}} =$$

$$= \frac{\sqrt{2m}}{\sqrt{\frac{1}{2}k}} \int_a^b \frac{dx}{\sqrt{\frac{2E_0}{k} - x^2}} = 2\sqrt{\frac{m}{k}} \pi = \frac{2\pi}{\omega}$$

T perioda je neodvisna od energije
(izokronono gibanje)

Primer: $U = ax^2 + bx^{-2}$

$[U]$ dimenzija \leftarrow energija

$[U] = [ax^2] = [a][x^2] = [a]L^2$

$[U] = [bx^{-2}] = [b]L^{-2}$

$\Rightarrow [a]L^2 = [b]L^{-2}$

$L^4 = \frac{[b]}{[a]} \Rightarrow L = \sqrt[4]{\frac{[b]}{[a]}}$

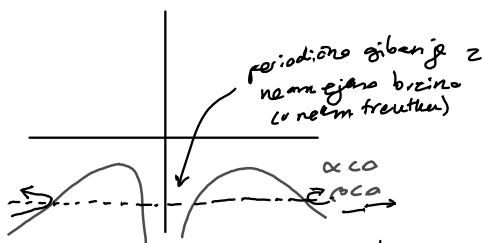
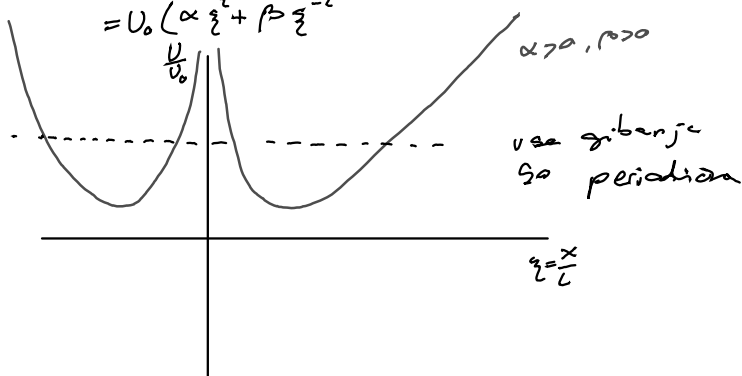
$x = L \xi \quad L = \left(\frac{b}{a}\right)^{\frac{1}{4}}$

τ brezdimenzijski

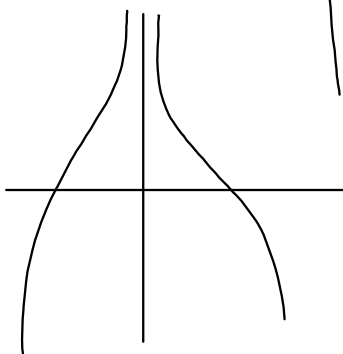
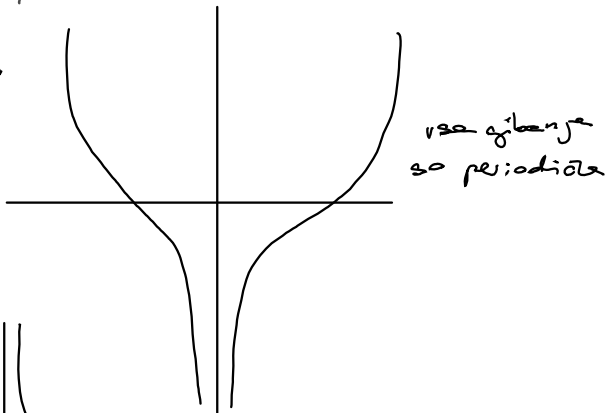
$U = a \sqrt{\frac{|b|}{|a|}} \xi^2 + b \sqrt{\frac{|a|}{|b|}} \xi^{-2} = \underbrace{\sqrt{|a||b|}}_{U_0} (\alpha \xi^2 + \beta \xi^{-2}) =$

$a = \underbrace{\frac{a}{|a|}}_{\alpha} |a| \quad b = \underbrace{\frac{b}{|b|}}_{\beta} |b|$

$= U_0 (\alpha \xi^2 + \beta \xi^{-2})$



$\alpha > 0, \beta < 0$



$\alpha < 0, \beta > 0$

vsa gibanja so neskončna

Računajmo periode za $\alpha, \beta > 0$

$$U = \sqrt{|a||b|} \left(\xi^2 + \xi^{-2} \right)$$

U_0 ← *težilo*
 $E_0 = U_0 \epsilon_0$

$$T = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - (ax^2 + bx^{-2})}} = \sqrt{2m} \int_{\xi_1}^{\xi_2} \frac{L d\xi}{\sqrt{U_0 \epsilon_0 - U_0 (\xi^2 + \xi^{-2})}} =$$

$$= \sqrt{2m} \frac{L}{\sqrt{U_0}} \int_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{\epsilon_0 - \xi^2 - \xi^{-2}}} d\xi = \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1}^{\xi_2} \frac{\xi d\xi}{\sqrt{\xi^2 \epsilon_0 - \xi^4 - 1}} d\xi$$

$$u = \xi^2$$

$$= \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1^2}^{\xi_2^2} \frac{\frac{1}{2} du}{\sqrt{u^2 + u \epsilon_0 - 1}} = \sqrt{2m} \frac{L}{\sqrt{U_0}} \frac{1}{2} \pi$$

↑
 v krajnjem položaju
 imamo
 horizontalno koren
 $z = a_2 = -1$

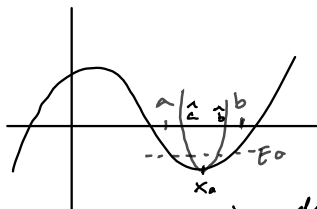
$$U_0 = \sqrt{|a||b|}$$

$$\frac{L}{\sqrt{U_0}} = \frac{|b|^{\frac{1}{4}}}{|a|^{\frac{1}{4}} |a^{\frac{1}{2}}| |b|^{\frac{1}{4}}} = 2 \sqrt{\frac{1}{|a|}}$$

$$T = \sqrt{\frac{m}{2|a|}} \pi \quad \text{opet je izokronično (neodvisno od energije)}$$

in notri u i b

Harmonična optimizacija periode



$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

= 0 ker je lok. minimum.

$$U(x) = U(x_0) + \frac{dU}{dx}(x_0)(x - x_0) + \frac{1}{2} \frac{d^2U}{dx^2}(x_0)(x - x_0)^2$$

$$T \approx \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U_0 - \frac{1}{2} U''(x_0)(x - x_0)^2}} =$$

$$= \sqrt{2m} \frac{1}{\sqrt{\frac{1}{2} U''(x_0)}} T = 2\pi \sqrt{\frac{m}{U''(x_0)}}$$

$$E_0 = U(x) + \frac{1}{2} m g(x) \dot{x}^2$$

$$u(x) = \int_{x_0}^x \sqrt{g(x)} dx$$

$$\dot{u} = \dot{x} \sqrt{g(x)}$$

$$u^2 = \dot{x}^2 g(x)$$

$$\frac{1}{2} m \dot{u}^2 + U(x) = E_0$$

$$\hat{U}(u) = U(x(u))$$

$$U(x) = \hat{U}(u) = \sqrt{g(x)}$$

$$U'(x) = \frac{d\hat{U}}{du} \frac{du}{dx} \quad \left. \frac{d\hat{U}}{du} \right|_{u_0} = 0$$

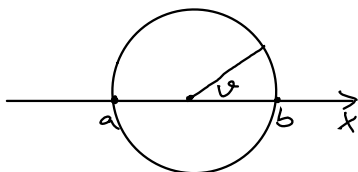
$$U''(x_0) = \frac{d^2 \hat{U}}{du^2} \left(\frac{du}{dx} \right)^2 + 0 = \frac{d^2 \hat{U}}{du^2}(u_0) \cdot g(x_0)$$

\leadsto harmonische Approximation

$$T = 2\pi \sqrt{\quad}$$

Libracijske aproksimacija periode

$$t - t_0 = \operatorname{sgn}(\dot{x}) \int_{x_0}^x \frac{dx}{\sqrt{(E_0 - U(x)) \cdot \frac{2}{m}}}$$



$$x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$$

$$E_0 - U(x) = (b-x)(x-a)\chi(x)$$

$$b-x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$

↳ vemo da me n:zile
a in b, ostane

$\rho \sim \sin \chi(x)$

$$= \operatorname{sgn} \dot{x} \int \frac{-\frac{1}{2}(b-a)\sin\vartheta d\vartheta}{\sqrt{\frac{1}{2}(b-a)^2 \sin^2 \chi(\vartheta)}}$$

$$\leq -\operatorname{sgn}(\dot{x}) \int_{\vartheta_0}^{\vartheta} \frac{\sin\vartheta d\vartheta}{|\sin\vartheta| \sqrt{\chi(\vartheta)}}$$

$$\vartheta \in (0, \pi) \Rightarrow \dot{x} < 0, \sin\vartheta > 0$$

$$= \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$$

$$\vartheta \in (\pi, 2\pi) \Rightarrow \dot{x} > 0 \text{ in } \vartheta < 0$$

$$= \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$$

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{\vartheta_a}^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$$

$$\Rightarrow T = 2\sqrt{\frac{m}{2}} \int_a^T \frac{d\vartheta}{\sqrt{\chi(\vartheta)}} \stackrel{\text{trapezna formula}}{=} \dots$$

$$\approx 2\sqrt{\frac{m}{2}} \frac{1}{2}\pi \left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right) = \pi \sqrt{\frac{m}{2}} \left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right)$$

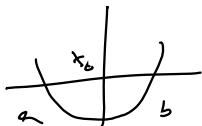
$$U(a) = E_0$$

$$U(b) = E_0$$

$$\chi(x) = \frac{E_0 - U(x)}{(b-x)(x-a)}$$

$$\chi(a) = \lim_{x \rightarrow a} \chi(x) = -\frac{U'(a)}{b-a}$$

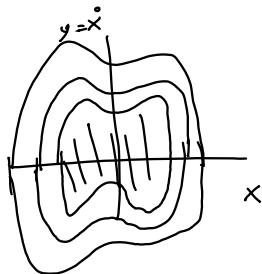
$$\chi(b) = \frac{U'(b)}{b-a}$$



$$T = \pi \sqrt{\frac{m}{2}} \sqrt{b-a} \left(\frac{1}{\sqrt{-U'(a)}} + \frac{1}{\sqrt{U'(b)}} \right) \approx 2\pi \sqrt{\frac{m}{U''(x_0)}}$$

$$U'(a) = U''(x_0) + U'''(x_0)(b-a)$$

Fazni portret



$$\frac{1}{2} m \dot{x}^2 + U(x) = E_0 \Rightarrow$$

$$\frac{1}{2} m \dot{y}^2 + U(x) = E_0$$

nivojnica
u formi ravnni

$$y = \pm \sqrt{\frac{2}{m} (E_0 - U(x))}$$

$$\frac{dy}{dx} = \pm \sqrt{\frac{2}{m}} \cdot \frac{1}{2} \frac{1 - (-U'(x))}{\sqrt{E_0 - U(x)}}$$

← položajne

$$A(E) = 2 \int_{a(E)}^{b(E)} \sqrt{\frac{2}{m} (E_0 - U(x))} dx$$

$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E - U(x)}}$$

ustavimo
b u integralu
a

$$\frac{dA}{dE} = 2b'(E) \cdot 0 - 2a'(E) \cdot 0 + 2\sqrt{\frac{2}{m}} \int_a^b \frac{1}{2} \frac{dx}{\sqrt{E - U(x)}} = \frac{1}{m} T$$

$$\Rightarrow T = m \frac{dA}{dE}$$

Primer: Harmonični oscilator

$$U = \frac{1}{2} kx^2 \text{ (hookov potencijal)}$$

$$\frac{1}{2} m y^2 + \frac{1}{2} k x^2 = E_0$$

$$\left(\frac{y^2}{\sqrt{\frac{2E_0}{m}}} \right)^2 + \frac{x^2}{\sqrt{\frac{2E_0}{k}}} = 1$$

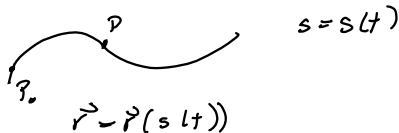
dip se ~~zapr.~~
s polosemama

plastična
s

$$A = \pi \cdot \frac{2E_0}{\sqrt{km}} \Rightarrow \frac{dA}{dE_0} = \frac{2\pi}{\sqrt{km}} \Rightarrow T = 2\pi \sqrt{\frac{m}{k}}$$

izobličena
perioda

Gibanje po krivulji:



$$\vec{v} = \frac{d\vec{s}}{dt} = \left(\frac{dr}{ds}\right) \dot{s} = \dot{s} \vec{e}_t \quad |\vec{v}| = |\dot{s}|$$

$\frac{1}{\dot{s}} \vec{e}_t$

$$\vec{a} = \frac{d}{dt} (\dot{s} \vec{e}_t) = \ddot{s} \vec{e}_t + \dot{s} \frac{d\vec{e}_t}{ds} = \ddot{s} \vec{e}_t + \frac{d\vec{e}_t}{ds} \dot{s}$$

$$\frac{d\vec{e}_t}{ds} = \kappa \vec{e}_n$$

$$\frac{d\vec{e}_t}{ds} \perp \vec{e}_t$$

$$\frac{d\vec{e}_t}{ds} = \frac{d\vec{e}_t}{ds} \left| \frac{d\vec{e}_t}{ds} \right|$$

← normalni vektor

← $\kappa = \frac{1}{R}$ R... krivuljni
polmer

$$|\vec{v} \times \vec{a}| = \kappa |\dot{s}|^3 \Rightarrow \kappa = \frac{|\vec{v} \times \vec{a}|}{|\dot{s}|^3}$$

$$\vec{a} = \ddot{s} \vec{e}_t + \kappa \dot{s}^2 \vec{e}_n \quad \text{tangenti + normalni pospešek}$$

$$\vec{e}_b = \vec{e}_+ \times \vec{e}_m \quad \dots, \text{binormala}$$

$$m\vec{a} = \vec{F} \quad m\vec{a} = \vec{F} + \vec{S} \quad \begin{matrix} \uparrow \\ \text{rezultante} \\ \text{aktivnih sil} \end{matrix} \quad \leftarrow \text{rezultanta vseh}$$

$$m\vec{S} = \vec{F}\vec{e}_+ + \vec{S}\vec{e}_+$$

$$m\vec{S}^2 = \vec{F}\vec{e}_n + \vec{S}\vec{e}_n \quad + \text{konstruiramo na} \\ \text{zvezo silo vseh}$$

$$0 = \vec{F}\vec{e}_b + \vec{S}\vec{e}_b$$

Gibanje pokrivulj: brez trenja: $\vec{S} \cdot \vec{e}_+ = 0$

Idealna krivulja, gladka krivulja

$$\vec{S} \cdot \vec{e}_+ = -\frac{\vec{v}}{|\vec{v}|} \cdot \sqrt{|\vec{S} - \vec{e}_n|^2 + |\vec{S} + \vec{e}_s|^2} \cdot \vec{e}_+$$

Gibanje po gladki krivulji

$$\vec{s} \cdot \vec{e}_t = 0$$

$$m\ddot{\vec{s}} = \vec{F} \cdot \vec{e}_x$$

$$F = -\text{grad} U$$

$$U(s) = U(\vec{r}(s))$$

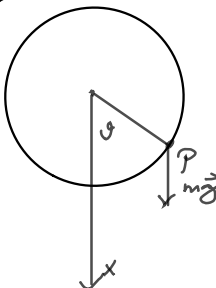
$$\frac{dU}{ds} = \text{grad} U \frac{d\vec{s}}{ds} = -\vec{e}_t$$

$$m\ddot{\vec{s}} \cdot \vec{s} = \vec{F} \cdot \vec{e}_x \cdot \vec{s} \rightarrow \frac{1}{m} \dot{\vec{s}}^2 + U(s) = E_0$$

gibanje po gladki krivulji je integrabilno

Primer:

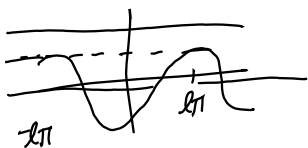
Matematično nihalo = gibanje po gladki krivulji pod vplivom sile teže



$$\vec{F}_g = m\vec{g}$$

$$U = -m\vec{g} \cdot \vec{r} = -m g l (\cos \varphi \vec{i} + \sin \varphi \vec{j}) = -m g l \cos \varphi$$

$$s = l\varphi \quad U(s) = -m g l \cos\left(\frac{s}{l}\right)$$



Reiada matematičnega nihala

$$T = \sqrt{2m} \int_{-s_0}^{s_0} \frac{ds}{\sqrt{E_0 - U(s)}} = \quad E_0 = -m g l \cos\left(\frac{s_0}{l}\right)$$

$$= 2\sqrt{2m} \frac{1}{\sqrt{mgl}} \int_0^{s_0} \frac{ds}{\sqrt{4 \cos^2\left(\frac{s}{l}\right) - \cos^2\left(\frac{s_0}{l}\right)}} \quad s = l\varphi$$

$$= \frac{2\sqrt{2}}{\sqrt{gl}} \int_0^{\varphi_0} \frac{l d\varphi}{\sqrt{\cos^2 \varphi - \cos^2 \varphi_0}} = 2\sqrt{2} \sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos^2 \varphi - \cos^2 \varphi_0}}$$

$$1 = \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}$$

$$\cos^2 \frac{\varphi}{2} = \cos^2 \frac{\varphi_0}{2} - \sin^2 \frac{\varphi}{2}$$

$$1 - \cos^2 \frac{\varphi}{2} = \sin^2 \frac{\varphi}{2} = \sin^2 \frac{\varphi_0}{2} - \sin^2 \frac{\varphi}{2}$$

$$= 2\sqrt{2} \sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2 \sin^2 \frac{\varphi}{2} - 2 \sin^2 \frac{\varphi_0}{2}}} =$$

$$u = \frac{\sin \frac{\varphi}{2}}{\sin \frac{\varphi_0}{2}}$$

$$\sin^2 \frac{\varphi}{2} = \sin^2 \frac{\varphi_0}{2} u^2$$

$$du = \frac{1}{\sin \frac{\varphi_0}{2}} \cos \frac{\varphi}{2} \cdot \frac{1}{2} d\varphi$$

$$= 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2 du \cdot \sin \frac{\varphi_0}{2}}{\cos \frac{\varphi}{2} \sqrt{\sin^2 \frac{\varphi_0}{2} (1 - u^2)}} =$$

$$= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{du}{\sqrt{1 - \sin^2 \frac{\varphi_0}{2} u^2} \sqrt{1 - u^2}} =$$

$$\sin \frac{\varphi_0}{2} = k$$

$$4\sqrt{\frac{l}{g}} \int_0^1 \frac{du}{\sqrt{1 - k^2 u^2} \sqrt{1 - u^2}} = K(k^2)$$

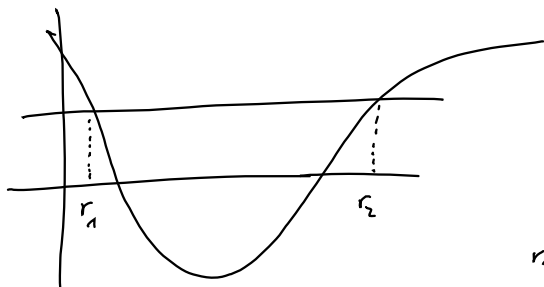
to je popolni eliptični integral prve vrste

$$T = 4\sqrt{\frac{l}{g}} K(k^2)$$

5.10

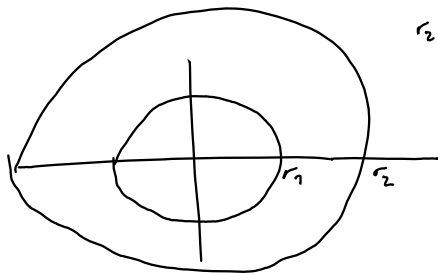
✓

ene wa - ene wa-pakurizi



r_1 pericenter

r_2 apocenter



$$\vec{r} = \vec{r}(v)$$

$$\frac{d\vec{r}}{dt} = \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta$$

Torej se letiše obkroži
krožnic

...

tir je središčni

Mege de Filipari zpidu 232

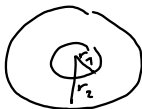
12.11.25

 $\frac{0.9}{\pi} \in \mathbb{Q} \Leftrightarrow$ tir je zaprti

 periodično
gibanje
koordinat r

Trditev: Nezaprti tir "omejenega" gibanja je gosta množica nad absolutnim radijem

Dokaz:



Dovolj je dokazati, da so presečišča tira s vsako krožnico med absolutnim ^{radijem} ~~razdaljami~~ radijem te krožnice gosta množica

Bertrandov izrek:

Edine potenciale za katera so vsi tiri v doli: krožnega tira zaprti sta geometrični potencial in Hookov potencial

$$\hookrightarrow U = \frac{1}{2}kr^2$$

 ??
Melnik
ker je ta
bistveno

Relativno gibanje

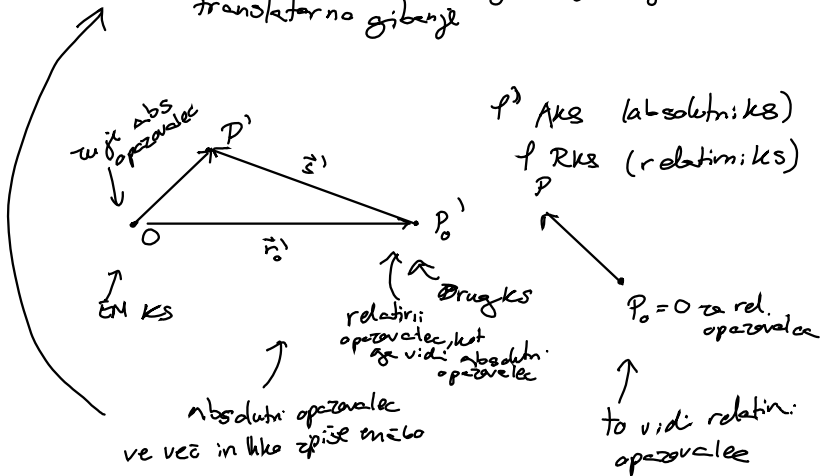
koordinatni sistem $\varphi(t, P)$ se giblje glede na koordinatni sistem $\varphi'(t', P')$, če \exists trajica

$(P_0 \in E, P_0': \mathbb{R} \rightarrow E, U: \mathbb{R} \rightarrow SO(3))$ take da velja

$t = t'$ \leftarrow pri analizi je bil Q konstanten

$$P' = P_0'(t) + Q(t)(P - P_0)$$

\nwarrow rotacijsko gibanje
 \nearrow translacijsko gibanje



Trditev: Rotacijski del gibanja je neodvisen od trajnice

Dokaz: $\tilde{P}_0, \tilde{P}_0', \tilde{Q}$

$$P_0' + Q(P - P_0) = \tilde{P}_0 + \tilde{Q}(P - \tilde{P}_0) \quad \forall P$$

$$P = P_0 \Rightarrow P_0' = \tilde{P}_0 + Q(P_0 - \tilde{P}_0)$$

$$\tilde{P}_0' + \tilde{Q}(P - \tilde{P}_0) + Q(P - P_0) = \tilde{P}_0' + \tilde{Q}(P - \tilde{P}_0)$$

$$\tilde{P}_0 + \tilde{Q}(P_0 - \tilde{P}_0) + Q(P - P_0) = \tilde{P}_0' + Q(P - \tilde{P}_0) \Rightarrow$$

$$Q(P - P_0) = \tilde{Q}(P - \tilde{P}_0) - \tilde{Q}(P_0 - \tilde{P}_0) =$$

$$\tilde{Q}((P - \tilde{P}_0) - (P_0 - \tilde{P}_0)) = \tilde{Q}(P - P_0)$$

$$\Rightarrow Q = \tilde{Q}$$

$$P = P(t) \Rightarrow$$

$$\dot{P}' = \dot{\tilde{P}}_0' + \dot{Q}(P - P_0) + Q(\dot{P})$$

\uparrow $\vec{v}_{rel} = \dot{P}$ relativna hitrost
 \vec{v}_0' translacijska hitrost $\xrightarrow{\text{rotacijske hitrost}}$

$$Q^T Q = I = Q Q^T$$

$$(Q^T)' Q + Q^T \dot{Q} = 0$$

$$\dot{Q}^T Q + Q^T \dot{Q} = 0$$

$$W = Q^T \cdot \dot{Q}$$

$$W^T = \dot{Q}^T Q$$

$$W^T = -W \Rightarrow W \text{ je po\u0161evno simetri\u010dni tenzor}$$

Linearna preslika \rightarrow **Tenzer** (drugega reda)

Trditev: $W = Q^T \dot{Q}$ je poševno simetrični tenzor
 Trditev: Naj bo W poševno simetrični tenzor (p.s.t.)
 definiran na trirazsežnem vektorskem prostoru.
 Potem obstaja vektor \vec{w} da je $W\vec{a} = \vec{w} \times \vec{a}$
 za vsi \vec{a}

Dokaz:

$$W\vec{p} = \lambda \vec{p}$$

$$\vec{p} \cdot W\vec{p} = \lambda |\vec{p}|^2$$

$$W^T \vec{p} \cdot \vec{p}$$

$$= -W\vec{p} \cdot \vec{p} =$$

$$= -\lambda |\vec{p}|^2 = \lambda |\vec{p}|^2 \Rightarrow \lambda = 0$$

$\{\vec{p}, \vec{q}, \vec{r}\}$ ONB

$$W\vec{p} = \vec{0}$$

$$W\vec{q} = \underbrace{(W\vec{q} \cdot \vec{p})}_{=0} \vec{p} + \underbrace{(W\vec{q} \cdot \vec{q})}_{=0} \vec{q} + (W\vec{q} \cdot \vec{r}) \vec{r}$$

$$W\vec{r} = W((\vec{q} \times \vec{p}) \vec{p} + (\vec{q} \times \vec{q}) \vec{q} + (\vec{q} \times \vec{r}) \vec{r}) = \vec{q} \times ((W\vec{q} \cdot \vec{p}) \vec{r} +$$

$$\vec{q} \times (W\vec{p} \cdot \vec{q}) \vec{q} +$$

$$\vec{q} \times (W\vec{r} \cdot \vec{q}) \vec{q}$$

$$\vec{p} \times \vec{q} = \vec{r} \text{ in } \vec{r} \times \vec{p} = \vec{q} \text{ ker ONB}$$

$$= (W\vec{q} \cdot \vec{r}) [\vec{q} \times (\vec{p} \times \vec{q}) - (\vec{q} \times \vec{r}) (\vec{r} \times \vec{p})] =$$

$$= (W\vec{q} \cdot \vec{r}) [\vec{p} \times (\vec{q} \times \vec{q}) \vec{q} + (\vec{q} \times \vec{r}) \vec{r}] = (W\vec{q} \cdot \vec{r}) (\vec{p} \times \vec{q})$$

$$(\vec{q} \cdot \vec{p}) \vec{p} = 0$$

$$\vec{q}$$

$$\vec{w} = (W\vec{q} \cdot \vec{r})$$

Vektorja $\vec{\omega}$ pravimo osni vektor p.s.t W

$$\vec{\omega} = \vec{\omega}(W)$$

$$e_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is a permutation of } 123 \\ -1 & \text{if } ijk \text{ is l'ike} \\ 0 & \text{if } ijk \text{ is not a permutation} \end{cases}$$

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^3 e_{ijk} a_j b_k \vec{e}_i = e_{ijk} a_j b_k \vec{e}_i$$

↑ pogost zapis brez vsote

sumacijska konjunktija

če se indeks ponovi, se sumira

$$\vec{a} = \sum a_i \vec{e}_i = a_i \vec{e}_i$$

$$\vec{a} \cdot \vec{b} = a_i b_i$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{ii} = 1$$

$$W_{ij} = -e_{ijk} w_k \quad \leftarrow \begin{array}{l} \text{seštevanje} \\ \text{samo po } k \end{array}$$

$$W \vec{a} = W_{ij} a_j \vec{e}_i$$

$$w \times \vec{a} = e_{ijk} w_j a_k \vec{e}_i$$

$$W_{ij} a_j = e_{ijk} w_j a_k \quad \forall a$$

$$W_{ij} a_j = e_{ikj} w_k a_j \quad \leftarrow \text{jik je zamenjano}$$

$$\Rightarrow W_{ij} = e_{ikj} w_k = -e_{ijk} w_k$$

$$e_{ijk} e_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \quad \text{velja (pake ne bo)}$$

$$e_{ijk} e_{njk} = \delta_{in} \delta_{jj} - \delta_{ij} \delta_{jn} = 3\delta_{in} - \delta_{in} = 2\delta_{in}$$

$$e_{ijk} e_{ijk} = 6$$

$$e_{ijp} w_{ij} = -e_{ijp} e_{ijk} w_k = -2\delta_{pk} w_k = -2w_p$$

$$w_p = -\frac{1}{2} e_{ijp} w_{ij}$$

Trilinear $A \in \mathcal{L}(U, V)$

$$A^*(\vec{a} \times \vec{b}) = (A\vec{a}) \times (A\vec{b})$$

$$A^* \text{ adjungiranta } A\text{-je} \quad A^{-1} = \frac{1}{\det A} (A^*)^T$$

Dikaz: $\det A \neq 0$ $\vec{a}, \vec{b}, \vec{c}$ lin neod

$$\det A = \frac{[A\vec{a}, A\vec{b}, A\vec{c}]}{[\vec{a}, \vec{b}, \vec{c}]}$$

$$(A\vec{a} \times A\vec{b})\vec{c} = (A\vec{a} \times A\vec{b})(AA^{-1}\vec{c}) = [A\vec{a}, A\vec{b}, AA^{-1}\vec{c}] =$$

$$= \det A [\vec{a}, \vec{b}, A^{-1}\vec{c}] = \det A (\vec{a} \times \vec{b}) A^{-1}\vec{c} =$$

$$= (\det A) A^T (\vec{a} \times \vec{b}) \vec{c} \Rightarrow A\vec{a} \times A\vec{b} = \det(A) A^{-T} (\vec{a} \times \vec{b})$$

$$A \xrightarrow{\det A} A \quad A\vec{a} \times A\vec{b} = A^* (\vec{a} \times \vec{b})$$

$$\det A = 0 \Rightarrow \forall \varepsilon > 0. \exists A_\varepsilon. \det A_\varepsilon = 0. |A - A_\varepsilon| < \varepsilon$$

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon \vec{a} \times A_\varepsilon \vec{b} = A^* (\vec{a} \times \vec{b})$$

$$A_\varepsilon \rightarrow A \quad A_\varepsilon^* \xrightarrow{?} A^*$$

cc p = so lin odron

$$[\vec{a}, \vec{b}, \vec{c}] = \det A [A\vec{a}, A\vec{b}, A\vec{c}] \text{ holivels}$$

$$A = Q \in O(3)$$

$$Q^* = \det(Q) (Q^{-1})^T = \det Q Q$$

$$\parallel \\ (Q^T)^T = Q$$

$$Q \in SO(3) \Rightarrow Q(\vec{a} \times \vec{b}) = (Q\vec{a}) \times (Q\vec{b})$$

$$W = Q^T \dot{Q} \quad \text{osnovni vektorju p.s.t pravimo} \\ \text{rotacijski vektor}$$

$$\text{Vektorju } \vec{\omega}' = Q \vec{\omega} \text{ pa pravimo vektor kotne hitrosti:} \\ \text{rotacije } Q$$

$$Q^T \dot{Q} \vec{a} = \omega \times \vec{a}$$

$$\dot{Q} \vec{a} = Q(\omega \times \vec{a}) = \vec{\omega}' \times Q \vec{a}$$

tradicij

Osn. vektor p.s.t. $W = \dot{Q}Q^T$ je vektor konstantne hitrosti rotacije Q

$$\text{Dakle: } \dot{Q}Q^T \vec{a} = \vec{\omega} \times \vec{a}$$

$$\dot{Q} \vec{a} = \vec{\omega} \times Q \vec{a}$$

$$\dot{Q}Q^T \vec{a} = \vec{\omega} \times Q(Q^T \vec{a}) = \vec{\omega} \times \vec{a}$$

$$\left. \begin{array}{l} \dot{Q} \vec{a} = \vec{\omega} \times Q \vec{a} \\ \dot{Q}Q^T \vec{a} = \vec{\omega} \times \vec{a} \end{array} \right\} \Rightarrow \dot{Q} = \vec{\omega}$$

$Q \in SO(3)$

W_1, W_2 ni pravi simetrični tenzor

$[W_1, W_2] = W_1 W_2 - W_2 W_1$ je pravno sim. tenz

↑ komutator

$(W, +, [\cdot])$ je algebra

↑
množica p.s.t.

$(V, +, \times)$ je tudi algebra
↑
prostor 3D vektorejv
(Lijeva algebra)

Triditer: $\omega([w_1, w_2]) = \omega(w_1) \times \omega(w_2)$

$\omega \mapsto W(\omega)$ je homomorfizem algebr

$$\begin{aligned} \text{Dokaz } [w_1, w_2] \vec{a} &= (w_1 w_2 - w_2 w_1) \vec{a} = \\ &= \vec{w}_1 \times (\vec{w}_2 \times \vec{a}) - \vec{w}_2 \times (\vec{w}_1 \times \vec{a}) = \\ &= (\vec{w}_1 \vec{a}) \vec{w}_2 - \cancel{(\vec{w}_1 \vec{w}_2) \vec{a}} - ((\vec{w}_2 \vec{a}) \vec{w}_1 - \cancel{(\vec{w}_2 \vec{w}_1) \vec{a}}) = \\ &= (\vec{w}_1 \times \vec{w}_2) \times \vec{a} \end{aligned}$$

Izreki: Najbo W p.s.t z enoskim osnim vektorjem \hat{e} .

Potem je e^{vW} rotacija okrog osi \hat{e} za kot v

Izreč: KS $\varphi(t, P)$ in $\varphi'(t, P')$ sta v istem razredu KS

$\Leftrightarrow P' = P_0' + Q(P - P_0)$ predstavlja vsako

premočrtno gibanje s konstantno brzino

v KS φ z premočrtno s konstantno brzino

v KS φ'

Dokaz

(\Rightarrow) očitno

$$(\Leftarrow) \varphi = P_0 + \vec{v}t \mapsto \varphi' = P'_0(t)$$

$$\vec{0} = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_{re}$$

$$\vec{0} = \vec{a}_0 + t \dot{\vec{\omega}} \times \vec{v} + t \vec{\omega} \times (\dot{\vec{\omega}} \times \vec{v}) + 2\vec{\omega} \times \vec{v} \quad / \cdot \vec{v}$$

$$\vec{0} = \vec{a}_0 \cdot \vec{v} + t (\dot{\vec{\omega}} \times (\dot{\vec{\omega}} \times \vec{v})) \cdot \vec{v}$$

$$\vec{0} = \vec{a}_0 \cdot \vec{v} - t (\vec{v} \times \vec{\omega})(\vec{\omega} \times \vec{v}) = \vec{a}_0 \cdot \vec{v} - t \|\vec{v} \times \vec{\omega}\|_2$$

$$\vec{v} = \lambda \vec{v}_0$$

$$0 = \lambda \vec{a}_0 \cdot \vec{v}_0 - t \lambda^2 \|\vec{v}_0 \times \vec{\omega}\|_2 \quad \forall \lambda$$

$$\Rightarrow \vec{a}_0 \cdot \vec{v}_0 = 0 \quad \wedge \quad \vec{v}_0 \times \vec{\omega} = 0 \quad \forall \vec{v}_0$$

$$\vec{a}_0 = 0 \quad \Rightarrow \quad \vec{\omega} = \vec{0}$$



\Leftrightarrow
 Q je konstanten

$$\Rightarrow \varphi_0(t) = P_0' + \vec{c}t$$

Posledica (Prvi Newtonov zakon)

$KS \text{ je } IKS \Leftrightarrow$ V prosti materialna točka se
giblje premočno s konstantno brzino

↙ mora obstajati
prosta materialna točka

Dokaz

$(\Rightarrow) \checkmark$

Prosta materialna točka - izdružena
od vpliva ostalih

$(\Leftarrow) \checkmark$

Lahko bi namesto proste rekel:
masno težišče
(prosta točka mora obstajati)

Newtonova enačba v relativnem koordinatnem sistemu ^{RKS}

VKS $m\vec{a}' = \vec{F}'$

inercialna sila
rotacijske
centrifugalna sila

$$m\vec{a}_{re} = \vec{F}' - m\vec{a}_1' - m\vec{\omega}' \times \vec{r}' - m\vec{\omega}' \times (\vec{\omega}' \times \vec{r}') -$$

inercialna sila

fiktivne sile

$-2m\vec{\omega}' \times \vec{v}_{re} = \vec{F}'_{rel}$

Coriolisova sila

$$m\vec{a}_{re} = \vec{F}_{re}$$

$$\vec{F} = Q^T \vec{F}' \quad F_{re} = Q^T F'_{rel}$$

Trditiv: F' potencijalna $\Rightarrow F$ potencijalna

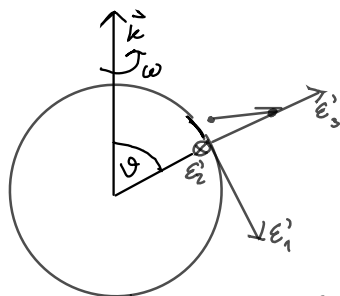
Dokaz: $\vec{F}' = -\text{grad}_{P'} U' = \left(\frac{\partial U}{\partial p'} \right)^T = - \left(\frac{\partial U}{\partial p} \frac{\partial p}{\partial p'} \right)^T =$

$$U(P) = U'(P') = U'(P_0' + Q(P - P_0'))$$

$$P = Q^T (P' - P_0') + P_0$$

$$\rightarrow = -Q \left(\frac{\partial U}{\partial p} \right)^T = Q \vec{F}' \Rightarrow \vec{F} = - \left(\frac{\partial U}{\partial p} \right)^T = -\text{grad}_p U$$

Foucaultovo nihalo



$$m \vec{a}_{rel} = F_g + \vec{T} - m \vec{a}_o - \cancel{m \dot{\vec{\omega}} \times \vec{r}} - \cancel{m \vec{\omega} \times (\dot{\vec{\omega}} \times \vec{r})} - 2m \dot{\vec{\omega}} \times \vec{v}_{rel}$$

\swarrow sila vrvica \swarrow rotacija o konst.

$$\vec{\omega}' = \vec{\omega} \vec{k}$$

$$\omega = \frac{2\pi}{24 \cdot 3600} \text{ s} < 10^{-4}$$

$$\uparrow 10^{-8}$$

$$\vec{a}_o = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a} = \vec{a}_o + \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

\uparrow \parallel \parallel \parallel \parallel
 \quad o o o o

pospešek KS nihala

glede na KS, ki se vrti skupaj z zemljo

$$F_g - m \vec{a}_o = m \vec{g}$$

$$\vec{T} = T \cdot \frac{\vec{r}}{r} = \frac{L_1}{l} \vec{e}_1 + \frac{l - L_1}{l} \vec{e}_3 - \frac{L_2}{l} \vec{e}_2$$

itd. neli

Sistem materialnih točk

$$P_1, \dots, P_n$$

$$m_i \ddot{\vec{P}}_i = \vec{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^N \vec{F}_{ji} \quad \vec{F}_{ji} = -\vec{F}_{ij}$$

$$\sum_{i=1}^n m_i \ddot{\vec{P}}_i = \sum_{i=1}^N \vec{F}_i + \sum_{i \neq j}^N \vec{F}_{ji}$$

||
0

$$\vec{P}_* = 0 + \sum_{i=1}^N m_i (\vec{P}_i - 0)$$

$$m = \sum_{i=1}^N m_i$$

$$\ddot{\vec{P}}_* = \frac{1}{m} \sum_{i=1}^N m_i \ddot{\vec{P}}_i \quad \ddot{\vec{P}}_* = \frac{1}{m} \sum_{i=1}^N m_i \ddot{\vec{P}}_i$$

$$m \ddot{\vec{P}}_* = \vec{F} = \sum_{i=1}^N m_i \ddot{\vec{P}}_i \quad \text{Enačba gibanja masnega središča}$$

