Navadna diferencialna enačba (NDE) je enačba oblike

$$f(x,y,y')=0 \quad \text{iščemo } y(x) \text{ da velja}$$

$$f(x,y(x),y'(x)) \quad \text{za } \forall x \in D_y$$

(To je v implicitni obliki)

Mi se bomo večinoma ukvarjali s temi ki je podano v eksplicitni obliki, torej ko je $y'=f(x,y)$

Enačba reda $n \in \mathbb{N}$ je oblike $G(x,y,y',\dots,y^{(n)})=0$

Enačba je avtonomna, če funkcija G ni odvisna od x. Sicer je neavtonomna.

7.10

1. Za dano družino funkcij poišči pripadajočo DE

a) $y = ce^x$

b) $y^2 = cx$

c) $y = c(x-c)$

$\Bigg\}$ $f(x, y, c) = 0$, $c \in \mathbb{R}$

$y' = ce^x \Rightarrow y = y'$

$y^2 = cx$

$2yy' = c \implies y^2 = 2yy'x$

$y = 2y'x$ za $y \neq 0$

$y' = c$

$y = y'(x-c)$

2) ugani rešitev naslednjih DE

a) $y'' = -y \Rightarrow \{\alpha \cos x + \beta \sin x\}$

b) $y + xy' = \cos x$

c) $xy' = ny$

b)

$$(xy)' = \cos x$$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

c) $xy' = ny$

$$xy' + y = n\cancel{y} + x$$

$$(xy)' = (n+1)y$$

$$y = x^n$$

Metoda izoklin

$y_c = y = y(x,c)$ $x, c \in \mathbb{R}$

$y \in \mathcal{C}^1$

Izoklina je krivulja vzdolž katere ima vsaka članica družine y enak odvod po x $(y_c'(x))$

$I_\alpha = \{(x,y) ; y = y_c(x)$ potem je $y_c'(x) = \alpha$

Če imamo DE oblike $y' = f(x,y)$ in pripadajočo družino rešitev $y_c(x)$, potem so izokline ravno nivojnice $\{f = \alpha\}$

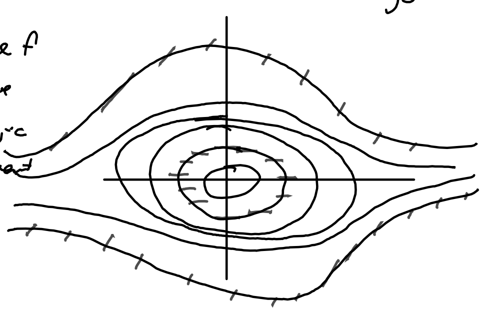
t.j. $y_c'(x) = f(x, y_c(x)) = \alpha$

(črno so nivojnice)
mi iščemo funkcijo

Postopek:

1. Skiciramo funkcije f

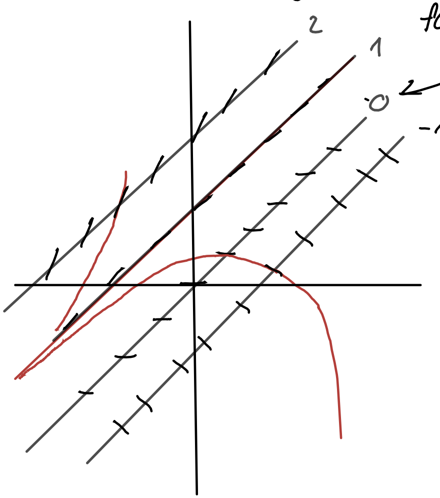
2) vzdolž vsake nivojnice narišemo nekaj daljic ki imajo smerni koeficient enak ~~nivojnici na~~ vrednosti f na nivojnici



3) narišemo krivuljo ki so v presečiščih nivojnic tangentne na daljico

3. Približno skiciraj potek enačbe $y' = y - x$

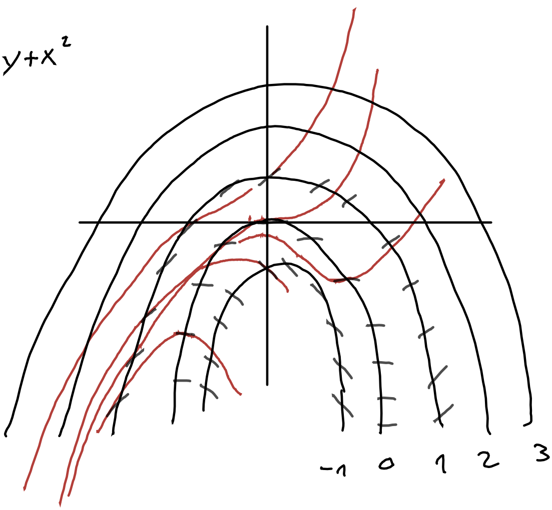
$f(x,y) = y - x$



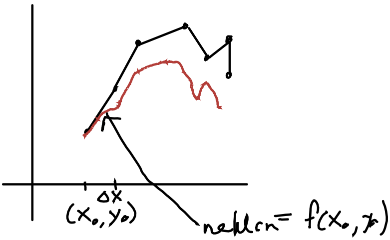
Rešitev 1:

$y = x + 1$

$y' = y + x^2$



Eulerjeva metoda



$x_1 = x_0 + \Delta x$

$y_1 = y_0 + \Delta x \; f(x, y_0)$

$x_2 = x_1 + \Delta x$

$y_2 = y_1 + \Delta f(x, y_0)$

Izberemo si Δx in iterativno definiramo

$x_{n+1} = x_n + \Delta x$

$= x_0 + (n+1)\Delta x$

$y_{n+1} = y_n + f(x_n y_n)\Delta x$

Zakaj ima to smisel?

Če privzamemo da je rešitev zvezno odvedljiva, velja, da je

$y(x + \Delta x) = y(x) + y'(x)\Delta x + o(\Delta x)$

5) Z Eulerjevo metodo dobimo rešitev DE

$y' = f(x)$ kjer je f zvezna pri pogoju $y(0)=0$

neka točka

$y(A) = ?$

$\Delta x = \frac{A}{m}$ za nek $m \in \mathbb{N}$

$A = x_m$

$x_n = 0 + \Delta x_n n$

$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x \cancel{+} y_0 =$

$= y_{n-1} + f(x_{n-1}) \Delta x + y_0$

$= (f(x_{n-2}) + f(x_{n-1})) \Delta x + y_0$ $\overset{0}{\parallel}$

Riemanova vsota

$$y_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

za $\int_0^A f(x)\,dx$

$$y_m = \sum_{i=0}^{m-1} f(x_i) \Delta x$$

To je očitno rešitev

Vaja (DN):

Najdi rešitev za $y'=2y$ z eulerjevo metodo

$$\Delta x = \frac{A}{n}$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\Delta x = y_{n-1} + 2y_{n-1}\frac{A}{n} =$$

$$= y_{n-1}\left(1+\frac{2A}{n}\right)$$

$$y_n = y_{n-2}\left(1+\frac{2A}{n}\right)^2 = y_1\left(1+\frac{2A}{n}\right)^{n-1}$$

$$\lim_{n\to\infty}\left(1+\frac{2A}{n}\right)^{n-1} = \lim_{u\to\infty}\frac{\left(\left(1+\frac{1}{u}\right)^u\right)^{2A}}{\left(1+\frac{1}{u}\right)} = \underline{\underline{e^{2A}}}$$

$$\frac{1}{u} = \frac{2A}{n} \Rightarrow n = 2Au$$

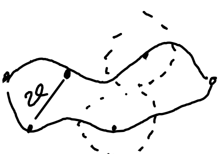
$$y = e^{2x}$$

Fazni prostor je prostor vseh možnih stanj sistema

$y' = f(x,y) \qquad f: \mathbb{R}^2 \to \mathbb{R}$

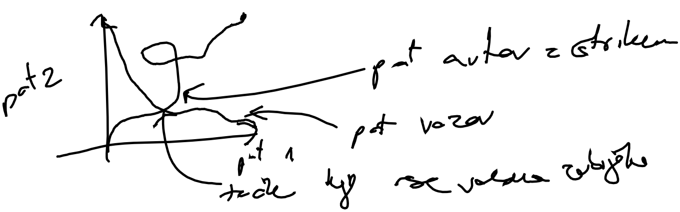
↳ fazni prostor je $\mathbb{R}^3$

6) iz Ljubljane v Maribor vodita dve cesti po katerih lahko iz Ljubljane do Maribora pripeljeta avtomobila kista eden na drugega privezana z vrvjo dolžine $<2l$, ne da bi jo pretrgala. Ali se lahko vozova krožne dolžine radia l, ki vozita vsak v svojo smer srečata, ne da bi trčila



$a \in S$

$[0,1] \times [0,1]$



1)

14.10

a) $y' = \frac{x^2}{y} = \frac{dy}{dx}$

$$x^2 dx = y dy$$

$$\frac{1}{3}x^3 + c = y^2$$

$$y = \pm\sqrt{\frac{1}{3}x^3 + c}$$

b)

$$2x^2 y y' + y^2 = 2$$

$$2x^2 = \frac{2-y^2}{yy'} = \frac{2-y^2}{y\frac{dy}{dx}} = \frac{2-y^2}{y\,dy}dx$$

$$\frac{dx}{2x^2} = \frac{y\,dy}{2-y^2}$$

$u = 2-y^2$
$du = -2y\,dy$

$$-\frac{1}{2}\frac{1}{x} = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}(\ln u + c)$$

$$\frac{1}{x} = \ln(2-y^2) + c$$

$$ce^{\frac{1}{x}} = 2-y^2$$

$$y = \pm\sqrt{2-ce^{\frac{1}{x}}}$$

c) $(1+x^2)y' = y$

$\parallel$

$(1+x^2)\frac{dy}{dx} = y$

$\frac{dy}{y} = \frac{dx}{1+x^2}$

$\ln y = \arctan x + c$

$y = ce^{\arctan x}$

$y \equiv 0$

c') $(1+x^3)y' = y$

$y_1 \equiv 0$

2) $\frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)}$

$B = -A$

$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$

x^2: $B + A = 0$

x: $B + C - A = 0$

1: $A + C = 1$

$C = 1 - \frac{1}{3}$

$2B + C = 0$

$\Rightarrow C = -2B = 2A$

$3A = 1$

$\frac{dy}{y} = \frac{1}{3}\frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1}dx$

$\int \frac{x-2}{x^2-x+1}dx = \int \frac{}{(x-\frac{1}{2})^2+\frac{3}{4}}$

2)

a) $y' = \tan(2x+3y-1)$

$y' = \tan z$

$z = 2x+3y-1$

$z' = 2+3y'$

$y' = \frac{z'-2}{3}$

$\tan z = \frac{z'-2}{3}$

$z' = 3\tan z + 2$

1. rešitev : $z' = 0 \Rightarrow z = \arctan(-\frac{2}{3})$

$\Downarrow$

$y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan(-\frac{2}{3}) - 2x + 1}{3}$

$\frac{dz}{dx} = 3\tan z + 2$

$\frac{dz}{3\tan z + 2} = dx$

$u = \tan z$

$du = \frac{1}{\cos^2 z} dz \Rightarrow dz = \frac{du}{1-u^2}$

$x + C = \int \frac{du}{(3u+2)(1-u^2)}$

$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2}$

$u^2: B - A = 0 \quad A = B$

$u: 3C + 2B = 0$

$1: 2C + A = 1 \Rightarrow A = 1-2C$

$3C + 2 - 4C = 0$

$C = 2 \Rightarrow A = -3 = B$

$= \int\left(\frac{-3}{3u+2} + \frac{-3u+2}{1-u^2}\right)du =$

$= -\ln(u+\frac{2}{3}) +$

$\frac{A}{1-u} + \frac{B}{1+u} =$

$u: A - B = -3$

$1: A + B = 2$

$+ \int \frac{1}{2}\frac{1}{u-1} + \frac{5}{2}\frac{1}{u+1} =$

$A = B-3$

$2B - 3 = 2$

$2B = 5$

$B = \frac{5}{2}$

$A = -\frac{1}{2}$

$= -\ln(u+\frac{2}{3}) + \frac{1}{2}\ln(u-1) + \frac{5}{2}\ln(u+1)$

$$y''y - y' = xy'' - (y')^2$$

$$y''(y-x) = y'(1-y')$$

$$y''(y-x) + y'(y'-1) = 0 \qquad y \equiv 0$$

$$a = y'$$

$$b = y - x \qquad b' = y' - 1$$

$$a'b + ab' = 0$$

$$(ab)' = 0$$

$$ab = C \in \mathbb{R}$$

$$y'(y-x) = C$$

$$y' = \frac{C}{y-x} \qquad z = y - x$$

$$z' = y' - 1 = \frac{C}{z} - 1$$

$$\cancel{z = \int\left(\frac{C}{z} - 1\right)dz = C\ln z - z + D}$$

$$\frac{dz}{dx} = \frac{C}{z} - 1$$

$$\frac{dz}{\frac{C}{z} - 1} = dx \qquad x + D = \int \frac{z}{C - z}\,dz = -\int\left(\frac{C}{t} - 1\right)dt =$$

$$t = C - z$$

$$dt = -dz \qquad = -C\ln(C-z) + C - z$$

$$z = y - x$$

$$\cancel{x} + D = -C\ln(C - y + x) + C - y + \cancel{x}$$

$$y = D e^C e^{\ln(C-y+x)+1}$$

Homogene enačbe

$$F(x,y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xz = y \Rightarrow y' = z + xz' = F(1,z)$$

$$z' = \frac{F(1,z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

A) $y^2 + x^2 y' = xyy'$ $\qquad z = \frac{y}{x}$

$$y'(xy - x^2) = y^2$$

$$y' = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z-1}$$

$$y = zx$$

$$y' = z + z'x = \frac{z^2}{z-1}$$

$$z'x = \frac{z^2 - z^2 + z}{z - 1} = \frac{z}{z-1}$$

$$\frac{dz}{dx} x = \frac{z}{z-1} \Rightarrow \frac{z-1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{y}{x}$$

b) $y = xy' - \sqrt{x^2+y^2}$

$$y' = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} + \overset{\operatorname{sgn} x}{\cancel{\ }}\sqrt{1+\frac{y^2}{x^2}} = z + \overset{\operatorname{sgn}}{\ }\sqrt{1+z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = \cancel{z} + z'x = \cancel{z} + \sqrt{1+z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1+z^2}}{x}$$

$z = \operatorname{sh} u \quad dz = -\operatorname{ch} u\,du$

$$\operatorname{sgn} x \ln x + D = \int \frac{1}{\sqrt{1+z^2}}\,dz = \int \frac{\cancel{\ } - \operatorname{ch} du}{\operatorname{ch} u} =$$

$$-\operatorname{arcsh}\frac{y}{x}$$

$$\text{"}$$

$$\operatorname{arch}\frac{y}{x}$$

$$y = x \operatorname{ch}(\overset{\operatorname{sgn} y}{\ln} x + D)$$

a)

$$ma = F = m\cdot g - kv^2$$

$$m\dot{v} = mg - kv^2$$

$$\dot{v} = g - \frac{k}{m}v^2 = g\left(1 - \underbrace{\frac{k}{mg}}_{\alpha} v^2\right) = g(1-\alpha v^2)$$

$$\frac{dv}{dt} = g(1-\alpha v^2)$$

$$\frac{dv}{1-\alpha v^2} = g\,dt \quad /\int$$

$$gt =$$

$$\frac{A}{1-\sqrt{\alpha}v} + \frac{B}{1+\sqrt{\alpha}v} :$$

$v: \sqrt{\alpha}A - \sqrt{\alpha}B = 0$

$A = B$

$1: \quad A + B = 1$

$\Rightarrow A = \frac{1}{2}$

$B = \frac{1}{2}$

$$gt = \frac{1}{\sqrt{\alpha}2}\left(\ln(1+\sqrt{\alpha}v) - \ln(\sqrt{\alpha}v - 1)\right) =$$

$$\frac{1}{2\sqrt{\alpha}}\ln\left(\frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v}\right) = gt + C$$

$$V_0 = 0 \Rightarrow \quad \cancel{1+\sqrt{\alpha}}v = \cancel{1} - \sqrt{\cancel{\alpha}}v + C$$

$$\Rightarrow 0 = C$$

$$\frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v} = \underbrace{e^{2\sqrt{\alpha}gt}}_{D}$$

$$1+\sqrt{\alpha}v = D - \sqrt{\alpha}Dv$$

$$v\sqrt{\alpha}(1-\sqrt{\alpha}D) = D-1$$

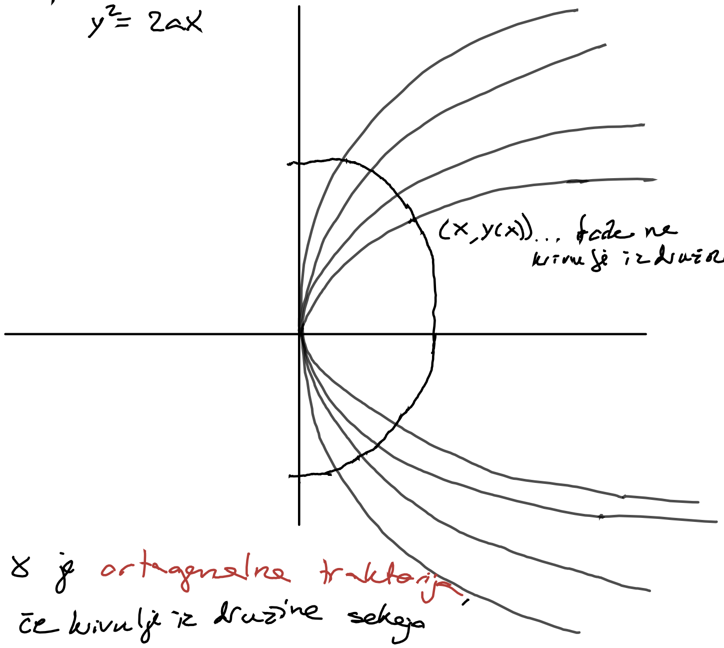
$$v = \frac{D-1}{\sqrt{\alpha}(1-\sqrt{\alpha}D)}$$

V merkenje

+ zamdude 20 min 21.10

2) Poišči družino ortagonalnih trajektorij na družino parabol

$y^2 = 2ax$



γ je ortagonalna traktorija, če krivulje iz družine sekajo γ pod pravim kotom

$2yy' = 2a \rightsquigarrow y^2 = 2yy'x$

v (x_0, y_0) velja $x_0 2y_0 y' = y_0^2$

$y' = \frac{y_0}{2x_0}$ smerni koeficient traktorije $-\frac{2x_0}{y_0}$

$y' = f(x,y)$ družina krivulj $\Rightarrow -\frac{1}{y'} = f(x,y)$ družina ort. t.

$y^2 = 2yy'x \rightsquigarrow y^2 = -\frac{2yx}{y'}$ iščemo y

1 možnost $y \equiv 0$ ✓

2. možnost

$y' = -\frac{2x}{y} = \frac{dy}{dx}$

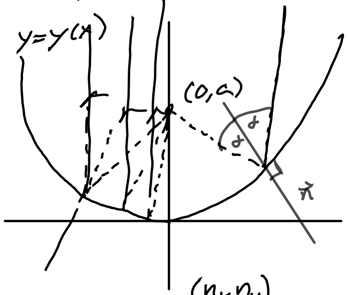
$2x\,dx = y\,dy \;/ \int$

$-\frac{2}{2}x^2 = \frac{1}{2}y^2 + C$

$x^2 + \frac{1}{2}y^2 + C = 0$

$x^2 + \frac{1}{2}y^2 = -C$ elipsa

3) Pri izdelavi žarometa mora biti odbojna površina ki obdaja žarnico take oblike, da vse svetlobne žarke odbije v isto smer. Predpostavimo, da je odbojna površina rotacijska ploskev grafa f_n



(n_x, n_y)

$$(-x, -y(x)+a)\cdot \vec{n} = -x n_x - y n_y - a n_y = \cos\alpha \sqrt{x^2+(y+a)^2}\sqrt{n_x^2+n_y^2}$$

$$\vec{n} = \left(1, -\frac{1}{y'}\right)$$

$$\cos\alpha \sqrt{x^2+(y+a)^2}\sqrt{1+\frac{1}{y'^2}} = -x + \frac{-y-a}{y'}$$

$$\cos\alpha = \frac{-xy'-y-a}{\sqrt{x^2+(y+a)^2}\sqrt{1+y'^2}}$$

$$\vec{n}\cdot(0,1) = \cos\alpha\, |\vec{n}|\cdot 1$$

$$n_y = \cos\alpha \sqrt{1+\frac{1}{y'^2}}$$

$$n_y = \frac{\cancel{\frac{1}{y'}}(-xy'-y-a)}{\sqrt{x^2+(y+a)^2}} = \cancel{\frac{1}{y'}}$$

???

$$-xy'+y-a = \sqrt{x^2+(y+a)^2}$$

$z = y-a$
$z' = y'$

$$xz'+z = \sqrt{x^2+z^2+z}$$

??

$$z' = \frac{\sqrt{x^2+z^2}}{x} = \sqrt{\frac{x^2+z^2}{x^2}} = \sqrt{1+\frac{z^2}{x^2}} + \frac{z}{x}$$

$u = \frac{z}{x}$

$z' = xu'$

$$\cancel{u} + xu' = \sqrt{1+u^2} + \cancel{u}$$

$$\frac{u'}{\sqrt{1+u^2}} = \frac{1}{x}$$

$v = \sqrt{1+u^2}$

$$dv = \frac{u\,du}{\sqrt{1+u^2}} = \frac{1-v^2}{\sqrt{1+u^2}}\,du$$

$$\ln x = \int \frac{1-v^2}{v}\,dv = \ln v - \frac{1}{2}v^2 + C$$

???

$$y = a + \frac{c}{2}x^2 - \frac{1}{2c}$$

Cauchy naloge

$y' = f(y) \qquad y(x_0) = y_0$

a) globi. ∃ reši DE za poljubni začetni pogoj

b) f Lipšicova tedaj kaj reštev iz a enolična

$$\frac{dy}{dx} = f(y)$$

če $f(y) \neq 0$

$$\frac{dy}{f(y)} = dx \quad / \int$$

Kaj pa če je $f(y_0) = 0$

Poskusimo $f(y) \equiv x_0$

$$F(y) = x + C$$

$y'(x) = 0$

$$\int \frac{d\cancel{y}}{f(\cancel{y})}$$

$f(y(x)) = f(y_0) = 0$

$$y'(x) = f(y(x))$$

$$y(x) = y(x_0) + \int_{x_0}^{x} y'(t)\,dt$$

daš vse na eno stran in integriraš

$$\underbrace{\int \frac{y'(t)\,dt}{f(y(t))}} = \int_{x_0}^{x} 1\,dt = x - x_0$$

$u = y$

$du = y'\,dt$

$$\int_{y_0}^{y(x)} \frac{du}{f(u)} = F(y(x)) - F(y_0) = x - x_0$$

$$F(y(x)) = x - x_0 + F(y_0)$$

$$y(x) = F^{-1}(x - x_0 + F(y_0))$$

$$y(x_0) = F^{-1}(F(y_0)) = y_0$$

$$y'(x) = \frac{1}{F'(F^{-1}(x - x_0 + F(y_0)))} = f(\cancel{F^{-1}(x - x_0 + F(y_0))})$$

$= f(y(x))$???

$$F'(F^{-1}(x)) = \frac{1}{(F^{-1})'(x)}$$

b) enoličnost

$$\int_{y_0}^{y} \frac{dy}{f(y)} = x - x_0$$

$$|f(x) - f(y)| \leq C|x-y|$$

recimo da $f(x)=0$ $\to$ $|f(y)| \leq C|x-y|$

$f \geq 0$ $y > y_0$, $f(y_0) = 0$

$$\int_{y_0}^{y} \frac{dy}{f(y)} \geq \frac{dy}{C|y-y_0|} = \infty$$

Recimo da y reši DE da je

$f(y_1) = 0$ in y ni konstanten

$x_1 > x_0 : f(y(x_1)) > 0$ BŠZ

potem je y blizu x_1 zvezna

$$\int_{x_1}^{x} \frac{y'(x)dx}{f(y(x))} = x - x_1$$

$$\| $$

$$\int_{y_1}^{y} \frac{dy}{f(y)}$$

izberemo si $x_2 \in [x_0, x_1]$, da velja

$f(y(x_2)) = 0$ in $f(y(x)) > 0$ za $\forall x \in (x_2, x_1]$

$$\lim_{x \to x_2} \int_{x_1}^{x} \frac{y'(x)}{f(y(x))} = \lim_{x \to x_2} x - x_1 = x_2 - x_1$$

$$= \lim_{y \to y_2} \int_{y_1}^{y} \frac{dy}{f(y)} \geq \lim_{y \to y_2} \int_{y_1}^{y} \frac{dy}{C(y-y_0)} = \infty$$

$\times$

$y' = \frac{f(y)}{g(x)}$ f, g brez ničel

a) Pokaži da vsaka rešitev da tokovno krivuljo polja $v(x,y) = (g(x), f(y))$

b) Dokaži za poljuben začetni pogoj dobimo natanko eno rešitev

Tokovnica ali integralna krivulja je krivulja, ki ima v vsaki točki odvod enak vektorskemu polju

$$F(y) - F(y_0) = \int_{y_0}^{y} \frac{dy}{f(y)} = t - t_0 \qquad x = G^{-1}(t - t_0 + G(x_0))$$

$$G(x) - G(x_0) = \int_{x_0}^{x} \frac{dx}{g(x)} = t - t_0 \rightsquigarrow$$

$\gamma = (x(t), y(t))$

$\dot{\gamma}(t) = V(x(t), y(t))$

$\dot{x}(t) = g(x)$

$\dot{y}(t) = f(y)$

$\frac{\partial}{\partial t} y(x(t)) =$

$= \frac{\partial y}{\partial t}(x(t))\, \dot{x} =$

$= \frac{f(y(t))}{g(x(t))} \frac{\partial x}{\partial t}$

$\rightsquigarrow \dot{x} = g(x)$???

Lin. dif. enačbe (LDE)

$y' + a(x)y = b(x) \qquad a, b \in \mathcal{C}([\alpha, \beta])$

i) $b \equiv 0 \Rightarrow$ LDE homogena $\Rightarrow$ ločljive spremenljivke

$$y = y_0 e^{-\int a(t)dt} = Ce^{-\int a dx}$$

ii) $b \not\equiv 0 \Rightarrow$ LDE nehomogena $\Rightarrow$ variacija konstante

$$y = e^{-\int a dx} \int b(x) e^{\int a(x)dx} dx + Ce^{-\int a dx}$$

•) LDE homogena: y_1, y_2 rešitvi $\Rightarrow y_1 + \lambda y_2$ tudi rešitev

če vemo homogeno rešitev $\bar{y}$ so vse druge rešitve oblike $C \cdot \bar{y}$ za $C \in \mathbb{R}$

••) LDE nehomogena

Partikularno rešitev najdemo z variacijo konstante

Homogeni del: $y' + ay = 0 \Rightarrow y_h$ rešitev

Partikularni del: $y_p = C(y) y_h$

Splošna rešitev: $\tilde{y} = y_p + Dy_h$

Odvajamo in vstavimo v DE:

$$y_p' = C'y_h + Cy_h' \Rightarrow C'y_h + Cy_h' + aCy_h = b$$

$$C'y_h = b \quad \Rightarrow \quad C(x) = \int \frac{b}{y_h} dx$$

1) Poišči splošno rešitev:

a) $xy' + 2y = (3x+2)e^{3x} \quad / : x$

$$y' + y\frac{2}{x} = \underbrace{\frac{3x+2}{x}e^{3x}}_{b}$$

$x=0 \Rightarrow 2y = 2 \Rightarrow y = 1$

$x = 0 \Rightarrow$

homogeno: $y' + y\frac{2}{x} = 0$

$$\frac{dy}{dx} + y\frac{2}{x} = 0 \quad / \cdot \frac{dx}{y}$$

$$\frac{dy}{y} = -\frac{2dx}{x} \quad / \int$$

$$\ln y = \ln x^{-2} + C$$

$$y = \frac{1}{x^2} \cdot D$$

partikularno:

$$y_p = D(x)x^{-2}$$

$$y_p' = D'(x)x^{-2} - 2D(x)x^{-3}$$

$$D'(x)x^{-2} - \cancel{2D(x)x^{-3}} + \cancel{2D(x)x^{-3}} = \frac{(3x+2)e^{3x}}{x}$$

$$D'(x) = (3x^2 + 2x)e^{x}$$

$$\boxed{\begin{gathered}\int p(x)e^{\lambda x}dx = g(x)e^{\lambda x} + C \\ p, g \in \mathbb{R}[x], \ \operatorname{st} p = \operatorname{st} g\end{gathered}}$$

$$\left((Ax^2 + Bx + C)e^{3x}\right)' = (3x^2 + 2x)e^{3x}$$

$$(3Ax^2 + 3Bx + 3C + 2Ax + B)e^{3x} \rightsquigarrow$$

$$\begin{aligned} 3A &= 3 && \Rightarrow A = 1 \\ 3B + 2A &= 2 && \Rightarrow B = 0 \\ 3C + B &= 0 && \Rightarrow C = 0 \end{aligned}$$

$$D(x) = x^2 e^{3x}$$

$$\rightsquigarrow y_p = x^2 e^{3x} \cdot x^{-2} = e^{3x}$$

$$y_s = y_h + y_p = D\frac{1}{x^2} + e^{3x}$$

b) $y'' = \sin x + y'$

$z = y'$

$z' = \sin x + z$

$z' - z = \sin x$

$a(x) = -1 \quad b(x) = \sin z$

$b \equiv 0 \Rightarrow z' - z = 0$

$z' = z$

1) $z \equiv 0 \Rightarrow y = C \in \mathbb{R}$

2) $\int \frac{dz}{z} = dx \Rightarrow \ln z = x + c$

$\Rightarrow z = ce^x \Rightarrow y = ce^x + D$

$b \not\equiv 0 \Rightarrow \quad z_h = e^x$

$z_p = C(x) z_h = C(x) e^x$

$z = z_p + D y_h = C(x) e^x + De^x$

$z' = C'(x) e^x + C(x) e^x + De^x$

$z' - z = C'(x) e^x + C(x)(e^x - e^x) = \sin x$

$$C'(x) = \frac{\sin x}{e^x} = e^{-x} \sin x$$

$\boxed{\text{nastavek: } C(x) = Ae^{-x} \sin x + Be^{-x} \cos x}$

$C(x) = e^{-x}(A \sin x + B \cos x)$

$C'(x) = -e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)$

$= e^{-x}(\sin x(-A - B) + \cos(x)(-B + A)) = e^{-x} \sin x$

$A + B = -1$

$A - B = 0 \Rightarrow A = B \Rightarrow 2A = -1 \Rightarrow A = B = -\frac{1}{2}$

$\Rightarrow C(x) = -\frac{1}{2} e^{-x}(\sin x + \cos x) =$

$z = -\frac{1}{2}(\sin x + \cos x) + De^x = y'$

$y = -\frac{1}{2} \int \sin x + \cos x \, dx = \frac{1}{2} \cos x - \frac{1}{2} \sin x + De^x + C$

2)

$y' + 2xy = 2x^3y^3$

1) $y \equiv 0$ je rešitev

2) $y \not\equiv 0 \Rightarrow \frac{y'}{y^3} + \frac{2x}{y^2} = 2x^3$

$z' = \frac{y'}{y^3} \Rightarrow \int dz = \int y^{-3} dy \Rightarrow z = -\frac{1}{2}y^{-2} = -\frac{1}{2y^2}$

> **Bernullijeva DE**
> $y' + ay = by^\alpha \quad \alpha \notin \{0,1\}$
> Nastavek: $z = y^{1-\alpha}$

$\frac{2x}{y^2} = \frac{4x}{2y^2} = -4xz$

enačba: $z' - 4xz = 2x^3 \qquad a(x) = -4x \quad b(x) = 2x^3$

homogeno: $z' - 4xz = 0$

$\int \frac{dz}{z} = \int 4x\,dx \Rightarrow \ln z = 2x^2 + c \Rightarrow z = Ce^{2x^2}$

partikularno: $z_h = e^{2x^2} \qquad z_p = C(x)e^{2x^2}$

$z = z_p + z_h$

$\rightsquigarrow$ enačba: $C'(x)e^{2x^2} + 4xe^{2x^2} \cdot C - 4xCe^{2x^2} = 2x^3$

$C'(x) = e^{-2x^2} \cdot 2x^3$

Nastavek:

$C(x) = \int 2x^3 e^{-2x^2} dx = \qquad \int xe^x dx = (Ax+B)e^x$

$u = -2x^2$

$du = -4x\,dx$

$= -\frac{1}{2}\int e^u u\,du = -\frac{1}{2}(Au+B)e^u$

$((Au+B)e^u)' = e^u(Au+B+A)$

$A+B=0 \quad A=1 \Rightarrow B=-1$

$C(x) = -\frac{1}{2}(1u-1)e^u \rightsquigarrow$

$C(x) = e^{-2x^2}(x^2 - \frac{1}{2}) + D$

$z = x^2 - \frac{1}{2} + De^{2x^2} = -\frac{1}{y^2} \rightsquigarrow y = \frac{1}{\sqrt{x^2 - \frac{1}{2} + De^{2x^2}}}$

3. naloga

$$xy' + y = y^2 \ln x$$

1) $y \equiv 0 \Rightarrow$ je rešitev

2) $y \not\equiv 0 \Rightarrow$
$x \neq 0 \Rightarrow$

$$\frac{y'}{y^2} + \frac{1}{xy} = \ln x$$

$$z' = \frac{y'}{y^2} \rightsquigarrow z = -\frac{1}{y}$$

$$\boxed{z' - \frac{z}{x} = \ln x} \quad \begin{array}{l} a = -1 \\ b = \ln x \end{array}$$

homogeno:

$$z' - \frac{z}{x} = 0$$

$$\frac{dz}{z} = \frac{dx}{x} \rightsquigarrow$$

$$\ln z = \ln x + C$$

$$z = e^{\ln x} \cdot e^{C} \rightsquigarrow z = C x$$

$$z_h = x$$

$$\rightsquigarrow \quad C'x + C - C = \ln x$$

$$C' = \frac{\ln x}{x} \qquad C = \int \frac{\ln x}{x} dx = \int u \, du = \frac{1}{2}u^2 + D$$

$$u = \ln x$$
$$du = \frac{dx}{x} \qquad C = \frac{1}{2}(\ln x)^2 + D$$

$$z = \frac{1}{2}(\ln x)^2 x + Dx = -\frac{1}{y}$$

$$y = -\frac{2}{(\ln x)^2 x + Dx}$$

4 naloga

f omejena zvezna

$$y' + y = f(x)$$

a) To ima natanko eno rešitev DE

b) f periodična s periodo $\omega \Rightarrow \bar{y}$ periodična s periodo ω

a) $y' + y = f(x)$

homogeno:

$$\frac{dy}{dx} = -y \rightsquigarrow \frac{dy}{y} = -dx \rightsquigarrow y = e^{-x}$$

partikularna: $y_p = C(x)e^{-x}$

$$C'(x)e^{-x} = f(x) \qquad C(x) = \int_0^x f(x)e^{x}dx = F(x) + D$$

$$\bar{y} = F(x)e^{-x} + Ce^{-x} = e^{-x}(F(x)+C) = e^{-x}\int_0^x f(t)e^{-t}dt$$

b)

$$\bar{y}(x+\omega) = e^{-x-\omega}\int_0^{x+\omega} f(t)e^{-t}dt = e^{-x}e^{-\omega}\int_0^x f(t)e^{-t}dt + e^{-\omega}e^{-x}\int_x^{\omega} f(t)e^{-t}dt =$$

$$= e^{-\omega}\bar{y} + e^{-\omega}e^{-x}\int_0^{\omega} f(t)dt - e^{-\omega}e^{-x}\int_0^x f(t)e^{-t}dt =$$

$$= e^{-\omega}\bar{y}(x) + e^{-x}\bar{y}(\omega) - e^{-\omega}\bar{y}(x) = e^{-x}\bar{y}(\omega) = e^{-\omega}\bar{y}(x)$$

$$\bar{y}(x+\omega) = e^{-x-\omega}\int_{-\infty}^{x+\omega} f(t)e^{-t}dt = e^{-x-\omega}\int_{-\infty}^{\omega} f(t)e^{-t}dt + e^{-x-\omega}\int_{\omega}^{x+\omega} f(t)e^{-t}dt =$$

$$= e^{-x-\omega}\left(\int_{\infty}^{\omega} f(t)e^{-t}dt + \int_0^x f(t)e^{-t}dt\right) =$$

$$= e^{-x}\bar{y}(\omega) + e^{-\omega}\bar{y}(x)$$

???

b) $\bar{y}(x) = e^{-x}\int_{-\infty}^{x} f(t)e^{t}\,dt = \int_{-\infty}^{x} f(t)e^{t-x} = \int_{-\infty}^{0} f(u+x)e^{u}\,du$

$u = t - x$

$\rightsquigarrow t = u + x$

$$\bar{y}(\omega+x) = \int_{-\infty}^{0} f(u+x+\omega)e^{u}\,du = \int_{-\infty} f(u+x)e^{u}\,du = \bar{y}(x)$$

5 naloga

$$3y' + y^2 + \frac{2}{x^2} = 0$$

Uganemo rešitev: $y = \frac{1}{x}$ $y' = -\frac{1}{x^2}$

$$-\frac{3}{x^2} + \frac{1}{x^2} + \frac{2}{x^2} = 0 \quad \checkmark$$

Splošna rešitev: $y = \frac{1}{x} + u(x)$

$$y' = -\frac{1}{x^2} + u'(x)$$

$$-\frac{3}{x^2} + 3u'(x) + \frac{1}{x^2} + \frac{2u(x)}{x} + u^2(x) + \frac{2}{x^2} = 0$$

$$3u'(x) + \frac{2u(x)}{x} + u^2(x) = 0 \quad /:u^2(x)$$

$$\frac{3u'(x)}{u^2(x)} + \frac{2}{xu(x)} + 1 = 0$$

$$z' = \frac{3u'(x)}{u^2(x)} \qquad z = 3\int u^{-2} du = -3u^{-1} = -\frac{3}{u}$$

$$\Rightarrow u = -\frac{3}{z}$$

$$z' - \frac{2}{3}\frac{z}{x} = -1$$

$$a(x) = \frac{2}{3x}$$

$$b(x) = -1$$

homogeni del:

$$\frac{dz}{z} = \frac{2}{3}\frac{dx}{x}$$

$$\ln z = \ln x^{\frac{2}{3}} + C \rightsquigarrow z = \frac{2}{3}Cx$$

partikularni:

$$C'(x)\frac{2}{3}x = -1$$

$$C(x) = -\frac{3}{2}\int \frac{1}{x}dx = \ln x^{-\frac{3}{2}}$$

$$z = x\ln x + \frac{2}{3}Cx$$

$$u = -\frac{3}{x\ln x + \frac{2}{3}Cx} \Rightarrow y = \frac{1}{x} - \frac{3}{x\ln x + \frac{2}{3}Cx}$$

Ricattijeva enačba

$y' = ay^2 + by + c \quad a, b, c$ funkcije x

1) Ugenemo rešitev y_p

2) Splošno rešitev poiščemo z nastavkom

$$y = y_p + u(x)$$

3) Vstavimo nastavek z začetno enačbo

4) Dobimo Bernullijevo enačbo

5) $y = y_p + \frac{1}{u^2(x)} \Rightarrow LDE$

4.11.

1. $(1-x^2)y' = 1-y^2$

Riccatijeva enačba

$y' = ay^2+by+c$: a,b,c func. od x

1) Ugenimo rešitev y_p

2) splošna rešitev kot $y = y_p + u(x)$

3) Dobimo Bernolijeva enačba

$y = y_p + \frac{1}{u(x)^2} \Rightarrow LDE$

$$y' = \frac{1-y^2}{1-x^2}$$

1. $y_p = x$: $1 = \frac{1-x^2}{1-x^2}$ ✓

2. $y = v + y_p = v+x$ v je neznana funkcija

$$y' = v'+1 = \frac{1+(v+1)^2}{1-x^2} = \dots = 1 + \frac{-2vx-v^2}{1-x^2}$$

$$v' = \frac{-2vx-v^2}{1-x^2} = -\left(\frac{1}{1-x^2}v^2 + \frac{2x}{1-x^2}v\right)$$

$$\frac{v'}{v^2} = -\frac{1}{1-x^2} - \frac{2x}{1-x^2}\frac{1}{v}$$

$z = \frac{1}{v}$ $\quad z' = \frac{v'}{v^2}$

$$z' = -\frac{1}{1-x^2} - \frac{2x}{1-x^2}z$$

homogeni del:

$$z' = -\frac{2x}{1-x^2}z$$

$$\int\frac{dz}{z} = -\int\frac{2x}{1-x^2}dx$$

$$\ln|z| = -\ln|1-x^2| + C$$

$$z_H = \frac{C}{x^2-1}$$

$$\frac{C'(x)}{x^2-1} + \cancel{\frac{C}{\dots}} = \cancel{\frac{C}{\dots}} - \frac{1}{x^2-1}$$

$C'(x) = -1 \quad C = -x$

$$z = z_h + z_p = \frac{C-x}{x^2-1}$$

$$v = \frac{x^2-1}{C-x}$$

$$y = \dots = \frac{1-Cx}{x-C}$$

Prvi integral enačbe $y' = f(x,y)$ je funkcija, ki je konstantna vzdolž vsake rešitve

odvod po c

Če imamo splošno rešitev $y = \varphi(x,C)$ in je $\varphi_C \neq 0$

$$\Rightarrow \quad C = C(x,y) \text{ je prvi integral}$$

1) DE zapišemo v obliki $P\,dx + Q\,dy = 0$ (*)

Opomba: (*) je ekvivalentna DE $P + Q y' = 0$

$$\langle (P,Q), (1,y') \rangle = 0$$

če velja $Q_x = \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = P_y$

pravimo da je enačba eksaktna.

Njen prvi (eksaktni integral) poiščemo z integracijo $u = \int$

$\big((P,Q)$ je potencialno $\Leftrightarrow (P,Q) = \nabla u\big)$

C poiščemo z odvajanjem

$$u_y = \frac{\partial}{\partial y}\int P\,dx + C'(y) = Q$$

2) Poišči kak nekonstanten prvi integral

a) $x\sqrt{x^2-y^2+y})dx + (y\sqrt{x^2+y^2}+x)dx = 0$

b) $3x^2(1+\ln y)dx - (2y - \frac{x^3}{y})dy = 0$

a)

$P = x\sqrt{x^2+y^2} + y$

$Q = y\sqrt{x^2+y^2} + x$

Ali je eksaktna?

$P_x = \sqrt{x^2+y^2} + \frac{xy}{\sqrt{x^2+y^2}}$

$Q_y = -\|-$

$u = \int (x\sqrt{x^2+y^2} + \cancel{xy}\, y)dx + C(y) =$

$t = x^2+y^2$

$dt = 2x dx$

$= \frac{1}{2}\int t^{\frac{1}{2}} dt + xy + C(y) = \frac{1}{3}(x^2+y^2)^{\frac{3}{2}} + xy + C(y)$

$u_y = \cancel{y(x^2+y^2)^{\frac{1}{2}}} + \cancel{x} + C'(y) = \cancel{y\sqrt{x^2+y^2}} + \cancel{x}$

$C'(y) = 0 \Rightarrow C$ je konstant

$u = \frac{1}{3}(x^2+y^2)^{\frac{3}{2}} + xy$ izberemo da $\tilde{y} < 0$

$\uparrow$ prvi integral

splošno rešitev dobimo če iz $u(x,y) = C$

izrazimo y $\tilde{y} = \tilde{y}(x)$

$u(x, \tilde{y}(x)) \equiv C$

Da je $Pdx + Qdy = 0$ $(*)$ mora veljati

$$y' = -\frac{P}{Q} = -\frac{\mu P}{\mu Q} \rightsquigarrow \mu P dx + \mu Q dy = 0 \quad (**)$$

Ti dve sta ekvivalentni za poljubno rešitev

če $*$ ni eksakten mora biti $**$ eksakten

faktor μ se imenuje integrirajoči množitelj

V praksi ga skušamo uganiti z nastavki

$\mu(x)$, $\mu(y)$, $\mu(x,y)$

$\mu(x^n \pm y^m)$, $\mu(x^2+y^2)$ ...

3) Podana je DE

$$P\,dx + Q\,dy = 0$$

a) Dokaži $\mu = \mu(y) \Leftrightarrow \dfrac{Q_x - P_y}{P} = g(y)$

$\Rightarrow$ $\mu P\,dx + \mu Q\,dy = 0$ in μ integrirajoči množitelj

DE pa je eksaktna

$$(\mu P)_y = (\mu Q)_x$$

$$\mu_y P + \mu P_y = \mu Q_x$$

$$\frac{\mu(Q_x - P_y)}{P} = \mu'(y) \quad /:\mu$$

$$\frac{Q_x - P_y}{P} = \frac{\mu'(y)}{\mu(y)} \Big\} \text{ odvisno samo od } y$$

$\Leftarrow$ $\dfrac{Q_x - P_y}{P} = g(y)$ recimo da je

$$\ln \mu = \int \frac{Q_x - P_y}{P}\,dy \Rightarrow \mu = e^{\int \frac{Q_x - P_y}{P}dy}$$

Potem je $(\mu P)_y = \dfrac{Q_x - P_x}{P} P + \dots$ itd itd

pride vredu

b) $\underbrace{(xy^2 - y^3)}_{P}dx + \underbrace{(1-xy^2)}_{Q}dy = 0$

$Q_x = -y^2 \quad P_y = 2xy - 3y^2$

$\frac{Q_x - P_y}{P} = \frac{2y^2 - 2xy}{xy^2 - y^3} = \frac{2(y-x)}{y(x-y)} = -\frac{2}{y}$

$\mu = e^{-\int \frac{2}{y}dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$

$\mu P dx + \mu Q dy = 0$ je [illegible]

$u = \int \mu P dx + C(y) =$

$= \int (x-y)dx + C(y) = \frac{1}{2}x^2 - xy + C(y).$

$u_y = -x + C'(y) = \frac{1-xy^2}{y^2} = \frac{1}{y^2} - x$

$C'(y) = \frac{1}{y^2} \rightsquigarrow C = -\frac{1}{y}$

$u = \frac{x^2}{2} - xy - \frac{1}{y}$

μ

4) $y(x^2+y^2+1) - x(x^2+y^2-1)\,dx = 0$

$\mu = \mu(xy)$

$P_y = x^2 + 3y^2 + 1$ $\qquad \mu P\,dx + \mu Q\,dy = 0$

$Q_x = -3x^2 - y^2 + 1$ $\qquad \mu = \mu(xy)$

$$\left.\begin{aligned}(\mu P)_y &= x\mu' P + \mu P_y \\ (\mu Q)_x &= y\mu' Q + \mu Q_x\end{aligned}\right\} \overset{?}{+} \quad \mu'(xP - yQ) + \mu(P_y - Q_x) = 0$$

$$\frac{\mu'}{\mu} = \frac{-P_y + Q_x}{xP - yQ} = \frac{-4x^2 - 4y^2}{xy(2x^2+2y^2)} = \frac{-2}{xy}$$

$z = xy$ $\qquad \frac{\mu'(z)}{\mu(z)} = -\frac{2}{z} \quad / \int$

$$\frac{d\mu}{\mu(z)} = -\frac{2\,dz}{z} \quad / \int$$

$$|\mu(z)| = |z|^{\frac{1}{z}}$$

$$\mu(z) = \frac{1}{z^2} = \frac{1}{(xy)^2}$$

$$u = \int \mu P\,dx + c(y) = \int \frac{\cancel{y}(x^2+y^2+1)}{x^2 y^{\cancel{2}}}\,dx + c(y)$$

$$= \int \frac{1}{y} + \frac{y}{x^2} + \frac{1}{x^2 y} = \frac{x}{y} - \frac{y}{x} - \frac{1}{xy} + c(y)$$

$$u_y = -\frac{x}{y^2} - \frac{1}{x} + \frac{1}{y^2 x} + c'(y) =$$

4)

$$y\,dx - x\,dy = 2x^3 \tan\left(\frac{y}{x}\right)dx$$

$$z = \frac{y}{x} \qquad dz = \frac{dy}{x} \overset{-}{\cancel{+}} \frac{y}{x^2}dx \qquad dF = F_x dx + F_y dy$$

$$-x^2 dz = -x\,dy + y\,dx$$

$$-dz = 2x \tan(z)\,dx$$

$$-\frac{dz}{\tan(z)} = 2x\,dx$$

$$-\int \frac{\cos(z)}{\sin(z)}dz = \int 2x\,dx$$

$$-\ln|\sin(z)| = x^2 + C$$

$$-\ln \sin\frac{y}{x} = x^2 + c$$

$C = -x^2 - \ln\left|\sin\frac{y}{x}\right| = u(x,y)$ = prvi integra

Parametrično reševanje implicitno podanih DE

Do zdaj so enačbe bile oblike $y' = F(x,y)$

Lahko pa je v $F(x,y,y') = 0$

Poanta: $F(x,y,p) = 0$ podaja ploskev v $\mathbb{R}^3$

(če $F \in \mathcal{C}^1$ in $\nabla F \neq 0$)

Vsaka rešitev bo potekala po neki krivulji:

$\gamma(t) = (x(t), y(t), p(t))$ da je $F(\gamma(t)) = 0$

Da γ res podaja rešitev DE v parametrični obliki, mora veljati

$$\dot{y}(t) = p(t) \cdot \dot{x}(t)$$

Preprost primer:

$F(x, y') = 0$

$\Downarrow$

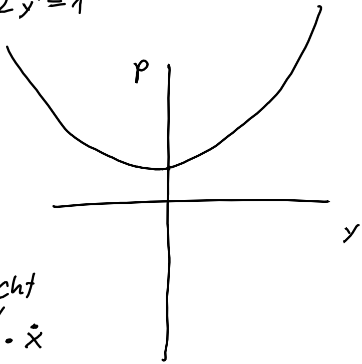
$F(x, p)$ podaja krivuljo v ravnini;

To krivuljo parametriziramo in poiščemo manjkajoči predpis za y komponento

6) Reši de

a) $(y')^2 - 2y^2 = 1$

b) $x^2 + 1 + (y')^2 - 2y' = 1$



$p^2 - 2y^2 = 1$

$p = \operatorname{ch} t$

$y = \frac{1}{\sqrt{2}} \operatorname{sh}(t)$

$\dot{y} = \frac{\operatorname{ch}(t)}{\sqrt{2}} = \overset{\operatorname{ch} t}{\overset{\shortparallel}{p}} \cdot \dot{x}$

$\dot{x} = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{2}{\sqrt{2}} t + c \qquad t = \sqrt{2}x$

krivulja je $\left(\frac{\sqrt{2}}{2} t + c, \frac{\sqrt{2}}{2} \operatorname{sh} t\right)$

$y = \frac{\operatorname{sh}(\sqrt{2}(x + c))}{\sqrt{2}}$

še druga rešitev $\frac{\operatorname{sh}(\sqrt{2}(c - x))}{\sqrt{2}}$ druga krivulja

b) $x^2+1+(y')^2-2y'=1$

$x^2+(p-1)^2=1$

$x=\cos t$

$p=1+\sin t$

$\dot y = p\dot x = (1+\sin t)(-\sin t) = -\sin^2 t-\sin t$

$y = \int \dot y\, dt = \int \frac{1-\cos 2t}{2}\, dt + \cos t =$

$= \frac{1}{2}t - \frac{1}{2}\sin 2t + \underbrace{\cos t}_{x}$

$y = \frac{1}{2}\arccos x - x\sin(\arccos x) + x$

izh: primeri

a):

p^2-2y^2-1

ēz $F_p=0$ $F_p: 2p=0 \Rightarrow p=0$

$\Rightarrow y$ konstantna

$y^2=-\frac{1}{2}$: realnih rešitev

b)

Od zadnjič: Parametrično reševanje DE 11.11
oblike $F(x,y,y')=0$

Splošni postopek
1. $p=y'$
2. izračunamo $DF(\underbrace{x,y,p}_{\text{neodvisne spremenljivke}})$
3. s pomočjo nastavka $dy=pdx$ dobimo diferencialno enačbo $x=x(p)$ ali $y=y(p)$.
Manjkajočo komponento dobimo iz prvotne enačbe

Včasih ima DE tudi singularne rešitve
1) $G(x,y,C)=0$ in enakosti $\frac{\partial G}{\partial C}=0$ eliminiramo C
2) parametriziramo množico $\{F=F_p=0\}$ in preverimo ali reši DE

odvod po p

1. naloga: Poišči splošne in morebitne sing. rešitve

$x^2(y')^3 - xy' + y = 0$

$y = xy' - x^2(y')^3 \qquad p = y' \qquad dy = p\,dx$

$y = xp - x^2p^3$

$dy = p\,dx + x\,dp - 2xp^3dx - 3x^2p^2dp$

$\cancel{p\,dx} = \cancel{p\,dx} - 2xp^3dx + (x - 3x^2p^2)dp$

$2xp^3dx = (x - 3x^2p^2)dp$

$x = 0 \Rightarrow y = 0$

$x \neq 0 \Rightarrow$

$2p^3dx = (1 - 3xp^2)dp$

$2p^3x'(p) = 1 - 3xp^2 \qquad p \equiv 0 \Rightarrow y \equiv 0$

$\dot{x} = \frac{1}{2p^3} - \frac{3}{2}\frac{x}{p^2}$

$\dot{x} + \frac{3}{2}\frac{x}{p^2} = \frac{1}{2p^3}$

homogeni del:

$\frac{dx}{x} = -\frac{3}{2}\frac{dp}{p}$

$|x| = C|p|^{-\frac{3}{2}}$

partikularni

$C = C(p)$

$\dot{x} = C'(p)p^{-\frac{3}{2}} = \frac{1}{2}p^{-3}$

$\int_0^p \frac{1}{2}p^{\frac{3}{2}-3} = \frac{1}{2}\int_0^p x^{-\frac{3}{2}}dx = -p^{-\frac{1}{2}}$

$C(p) = -p^{-\frac{1}{2}}$

$x = Cp^{-\frac{3}{2}} - p^{-2}$

$y = xy' - x^2(y')^3 = Cp^{-\frac{1}{2}} - p^{-1} - (C^2p^{-3} - 2Cp^{-\frac{3}{2}-2} + p^{-4})p^3$

$= Cp^{-\frac{1}{2}} - p^{-1} - C^2 + 2Cp^{-\frac{1}{2}} - p^{-1} =$

$= 3Cp^{-\frac{1}{2}} - 2p^{-1} - C^2$

Dobili smo rešitev v parametrični obliki $(x(p), y(p))$

kjer je $p = \frac{\dot{y}(p)}{\dot{x}(p)} = y'(x)$

Še singularne rešitev

$$x=-p^{-2}+Dp^{-\frac{3}{2}}$$

$$y=-2p^{-1}+3Dp^{-\frac{1}{2}}-D^2$$

$$F(x,y,p)=y-xp+x^2p^3=0$$

$$F_p=-x+3x^2p^2=x(1+3xp^2)$$

$x=0 \Rightarrow y=0$ ena točka ni ogrinjača

$$1+3xp^2=0 \Rightarrow p^2=+\frac{1}{3x} \Rightarrow p=\pm\sqrt{+\frac{1}{3x}}$$

$x>0$

$$y-x\left(\pm\sqrt{\frac{1}{3x}}\right)+x^2\frac{1}{3x}\left(\pm\sqrt{\frac{1}{3x}}\right)=0$$

$$y=\pm\sqrt{\frac{1}{3x}}\left(x-\frac{1}{3}x\right)=\pm\sqrt{\frac{1}{3x}}\cdot\frac{2}{3}x=\pm\frac{2}{3}\sqrt{\frac{x}{3}}$$

$$y'=\pm\frac{2}{3}\frac{\frac{1}{3}}{2\sqrt{\frac{x}{3}}}=\pm\frac{1}{9}\sqrt{\frac{3}{x}}$$

Ali reši enačbo

$$\pm\frac{2}{3}\sqrt{\frac{x}{3}}-x\left(\pm\frac{1}{9}\sqrt{\frac{3}{x}}\right)+x^2\left(\pm\frac{1}{9^3}\frac{3}{x}\cdot\sqrt{\frac{3}{x}}\right)=$$

$$=\pm\frac{2}{3}\sqrt{\frac{x}{3}}\mp\frac{1}{9}\sqrt{3x}\pm\frac{3}{9^3}\sqrt{3x}\neq 0$$

$$=\pm\frac{2}{3}\sqrt{\frac{x}{3}}\mp\frac{78}{9^3}\sqrt{3x}=\sqrt{x}\left(\pm\frac{2}{3\sqrt{3}}\mp\frac{78\sqrt{3}}{9^3}\right)=$$

$$=\sqrt{3x}\left(\pm\frac{2}{9}\right.\quad\ldots$$

2. način primerjamo y' in p

$$\pm\frac{1}{9}\sqrt{\frac{3}{x}}=\pm\sqrt{\frac{1}{3x}}$$

$$\pm\frac{\sqrt{3}\,3}{9\sqrt{3}}=\pm\frac{1}{\sqrt{3}}\quad\times$$

Zneloge

Dala si rešitve

$$(y')^2 - xy' - y + \frac{x^2}{2} = 0$$

$$y = (y')^2 - xy' + \frac{x^2}{2}$$

$$y = p^2 - xp + \frac{x^2}{2}$$

$$dy = 2p\,dp - x\,dp - p\,dx + x\,dx \qquad dy = p\,dx$$

$$2p\,dx - x\,dx = 2p\,dp - x\,dp$$

$$(2p - x)dx = (2p - x)dp$$

1. $2p - x = 0$

$$2p = x \Rightarrow p = \frac{x}{2}$$

$$y = p^2 - 2p^2 + 2p^2 = p^2$$

2. $2p - x \neq 0$

$$dx = dp \qquad\qquad y = 1 - x + \frac{x^2 + 2xc + c^2}{2} =$$

$$p'(x) = 1 \qquad\qquad = \frac{x^2 + 2x(c-1) + c^2 + 2}{2} =$$

$$p(x) = x + c \qquad\qquad = \frac{x^2}{2} + x(c-1) + c^2 + 2$$

??

$$y = \frac{x^2}{2} + xc + c^2$$

$\cancel{\text{Ali}}$ $\cancel{0 = x + 2c \rightsquigarrow c = -\frac{x}{2}}$

$\cancel{\frac{x^2}{2} - \frac{x^2}{2} - x + \frac{x^2}{4} + 2}$

singularne:

$$0 = x + 2c$$

$$c = -\frac{x}{2} \quad \rightsquigarrow y = \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{4} = \frac{x^2}{4}$$

Clairautova enačba

$$y = xy' + \psi(y')$$

Videli bomo, da ogrinjačo najdemo že sproti

3. naloga

a) $y = xy' + \frac{1}{y'}$

b) $y = xy' + (y')^2 + 1$

a) $y = xy' + \frac{1}{y'}$ $\qquad \psi(t) = \frac{1}{t}$ $\qquad y' = p$

$$y = px + \frac{1}{p}$$

$$dy = p\,dx + x\,dp - \frac{1}{p^2}dp \qquad p\,dx = dy$$

$$\frac{1}{p^2}dp = x\,dp$$

$$0 = \left(\frac{1}{p^2} - x\right)dp$$

Clairotova enačba se vedno reducira na obliko $f(x,p)dp = 0$

$dp = 0$ .... splošna rešitev

$f(x,p) = 0$ .... ogrinjača

1) $dp = 0 \Rightarrow p \equiv C$

$$y = xC + \frac{1}{C}$$

2) $\frac{1}{p^2} = x \Rightarrow p = \pm\sqrt{\frac{1}{x}}$

$$y = \pm\sqrt{x} \pm \sqrt{x} = \pm 2\sqrt{x}$$

$$y' = \mp\frac{1}{\sqrt{x}}$$

b) $y = xp + p^2 + 1$

$dy = x\,dp + p\,dx + 2p\,dp \qquad dy = p\,dx$

$(x + 2p)\,dp = 0$

$dp = 0 \Rightarrow p \equiv C \Rightarrow y = xC + C^2 + 1$

$x = -2p \Rightarrow p = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x^2 + \frac{1}{4}x^2 + 1 = -\frac{1}{4}x^2 + 1$

$y' = -\frac{1}{2}x$

Opombe o singularnih rešitvah Clairotovih enačb:

$y = xy' + \psi(y')$

$F(x,y,p) = xp + \psi(p) - y = 0$

$F_p = x + \psi'(p) = 0$

Splošna rešitev

$0 = G(x,y,C) = xC + \psi(C) - y = F(x,y,C) = 0$

Torej odvajanje v p ti vedno da ~~...~~ ogrinjačo pri Clairotovi enačbi

Lagrangeeva enacba

$$y = x\varphi(y') + \psi(y')$$

4. Reši

$$y = \frac{xy'}{2} + \frac{2}{y'} \qquad \varphi(y') = \frac{y'}{2}$$

$$dy = \frac{x}{2}dp + \frac{p}{2}dx - \frac{2}{p^2}dp \qquad dy = p\,dx$$

$$\frac{1}{2}p\,dx = \frac{x}{2}dp - \frac{2}{p^2}dp$$

$$p\,dx = \left(x - \frac{4}{p^2}\right)dp$$

$$\dot{x} = \frac{dx}{dp} = \frac{x}{p} - \frac{4}{p^3} \qquad \text{LDE za} \quad x = x(p)$$

$$\dot{x} - \frac{x}{p} = -\frac{4}{p^3}$$

homogeno:

$$\frac{dx}{x} = \frac{1}{p}dp$$

$$|x| = C|p| \qquad x = C\cdot p$$

parti:

$$C'(p)\cdot p = -\frac{4}{p^3} \Rightarrow C'(p) = -\frac{4}{p^4}$$

$$C = 3\cdot 4\; p^{-3}$$

$$x = 12\cdot p^{-3}\cdot p + C\cdot p \Rightarrow x = \frac{4}{p^2\cdot 3} + C\cdot p$$

$$y = \frac{1}{2}\left(\frac{4}{3p} + Cp^2\right) + \frac{2}{p} = \frac{1}{p}\left(\frac{2}{3} + \frac{1}{2}\right) + \frac{C}{2}p^2 = \frac{8}{3} + \frac{C}{2}p$$

Singularna rešitev: $0 = \frac{x}{2} - \frac{2}{p^2} = F_p$

$$\frac{2}{p^2} = \frac{x}{2}$$

$$p^2 = \frac{4}{x}$$

$$\Rightarrow y = \frac{2}{p^2}\cdot p + \frac{2}{p} = \frac{4}{p}$$

Ali je res $dy = p\,dx$?

$$\dot{y} = -\frac{4}{p^2} \qquad \dot{x} = -\frac{8}{p^3}$$

Ali velja $\dot{y} = p\cdot\dot{x}$

$$p\cdot\left(-\frac{8}{p^3}\right) = -\frac{8}{p^2} \neq -\frac{4}{p^2}$$

$\Rightarrow$ To ni singularna rešitev

5. Reši $y = \frac{x(y')^2}{2} + \frac{2}{y'}$.

Povzetek (o reševanju impl DE)

$F(x,y,y')=0 \quad \nabla F \neq 0$ F dovolj gladka

$F(x,y,p)$ je ploskev

$P(x_0,y_0,p_0) \in \mathbb{R}^3$

1) $F_p(P) \neq 0 \Rightarrow p = p(x,y)$

$F(x,y,p(x,y))=0$ za x,y blizu x_0,y_0

$p(x_0,y_0)=p_0$

če je $x \mapsto y(x)$ rešitev začetnega problema $F(x,y,y')$, $y(x_0)=y_0$

Potem je $y'=p(x,y)$

tako dobimo splošno rešitev

2) $F_p(P)=0 \Rightarrow$

dobimo množico $\{F=F_p=0\}$ ← unija točk in krivulj

če je P_0 izolirana točka:

na neki okolici U od P_0 velja

$\nabla F \neq 0$ za $P \in U - \{P_0\}$

dobimo splošno rešitev, vendar se lahko zgodi, da več članic splošne rešitve poteka skozi P_0

če je $\{F=F_p=0\}$ krivulja, jo parametriziramo npr kot $(x(p), y(p), p)$; če velja $p \cdot \dot{x}(p) = \dot{y}(p)$, potem je to singularna rešitev (in hkrati ogrinjača splošne rešitve

18.11

Primer od zadnjič

$y = \frac{xy'^2}{2} + \frac{2}{y'}$

1) $F_p = 0$

$F(x,y,p) = \frac{xp^2}{2} + \frac{2}{p} - y$

$F_p = 0 \Rightarrow xp - \frac{2}{p^2} = 0 \Rightarrow x = \frac{2}{p^3}$

$y = \frac{xp^2}{2} + \frac{2}{p} = \frac{2}{p^3} \cdot \frac{p^2}{2} + \frac{2}{p} = \frac{3}{p}$

$\dot{y} = -\frac{3}{p^2}$

$p\dot{x} = p\left(-\frac{6}{p^4}\right) = -\frac{6}{p^3} \neq \dot{y} \Rightarrow G$ ni ogrinjača

oz ni sing. reš

2) $dy = p\,dx$

$dy = xp\,dp + \frac{p^2}{2}dx - \frac{2}{p^2}dp$

...

$\left(\frac{p^2}{2} - p\right)dx + \left(xp - \frac{2}{p^2}\right)dp = 0$

Ker nas zanimajo rešitve v obliki grafov funkcij $x \mapsto y(x)$ lahko predpostavimo, da $dx \neq 0$

$\Rightarrow$ formalno lahko delimo z dx; dobimo DE za

$p = p(x)$.

Običajno je lažje reševati DE za $x = x(p)$

želimo deliti z dp, da dobimo $\frac{dx}{dp} = \dot{x}(p)$

Ali je lahko $dp = 0$? $\Rightarrow p \equiv C = y' \Rightarrow y(x) = Cx + D$ zadostuje D

Sicer $\Rightarrow \left(\frac{p^2}{2} - p\right)\frac{dp}{dx} + xp - \frac{2}{p^2} = 0$

Opazimo: $dp = 0 \Rightarrow \frac{p^2}{2} - p = 0$

$dp \neq 0 \Rightarrow$ dobimo DE, če $\frac{p^2}{2} - p \neq 0$

če $\frac{p^2}{2} - p = 0 \Rightarrow p = 0 \vee p = 2$

ni definirana

$\rightarrow y = 2x + 1$

Ali je rešitev singularna?

$F_p = xp - \frac{2}{p^2}$

Premica $y = 2x + 1$ ustreza krivulji $x \mapsto (x, y(x), y'(x)) = (x, 2x+1, 2)$

$F_p = 2x - \frac{1}{2} \Rightarrow$

Povsod razen v $x = \frac{1}{4}$ je ta izraz $\neq 0$

Ta premica torej pripada splošni rešitvi

$x = \frac{1}{4}$ pa je lahko singularna točka

Smo poiskali zadnjič

Povzetek povzetka

1) $F_p = 0$, parametriziramo/eliminiramo p iz enačb $F = F_p = 0$, preverimo če je rešitev
↳ singularna

2) Splošno rešitev iščemo s pomočjo nastavka $dy = p\,dx$

Naj bo $y = \Phi(x, C)$ splošna rešitev DE $F(x, y, y') = 0$ in ψ ogrinjača. Pokaži da skozi $\forall$ točko na grafu ψ poteka neskončno mnogo rešitev

DN

Reši DE

$y = xp^2 - 2p^3$

$y = (xy'^2) - 2(y')^3$

$dy = p\,dx = p^2 dx + 2px\,dx - 6p^2 dp$

$dx(p^2 + p) + 6p^2 dp = 0$

1) $dp \equiv 0 \Rightarrow p = c,\ y = c^2 x - 2c^3$

$y' = c^2 = p = c$

$c^2 = c \Rightarrow c = 0 \lor c = 1$

$\swarrow$ $y = c$ sing. $\qquad \searrow y = x - 2$ ni singularna

$F_p = F = 0$

$-2xp + 6p^2 = 0 \qquad p = 0 \Rightarrow$

$-2p(x - 3p) = 0 \qquad x = 3p \Rightarrow$

$F = 0 \Rightarrow 3p^3 - 2p^3 = 0$

$p^3 = 0$

$\dot{x}(p) = 3$

$\dot{y}(p) = 3p^2$

$p\dot{x} \neq \dot{y} \Rightarrow$ to ni rešitev

$dp \neq 0 \Rightarrow$

$\dot{x}(p - p^2) + 6p^2 - 2px = 0 \quad / : p(1-p)$ smo že obravnavali koje to 0

$\dot{x} + x\,\frac{-2p}{p - p^2} = \frac{6p^2}{p^2 - p}$

homogeni del:

$\frac{dx}{x} = \frac{-2p\,dp}{p - p^2} = \frac{-2}{1-p}\,dp$

$\ln|x| = -2\ln|1-p|$

$x = D(1-p)^{-2}$

partikularni:

$D'(1-p)^{-2} = \frac{6p}{p-1}$

$D' = 6p(p-1) = 6p^2 - 6p$

$D = 2p^3 - 3p^2 + C$

Splošno: $x = \frac{(2p^3 - 3p^2 + C)}{(1-p)^2}$

Enačbe višjega reda

$F(x, y^{(k)}, \dots, y^{(n)}) = 0 \qquad 0 \le k \le n$

Lahko znižamo red z nastavkom: $z = y^{(k)}$

$\hookrightarrow F(x, z, \dots, z^{(n-k)}) = 0$

$xy''' = y''$

$z = y''$

$xz' = z$

$\frac{dz}{z} = C\frac{dx}{x} \rightsquigarrow |x| = C|z| \Rightarrow x = C|z| = |y''|$

$z = C \cdot x = y''$

± skrijemo v konstanto

$\iint (Cx\,dx)\,dx = \int C\frac{1}{2}x^2 + D\,dx = \int Cx^2\,dx = C\frac{1}{3}x^3 + Dx + E =$

$= Cx^3 + Dx + E$

(2)

$F(y, y', \dots y^{(k)}) = 0$ $\hookrightarrow$ F neodvisen od x

"$z(y) = y'(y)$" zody je yodvod od y

Interpretiramo kot:

$z(y) = y'(x(y))$ $\qquad$ $z(y(x)) = y'(x)$

$y \mapsto x(y)$ invez rešitve y
(obstaja ker $y'(x) \neq 0$)

Reši enačbo

$yy'' = y'(y'+1)$ $\leftarrow$ Možnost ko ne moremo uporabiti nastavka

$y' = 0 \Rightarrow yy'' = 0$

1) $y = 0$

2) $y'' = 0 \Rightarrow$

$y' = c = 0 \Rightarrow y = c$

$\Rightarrow y = c \in \mathbb{R}$

$z = y'(x(y))$

$z(y(x)) = y'(x)$

$\frac{dz}{dy} \cdot y' = y''(x)$ $\qquad$ $\dot{z} y' = y''$

$\dot{z} \cdot z = y''$

Nesemo v enačbo

$y \cdot \dot{z} \cdot z = z(z+1)$

1) $z = 0$

2) $z \neq 0 \Rightarrow$ lahko delimo

2) $y \dot{z} = z + 1$

hom: $\frac{dz}{dy} \cdot y = z$ $\qquad$ $z = Cy$

partikularna:

$y \cdot C' y = 1$

$C' = \frac{1}{y^2} \rightsquigarrow C = -\frac{1}{y} + D$

$z = -1 + Dy$

$\frac{y'}{-1+Dy} = 1$

$x = \int \frac{dy}{-1+Dy} = \frac{1}{D} \ln(-1+Dy) + c$

$Dx + c = \ln(Dy - 1)$

$\frac{ce^{Dx} + 1}{D} = y$

$D = 0 \rightarrow$

$\frac{y'}{-1} = 1$

$y' = -1$

$y = c - x$

$F(x, y, y', \ldots) = 0$

F homogena v $y, y', \ldots$

$F(x, \lambda y, \lambda y', \ldots) = \lambda^k F(x, y, y' \ldots)$ za nek fiksen k in za $\forall \lambda \in \mathbb{R}$

... uvedemo $z(x) = \frac{y'}{y}$

$yy'' = x^2y^2 + (y')^2$

$z = \frac{y'}{y}$ $\quad yz = y'$ $\quad y'' = y'z + z'y = z^2y + z'y$

$y(yz^2 + z'y) = x^2y^2 + (yz)^2$

$y = 0 \Rightarrow$ je rešitev

$y \neq 0 \Rightarrow$

$\cancel{z^2} + z' = x^2 + \cancel{z^2}$

$z = \frac{1}{3}x^3 + C = \frac{y'}{y} \Rightarrow \ln|y| = \frac{x^4}{12} + Cx + D$

$y = De^{\frac{x^4}{12} + Cx}$

Eksistenčni izrek

$f: D \subset \mathbb{R}^2 \to \mathbb{R}$ zvezna in Lipschitzeva v y

$|f(x,y_0) - f(x,y_1)| < L|y_0 - y|$ za nek $L \geq 0$

$\Rightarrow \exists!$ rešitev $y' = f(x,y)$, $y(x_0) = y_0$ $\forall (x_0,y_0) \in D$, ki je definirana na intervalu $(x_0-\alpha, x_0+\alpha)$, kjer je $\alpha = \min\{a, \frac{b}{M}, \frac{1}{L}\}$, kjer sta $a,b>0$ taka, da je

$\underbrace{[x_0-a, x_0+a] \times [y_0-b, y_0+b]}_{C_{a,b}} \subset D$

in je ~~M maksimum~~

$M = \max_{x,y \in C_{ab}} |f(x_0,y_0)|$

opomba: $f \in C^1[a,b] \Rightarrow f$ je Lipschitzova

Podana je Cauchyjeva naloga

$y' = y + \cos y \qquad y(0)=0$

a) Utemelji da obstaja natanko ena rešitev in določi Taylorjev polinom stopnje 3 v $x=0$

b) določi α

c) Dokaži, da se rešitev razširi na $\mathbb{R}$

d) Dokaži da velja $\lim_{x\to\infty} y(x) = \infty$

a) F je zvezna na $\mathbb{R}^2$

Ali je Lipschitzeva? da ker je v C^1

$\Rightarrow$ Rešitev obstaja in je enolično določena

$y'' = y' - \sin(y)y' = (y+\cos y)(1-\sin(y))$

$y''' = (y' - y'\sin(y))(1-\sin(y)) + (y+\cos y)(1-\cos(y)y') =$

$= (y+\cos y)(1-\sin(y))^2 + (y+\cos(y))(1-y\cos y+\cos^2 y)$

$$y(x) = y(0) + \overset{1}{\overset{\shortparallel}{y'(0)}}x + \frac{1}{2}\underset{\frac{1}{2}x^2}{\underset{\shortparallel}{y''(0)}}x^2 + \frac{1}{6}\underset{0}{\underset{\shortparallel}{y'''(0)}} + o(x^3) =$$

$= x + \frac{1}{2}x^2 + o(x^3)$

b) $L = \max_{C(a,b)} \left|\frac{\partial f}{\partial y}\right|$ $f(x,y) = y + \cos y$

$\left|\frac{df}{dy}\right| = |1 - \sin(y)| \leq 2$ $L = 2$

$\alpha = \min(a, \frac{b}{M}, \frac{1}{2})$

$\max_{C(a,b)} f(x,y) = \max_{[-b,b]} f(\cdot, y) = M$

$|f(x,y)| = |y + \cos y| \leq |y| + |\cos y| \leq b + 1$

$\frac{b}{M} \geq \frac{b}{b+1}$

$\alpha \geq \min\{a, \frac{b}{b+1}, \frac{1}{2}\}$

če izberemo a in b dovolj velika, je

lahko $\alpha > \frac{1}{2}$

$\Rightarrow$ rešitev na $(-\frac{1}{2}, \frac{1}{2})$

c) Pogoj iz b velja za vsako točko.
Torej če si postopoma izbiramo točke vedno
tudi na robovih že znanega, lahko
razširimo intervale na cel $\mathbb{R}$

razen namesto b spodaj pišemo $b + \beta b$

Sklep: če lahko α omejimo navzdol neodvisno
ad začetnega pogoja, potem se rešitev
razširi na $\mathbb{R}$

V posebnem to velja: če sta f in f_y omejena

d) $\lim\limits_{y\to\infty} y(x)=\infty$

y naraščajoča in večja od 1

$y' > 0$ velja za $t>1$

$1-\sin t \geq 0$ za $t \geq 0 \Rightarrow$ fun narašča

Dokažemo da je $y(x)>0$ $\forall x>0$

Recimo da $y(\tilde{x})=0$ za nek $x'>0$

$y'(\tilde{x}) = 0+\cos(0)=1$

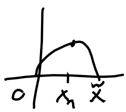
Torej je mogla biti prej funkcija negativna,

$Z = \{x, >0, x(x)=0)$

$y(0)=1$ $\quad y=f(x)$

$\Rightarrow f(x)>0$

za $x\in(0,\varepsilon)$ $\varepsilon>0$



Rolle: $x_1\in(0,\tilde{x})$

$f'(x_1)=0$

$f'(x_1)=\underset{\substack{\vee\\0}}{f(x_1)}+\cos(f(x_1))=0$

$\Rightarrow \cos f(x_1)<0$

$f(x_1)>\frac{\pi}{2}$

$f'(x_1)>\frac{\pi}{2}+\cos(x_1)>0$ $\quad *$

$\Rightarrow f(x)>0$ za $\forall x>0$

$\Rightarrow f'(0)$ je naraščajoča

Eksistenčni izrek

$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ zvezna in Lipschitzova

$\Rightarrow \exists!$ rešitev $y' = f(x,y)$ $y(x_0) = y_0$ za $\forall (x_0, y_0)$,
ki je definirana na intervalu $(x_0 - \alpha, x_0 + \alpha)$,
kjer je $\alpha = \min\{a, \frac{b}{M}, \frac{1}{L}\}$; $M = \max|f|$, L lipsch.,
$a, b > 0$. $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset D$

$f \in C^1[a,b] \Rightarrow f$ je lipschitzova

$F(x, y, y', \dots) = 0$

F homogena v y, y' $\left(F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y', \dots) \right)$

$\Rightarrow$ uvedemo $z(x) = \frac{y'}{y}$

Enačba višjega reda $F(x, y^{(k)}, \dots y^{(n)})$ $\rightsquigarrow z = y^{(k)}$

Reševanje impl. DE

$F(x, y, y') = 0$ $\nabla F \neq 0$

1) $F(x, y, p) = 0$ je ploskev

$P = (x_0, y_0, z_0)$

2.1) $F_p(P) \neq 0 \Rightarrow p = p(x,y)$

$F(x, y, p(x,y)) = 0$ v okolici P

$p(x_0, y_0) = p_0$

$y(x)$ rešitev $\Rightarrow y' = p(x,y)$

Dobimo splošno rešitev

2.2) $F_p(P) = 0 \Rightarrow$ $\{F = F_p = 0\}$ je unija točk in krivulj

P_0 izolirana točka $\Rightarrow \exists U$ okolica P. $P \in U - \{P_0\}$. $\nabla F \neq 0$

Dobimo splošno rešitev

$\{F = F_p = 0\}$ je krivulja $\Rightarrow$ parametriziramo
npr $(x(p), y(p), p)$.

$p \cdot \dot{x}(p) = \dot{y}(p) \Rightarrow$ Singularna rešitev

Ogrinjača splošne rešitve

Clairautova enačba

$y = xy' + \psi(y') \rightsquigarrow y' = p \rightsquigarrow p\,dx = dy$

Dobimo: $f(x,p)\,dp = 0$

$dp = 0$ ... singularna rešitev

$f(x,p) = 0$ ... ogrinjača

Odvajanje po p, ti vedno da ogrinjačo pri tej enačbi

Prvi integral enačbe $y' = f(x,y)$ je funkcija, ki je konst. vzdolž vsake rešitve

$y = \varphi(x,C) \wedge \varphi_C \neq 0 \Rightarrow C = C(x,y)$

1) DE zapišemo kot $P\,dx + Q\,dy = 0$

$$Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y \Rightarrow (P,Q) \text{ je potencialno}$$

$$Q = \frac{\partial}{\partial y}\int P\,dx + C'(x,y)$$

Riccatijeva enačba

$y' = a^2 y + by + c$

1) Uganemo rešitev y_p

2) Splošna rešitev je $y = y_p + u(x)$

3) Dobimo Bernullijevo enačbo $y = y_p + \frac{1}{u^2(x)} \Rightarrow$ LDE

Bernullijeva enačba

$y' + ay = by^{\alpha} \quad \alpha \notin \{0,1\} \rightsquigarrow z = y^{1-\alpha}$

Od zadnjič naprej $\quad y' = y + \cos y \quad y(0) = 0$

Rešitev φ je definirana za $\forall x \in \mathbb{R}$

$y > 0$ za $\forall x \geq 0$

$$Z = \{x > 0,\ \varphi(x) = 0\}$$

$$\varphi'(0) = \varphi(0) + \cos(0) = 0 + \cos 0 = 1$$

$\Rightarrow$ za nek $\varepsilon > 0$ je $\varphi(x) > 0 \quad \forall x \in (0, \varepsilon)$

označimo $x_0 = \inf Z > 0$

φ zvezna $\Rightarrow \varphi(x_0) = 0$

Rolle: $\exists x_1 \in (0, x_0).\ \varphi'(x_1) = 0 \ \wedge\ \varphi(x) > 0 \quad \forall x \in (0, x_0)$

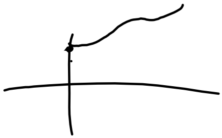
φ reši DE $\Rightarrow \varphi'(x_1) = \underset{\overset{\vee}{0}}{\varphi(x_1)} + \cos(\varphi(x_1))$

$\Rightarrow \varphi(x_1) > \frac{\pi}{2}$

$\Rightarrow \varphi(x_1) - \cos(\varphi(x_1)) > 0$

$\Rightarrow Z = \emptyset \quad$ ↯

$g(t) = t + \cos t \quad g(0) = 1 \quad g'(t) = 1 - \sin t \geq 0 \ \forall t$



$g(t) \geq 1.\ \forall\ t \geq 0$

$$\varphi'(x) = g(\underset{\overset{\vee\!|}{0}}{\varphi(x)}) \geq 1 \quad \forall x \geq 0$$

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(t)\,dt \geq \int_0^x 1\,dt = x \xrightarrow{x \to \infty} \infty$$

1) Dana je Cauchyjeva naloga

$$y' = \frac{y}{1+x^2+y^2} \qquad y(0)=1$$

Dokaži da obstaja ! rešitev, ki je definirana na $\mathbb{R}$.

Ali ima φ ničlo?

$f(x,y)$ mora biti zvezna in lipsicheva v y

$$f(x,y) = \frac{y}{1+x^2+y^2}$$ zveznost zaradi elementarnosti

Naredimo pravokotnik $[a,a] \times [b, b]$

Ta m je lipscicheva, ker je odvod omejen

$\Rightarrow$ φ obstaja in je enolično določena na $[-\alpha, \alpha]$

kjer je $\alpha = \min\{a, \frac{b}{M}, \frac{1}{L}\}$

a, b poljubno velika

$$M = \max_{[a,b]} |f(x,y)| = \max \frac{y}{1+y^2} = \{|\pm \tfrac{1}{2}|\} = \frac{1}{2}$$

$x^2 = 0$

$$\hookrightarrow \frac{(1+y^2) - 2y^2}{(1+y^2)^2} = \frac{1-y^2}{1+y^2}$$

$$L = \max \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{1+x^2+y^2-2y^2}{(1+x^2+y^2)} = \frac{1+x^2-y^2}{(1+x^2+y^2)}$$

L je tudi omejena

Dobili smo α, ki je isti za vse točke in neodvisen od začetnega pogoja, tako da lahko z eksistenčnim izrekom razširimo na vse točke

b) $y' = 0 \Leftrightarrow y = 0$

Recimo da je $\varphi(x_0) = 0$

$z \equiv 0$ $\quad z' = f(x,y)$ reši $z(x_0) = 0$

φ tudi reši to nalogo $\Rightarrow \varphi = z$ (zaradi enoličnosti)

$\times$, ker je $\varphi(0) = 1$

Trditev: Za α lahko vzamemo $\alpha = \min\{a, \frac{b}{M}$

L mora pa obstajat na tem pravokotniku

2)

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ zvezna

$$y'(x) = 1 + \frac{y(x)^2}{2} \sin f(x) \quad ; x \in [0,2]$$

$$y(0) = 0$$

Dokaži da je

$$\max_{x \in [0,2]} y(x) \geq 1$$

Recimo da $\max y(x) < 1 \Rightarrow$

$$y'(x) \geq 1 + \frac{y(x)^2}{2} \geq \frac{1}{2}, \text{če } y \in [0,1]$$

$y \geq 0$

$x_0 = \inf\{x; y(x_0) = 0\}$

Recimo da $\exists x_0 > 0$ da je $y(x_0) = 0$

Potem po Rollovem izreku, $\exists x_1 \in (0, x_0)$ da je $y'(x_1) = 0$

$$y'(x_1) = 1 + \frac{(y(x_1))^2}{2} \sin(f(x_1))$$

$$0 \leq y(x_1) \leq 1$$

$$y'(x_1) \geq 1 - \frac{(y(x_1))^2}{2} \geq \frac{1}{2}$$

$\Rightarrow y > 0$ za $\forall x \in \mathbb{R}_{>0} [0,2]$ ker jo odvod v 0 pozitiven

$$\Rightarrow y'(x) \geq 1 - \frac{1}{2} \geq \frac{1}{2}$$

Potem je $y(2) = y(0) + \int_0^2 y'(t)\,dt \geq 0 + 2 \cdot \frac{1}{2} = 1$ $\times$

$\Rightarrow \max y \geq 1$

3.

$y' = f(y) \qquad f \in C^1(\mathbb{R})$

Pokaži da je $\forall$ periodična rešitev konstantna

Recimo da $\exists\, \omega > 0.\ y(x+\omega) = y(x)\ \forall x$

$$\int_0^\omega y' dx = \int_0^\omega f(y) dx = c$$

y periodična $\Rightarrow$ $f(y)$ periodična

če je $f(c) = 0$, potem je $y \equiv c$ rešitev

$f(y)$ je odvod $\Rightarrow$ f ima ničlo (ker je y periodična)

$\Rightarrow \exists x_0 \in \mathbb{R}.\ f(y(x_0))$ $\quad y \equiv y(x_0)$ je ena rešitev

sistema in zaradi enoličnosti je to edina

4.

$y' = y + e^y \qquad y(0) = 1$

a) Dokaži, da je definicijski interval rešitve, ki ga določa eksistenčni izrek omejen

b) Dokaži da je φ naraščajoča v $\forall$ točki $x \geq 0$, kjer je definirana

c) Dokaži da ima φ ničlo

d) $\exists\ \omega .\ 0 < \omega < \infty .\ \lim\limits_{x \to \omega} \varphi(x) = \pm\infty$

---

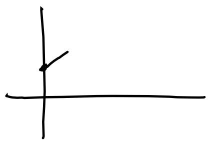
a) $y + e^y$ je zvezno odvedljiva $\Rightarrow \exists$ enolična rešitev na $[\alpha, -\alpha]$ $\quad \alpha = \min\{a, \frac{b}{M}, \cancel{\frac{1}{L}}\}$

$y \in [1-b, 1+b] \Rightarrow y + e^y \leq 1 + b + e^{1+b}$

$M = 1 + b + e^{1+b} \Rightarrow \frac{b}{M} = \frac{b}{1+b+e^{1+b}} \leq C \ \forall b > 0$ ta funkcija je omejena

b) $y' = y + e^y$

$\underline{y' \geq 0} \qquad y(0) = 1$



Recimo da je $y + e^y = 0$

$$\begin{array}{c} e^y = -y \\ \vee \\ 0 \end{array}$$

če je $y = 0$ potem je $y' = 1$ ker pomeni da v vsaki ničli narašča, ker pa ne more biti res razen za eno ničlo

c) f ima ničle

Recimo da nima ničle. potem je $f(x)>0 \ \forall x\in\mathbb{R}$,
potem je pa f naraščajoča
in odvod je večji od 1. potem pa ne more sekati x os ali pa recimo naj je ne moremo razširiti če je pd, ne more biti naglnen, tako je pozitiven, potem je pa odvod

<0 $\cancel{\ }$

$$f(x)=f(c)+\int_a^x f'\,dt \leq f(c)+x \xrightarrow{x\to-\infty} -\infty$$

d) Dokazujemo:

$\exists\omega>0.\ \lim_{x\to\infty} f(x)=\infty$

Dokažimo da je inverz omejen. (inverz obstaja ker je f strogo naraščajoča na $x\geq 0$

$$\frac{1}{z'}=y+e^y \Rightarrow z'=\frac{1}{y+e^y}$$

dostaja tak M

$$z(y)=\int_1^y \frac{1}{t+e^t}\,dt \leq \int_1^y \frac{1}{e^t}\,dt = e^{-1}-e^{-y} \leq M$$

$t\geq 0$

Naj bo $f \in C^1(\mathbb{R}^2)$, zadošča predpostavkam eksistenčnega izreka. Recimo, da se naše rešitve φ ne da razširiti čez interval (α, ω); $-\infty < \alpha < x_0 < \omega < \infty$, potem velja $\lim_{x \to \omega} y(x) = \pm\infty$ in $\lim_{x \to \alpha} y(x) = \pm\infty$

5)

$$y' = \frac{1}{x^2+y^2} \quad ; \quad y(0) > 0$$

a) Pokaži, da $\exists!\ \varphi$ rešitev, ki je definirana na $\mathbb{R}$

b) pokaži $\lim_{x \to \infty} \varphi(x) < \infty$

$f(x,y) = \frac{1}{x^2+y^2}$ je omejena razen v okolici 0

$f_y(x,y) = \frac{2y}{(x^2+y^2)^2}$ je omejen razen na okolici 0

Dokažimo da je rešitev omejena na vsakem končnem intervalu

v 0 je rešitev pozitivna in narašča

$$\varphi(x) = \varphi(0) + \int_0^x \frac{1}{t^2+\varphi(t)^2}\,dt = \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2+(\varphi(t))^2}\,dt \leq$$

$$\leq \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2}\,dt = \varphi(x_0) - \frac{1}{x} + \frac{1}{x_0} \leq M$$

omejeno

$\Rightarrow$ Po trditvi je $\omega = \infty$

Popravimo f v $\tilde{f}$ tako, da nima več pola v $(0,0)$ in velja $f = \tilde{f}$ izven neke majhne okolice $(0,0)$ ki je graf φ ne seka

v ~~...~~ $-\infty$ smer je simetrična $y(-x)$

b)

Pokazali smo da je $\varphi(x)$ naraščajoča in omejena $\Rightarrow$ ima limito

Vektorsko polje na $\mathbb{R}^n$ je preslikava $V : \mathbb{R}^n \to \mathbb{R}^n$.

Tokovnica polja V je krivulja $\gamma : (a, b) \to \mathbb{R}^n$, ki
$\overset{\Psi}{0}$

zadošča, da je $\dot{\gamma} = V(\gamma(t))$

$$\gamma(t) = (x_1 \ldots x_n)$$
$$\dot{x}_1 = V_1(\gamma(t))$$
$$\vdots$$
$$\dot{x}_n = V_n(\gamma(t))$$

Do zdaj: obravnavali smo primere za $n = 0$, vendar v

povečini oblike $y'(x) = f(x, y(x))$

$\dot{\gamma}$ ... γ neavtonomni sistemi

Enačbi $y' = f(x, y)$ priredimo sistem oz avtonomno

vektorsko polje

$$\dot{x} = 1$$
$$y = f(x, y)$$