Navadna diferencialna enačba (NDE) je enačba oblike

$f(x, y, y') = 0$ iščemo $y(x)$ da velja

$f(x, y(x), y'(x))$ za $\forall x \in D_y$

(To je v implicitni obliki)

Mi se bomo večinoma ukvarjali s temi ko je podano v eksplicitni obliki, torej ko je $y' = f(x, y)$

Enačba reda $n \in \mathbb{N}$ je oblike $G(x, y, y', \ldots, y^{(n)}) = 0$

Enačba je autonomna, če funkcija G ni odvisna od x. Sicer je neavtonomna.

7.10

1. Za dano družino funkcij poišči pripadajočo DE

a) $y = ce^x$

b) $y^2 = cx$

c) $y = c(x-c)$

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ $f(x,y,c)=0$, $c \in \mathbb{R}$

$y' = ce^x \Rightarrow y = y'$

$y^2 = cx$

$2yy' = c \implies y^2 = 2yy'x$

$y = 2y'x$ za $y \neq 0$

$y' = c$

$y = y'(x-c)$

2) ugani rešitev naslednjih DE

a) $y'' = -y \Rightarrow \{\alpha \cos x + \beta \sin x\}$

b) $y + xy' = \cos x$

c) $xy' = ny$

b)

$$(xy)' = \cos x$$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

c) $xy' = ny$

$$xy' + y = n\cancel{y} + x$$

$$(xy)' = (n+1)y$$

$$y = x^n$$

Metoda izoklin

$y_c = y = y(x, c) \quad x, c \in \mathbb{R}$

$y \in \mathcal{C}^1$

Izoklina je krivulja vzdolž katere ima vsaka člnica družine y enak odvod po x $(y_c'(x))$

$I_\alpha = \{(x,y);\ y = y_c(x) \text{ potem je } y_c'(x) = \alpha$

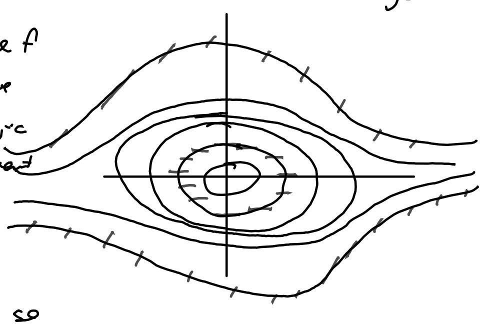
Če imamo DE oblike $y' = f(x, y)$ in pripadajočo družino rešitev $y_c(x)$, potem so izokline ravno nivojnice $\{f = \alpha\}$

t.j. $y_c'(x) = f(x, y_c(x)) = \alpha$

(črne so nivojnice) mi iščemo funkcijo

Postopek:

1. Skiciramo funkcije f



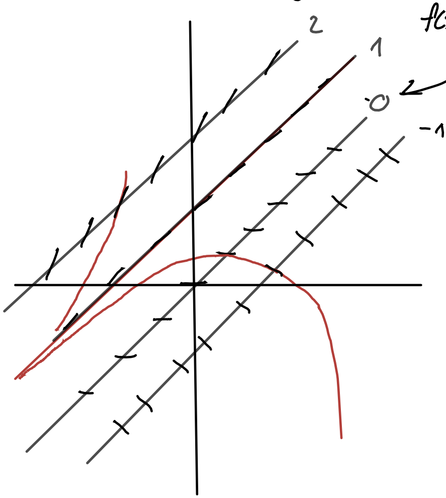
2) vzdolž vsake nivojnice narišemo nekaj koljic ki imajo smerni koeficient enak ~~nivojnica na~~ vrednosti f na nivojnici

3) narišemo krivuljo ki so v presečiščih nivojnic tangentne nanavpičnice

3. Približno skiciraj potek enačbe $y' = y - x$

$f(x,y) = y - x$

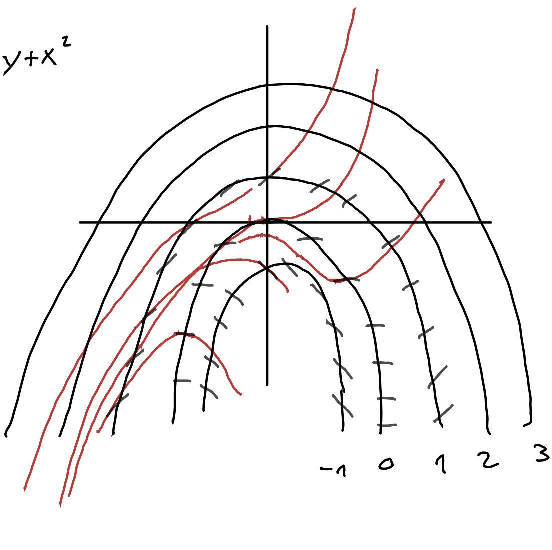
vrednost odvoda



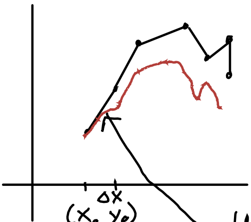
Reške 1:

$y = x + 1$

$y' = y + x^2$



Eulerjeva metoda



$x_1 = x_0 + \Delta x$

$y_1 = y_0 + \Delta x\, f(x, y_0)$

$x_2 = x_1 + \Delta x$

$y_2 = y_1 + \Delta f(x, y_0)$

Izberemo si Δx
in iterativno definiramo

$x_{n+1} = x_n + \Delta x$
$= x_0 + (n+1)\Delta x$

$y_{n+1} = y_n + f(x_n, y_n)\Delta x$

Zakaj ima to smisel?
Če privzamemo da je rešitev dvakrat odvedljiva, velja, da je

$y(x + \Delta x) = y(x) + y'(x)\Delta x + o(\Delta x)$

5) Z Eulerjevo metodo dobi reševan DZ

$y' = f(x)$ kjer je f zvezna pri pogoju $y(0)=0$

neka točka

$y(A) = ?$

$\Delta x = \frac{A}{m}$ za nek $m \in \mathbb{N}$

$A = x_m$

$x_n = 0 + \Delta x_n n$

$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\Delta x \cancel{+} y_0 =$

$= y_{n-1} + f(x_{n-1})\Delta x + y_0$

$= (f(x_{n-2}) + f(x_{n-1}))\Delta x + y_0$ $\overset{0}{\|}$

Riemanova vsota

$$y_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

za $\int_0^A f(x)\,dx$

$$\rightarrow y_m = \sum_{i=0}^{m-1} f(x_i)\Delta x$$

To je očitno rešitev

Vaja (DN):

Najdi rešitev za $y' = 2y$ z eulerjevo metodo

$$\Delta x = \frac{A}{n}$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\Delta x = y_{n-1} + 2y_{n-1}\frac{A}{n} =$$

$$= y_{n-1}\left(1 + \frac{2A}{n}\right)$$

$$y_n = y_{n-2}\left(1 + \frac{2A}{n}\right)^2 = y_1\left(1 + \frac{2A}{n}\right)^{n-1}$$

$$\lim_{n\to\infty}\left(1 + \frac{2A}{n}\right)^{n-1} = \lim_{u\to\infty}\frac{\left(\left(1 + \frac{1}{u}\right)^{u}\right)^{2A}}{\left(1 + \frac{1}{u}\right)} = \underline{\underline{e^{2A}}}$$

$$\frac{1}{u} = \frac{2A}{n} \Rightarrow n = 2Au$$

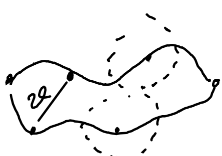
$$y = e^{2x}$$

Fazni prostor je prostor vseh možnih stanj sistema

$y' = f(x,y)$ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

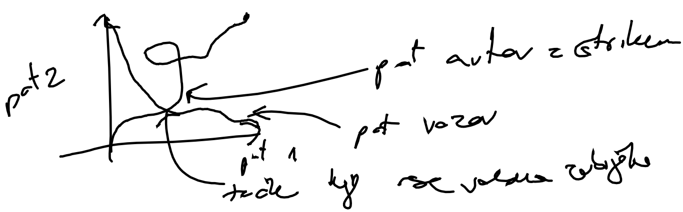
↳ fazni prostor je $\mathbb{R}^3$

6) iz Ljubljane v Maribor vodita dve cesti po katerih lahko iz Ljubljane do maribora pripeljete avtomobila ki sta eden na drugega privezana z vrvjo dolžine $<2l$, ne da bi jo pretrgala. Ali se lahko vozova krožne dolžine radia l, ki vozita vsak v svojo smer srečata, ne da bi trčila



ves

$[0,1] \times [0,1]$



1) 14.10

a) $y' = \frac{x^2}{y} = \frac{dy}{dx}$

$$x^2 dx = y dy$$

$$\frac{1}{3}x^3 + c = y^2$$

$$y = \pm\sqrt{\frac{1}{3}x^3 + c}$$

b)

$2x^2 y y' + y^2 = 2$

$$2x^2 = \frac{2-y^2}{yy'} = \frac{2-y^2}{y\frac{dy}{dx}} = \frac{2-y^2}{y\,dy}\,dx$$

$$\frac{dx}{2x^2} = \frac{y\,dy}{2-y^2}$$

$u = 2-y^2$
$du = -2y\,dy$

$$-\frac{1}{2}\frac{1}{x} = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}(\ln u + c)$$

$$\frac{1}{x} = \ln(2-y^2) + c$$

$$ce^{\frac{1}{x}} = 2-y^2$$

$$y = \pm\sqrt{2-ce^{\frac{1}{x}}}$$

c) $(1+x^2)y' = y$

$\parallel$

$(1+x^2)\frac{dy}{dx} = y$

$\frac{dy}{y} = \frac{dx}{1+x^2}$

$\ln y = \arctan x + c$

$y = ce^{\arctan x}$

$y \equiv 0$

c') $(1+x^3)y' = y$

$y_1 \equiv 0$

2) $\frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)}$

$B = -A$

$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$

x^2: $B + A = 0$

x: $B + C - A = 0$

1: $A + C = 1$

$C = 1 - \frac{1}{3}$

$2B + C = 0$

$\Rightarrow C = -2B = 2A$

$3A = 1$

$\frac{dy}{y} = \frac{1}{3}\frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1}dx$

$\int \frac{x-2}{x^2-x+1}dx = \int \frac{}{(x-\frac{1}{2})^2+\frac{3}{4}}$

2)

a) $y' = \tan(2x+3y-1)$

$y' = \tan z$

$z = 2x+3y-1$

$z' = 2+3y'$

$y' = \frac{z'-2}{3}$

$\tan z = \frac{z'-2}{3}$

$z' = 3\tan z + 2$

1. rešitev : $z' = 0 \Rightarrow z = \arctan(-\frac{2}{3})$

$\Downarrow$

$y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan(-\frac{2}{3}) - 2x + 1}{3}$

$\frac{dz}{dx} = 3\tan z + 2$

$\frac{dz}{3\tan z + 2} = dx$

$u = \tan z$

$du = \frac{1}{\cos^2 z} dz \Rightarrow dz = \frac{du}{1-u^2}$

$x + C = \int \frac{du}{(3u+2)(1-u^2)}$

$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2}$

$u^2: B - A = 0 \qquad A = B$

$u: 3\cancel{C} \cancel{-} + 2B = 0$

$1: 2C + A = 1 \Rightarrow A = 1-2C$

$3C + 2 - 4C = 0$

$C = 2 \Rightarrow A = -3 = B$

$= \int \left(\frac{-3}{3u+2} + \frac{-3u+2}{1-u^2} \right) du =$

$= -\ln(u+\frac{2}{3}) +$

$\frac{A}{1-u} + \frac{B}{1+u} =$

$u: A - B = -3$

$1: A + B = 2$

$+ \int \frac{1}{2}\frac{1}{u-1} + \frac{5}{2}\frac{1}{u+1} =$

$A = B - 3$

$2B - 3 = 2$

$2B = 5$

$B = \frac{5}{2}$

$A = -\frac{1}{2}$

$= -\ln(u+\frac{2}{3}) + \frac{1}{2}\ln(u-1) + \frac{5}{2}\ln(u+1)$

$$y''y - y' = xy'' - (y')^2$$

$$y''(y-x) = y'(1-y')$$

$$y''(y-x) + y'(y'-1) = 0 \qquad y \equiv 0$$

$$a = y'$$

$$b = y - x \qquad b' = y' - 1$$

$$a'b + ab' = 0$$

$$(ab)' = 0$$

$$ab = c \in \mathbb{R}$$

$$y'(y-x) = c$$

$$y' = \frac{c}{y-x} \qquad z = y - x$$

$$z' = y' - 1 = \frac{c}{z} - 1$$

$$\cancel{z = \int\left(\frac{c}{z} - 1\right)dz = c\ln z - z + D}$$

$$\frac{dz}{dx} = \frac{c}{z} - 1$$

$$\frac{dz}{\frac{c}{z} - 1} = dx \qquad x + D = \int \frac{z}{c-z}\,dz = -\int\left(\frac{c}{t} - 1\right)dt =$$

$$t = c - z$$

$$dt = -dz \qquad = -c\ln(c-z) + c - z$$

$$z = y - x$$

$$\cancel{x} + D = -c\ln(c - y + x) + c - y + \cancel{x}$$

$$y = De^{c}e^{\ln(c-y+x)+1}$$

Homogene enačbe

$$F(x,y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xz = y \Rightarrow y' = z + xz' = F(1,z)$$

$$z' = \frac{F(1,z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

A) $y^2 + x^2 y' = xyy'$ $\qquad z = \frac{y}{x}$

$$y'(xy - x^2) = y^2$$

$$y' = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z-1}$$

$$y = zx$$

$$y' = z + z'x = \frac{z^2}{z-1}$$

$$z'x = \frac{z^2 - z^2 + z}{z - 1} = \frac{z}{z-1}$$

$$\frac{dz}{dx} x = \frac{z}{z-1} \Rightarrow \frac{z-1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{x}{x}$$

b) $y = xy' - \sqrt{x^2+y^2}$

$$y' = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} + \overset{\text{sgn}\,x}{\cancel{\ }}\sqrt{1+\frac{y^2}{x^2}} = z + \overset{\text{sgn}}{}\sqrt{1+z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = \cancel{z} + z'x = \cancel{z} + \sqrt{1+z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1+z^2}}{x} \qquad z = \text{sh}\,u \quad dz = -\text{ch}\,u\,du$$

$$\text{sgn}\,x \ln x + D = \int \frac{1}{\sqrt{1+z^2}}\,dz = \int \frac{\cancel{\ }-\text{ch}\,du}{\text{ch}\,u} =$$

$$-\text{arcsh}\,\frac{y}{x}$$

$$\parallel$$

$$\text{arch}\,\frac{y}{x}$$

$$y = x\,\text{ch}(\overset{\text{sgn}\,x}{}\ln x + D)$$

a)

$$m a = F = m \cdot g - k v^2$$

$$m \dot{v} = m g - k v^2$$

$$\dot{v} = g - \frac{k}{m} v^2 = g\left(1 - \underbrace{\frac{k}{mg}}_{\alpha} v^2\right) = g(1 - \alpha v^2)$$

$$\frac{dv}{dt} = g(1 - \alpha v^2)$$

$$\frac{dv}{1 - \alpha v^2} = g\,dt \quad // \int$$

$$g t =$$

$$\frac{A}{1 - \sqrt{\alpha} v} + \frac{B}{1 + \sqrt{\alpha} v} :$$

$v: \sqrt{\alpha} A - \sqrt{\alpha} B = 0$

$A = B$

$1: \quad A + B = 1$

$\Rightarrow A = \frac{1}{2}$

$B = \frac{1}{2}$

$$g t = \frac{1}{\sqrt{\alpha} 2}\left(\ln(1 + \sqrt{\alpha} v) - \ln(\sqrt{\alpha} v - 1)\right) =$$

$$\frac{1}{2\sqrt{\alpha}} \ln\left(\frac{1 + \sqrt{\alpha} v}{1 - \sqrt{\alpha} v}\right) = g t + C$$

$$V_0 = 0 \Rightarrow \quad \cancel{1} \cancel{+\sqrt{\alpha}} v = \cancel{1} - \cancel{\sqrt{\alpha}} v + C$$

$$\Rightarrow 0 = C$$

$$\frac{1 + \sqrt{\alpha} v}{1 - \sqrt{\alpha} v} = \underbrace{e^{2\sqrt{\alpha} g t}}_{D}$$

$$1 + \sqrt{\alpha} v = D - \sqrt{\alpha} D v$$

$$v \sqrt{\alpha}(1 - \sqrt{\alpha} D) = D - 1$$

$$v = \frac{D - 1}{\sqrt{\alpha}(1 - \sqrt{\alpha} D)}$$