

ket: elemen^h kelab^h $\mathbb{Z}[i]$

6.10

\Rightarrow asosiasi: $m+ni$

$$x = u(m+ni) = um + un i$$

ket: so cari jwb:

$$(x+yi)(a+bi) = 1$$

$$xa - yb + i(ay + xb) = 1$$

$$ay + xb = 0 \quad xa - yb = 1$$

$$x = \frac{-ay}{b} \quad -\frac{a^2 y}{b} - yb = 1$$

$$y = \frac{-b}{a^2 + b^2} \quad -\frac{y}{b}(a^2 + b^2) = 1$$

$$x = \frac{a}{a^2 + b^2}$$

$$|a| \leq |a^2| \Rightarrow$$

$$a^2 + b^2 \in \{0, 1\} \Rightarrow a^2 + b^2 = 1$$

$$\Rightarrow a \in \{0, 1\}$$

$$b \in \{0, 1\}$$

$$\text{dan } i \in \{1, -1, i, -i\}$$

$$\text{asoc: } \{m+ni, -m-ni, n-mi, -n+mi\}$$

$$d \in \mathbb{Z}$$

$$\mathbb{Z}[\sqrt{d}] = \{m+n\sqrt{d}; m, n \in \mathbb{Z}\}$$

- 1) Pokaži $\mathbb{Z}[\sqrt{d}]$ je podkolebar \mathbb{C}
- 2) Množica $\mathbb{Q}[\sqrt{d}] = \{g+r\sqrt{d}; g, r \in \mathbb{Q}\}$ je podpodje \mathbb{C} generirano z $\mathbb{Z}[\sqrt{d}]$
-

1) $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{C}$

zapišot z seštevanje, množenje, evle
z množenje:

$$(m+n\sqrt{d})(x+y\sqrt{d}) = mx+nyd+\sqrt{d}(nx+ym)$$

$$1 \cdot 1 = 1 = 1+0 \cdot \sqrt{d}$$

$$\text{seštevanje: } (m+n\sqrt{d}) - (x+y\sqrt{d}) = (m-x) + (n-y)\sqrt{d}$$

2) množenje

$$\frac{m+n\sqrt{d}}{x+y\sqrt{d}} = \frac{(m+n\sqrt{d})(x-y\sqrt{d})}{x^2-yd} = \frac{c+e\sqrt{d}}{x^2-yd} \checkmark$$

↙ faktor

Automorfizmi $\mathbb{Z}[\sqrt{d}] = \{id; \sqrt{d} \mapsto \pm\sqrt{d}\}$

Norma $N(x) = x\sigma(x)$

$$N(g+r\sqrt{d}) = (g+r\sqrt{d})(g-r\sqrt{d}) = g^2 - r^2d$$

$$3) \quad \forall x, y \in \mathbb{Z}[\sqrt{a}] : N(xy) = N(x)N(y)$$

$$\begin{aligned} N(xy) &= xy\sigma(xy) = xy\sigma(x)\sigma(y) = x\sigma(x)y\sigma(y) \\ &= N(x) \cdot N(y) \end{aligned}$$

$$4) \quad x \in \mathbb{Z}[\sqrt{a}] \text{ ist invertierbar} \Leftrightarrow N(x) = \pm 1$$

$$xy = 1 \Rightarrow N(xy) = N(1)$$

$$N(x)N(y) = 1$$

$$N(x) = \frac{1}{N(y)}$$

$$N(y) \in \mathbb{Z} \wedge N(x) \in \mathbb{Z}$$

$$\Rightarrow N(y) \in \{\pm 1\}$$

$$\Rightarrow N(x) \in \{\pm 1\}$$

6)

$$p \in \mathbb{P}; \quad N(x) = \pm p \Rightarrow x \text{ nicht invertierbar}$$

in $N(x) = \pm p$

rechen da x invertierbar aber nicht invertierbar

$$x = ab$$

$$N(x) = N(a)N(b) \neq \pm p \Rightarrow N(a) = 1 \vee N(b) =$$

$$4) \quad \Leftarrow N(x) = 1, \quad x\sigma(x) = \pm 1 \Rightarrow x^{-1} = \sigma(x)$$

5) zu prüfen

hier da $x \in \mathbb{Z}[\sqrt{a}] \Rightarrow a$ invertierbar.

$d \mid -1 \Rightarrow 1, -1$ sta edine obrnljive v $\mathbb{Z}[\sqrt{d}]$

$$x \text{ obrnljiv} \Rightarrow N(x) N(y) = \pm 1$$

$$N(a+b\sqrt{d}) = (a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - b^2d = a^2 + b^2|d|$$

\downarrow
1

$$\text{če je } b^2 > 1 \Rightarrow N > 1$$

$$\Rightarrow b = 0$$

a^2 mora biti 1 $\Rightarrow a = \pm 1$ ote edini;

možnosti
in ueno da sta deliljiva to ne je rezultat

Pokaži da so $1+i$, $7+8i$, 3 nerazcepni v $\mathbb{Z}[i]$

Recimo da so razcepni:

$$1+i = xy \quad N(xy) = x\sigma(x)y\sigma(y) = 2$$

prejeto

$p \rightarrow$ nerazcepno

$$N(7+8i) = 49+64 = \text{velika} = 113 \text{ preostalo}$$

~~Preostalo~~ Nema je 9

$$3 = xy \quad N(xy) = x\sigma(x)y\sigma(y) = 9$$

vsej 2 mora biti 1

$$\sigma(\sigma(x)) = N(x) = N(\sigma(x)) = \sigma(x)\sigma(\sigma(x)) = x\sigma(x)$$

$$N(x) = n \Rightarrow n \text{ razcepa}$$

$$N(x) = n = x\sigma(x)$$

6) Pāšāvisē delīvējē elementa $2 \in \mathbb{Z}[i]$

$$x|2 \Leftrightarrow 2 = xy \text{ ar } nd \neq y$$

$$N(x, y) = 4 \quad N(x)N(y) = 4$$

$$x\sigma(x)y\sigma(y) = 4 = 2 \cdot 2 = xy\sigma(xy) = 2\sigma(xy)$$

$$\Rightarrow \sigma(xy) = 2 = \sigma(x)\sigma(y) \Rightarrow$$

$$N(x) = 2 \quad x = \pm 1 \pm i$$

$$x = \pm 2$$

$$x = \pm 2i$$

go use normu

1)

$$a = 3 + 4i \quad \mathcal{J} = m^2 + n^2$$

$$b = 1 - 3i$$

$$|a| = 3 + 16 = 19$$

$$|b| = 1 + 9 = 10$$

magari dell'10

$$a = kb + r$$

$$\frac{a}{b} = k + \frac{r}{b}$$

$$\in \mathbb{Z}[i] \quad \in \mathbb{Q}[i]$$

$$a = 3 - 4i$$

$$b = 1 - 3i$$

$$\frac{a}{b} = \frac{(3-4i)(1+3i)}{10} = \frac{3+12+i(3-4)}{10} = 1 + \frac{1}{2} + \frac{1}{2}i$$

$$\frac{1}{2}(1+i)(1-3i) = \frac{1}{2}(1+3+i(1-3)) = 2-i$$

$$a = b + 2-i \quad /: (2-i)$$

$$\frac{a}{2-i} = \frac{b}{2-i} + 1 \quad \Rightarrow$$

$$\frac{a}{2-i} - \frac{b}{2-i} = 1 \quad \Rightarrow \quad a - b = 2-i$$

2)

$$a=6$$

$$\mathbb{Z}[\sqrt{-5}]$$

$$b=2+\sqrt{-5}$$

$$c=2 \Rightarrow \frac{N(a)}{2} = \frac{36}{2} = 3 \quad \frac{N(b)}{2} = \frac{4+4\cdot 5}{2} = \frac{4\cdot 6}{1+\sqrt{5}}$$

$$\text{E.g. } N(a) \nmid N(b) \Rightarrow \text{cancel out } d, d|a, d|b \Rightarrow \begin{matrix} 6 \cdot 2 \cdot 3 \\ \parallel \quad \parallel \\ 6^2 \quad 6 \cdot 4 \\ \parallel \quad \parallel \\ 6^2 \quad 6 \cdot 2 \cdot 3 \end{matrix}$$

$$d(c) = 1 + 5 = 6$$

$$N(\gcd(a, b)) = \gcd(N(a), N(b)) = \gcd(36, 24) = 12 = x^2 \cdot 5y^2$$

$$y^2 \in \{1, 2, 3\}$$

$$\downarrow$$

$$x^2 = 2 \cdot 3 \quad x^2 = x$$

Take ~~negative~~

3) UFD \Rightarrow gcd exists

$$a = \prod_{i \in I} p_i^{k_i}$$

$$b = \prod_{i \in I} p_i^{l_i}$$

$$d = \gcd(a, b) = \prod_{i \in I} p_i^{\min\{k_i, l_i\}}$$

$$d|a \wedge d|b$$

Not b/c $c = \prod_{i \in I} p_i^{m_i}$ $c|a \wedge c|b$

$$a = c\alpha \quad \alpha = \prod p_i^{n_i}$$

$$\text{Q} \quad a = \prod p_i^{n_i + m_i} \Rightarrow \prod p_i^{k_i} \Rightarrow m_i \leq k_i$$

$\forall i, m_i < l_i \Rightarrow f_i$
 $c|d$

$$\mathbb{Z}[\sqrt{-2}] \text{ additiv}$$

a) $\forall a, b \in \mathbb{Z}[\sqrt{-2}]$ $a = gb + r$ $\wedge r = 0 \vee \sigma(r) < \sigma(b)$

b) $\sigma(a) \leq \sigma(g, b) \forall a, b \in \mathbb{Z}[\sqrt{-2}]$

$$\sigma: m + \sqrt{-2}n = m^2 + 2n^2$$

Vergleiche b und a

$a, b \in \mathbb{Z}[\sqrt{-2}]$ nehmen $a = x + \sqrt{-2}y$ $b = m + \sqrt{-2}n$

$$\frac{a}{b} = g_1 + g_2 \sqrt{-2}$$

$$a = ([g_1] + [g_2]\sqrt{-2})b + ([g_1 - [g_1]] + ([g_2 - [g_2]]\sqrt{-2}))b$$

DN:

Naj bo $K[X]$ UFD

Potem me polinom n^{ek} endično faktoriziramo
Torej je tudi $K[X]$ UFD

Naj bo K UFD

Naj bo poljuben polinom

$$\begin{aligned} \text{Razcepimo pol} &= \prod_{k=1}^r \left(\sum_{i=1}^n a_{ik} x^i \right) = \\ &= \prod_{k=1}^r (a_{nk} x^n + \dots + a_{0k}) = x^n \underbrace{\left(\prod_{k=1}^r a_{nk} \right)} + \dots \end{aligned}$$

$$\text{cont}(fg) = \text{cont}(f) \text{cont}(g)$$

$$p \mid \text{cont}(f, g) \Rightarrow \gcd(a_0, \dots, a_n) = p$$

$$fg = 0 \vee K/p[X]$$

$$\Rightarrow p=0 \vee g=0$$

$$(ax+tb)(cx+td) = acx^2 + x(cb+ad) + btd$$

$$\Rightarrow p \mid \text{cont } f \vee p \mid \text{cont } g$$

Remarque importante: FOXT

plus l'élément est représenté $a = p \cdot q \in \mathbb{K}^*$

Nous avons donc $\text{conf}(a) = \text{ker } a = p \cdot q / \text{conf}$

$$\text{conf}(a) = \text{conf}(p) \cdot \text{conf}(q) = a \cdot \text{conf}$$

$$\text{conf}(a) = a \cdot \text{conf} \Rightarrow \text{conf}(a) = a \cdot \text{conf}$$

$$a \mid \text{conf}(p \cdot q) \Rightarrow \frac{p}{x} \cdot \frac{q}{y} = \frac{pq}{xy} = f$$

$$\frac{p}{x} \mid \frac{q}{y}$$

\Rightarrow

$$f \mid p \cdot q \Rightarrow f \mid p \vee f \mid q$$

$$p \cdot q = p \cdot q \quad \text{conf}(p \cdot q) = \text{conf}(p) \cdot \text{conf}(q)$$

$$p \cdot q \text{ est premier} \Rightarrow f \mid p \vee f \mid q$$

$$\Rightarrow \text{si } p \cdot q \text{ est premier}$$

2) $K[X]$ glavni idealni prsten $\Leftrightarrow K$ je polje

$$a \in K - \{0\}$$

$$\exists b \text{ } ab = 1$$

$$(a) \text{ je ideal } (a) = \{af : f \in K[X]\}$$

↑ $1+\sqrt{3}$ nerazcepen v $\mathbb{Z}[\sqrt{-3}]$,
ampak ni prazemant

$1+\sqrt{3}$ nerazcepen

Recimo da je $(1+\sqrt{3}) = \alpha \cdot \beta$

$$N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta) = 1+3=4$$

$$N(\alpha) = N(\beta) = \pm 2$$

če bi bila 1 bi bila deljiva

$$N(a+\sqrt{3}b) = a^2 + 3b^2 = \pm 2$$

$$b=0$$

$$a^2 = \pm 2 \text{ ne obstaja}$$

Ni prazemant

$$(a+b\sqrt{3})(1+\sqrt{3}) = a + b\sqrt{3} + a\sqrt{3} + b =$$

Vsek dement deli svojo normo

$$(1+\sqrt{3})(1-\sqrt{3}) = 4 = 2 \cdot 2$$

$$(1+\sqrt{3}) \nmid 2$$

$$\text{Recimo da } k(1+\sqrt{3}) = 2$$

$$N(2) = 4 \Rightarrow$$

$$N(k) = \pm 1$$

$$\Rightarrow k = \pm 1$$

Domena relage
1)

$$w = \frac{(1+\sqrt{-19})}{2} \quad \mathbb{Z}[w] = \{a+bw : a, b \in \mathbb{Z}\}$$

$$N(a+bw) = |a+bw|^2 = a^2 + ab + 5b^2$$

2 in 3 nerazcepná

$$N(2) = 4$$

$$2 = ab, a, b \text{ neobr.} \Rightarrow N(a) = N(b) = 2$$

$$a = a + bw$$

$$N(a) = a^2 + ab + 5b^2 = 2$$

$$b \neq 0 \Rightarrow 5b^2 > 2 \Rightarrow N(a) > 2 \quad \times$$

$$N(3) = 9 \quad 9 = x \cdot y \quad 9 = 3 \cdot 3 \Rightarrow N(x) = N(y) = 3$$

$$N(x) = 3 \quad x = a + bw$$

$$a^2 + ab + 5b^2 = 3$$

$$b \neq 0 \Rightarrow 5b^2 > 3 \Rightarrow N(x) > 3 \quad \times$$

$$N(x) = 1 \Rightarrow x \in \underline{\underline{\mathbb{Z}\{1, -1\}}}$$

$$a^2 + ab + 5b^2 = 1$$

$$b \neq 0 \Rightarrow 5b^2 > 1 \quad \times \Rightarrow b = 0$$

$$a^2 = 1 \Rightarrow a \in \mathbb{Z}\{1, -1\} \Rightarrow x \in \mathbb{Z}\{1, -1\}$$

Modul:

$K = \{ \text{zgodnje tritelne matrice nad } F \}$

$M = \begin{bmatrix} F \\ F \end{bmatrix}$ je lev K -modul
istočasno podmodule od M

$$KM \subseteq M$$

$$N = \{ \lambda \begin{bmatrix} u \\ v \end{bmatrix} ; \lambda \in F \} \text{ za } u, v \in F$$

$$\begin{bmatrix} a & b \\ c & c \end{bmatrix} \cdot \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} au + bv \\ cu \end{bmatrix} \in N$$

$$\lambda au + \lambda bv = \mu u$$

$$\lambda cv = \mu v \Rightarrow \lambda c = \mu$$

$$\lambda au + \lambda bv = \lambda cu$$

$$au - cu + bv = 0$$

$$bv = (c-a)u$$

DN negrej

$\mathbb{Z}[\omega]$ n: euclidisk

$$(\exists \delta: \mathbb{Z}[\omega] - \{0\} \rightarrow \mathbb{N} \cup \{0\})$$

$$\omega = gx + r \Rightarrow r \in \{1, -1, 0\}$$

$$r=0 \Rightarrow \omega = gx \Rightarrow x \text{ aso d'iren } z \omega$$

$$N(x) = N(\omega) = 5$$

$$2 = g'x \pm 1 \Rightarrow x \in \{\pm 2, \pm 3\} \Rightarrow N(x) \in \{4, 9\}$$

$$r=1 \Rightarrow \omega = gx + 1$$

~~X~~

$$-1 + \omega = gx \Rightarrow 1 - 1 + 5 = N(g) N(x) = 5$$

$$r=-1 \Rightarrow 1 + \omega = gx \Rightarrow 7 = N(g) N(x) \Rightarrow N(x) = 5 \quad \text{N(x)=5} \quad \text{X}$$

$$\Rightarrow N(x) = 7 \quad \text{X}$$

$\mathbb{Z}[W]$ je glavni

$$(\forall \alpha \in \mathbb{D} - \mathbb{Z}[W], \exists p, q \in \mathbb{Z}[W], 0 < |p\alpha - q| < 1) \\ \Rightarrow \mathbb{Z}[W] \text{ glavnikolobar}$$

Mej $b \in I$ poljuken : del. Mej $b \in I$ tak da

$$\forall x \in I, N(x) \geq N(b)$$

$$\text{Mej } b \in I \text{ poljuken } \alpha = \frac{a}{b} \quad a = \alpha b$$

$$\alpha \in \mathbb{Z}[W] \Rightarrow b|a \quad \forall a \in I \Rightarrow I = (b)$$

$$\alpha \in \mathbb{Z}[W] \text{ in } \{\text{pozj}\} \Rightarrow 0 < |p\frac{a}{b} - q|^2 < 1 \quad / |b|$$

$$0 < |p\underbrace{a}_{\in I} - q\underbrace{b}_{\in I}|^2 < |b|^2$$

$\in I \quad \in I \quad \times$

DN ze teore:

$$d = 4k + 1$$

$$(a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2 = 4(2 - k)$$

$$b^2 = 1:$$

$$a^2 - 4k - 1$$

$$a = 3$$

$$8 - 4k = 4(2 - k)$$

$$\begin{array}{l} \uparrow \\ b^2 = 1 \\ a^2 = 9 \end{array}$$

$$\{(x-1)p(x)\} = \sum_{n=0}^{\infty} a_n x^n = 0$$

Tide is ideal

$$\sum_{n=0}^k a_n x^n$$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 : (x-1) = a_n x^{n-1} + a_{n-1} x^{n-2} + \dots + a_1$$

ind. prep. edge of pol. result

$$ax - a = a(x-1) \quad \checkmark$$

Assume for $j \in \mathbb{N}$ $\exists B_j \subseteq \{t_1, \dots, t_j\}$

Let B_j be the complement

$B_j = \{t_1, \dots, t_j\} \setminus B_j$

$$\sqrt{t_1 + \dots + t_j}$$

$$\sqrt{t_1 + \dots + t_j} \leq \sqrt{t_1 + \dots + t_j} \sqrt{t_1 + \dots + t_j}$$

$$\sqrt{t_1 + \dots + t_j}$$

$$M = \mathbb{Z} \mathbb{Z}_{12}$$

$$\mathbb{Z}_2 = \{0, 6\} \quad \text{and } 6 \text{ are}$$

$$\mathbb{Z}_3 = \{0, 4, 8\} \quad \text{and } 4 \text{ are}$$

$$\mathbb{Z}_4 = \{0, 3, 6, 9\} \supseteq \mathbb{Z}_2$$

$$\mathbb{Z}_6 = \{0, 2, 4, 6, 8, 10\} \supseteq \mathbb{Z}_2, \mathbb{Z}_3$$

$$3) \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{12}), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_4), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_{12}, \mathbb{Z}_{12})$$

$$\mathbb{Z} \longrightarrow \mathbb{Z}_{12}$$

$$0 \mapsto 0 \quad 1 \text{ netenke dobi}$$

$$1 \mapsto k \text{ kedi} \Rightarrow \text{ima } 12 \text{ konstanta}$$

K klobber, KM modul \Rightarrow

$$\operatorname{Hom}(K, M) = \{x \mapsto xM : m \in M\} \cong {}_K M$$

$$\mathbb{Z}_3 \rightarrow \mathbb{Z}_4$$

$$0 \mapsto 0$$

1.

$$1 \mapsto 1$$

$$2 \mapsto 2$$

$$3 \mapsto 0$$

$$4 \mapsto 1$$

2.

$$1 \mapsto 2$$

$$2 \mapsto 1$$

$$3 \mapsto 0$$

$$4 \mapsto 2$$

3.

$$n \mapsto 0$$

$$\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$$

$$0 \mapsto 0$$

$$1 \mapsto a$$

$$12 \mapsto 12, \text{ mod } 12$$

K kolobar. K enasteven $\Leftrightarrow K$ obzno
 \Rightarrow Redukcija je enostaven, a enostaven
 \Rightarrow obratno

$$Ka = K \Rightarrow \exists k \in K. ka = 1$$

\Leftarrow K je podoben m

Vaje za rezej

1. Naj bo M K -modul $M \neq \{0\}$ enostaven $\Leftrightarrow M = K_m$

$\forall m \in M$

Naj bo M enostaven. Edina podmodula sta $\{0\}$ in M

\Rightarrow Naj bo $m \in M$ poljuben K_m je podmodul!

$$\alpha, \beta \in K \quad \alpha k m - \beta h m = (\alpha k - \beta h) m \in K_m$$

$\Leftarrow M = K_m$ za $\forall m \in M$.

Naj bo M podmodul.

$\forall n \in \mathbb{N}$. $K_n = M$ torej $N = M$

2. $n \log n$

levi: k moduli M monostaven $\Leftrightarrow M \equiv k \pmod{n}$, gdje r je
 \downarrow sk. maksimalen levi ideal

$\Rightarrow M_j \in M$ enostaven, $a \in M$

$$\varphi: K \longrightarrow M \quad \varphi \text{ surjektiven po prejšnji nalogi}$$

$k \in \rho = \{k; k_a = 0\} =_k I$ maksimale

$$I \subsetneq J \subsetneq K$$
$$K \rightarrow M$$
$$J \xrightarrow{\text{sur}} \mathcal{L}_*(J)$$
$$I \rightarrow 0$$

$\varphi_*(J) = M \Rightarrow m = ja$
 $\Rightarrow j = ja \Rightarrow j(1-a) = 0$
 $1-a \in \ker \varphi$
 $\Rightarrow 1-a \in J$
 $(1-a) + a = 1 \in J$
 $\Rightarrow J = R$

$$\Leftarrow \text{Naj } b \in {}_K M \cong \frac{K}{I} : I \text{ maksimalen levi ideal}$$
$$\begin{array}{ccccc} O & < & N & < & M \\ \uparrow & & \uparrow & & \uparrow \\ I & < & J & < & K \end{array} \quad \begin{array}{l} \varphi \text{ surj} \\ J = \varphi^*(N) \end{array}$$
$$N \subseteq \varphi_*(J) = \varphi_*(\varphi^*(N)) \Rightarrow \varphi_*(J) = N$$
$$K_A \rightarrow M$$
$$1 + \Gamma \mapsto 0 \in M - \{0\}$$
$$1, I \mapsto e \in M - \{0\} \Rightarrow k \mapsto ke$$
$$x + I = x(1 + I) \mapsto xe$$

Menostaven

Menostaven $I = j \quad V \quad j = k$

$$\begin{array}{ccccccc} I & & J_2 & & J_1 & & K \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & \varphi_*(J_2) & = & \varphi_*(J_1) & < & M \end{array} \quad \text{by } \varphi(J_2)$$
$$0 = \frac{\varphi_*(J_1)}{\varphi_*(J_2)} < M/\varphi(J_2)$$
 $\zeta \zeta$

Fine nebože
oparte

Od zadnjic:

$$\text{End}(M \oplus N) \cong \begin{bmatrix} \text{End}(M) & \text{Hom}(N, M) \\ \text{Hom}(M, N) & \text{End}(N) \end{bmatrix}$$

1) določi kalobarje endomorfizmov

$$\text{End}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3), \text{End}_{\mathbb{Z}}(\mathbb{Z}_6 \oplus \mathbb{Z}_3)$$

$$\text{End}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) = \begin{bmatrix} \text{End } \mathbb{Z}_2 & \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_2) \\ \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) & \text{End } \mathbb{Z}_3 \end{bmatrix}$$

$$\text{End } \mathbb{Z}_2: \begin{array}{cc} 0 \mapsto 0 & 1 \mapsto 0 \\ 1 \mapsto 1 & 0 \mapsto 1 \end{array} \cong \mathbb{Z}_2$$

$$\text{Splošno: } \text{End } \mathbb{Z}_p \cong \mathbb{Z}_p \Rightarrow \mathbb{Z}_3 \cong \mathbb{Z}_3$$

$$\text{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) \cong \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \xrightarrow{1 \mapsto \text{red demerita 3 ne abstrakcijski}} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$$

$$\text{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3): \text{End}(\text{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus \mathbb{Z}_3) =$$

$$= \begin{bmatrix} \text{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) & \text{Hom} \\ \text{Hom} & \text{End } \mathbb{Z}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_6 & \text{Hom} \\ \text{Hom} & \mathbb{Z}_3 \end{bmatrix}$$

$$\text{Hom}(\mathbb{Z}_6, \mathbb{Z}_3) \cong_{\mathbb{Z}} \text{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_3) \cong$$

$$\cong \underbrace{\text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3)}_0 \oplus \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_3) \cong \mathbb{Z}_3$$

$$\text{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_6 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{1 \mapsto 2} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \text{ se ne izide}$$

$$\begin{array}{c} x \mapsto (2, x \bmod 3) \\ \mathbb{Z}_3 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3 \rightarrow \mathbb{Z}_6 \\ 1 \mapsto (1, 0) \mapsto 4 \end{array} \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \begin{array}{c} \mathbb{Z}_6 \rightarrow \mathbb{Z}_2 + \mathbb{Z}_3 \rightarrow \mathbb{Z}_3 \\ (1, 0) \mapsto 2 \\ x \mapsto 2x \bmod 3 \end{array} \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$$

$$= \left[\begin{array}{cc} x \mapsto 2x \mapsto (2x, 0) = 4x \\ \underline{2 \mapsto 8 \bmod 3 = 2} \end{array} \right]$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\updownarrow \mathbb{Z}_6 \rightarrow \mathbb{Z}_3$$

$$\begin{bmatrix} 0 & x \mapsto 4x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x \mapsto 2x & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} \mathbb{Z}_3 \rightarrow \mathbb{Z}_6 \\ x \mapsto 8x \end{array} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\forall \mathbb{Z}_3 \rightarrow \mathbb{Z}_6$$

$$\mathbb{Z}_6 \rightarrow \mathbb{Z}_3$$

$$1 = 1 \mapsto (1, 0) \mapsto 4$$

$$k = 1 \mapsto k \bmod 3$$

$$2 = 1 \mapsto (2, 0) \mapsto 2$$

$$k = 1 \mapsto (k, 0) \mapsto \dots$$

Hilfje:

$$\mathbb{Z}_{18} = \mathbb{Z}_6 \oplus \mathbb{Z}_3 = \mathbb{Z}_2 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3)$$

$$\text{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_2 & \text{Hom}(\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_2) \\ \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3) & \text{End}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \cong \mathbb{Z}_2 \oplus M_2(\mathbb{Z}_3)$$

Bimoduli

$$K^{\circ p} = \{ x^{\circ p} \mid x \in K \} \quad x^{\circ p} \cdot y^{\circ p} = (yx)^{\circ p}$$

$$\text{Grupa: } G \cong G^{\circ p} \\ x^{\circ p} = x^{-1}$$

${}_K M \cong M_{{}_K K^{\circ p}}$ vsak levi modul lahko predstavimo kot desni K modul

Bimodul je ${}_K M_S$, povezuje prave asociativnosti: K, S kolobarja ${}_K M$ je levi K -modul M_S je desni S -modul zakon

$$x \in K, y \in S$$

$$m \in M \quad \underbrace{(x \cdot m)}_{\in M} y = x(m \cdot y)$$

Primeri

1) ${}_K K_K$ je bimodul

2) ${}_K M_{{}_K K^{\circ p}}$ ni bimodul (razen če K komut.)

$$\begin{aligned} m \cdot x^{\circ p} &= xm & x(m \cdot y^{\circ p}) &= xy m \\ & & \text{"} & \\ (xm) \cdot y^{\circ p} &= yxm \end{aligned}$$

3) ${}_K M_{{}_K \text{End}(M)^{\circ p}}$ je bimodul

$$(xm) \cdot \varphi^{\circ p} = \varphi(xm) = x \varphi(m) = x(m \cdot \varphi^{\circ p})$$

2) Maj boste ${}_K M_S$ in ${}_K N_R$ bimodule (K, S, R klobarji)

Premisi, da imajo homomorfizmi: $M \rightarrow N$ reverse
 strukturo (S, R) -bimodule \nwarrow homomorfizmi:
 kot levi K -module
 zgubi strukturo K -module

$${}_S \text{Hom}(M, N)_R$$

$$s \in S \quad s \cdot f := m \mapsto f(ms)$$

$$r \in R \quad f \cdot r := m \mapsto f(m)r$$

$\text{Hom}(M, N)$ je levi S modul

$$(s+t) \cdot f = (m \mapsto f(m(st+t))) = (m \mapsto f(ms) + m \mapsto f(mt)) \\ \parallel \\ f(ms) + f(mt) = sf + tf$$

$$1 \cdot f = (m \mapsto f(m)) = f$$

$$S(f+g) = \checkmark$$

$$\forall m. (st) \cdot f(m) = f(mst) = f((ms)t) =$$

$$= (tf)(ms) = s(tf(m)) = s(tf)(m)$$

$$\uparrow \\ \text{al: je to vredn} \quad \underline{(tf)(xm) = x((tf)(m))}$$

$$(tf)(xm) = f(xm)t = f(x(mt)) = x f(mt)$$

Podobno za desni R -modul

$$S(fr)(m) = (fr)(ms) = f(ms)r = (sf(m))r$$

Lev K -modul ${}_K M$. **Dual** je $M_K^* = \text{Hom}({}_K M, {}_K K)$
 K je domena bimodul, je tudi: $M_K^* \xleftarrow{\text{Funktional desni } K \text{ modul}}$

$$M = \begin{bmatrix} F \\ F \end{bmatrix}$$

3) Dobi $(M_2(F) M)^*$

$$M_{M_2(F)}^* = \text{Hom}(M_2(F) M, M_2(F)_{M_2(F)}) \cong$$

$$\begin{bmatrix} F & F \\ F & F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & a \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \cong M \oplus M$$

$$\cong \text{Hom}(M, M \oplus M) = \text{End}(M) \oplus \text{End}(M) \cong F^2$$

$\uparrow \quad \nearrow$
 To sta obsega (shkrova lena)

$$M = \begin{bmatrix} F & \\ & F \end{bmatrix} \quad M \cong \begin{bmatrix} F & F \\ & F \end{bmatrix}$$

$$K = \begin{bmatrix} F & F \\ F & F \end{bmatrix} \quad K \cong M \oplus M \text{ ristična + vristica}$$

$$\begin{bmatrix} 0 & F \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$$

struktura
desnega
K-modula

$$M^* = \text{Hom}(M, K) \cong \text{Hom}(M, M) \oplus \text{Hom}(M, M) \cong F \oplus F$$

↑
ni desni K-modul.

Izgubimo strukturo modula

$$(*) \quad F \cong \text{Hom}(M, M) = \text{End } M$$

$\text{End}(M)$ obseg, če M enostaven

$$F \cong \text{End}(M)$$

2-dim vektorski prostor

$$\geq \text{V posebenem: } \varphi \in \text{Hom}_K(M, M) \subseteq \text{Hom}_F(M, M) \cong M_2(F)$$

$$\varphi(Ax) = A\varphi(x) \quad \varphi = B \Rightarrow$$

$$BAx = ABx \text{ za } \forall A \in K \text{ in } \forall x \in M \Rightarrow BA = AB \Leftrightarrow \forall A \in K.$$

$$B \in Z(M_2(F)) = F \cdot I \quad \begin{matrix} \swarrow \text{diagonalne} \\ \text{matrice} \\ \cong F \end{matrix}$$

≅ očitno

$$(**) \quad M^*_K \text{ je } (F \oplus F)_K$$

$$\varphi \in \text{Hom}(M, K)$$

$$\varphi(m) = (\varphi_1(m), \varphi_2(m)) = (\varphi(m) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \varphi(m) \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

← očitno

$$\varphi(m) = \begin{bmatrix} \varphi_1(m) \\ \varphi_2(m) \end{bmatrix}$$

$$\varphi(Am) = \begin{bmatrix} \varphi_1(Am) \\ \varphi_2(Am) \end{bmatrix}$$

$$A\varphi(m) = \begin{bmatrix} A\varphi_1(m) \\ A\varphi_2(m) \end{bmatrix} \quad \varphi_{1,2} \text{ skalarne vektorske identitete}$$

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \varphi.A = (m \mapsto \varphi(m).A) \quad \begin{matrix} \swarrow M^* \\ \text{dano po} \\ \text{definiciji} \end{matrix}$$

$$[\varphi_1, \varphi_2].A = \varphi_1.a + \varphi_2.b + \varphi_1.c + \varphi_2.d = \varphi_1(a+c) + \varphi_2(b+d)$$

↑ tako sklopamo

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \varphi(m) = [\varphi_1(m), \varphi_2(m)] \quad m = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{matrix} \swarrow \text{skalarje} \\ = [\lambda_1 m, \lambda_2 m] = \begin{bmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y & \lambda_2 y \end{bmatrix} \end{matrix}$$

$$\varphi(m).A = \begin{bmatrix} a\lambda_1 x + b\lambda_2 x & c\lambda_1 x + d\lambda_2 x \\ a\lambda_1 y + b\lambda_2 y & c\lambda_1 y + d\lambda_2 y \end{bmatrix} =$$

$$= \begin{bmatrix} x(a\lambda_1 + b\lambda_2) & x(c\lambda_1 + d\lambda_2) \\ y(a\lambda_1 + b\lambda_2) & y(c\lambda_1 + d\lambda_2) \end{bmatrix}$$

$$\varphi = [\lambda_1, \lambda_2] \quad \varphi.A = [\lambda_1 a + \lambda_2 b, c\lambda_1 + d\lambda_2]$$

$$\text{Lajze: } F \oplus F \rightarrow \text{Hom}(M, M)$$

$$[\lambda, \mu] \mapsto ([x] \mapsto [\lambda, \mu] \begin{bmatrix} x \\ y \end{bmatrix})$$

$$(\varphi.A) \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{([\lambda, \mu] \begin{bmatrix} x \\ y \end{bmatrix})}_\varphi A = [\lambda, \mu] \begin{bmatrix} x \\ y \end{bmatrix} A$$

1) Dokazi dual \mathbb{Z}_n^*

$$= \text{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0$$

$$1 \mapsto x \quad \text{red } 1 \mid \text{red } x \Rightarrow x = 0$$

2) K cel kolobar (brez del.niš, komut)

M K -modul definiramo

$$\text{tor}(M) = \{m \in M; xm = 0, x \in K - \{0\}\}$$

$$\text{tor}\left(\bigoplus_{n=2}^{\infty} \mathbb{Z}_n\right) = \bigoplus_{n=2}^{\infty} \mathbb{Z}_n; \text{ a mak ne obstaja element, ki bi uničil vse}$$

Dokazi da je $\text{tor}(M) \leq M$ podmodul in

$$\varphi: M \rightarrow N$$

$$\varphi_*(\text{tor}(M)) \leq \text{tor}(N)$$

a) $xm = 0 \quad ym = 0 \quad m, n \in M \quad x, y \in K$

$$\underbrace{xy}_{\neq 0}(m+n) = \underbrace{xy}_0 m + \underbrace{xy}_0 n = yxm = 0$$

$$\alpha \in K \quad xm = 0$$

$$x \alpha m = \alpha xm = 0$$

b) φ homomorfizem

$$x\varphi(m) = \varphi(xm) = 0 \rightarrow m \in \text{tor}(M) \Rightarrow \varphi(m) \in \text{tor}(N)$$

$$\Rightarrow \varphi_*(\text{tor}(M)) \leq \text{tor}(N)$$

$$\text{je podmodul: } x\varphi(\alpha\varphi(m) + \beta\varphi(n)) =$$

$$\underbrace{\alpha y \varphi(xm)}_0 + \underbrace{\beta x \varphi(n)}_0 = 0$$

$$\text{tor}(M/\text{tor}(M)) = 0 \quad \text{DN}$$

$$\chi(m + \text{tor}M) = \chi m + \text{tor}M$$

$$\underbrace{= 0} \Rightarrow m \in \text{tor}M \rightarrow m + \text{tor}M = 0$$

$$\neq 0 \Rightarrow m \notin \text{tor}M$$

3. M je prost $\overset{\text{K cel}}{\Rightarrow} \text{tor}(M)=0$

$$\varphi: M \xrightarrow{\cong} \bigoplus_{i \in I} K$$

$$\varphi(\text{tor} M) \leq \text{tor} \left(\bigoplus_{i \in I} K \right)$$

$$\varphi \text{ inj} \Rightarrow \text{dovolj je dokazati } \text{tor} \left(\bigoplus_{i \in I} K \right) = 0$$

(Dovolj je inj vložitev, ne izmeritvam)

$$\bigoplus_{i \in I} K = \left\{ (x_i)_{i \in I} ; x_i \in K \quad x_i = 0 \text{ za vse razen končno mnogo} \right\}$$

$$(x_1, x_2, \dots, x_i, \dots)$$

$$(x_i)_{i \in I} \in \text{tor} \left(\bigoplus_{i \in I} K \right)$$

$$\gamma(x_i)_{i \in I} = (\gamma x_i)_{i \in I} \Leftrightarrow \overset{0}{\neq} \gamma x_i = 0 \quad \forall i$$

nevel, a za uoban element razen 0

$$\Rightarrow x_i = 0 \quad \forall i$$

4) Ali je \mathbb{Z}_n prost \mathbb{Z} -modul?

NE, ker ima torzijo

Izrek: M kanono generiran \mathbb{Z} -modul.

$$M \text{ prost} \Leftrightarrow M \text{ torzijsko prost}$$

5) Pokaži

${}_{\mathbb{Z}}\mathbb{Q}$ ni prost \mathbb{Z} -modul

$$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$$

$$(cb) \frac{a}{b} - (ad) \frac{c}{d} = 0 \quad \text{netrivialna linearna kombinacija}$$

Ali: jlahke baze?

$$\mathbb{Q} = \left(\frac{c}{b} \right) \quad \frac{1}{b^2} \text{ ni delilke } \propto \frac{a}{b}$$

Ali: je (\mathbb{Q}^*, \cdot) prost \mathbb{Z} modul

$$-1 \in \mathbb{Q}^* \quad (-1)(-1)=1 \Rightarrow (-1) \in T(\mathbb{Q}^*)$$

\Rightarrow ima torzijo, torej ni prost

Kaj pa (\mathbb{Q}_+^*, \cdot)

$$\sum \left(\frac{a_i}{b_i} \right)^{n_i} > 0 \neq 0$$

Eksaktna zaporedja

kratka = kvocienti

$$0 \xrightarrow{\phi} L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\chi} 0$$

$\text{im } \phi = \ker \varphi$ $\text{im } \varphi = \ker \psi = M$
 $\text{im } \chi = \ker 0 = N$
 $\text{im } \phi = \ker \varphi$ ψ surjektivna
 φ injektivna

$$\frac{M}{\varphi(L)} \cong N$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}_2 \rightarrow 0$$

$x \mapsto 2x$ $x \mapsto x \bmod 2$
 inj $\text{dim } \text{može}$
 preslikava

$$\ker \psi = 2\mathbb{Z} = \text{im } \varphi$$

$$0 \xrightarrow{\text{inj}} \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \xrightarrow{\text{sur}} 0$$

← enotna krožnica

$n \mapsto \begin{bmatrix} n \\ x \end{bmatrix} \mapsto 1$
 $\mapsto e^{2\pi i x}$

P je **projektivnen** ce \forall eksaktno zaporedje $0 \rightarrow L \rightarrow M \rightarrow P \rightarrow 0$ razpade

kratko eksaktno zaporedje razpade oz. je ekvivalentno $0 \rightarrow L \rightarrow L \oplus N \rightarrow N \rightarrow 0$

P je projektiven $\Leftrightarrow 1) \Leftrightarrow 2)$

$$\Leftrightarrow 1) \quad \begin{array}{ccc} & P & \\ \exists f \nearrow & \downarrow \text{vf} & \\ A & \xrightarrow{\text{sur}} B \rightarrow 0 & \end{array}$$

$\Leftrightarrow 2) P$ je direktni sumand prostega modula $K^{(I)}$

$$\exists N. P \oplus N \cong K^{(I)} = \bigoplus_{i \in I} K$$

↑
kolobar

1. naloge

1.12

$\begin{bmatrix} F \\ F \end{bmatrix}$ je projektiven in ne prost nad $M_2(F)$

$$2) M_2(F) \cong \begin{bmatrix} F \\ F \end{bmatrix} \oplus \begin{bmatrix} F \\ F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$$

\nwarrow zunanji \nearrow notranji

$$\begin{bmatrix} x & z \\ y & w \end{bmatrix} \mapsto \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right) \mapsto \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & z \\ 0 & w \end{bmatrix}$$

je očiten izomorfizem

\Rightarrow je projektiven

Recimo da je prost, potem

$$\begin{bmatrix} F \\ F \end{bmatrix} = \begin{cases} M_2(F) = \dim 4 & \text{vse dimenzije} \\ M_2(F) \oplus M_2(F) = \dim 8 & \text{so pravele} \end{cases}$$

$$\Rightarrow \begin{bmatrix} F \\ F \end{bmatrix} \text{ ni prost}$$

2. način:

$$P \text{ proj} \Rightarrow P \text{ oz } \text{prost}$$

$$P \rightarrow M \text{ homomorfizem}$$

$$P(\text{tor}(P)) \subseteq \text{tor}(M)$$

Kategorije $\text{Top}, \text{Grp}, \text{Ab}, \text{Vect}_F, K\text{-Mod}, \dots$

$K\text{ mod}$:

objekti: K -moduli

morfizmi: homomorfizmi K -modulov

$$\text{Hom}_K(P, _): K\text{ mod} \rightarrow \text{Ab}$$

$$K M \mapsto \text{Hom}(P, M)$$

$$(M \xrightarrow{f} N) \mapsto \text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$$

$$f \mapsto f \circ \varphi$$

3. relacije

$\text{Hom}(P, _)$ je funktor

- ohranja kompozitum
- ohranja identiteto

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$F(g \circ f) = (\varphi \mapsto \varphi \circ g \circ f)$$

$$\begin{aligned} F(g) \circ F(f)(x) &= (\varphi \mapsto \varphi \circ g)(\varphi \mapsto \varphi \circ f)(x) = \\ &= (\varphi \mapsto \varphi \circ g) \circ (\varphi \mapsto \varphi \circ f) = x \circ f \circ g = (\varphi \mapsto \varphi \circ (f \circ g)) = F(f \circ g) \end{aligned}$$

u.p.s., zamenjati vrstni red

$$F(\text{id}) = (\varphi \mapsto \text{id} \circ \varphi) = \text{id}$$

P proj $\Leftrightarrow \underbrace{\text{Hom}(P, -)}$ eksakten

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{\psi} N \rightarrow 0$$

$$0 \rightarrow \text{Hom}(P, L) \xrightarrow{F(f)} \text{Hom}(P, M) \xrightarrow{F(\psi)} \text{Hom}(P, N) \rightarrow 0$$

$\text{Hom}(P, -)$ je **eksakten** če slike eksaktne zaporedja v eksaktne

$$\Rightarrow 1) P = K \quad \text{Hom}(K, \frac{M}{f(M)}) \cong_K M$$

$$p \mapsto p \circ f \quad f \mapsto f(1)$$

$$\begin{array}{ccc} \text{Hom}(P, L) & \xrightarrow{F(f)} & \text{Hom}(P, M) \\ \parallel & & \parallel \downarrow \\ L & \xrightarrow{f} & M \end{array}$$

$$2) P \text{ je prost } P \cong K^{(I)}$$

$$(\text{Hom}, -) = \text{Hom}(\bigoplus_I K, -) = \bigoplus_I \text{Hom}(K, -)$$

$$0 \rightarrow \bigoplus \text{Hom}(K, L) \rightarrow \bigoplus \text{Hom}(K, M) \rightarrow \dots$$

$$(l_1, \dots, l_n) \mapsto (f(l_1), f(l_2), \dots)$$

8.12

Y 1. arc

Tenzorski produkt

${}_K M, {}_K N$ moduli, K komutativan
 n : nujno, ampak je bolj enostavno če je

$$M \otimes_K N = \left\{ \sum_{i=1}^p m_i \otimes n_i \mid m_i \in M, n_i \in N, p \in \mathbb{N}_0 \right\}$$

1) bilinearnost

$$(m+m') \otimes n = m \otimes n + m' \otimes n$$

$$m \otimes (n+n') = m \otimes n + m \otimes n'$$

2) asociativnost ~~xxx~~

$$x \sum m_i \otimes n_i = \sum x(m_i \otimes n_i) = \sum (x m_i) \otimes n_i = \sum m_i \otimes (x n_i)$$

$$f: {}_K M \times {}_K N \xrightarrow{\text{bilin}} {}_K L$$

$$\begin{array}{ccc} (m, n) & \xrightarrow{\quad} & f(m, n) \\ \downarrow & \nearrow & \uparrow \\ m \otimes n & \xrightarrow{\quad} & f(m, n) \end{array}$$

bilinearna
 $f(xm, n) = x f(m, n) = f(m, xn)$
 $f \circ \tilde{f}$

U, V vek. prostora $U = \text{span}\{e_i\}$ $V = \text{span}\{f_j\}$
 \downarrow baza

$$U \otimes V = \text{span}\{e_i \otimes f_j\}$$

1. Zsp: \exists

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes (1, 1) + \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 1) + (-1, 0) \otimes \left(\frac{1}{2}, 1\right) \in \mathbb{R}^2 \otimes \mathbb{R}^2$$

ist einsteu. tensor $(u \otimes v)$

$$\begin{aligned} & \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (1, 0) + \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 1) + \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 1) + \frac{1}{2}(-1, 0) \otimes \left(\frac{1}{2}, 1\right) + \\ & \quad (-1, 0) \otimes (0, 1) - \\ &= \left(0, \frac{1}{2}\right) \otimes (1, 0) + (0, 1) \otimes (0, 1) = \\ &= \underline{\underline{(0, 1) \otimes \left(\frac{1}{2}, 1\right)}} \end{aligned}$$

... ..

2.

Poisson prime vek. prost. U, V in tenzorski

$x \in U \otimes V$, ki ni enostaven

$$U, V = \mathbb{R}^2$$

$$U', V' = \mathbb{R}$$

$$U', V'$$

$$\sum (x_i \otimes y_i) = \sum x_i y_i (1 \otimes 1) = \boxed{1} \left(1 \otimes \sum x_i y_i (1 \otimes 1) \right)$$

~~enostaven~~

$$M \otimes_k K \cong M$$

$$(1, 0) \otimes (1, 0) + (0, 1) \otimes (0, 1) = x \otimes y$$

$$x = (x_1, x_2)$$

$$y = (y_1, y_2)$$

$$x \otimes y = (x_1(1, 0) + x_2(0, 1)) \otimes (y_1(1, 0) + y_2(0, 1)) =$$

$$= x_1 y_1 (1, 0) \otimes (1, 0) + x_2 y_1 (0, 1) \otimes (1, 0) + x_1 y_2 (1, 0) \otimes (0, 1) +$$

$$+ x_2 y_2 (0, 1) \otimes (0, 1)$$

$$x_1 y_1 \neq 0 \Rightarrow x_1 y_1 \neq 0$$

$$x_2 y_1 = 0 \Rightarrow x_2 = 0$$

$$x_1 y_2 = 0$$

$$x_2 y_2 \neq 0 \Rightarrow x_2 \neq 0$$

$$L^N = \text{Hom}(N, L)$$

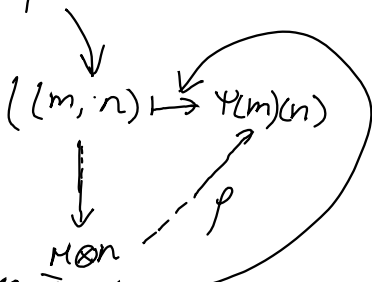
At: valje $(L^N)^M \cong L^{N \times M}$

3. Pokazi $\text{Hom}(M \otimes N, L) \cong \text{Hom}(M, \text{Hom}(N, L))$

$$\begin{aligned} \Phi: \text{Hom}(M \otimes N, L) &\longrightarrow \text{Hom}(M, \text{Hom}(N, L)) \\ (f: M \otimes N, L) &\longmapsto (\psi: m \mapsto (n \mapsto f(m \otimes n))) \end{aligned}$$

$$\Psi: \text{Hom}(M, \text{Hom}(N, L)) \longrightarrow \text{Hom}(M \otimes N, L)$$

$$\psi \longmapsto f$$



Dokazati mereno $M \otimes N$ se de je bilinearno

$$\psi(m+m')(n) = (\psi(m) + \psi(m'))(n) = \psi(m)(n) + \psi(m')(n)$$

\uparrow ψ homomorfizam

$$\psi(m)(n+n') = \psi(m)(n) + \psi(m)(n')$$

$\psi(m)$ je homomorfizam

$$\Phi \circ \Psi = \text{id}$$

$$(\Phi \circ \Psi)(\psi)(m)(n)$$

$$\Phi(\psi(m)(n)) = \Phi(m \otimes n) \mapsto \psi(m)(n) = m \mapsto n \mapsto \psi(m, n)(n)$$

$$\psi(m, n)$$

\forall pamembna
razloga

...

zatoj iz te razloga sledi

$$(\oplus M_i) \otimes N \cong \oplus (M_i \otimes N)$$