Osnove Newtonove mehanike

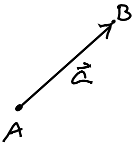
Def: Afini prostor $\mathcal{A}$ nad vektorskim prostorom V je množica $\mathcal{A}$ z binarno operacijo $\mathcal{A}\times V \to \mathcal{A} : (A,\vec a)\mapsto A+\vec a$ z lastnostmi:

i) $(A+\vec a)+\vec b = A+(\vec a+\vec b)$

ii) $\forall A,B\in\mathcal{A}\ \exists!\, \vec a\in V :\ B=A+\vec a$

$\dim \mathcal{A} = \dim V$

Primeri:



Def: Definiramo operacijo $\mathcal{A}\times\mathcal{A}\to V$.
s predpisom $B-A=\vec a \Leftrightarrow \vec B = A+\vec a$

Trditev:

i) $A-A=\vec 0$

ii) $(A-B)+(B-A)=\vec 0$

iii) $(A-B)+(B-C)+(C-A)=\vec 0$

iv) $(A-B)+\vec a = (A+\vec a)-B$

v) $(A+B)-C = B+(A-C)$

Dokaz:

i) $A-A=\vec a \Leftrightarrow A=A+\vec a \Leftrightarrow A=(A+\vec a)+\vec a = A+2\vec a$

$\Rightarrow \vec a = 2\vec a$ (ker je $\vec a$ natanko določen)

$\Rightarrow \vec a=\vec 0$

v) $A-C=\vec a \Rightarrow$ $A=C+\vec a$; $B+(A-C)=B+\vec a = C+(\vec b+\vec a)$

$B-C=\vec b \Rightarrow B=C+\vec b$

$(A+B)-C = (\overbrace{(C+\vec a)}^{\text{vektor}}+(C+\vec b))-C =$, zato lahko zamenjamo oklepaje

$= C+\vec a+\underbrace{(C+\vec b-C)}_{\text{vektor}} = \vec C+\vec a+\vec b$

$(\mathcal{A}, V)$ $(\mathcal{A}', V')$

$g: \mathcal{A} \to \mathcal{A}'$

Def: Preslikava $g: \mathcal{A} \to \mathcal{A}'$ je afina če obstaja

$d_g \in \mathcal{L}(V, V')$ tako da velja $g(A) - g(B) = d_g(A-B)$

za $\forall A, B \in \mathcal{A}$

$$g(A) = g(B) + d_g(A-B)$$

$g(A) = g(O) + d_g(A-O)$; O je pol afine preslikave

Izbira pola je poljubn

$$\tilde{g}(A) = g(\tilde{O}) + d_g(A-\tilde{O}) = g(O) + d_g(\tilde{O}-O) + d_g(A-\tilde{O})$$

$$= g(O) + d_g(\underbrace{(\tilde{O}-O) + (A-\tilde{O})}_{A-O}) = g(A)$$

Izberimo $O \in \mathcal{A}$ $\quad \vec{a} = A - O$

$$A \in \mathcal{A} \ ; \ A = O + (A-O) = O + \vec{a}$$

$\forall A \in \mathcal{A}$ lahko identificiramo z vektorjem $\vec{a} \in V$

Definicija: Galilejeva struktura G je trojica $(\mathcal{A}, \tau, d)$, kjer je $\mathcal{A}$ štirirazsežen afini prostor, $\tau \in \mathcal{L}(V, \mathbb{R})$ in d evklidska razdalja nad prostorom istočasnih dogodkov.

linearna preslikava

Funkcionalu τ pravimo časovnost. Dogodka $A, B \in \mathcal{A}$ sta istočasna, če $A-B \in \ker \tau$

Definicija: Galilejevi strukturi $G(\mathcal{A}, \tau, d)$ in $\tilde{G}(\tilde{\mathcal{A}}, \tilde{\tau}, \tilde{d})$ sta ekvivalentni, če obstaja afina bijekcija $g: \mathcal{A} \to \tilde{\mathcal{A}}$, ki ohranja časovnost in razdaljo istočasnih dogodkov

$$\tilde{\tau}(g(A)-g(B)) = \tau(A-B)$$

A, B istočasna $\Leftrightarrow$ $g(A), g(B)$ istočasna

$$d(A, B) = \tilde{d}(g(A), g(B))$$

Definicija

$\mathbb{R}\times E$ afini prostor, kjer je E trorazsežni evklidski prostor

Na $\mathbb{R}\times E$ vpeljemo naravno Galilejevo strukturo.

$$A\in \mathbb{R}\times E \Rightarrow A=(t,P) \quad t\in\mathbb{R},\ P\in E$$

ima normo porojeno s skalarnim produktom

$$\tau(A_2-A_1)=t_2-t_1$$

$$d(A_1,A_2)=\|P_1-P_2\|$$

Tej strukturi pravimo naravna Galilejeva struktura

Definicija: Koordinatni sistem na afinem prostoru $\mathcal{A}$ je bijektivna preslikava $\varphi:\mathcal{A}\to\mathbb{R}$

$$A\longmapsto \varphi(A)=(\pi_t\varphi(A),\pi_p\varphi(A))$$
$$=\varphi_t(A),\varphi_p(A))$$

za katero je φ_t afina preslikava

$$(\mathbb{R}\times E,\ t,\ \|\ \|)$$

$$\tau(A-B)=t(\varphi_t(A)-\varphi_t(B))$$

$$d(A,B)=\|\varphi_p(A)-\varphi_p(B)\|$$

Lahko merimo razdaljo tudi med neistočasnimi dogodki

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi"] \arrow[dr, "\tilde{\varphi}"'] & \mathbb{R}\times E \arrow[d, leftrightarrow] \\
 & \mathbb{R}\times E
\end{tikzcd}$$

Kdaj sta dve naravni galilejevi strukturi ekvivalentni

$$g : \mathbb{R}\times E \longrightarrow \mathbb{R}\times E$$

$$A = \begin{bmatrix} t \\ P \end{bmatrix} \longmapsto g(A) = \begin{bmatrix} t' \\ P' \end{bmatrix} = g(O) + d_g(A-O)$$

$$= \begin{bmatrix} t_0' \\ P_0' \end{bmatrix} + \underbrace{\left[\begin{array}{c:c} \alpha & \vec{a}^T \\ \hdashline \vec{c} & Q \end{array}\right]}_{4\times 4 \text{ matrika}} \begin{bmatrix} t - t_0 \\ P - P_0 \end{bmatrix}$$

$$O = \begin{bmatrix} t_0 \\ P_0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} t_1 \\ P_1 \end{bmatrix} \quad A_2 = \begin{bmatrix} t_2 \\ P_2 \end{bmatrix}$$

$$t\big(g(A_2) - g(A_1)\big) = t(A_2 - A_1)$$

$$g(A_2) - g(A_1) = \begin{bmatrix} \alpha & \vec{a}^T \\ \vec{c} & Q \end{bmatrix} \begin{bmatrix} t_2 - t_1 \\ P_2 - P_1 \end{bmatrix} = \begin{bmatrix} \alpha(t_2 - t_1) + \vec{a}(P_2 - P_1) \\ (t_2 - t_1)\vec{c} + Q(P_2 - P_1) \end{bmatrix}$$

$$\Rightarrow \alpha(t_2 - t_1) + \vec{a}(P_2 - P_1) = t_2 - t_1 \Rightarrow \alpha = 1,\ \vec{a} = \vec{0}$$

To velja da se ohranja časovnost

ohranjanje razdalje med istočasnimi dogodki:

$$d(g(A_1), g(A_2)) = d(A_2, A_1) \quad \text{za } t_1 = t_2$$

$$\underset{t_2 - t_1 = 0}{\|} \qquad\qquad \|$$

$$\|Q(P_2 - P_1)\| \qquad\qquad \|P_2 - P_1\|$$

$$\Rightarrow Q \in O(3)$$

ortogonalna matrika 3×3

Definicija: Preslikava, ki ohranja Galilejevo strukturo pravimo Galilejeva preslikava

Trditev: Galilejeve preslikave med naravna Galilejevima strukturama $\mathbb{R}\times E$ je oblike

$$\begin{bmatrix} t \\ p \end{bmatrix} \longmapsto \begin{bmatrix} t' \\ p' \end{bmatrix} = \begin{bmatrix} t_0' + t - t_0 \\ P_0' + \vec{c}(t-t_0) + Q(P-P_0) \end{bmatrix} =$$

kjer je $Q \in O(3)$, $\vec{c}$ poljuben vektor, t_0 poljubno števila in P_0' poljubna točka

$$= \begin{bmatrix} \tilde{t}_0 + t \\ \tilde{P}_0 + \vec{c}t + Q(P-P_0) \end{bmatrix}$$

$t_0' - t_0 = \tilde{t}_0$

Dva opazovalca: $\varphi(t,P)$, $\varphi'(t',P')$

gibanje $t \longmapsto P(t)$ trajektorija točke

$\vec{v} = \dfrac{dP}{dt}$ vektor hitrosti $\lim\limits_{h \to 0} \dfrac{P(t+h) - P(t)}{h}$

$\ddot{P} = \dot{\vec{v}} = \vec{a} = \dfrac{d\vec{v}}{dt}$ vektor pospeška

$|\vec{v}| = v$ brzina

$$P'(t') = P_0' + \vec{c}(t' - t_0') + Q\overbrace{(P(t) - P_0)}^{= Q(P(t'-t_0') - P_0)}$$ trajektorija v φ'

$$\vec{v}' = \frac{dP'}{dt'} = \vec{c} + Q\frac{dP}{dt}(t' - t_0') = \vec{c} + QP(t)$$

$$\vec{v}' = \dot{P}' \qquad \dot{P}'(t') = \vec{c} + Q\dot{P}(t)$$

najprej črtica, nato odvod

$$\ddot{P}'(t') = \vec{a}' = \frac{dv'}{dt'} = Q\ddot{P}(t)$$

	φ	φ'
	P	$P' = P_0' + \vec{c}\,t + Q(P - P_0)$
$\vec{a} = P_2 - P_1$		$\vec{a}' = P_2' - P_1' = Q(P_2 - P_1) = Q\vec{a}$

Definicija: vektor $a = P_2 - P_1$ je koordinatno neodvisen

$A \in \mathcal{L}(V, V)$ $\quad A' = Q^T A Q$

$\lambda \qquad \lambda' = \lambda$

Sistem materialnih točk $(P_1, \ldots P_n) = \underline{P}$

$$\underline{P}' = \underline{P_0}' + \underline{\vec{c}}\,t + Q\,(\underline{P} - \underline{P_0})$$

$$\underline{P_0}' = (P_0', \ldots P_0') \qquad Q(\underline{P} - \underline{P_0}) = \left(Q\,(P_1 - P_0), \ldots Q(P_n - P_0)\right)$$

$$\underline{\vec{c}} = (\vec{c}, \ldots \vec{c})$$

$$\Rightarrow \underline{\dot{P}}' = \vec{c} + Q\,\underline{\dot{P}}$$

$$\underline{\ddot{P}}' = Q\,\underline{\ddot{P}}$$

Princip determiniranosti

V danem KS (koordinatni sistem) je trektorija sistema materialnih točk natanko določena z začetnim položajem in začetno hitrostjo.

To specialno pomeni, da obstaja funkcija interakcije $\underline{\vec{f}}$ tako da je $\underline{\ddot{P}} = \underline{\vec{f}}(t, \underline{P}, \underline{\dot{P}})$

$$\left(\ddot{P}(t) = \vec{f}(t, \underline{P(t)}, \dot{P}(t)) \quad \text{nadalje}\right)$$

8.10

Princip relativnosti Obstaja razred koordinatnih sistemov v katerem je funkcija interakcije invariantna za Galilejeve transformacije. Koordinatnim sistemom iz tega pravimo **inercialni koordinatni sistem**

$(t, \underline{P}) \xrightarrow{GT} (t', \underline{P}')$ Galilejeva tranformacija.

Potem je $\underline{f} = \underline{f}'$ $\quad (\ddot{\underline{P}} = \ddot{\underline{P}}')$

$$\underset{\parallel}{\ddot{\underline{P}}'} = \vec{\underline{f}}(t', \underline{P}', \dot{\underline{P}}')$$
$$Q\ddot{\underline{P}} = Q\vec{\underline{f}}(t, \underline{P}, \dot{\underline{P}}) = \vec{\underline{f}}(t_0' + t, P_0' + \vec{c}t + Q(P - P_0), \vec{c} + Q\dot{\underline{P}})$$

i) $\vec{c} = \vec{0}$ $\quad Q = I$ $\quad P_0' = P_0$

$$\vec{\underline{f}}(t, \underline{P}, \dot{\underline{P}}) = \vec{\underline{f}}(t_0' + t, \underline{P}, \dot{\underline{P}}) \text{ za } \forall t_0'$$

$\Rightarrow f$ ni eksplicitno odvisna od časa (t)

(**homogenost časa**)

ii) $\vec{c} = 0$, $Q = I$, $\dot{P}' = P_0 + \vec{a}$

$$\vec{f}(P, \dot{P}) = \vec{f}(\underline{P} + \underline{\vec{a}}, \dot{P}) \text{ za } \forall \vec{a}$$
$$\underline{P} + \underline{\vec{a}} = (P_1 + \vec{a}, \ldots, P_n + \vec{a})$$

$\Rightarrow \underline{\vec{f}}$ je odvisna samo od relativnih položajev

$$\vec{f}(P, \dot{P}) = \vec{f}(P_i - P_j, \underline{\dot{P}})$$

$i \neq j$ in vse kombinacije tega

(**homogenost prostora**)

iii) $Q = I$

$$f(P_i - P_j, \dot{P}) = f(P_i - P_j, \vec{c} + \dot{P}) \Rightarrow \vec{f}(P_i - P_j, \dot{P}_k - \dot{P}_\ell)$$

(**homogenost prostora hitrost**)

iv) Q poljuben

$$Q\vec{f}(P_i - P_j, \dot{P}_k - \dot{P}_\ell) = f(Q(P_i - P_j), Q(\dot{P}_k - \dot{P}_\ell))$$

f je **izotropična funkcija**

$$Q\vec{g}(\vec{a}) = \vec{g}(Q\vec{a}) \text{ za } \forall Q \in O(3)$$

Posebni primer : $N=1$ (izolirana točka)

$$\ddot{P} = \vec{f}(\,)$$ brez argumentov (konstanta)

še vedno velja izotropičnost $Q\ddot{P} = Q\vec{f} = \vec{f} \Rightarrow \vec{f} = 0$

$$\Rightarrow \ddot{P} = \vec{0}$$

v IKS (inercialni koordinatni sistem) se izolirana materialna točka giblje premočrtno s konstantno hitrostjo. $P = v_0 t + P_0 (t=0)$

$N=2 \Rightarrow$

$$\ddot{P}_1 = \vec{f}_1(P_1 - P_2, \dot{P}_1 - \dot{P}_2)$$
$$\ddot{P}_2 = \vec{f}_1(P_2 - P_1, \dot{P}_2 - \dot{P}_1)$$

če se točki gibljeta po premici proti drugi bosta vedno ostali na tej premici

(Podobno tri: točke ostanejo na ravnini:)

$$\ddot{P}_1 = f_1(P_1 - P_2, P_1 - P_3, \underline{\underline{P_2 - P_3}}, \ldots$$

tudi to je pomembno

Definicija: Interakcija $\ddot{P}_i = f_i(\ldots)$ je parska
če je odvisna samo od relativnih položajev
in hitrosti glede na P_i in je delovanje
točk avtonomno

$$\vec{f}_i(\underset{j \neq i}{P_i - P_j}, \underset{k \neq i}{\dot{P}_i - \dot{P}_k})$$

$$\vec{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{f}_{ji}(P_i - P_j, \dot{P}_i - \dot{P}_j)$$

Def: Interakcija $\vec{f}_{ji} = \vec{f}_{ji}(P_i - P_j, \dot{P}_i - \dot{P}_j)$ je lokalna
če velja $\lim\limits_{|P_i - P_j| \to \infty} f_{ji} = \vec{0}$

Princip sorazmernosti

V IKS za sistem materialnih točk $P_1 \dots P_N$ obstajajo natanko določene konstante α_{ij} tako, da ne glede na interakcije $\vec{f}_i$ velja $\vec{f}_i = -\sum_{j=1}^{N} \alpha_{ij} \vec{f}_j$

Lema: Za konstante α_{ij} velja

i) $\alpha_{ij}\alpha_{ji} = 1$

ii) $\alpha_{ij}\alpha_{jk}\alpha_{ki} = 1$

Dokaz:

$\vec{f}_i = \vec{f}_i(P_i - P_k) = \vec{f}_i(P_i - P_1, P_i - P_2 \dots P_i - P_N)$

in naj bodo f_i lokalni

P_k pošljemo proti ∞ razen P_i in P_j

$$\vec{f}_i = -\sum_{\substack{k=1 \\ k \neq i}}^{N} \alpha_{ki} \vec{f}_k = -\alpha_{ji} \vec{f}_j$$

$$\vec{f}_j = -\sum_{\substack{k=1 \\ k \neq j}}^{N} \alpha_{kj} \vec{f}_k = -\alpha_{ij} f_i$$

$$f_i = -\alpha_{ji} f_j = \alpha_{ji}\alpha_{ij} f_i \Rightarrow \alpha_{ji}\alpha_{ij} = 1$$

P_i, P_j, P_k ostanejo, ostale gredo proti ∞

$$\vec{f}_i = -\alpha_{ji} f_j - \alpha_{ki} f_k = -\alpha_{ji}(-\alpha_{ij} f_i - \alpha_{kj} \vec{f}_k) - \alpha_{ki} f_k =$$

$$\vec{f}_j = -\alpha_{ij}\vec{f}_i - \alpha_{kj} f_k$$

$$f_i = \underbrace{\alpha_{ji}\alpha_{ij}}_{1} f_i + \alpha_{ji}\alpha_{kj} f_k - \alpha_{ki} f_k$$

$$\alpha_{ki} f_k = \alpha_{ji}\alpha_{kj} \vec{f}_k \quad / \alpha_{ik}$$

$$f_k = \alpha_{ji}\alpha_{kj}\alpha_{ik} f_k \Rightarrow \alpha_{ik}\alpha_{kj}\alpha_{ji} = 1$$

Lema: Naj za pozitivna števila α_{ij} velja

i) $\alpha_{ij} \cdot \alpha_{ji} = 1$

ii) $\alpha_{ij}\alpha_{jk}\alpha_{ki} = 1$.

Potem $\exists$ pozitivna števila m_i, tako da velja da je

$$\alpha_{ji} = \frac{m_j}{m_i}$$

Števila m_i so določena do sorazmernostnega faktorja natančnost. Številom m_i pravimo

inercijske mase

Dokaz: $\alpha_{ii} = 1$

$$\alpha_{ij}\alpha_{jj}\alpha_{ji} = 1 \Rightarrow \alpha_{jj} = 1$$

$l_{ij} = \log \alpha_{ij}$ $\qquad l_{ij} + l_{jk} + l_{ki} = 0$

$l_{ij} = -l_{ji}$

$$l_{i_0 j} + l_{jk} + l_{k i_0} = 0$$

$$l_{ij} - l_{i_0 j} + l_{ki} - l_{k i_0} = 0 \Rightarrow$$

$$l_{ij} - l_{i_0 j} = -l_{ki} + l_{k i_0} = l_{ik} - l_{i_0 k} \quad \forall j,k$$

$$\Rightarrow l_{ij} - l_{i_0 j} = n_{i i_0}$$

$$\left.\begin{array}{l} l_{ij} = n_{i i_0} + l_{i_0 j} \\ j = i \end{array}\right\} \Rightarrow 0 = n_{i i_0} + l_{i_0 i} \Rightarrow l_{i i_0} = n_{i i_0}$$

$$l_{ij} = l_{i i_0} - l_{j i_0}$$

$l_{i i_0} = \log m_i$ ← definiram

$$\Rightarrow l_{ij} = \log m_i - \log m_j = \log \frac{m_i}{m_j} \Rightarrow \alpha_{ij} = \frac{m_i}{m_j}$$

Izrek: V inercialnem koordinatnem sistemu velja

$$m_1\ddot{P}_1 + m_2\ddot{P}_2 + \dots + m_n\ddot{P}_n = \vec{0}$$

Dokaz:

$$f_i = -\sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_{ji} f_j = -\sum_{j=1}^{N} \frac{m_j}{m_i} f_j$$

$$\ddot{P}_i = -\sum_{j=1}^{N} \frac{m_j}{m_i} \ddot{P}_j \qquad / m_i$$

$$\sum_{i=1}^{N} m_i \ddot{P}_i = \vec{0}$$

$$P_* = \frac{1}{m}\sum_{i=1}^{N} m_i P_i \,, \quad m = \sum_{i=1}^{N} m_i$$ masno središče

seštevanje točk:

$$\sum_{i=1}^{N} P_i = \sum \left(m_i(P_i - O) + m_i O\right) = \sum_{i=1}^{N} m_i (P_i - O) + \underbrace{\sum_{i=1}^{N} m_i O}_{O}$$

vektor

$$P_* = O + \frac{1}{m}\sum_{i=1}^{N} m_i (P_i - O)$$

$$m\ddot{P}_* = \vec{0}$$

Definicija: sila na P_i je produkt interakcije f_i z maso m_i:

$$m_i \ddot{P}_i = m_i f_i(\dots) =: \vec{F}_i$$

Princip o masi masa materialnih točk je enaka v vseh koordinatnih sistemih

$$\ddot{P}_1 = f_1(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \qquad m_1 \ddot{P}_1 = F_1$$

$$\ddot{P}_2 = f_2(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \qquad m_2 \ddot{P}_2 = F_2$$

$$m_1 \ddot{P}_1 + m_2 \ddot{P}_2 = F_1 + F_2 = 0$$

$$F_2 = -F_1$$

$$\vec{F}_1 = \vec{F}_{21}$$

$$\vec{F}_2 = \vec{F}_{12} \qquad \vec{F}_{12} = -\vec{F}_{21}$$

Trditev: Če so vse sile parske in lokalne velja zakon akcije in reakcije (3NZ)

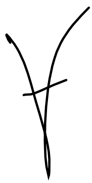
Dokaz: $f_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} f_{ji}(\vec{r}_j - \vec{r}_i, \dot{\vec{r}}_j - \dot{\vec{r}}_i)$

$\vec{r}_j \to \infty \quad j \notin \{i, k\}$

$m_i f_i = m_i \ddot{\vec{r}}$

$m_i f_{ji} + m_j f_{ij} = \vec{0}$

$\vec{F}_{ji} = -\vec{F}_{ij}$



10.10
zamudila prvo uro

odvod vzdolž trajektorije $\frac{d}{dt} E(t, \vec{p}(t), \dot{\vec{p}}(t)) =$

$$\frac{\partial E}{\partial t} + \left(\frac{\partial E}{\partial \vec{p}}\right)^T \dot{\vec{p}} + \left(\frac{\partial E}{\partial \dot{\vec{p}}}\right)^T \ddot{\vec{p}}$$

Izrek o množenju: Naj bo sila $\vec{F}$ potencialna s potencialom $U(t, \vec{p}, \dot{\vec{p}})$. Potem je vsota kinetične in potencialne energij konstanta gibanja (= njen odvod vzdolž trajektorije je 0) $\Longleftrightarrow$ moč sil je nasprotna enaka odvodu potencialne energije vzdolž trajektorije

Dokaz: $\Rightarrow$ $E_0 = T + U = \frac{1}{2} m |\dot{\vec{p}}|^2 + U$

$$0 = \frac{\partial E_0}{\partial t} - m \ddot{\vec{p}} \dot{\vec{p}} + \vec{F} \dot{\vec{p}} + \frac{dU}{dt} \Rightarrow \frac{dU}{dt} = -\vec{F} \dot{\vec{p}}$$

$\Leftarrow$

$$A = T_2 - T_1$$

$$A = \int_{t_1}^{t_2} \vec{F} \cdot \dot{\vec{p}} \, dt = -\int_{t_1}^{t_2} \frac{dU}{dt} dt = \underset{U_1}{U(t_1)} - \underset{U_2}{U(t_2)}$$

$$\Rightarrow U_1 + T_1 = U_2 + T_2 = E_0$$

Slila je konzervativna, če je potencialna in odvisna samo od položaja

Posledica: če je sila konzervativna velja izrek o energiji

Dokaz: $\frac{d}{dt}U(P(t)) = \left(\frac{\partial U}{\partial P}\right)^T \dot{P} = -F\dot{P}$

$$\vec{F}\dot{P} = -\frac{dU}{dt}$$

$$F'\dot{P}' = QF(\vec{c} + Q\dot{P}) = \underset{\substack{\parallel \\ F'}}{QF\cdot\vec{c}} + \underbrace{QF\cdot Q\dot{P}}_{\vec{F}\cdot\dot{P} = -\frac{dU}{dt}} = -\frac{dU}{dt} + m\ddot{P}'\vec{c} =$$

$$= -\frac{d}{dt}\left(U - m\dot{P}'\vec{c}\right)$$

ortogonalna preslikava ohranja skalarni produkt

$$\tilde{U} = U - m\dot{P}'\vec{c}$$

$$\tilde{U}(P', \dot{P}') = U'(P') - m\dot{P}'\vec{c}$$

Izrek: Izrek o energiji je invarianten za Galilejeve transformacije

$$t_o' = T' + \tilde{U}' = \frac{1}{2} m |\vec{c}|^2 + m c Q \dot{P} + T + \tilde{U} - m \dot{P}' \vec{c} =$$

$$= E_o + \frac{1}{2} m |\vec{c}|^2 + m \vec{c} (Q\dot{P} - \dot{P}') = E_o - \frac{1}{2} m |\vec{c}|^2$$

$$T' = \frac{1}{2} m \dot{P}' \dot{P}' = \frac{1}{2} m (\vec{c} + Q \dot{P})(\vec{c} + Q\dot{P}) = \frac{1}{2} m |\vec{c}|^2 + m \vec{c} Q \dot{P} + \frac{1}{2} m |\dot{P}|^2$$

$$\dot{P}' = \vec{c} + Q \dot{P}$$

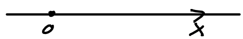
Premočrtno gibanje

= pospešek ima konstantno smer

$$\vec{a} = a(t)\,\vec{c}$$

$$\vec{v} = \int_{t_0}^{t} \vec{a}(t)\,dt + \vec{v}(t_0) = \Big(\int_{T_0}^{t} a(t)\,dt\; \vec{e} + \vec{c}(t_0)$$

Obstaja KS v katerem tir leži na premici



$p = x \quad \dot{p} = \dot{x} \quad \ddot{p} = \ddot{x}$

$v = \dot{x}$ je lahko zdaj tudi negativen

$$m\ddot{x} = f(t, x, \dot{x})$$

omejimo se ko bo f konzervativna

$$m\ddot{x} = f(x)$$

kdaj je sila v odvisnosti položaja potencialna?

(potencialna ⇒ konzervativ

$$f = -\frac{dU}{dx} \Rightarrow U = \int_{x_0}^{x} f(\xi)\,d\xi + U_0$$

sila $f(x)$ je potencialna če je f zvezna

če je $f(x)$ zvezna ⇒ velja izrek o energiji

$m\ddot{x} = f(x)$

$\frac{1}{2}m\dot{x}^2 + U(x) = E_0$

$$\dot{x}^2 = \frac{2}{m}(E_0 - U(x))$$

$$\frac{dx}{dt} = \dot{x} = \pm\sqrt{\frac{2}{m}(E_0 - U(x))}$$

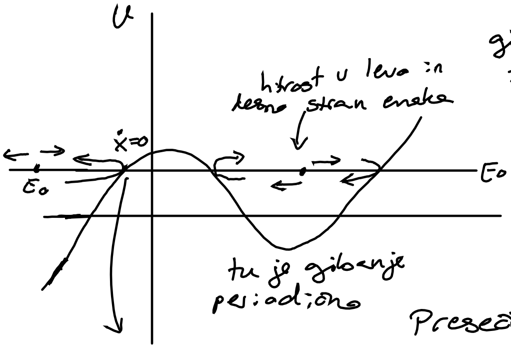
$$\pm\frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = dt$$

$$\operatorname{sgn}\dot{x}\int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \int_{t_0}^{t}dt = t - t_0 \Rightarrow t = t(x) \Rightarrow x = x(t)$$

Lahko dobimo ko $\dot{x} \neq 0$ (sgn $\dot{x}$ je konstanten)

$$t = t_0 \pm \int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

Kvalitativna obravnava gibanja

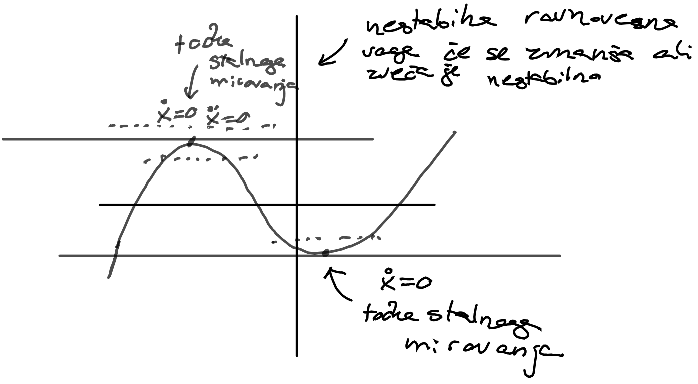


gibanje je možno tam kjer je energijska nivojnica nad potencialom

E_0 nad grafom

$$0 = \dot{x} = \frac{f(x)}{m} = -\frac{dU}{dx}$$

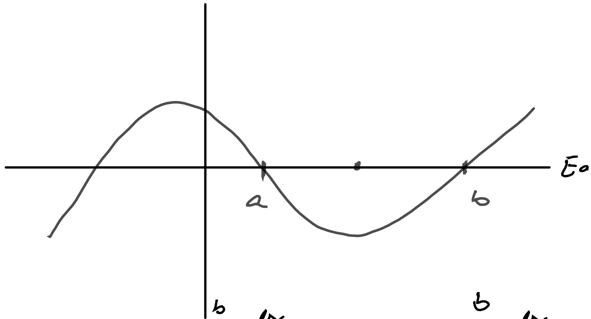
Presečišče je točka trenutnega mirovanja (točka obrata gibanja



Lokalni minimum je stabilna ravnovesna lega (če se energija malo premakne pride do majhnega periodičnega gibanja (majhen odmik od ravnovesne lege)

Prevoj je tudi nestabilna lega

Kakšna je perioda periodičnega gibanja



gibanje: $t_1 = \int_{x_0}^{b} \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}}$ $\quad t_2 = -\int_{a}^{b} \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}}$

$t_3 = \int_{a}^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}}$

$T = t_1+t_2+t_3 = 2\int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0-U(x))}} = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0-U(x)}}$

$\uparrow$
perioda

Ali je res periodično in se ne ustavi prej?

$\int_a^{a+d} \frac{dx}{\sqrt{E_0-U(x)}}$ $\qquad U(x) = U(a) + \frac{dU}{dx}(\xi)(x-a)$

$\frac{dU}{dx}(a) < 0$ a je presečišče

$\exists \delta > 0.\ \xi \in (a, a+\delta) \Rightarrow 2\frac{dU}{dx}(\xi) < \frac{1}{2}\frac{dU}{dx}(a)$

$E_0 - U(x) = -\frac{dU}{dx}(\xi)(x-a) < -2\frac{dU}{dx}(a)(x-a)$

$\Rightarrow$ integral je konvergenten

ker $\int_a^{a+\delta} \frac{dx}{\sqrt{E_0-U(x)}} \leq \frac{1}{\sqrt{-\frac{1}{2}U'(x)(a)}} \int_a^{a+\delta} \frac{dx}{\sqrt{x-a}} < \infty$

Če se energijska nivojnica dotika lokalnega maksimuma, potem ne pridemo do dotikališča v končnem času

Lema $\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi$

Dokaz: $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$

$$b-x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$

$$(b-x)(x-a) = \left(\frac{1}{2}(b-a)\right)^2 \underbrace{(1-\cos^2\vartheta)}_{\sin^2\vartheta} = \left(\frac{1}{2}(b-a)\right)^2$$

$$dx = -\frac{1}{2}(b-a)\sin\vartheta\, d\vartheta$$

$$I = -\int_\pi^0 \frac{\frac{1}{2}(b-a)\sin\vartheta\, d\vartheta}{\frac{1}{2}(b-a)\sin\vartheta} = -\int_\pi^0 d\vartheta = \pi$$

Primer:

Harmonični oscilator $\quad U = \frac{1}{2}kx^2$

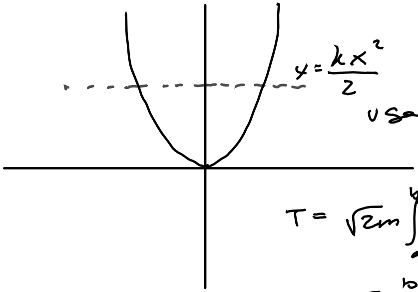
$$f = -kx$$

$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \frac{k}{m} = \omega^2$$

$$\ddot{x} + \omega x = 0$$

$$x = A\cos(\omega t - \delta)$$



vsa gibanja so periodična

$$T = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0 - \frac{1}{2}kx^2}} =$$

$$= \frac{\sqrt{2m}}{\sqrt{\frac{1}{2}k}}\int_a^b \frac{dx}{\sqrt{\frac{2E_0}{k} - x^2}} = 2\sqrt{\frac{m}{k}}\,\pi = \frac{2\pi}{\omega}$$

T perioda je neodvisna od energije (izokronično gibanje)

Primer: $U = ax^2 + bx^{-2}$

$[U]$ dimenzija enota

$$[U] = [ax^2] = [a][x^2] = [a] \underset{\text{enota}}{L^2}$$

$$[U] = [bx^{-2}] = [b] L^{-2}$$

$$\Rightarrow [a] L^2 = [b] L^{-2}$$

$$L^4 = \frac{[b]}{[a]} \Rightarrow L = \sqrt[4]{\frac{[b]}{[a]}}$$

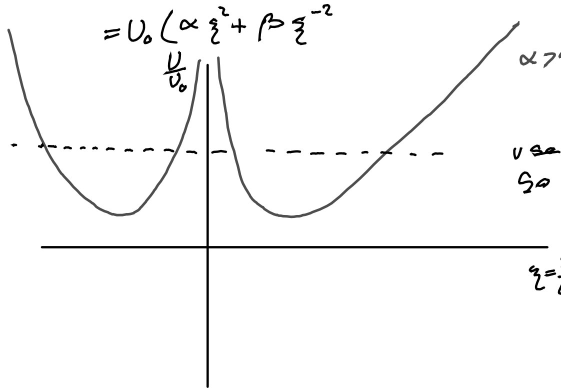
$x = L\xi \qquad L = \left(\frac{b}{a}\right)^{\frac{1}{4}}$

brezdimenzijski ξ

$$U = a\sqrt{\frac{|b|}{|a|}}\,\xi^2 + b\sqrt{\frac{|a|}{|b|}}\,\xi^{-2} = \underbrace{\sqrt{|a||b|}}_{U_0}(\alpha\xi^2 + \beta\xi^{-2}) =$$

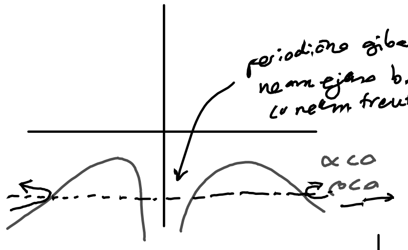
$$a = \underbrace{\frac{a}{|a|}}_{\alpha}|a| \qquad b = \underbrace{\frac{b}{|b|}}_{\beta}|b|$$

$$= U_0(\alpha\xi^2 + \beta\xi^{-2})$$

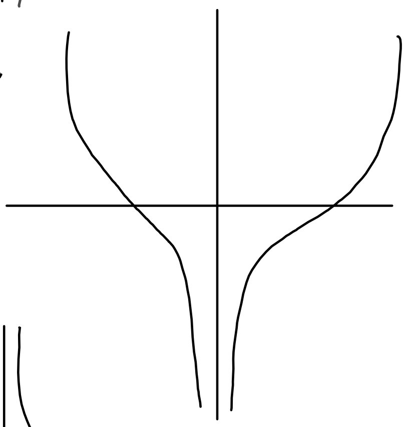


$\alpha > 0, \beta > 0$

vsa gibanje so periodična

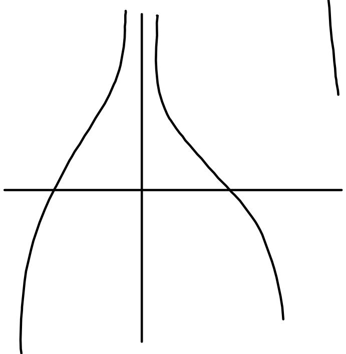


$\alpha < 0$
$\beta < 0$



$\alpha > 0 \ \beta < 0$

vsa gibanje so periodična



$\alpha < 0$
$\beta > 0$

vsa gibanje so neomejena

Računajmo periodo za $\alpha, \beta > 0$

$$U = \sqrt{|a||b|}\left(\xi^2 + \xi^{-2}\right)$$

$$\underset{U_0}{\shortparallel}$$

število

$$E_0 = U_0 e_0$$

$$T = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - (ax^2 + bx^{-2})}} = \sqrt{2m}\int_{\xi_1}^{\xi_2} \frac{L\,d\xi}{\sqrt{U_0 e_0 - U_0(\xi^2 - \xi^{-2})}} =$$

$$= \sqrt{2m}\frac{L}{\sqrt{U_0}}\int_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{e_0 - \xi^2 - \xi^{-2}}} d\xi = \frac{\sqrt{2m}}{\sqrt{U_0}} L \int \frac{\xi\,d\xi}{\sqrt{\xi^2 e_0 - \xi^4 - 1}} d\xi$$

$$u = \xi^2$$

$$= \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1^2}^{\xi_2^2} \frac{\frac{1}{2}\,du}{\sqrt{-u^2 + ue_0 - 1}} = \sqrt{2m}\frac{L}{\sqrt{U_0}}\frac{1}{2}\pi$$

v krajščih jena a in imam kvadratni koren z $a_2 = -1$

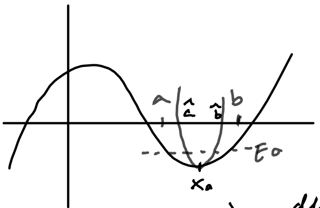
$$U_0 = \sqrt{|a||b|}$$

$$\frac{L}{\sqrt{U_0}} = \frac{|b|^{\frac{1}{4}}}{|a|^{\frac{1}{4}}\,|a|^{\frac{1}{4}}|b|^{\frac{1}{4}}} = 2\sqrt{\frac{1}{|a|}}$$

$$T = \sqrt{\frac{m}{2|a|}}\,\pi$$ opet je izokronično (neodvisna od energije)

in notri ni b

Harmonicna optimizacija periode



$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

$$U(x) = U(x_0) + \underbrace{\frac{du}{dx}(x_0)}_{=0 \text{ ker je lok. minimum.}}(x - x_0) + \frac{1}{2}\frac{du}{dx}(x_0)(x - x_0)^2$$

$$T \approx \sqrt{2m} \int_{\hat{a}}^{\hat{b}} \frac{dx}{\sqrt{E_0 - U_0 - \frac{1}{2}\left(U''(x_0)\right)(x - x_0)^2}} =$$

$$= \sqrt{2m} \frac{1}{\sqrt{\frac{1}{2}U''(x_0)}} \pi = 2\pi \sqrt{\frac{m}{U''(x_0)}}$$