Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x\,dx$$

FV(f) konvergira k f, če je f zvezna v x, če pa ni, pa konvergira k $\frac{f(x-)+f(x+)}{2}$

$$\int x\cos(nx)\,dx = \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int x\sin(nx)\,dx = -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

f sode $\Rightarrow b_n = 0$

f liha $\Rightarrow a_n = 0$

$$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$$

$$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$$

2

$f(x) = |x|$ razvij v Fourierovo vrsto in seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\,dx \underset{\text{sodost}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right)\Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(nx) \overset{\text{lihost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2}\cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi}\cdot\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{1}\frac{1}{(2k+1)^2} = -\frac{\pi}{2}\cdot\frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \overset{?}{=} S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

S ... lih.

S' ... ostalo

$$= S + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) =$$

$$= S + \frac{1}{4}\Sigma = \Sigma$$

$$S = \frac{3}{4}\Sigma$$

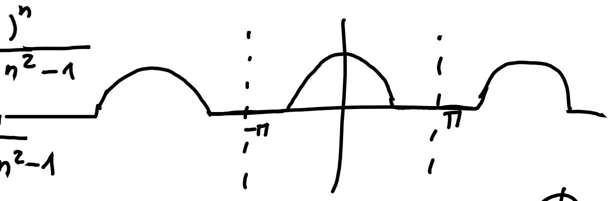
$$\Sigma = \frac{4}{3}S = \frac{\pi^2}{6}$$

③

$f(x) = \max(\cos x, 0)$

$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$

$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$



$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(x) = \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

ničelnost $f(x)$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos(nx) \cancel{+ 2\int_{\frac{\pi}{2}} \cos(nx)\,dx}$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right) =$

$n \neq 1$

$\frac{1}{\pi}\left[\frac{1}{n+1}\sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1}\sin\left((n-1)\frac{\pi}{2}\right)\right]$

$n = 4k: \frac{1}{\pi}\frac{1}{n+1}\cdot 1 + \frac{1}{n-1}(-1)\frac{1}{\pi} = \frac{-2}{n^2-1}\cdot\frac{1}{\pi} \quad \frac{-2}{16k^2-1}\cdot\frac{1}{\pi}$

n lih: 0

$n = 4k+2: \frac{1}{\pi}\frac{1}{4k+3}\sin\frac{3\pi}{2} + \frac{1}{\pi}\frac{1}{4k+1}\sin\frac{\pi}{2} =$

$= \frac{1}{\pi}\left(-\frac{1}{4k+3} + \frac{1}{4k+1}\right)$

$= \frac{1}{\pi}\begin{cases} 0 & ; n \text{ liho} \\ \frac{(-1)^{m+1}\cdot 2}{(2n)^2-1} & ; n = 2m \end{cases}$

$a_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{1}{\pi} + \frac{\sin(2x)}{2}\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}$

$b_n = 0$ ker f soda

$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\cdot 2}{(2m)^2-1}\cos(2mx)$

$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}S_1 \Rightarrow$

$S_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)\cdot\frac{\pi}{2}(-1) = \frac{1}{2} - \frac{\pi}{4}$

$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2-1}\cdot\underset{\substack{\parallel \\ (-1)^m}}{\cos(m\pi)} =$

$= \frac{1}{\pi} - \frac{2}{\pi}\underset{\substack{\parallel \\ S_2}}{\sum_{m=1}^{\infty} \frac{1}{4m^2-1}}$

$S_2 = -\frac{1}{\pi}\cdot\frac{\pi}{2} = -\frac{1}{2}$

(4)

$f(x) = x^2 \qquad f: [0,\pi] \longrightarrow \mathbb{R}$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v Sinusno FV in skiciraj njen graf

1) Če f razširimo do sode funkcije

$f_s : [-\pi,\pi] \longrightarrow \mathbb{R}$

$x<0 \Rightarrow f_s(x) = f(-x)$

$FV_{\cos}(f)(x) = FV(f)(x)$

2) $f_l : [-\pi,\pi] \longrightarrow \mathbb{R} \qquad f_l(x) = -f(-x)$

$FV_{\sin}(f)(x) = FV(f_l)(x)$

1) $f_s(x) = x^2$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$b_n = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$

Poglejmo:

$\int x^2 e^{inx} \qquad =$

$\boxed{\int x e^{inx} = -\frac{i\cdot x\cdot e^{ix}}{n} + \frac{e^{inx}}{n^2}}$

$u = x^2 \qquad dv = e^{inx}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{in}e^{inx}$

$= x^2\cdot\frac{e^{inx}}{in} - \int 2x\,\frac{e^{inx}}{in}\,dx \overset{\text{od prej}}{=}$

$= x^2\cdot\frac{e^{inx}}{in} - \frac{2}{in}\left(\frac{ixe^{inx}}{n} + \frac{e^{inx}}{n^2}\right) + C$

$= e^{inx}\left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3}\right)$

$\overset{\shortparallel}{\cos nx + i\sin nx}$

$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$

$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$

$a_n = \frac{1}{\pi}\left(\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) + \frac{2}{n^3}\sin(nx)\right)\Big|_{-\pi}^{\pi} =$

$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right) = \frac{4(-1)^n}{n^2}$

$FV\cos(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(x$

b) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\text{soda}}{f_p(x)\sin(x)} = \frac{2}{\pi} \int_0^{\pi} \underbrace{f_p(x)\sin(nx)}$

$= f(x)$ na $[0,\pi] = x^2$

$$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2}{n^2}\sin nx + \frac{2}{n^3}(\cos(nx))\right]\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right)$$

$$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left((-1)^{n+1}\frac{\pi^2}{n} + \frac{2(1-(-1)^n)}{n^3}\right)\sin(nx)$$

$f(x) = x(\pi - x)$ [illegible]

⑥

$f(x) = \sin^3 x$ razvij v FV

Pred premislek:

$f(x) = \sin 2x$ je že FV

$b_2 = 1$, ostali so 0

$f(x) = \sin^2 x = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) =$

$= \frac{1}{2}(\cos(0) - \cos(2x)) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2}\sin x - \frac{1}{2}\cos 2x \sin x =$

$= \frac{1}{2}\sin x - \frac{1}{2}\cdot\frac{1}{2}(\sin(3x) - \sin(x)) =$

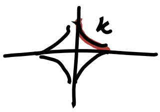
$= \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$

polinom

$\forall p(\sin x, \cos x)$ ima končno furierovo vrsto

$p \in \mathbb{R}[x,y]$

$a>0$ Izračunaj težišče homogenega

loka astroide 

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

$$x_T=\frac{\int_K x\,dm}{m(K)}=\frac{\int_K x\rho\,ds}{\int_K \rho\,ds}\quad = l(K)$$

$\vec{r}(t)$ ... parametrizacija

$$ds=|\dot{\vec{r}}(t)|\,dt=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt$$

u .... skalarno polje

$$\int_K u\,ds=\int_a^b u(\vec{r}(t))\,|\dot{\vec{r}}(t)|\,dt$$

$$x=a\cos^3 t\qquad y=a\sin^3 t$$

$t\in[0,\frac{\pi}{2}]$ ← ker je vse pozitivno (gledamo pravilok)

$$\dot{\vec{r}}(t)=(-3a\cos^2 t\cdot\sin t,\ 3a\sin^2 t\cos t)$$

$$|\dot{\vec{r}}|=3a\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}=$$
$$=3a\cos t\sin t$$

$$l(K)=\int_0^{\frac{\pi}{2}}|\dot{\vec{r}}(t)|\,dt=\int_0^{\frac{\pi}{2}}3a\cos t\sin t\,dt=$$

$u=\sin t\,dt$
$du=\cos t\,dt$

$$=3a\int_0^1 u\,du=\frac{3}{2}a$$

$$\int_K x\,ds=\int_0^{\frac{\pi}{2}}a\cos^3 t\ 3a\cos t\sin t\,dt=$$

$\cos t=u\qquad du=-\sin t$

$$=3a^2\int_0^1 u^4\,du=\frac{3}{5}a^2$$

$$y_T=\frac{3}{5}a^2$$

$a>0 \quad \alpha\in[0,2\pi]$

$K = S(0,a)\cap\Pi$

$\Pi: y = x\tan\alpha$

$I: \int_K (y-z) + (z-x)dy + (x-y)dz$

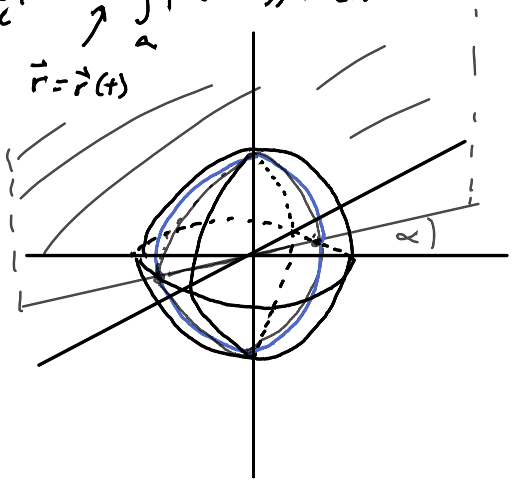
$\vec f(x,y,z) = (y-z,\ z-x,\ x-y)$ vektorsko polje

$I = \int_K \vec f\, d\vec r$

$\vec r = (x,y,z) \Rightarrow d\vec r$

$\int_K \vec f\, d\vec r = \int_a^b \vec f(\vec r(t))\, \dot{\vec r}(t)\, dt$

$\vec r = \vec r(t)$



Parametrizacja K

$x = a\cos\alpha\cdot\cos\vartheta$

$y = a\sin\alpha\cos\vartheta$

$z = a\sin\vartheta$

$r(\vartheta) = (a\cos\vartheta\cos\alpha,\ a\cos\vartheta\sin\alpha,\ a\sin\vartheta)$

$\vartheta\in[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\dot{\vec r}(\vartheta) = (-a\sin\vartheta\cos\alpha,\ -a\sin\vartheta\sin\alpha,\ a\cos\vartheta)$

$\vec f(\vec r(\vartheta)) = a(\cos\vartheta\sin\alpha - \sin\vartheta,\ \sin\vartheta - \cos\vartheta\cos\alpha,$
$\cos\vartheta\sin\alpha - \cos\vartheta\cos\alpha)$

$\vec f\cdot\dot{\vec r} = a^2(-\cancel{\cos\alpha\cos\vartheta\sin\alpha\sin\vartheta} + \sin^2\vartheta\cos\alpha\ \cancel{\cos}$

$-\sin^2\vartheta\sin\alpha + \cancel{\cos\vartheta\cos\alpha\sin\vartheta\sin\alpha} +$

$+\cos^2\vartheta\sin\alpha - \cos^2\vartheta\cos\alpha) =$

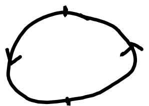
$= a^2(\cos 2\vartheta\sin\alpha - \cos 2\vartheta\cos\alpha)$ ✗

Ne

$= a^2(\cos\alpha - \sin\alpha$

$\int_K \vec f\, d\vec r = \int_0^{2\pi} a^2(\cos\alpha - \sin\alpha)\, d\vartheta =$

$2\pi a^2(\cos\alpha - \sin\alpha)$

↑ pri orientaciji 

$a, b, c \in \mathbb{R}$

$$I = \int_S a\, dy dz + b\, dx dz + c\, dx dy$$

$$S = \{(x,y,z) \; ; \; x^2+y^2+z^2 = 1, \; x, y, z \geq 0\}$$

$$I = \int \vec{f}\, d\vec{S}$$

$$\vec{f} = (a, b, c)$$

enotske normale

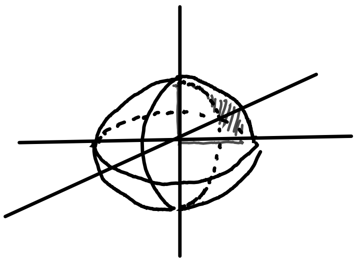
$$\int_S (\vec{f} \cdot \vec{N})\, dS$$

orientacija ploskve je usmerjena s smerjo normale

$$\int_S \vec{f}\, d\vec{S} = \int_D \vec{f}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du dv$$

$r = \vec{r}(u, v)$

smer se ujema s predpisano orientacijo



Sferične koordinate

$x = \cos\vartheta \cos\varphi \qquad \vartheta \in [0, \frac{\pi}{2}]$

$y = \cos\vartheta \sin\varphi \qquad \varphi \in [0, \frac{\pi}{2}]$

$z = \sin\vartheta$

$$\vec{r}(\vartheta, \varphi) = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, \sin\vartheta)$$

$$\vec{r}_\vartheta = (-\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\vartheta)$$

$$\vec{r}_\varphi = (-\cos\vartheta\sin\varphi, \cos\vartheta\cos\varphi, 0)$$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta\cos^2\varphi - \cos\vartheta\sin\vartheta\sin^2\varphi) =$$

$$= (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta) =$$

$$= -\cos\vartheta\,\underbrace{(\cos\vartheta\cos\varphi, -\cos\vartheta\sin\varphi, \sin\vartheta)}$$

$$\int_S \vec{f}\, d\vec{S} = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (-a\cos^2\vartheta\cos\varphi + b\cos^2\vartheta\sin\varphi - c\sin\vartheta\cos\vartheta)\, d\vartheta$$

$$= \int_0^{\frac{\pi}{2}} \int \left((-a\cos\varphi + b\sin\varphi)\cos^2\vartheta - \frac{c}{2}\sin 2\vartheta\right) d\vartheta =$$

Prevedemo na B funkcijo

$$= \int_0^{\frac{\pi}{2}} (-a\cos\varphi + b\sin\varphi)\frac{\pi}{4} - \frac{c}{2}) =$$

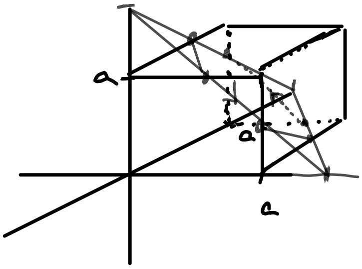
$$= \left(-\frac{\pi a}{4}\sin\varphi + \frac{\pi}{4} b\cos\varphi - \frac{c}{2}\varphi\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4}a - \frac{\pi}{4}b - \frac{\pi}{4}c = -\frac{\pi}{4}(a+b+c)$$

$a > 0$

$\vec{f}(x,y,z) = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z = \frac{3a}{2}$



krivulja 6 kotnik

cirkulacija.... integral vektorskega polja vzdolž sklenjene krivulje

K_1:

$x = x$

$y = 0$

$z = \frac{3a}{2} - x$

$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$

$$\int_{K_1} \vec{f} d\vec{r}$$

$\dot{\vec{r}}(x) = (1, 0, -1)$

$\vec{f}(r(x)) = (0 - (\frac{3}{2} - x)^2, (\frac{3}{2}a - x)^2 - x^2, x^2 - 0)$

$\vec{f} \cdot \dot{\vec{r}} = -(\frac{3}{2}a - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$

$$\int_{a/2}^{a} (-(\tfrac{3}{2}a - x)^2 - x^2)\, dx =$$

$$-\frac{1}{3}(\tfrac{3}{2}a - x)^3 \Big|_{a/2}^{a} - \frac{1}{3}x^3 \Big|_{a/2}^{a} =$$

$$= \cancel{-\frac{1}{3}\frac{a^3}{8}} + \cancel{\frac{1}{3}a^3} - \cancel{\frac{1}{3}a^3} + \cancel{\frac{1}{3}\frac{a^3}{8}} = 0$$