Navadna diferencialna enačba (NDE) je enačba oblike

$f(x,y,y')=0$ iščemo $y(x)$ da velja

$f(x,y(x),y'(x))$ za $\forall x \in D_y$

(To je v implicitni obliki)

Mi se bomo večinoma ukvarjali s temi, ki je podano v eksplicitni obliki, torej ko je $y'=f(x,y)$

Enačba reda $n \in \mathbb{N}$ je oblike $G(x,y,y',\ldots,y^{(n)})=0$

Enačba je **avtonomna**, če funkcija G ni odvisna od x. Sicer je neavtonomna.

7.10

1. Za dano družino funkcij poišči pripadajočo DE

a) $y = ce^x$

b) $y^2 = cx$

c) $y = c(x-c)$

$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad f(x,y,c)=0 \quad c \in \mathbb{R}$

$y' = ce^x \Rightarrow y = y'$

$y^2 = cx$

$2yy' = c \Longrightarrow y^2 = 2yy'x$

$y = 2y'x$ za $y \neq 0$

$y' = c$

$y = y'(x-c)$

2) ugani rešitev naslednjih DE

a) $y'' = -y \Rightarrow \{\alpha \cos x + \beta \sin x\}$

b) $y + xy' = \cos x$

c) $xy' = ny$

b)

$(xy)' = \cos x$

$xy = \sin x + C$

$y = \frac{\sin x + C}{x}$

c) $xy' = ny$

$xy' + y = n\cancel{y} + x$

$(xy)' = (n+1)y$

$y = x^n$

Metoda izoklin

$y_c = y = y(x,c) \quad x, c \in \mathbb{R}$

$y \in \mathcal{C}^1$

Izoklina je krivulja vzdolž katere ima vsaka članica družine y enak odvod po x $(y_c'(x))$

$I_\alpha = \{(x,y) ; y = y_c(x)$ potem je $y_c'(x) = \alpha$

Če imamo DE oblike $y' = f(x,y)$ in pripadajočo družino rešitev $y_c(x)$, potem so izokline ravno nivojnice $\{f = \alpha\}$

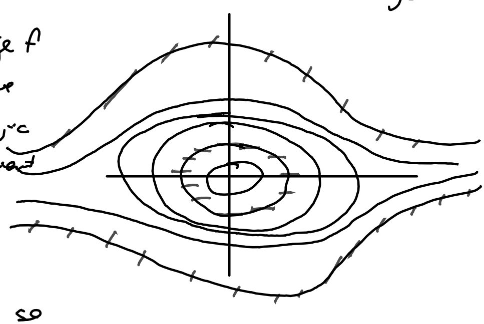
tj. $y_c'(x) = f(x, y_c(x)) = \alpha$

(črno so nivojnice) mi iščemo funkcijo

Postopek:

1. Skiciramo funkcije f

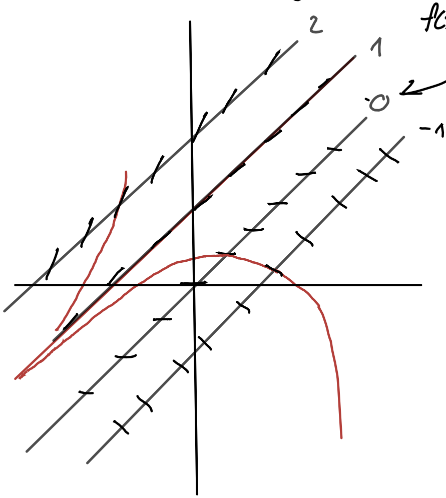
2) vzdolž vsake nivojnice narišemo nekaj daljic ki imajo smerni koeficient enak ~~nivojnica na~~ vrednosti f na nivojnici



3) narišemo krivuljo ki so v presečiščih nivojnic tangentne nanavnio

3. Približno skiciraj potek enačbe $y' = y - x$

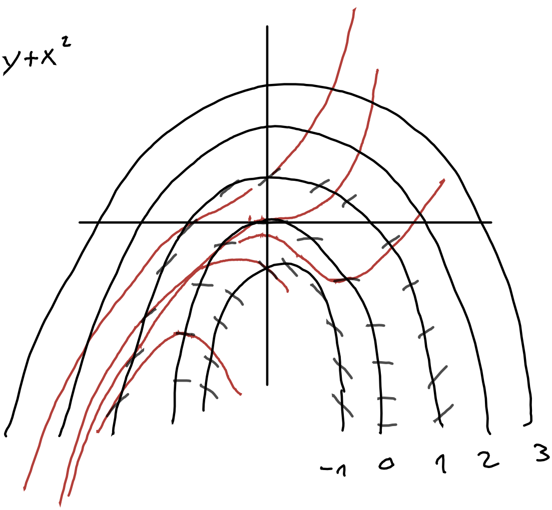
$f(x,y) = y - x$



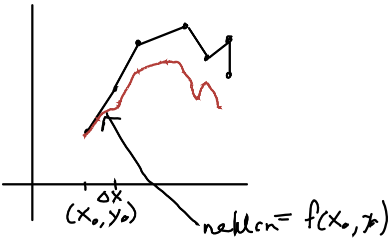
Rešitev 1:

$y = x + 1$

$y' = y + x^2$



Eulerjeva metoda



$x_1 = x_0 + \Delta x$

$y_1 = y_0 + \Delta x \, f(x, y_0)$

$x_2 = x_1 + \Delta x$

$y_2 = y_1 + \Delta f(x, y_0)$

Izberemo si Δx in iterativno definiramo

$x_{n+1} = x_n + \Delta x$

$= x_0 + (n+1)\Delta x$

$y_{n+1} = y_n + f(x_n, y_n)\Delta x$

Zakaj ima to smisel? Če privzamemo da je rešitev zvezno odvedljiva, velja, da je

$y(x + \Delta x) = y(x) + y'(x)\Delta x + o(\Delta x)$

5) Z Eulerjevo metodo izpelji rešitev DE

$y' = f(x)$ kjer je f zvezna pri pogoju $y(0)=0$

neka točka

$y(A) = ?$

$\Delta x = \frac{A}{m}$ za nek $m \in \mathbb{N}$

$A = x_m$

$x_n = 0 + \Delta x_n n$

$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x + y_0 =$

$= y_{n-1} + f(x_{n-1}) \Delta x + y_0$

$= \left(f(x_{n-2}) + f(x_{n-1})\right) \Delta x + y_0$ $\overset{0}{=}$

Riemanova vsota

$y_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$

za $\int_0^A f(x)\,dx$

$y_m = \sum_{i=0}^{m-1} f(x_i) \Delta x$

To je očitno rešitev

Vaja (DN):

Najdi rešitev za $y' = 2y$ z eulerjevo metodo

$$\Delta x = \frac{A}{n}$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\,\Delta x = y_{n-1} + 2y_{n-1}\frac{A}{n} =$$

$$= y_{n-1}\left(1 + \frac{2A}{n}\right)$$

$$y_n = y_{n-2}\left(1+\frac{2A}{n}\right)^2 = y_1\left(1+\frac{2A}{n}\right)^{n-1}$$

$$\lim_{n\to\infty}\left(1+\frac{2A}{n}\right)^{n-1} = \lim_{u\to\infty}\frac{\left(\left(1+\frac{1}{u}\right)^u\right)^{2A}}{\left(1+\frac{1}{u}\right)} = \underline{\underline{e^{2A}}}$$

$$\frac{1}{u} = \frac{2A}{n} \Rightarrow n = 2Au$$

$$y = e^{2x}$$

Fazni prostor je prostor vseh možnih stanj sistema

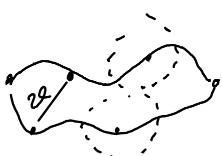
$y' = f(x,y) \quad f: \mathbb{R}^2 \to \mathbb{R}$

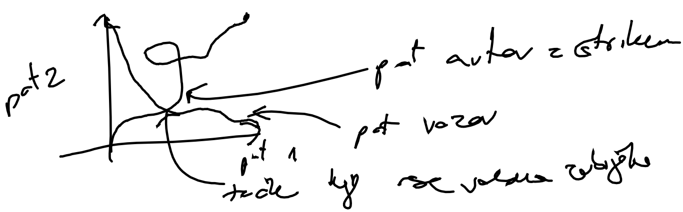
$\hookrightarrow$ fazni prostor je $\mathbb{R}^3$

6) iz Ljubljane v Maribor vodita dve cesti po katerih lahko iz Ljubljane do maribora pripeljete avtomobila
kista eden na drugega privezana z vrvjo dolžine $<2l$, ne da bi jo pretrgala. Ali se lahko vozova
krožne dolžine radia l, ki vozita vsak v svojo smer srečata, ne da bi trčila

$a \in S$

$[0,1] \times [0,1]$





1) 14.10

a) $y' = \frac{x^2}{y} = \frac{dy}{dx}$

$$x^2 dx = y dy$$

$$\frac{1}{3}x^3 + c = y^2$$

$$y = \pm\sqrt{\frac{1}{3}x^3 + c}$$

b)

$$2x^2yy' + y^2 = 2$$

$$2x^2 = \frac{2-y^2}{yy'} = \frac{2-y^2}{y\frac{dy}{dx}} = \frac{2-y^2}{y\,dy}dx$$

$$\frac{dx}{2x^2} = \frac{y\,dy}{2-y^2}$$

$u = 2-y^2$

$du = -2y\,dy$

$$-\frac{1}{2}\frac{1}{x} = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}(\ln u + c)$$

$$\frac{1}{x} = \ln(2-y^2) + c$$

$$ce^{\frac{1}{x}} = 2-y^2$$

$$y = \pm\sqrt{2-ce^{\frac{1}{x}}}$$

c) $(1+x^2)y' = y$

$$\parallel$$

$$(1+x^2)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x^2}$$

$$\ln y = \arctan x + c$$

$$y = ce^{\arctan x}$$

$y \equiv 0$

c') $(1+x^3)y' = y$

$y_1 \equiv 0$

2) $$\frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)}$$

$$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

$B = -A$

x^2: $B + A = 0$

x: $B + C - A = 0$

1: $A + C = 1$

$C = 1 - \frac{1}{3}$

$2B + C = 0$

$\Rightarrow C = -2B = 2A$

$3A = 1$

$$\frac{dy}{y} = \frac{1}{3}\frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1}dx$$

$$\int \frac{x-2}{x^2-x+1}dx = \int \frac{}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

2)

a) $y' = \tan(2x+3y-1)$

$$y' = \tan z$$
$$z = 2x+3y-1$$
$$z' = 2+3y'$$
$$y' = \frac{z'-2}{3}$$
$$\tan z = \frac{z'-2}{3}$$
$$z' = 3\tan z + 2$$

1. rešitev : $z' = 0 \Rightarrow z = \arctan\left(-\frac{2}{3}\right)$

$\Downarrow$

$y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan\left(-\frac{2}{3}\right) - 2x + 1}{3}$

$$\frac{dz}{dx} = 3\tan z + 2$$

$$\frac{dz}{3\tan z + 2} = dx \qquad u = \tan z$$

$$du = \frac{1}{\cos^2 z}\,dz \Rightarrow dz = \frac{du}{1-u^2}$$

$$x + C = \int \frac{du}{(3u+2)(1-u^2)}$$

$$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2}$$

u^2: $B - A = 0 \qquad A = B$

u: $3C + 2B = 0$

1: $2C + A = 1 \Rightarrow A = 1-2C$

$3C + 2 - 4C = 0$

$C = 2 \Rightarrow A = -3 = B$

$$= \int\left(\frac{-3}{3u+2} + \frac{-3u+2}{1-u^2}\right) du =$$

$$\frac{A}{1-u} + \frac{B}{1+u} = \qquad u: A - B = -3 \qquad 1: A + B = 2$$

$A = B - 3$

$2B - 3 = 2$

$2B = 5$

$B = \frac{5}{2}$

$A = -\frac{1}{2}$

$$= -\ln\left(u + \frac{2}{3}\right) +$$

$$+ \int \frac{1}{2}\frac{1}{u-1} + \frac{5}{2}\frac{1}{u+1} =$$

$$= -\ln\left(u+\frac{2}{3}\right) + \frac{1}{2}\ln(u-1) + \frac{5}{2}\ln(u+1)$$

$$y''y - y' = xy'' - (y')^2$$

$$y''(y-x) = y'(1-y')$$

$$y''(y-x) + y'(y'-1) = 0 \qquad y \equiv 0$$

$$a = y'$$

$$b = y - x \qquad b' = y' - 1$$

$$a'b + ab' = 0$$

$$(ab)' = 0$$

$$ab = c \in \mathbb{R}$$

$$y'(y-x) = c$$

$$y' = \frac{c}{y-x} \qquad z = y - x$$

$$z' = y' - 1 = \frac{c}{z} - 1$$

$$\cancel{z = \int \left(\frac{c}{z} - 1\right) dz = c \ln z - z + D}$$

$$\frac{dz}{dx} = \frac{c}{z} - 1$$

$$\frac{dz}{\frac{c}{z} - 1} = dx \qquad x + D = \int \frac{z}{c-z} dz = -\int \left(\frac{c}{t} - 1\right) dt =$$

$$t = c - z$$

$$dt = -dz \qquad = -c \ln(c-z) + c - z$$

$$z = y - x$$

$$\cancel{x} + D = -c \ln(c - y + x) + c - y + \cancel{x}$$

$$y = D e^{c} e^{\ln(c-y+x)+1}$$

Homogene enačbe

$$F(x,y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xz = y \Rightarrow y' = z + xz' = F(1,z)$$

$$z' = \frac{F(1,z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

A) $y^2 + x^2 y' = xyy'$ $\qquad z = \frac{y}{x}$

$$y'(xy - x^2) = y^2$$

$$y' = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z-1}$$

$$y = zx$$

$$y' = z + z'x = \frac{z^2}{z-1}$$

$$z'x = \frac{z^2 - z^2 + z}{z-1} = \frac{z}{z-1}$$

$$\frac{dz}{dx} x = \frac{z}{z-1} \Rightarrow \frac{z-1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{y}{x}$$

b) $y = xy' - \sqrt{x^2+y^2}$

$$y' = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} + \overset{\operatorname{sgn} x}{}\sqrt{1+\frac{y^2}{x^2}} = z \overset{\operatorname{sgn}}{+} \sqrt{1+z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = \cancel{z} + z'x = \cancel{z} + \sqrt{1+z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1+z^2}}{x} \qquad z = \operatorname{sh} u \quad dz = -\operatorname{ch} u\, du$$

$$\operatorname{sgn} x \ln x + D = \int \frac{1}{\sqrt{1+z^2}}\, dz = \int \frac{-\operatorname{ch} du}{\operatorname{ch} u} =$$

$$-\operatorname{arcsh}\frac{y}{x}$$

$$\parallel$$

$$\operatorname{arch}\frac{y}{x}$$

$$y = x \operatorname{ch}(\overset{\operatorname{sgn} y}{}\ln x + D)$$

a)

$$ma = F = m \cdot g - kv^2$$

$$m\dot{v} = mg - kv^2$$

$$\dot{v} = g - \frac{k}{m}v^2 = g\left(1 - \underbrace{\frac{k}{mg}}_{\alpha} v^2\right) = g(1 - \alpha v^2)$$

$$\frac{dv}{dt} = g(1 - \alpha v^2)$$

$$\frac{dv}{1 - \alpha v^2} = g\,dt \quad / \int$$

$$gt =$$

$$\frac{A}{1 - \sqrt{\alpha} v} + \frac{B}{1 + \sqrt{\alpha} v} :$$

$v: \sqrt{\alpha} A - \sqrt{\alpha} B = 0$

$A = B$

$1: \quad A + B = 1$

$\Rightarrow A = \frac{1}{2}$

$B = \frac{1}{2}$

$$gt = \frac{1}{\sqrt{\alpha} 2}\left(\ln(1 + \sqrt{\alpha} v) - \ln(\sqrt{\alpha} v - 1)\right) =$$

$$\frac{1}{2\sqrt{\alpha}} \ln\left(\frac{1 + \sqrt{\alpha} v}{1 - \sqrt{\alpha} v}\right) = gt + C$$

$$V_0 = 0 \Rightarrow \quad \cancel{1 + \sqrt{\alpha}} v = \cancel{1} - \sqrt{\cancel{\alpha}} v + C$$

$$\Rightarrow 0 = C$$

$$\frac{1 + \sqrt{\alpha} v}{1 - \sqrt{\alpha} v} = \underbrace{e^{2\sqrt{\alpha} gt}}_{D}$$

$$1 + \sqrt{\alpha} v = D - \sqrt{\alpha} D v$$

$$v\sqrt{\alpha}(1 - \sqrt{\alpha} D) = D - 1$$

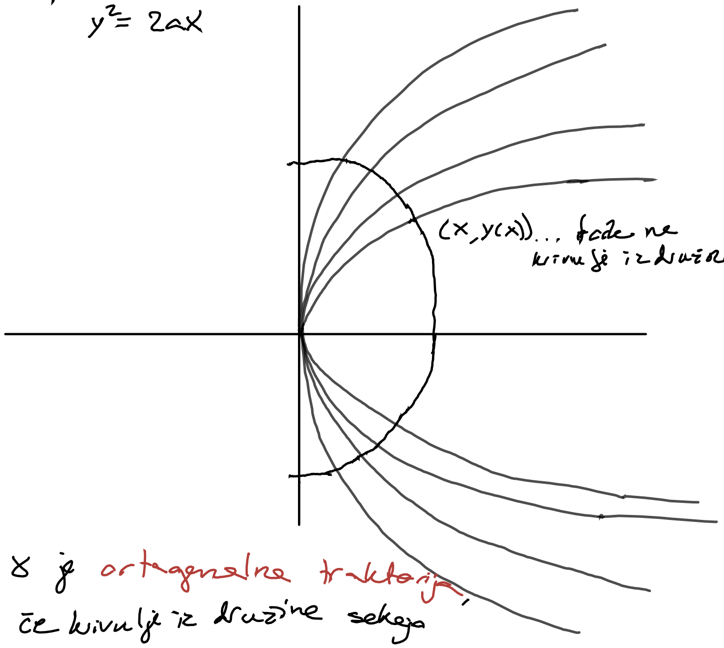
$$v = \frac{D - 1}{\sqrt{\alpha}(1 - \sqrt{\alpha} D)}$$

V menkenje

+ zamdude 20 min 21.10

2) Poišči družino ortagonalnih trajektorij na družino parabol

$y^2 = 2ax$



γ je ortagonalna trajektorija, če krivulje iz družine sekajo γ pod pravim kotom

$2yy' = 2a \rightsquigarrow y^2 = 2yy'x$

v (x_0, y_0) velja $x_0 2 y_0 y' = y_0^2$

$y' = \frac{y_0}{2x_0}$ smerni koeficient trajektorije $-\frac{2x_0}{y_0}$

$y' = f(x,y)$ družina krivulj $\Rightarrow -\frac{1}{y'} = f(x,y)$ družina ort. t.

$y^2 = 2yy'x \rightsquigarrow y^2 = -\frac{2yx}{y'}$ iščemo y

1 možnost $y \equiv 0$ ✓

2. možnost

$y' = -\frac{2x}{y} = \frac{dy}{dx}$

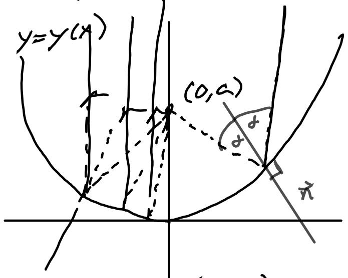
$2x\,dx = y\,dy \;/ \int$

$-\frac{2}{2}x^2 = \frac{1}{2}y^2 + C$

$x^2 + \frac{1}{2}y^2 + C = 0$

$x^2 + \frac{1}{2}y^2 = -C$ elipse

3) Pri izdelavi žarometa mora biti odbojna površina ki obsega žarnico take oblike, da vse svetlobne žarke odbije v isto smer. Predpostavimo, da je odbojna površina rotacijska ploskev grafa f_n



(n_x, n_y)

$$(-x, -y(x)+a)\cdot \vec{n} = -xn_x - yn_y - an_y = \cos\alpha \sqrt{x^2+(y+a)^2}\sqrt{n_x^2+n_y^2}$$

$$\vec{n} = \left(1, -\frac{1}{y'}\right)$$

$$\cos\alpha \sqrt{x^2+(y+a)^2}\sqrt{1+\frac{1}{y'^2}} = -x + \frac{-y-a}{y'}$$

$$\cos\alpha = \frac{-xy'-y-a}{\sqrt{x^2+(y+a)^2}\sqrt{1+y'^2}}$$

$$\vec{n}\cdot(\cancel{1},\cancel{0},\cancel{0}) = \cos\alpha\, |\vec{n}|\cdot 1$$

$$n_y \cancel{n_x} = \cos\alpha \sqrt{1+\frac{1}{y'^2}}$$

$$n_y \cancel{n_x} = \frac{\cancel{\frac{1}{y'}}(-xy'-y-a)}{\sqrt{x^2+(y+a)^2}} = \frac{1}{\cancel{y'}}$$

???

$$-xy'+y-a = \sqrt{x^2+(y+a)^2}$$

$z = y-a$

$z' = y'$

$$\cancel{+}xz'+z = \sqrt{x^2+z^2}+z$$

??

$$z' = \frac{\sqrt{x^2+z^2}}{\cancel{m}x} = \sqrt{\frac{x^2+z^2}{x^2}} = \sqrt{1+\frac{z^2}{x^2}}+\frac{z}{x}$$

$u = \frac{z}{x}$

$z' = xu'$

$$\cancel{u}+xu' = \sqrt{1+u^2}+\cancel{u}$$

$$\frac{u'}{\sqrt{1+u^2}} = \frac{1}{x}$$

$v = \sqrt{1+u^2}$

$$dv = \frac{u\,du}{\sqrt{1+u^2}} = \frac{1-v^2}{\sqrt{1+u^2}}\,du$$

$$\ln x = \int \frac{1-v^2}{v}\,dv = \ln v - \frac{1}{2}v^2 + C$$

???

$$y = a + \frac{c}{2}x^2 - \frac{1}{2c}$$

Cauchy naloge

$y' = f(y) \qquad y(x_0) = y_0$

a) globalni $\exists$ reši DE za poljubni začetni pogoj

b) f Lipšicova tako kaj rešitev iz a enolična

$$\frac{dy}{dx} = f(y)$$

če $f(y) \neq 0$

$$\frac{dy}{f(y)} = dx \quad / \int$$

Kaj pa če je $f(y_0) = 0$

Poskusimo $f(y) \equiv x_0$

$$F(y) = x + C$$

$$y'(x) = 0$$

$$\int \frac{dy}{f(y)}$$

$$f(y(x)) = f(y_0) = 0$$

$$y'(x) = f(y(x))$$

$$y(x) = y(x_0) + \int_{x_0}^{x} y'(t)\,dt$$

daš vse na eno stran in integriraš

$$\underbrace{\int \frac{y'(t)\,dt}{f(y(t))}} = \int_{x_0}^{x} 1\,dt = x - x_0$$

$u = y$

$du = y'\,dt$

$$\int_{y_0}^{y(x)} \frac{du}{f(u)} = F(y(x)) - F(y_0) = x - x_0$$

$$F(y(x)) = x - x_0 + F(y_0)$$

$$y(x) = F^{-1}(x - x_0 + F(y_0))$$

$$y(x_0) = F^{-1}(F(y_0)) = y_0$$

$$y'(x) = \frac{1}{F'(F^{-1}(x - x_0 + F(y_0)))} = \cancel{\frac{1}{f(x - x_0 + F(y_0))}}$$

$$= f(y(x)) \quad ???$$

$$F'(F^{-1}(x)) = \frac{1}{(F^{-1})'(x)}$$

b) enoličnost

$$\int_{y_0}^{y} \frac{dy}{f(y)} = x - x_0$$

$$|f(x) - f(y)| \leq C|x-y|$$

recimo da $f(x)=0$ $\longrightarrow |f(y)| \leq C|x-y|$

$f \geq 0 \quad y > y_0, f(y_0) = 0$

$$\int_{y_0}^{y} \frac{dy}{f(y)} \geq \frac{dy}{C|y-y_0|} = \infty$$

Recimo da y reši DE da je $f(y_1) = 0$ in y ni konstanten

$x_1 > x_0 : f(y(x_1)) > 0$ BŠZ

potem je y blizu x_1. zaradi

$$\int_{x_1}^{x} \frac{y'(x)dx}{f(y(x))} = x - x_1$$

$$\| $$

$$\int_{y_1}^{y} \frac{dy}{f(y)}$$

izberemo si $x_2 \in [x_0, x_1]$, da velja

$f(y(x_2)) = 0$ in $f(y(x)) > 0$ za $\forall x \in (x_2, x_1]$

$$\lim_{x \to x_2} \int_{x_1}^{x} \frac{y'(x)}{f(y(x))} = \lim_{x \to x_2} x - x_1 = x_2 - x_1$$

$$= \lim_{y \to y_2} \int_{y_1}^{y} \frac{dy}{f(y)} \geq \lim_{y \to y_2} \int_{y_1}^{y} \frac{dy}{C(y-y_0)} = \infty$$

$\times$

$$y' = \frac{f(y)}{g(x)} \qquad f, g \text{ brez ničel}$$

a) Pokaži da vsaka rešitev da tokovno krivuljo polja $v(x,y) = (g(x), f(y))$

b) Dokaži za poljuben začetni pogoj dostopno velikost eno rešitev

Tokovnica ali integralna krivulja je krivulja, ki ima v vseh točkah odvod enak vektorskemu polju

$$F(y) - F(y_0) = \int_{y_0}^{y} \frac{dy}{f(y)} = t - t_0 \qquad x = G^{-1}(t - t_0 + G(x_0))$$

$$G(x) - G(x_0) = \int_{x_0}^{x} \frac{dx}{g(x)} = t - t_0 \rightsquigarrow$$

$$\gamma = (x(t), y(t))$$
$$\dot\gamma(t) = V(x(t), y(t))$$
$$\dot x(t) = g(x)$$
$$\dot y(t) = f(y)$$

$$\frac{\partial}{\partial t} y(x(t)) =$$
$$= \frac{\partial y}{\partial t}(x(t))\, \dot x =$$
$$= \frac{f(y(t))}{g(x(t))} \frac{\partial x}{\partial t}$$
$$\rightsquigarrow \dot x = g(x) \quad ???$$

ZZZ

Lin. dif. enačbe (LDE)

$y' + a(x)y = b(x) \quad a,b \in \mathcal{C}([\alpha,\beta])$

i) $b \equiv 0 \Rightarrow$ LDE homogena $\Rightarrow$ Ločljive spremenljivke

$$y = y_0 e^{-\int a(t)dt} = C e^{-\int a dx}$$

ii) $b \not\equiv 0 \Rightarrow$ LDE nehomogena $\Rightarrow$ variacija konstante

$$y = e^{-\int a dx} \int b(x) e^{\int a(x) dx} dx + C e^{-\int a dx}$$

•) LDE homogena: y_1, y_2 rešitvi $\Rightarrow y_1 + \lambda y_2$ tudi rešitev

Če vemo homogeno rešitev $\bar{y}$ so vse druge rešitve oblike $C \cdot \bar{y}$ za $C \in \mathbb{R}$

••) LDE nehomogena

Partikularno rešitev najdemo z variacijo konstante

Homogeni del: $y' + ay = 0 \Rightarrow y_h$ rešitev

Partikularni del: $y_p = C(y) y_h$

Splošna rešitev: $\tilde{y} = y_p + D y_h$

Odvajamo in vstavimo v DE:

$$y_p' = C' y_h + C y_h' \Rightarrow C' y_h + C y_h' + a C y_h = b$$

$$C' y_h = b \Rightarrow C(x) = \int \frac{b}{y_h} dx$$

1) Poišči splošno rešitev:

a) $xy' + 2y = (3x+2)e^{3x} \quad /:x$

$$y' + y\frac{2}{x} = \underbrace{\frac{3x+2}{x}e^{3x}}_{b}$$

$x=0 \Rightarrow 2y = 2 \Rightarrow y=1$

$x \neq 0 \Rightarrow$

homogena: $y' + y\frac{2}{x} = 0$

$$\frac{dy}{dx} + y\frac{2}{x} = 0 \quad / \cdot \frac{dx}{y}$$

$$\frac{dy}{y} = -2\frac{dx}{x} \quad / \int$$

$$\ln y = \ln x^{-2} + C$$

$$y = \frac{1}{x^2} \cdot D$$

partikularno:

$$y_p = D(x)x^{-2}$$

$$y_p' = D'(x)x^{-2} - 2D(x)x^{-3}$$

$$D'(x)x^{-2} \cancel{- 2D(x)x^{-3}} \cancel{+ 2D(x)x^{-3}} = \frac{(3x+2)e^{3x}}{x}$$

$$D'(x) = (3x^2+2x)e^{x}$$

$$\boxed{\begin{gathered}\int p(x)e^{\lambda x}dx = g(x)e^{\lambda x} + C \\ p,g \in \mathbb{R}[x], \ \operatorname{st} p = \operatorname{st} g\end{gathered}}$$

$$\left((Ax^2+Bx+C)e^{3x}\right)' = (3x^2+2x)e^{3x}$$

$$(3Ax^2+3Bx+3C+2Ax+B)e^{3x} \rightsquigarrow$$

$3A = 3 \Rightarrow A = 1$
$3B + 2A = 2 \Rightarrow B = 0$
$3C + B = 0 \Rightarrow C = 0$

$$D(x) = x^2e^{3x}$$

$$\rightsquigarrow y_p = x^2e^{3x} \cdot x^{-2} = e^{3x}$$

$$y_s = y_h + y_p = D\frac{1}{x^2} + e^{3x}$$

b) $y'' = \sin x + y'$

$z = y'$ $\qquad a(x) = -1 \quad b(x) = \sin z$

$z' = \sin x + z$ $\qquad b \equiv 0 \Rightarrow z' - z = 0$

$\qquad z' = z$

$z' - z = \sin x$

1) $z \equiv 0 \Rightarrow y = C \in \mathbb{R}$

2) $\int \frac{dz}{z} = dx \Rightarrow \ln z = x + C$

$\Rightarrow z = Ce^x \Rightarrow y = Ce^x + D$

$b \not\equiv 0 \Rightarrow \quad z_h = e^x$

$z_p = C(x) z_h = C(x) e^x$

$z = z_p + D y_h = C(x) e^x + D e^x$

$z' = C'(x) e^x + C(x) e^x + D e^x$

$z' - z = C'(x) e^x + C(x)(e^x - e^x) = \sin x$

$C'(x) = \frac{\sin x}{e^x} = e^{-x} \sin x$

$$\boxed{\text{nastavek: } C(x) = Ae^{-x} \sin x + Be^{-x} \cos x}$$

$C(x) = e^{-x}(A \sin x + B \cos x)$

$C'(x) = -e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)$

$= e^{-x}(\sin x(-A - B) + \cos(x)(-B + A)) = e^{-x} \sin x$

$A + B = -1$

$A - B = 0 \Rightarrow A = B \Rightarrow 2A = -1 \Rightarrow A = B = -\frac{1}{2}$

$\Rightarrow C(x) = -\frac{1}{2} e^{-x} (\sin x + \cos x) =$

$z = -\frac{1}{2}(\sin x + \cos x) + De^x = y'$

$y = -\frac{1}{2} \int \sin x + \cos x \, dx = \frac{1}{2} \cos x - \frac{1}{2} \sin x + De^x + C$

2)

$y' + 2xy = 2x^3 y^3$

1) $y \equiv 0$ je rešitev

2) $y \not\equiv 0 \Rightarrow \frac{y'}{y^3} + \frac{2x}{y^2} = 2x^3$

$z' = \frac{y'}{y^3} \Rightarrow \int dz = \int y^{-3} dy \Rightarrow z = -\frac{1}{2} y^{-2} = -\frac{1}{2y^2}$

$$\boxed{\begin{array}{l}\text{Bernullijeva DE} \\ \quad y' + ay = by^{\alpha} \quad \alpha \notin \{0,1\} \\ \quad \text{Nastavek: } z = y^{1-\alpha}\end{array}}$$

$\frac{2x}{y^2} = \frac{4x}{2y^2} = -4xz$

enačba: $z' - 4xz = 2x^3$ $\quad a(x) = -4x \quad b(x) = 2x^3$

homogeno: $z' - 4xz = 0$

$\int \frac{dz}{z} = \int 4x\, dx \Rightarrow \ln z = 2x^2 + c \Rightarrow z = Ce^{2x^2}$

partikularno: $z_h = e^{2x^2}$ $\quad z_p = C(x) e^{2x^2}$

$z = z_p + z_h$

$\rightsquigarrow$ enačba: $C'(x) e^{2x^2} + 4xe^{2x^2} \cdot C - 4xCe^{2x^2} = 2x^3$

$C'(x) = e^{-2x^2} \cdot 2x^3$

Nastavek:

$C(x) = \int 2x^3 e^{-2x^2} dx =$ $\quad \int xe^x dx = (Ax + B)e^x$

$u = -2x^2$

$du = -4x\, dx$

$= -\frac{1}{2} \int e^u u\, du = -\frac{1}{2}(Au + B)e^u$

$((Au + B)e^u)' = e^u (Au + B + A)$

$A + B = 0 \quad A = 1 \Rightarrow B = -1$

$C(x) = -\frac{1}{2}(1u - 1)e^u \rightsquigarrow$

$C(x) = e^{-2x^2}(x^2 - \frac{1}{2}) + D$

$z = x^2 - \frac{1}{2} + De^{2x^2} = -\frac{1}{y^2} \rightsquigarrow y = \frac{1}{\sqrt{x^2 - \frac{1}{2} + De^{2x^2}}}$

3. naloga

$xy' + y = y^2 \ln x$

1) $y \equiv 0 \Rightarrow$ je rešitev

2) $y \not\equiv 0 \Rightarrow$
$x \neq 0 \Rightarrow$

$$\frac{y'}{y^2} + \frac{1}{xy} = \ln x$$

$$z' = \frac{y'}{y^2} \rightsquigarrow z = -\frac{1}{y}$$

$$\boxed{z' - \frac{z}{x} = \ln x} \quad \begin{matrix} a = -1 \\ b = \ln x \end{matrix}$$

homogeno:

$$z' - \frac{z}{x} = 0$$

$$\frac{dz}{z} = \frac{dx}{x} \rightsquigarrow$$

$$\ln z = \ln x + C$$

$$z = e^{\ln x} \cdot e^{C} \rightsquigarrow z = C\cancel{e}x$$

$$z_h = x$$

$$\rightsquigarrow \quad C'x + C - C = \ln x$$

$$C' = \frac{\ln x}{x} \qquad C = \int \frac{\ln x}{x} dx = \int u\, du = \frac{1}{2}u^2 + \cancel{e}D$$

$$u = \ln x \qquad du = \frac{dx}{x} \qquad C = \frac{1}{2}(\ln x)^2 + D$$

$$z = \frac{1}{2}(\ln x)^2 x + Dx = -\frac{1}{y}$$

$$y = -\frac{2}{(\ln x)^2 x + Dx}$$

4 naloga

f omejena zvezna

$y' + y = f(x)$

a) To ima natanko eno rešitev DE

b) f periodična s periodo $\omega \Rightarrow \bar{y}$ periodična s periodo ω

a) $y' + y = f(x)$

homogeno:

$$\frac{dy}{dx} = -y \rightsquigarrow \frac{dy}{y} = -dx \rightsquigarrow y = e^{-x}$$

partikularna: $y_p = C(x) e^{-x}$

$$C'(x) e^{-x} = f(x) \quad C(x) = \int_0^x f(x) e^x dx = F(x) + D$$

$$\bar{y} = F(x) e^{-x} + C e^{-x} = e^{-x}(F(x) + C) = e^{-x} \int_0^x f(t) e^{-t} dt$$

b) $$\bar{y}(x+\omega) = e^{-x-\omega} \int_0^{x+\omega} f(t) e^{-t} dt = e^{-x} e^{-\omega} \int_0^x f(t) e^{-t} dt + e^{-\omega} e^{-x} \int_x^{\omega} f(t) e^{-t} dt =$$

$$= e^{-\omega} \bar{y} + e^{-\omega} e^{-x} \int_0^{\omega} f(t) dt - e^{-\omega} e^{-x} \int_0^x f(t) e^{-t} dt =$$

$$= e^{-\omega} \bar{y}(x) + e^{-x} \bar{y}(\omega) - e^{-\omega} \bar{y}(x) = e^{-x} \bar{y}(\omega) = e^{-\omega} \bar{y}(x)$$

$$\bar{y}(x+\omega) = e^{-x-\omega} \int_{-\infty}^{x+\omega} f(t) e^{-t} dt = e^{-x-\omega} \int_{-\infty}^{\omega} f(t) e^{-t} dt + e^{-x-\omega} \int_{-\omega}^{x+\omega} f(t) e^{-t} dt =$$

$$= e^{-x-\omega} \left(\int_{\infty}^{\omega} f(t) e^{-t} dt + \int_0^x f(t) e^{-t} dt \right) =$$

$$= e^{-x} \bar{y}(\omega) + e^{-\omega} \bar{y}(x)$$

???

b) $\bar{y}(x) = e^{-x}\int_{-\infty}^{x} f(t)e^{t}\,dt = \int_{-\infty}^{x} f(t)e^{t-x} = \int_{-\infty}^{0} f(u+x)e^{u}\,du$

$u = t-x$

$\rightsquigarrow t = u+x$

$\bar{y}(\omega+x) = \int_{-\infty}^{0} f(u+x+\omega)e^{u}\,du = \int_{-\infty} f(u+x)e^{u}\,du = \bar{y}(x)$

5 naloga

$$3y' + y^2 + \frac{2}{x^2} = 0$$

Uganemo rešitev: $\quad y = \frac{1}{x} \quad y' = -\frac{1}{x^2}$

$$-\frac{3}{x^2} + \frac{1}{x^2} + \frac{2}{x^2} = 0 \quad \checkmark$$

Splošna rešitev: $y = \frac{1}{x} + u(x)$

$$y' = -\frac{1}{x^2} + u'(x)$$

$$-\frac{3}{x^2} + 3u'(x) + \frac{1}{x^2} + \frac{2u(x)}{x} + u^2(x) + \frac{2}{x^2} = 0$$

$$3u'(x) + \frac{2u(x)}{x} + u^2(x) = 0 \quad / : u^2(x)$$

$$\frac{3u'(x)}{u^2(x)} + \frac{2}{x\,u(x)} + 1 = 0$$

$$z' = \frac{3u'(x)}{u^2(x)} \qquad z = 3\int u^{-2}du = -3u^{-1} = -\frac{3}{u}$$

$$\Rightarrow u = -\frac{3}{z}$$

$$z' - \frac{2}{3}\frac{z}{x} = -1$$

$$a(x) = \frac{2}{3x}$$
$$b(x) = -1$$

homogeni del:

$$\frac{dz}{z} = \frac{2}{3}\frac{dx}{x}$$

$$\ln z = \ln x^{\frac{2}{3}} + C \rightsquigarrow z = \frac{2}{3}Cx$$

partikularni:

$$C'(x)\frac{2}{3}x = -1$$

$$C(x) = -\frac{3}{2}\int \frac{1}{x}dx = \ln x^{-\frac{3}{2}}$$

$$z = x\ln x + \frac{2}{3}Cx$$

$$u = -\frac{3}{x\ln x + \frac{2}{3}Cx} \quad \Rightarrow y = \frac{1}{x} - \frac{3}{x\ln x + \frac{2}{3}Cx}$$

Ricattijeva enačba

$y' = ay^2 + by + c \quad a, b, c$ funkcije x

1) Uganemo rešitev y_p
2) Splošno rešitev poiščemo z nastavkom
$$y = y_p + u(x)$$
3) Vstavimo nastavek z začetno enačbo
4) Dobimo Bernullijevo enačbo
5) $y = y_p + \frac{1}{u^2(x)} \Rightarrow LDE$

4.11.

1. $(1-x^2)y' = 1-y^2$

Riccatijeva enačba

$y' = ay^2 + by + c$: a, b, c func. od x

1) Uganimo rešitev y_p

2) splošna rešitev kot $y = y_p + u(x)$

3) Dobimo Bernoulijeva enačba

$y = y_p + \frac{1}{u(x)^2} \Rightarrow LDE$

$$y' = \frac{1-y^2}{1-x^2}$$

1. $y_p = x$: $1 = \frac{1-x^2}{1-x^2}$ ✓

2. $y = v + y_p = v+x$ v je neznana funkcija

$$y' = v' + 1 = \frac{1+(v+1)^2}{1-x^2} = \dots = 1 + \frac{-2vx - v^2}{1-x^2}$$

$$v' = \frac{-2vx - v^2}{1-x^2} = -\left(\frac{1}{1-x^2} v^2 + \frac{2x}{1-x^2} v\right)$$

$$\frac{v'}{v^2} = -\frac{1}{1-x^2} - \frac{2x}{1-x^2}\frac{1}{v}$$

$z = \frac{1}{v}$ $\quad z' = \frac{v'}{v^2}$

$$z' = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} z$$

homogeni del:

$$z' = -\frac{2x}{1-x^2} z$$

$$\int \frac{dz}{z} = -\int \frac{2x}{1-x^2} dx$$

$$\ln|z| = -\ln|1-x^2| + C$$

$$z_H = \frac{C}{x^2-1}$$

$$\frac{C'(x)}{x^2-1} + \cancel{\frac{C}{\dots}} = \cancel{\frac{C}{\dots}} - \frac{1}{x^2-1}$$

$C'(x) = -1 \quad C = -x$

$$z = z_h + z_p = \frac{C-x}{x^2-1}$$

$$v = \frac{x^2-1}{C-x}$$

$$y = \dots = \frac{1-Cx}{x-C}$$

Prvi integral enačbe $y' = f(x,y)$ je funkcija, ki je konstantna vzdolž vsake rešitve

Če imamo splošno rešitev $y = \varphi(x, C)$ in je $\varphi_C \neq 0$ (odvod po C)

$\Rightarrow$ $C = C(x,y)$ je prvi integral

1) DE zapišemo v obliki $P\,dx + Q\,dy = 0$ (*)

Opomba: (*) je ekvivalentna DE $P + Qy' = 0$

$\langle (P,Q), (1, y') \rangle = 0$

če velja $Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y$

pravimo da je enačba eksaktna.

Njen prvi (eksaktni integral) poiščemo z integracijo $u = \int$

$\left((P,Q) \text{ je potencialno} \Leftrightarrow (P,Q) = \nabla u \right)$

C poiščemo z odvajanjem

$u_y = \frac{\partial}{\partial y} \int P\,dx + C'(y) = Q$

2) Poišči kak nekonstanten prvi integral

a) $x\sqrt{x^2-y^2+y})\,dx + (y\sqrt{x^2+y^2}+x)\,dx = 0$

b) $3x^2(1+\ln y)\,dx - \left(2y - \frac{x^3}{y}\right)dy = 0$

a)

$$P = x\sqrt{x^2+y^2} + y$$

$$Q = y\sqrt{x^2+y^2} + x$$

Ali je eksaktna?

$$P_x = \sqrt{x^2+y^2} + \frac{xy}{\sqrt{x^2+y^2}}$$

$$Q_y = -\|-$$

$$u = \int \left(x\sqrt{x^2+y^2} + \cancel{xy}\,y\right)dx + C(y) =$$

$$t = x^2+y^2$$
$$dt = 2x\,dx$$

$$= \frac{1}{2}\int t^{\frac{1}{2}}dt + xy + C(y) = \frac{1}{3}(x^2+y^2)^{\frac{3}{2}} + xy + C(y)$$

$$u_y = \cancel{y(x^2+y^2)^{\frac{1}{2}}} + \cancel{x} + c'(y) = \cancel{y\sqrt{x^2+y^2}} + \cancel{x}$$

$$c'(y) = 0 \Rightarrow c \text{ je konstant}$$

$$u = \frac{1}{3}(x^2+y^2)^{\frac{3}{2}} + xy$$

izberemo da $\tilde{y} < 0$

↑ prvi integral

splošno rešitev dobimo če iz $u(x,y) = c$

izrazimo y $\quad \tilde{y} = \tilde{y}(x)$

$$u(x, \tilde{y}(x)) \equiv c$$

\

Da je $Pdx + Qdy = 0$ (*) mora veljati

$$y' = -\frac{P}{Q} = -\frac{\mu P}{\mu Q} \rightsquigarrow \mu P dx + \mu Q dy = 0 \quad (**)$$

Ti dve sta ekvivalentni za poljubno rešitev

če * ni eksakten mora biti ** eksakten

faktor μ se imenuje integrirajoči množitelj

V praksi ga skušamo ugeniti z nastavki

$\mu(x), \mu(y), \mu(x,y)$

$\mu(x^n \pm y^m), \mu(x^2+y^2) \ldots$

3) Podana je DE

$$P\,dx + Q\,dy = 0$$

a) Dokaži $\mu = \mu(y) \Leftrightarrow \dfrac{Q_x - P_y}{P} = g(y)$

$\Rightarrow$ $\mu P\,dx + \mu Q\,dy = 0$ in μ integrirajoči množitelj
DE pa je eksaktna

$$(\mu P)_y = (\mu Q)_x$$

$$\mu_y P + \mu P_y = \mu Q_x$$

$$\frac{\mu (Q_x - P_y)}{P} = \mu'(y) \quad /:\mu$$

$$\frac{Q_x - P_y}{P} = \frac{\mu'(y)}{\mu(y)} \Big\}\text{ odvisno samo od } y$$

$\Leftarrow$ $\dfrac{Q_x - P_y}{P} = g(y)$ recimo da je

$$\ln \mu = \int \frac{Q_x - P_y}{P}\,dy \Rightarrow \mu = e^{\int \frac{Q_x - P_y}{P}\,dy}$$

Potem je $(\mu P)_y = \dfrac{Q_x - P_x}{P} P + \ldots$ itd itd
pride vredu

b) $\underbrace{(xy^2-y^3)}_{P}dx + \underbrace{(1-xy^2)}_{Q}dy = 0$

$Q_x = -y^2 \quad P_y = 2xy - 3y^2$

$\frac{Q_x - P_y}{P} = \frac{2y^2 - 2xy}{xy^2 - y^3} = \frac{2(y-x)}{y(x-y)} = -\frac{2}{y}$

$\mu = e^{-\int \frac{2}{y}dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$

$\mu P dx + \mu Q dy = 0$ je [illegible]

$u = \int \mu P dx + C(y) =$

$= \int (x-y)dx + C(y) = \frac{1}{2}x^2 - xy + C(y).$

$u_y = -x + C'(y) = \frac{1-xy^2}{y^2} = \frac{1}{y^2} - x$

$C'(y) = \frac{1}{y^2} \rightsquigarrow C = -\frac{1}{y}$

$u = \frac{x^2}{2} - xy - \frac{1}{y}$

μ

4) $y(x^2+y^2+1) - x(x^2+y^2-1)\,dx = 0$

$\mu = \mu(xy)$

$P_y = x^2+3y^2+1$ $\qquad$ $\mu P\,dx + \mu Q\,dy = 0$

$Q_x = -3x^2-y^2+1$ $\qquad$ $\mu = \mu(xy)$

$$\left.\begin{aligned}(\mu P)_y &= x\mu' P + \mu P_y \\ (\mu Q)_x &= y\mu' Q + \mu Q_x\end{aligned}\right\} + \quad \mu'(xP - yQ) + \mu(P_y - Q_x) = 0$$

$$\frac{\mu'}{\mu} = \frac{-P_y + Q_x}{xP - yQ} = \frac{-4x^2-4y^2}{xy(2x^2+2y^2)} = \frac{-2}{xy}$$

$z = xy$ $\qquad \dfrac{\mu'(z)}{\mu(z)} = -\dfrac{2}{z} \;/\int$

$$\frac{d\mu}{\mu(z)} = -\frac{2\,dz}{z} \;/\int$$

$$|\mu(z)| = |z|^{\frac{1}{2}}$$

$$\mu(z) = \frac{1}{z^2} = \frac{1}{(xy)^2}$$

$$u = \int \mu P\,dx + c(y) = \int \frac{\cancel{y}(x^2+y^2+1)}{x^2 y^{\cancel{2}}}\,dx + c(y)$$

$$= \int \frac{1}{y} + \frac{y}{x^2} + \frac{1}{x^2 y} = \frac{x}{y} - \frac{y}{x} - \frac{1}{xx} + c(y)$$

$$u_y = -\frac{x}{y^2} - \frac{1}{x} + \frac{1}{y^2 x} + c'(y) =$$

4)

$$y\,dx - x\,dy = 2x^3 \tan\left(\frac{y}{x}\right) dx$$

$$z = \frac{y}{x} \quad dz = \frac{dy}{x} - \frac{y}{x^2}dx \qquad dF = F_x dx + F_y dy$$

$$-x^2 dz = -x\,dy + y\,dx$$

$$-dz = 2x \tan(z)\,dx$$

$$-\frac{dz}{\tan(z)} = 2x\,dx$$

$$-\int \frac{\cos(z)}{\sin(z)} dz = \int 2x\,dx$$

$$-\ln|\sin(z)| = x^2 + C$$

$$-\ln \sin\frac{y}{x} = x^2 + c$$

$$C = -x^2 - \ln\left|\sin\frac{y}{x}\right| = u(x,y) = \text{prvi integra}$$

Parametrično reševanje implicitno podanih DE

Do zdaj so enačbe bile oblike $y' = F(x,y)$

Lahko pa je v $F(x,y,y')=0$

Poanta: $F(x,y,p)=0$ podaja ploskev v $\mathbb{R}^3$

(če $F \in \mathcal{C}^1$ in $\nabla F \neq 0$)

Vsaka rešitev bo potekala po neki krivulji:

$\gamma(t) = (x(t), y(t), p(t))$ da je $F(\gamma(t))=0$

Da γ res podaja rešitev DE v parametrični obliki, mora veljati

$\dot{y}(t) = p(t) \cdot \dot{x}(t)$

Preprost primer:

$F(x, y') = 0$

$\Downarrow$

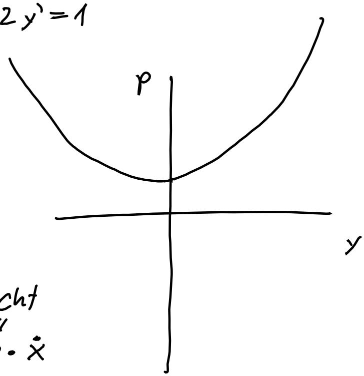
$F(x, p)$ podaja krivuljo v ravnini

To krivuljo parametriziramo in poiščemo manjkajoči predpis za y komponento

6) Reši de

a) $(y')^2 - 2y^2 = 1$

b) $x^2 + 1 + (y')^2 - 2y' = 1$



$p^2 - 2y^2 = 1$

$p = \operatorname{ch} t$

$y = \frac{1}{\sqrt{2}} \operatorname{sh}(t)$

$\dot{y} = \frac{\operatorname{ch}(t)}{\sqrt{2}} = \overset{\operatorname{ch} t}{\overset{\shortparallel}{p}} \cdot \dot{x}$

$\dot{x} = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{2}{\sqrt{2}} t + c \qquad t = \sqrt{2} x$

krivulja je $\left(\frac{\sqrt{2}}{2} t + c, \frac{\sqrt{2}}{2} \operatorname{sh} t\right)$

$y = \frac{\operatorname{sh}(\sqrt{2}(x+c))}{\sqrt{2}}$

še druga rešitev $\frac{\operatorname{sh}(\sqrt{2}(c-x))}{\sqrt{2}}$ druga krivulja

b) $x^2+1+(y')^2-2y'=1$

$x^2+(p-1)^2=1$

$x=\cos t$

$p=1+\sin t$

$\dot y = p\dot x = (1+\sin t)(-\sin t) = -\sin^2 t - \sin t$

$$y = \int \dot y\, dt = \int \frac{1-\cos 2t}{2}\, dt + \cos t =$$

$$= \frac{1}{2}t - \frac{1}{2}\sin 2t + \underbrace{\cos t}_{x}$$

$$y = \frac{1}{2}\arccos x - x\sin(\arccos x) + x$$

izbr: primeri

a):

p^2-2y^2-1

ez $F_p=0$ $F_y: 2p=0 \Rightarrow p=0$

$\Rightarrow y$ konstantna

$y^2=-\frac{1}{2}$ ni: nedivih vostwo

b)

Od zadnjič: Parametrično reševanje DE 11.11
oblike $F(x,y,y')=0$

Splošni postopek
1. $p = y'$
2. izračunamo $DF(\underbrace{x,y,p}_{\text{neodvisne spremenljivke}})$
3. s pomočjo nastavka $dy = p\,dx$ dobimo
diferencialno enačbo $x=x(p)$ ali $y=y(p)$.
Manjkajočo komponento dobimo iz prvotne enačbe

Včasih ima DE tudi singularne rešitve
1) $G(x,y,C)=0$ in enakosti $\frac{\partial G}{\partial C}=0$ eliminiramo C
2) parametriziramo množico $\{F = F_p = 0\}$ in preverimo
ali reši DE

odvod po p

1. naloga: Poišči splošne in morebitne sng. rešitve

$x^2(y')^3 - xy' + y = 0$

$y = xy' - x^2(y')^3 \qquad p = y' \qquad dy = p\,dx$

$y = xp - x^2p^3$

$dy = p\,dx + x\,dp - 2xp^3\,dx - 3x^2p^2\,dp$

$\cancel{p\,dx} = \cancel{p\,dx} - 2xp^3\,dx + (x - 3x^2p^2)\,dp$

$2xp^3\,dx = (x - 3x^2p^2)\,dp$

$x = 0 \Rightarrow y = 0$

$x \neq 0 \Rightarrow$

$2p^3\,dx = (1 - 3xp^2)\,dp$

$2p^3x'(p) = 1 - 3xp^2 \qquad p \equiv 0 \Rightarrow y \equiv 0$

$\dot{x} = \frac{1}{2p^3} - \frac{3}{2}\frac{x}{p^2}$

$\dot{x} + \frac{3}{2}\frac{x}{p^2} = \frac{1}{2p^3}$

homogeni del:

$\frac{dx}{x} = -\frac{3}{2}\frac{dp}{p}$

$|x| = C|p|^{-\frac{3}{2}}$

partikularni

$C = C(p)$

$\dot{x} = C'(p)\,p^{-\frac{3}{2}} = \frac{1}{2}p^{-3}$

$\int_0^p \frac{1}{2}p^{\frac{3}{2}-3} = \frac{1}{2}\int_0^p p^{-\frac{3}{2}}\,dx = -p^{-\frac{1}{2}}$

$C(p) = -p^{-\frac{1}{2}}$

$x = Cp^{-\frac{3}{2}} - p^{-2}$

$y = xy' - x^2(y')^3 = Cp^{-\frac{1}{2}} - p^{-1} - (C^2p^{-3} - 2Cp^{-\frac{3}{2}-2} + p^{-4})p^3$

$= Cp^{-\frac{1}{2}} - p^{-1} - C^2 + 2Cp^{-\frac{1}{2}} - p^{-1} =$

$= 3Cp^{-\frac{1}{2}} - 2p^{-1} - C^2$

Dobili smo rešitev v parametrični obliki $(x(p), y(p))$

kjer je $p = \frac{\dot{y}(p)}{\dot{x}(p)} = y'(x)$

Še singularne rešitev

$$x = -p^{-2} + Dp^{-\frac{3}{2}}$$

$$y = -2p^{-1} + 3Dp^{-\frac{1}{2}} - D^2$$

$$F(x,y,p) = y - xp + x^2p^3 = 0$$

$$F_p = -x + 3x^2p^2 = x(1+3xp^2)$$

$x=0 \Rightarrow y=0$ ena točka ni ogrinjača

$$1+3xp^2=0 \Rightarrow p^2 = +\frac{1}{3x} \Rightarrow p = \pm\sqrt{+\frac{1}{3x}}$$

$$x < 0$$

$$y - x\left(\pm\sqrt{\frac{1}{3x}}\right) + x^2\frac{1}{3x}\left(\pm\sqrt{\frac{1}{3x}}\right) = 0$$

$$y = \pm\sqrt{\frac{1}{3x}}\left(x - \frac{1}{3}x\right) = \pm\sqrt{\frac{1}{3x}}\cdot\frac{2}{3}x = \pm\frac{2}{3}\sqrt{\frac{x}{3}}$$

$$y' = \pm\frac{2}{3}\frac{\frac{1}{3}}{2\sqrt{\frac{x}{3}}} = \pm\frac{1}{9}\sqrt{\frac{3}{x}}$$

Ali reši enačbo

$$\pm\frac{2}{3}\sqrt{\frac{x}{3}} - x\left(\pm\frac{1}{9}\sqrt{\frac{3}{x}}\right) + x^2\left(\pm\frac{1}{9^3}\frac{3}{x}\cdot\sqrt{\frac{3}{x}}\right) =$$

$$= \pm\frac{2}{3}\sqrt{\frac{x}{3}} \mp \frac{1}{9}\sqrt{3x} \pm \frac{3}{9^3}\sqrt{3x} \neq 0$$

$$= \pm\frac{2}{3}\sqrt{\frac{x}{3}} \mp \frac{78}{9^3}\sqrt{3x} = \sqrt{x}\left(\pm\frac{2}{3\sqrt{3}} \mp \frac{78\sqrt{3}}{9^3}\right) =$$

$$= \sqrt{3x}\left(\pm\frac{2}{9}\right. \quad \ldots$$

2. način primerjamo y' in p

$$\pm\frac{1}{9}\sqrt{\frac{3}{x}} = \pm\sqrt{\frac{1}{3x}}$$

$$\pm\frac{\sqrt{3}\,3}{9\sqrt{3}} = \pm\frac{1}{\sqrt{3}} \quad \times$$

Znaloge

Določi rešitve

$$(y')^2 - xy' - y + \frac{x^2}{2} = 0$$

$$y = (y')^2 - xy' + \frac{x^2}{2}$$

$$y = p^2 - xp + \frac{x^2}{2}$$

$$dy = 2p\,dp - x\,dp - p\,dx + x\,dx \qquad dy = p\,dx$$

$$2p\,dx - x\,dx = 2p\,dp - x\,dp$$

$$(2p - x)\,dx = (2p - x)\,dp$$

1. $2p - x = 0$

$$2p = x \Rightarrow p = \frac{x}{2}$$

$$y = p^2 - 2p^2 + 2p^2 = p^2$$

2. $2p - x \neq 0$

$$dx = dp$$

$$p'(x) = 1$$

$$p(x) = x + c$$

$$y = 1 - x + \frac{x^2 + 2xc + c^2}{2} = \frac{x^2 + 2x(c-1) + c^2 + 2}{2} = \frac{x^2}{2} + x(c-1) + c^2 + 2$$

??

$$y = \frac{x^2}{2} + xc + c^2$$

$$\cancel{0 = x + 2c \rightsquigarrow c = -\frac{x}{2}}$$

$$\cancel{\frac{x^2}{2} - \frac{x^2}{2} - x + \frac{x^2}{4} + 2}$$

singularne:

$$0 = x + 2c$$

$$c = -\frac{x}{2} \rightsquigarrow y = \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{4} = \frac{x^2}{4}$$

Clairautova enačba

$$y = xy' + \psi(y')$$

Videli bomo, da ogrinjačo najdemo že sproti

3. naloga

a) $y = xy' + \frac{1}{y'}$

b) $y = xy' + (y')^2 + 1$

a) $y = xy' + \frac{1}{y'}$ $\quad \psi(t) = \frac{1}{t}$ $\quad y' = p$

$$y = px + \frac{1}{p}$$

$$dy = p\,dx + x\,dp - \frac{1}{p^2}dp \qquad p\,dx = dy$$

$$\frac{1}{p^2}dp = x\,dp$$

$$0 = \left(\frac{1}{p^2} - x\right)dp$$

Clairotova enačba se vedno reducira na obliko $f(x,p)\,dp = 0$

$dp = 0$ .... splošna rešitev

$f(x,p) = 0$ .... ogrinjača

1) $dp = 0 \Rightarrow p \equiv C$

$$y = xC + \frac{1}{C}$$

2) $\frac{1}{p^2} = x \Rightarrow p = \pm\sqrt{\frac{1}{x}}$

$$y = \pm\sqrt{x} \pm \sqrt{x} = \pm 2\sqrt{x}$$

$$y' = \mp \frac{1}{\sqrt{x}}$$

b) $y = xp + p^2 + 1$

$dy = x\,dp + p\,dx + 2p\,dp \qquad dy = p\,dx$

$(x + 2p)\,dp = 0$

$dp = 0 \Rightarrow p \equiv C \Rightarrow y = xC + C^2 + 1$

$x = -2p \Rightarrow p = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x^2 + \frac{1}{4}x^2 + 1 = -\frac{1}{4}x^2 + 1$

$y' = -\frac{1}{2}x$

Opombe o singularnih rešitvah Clairotovih enačb:

$y = xy' + \psi(y')$

$F(x, y, p) = xp + \psi(p) - y = 0$

$F_p = x + \psi'(p) = 0$

Splošna rešitev

$0 = G(x, y, C) = xC + \psi(C) - y = F(x, y, C) = 0$

Torej odvajanje v p ti vedno da ~~..~~ ogrinjačo pri Clairotovi enačbi

Lagrangeeva enačba

$$y = x\varphi(y') + \psi(y')$$

4. Reši

$$y = \frac{xy'}{2} + \frac{2}{y'} \qquad \varphi(y') = \frac{y'}{2}$$

$$dy = \frac{x}{2}dp + \frac{p}{2}dx - \frac{2}{p^2}dp \qquad dy = \cancel{dp}\, p\,dx$$

$$\frac{1}{2}p\,dx = \frac{x}{2}dp - \frac{2}{p^2}dp$$

$$p\,dx = \left(x - \frac{4}{p^2}\right)dp$$

$$\dot{x} = \frac{dx}{dp} = \frac{x}{p} - \frac{4}{p^3}$$ LDE za $x = x(p)$

$$\dot{x} - \frac{x}{p} = -\frac{4}{p^3}$$

homogeno:

$$\frac{dx}{x} = \frac{1}{p}dp$$

$$|x| = C|p| \qquad x = C\cdot p$$

parti:

$$C'(p)\cdot p = -\frac{4}{p^3} \Rightarrow C'(p) = -\frac{4}{p^4}$$

$$C = \overline{3}\cdot 4\; p^{-3}$$

$$x = 12\cdot p^{-3}\cdot p + C\cdot p \Rightarrow x = \frac{\cancel{12}\,4}{p^2\cdot 3} + C\cdot p$$

$$y = \frac{1}{2}\left(\frac{4}{3p} + Cp^2\right) + \frac{2}{p} = \frac{1}{p}\left(\frac{2}{3} + \frac{1}{2}\right) + \frac{C}{2}p^2 = \frac{8}{3} + \frac{C}{2}p$$

Singularna rešitev: $0 = \frac{x}{2} - \frac{2}{p^2} = F_p$

$$\frac{2}{p^2} = \frac{x}{2}$$

$$p^2 = \frac{4}{x}$$

$$\Rightarrow y = \frac{2}{p^2}\cdot p + \frac{2}{p} = \frac{4}{p}$$

Ali je res $dy = p\,dx$?

$$\dot{y} = -\frac{4}{p^2} \qquad \dot{x} = -\frac{8}{p^3}$$

Ali velja $\dot{y} = p\cdot\dot{x}$

$$\cancel{-\frac{4}{p^2}} \qquad p\cdot\left(-\frac{8}{p^3}\right) = -\frac{8}{p^2} \neq -\frac{4}{p^2}$$

$\Rightarrow$ To ni singularna rešitev

5. Reši $y = \frac{x(y')^2}{2} + \frac{2}{y'}$.

Povzetek (o reševanju impl DE)

$F(x,y,y')=0 \quad \nabla F \neq 0$ F dovolj gladka

$F(x,y,p)$ je ploskev

$P(x_0,y_0,p_0) \in \mathbb{R}^3$

1) $F_p(P) \neq 0 \Rightarrow p = p(x,y)$

$F(x,y,p(x,y))=0$ za x,y blizu x_0,y_0

$p(x_0,y_0)=p_0$

če je $x \mapsto y(x)$ rešitev začetnega problema $F(x,y,y')$
$y(x_0)=y_0$

Potem je $y'=p(x,y)$

tako dobimo splošno rešitev

2) $F_p(P)=0 \Rightarrow$

dobimo množico $\{F=F_p=0\}$ ← unija točk in krivulj

če je P_0 izolirana točka:

na neki okolici U od P_0 velja

$\nabla F \neq 0$ za $P \in U - \{P_0\}$

dobimo splošno rešitev, vendar se lahko zgodi, da več družin splošne rešitve poteka skozi P_0

če je $\{F=F_p=0\}$ krivulja, jo parametriziramo npr kot $(x(p),y(p),p)$; če velja $p \cdot \dot{x}(p)=\dot{y}(p)$, potem je to singularna rešitev (in hkrati ogrinjača splošne rešitve

18.11

Primer od zadnjič

$$y = \frac{xy'^2}{2} + \frac{2}{y'}$$

1) $F_p = 0$

$$F(x,y,p) = \frac{xp^2}{2} + \frac{2}{p} - y$$

$$F_p = 0 \Rightarrow xp - \frac{2}{p^2} = 0 \Rightarrow x = \frac{2}{p^3}$$

$$y = \frac{xp^2}{2} + \frac{2}{p} = \frac{2}{p^3} \cdot \frac{p^2}{2} + \frac{2}{p} = \frac{3}{p}$$

$$\dot{y} = -\frac{3}{p^2}$$

$$p\dot{x} = p\left(-\frac{6}{p^4}\right) = -\frac{6}{p^3} \neq \dot{y} \Rightarrow G \text{ ni ogrinjača}$$

oz ni sing. reš

2) $dy = p\,dx$

$$dy = xp\,dp + \frac{p^2}{2}dx - \frac{2}{p^2}dp$$

...

$$\left(\frac{p^2}{2} - p\right)dx + \left(xp - \frac{2}{p^2}\right)dp = 0$$

Ker nas zanimajo rešitve v obliki grafov funkcij $x \mapsto y(x)$ lahko predpostavimo, da $dx \neq 0$

$\Rightarrow$ formalno lahko delimo z dx; dobimo DE za

$p = p(x)$.

Običajno je lažje reševati DE za $x = x(p)$
želimo deliti z dp, da dobimo $\frac{dx}{dp} = \dot{x}(p)$

Ali je lahko $dp = 0$? $\Rightarrow p \equiv C = y' \Rightarrow y(x) = Cx + D$ zaustrezen D

Sicer $\Rightarrow \left(\frac{p^2}{2} - p\right)\frac{dp}{dx} + xp - \frac{2}{p^2} = 0$

Opazimo: $dp = 0 \Rightarrow \frac{p^2}{2} - p = 0$

$dp \neq 0 \Rightarrow$ dobimo DE, če $\frac{p^2}{2} - p \neq 0$

če $\frac{p^2}{2} - p = 0 \Rightarrow p = 0 \vee p = 2$

ni definirana

$\rightarrow y = 2x + 1$

Ali je rešitev singularna?

$$F_p = xp - \frac{2}{p^2}$$

Premica $y = 2x+1$ ustreza krivulji
$x \mapsto (x, y(x), y'(x)) =$
$(x, 2x+1, 2)$

$$F_p = 2x - \frac{1}{2} \Rightarrow$$

Povsod razen v $x = \frac{1}{4}$ je ta izraz $\neq 0$

Ta premica torej pripada splošni rešitvi
$x = \frac{1}{4}$ pa je lahko singularna točka

smo poiskali zadnjič

Povzetek povzetka

1) $F_p = 0$, parametriziramo/eliminiramo p iz enačb $F = F_p = 0$, preverimo če je rešitev singularna

2) Splošno rešitev iščemo s pomočjo nastavka $dy = p\,dx$

Naj bo $y = \Phi(x, C)$ splošna rešitev DE $F(x, y, y') = 0$ in ψ ogrinjača. Pokaži da skozi $\forall$ točko na grafu ψ poteka neskončno mnogo rešitev

DN

Reši DE

$$y = xp^2 - 2p^3$$

$$y = (xy'^2) - 2(y')^3$$

$$dy = p\,dx = p^2 dx + 2px\,dx - 6p^2 dp$$

$$dx(p^2 + p) + 6p^2 dp = 0$$

1) $dp \equiv 0 \Rightarrow p = c,\ y = c^2 x - 2c^3$

$$y' = c^2 = p = c$$

$$c^2 = c \Rightarrow c = 0 \vee c = 1$$

$y = c$ sing. $\qquad y = x - 2$ ni singularna

$F_p = F = 0$

$-2xp + 6p^2 = 0 \qquad p = 0 \Rightarrow$

$-2p(x - 3p) = 0 \qquad x = 3p \Rightarrow$

$$F = 0 \Rightarrow 3p^3 - 2p^3 = 0$$

$$p^3 = 0$$

$$\dot{x}(p) = 3$$

$$\dot{y}(p) = 3p^2$$

$p\dot{x} \neq \dot{y} \Rightarrow$ to ni rešitev

$dp \neq 0 \Rightarrow$

$\dot{x}(p - p^2) + 6p^2 - 2px = 0 \quad / : p(1-p)$ smo že obravnavali kdaj je to 0

$$\dot{x} + x\frac{-2p}{p - p^2} = \frac{6p^2}{p^2 - p}$$

homogeni del:

$$\frac{dx}{x} = \frac{-2p\,dp}{p - p^2} = \frac{-2}{1-p}dp$$

$$\ln|x| = -2\ln|1-p|$$

$$x = D(1-p)^{-2}$$

partikularni:

$$D'(1-p)^{-2} = \frac{6p}{p-1}$$

$$D' = 6p(p-1) = 6p^2 - 6p$$

$$D = 2p^3 - 3p^2 + C$$

Splošno: $x = \dfrac{(2p^3 - 3p^2 + C)}{(1-p)^2}$

Enačbe višjega reda

$$F(x, y^{(k)}, \dots, y^{(n)}) = 0 \qquad 0 \le k \le n$$

Lahko znižamo red z nastavkom: $z = y^{(k)}$

$$\hookrightarrow F(x, z, \dots, z^{(n-k)}) = 0$$

$$xy''' = y''$$

$$z = y''$$

$$xz' = z$$

$$\frac{dz}{z} = C\frac{dx}{x} \rightsquigarrow |x| = C|z| \Rightarrow x = C|z| = |y''|$$

$$z = C \cdot x = y''$$

± skrijemo v konstanto

$$\iint (Cx\,dx)\,dx = \int C\frac{1}{2}x^2 + D\,dx = \int Cx^2\,dx = C\frac{1}{3}x^3 + Dx + E =$$

$$= Cx^3 + Dx + E$$

(2)

$F(y, y', \dots y^{(k)}) = 0$ $\hookrightarrow$ F neodvisen od x

"$z(y) = y'(y)$" zady je yodvod od y

Interpretiramo kot:

$z(y) = y'(x(y))$ $\qquad$ $z(y(x)) = y'(x)$

$y \mapsto x(y)$ invez rešitve y
(obstaja ker $y'(x) \neq 0$)

Reši enačbo

$yy'' = y'(y'+1)$

$y' = 0 \Rightarrow yy'' = 0$ $\leftarrow$ Možnost ko ne moremo uporabiti nastavka

1) $y = 0$

2) $y'' = 0 \Rightarrow$

$y' = c = 0 \Rightarrow y = c$

$\Rightarrow y = c \in \mathbb{R}$

$z = y'(x(y))$

$z(y(x)) = y'(x)$

$\frac{dz}{dy} \cdot y' = y''(x)$ $\qquad$ $\dot{z} y' = y''$

$\dot{z} \cdot z = y''$

Nesemo v enačbo

$y \cdot \dot{z} \cdot z = z(z+1)$

1) $z = 0$

2) $z \neq 0 \Rightarrow$ lahko delimo

2) $y \dot{z} = z + 1$

hom: $\frac{dz}{dy} \cdot y = z$ $\qquad$ $z = Cy$

partikularna:

$y \cdot C' y = 1$

$C' = \frac{1}{y^2} \rightsquigarrow C = -\frac{1}{y} + D$

$z = -1 + Dy$

$\frac{y'}{-1+Dy} = 1$

$x = \int \frac{dy}{-1+Dy} = \frac{1}{D} \ln(-1+Dy) + c$

$Dx + c = \ln(Dy - 1)$

$\frac{ce^{Dx} + 1}{D} = y$

$D = 0 \rightarrow$

$\frac{y'}{-1} = 1$

$y' = -1$

$y = E - x$

$F(x, y, y', \dots) = 0$

F homogena v $y, y', \dots$

$F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y' \dots)$ za nek fiksen k in za $\forall \lambda \in \mathbb{R}$

... uvedemo $z(x) = \frac{y'}{y}$

$yy'' = x^2y^2 + (y')^2$

$z = \frac{y'}{y} \qquad yz = y' \qquad y'' = y'z + z'y = z^2y + z'y$

$y(yz^2 + z'y) = x^2y^2 + (yz)^2$

$y = 0 \Rightarrow$ je rešitev

$y \neq 0 \Rightarrow$

$\cancel{z^2} + z' = x^2 + \cancel{z^2}$

$z = \frac{1}{3}x^3 + C = \frac{y'}{y} \implies \ln|y| = \frac{x^4}{12} + Cx + D$

$y = De^{\frac{x^4}{12} + Cx}$

Eksistenčni izrek

$f: D \subset \mathbb{R}^2 \to \mathbb{R}$ zvezna in Lipschitzeva v y

$|f(x,y_0) - f(x,y_1)| < L|y_0 - y|$ za nek $L \geq 0$

$\Rightarrow \exists!$ rešitev $y' = f(x,y)$, $y(x_0) = y_0$ $\forall (x_0,y_0) \in D$, ki je definirana na intervalu $(x_0-\alpha, x_0+\alpha)$, kjer je $\alpha = \min\{a, \frac{b}{M}, \frac{1}{L}\}$, kjer sta $a,b>0$ taka, da je

$$\underbrace{[x_0-a, x_0+a]\times[y_0-b, y_0+b]}_{C_{a,b}} \subset D$$

in je ~~M maksimum~~

$$M = \max_{x,y \in C_{a,b}} |f(x_0,y_0)|$$

opomba: $f \in C^1[a,b] \Rightarrow f$ je Lipschitzova

Podana je Cauchyjeva naloga

$y' = y + \cos y \qquad y(0) = 0$

a) Utemelji da obstaja natanko ena rešitev in določi Taylorjev polinom stopnje 3 v $x=0$

b) določi α

c) Dokaži, da se rešitev razširi na $\mathbb{R}$

d) Dokaži da velja $\lim_{x\to\infty} y(x) = \infty$

a) F je zvezna na $\mathbb{R}^2$

Ali je Lipschitzeva? da ker je v C^1

$\Rightarrow$ Rešitev obstaja in je enolično določena

$$y'' = y' - \sin(y) y' = (y + \cos y)(1 - \sin(y))$$

$$y''' = (y' - y' \sin(y))(1 - \sin(y)) + (y + \cos y)(1 - \cos(y) y') =$$

$$= (y + \cos y)(1 - \sin(y))^2 + (y + \cos(y))(1 - y\cos y + \cos^2 y)$$

$$y(x) = y(0) + \underset{1}{\underset{\shortparallel}{y'(0)}} x + \underset{\frac{1}{2}x^2}{\underset{\shortparallel}{\tfrac{1}{2} y''(0)}} x^2 + \underset{0}{\underset{\shortparallel}{\tfrac{1}{6} y'''(0)}} + o(x^3) =$$

$$= x + \frac{1}{2}x^2 + o(x^3)$$

b) $L = \max_{C(a,b)} \left|\frac{\partial f}{\partial y}\right|$ $\qquad f(x,y) = y + \cos y$

$$\left|\frac{df}{dy}\right| = |1 - \sin(y)| \leq 2 \qquad L = 2$$

$$\alpha = \min\left(a, \frac{b}{M}, \frac{1}{2}\right)$$

$$\max_{C(a,b)} f(x,y) = \max_{[-b,b]} f(\cdot, y) = M$$

$$|f(x,y)| = |y + \cos y| \leq |y| + |\cos y| \leq b + 1$$

$$\frac{b}{M} \geq \frac{b}{b+1}$$

$$\alpha \geq \min\left\{a, \frac{b}{b+1}, \frac{1}{2}\right\}$$

če izberemo a in b dovolj velika, je

lahko $\alpha > \frac{1}{2}$

$\Rightarrow$ rešitev na $\left(-\frac{1}{2}, \frac{1}{2}\right)$

c) Pogoj iz b velja za vsako točko.
Torej če si postopoma izbiramo točke vedno bolj na robovih že znanega, lahko razširimo intervale na cel $\mathbb{R}$

razen namesto b spada pišemo $b + \mu b$

Sklep: če lahko α omejimo navzdol neodvisno od začetnega pogoja, potem se rešitev razširi na $\mathbb{R}$

V posebnem to velja: če sta f in f_y omejena

d) $\lim_{y \to \infty} y(x) = \infty$

y naraščajoča in večja od 1

$y' > 0$ velja za $t > 1$

$1 - \sin t \geq 0$ za $t \geq 0 \Rightarrow$ fun narašča

Dokažemo da je $y(x) > 0$ $\forall x > 0$

Recimo da $y(\tilde{x}) = 0$ za nek $x' > 0$

$y'(\tilde{x}) = 0 + \cos(0) = 1$

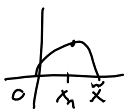
Torej je mogla biti prej funkcija negativna,

$Z = \{x, > 0, x(x) = 0)$

$y(0) = 1$ $\qquad y = f(x)$

$\Rightarrow f(x) > 0$

za $x \in (0, \varepsilon)$ $\varepsilon > 0$



Rolle: $x_1 \in (0, \tilde{x})$

$f'(x_1) = 0$

$f'(x_1) = \underset{0}{\underset{\vee}{f(x_1)}} + \cos(f(x_1)) = 0$

$\Rightarrow \cos f(x_1) < 0$

$f(x_1) > \frac{\pi}{2}$

$f'(x_1) > \frac{\pi}{2} + \cos(\lambda_1) > 0$ $\ast$

$\Rightarrow f(x) > 0$ za $\forall x > 0$

$\Rightarrow f'(0)$ je naraščajoča

Eksistenčni izrek

$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ zvezna in Lipschitzova

$\Rightarrow \exists!$ rešitev $y' = f(x,y)$ $y(x_0) = y_0$ za $\forall (x_0, y_0)$,
ki je definirana na intervalu $(x_0 - \alpha, x_0 + \alpha)$,
kjer je $\alpha = \min\{a, \frac{b}{M}, \frac{1}{L}\}$; $M = \max|f|$, L Lipsch.,
$a, b > 0$. $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset D$

$f \in C^1[a,b] \Rightarrow f$ je lipschitzova

$F(x, y, y', \dots) = 0$

F homogena v y, y' $(F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y', \dots))$

$\Rightarrow$ uvedemo $z(x) = \frac{y'}{y}$

Enačba višjega reda $F(x, y^{(k)}, \dots, y^{(n)}) \rightsquigarrow z = y^{(k)}$

Reševanje impl. DE

$F(x,y,y') = 0$ $\nabla F \neq 0$

1) $F(x,y,p) = 0$ je ploskev

$P = (x_0, y_0, z_0)$

2.1) $F_p(P) \neq 0 \Rightarrow p = p(x,y)$

$F(x, y, p(x,y)) = 0$ v okolici P

$p(x_0, y_0) = p_0$

$y(x)$ rešitev $\Rightarrow y' = p(x,y)$

Dobimo splošno rešitev

2.2) $F_p(P) = 0 \Rightarrow \{F = F_p = 0\}$ je unija točk in krivulj

P_0 izolirana točka $\Rightarrow \exists U$ okolica P. $P \in U - \{P_0\}$. $\nabla F \neq 0$

Dobimo splošno rešitev

$\{F = F_p = 0\}$ je krivulja $\Rightarrow$ parametriziramo
npr $(x(p), y(p), p)$.

$p \cdot \dot{x}(p) = \dot{y}(p) \Rightarrow$ Singularna rešitev

Ogrinjača splošne rešitve

Clairautova enačba

$y = xy' + \psi(y')$ ⇝ $y' = p$ ⇝ $p\,dx = dy$

Dobimo: $f(x,p)\,dp = 0$

$dp = 0$ ... singularna rešitev
$f(x,p) = 0$ ... ogrinjača

Odvajanje pop, ti vedno da ogrinjačo pri tej enačbi

Prvi integral

enačbe $y' = f(x,y)$ je funkcija, ki je konst. vzdolž vsake rešitve

$y = \varphi(x,c) \wedge \varphi_c \neq 0 \Rightarrow c = C(x,y)$

1) DE zapišemo kot $P\,dx + Q\,dy = 0$

$Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y \Rightarrow (P,Q)$ je potencialno

$Q = \frac{\partial}{\partial y}\int P\,dx + C'(x,y)$

Riccatijeva enačba

$y' = a^2y + by + c$

1) Uganemo rešitev y_p
2) Splošna rešitev je $y = y_p + u(x)$
3) Dobimo Bernullijevo enačbo $y = y_p + \frac{1}{u^2(x)} \Rightarrow$ LDE

Bernullijeva enačba

$y' + ay = by^\alpha \quad \alpha \notin \{0,1\}$ ⇝ $z = y^{1-\alpha}$

25.11

Od zadnjič naprej $y' = y + \cos y \quad y(0) = 0$

Rešitev φ je definirana za $\forall x \in \mathbb{R}$

$y > 0$ za $\forall x \geq 0$

$Z = \{ x > 0,\ \varphi(x) = 0 \}$

$\varphi'(0) = \varphi(0) + \cos(0) = 0 + \cos 0 = 1$

$\Rightarrow$ za nek $\varepsilon > 0$ je $\varphi(x) > 0 \quad \forall x \in (0, \varepsilon)$

označimo $x_0 = \inf Z > 0$

φ zvezna $\Rightarrow \varphi(x_0) = 0$

Rolle: $\exists x_1 \in (0, x_0).\ \varphi'(x_1) = 0 \wedge \varphi(x) > 0 \quad \forall x \in (0, x_0)$

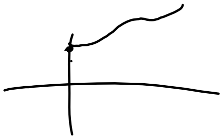
φ reši CN $\Rightarrow \varphi'(x_1) = \varphi(x_1) + \cos(\varphi(x_1))$

$\underset{0}{\vee} \quad \Rightarrow \varphi(x_1) > \frac{\pi}{2}$

$\Rightarrow \varphi(x_1) - \cos(\varphi(x_1)) > 0$

$\Rightarrow Z = \emptyset \qquad \times$

$g(t) = t + \cos t \quad g(0) = 1 \quad g'(t) = 1 - \sin t \geq 0 \ \forall t$



$g(t) \geq 1.\ \forall t \geq 0$

$\varphi'(x) = g(\underset{\substack{\vee\!| \\ 0}}{\varphi(x)}) \geq 1 \quad \forall x \geq 0$

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(t)\,dt \geq \int_0^x 1\,dt = x \xrightarrow{x \to \infty} \infty$$

1) Dana je Cauchyjeva naloga

$$y' = \frac{y}{1+x^2+y^2} \qquad y(0)=1$$

Dokaži da obstaja ! rešitev, ki je definirana na $\mathbb{R}$.

Ali ima φ ničlo?

$f(x,y)$ mora biti zvezna in lipschitzova v y

$$f(x,y) = \frac{y}{1+x^2+y^2}$$ zveznost zaradi elementarnosti

Naredimo pravokotnik $[a,a] \times [b, b]$

Ta mi je lipschitzova, ker je odvod omejen

$\Rightarrow$ φ obstaja in je enolično določena na $[-\alpha, \alpha]$

kjer je $\alpha = \min\{a, \frac{b}{M}, \frac{1}{L}\}$

a, b poljubno velika

$$M = \max_{[a,b]} |f(x,y)| = \max \frac{y}{1+y^2} = \{|\pm \tfrac{1}{2}|\} = \frac{1}{2}$$

$x^2 = 0$

$$\frac{(1+y^2) - 2y^2}{(1+y^2)^2} = \frac{1-y^2}{1+y^2}$$

$$L = \max \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{1+x^2+y^2-2y^2}{(1+x^2+y^2)} = \frac{1+x^2-y^2}{(1+x^2+y^2)}$$

L je tudi omejena

Dobili smo α, ki je isti za vse točke in neodvisen od začetnega pogoja, tako da lahko z eksistenčnim izrekom razširimo na vse točke

b) $y' = 0 \Leftrightarrow y = 0$

Recimo da je $\varphi(x_0) = 0$

$z \equiv 0$ $z' = f(x,y)$ reši $z(x_0) = 0$

φ tudi reši to nalogo $\Rightarrow \varphi = z$ (zaradi enoličnosti)

$\times$, ker je $\varphi(0) = 1$

Trditev: Za α lahko vzamemo $\alpha = \min\{a, \frac{b}{M}$

L mora pa obstajat na tem pravokotniku

2)

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ zvezna

$$y'(x) = 1 + \frac{y(x)^2}{2} \sin f(x) \quad ; x \in [0,2]$$
$$y(0) = 0$$

Dokaži da je

$$\max_{x \in [0,2]} y(x) \geq 1$$

Recimo da $\max y(x) < 1 \Rightarrow$

$$y'(x) \geq 1 + \frac{y(x)^2}{2} \geq \frac{1}{2}, \text{če } y \in [0,1]$$

$\underline{y \geq 0}$

$x_0 = \inf\{x ; y(x_0) = 0\}$

Recimo da $\exists x_0 > 0$ da je $y(x_0) = 0$

Potem po Rollovem izreku, $\exists x_1 \in (0, x_0)$ da je $y'(x_1) = 0$

$$y'(x_1) = 1 + \frac{(y(x_1))^2}{2} \sin(f(x_1))$$

$$0 \leq y(x_1) \leq 1$$

$$y'(x_1) \geq 1 - \frac{(y(x_1))^2}{2} \geq \frac{1}{2}$$

$\Rightarrow y > 0$ za $\forall x \in \mathbb{R}_{>0} [0,2]$ ker je odvod v 0 pozitiven

$$\Rightarrow y'(x) \geq 1 - \frac{1}{2} \geq \frac{1}{2}$$

Potem je $y(2) = y(0) + \int_0^2 y'(t)\,dt \geq 0 + 2 \cdot \frac{1}{2} = 1$ $\times$

$\Rightarrow \max y \geq 1$

3.

$y' = f(y) \qquad f \in C^1(\mathbb{R})$

Pokaži da je $\forall$ periodična rešitev konstantna

Recimo da $\exists\, \omega > 0.\ y(x+\omega) = y(x)\ \forall x$

$$\int_0^\omega y' dx = \int_0^\omega f(y) dx = c$$

y periodična $\Rightarrow f(y)$ periodična

če je $f(c) = 0$, potem je $y \equiv c$ rešitev

$f(y)$ je odvod $\Rightarrow f$ ima ničlo (ker je y periodična)

$\rightarrow \exists x_0 \in \mathbb{R}.\ f(y(x_0))$ $\quad y \equiv y(x_0)$ je ena rešitev

sistema in zaradi enoličnosti je to edina

4.

$y' = y + e^y \qquad y(0) = 1$

a) Dokaži, da je definicijski interval rešitve, ki ga določa eksistenčni izrek omejen

b) Dokaži da je φ naraščajoča v $\forall$ točki $x \geq 0$, kjer je definirana

c) Dokaži da ima φ ničlo

d) $\exists\ \omega.\ 0 < \omega < \infty.\ \lim_{x \nearrow \omega} \varphi(x) = \pm\infty$

---

a) $y + e^y$ je zvezno odvedljiva $\Rightarrow \exists$ enolična rešitev

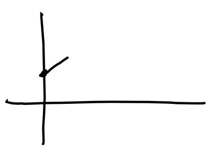
na $[\alpha, -\alpha]$ $\quad \alpha = \min\{a, \frac{b}{M}, \cancel{\frac{1}{L}}\}$

$y \in [1-b, 1+b] \Rightarrow y + e^y \leq 1 + b + e^{1+b}$

$M = 1 + b + e^{1+b} \Rightarrow \frac{b}{M} = \frac{b}{1+b+e^{1+b}} \leq C \quad \forall b > 0$ ta funkcija je omejena

b) $y' = y + e^y$

$\underline{y' \geq 0} \qquad y(0) = 1$



Recimo da je $y + e^y = 0$

$$\underset{0}{\overset{\vee}{e^y}} = -y$$

če je $y = 0$ potem je $y' = 1$ ker pomeni da v vsaki ničli narašča, kar pa ne more biti razen za eno ničlo

c) f ima ničlo

Recimo da nima ničle. potem je $f(x) > 0 \ \forall x \in \mathbb{R}$,

potem je pa f naraščajoča

in odvod je večji od 1. potem pa nima

sekati x os ali pa rešitve naloge ne moremo razširiti

če je pol, ne more biti negativen, torej je pozitiven, potem je pa odvod

< 0 $\cancel{\quad}$

$$f(x) = f(a) + \int_a^x f' \, dt \leq f(a) + x \xrightarrow{x \to -\infty} -\infty$$

d) Dokazujemo:

$\exists \omega > 0. \lim_{x \to \infty} f(x) = \infty$

Dokažimo da je inverz omejen. (inverz obstaja ker je f strogo naraščajoča na $x \geq 0$

$$\frac{1}{z'} = y + e^y \Rightarrow z' = \frac{1}{y+e^y}$$

$$z(y) = \int_1^y \frac{1}{t+e^t} \, dt \leq \int_1^y \frac{1}{e^t} \, dt = e^{-1} - e^{-y} \leq M$$

$t \geq 0$

obstaja tak M

Naj bo $f \in C^1(\mathbb{R}^2)$, zadošča predpostavkam eksistenčnega izreka. Recimo, da se naše rešitve φ ne da razširiti čez interval (α, ω); $-\infty < \alpha < x_0 < \omega < \infty$, potem velja $\lim_{x \to \omega} y(x) = \pm\infty$ in $\lim_{x \to \alpha} y(x) = \pm\infty$

5)

$$y' = \frac{1}{x^2+y^2} \quad ; \quad y(a) > 0$$

a) Pokaži, da $\exists ! \varphi$ rešitev, ki je definirana na $\mathbb{R}$

b) pokaži $\lim_{x \to \infty} \varphi(x) < \infty$

$f(x,y) = \frac{1}{x^2+y^2}$ je omejena razen v okolici 0

$f_y(x,y) = \frac{2y}{(x^2+y^2)^2}$ je omejen razen na okolici 0

Dokažimo da je rešitev omejena na vsakem končnem intervalu

v 0 je rešitev pozitivna in naraščajoča

$$\varphi(x) = \varphi(0) + \int_0^x \frac{1}{t^2+\varphi(t)^2} dt = \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2+(\varphi(t))^2} dt \leq$$

$$\leq \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2} dt = \varphi(x_0) - \frac{1}{x} + \frac{1}{x_0} \leq M$$

omejeno

$\Rightarrow$ Po trditvi je $\omega = \infty$

Popravimo f v $\tilde{f}$ tako, da nima več pola v $(0,0)$ in velja $f = \tilde{f}$ izven neke majhne okolice $(0,0)$ ki je graf φ ne seka

v $-\infty$ smer je simetrija $y(-x)$

b)

Pokazali smo da je $\varphi(x)$ naraščajoča in omejena $\Rightarrow$ ima limito

Vektorsko polje na $\mathbb{R}^n$ je preslikava $V:\mathbb{R}^n \to \mathbb{R}^n$.

Tokovnica polja V je krivulja $\gamma:(a,b) \to \mathbb{R}^n$, ki
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\cup}{0}$

zadošča, da je $\dot{\gamma} = V(\gamma(t))$

$$\gamma(t) = (x_1 \dots x_n)$$

$$\dot{x}_1 = V_1(\gamma(t))$$

$$\vdots$$

$$\dot{x}_n = V_n(\gamma(t))$$

Do zdaj: obravnavali smo primere za $n=0$, vendar v povečini oblike $y'(x) = f(x, y(x))$

$$\underset{\dot{\gamma}}{\wr} \quad\quad\quad \underset{\gamma}{\wr}$$

neavtonomni sistemi

Enačbi $y' = f(x,y)$ priredimo sistem oz avtonomno vektorsko polje

$$\dot{x} = 1$$

$$y = f(x,y)$$

2.12

V 1. uro

Mislim da je dokaj pomembno

3. naloga

$f \in \mathcal{C}^\infty(\mathbb{R})$

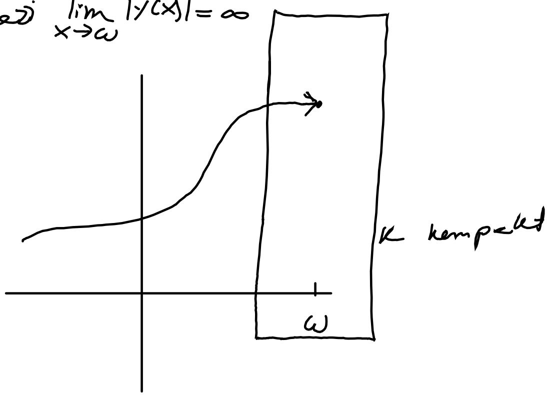
y rešitev

$y' = f(x,y)$

$y(x_0) = y_0$, taka da se ne razširi čez

(α, ω) $\omega < \infty$

Dokaži $\lim_{x \to \omega} |y(x)| = \infty$



Graf rešitve je vektorsko polje $V(1, f(x,y))$. Izberemo kompakt pravokotnik $[\omega-\delta, \omega+\delta] \times [-l, l]$

Izrek nam pove: ko je x dovolj blizu ω, je tokovnica bodisi pod $-l$ in nad $-l$

Sklep: za x dovolj blizu ω je $y(x) > l$ za $\forall l$

$\Longrightarrow \lim_{x \to \omega} |y(x)| = \infty$

$$y' = \frac{1}{x^2+y^2}$$

$y(0) > 0$

y je definirana na $\mathbb{R}$

Odvod je omejen, ker je $y' > 0 \ \forall x \in \mathbb{R}$, torej je y naraščajoča, večji kot je x^2, večji je y, vedno manjši je odvod. $\Rightarrow$

$$y' = \frac{1}{x^2+y^2} \leq \frac{1}{y^2} \leq \frac{1}{y_0^2} \quad \text{za } x \geq 0$$

$\rightsquigarrow y(x) \leq \frac{1}{y_0} x + y_0 \Rightarrow$ če je ω končen, $y(x)$ ne zapusti vsakega kompakta

Za $x < 0$. y vedno narašča

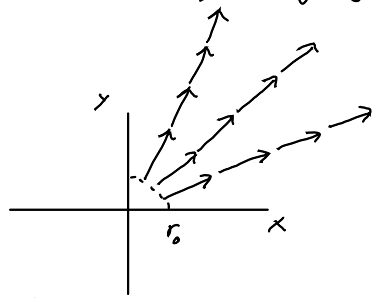
$$y' = \frac{1}{x^2+y^2} \leq \frac{1}{x^2} \rightsquigarrow 0 \Rightarrow y' \rightsquigarrow 0$$

$\Rightarrow y'$ je omejen

N4.

a) Izrazi polje $(1,0)$ in $(0,1)$, ki sta podani v polarnih koordinatah (r in φ)

b) Določi tok polja $V(x,y) = a(x^2+y^2)(y,-x)$, kjer je $a \in \mathcal{C}^\infty(\mathbb{R})$



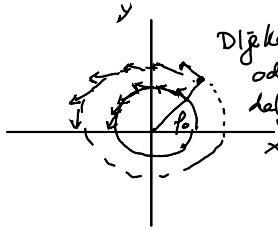
a)
$$\dot r = 1 \Rightarrow r(t) = r_0 + t$$
$$\dot\varphi = 0 \Rightarrow \varphi(t) = \varphi_0$$

$$x = r\cos\varphi \rightsquigarrow x(t) = r_0\cos\varphi_0 + t\cdot\cos\varphi_0$$
$$y = r\sin\varphi \rightsquigarrow y(t) = r_0\sin\varphi_0 + t\sin\varphi_0$$

$$\dot x(t) = \cos\varphi_0 = \cos\varphi(t)$$
$$\dot y = \sin\varphi_0 = \sin\varphi(t) \quad \overset{\shortparallel}{\varphi_0}$$

$$\left.\begin{aligned} \dot x &= \frac{x}{\sqrt{x^2+y^2}} \\ \dot y &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}\right\} W(x,y)$$

$$\dot r = 0 \rightsquigarrow r = r_0$$
$$\dot\varphi = 1 \rightsquigarrow \varphi = \varphi_0 + t$$



Dljekot bomo od izhodišča, daljše bodo puščice

$$x = r\cos\varphi = r_0\cos(\varphi_0+t)$$
$$y = r\sin\varphi = r_0\sin(\varphi_0+t)$$
$$\dot x = -r_0\sin(\varphi_0+t) = -y$$
$$\dot y = r_0\cos(\varphi_0+t) = x$$

$$Q(x,y) = (-y, x)$$

b)

$$a(x^2+y^2)\cdot(y,-x) \underset{\text{v polarnih koordinatah}}{=} a(r^2)(-(0,1)) = ar^2(-Q)$$

$\dot{r}=0$

$$\dot{r} = a(r^2)\cdot 0 = 0 \rightsquigarrow r = r_0$$
$$\dot{\varphi} = a(r^2)\cdot 1 = a(r^2) \rightsquigarrow \varphi(t) = \int a(r^2)\,dt = \varphi_0 + ta(r^2)$$
$$(x,y) = F(r,\varphi)$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = J_F \begin{pmatrix}\dot{r}\\ \dot{\varphi}\end{pmatrix} = J_F \cdot P(r,\varphi)$$

ne kapdje

$$x(t) = r(t)\cos\varphi(t) = r_0\cos(\varphi_0 - a(r_0^2)t)$$
$$y(t) = r(t)\sin\varphi(t) = r_0\sin(\varphi_0 - a(r_0^2)t$$

$$x(t) = \underbrace{r_0\cdot\cos\varphi_0}_{x_0}\cos(a(r_0^2)t) + \underbrace{r_0\sin\varphi_0}_{y_0}\sin(ar_0^2t) =$$

$$= x_0\cos(a(x_0^2+y_0^2)t) + y_0\sin(a(x_0^2+y_0^2)t)$$

DN $\dot{r} = \sin r$

$\dot{\varphi} = \cos r$

$$\dot{x} = A(t)\, x(t)$$

$A(t) = \text{konst}$

$\Rightarrow e^{At}$

$$e^{A} = 1 + A + \frac{A^2}{} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Določi A^{et} za

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{R}$

c) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

a) $$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(A t)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^k}{k!} & 0 \\ 0 & \frac{(2x)^k}{k!} \end{bmatrix} =$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix} \qquad = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

b) Jordanova forma

$$A = \lambda I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\cancel{A^k = (\lambda I + N) =}$$

komutira

Imamo obstojen id + At + $\frac{1^2}{2}$ + ...

$$e^{At} = e^{\lambda I t + Nt} = e^{\lambda I t} e^{Nt} =$$

$$= e^{\lambda t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} N^k\right) =$$

$$= e^{\lambda t}\left(I + \frac{t}{1} N + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) =$$

$$e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

c) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ Vsilsamo: to je množenje s kompleksni

$A \rightsquigarrow a+bi$

$$e^{At} = e^{(a+bi)t} = e^{at} e^{ibt} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

$A = aI + bJ$ $\quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad J^2 = -I$

$J^3 = -J \quad J^4 = I$

$$e^{At} = e^{(\cancel{a}I + bJ)t} = e^{bt} e^{bJt} =$$

$$= e^{bt}\left(I + btJ - \frac{b^2 I t^2}{2} - \frac{b^3 J t^3}{3!} + \frac{b^4 I t^4}{4!} + \frac{b^5 J t^5}{5!}\right) =$$

$$= e^{bt} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (bt)^k I + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (bt)^{2k+1} J \right) =$$

$$= e^{bt}(\cos bt\, I + \sin bt\, J) = e^{bt} \begin{bmatrix} \cos at & \sin bx \\ -\sin bt & \cos at \end{bmatrix}$$

Sistemi NDE s konstantnimi koeficienti

$\dot{\vec{x}} = Ax \qquad A \in M_{n\times n}$

Vsaka rešitev je oblike $\vec{x}(t) = \Phi(t)\vec{X}(0)$, kjer je

Φ fundamentalna rešitev (reši matrični sistem

$\dot{\Phi} = A\Phi \qquad \Phi(0) = id$

Kako poiščemo Φ?

$\varphi = e^{At} \qquad\qquad \varphi(0) = I$

$\dot{\varphi} = Ae^{At} = A\varphi(t) \qquad \checkmark$

$\Phi = \varphi$

$A = PJP$ ... jordanova forma

$$e^{At} = e^{(PJP^{-1})t} = \sum \frac{PJ^kP^{-1}t^k}{k!} = Pe^{Jt}P^{-1}$$

Predpostavljali bomo da ima A realne lastne vrednosti

Poišči fundamentalno rešitev sistema

$\dot{x} = x - y + 4z$

...

ne deluje

Poišči fundamentalno m. S

$$A=\begin{bmatrix}3&1&2\\3&3&4\\-2&-1&-1\end{bmatrix}\qquad\begin{bmatrix}3-\lambda&1&2\\3&3-\lambda&4\\-2&-1&-1-\lambda\end{bmatrix}$$

$\det(A-\lambda I)=$

$$(3-\lambda)(\lambda^2+\lambda(1-\overset{-3}{3})+4)-1(-3\lambda-3+8)+2(-3+6-2\lambda)$$

$$=(3-\lambda)(\lambda-1)^2+(-\lambda+1)=$$

$$=(\lambda-1)(-\lambda^2+\lambda(3+1)-3-1)=$$

$$=-(\lambda-1)(\lambda^2-4\lambda+4)=-(\lambda-1)(\lambda-2)^2$$

$$J=\begin{bmatrix}1&&\\&2&\overset{\{0,1\}}{?}\\&&2\end{bmatrix}\qquad P=\begin{bmatrix}0\\-2\\1\end{bmatrix}$$

$\lambda_1=1$:

$$\begin{bmatrix}2&1&2\\3&2&4\\-2&-1&-2\end{bmatrix}\sim\begin{bmatrix}2&1&2\\1&1&2\\0&0&0\end{bmatrix}\sim\begin{bmatrix}0&-1&-2\\1&1&2\end{bmatrix}\sim\begin{bmatrix}0&1&2\\1&0&0\end{bmatrix}$$

$$v_1'=\begin{bmatrix}-1\\1\\-1\end{bmatrix}\qquad x=0,\quad y=-2z$$

$\lambda_2=2$

$$\begin{bmatrix}1&1&2\\3&1&4\\-2&-1&-3\end{bmatrix}\sim\begin{bmatrix}1&1&2\\3&0&3\\0&1&1\end{bmatrix}\sim\begin{bmatrix}0&1&1\\1&0&1\\0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}$$

$\ker(A-2\lambda I)^2$:

$$\begin{bmatrix}0&0&0\\-2&0&-2\\1&0&1\end{bmatrix}\qquad x+z=0$$

$$\ker(A-2\lambda I)=\mathrm{Lin}\left\{\begin{bmatrix}x\\y\\-x\end{bmatrix}\right\}=\mathrm{Lin}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

$$v_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$$ ← je v jedru, ni pa lastni vektor

$$\vec{v}_1=(A-2I)\vec{v}_2$$

$$\vec{v}_1=\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$P=\begin{bmatrix}0&1&0\\-2&1&1\\1&-1&0\end{bmatrix}\qquad J=\begin{bmatrix}1&&\\&2&1\\&&2\end{bmatrix}$$

$$e^{Jt}=\begin{bmatrix}e^t&&\\&e^{2t}&te^{2t}\\&&e^{2t}\end{bmatrix}$$

Nehomogeni linearni sistemi NDE s konst. koef.

$$\dot{\vec{v}} = A\vec{v} + f(t)$$

$$\vec{v}_h = e^{At}\vec{v}_0 \qquad \vec{v}_p = e^{At} \cdot \underbrace{w(t)}_{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}$$

$$\dot{\vec{v}}_p = \underbrace{A e^{At} w}_{A\vec{v}_p} + e^{At} \cdot \dot{w} = A\vec{v}_p + f$$

$$e^{At} \cdot \dot{\vec{w}} = f$$

$$\dot{w} = e^{-At} \cdot f$$

$\Phi(t)$ fundamentalna rešitev $\Rightarrow \Phi$ obrnljiva za vse t, kjer je definirana

$$\Phi^{-1}(t) = \Phi(-t)$$

$$w(t) = \int_a^t e^{-A\xi} f(\xi)\, d\xi$$

$$v_p = e^{At} \cdot w(t) = \int_0^t e^{A(t-\xi)} f(\xi)\, d\xi$$

Poišči splošno rešitev sistema

$$\dot{x} = x - y + \frac{1}{\cos t}$$
$$\dot{y} = 2x - y$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = -(1-\lambda)(1+\lambda) + 2 = \lambda^2 - 1 + 2 = \lambda^2 + 1$$

$$\lambda = \pm i$$

uvedi nove koordinate $u = x$

$v = y - x$ $\quad \dot{v} = \dot{y} - \dot{x}$

$y = v + u$ $\quad \dot{y} = \dot{v} + \dot{u}$

$$\dot{u} = u - v - u + \frac{1}{\cos t} = -v + \frac{1}{\cos t}$$

$$\dot{v} \cancel{-v} + \frac{1}{\cos t} = 2u \cancel{-v} - u =$$

$$\dot{v} = u - \frac{1}{\cos t}$$

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad e^{Bt} = e^{it} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$w = \int_0^t e^{-i\xi} \begin{bmatrix} \frac{1}{\cos \xi} \\ -\frac{1}{\cos t} \end{bmatrix} d\xi = \qquad e^{-Bt} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$w = \begin{bmatrix} \int_0^t (1 - \tan \xi) d\xi \\ \int_0^t (-\tan - 1) d\xi \end{bmatrix} = \begin{bmatrix} t + \ln|\cos t| \\ -t - \ln|\cos t| \end{bmatrix}$$

$$v(t) = e^{Bt} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + e^{Bt} w(t)$$

Nej bo $A = (a_{ij})_{ij}$

[redacted]

ne stevel,

$x_j(t) = e^{\lambda_1 \dots t} \cdot p_i(t)$ $st(p_i) \leq alg.$

vecki kratne vrednosti

$\text{od}\ \lambda_i$

Vrda jo kul

12. 12

Spomnimo se

$$\dot{\vec{x}} = A \cdot \vec{x} + f(t)$$

$$\Phi = P e^{Jt} P^{-1}$$

$$x_h = \Phi x_0$$

$$x_p = \int_a^t \Phi(t) \Phi(s)^{-1} f(s) ds$$

N: Določi splošno rešitev sistema

$$\dot{x} = y - z + t$$
$$\dot{y} = x - z + 1$$
$$\dot{z} = 2x + 2y - 3z + t + 1$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix} \qquad f(t) = \begin{bmatrix} t \\ 1 \\ t-1 \end{bmatrix}$$

$$\det(A - \lambda I) =$$

$$\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -\lambda(3\lambda + \lambda^2 + 2) - 1(-3-\lambda+2) - 1(2+2\lambda)$$

$$= -\lambda((\lambda+1)(\lambda+2)) + (\lambda+1) - 2(\lambda+1) =$$

$$= (\lambda+1)(\lambda^2 + 2\lambda - 1) \dots \text{ ups} = -(\lambda+1)^3$$

$$A + I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \ker(A+I) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \mathrm{Lin}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\ker(A+I)^2 = \mathbb{R}^{2\times 3}$$ (ker se dimenzija jedra poveča za vsaj 1)

$\Rightarrow$ dopolnimo do baze: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

en od teh dveh je lahko prvi stolpec

$$J = \begin{bmatrix} -1 & \sim & \\ & -1 & 1 \\ & & -1 \end{bmatrix}$$

drugi stolpec

$$(A+I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \ker(A+I)$$

tretji stolpec

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \text{ ali pa } P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

oba sta vredu

(ker sta oba lin. neod. od $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$)

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \dots \quad P^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & \\ & -1 & 1 \\ & & -1 \end{bmatrix} \qquad e^{Jt} = \begin{bmatrix} e^{-t} & & \\ & e^{-t} & te^{-t} \\ & & e^{-t} \end{bmatrix}$$

$$\Phi(t) = e^{At} = P e^{Jt} P^{-1}$$

$$\Phi^{-1}(s) = P e^{-Js} P^{-1}$$

$$x_p = P e^{Jt} \cancel{P^{-1}} \int^t \cancel{P} e^{-Js} P^{-1} f(s)\, ds =$$

$$= P e^{Jt} \int_0^T e^s \begin{bmatrix} 1 & & \\ & 1 & -s \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} ds$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} 1 & & \\ & 1 & -s \\ & & 1 \end{bmatrix} \begin{bmatrix} s-1 \\ 1 \\ 0 \end{bmatrix} ds =$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} s-1 \\ 1 \\ 0 \end{bmatrix} ds = P e^{Jt} \begin{bmatrix} se^s - 2e^s \\ e^s \\ 0 \end{bmatrix} \Bigg|_0^t =$$

$$\int s e^s\, ds = se^s - \int e^s ds = se^s - e^s$$

$u = s \quad dv = e^s$

$du = 1 \quad v = e^s$

$$= P e^{Jt} \left(e^t \begin{bmatrix} t-2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) = x_p(t)$$

splošna rešitev:

$$X(t) = \Phi(t) x_0 + x_p(t)$$

Linearne enačbe reda n z konstantnimi koeficienti

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

sistem

$$x_1 = y$$
$$x_2 = y'$$
$$\vdots$$
$$x_n = y^{(n-1)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ -\frac{a_1}{a_0} x_{n-1} - \frac{a_2}{a_0} x_{n-2} + \dots + \frac{a_n}{a_0} x_1 \end{bmatrix}$$

$$= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Dobimo sistem oblike $\dot{\vec{v}} = A\vec{v}$

Komponente rešitev so oblike: $\sum p(t) e^{\lambda t}$

a)

$$y^{(n)} - 4y = 0$$

$$y = e^{\lambda x}$$

$$\lambda^n e^{\lambda x} - 4e^{\lambda x} = 0$$

$$\lambda^n - 4 = 0$$

$n = 4$

$$(\lambda^2-2)(\lambda^2+2) = (\lambda-\sqrt{2})(\lambda+\sqrt{2})(\lambda-\sqrt{2}i)(\lambda+\sqrt{2}i)$$

$$y_s = a_1 e^{\sqrt{2}x} + a_2 e^{-\sqrt{2}x} + a_3 e^{\sqrt{2}ix} + a_4 e^{-\sqrt{2}ix}$$

$$= a_1 e^{\sqrt{2}x} + a_2 e^{-\sqrt{2}x} + \cos(\sqrt{2}x)\underbrace{(a_3+a_4)}_{\text{eine Konstante}} + \sin(\sqrt{2}x)\underbrace{(a_3-a_4)}_{\text{eine Konstante}}$$

b)

$$y''' + y'' + y' + y = 0$$

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

$$\lambda^2(\lambda+1) + (\lambda+1) = (\lambda+1)(\lambda-i)(\lambda-i) = 0$$

$$y_s = a_1 e^{-x} + a_2 \cos x + a_3 \sin x$$

c) $y^{(4)} - 8y^{(3)} + 18y'' - 27y = 0$

$$\lambda^4 - 8\lambda^3 + 18\lambda^2 - 27 = 0$$

$$\begin{array}{r|rrrrr}
 & 1 & -8 & +18 & 0 & -27 \\
-1 & \downarrow & -1 & 9 & -27 & 27 \\
\hline
 & 1 & -9 & 27 & -27 & \| \; 0 \\
3 & \downarrow & 3 & -18 & 27 & \\
\hline
 & 1 & -6 & 9 & \| \; 0 & \\
+3 & \downarrow & +3 & -9 & & \\
\hline
 & 1 & -3 & \| \; 0 & &
\end{array}$$

$$= (\lambda+1)(\lambda+3)^3 \cancel{(\lambda+3)}$$

$$y_s = Ce^{-x} + (C_2x^2 + C_3x + C_4)e^{3x}$$

$$a_n x^n y^{(n)} + \dots a_0 y = 0$$

Dokaži ke se z $x = e^t$ pride so ELDE

$$a_n e^t y^{(n)} + \dots a_0 y =$$

$$z(t) = y(e^t)$$

$$z'(t) = e^t (y'(e^t))$$

$$z''(t) = e^t y'(e^t) + e^{2t} y''(e^t)$$

$$z'''(t) = e^t y'(e^t) + y''(e^t)(e^{2t} + e^{2t}) + e^{3t} y'''(e^t)$$

$$y'(e^t) = \frac{z'(t)}{e^t}$$

$$y''(e^t) = \frac{z''(t) - e^t y'(e^t)}{e^{2t}} = \frac{z''(t) - z'(t)}{e^{2t}}$$

$$y'''(e^t) = \underbrace{\frac{-z'(t) - 3z''(t)}{e^{3t}}}_{L(z''(t)} + z'''(t)$$

$$z^{(k)} = x^k y^{(k)} + L(z' \dots z^{(k-1)})$$

$$z^{(k+1)} = k x^k y^{(k)} + x^k y^{(k+1)} + L(z'' \dots z^{(k)})$$

$$z^{(k+1)} = k e^{kt} y^{(k)} +$$

Določi splošno rešitev

a) $x^3 y''' - 2xy' + 4y$

b) $x^4 y^{(4)} + 6x^3 y''' + 3xy' + y = 0$

a) $y = x^\lambda$

$y' = \lambda x^{\lambda-1}$ $\quad y'' = \lambda(\lambda-1)x^{\lambda-2}$ $\quad y''' = \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$

$x^3 \cdot \lambda(\lambda-1)(\lambda-2)x^{\lambda-3} - 2x\lambda x^{\lambda-1} + 4x^\lambda$

$(\lambda(\lambda-1)(\lambda-2) - 2\lambda + 4)x^\lambda = 0$

$(\lambda-2)(\lambda^2-\lambda-2) = (\lambda-2)(\lambda-2)(\lambda+1) = (\lambda-2)^2(\lambda+1)$

$y_1 = C_1 \frac{1}{x}$ $\quad y_2 = (C_2 \ln x + C_3)x^2$

b) $x^\lambda(\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 6\lambda(\lambda-1)(\lambda-2) + 3(\lambda-1) + 1) = 0$

$\lambda(\lambda-1)(\lambda-2)(\lambda-3+6) + \underbrace{3\lambda - 3 + 1}_{3\lambda-2}$

$\lambda(\lambda^2-3\lambda+2)(\lambda+3) + 3\lambda-2 =$

$= \lambda(\lambda^3 + \lambda^2(3-3) + \lambda(2-9) + 6 + 3\lambda - 2) =$

$= \lambda^4 - 7\lambda^2 + 3\lambda + 4$

Ups

$\lambda^4 + \cancel{8}2\lambda^2 + 1 = (\lambda^2+1)^2 = (\lambda-i)^2(\lambda+i)^2$

$(c_1 \ln x + c_2)x^i + (c_2 \ln(x) + c_3)x^{-i}$

$x^i = e^{i\ln x}$

$e^{i\ln x} = \cos(\ln x) + i\sin(\ln x)$

$\cos(\ln x)(c_1 \ln x + c_2 + c_3 \ln x + c_4) +$
$i\sin(\ln x)(c_1 \ln x + c_2 - c_3 \ln x - c_4) =$

$= \cos(\ln x)(\ln x(c_1+c_3) + c_2 + c_4) +$
$i\sin(\ln x)(\ln x(c_1-c_3) + c_2 - c_4) =$

i nesemo noter v konstante

Dobimo $(a_1 \ln x + a_2)\cos(\ln x) +$
$(a_3 \ln x + a_4)\sin(\ln x)$

$a_n y^{(n)} + \dots a_1 y' + a_0 y = e^{\mu x} p(x)$

če je y_p neka rešitev, je vsaka druga rešitev oblike

$y_h + y_p$

y_p iščemo z nastavkom $e^{\mu x} q(x) x^k$

q... polinom iste stopnje kot p

k... kratnost μ v $P(\lambda)$

karakteristični polinom homogenega dela

Poišč spl. reš.

a) $y'' - 5y' + 4y = 4x^2 e^{2x}$

H: $y'' - 5y' + 4y = 0$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$y_H = C_1 e^{x} + C_2 e^{4x}$$

P: $y_p = e^{2x}(Ax^2 + Bx + C)$

$$y_p' = 2e^{2x}(Ax^2 + Bx + C) + e^{2x}(2Ax + B) =$$

$$= e^{2x}\left(x^2(2A) + x(2B + 2A) + (2C + B)\right)$$

$$y_p'' = 2e^{2x}\left(x^2(2A) + x(2B + 2A) + 2C + B\right) +$$
$$+ e^{2x}(4xA + 2B + 2A)$$

x^2: $4A + 2A + A = 1 \quad \Rightarrow A = -2$

x: $(4A + 2B + 2A + 2B + 2A - B) = -5 \quad \Rightarrow B = 2$

1: $2A + 2B + 2B + 4C - 5 \cdot 2C - 5B + 4C \Rightarrow C = -3$

$$y_s = y_h + y_p = C_1 e^{x} + C_2 e^{4x} + e^{2x}(-2x^2 + 2x - 3)$$

$y'' + 3y' - 4y = e^{-4x} + xe^{-x}$

$H: \lambda^2 + 3\lambda - 4 = 0$

$(\lambda+4)(\lambda-1) = 0$

$y = C_1 e^{-4x} + C_2 e^{x}$

1. $-\|- = e^{-4x}$ $\qquad y_p = y_1 + y_2$

$y_1 = e^{-4x} C x$

ker je -4 ničla polinoma

$y_1' = -4e^{-4x}Cx + e^{-4x}C$

$y_1'' = 16e^{-4x}Cx - 8e^{-4x}C$

$x^2: \quad = 0$

$x: 16C + 3(-4)C - 4 \cdot C = 0$

$1: -8C + 3C \cancel{-4C} = 1$

$C = -\frac{1}{5}$

2 $-\|- = xe^{-x}$

$y_2 = e^{-x}(Ax + B)$

$y_2' = -e^{-x}(Ax+B) + e^{-x}A$

$y_2'' = e^{-x}(Ax+B) - Ae^{-x} - e^{-x}A$

$x: A - 3A - 4A = 1 \Rightarrow A = -\frac{1}{6}$

$1: B - 2A - \dots \Rightarrow B = -\frac{1}{36}$

$y = C_1 e^{-4x} + C_2 e^{x} + \frac{1}{5} x e^{-4x} - \frac{1}{6} e^{-x} (x + \frac{1}{6})$

c) $y'' + 2y' + 2y = e^{-x} \cos x$

H: $y'' + 2y' + 2y = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$D = 4 - 8 = -4$

$\lambda_{1,2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$y_h = e^{-x}(c_1 \cos x + c_2 \sin x)$

P:

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$y'' + 2y' + 2y = \frac{1}{2}\left(e^{x(-1+i)} + e^{x(-1-i)}\right)$

$y_1 = e^{x(i-1)} A x$

$y_1' = (i-1)e^{x(i-1)} A x + A e^{x(i-1)}$

$y_1'' = A(i-1)e^{x(i-1)} + (i-1)e^{x(i-1)} A + A(i-1)^2 e^{x(i-1)} x$

$x: \; \ldots \ldots = 0 \quad \Rightarrow A =$ poljuben

$1: \; \ldots \ldots = \frac{1}{2} \quad \Rightarrow A = \frac{1}{4i}$

$y_2:$

$\ldots\ldots\ldots$

$a_n x^n y^n + \ldots + a_n y = x^\mu p(\ln x)$ Eulerjeva DE

nastavek: $x^\mu g(\ln x)(\ln x)^k$ ← stopnja μ v karakterističnem polinomu

↑ iste stopnje kot p

H: $x^2 \lambda(\lambda-1)x^{\lambda-2} - 4x\lambda x^{\lambda-1} + 6x^\lambda = 0$

$\lambda(\lambda-1) - 4\lambda + 6 = 0$ ← $y = x^\lambda$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-2)(\lambda-3) = 0$

$y_h = C_1 x^2 + C_2 x^3$

P: $y_p = \underset{\swarrow x^\mu}{x}\,\underset{\swarrow g(\ln x)}{(A\ln x + B)}$

$y_p' = A\ln x + B + x\left(\frac{A}{x}\right) = A\ln x + A + B$

$y_p'' = \frac{A}{x}$

$x^2\left(A\frac{1}{x}\right) - 4x(A\ln x + A + B) + 6x(A\ln x + B) = 2x\ln x$

$(A + 6B + 4A + 4B)x + x\ln x(4A + 6A) = 2x\ln x$

$10B + 5A = 0$ $\qquad$ $10A = 2$

$10B = -1$ $\qquad$ $A = \frac{1}{5}$

$B = -\frac{1}{10}$

UPS: $\Rightarrow A = 1$

$B = \frac{3}{2}$

$y_s = y_h + y_p = c_1 x^2 + c_2 x^3 + x\left(\ln x + \frac{3}{2}\right)$

Dokaži da se uz $t = \arcsin x$

$(1-x^2)y'' - xy' - 4y = 0$ prevede na homogenu ^{2DE} s konst. koef.

$z(t) = y(\sin t)$ $\qquad x(t) = \sin t$

$z'(t) = \cos t \, y'(\sin t)$

$z''(t) = \cos^2 t \, y''(\sin t) - \sin(t) \, y'(\sin t)$

$\underbrace{\cos^2 t \, y'' - \sin t \, y'}_{z''(t)} - \underbrace{4y}_{4z(t)} = 0$

$z''(t) - 4z(t) = 0$

$z = e^{\lambda t}$

$\lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$

$z_1 = C_1 e^{2t} = C_1 e^{2\arcsin x} = y_1(x)$

$z_2 = C_2 e^{-2t}$ $\quad C_2 e^{-2\arcsin(x)} = y_2(x)$

Variacija konstante

$$a_0 y^{n} + \dots + a_n y = f$$

$$\Phi(x) = \begin{bmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n)} & & y_n^{(n)} \end{bmatrix}$$

$$\Phi \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \rightsquigarrow y_p = \Phi \begin{bmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{bmatrix}$$

$$y_p' = \Phi' \cdot C + \Phi C'$$

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\frac{a_n}{a_0} & \dots & 0 & -a_1/a_0 \end{bmatrix}$$

$$\varphi = \begin{cases} \varphi_1 = y \\ \varphi_2 = \dot{y} \\ \vdots \\ \varphi_n = y^{(n-1)} \end{cases}$$

$$\dot{\varphi} = A\varphi + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ f \end{bmatrix}}_{= \vec{z}}$$

$$\dot{\Phi} = A\Phi \qquad A\Phi C$$

$$y_p' = \Phi' C + \Phi C' = A\Phi C + \vec{z}$$

Dobimo: $\Phi C' = \vec{z}$

Sklep $C = C(x)$ dobimo iz enakosti

$$\begin{bmatrix} y_1 & \dots & y_n \\ \vdots & & \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ f/a_0 \end{bmatrix}$$

Določi splošno rešitev DE

$y'' + 3y' + 2y = \ln(1+e^x)$

homogeno:

$y'' + 3y' + 2y = 0$

nastavek $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^x$

$\lambda^2 + 3\lambda + 2 = 0 \qquad \lambda_1 = -1$

$(\lambda+1)(\lambda+2) = 0 \qquad \lambda_2 = -2$

$y = C_1 e^{-x} + C_2 e^{-2x}$

$y_1 = e^{-x} \qquad \dot{y}_1 = -e^{-x} \qquad \Phi = \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix}$

$y_2 = e^{-2x} \qquad \dot{y}_2 = -2e^{-2x}$

$\Phi \begin{bmatrix} \dot{C}_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \ln(1+e^x) \end{bmatrix}$

$$\left.\begin{aligned} e^{-x} C_1'(x) + e^{-2x} C_2'(x) &= 0 \\ -e^{-x} C_1'(x) - 2C_2'(x) e^{2x} &= \ln(1+e^x) \end{aligned}\right\} +$$

$-C_2' e^{-2x} = \ln(1+e^x)$

$C_2' = -e^{2x} \ln(1+e^x)$

$C_2 = \int -e^{2x} \ln(1+e^x) dx = \int e^u (e^u - 1) u \, dx$

$e^u \doteq 1 + e^x$

$e^u du = e^x dx$

$C_2 = \int e^{2u} u - e^u u \, du = ue^u - e^u - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u}$

$u = u \quad dv = e^u du$

$du = du \quad v = e^u$

$C_2(x) = (1+e^x)\ln(1+e^x) - (1+e^x) - \frac{\ln(1+e^x)}{2}(1+e^x)^2 + \frac{1}{4}(1+e^x)^2$

$\ln(1+e^x)\left(1+e^x - \frac{(1+e^x)^2}{2}\right) - 1 - e^x + \frac{1}{4}(1+2e^x+e^{2x}) =$

$= \ln(1+e^x)\left(\frac{1}{2}\left(1-e^{2x}\right)\right) - \cancel{\frac{3}{4}} - \frac{1}{2}e^x + \frac{1}{4}e^{2x}$

gre v konstanto C

$C_1' = -C_2' e^{-x} = e^x \ln(1+e^x)$

$u = \ln(1+e^x)$

$C_1 = \int ue^u - e^u = \ln(1+e^x)(1+e^x) - (1-e^x) =$

$(1+e^x)(\ln(1+e^x) - 1)$

$y_s = C_1 y_1 + C_2 y_2 + y_h =$

$(e^{-x}+1)(\ln(1+e^x) - 1) + \frac{1}{4} - \frac{1}{2}e^{-x} + \ln(1+e^x)\left(\frac{e^{-2x}-1}{2}\right)$

$+ a e^{-x} + b e^{-2x}$

Determinanta Wronskega

$y_1, \ldots, y_n \in \mathcal{C}^{n-1}([a,b])$

$$W = \det \begin{bmatrix} y_1 & \cdots & y_n \\ y_1' & & \vdots \\ \vdots & & \vdots \\ y_1^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}$$

V primeru ko sta u, v lin. neodvisn. rešitvi:

$y'' + p(x)y + g(x) = y$

$\rightsquigarrow W = u'v - uv'$

iz Liouvilove formule sledi, da je $W' + p(x)W = 0$

$\rightsquigarrow \ln W = -\int p \, dt$

$W = W_0 e^{-\int p \, dt}$

Podana je DE

$(1+x^2)y'' = 2y$

a) $y_1 = 1+x^2$

rešitev?

b) Določi splošno rešitev

a) $(1+x^2)\cdot 2 = 2(1+x^2)$ ✓

b) Naj bo y_2 lin. neod. od y_1 rešimo enačbo wronskega

$W = y_1 y_2' - y_2 y_1' \neq 0$

$(1+x^2)y'' - 0\cdot y' - \frac{2}{1+x^2}y = 0$

$W' + \overset{0}{p} W = 0$ ker

$W' = 0$

$W = C = 1$ ← izberemo si

$1 = y_1 y_2' - y_2 y_1' = y_2'(x^2+1) - y_2(2x)$

H: $y_2' = y_2 \frac{2x}{x^2+1}$

$\ln y_2 = \int \frac{2x}{x^2+1}dx = \ln(1+x^2) + C$

$y_2 = C(1+x^2)$

P: $C'(1+x^2)^2 = 1 \rightsquigarrow C = \int \frac{1}{(1+x^2)^2}dx$

$$C = A\arctan x + B\ln(1+x^2) + \frac{Dx+E}{1+x^2}$$

$\Downarrow$

$$C' = \frac{A}{1+x^2} + \frac{2Bx}{1+x^2} + \frac{D(1+x^2) - 2x(Dx+E)}{(1+x^2)^2}$$

$$= \frac{A(1+x^2) + 2B(x+x^3) + D(1+x^2) - 2Dx^2 - 2Ex}{(1+x^2)^2}$$

x^3: $2B = 0$

x^2: $A + D - 2D = A - D = 0 \Rightarrow A = D$

x : $2B - 2E = 0 \Rightarrow E = 0$

1: $A + D = 1 \Rightarrow A = D = \frac{1}{2}$

$C = \frac{1}{2}\arctan x + \frac{\frac{1}{2}x}{1+x^2} + D$

$y_p = C(1+x^2) = D(1+x^2) + \frac{1}{2}(1+x^2)\arctan x + \frac{1}{2}x$

$y_2 = (1+x^2)\arctan x + x$

Lahko množimo z 2, ker je itak konstanta spredaj

Naj bo $g \in L_1(\mathbb{R})$ in naj bo $y \in C^2(\mathbb{R})$ rešitev DE

$y'' + g\,y = 0$

a) Dokaži $\lim_{x\to\infty} y'(x) = 0$

b) Dokaži da $\exists$ neomejen rešitev DE

a) $y'(x) = \int_a^x y''(t)\,dt = \int_0^x -g\,y(t)\,dt + y'(0)$

$|y'(x)| \leq \int_a^x |g\,y(t)|\,dt + |y_0'(0)|$ $\qquad |y| \leq M$

$\leq M \int_a^x |g(t)|\,dt + y_a'(0) < \infty$ za

Dokazali bomo da je odvod od nekod naprej monoton $\Rightarrow$ gre proti 0

$\lim_{x\to\infty} y'(x) = \int_0^\infty -g\,y(t)\,dt + y'(0) = 0$ ker je y omejen

b) $\exists$ lin. neodvisna rešitev z

$w' + p\,w = 0 \Rightarrow w' = 0$
$\Rightarrow w'' = C$
Recimo 1

$\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} = yz' - zy' \neq 0$

$z(\infty) = y(\infty)z' - z(\infty)\underbrace{y'(\infty)}_{0}$

$y(\infty)\,z'(\infty) = 1$

$z'(\infty) \neq 0 \Rightarrow z$ neomejena

$$(1+x^2)y''+xy'-n^2y=0$$

a) $y_1=(x+\sqrt{1+x^2})^n$ je reš. ku

$$y_1'=\left(1+\frac{x}{\sqrt{1+x^2}}\right)n\left(x+\sqrt{1+x^2}\right)^{n-1}$$

$$y_1''=\left(\frac{1-\frac{x}{\sqrt{1+x^2}}}{1+x^2}\right)n\left(x+\sqrt{1+x^2}\right)^{n-1}+n\left(1+\frac{x}{\sqrt{1+x^2}}\right)(n-1)\left(1+\frac{x}{\sqrt{1+x^2}}\right)^{n-2}$$

za $n\geq 2$

za $n=1$: $y_1'=1+\frac{x}{\sqrt{1+x^2}}$ $\quad y_1''=\frac{1-\frac{x}{\sqrt{1+x^2}}}{1+x^2}$

$(1+x^2)$

$$y_1''=\left(1+\frac{x}{\sqrt{1-x^2}}\right)^{n-2}\left(n\frac{-\cancel{x}+\sqrt{1+x^2}}{\sqrt{1+x^2}}\cdot(x+\sqrt{1+x^2})+n(n-1)\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right.$$

$$=\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)^{n-2}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)\left(n(\sqrt{1+x^2}-x)+n(n-1)\right)$$

$$=n\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)^{n-1}\left(\sqrt{1+x^2}-x+n-1\right)$$

$$xy_1'=\frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}nx$$

$$n^2y_1=n^2(x+\sqrt{1+x^2})^n$$

$$(\sqrt{1+x^2}+x)^n\left(n(n-1)+\frac{n}{\sqrt{1+x^2}}+\frac{nx}{\sqrt{1+x^2}}+n^2\right)$$

$$=y_1\left(\frac{n(x+1)}{\sqrt{1+x^2}}+2n^2-n\right)$$

nekje je napaka

$$y_1'' = n(n-1)\frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\left(\frac{1}{(1+x^2)}\right) n\,(x+\sqrt{1+x^2})^{n-1}$$

$$= \frac{(x+\sqrt{1+x^2})^{n-1}}{\sqrt{1+x^2}}\left(n(n-1)(x+\sqrt{1+x^2}) + \frac{n}{1+x^2}\right) \quad \ldots$$

...

b) določi splošno rešitev

$$w = \begin{vmatrix} y_1 & y \\ y_1' & y' \end{vmatrix}$$

$$w' + p\,w = 0$$

$$p = \frac{x}{1+x^2}$$

$$\frac{dw}{w} = -\frac{x}{1+x^2}\,dx$$

$$\ln w = -\frac{1}{2}\ln(1+x^2) + c$$

$$y_1 y' - y y_1' = \frac{c}{\sqrt{1+x^2}} \qquad w = \frac{c}{(1+x^2)^{+\frac{1}{2}}} = \frac{c}{\sqrt{1+x^2}}$$

$$y' - y\frac{y_1'}{y_1} = \frac{c}{(1+x^2)^{\frac{3}{2}}}$$

$$y_1' = (x+\sqrt{1+x^2})^n\, n\frac{1}{\sqrt{1+x^2}}$$

$$y' - y\, n\frac{1}{\sqrt{1+x^2}} = \frac{1}{(1+x^2)^{\frac{3}{2}}} \qquad c = 1$$

Homogeno:

$$y' = y_1$$

vr. var konst:

$$y_2 = C(x+\sqrt{1+x^2})^n \cancel{\frac{1}{(1+x^2)^{\frac{3}{2}}}}$$

$$y_2' = C'(x)y_1 + C(x)y_1'$$

$$C'(x)y_1 + C(x)y_1' - \underbrace{C(x)\,y_1\, n\frac{1}{\sqrt{1+x^2}}}_{C(x)y_1'} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$C'(x)\,y_1^2 = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$C'(x) = \frac{1}{(1+x^2)^{\frac{3}{2}}(x+\sqrt{1+x^2})^{2n}}$$

$x = \operatorname{sh}\varphi$
$dx = \operatorname{ch}\varphi\, d\varphi$

$$C = \int \frac{\operatorname{ch} d\varphi}{(\operatorname{ch}^2)^{\frac{3}{2}}(\operatorname{sh}\varphi + \operatorname{ch}\varphi)^{2n}} =$$

$$= \cancel{(\operatorname{ch}+\operatorname{sh})^{2n}} \ldots = \int(\operatorname{sh} t - \operatorname{ch} t)^{2n}\,dt$$

$$= \int \frac{e^t - e^{-t} - e^t - e^{-t}}{2}\,dt = \int (e^{-t})^{2n} dt = -e^t =$$

$$-\frac{1}{2n}e^{-2nt} = -\frac{1}{2n}e^{-2n\operatorname{arcsh} x} =$$

$$= -\frac{1}{2n}(x - \sqrt{1+x^2})^{-2n}$$

$$y_2 = -\frac{1}{2n}(x-\sqrt{1+x^2})^{2n} y_1 = -\frac{1}{2n}(x-\sqrt{1+x^2})^{2n}(x+\sqrt{1+x^2})^n =$$

$$= -\frac{1}{2n}(-1)^n(x-\sqrt{1+x^2}) = \frac{(-1)^{n+1}}{2}(x-\sqrt{1+x^2})^n$$

$$y_2 = (x-\sqrt{1+x^2})^n$$

$y'' + py' + qy = 0$ ena rešitev je y_1

Dokaži splošno formulo za y_2 $w' + pw = 0$

$$y_1 y_2' - y_1' y_2 = w$$

$$C'(x) y_1^2 = w$$

$$C(x) = \int \frac{w}{y_1^2} \qquad y_2 = \int \frac{w}{y_1^2} dx \cdot y_1$$

b)

$$zy = y'$$

$$y'' = z'y + y'z$$

$$z'y + \underset{\substack{\parallel \\ yz^2}}{y'z} + pzy + qy = 0$$

$$z'y + yz^2 + pzy + qy = 0$$

$$z' + z^2 + z = -q \qquad \text{Riccatijeva enačba}$$

c) nastavek: $z = u + \frac{1}{v}$ $u = \frac{y_1'}{y_1}$

$$z' = u' - \frac{1}{v^2} v'$$

$$u' - \frac{1}{v^2} v' + u^2 + 2\frac{u}{v} + \frac{1}{v^2} + p\left(u + \frac{1}{v}\right) - q = 0$$

$$\underbrace{u' + pu + q}_{0} + \left(-\frac{v'}{v^2}\right) + 2\frac{u}{v} + \frac{\cancel{p}}{v^2} + \frac{\cancel{1}}{v} = 0$$

$$-v' + 2uv + v + 1 = 0$$

$$-v^2 + v(-2u - p) - 1 = 0$$

$$\frac{v'}{v} = 2u + p \qquad \ln v = \int 2u + p\, dt \qquad v = e^{\int 2u + p\, dt}$$

$$w' + wp = 0 \Rightarrow w = e^{-\int p\, dt}$$

$$v = \frac{e^{\int 2u\, dt}}{w}$$

$$e^{\int 2u\, dt} = e^{2\int \frac{y_1'}{y_1} dt} = y_1^2 \qquad v = \frac{y_1^2}{w} = \frac{1}{C'(x)}$$

$$D'(x) \frac{y_1^2}{w} = 1 \Rightarrow D = \int \frac{w}{y_1^2} = C(x)$$

$$v(x) = \int \frac{w}{y_1} dt \, \frac{y_1^2}{w} + c\, \frac{y_1^2}{w}$$

$$z' = \frac{y_1'}{y_1} + \frac{w}{y_1^2} \frac{1}{\int \frac{w}{y_1} dt} = \frac{y_1'}{y_1} + \frac{w}{y_1 y_2} = \frac{y_1' y_2 + y_1 y_2' - y_1' y_2}{y_1 y_2}$$

$$= \frac{y_2'}{y_2}$$

DN

Splošna rikatijeva $z' + az^2 + bz + c = 0$

$ayz = y'$ .... LDE ~~prvega~~ reda
drugega

$y'' + x^3 y = \sin x$

$y(0) = 1 \qquad y'(0) = 0$

za x na okolici 0

$y(x) = \sum c_k x^k \qquad \begin{matrix} c_0 = 1 \\ c_1 = 0 \end{matrix}$

$$\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} + \sum_{k=0}^{\infty} c_k x^{k+3} = \sum_{x=0} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

$$2c_2 + 6c_3 x + 12 c_4 x^2 + \sum_{k=3}^{\infty} (k+2)(k+1)(c_{k+2}, c_{k-3}) x^k$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{2k+1)}$$

$c_2 = 0$

$c_3 = \frac{1}{6}$

$c_4 = 0$

k lih:

$(k+2)(k+1)(c_{k+2} + c_{k-3}) = \pm 1 \; \frac{1}{k!}$

$k = 2l+1$:

$(2l+3)(2l+2)(c_{2l+3} + c_{2l-3}) = (-1)^{l} \frac{1}{(2l+1)!}$

$(2l+2)(2l+1) c_{2l+2} + 2_{2l+3} = 0$

$$c_{2l+2} = -\frac{c_{2l-3}}{(2l+3)(2l+2)}$$

$$c_{2l+2} = \frac{-c_{2l+3}}{(2l+2)(2l+1)}$$

Variacijski račun

$L: [a,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \quad \in \mathcal{C}^2$

$\mathcal{A} = \{ y \in \mathcal{C}^2([a,b]) \, ; \, y(a) = A, \, y(b) = B \}$

$\mathcal{L} : \mathcal{A} \longrightarrow \mathbb{R}$

$\mathcal{L}(y) = \int_a^b L(x, y(x), y'(x)) \, dx$

Iščemo kritične točke $\mathcal{L}$

$\hookrightarrow$ Če je $y_0 \in \mathcal{A}$ min/max $\mathcal{L}$, potem mora veljati $\frac{\partial}{\partial t}\Big|_{t=0} \mathcal{L}(y_0 + th) = 0$ (za poljubno dopustno variacijo h)

$\rightsquigarrow t \longmapsto \mathcal{L}(y_0 + th)$ je običajna funkcija ene realne sprem. t

Dopustna variacija h: $y_0 + th \in \mathcal{A}$ vsaj za $t \in (-\varepsilon, \varepsilon)$

$\rightsquigarrow h \in \mathcal{C}^2([a,b])$

$h(a) = h(b) = 0$, da bo $(y_0 + th)(a) = A$

$(y_0 + th)(b) = B$

$\forall t \in (-\varepsilon, \varepsilon)$

$h \in \mathcal{X}$

$\rightsquigarrow$ prostor dopustnih variacij

Na dnu je pomembna formula

Naj bo $L = L(x, t, s)$, kjer $t = y(x)$, $s = y'(x)$

Računamo $\frac{\partial}{\partial t}\Big|_{t=0} \int_a^b L(x, y+th, y'+th')dx = 0$

$$= \int_a^b \frac{\partial}{\partial t}\Big|_{t=0} L(\dots)dx = \int_a^b \Big(h \underbrace{L_t(x, y_0, y_0')}_{=\frac{\partial L}{\partial t}} + h' \underbrace{L_s(x, y_0, y_0')}_{=\frac{\partial L}{\partial s}}\Big)dx$$

$$=: \int_a^b (L_y \cdot h + L_{y'} h')\,dx = \int_a^b L_y h\,dx + \int_a^b L_{y'} h'\,dx =$$

$u = L_{y'}$ $\quad v = h$

$du = \frac{\partial}{\partial x} L_{y'} dx$ $\quad dv = h' dx$

$\frac{\partial}{\partial x} L_{y'}(x, y_0, y_0')$

$$\int L_{y'} h' dx = L_{y'} h \Big|_a^b - \int_a^b h L_{y'} dx$$

$$D\mathcal{L}_{y_0}(h) = L_{y'} h \Big|_a^b + \int_a^b \Big(L_y - \frac{\partial}{\partial x} L_{y'}\Big) h\,dx = 0$$

$$L_{y'}(b, y_0(b), y_0'(b)) \underbrace{h(b)}_{=0} - \dots\dots \underbrace{h(a)}_{=0}$$

$$\int_a^b \Big(L_y - \frac{\partial}{\partial x} L_{y'}\Big) h\,dx = 0 \quad \text{za } \forall h \in \mathcal{H}$$

osnovna lema o variacijskem računu $\Longrightarrow$ $L_y - \frac{\partial}{\partial x} L_{y'} \equiv 0$

to je Euler-Lagrangeev pogoj

Uredimo oznake:

$$EL = L_y - \frac{\partial}{\partial x} L_{y'}$$

$$L_y(x, y(x), y'(x)) - \frac{\partial}{\partial x} L_{y'}(x, y(x), y'(x)) = 0$$

$\hookrightarrow$ v splošnem je to DE II. reda

① Določi kritične točke

a) $\mathcal{L}(y) = \int_0^1 (y'^2 + yy' + 12xy)\,dx \qquad y(0)=y(1)=0$

b) $\mathcal{L}(y) = \int_0^1 (e^y + xy')\,dx \qquad y(0)=0 \quad y(1)=a \in \mathbb{R}$

c) $\int_{-1}^{1} \sqrt{1+x^2+y'^2}\,dx \qquad y(-1)=0 \quad y(1)=1$

a) $L = y'^2 + yy' + 12xy$

Delamo se da so y, y', x neodvisne, samo oznake

$L_y = \cancel{} + y' + 12x$

$L_{y'} = 2y' + y$

tu upoštevamo da je $y(x)$

$L_y - \frac{\partial}{\partial x} L_{y'} = 0$

$y' + 12x - 2y'' - y' = 0$

$y'' = 6x$

$y' = 3x^2 + D$

$y = x^3 + Dx + C$

$y(0) = C = 0$

$y(1) = 1 + D = 0 \Rightarrow D = -1$

$y = x^3 - x$

b) $\mathcal{L}(y) = \int_0^1 (e^y + xy')\,dx \quad y(0)=0 \quad y(1) = a \in \mathbb{R}$

$L = e^y + xy'$

$L_y - \frac{\partial}{\partial x} L_{y'} = 0$

$L_y = e^y$

$L_{y'} = x$

$e^y - 1 = 0$

$y = \ln 1 = 0$

$y \equiv 0$

$y(1) = a = 0$

$a = 0 \Rightarrow$ potem ime $\mathcal{L}$ kritično točko, drugače pa nima lokalnih ekstremov

c) $\int_{-1}^{1} \sqrt{1+x^2+y'^2}\,dx \qquad y(-1)=0 \quad y(1)=1$

$L = \sqrt{1+x^2+y'^2}$

$L_y = 0$

$L_{y'} = \dfrac{y'}{\sqrt{1+x^2+y'^2}}$

$y' \cdot y'$

$\parallel$

$x + y'^2$

$L_y - \frac{\partial}{\partial x} L_{y'} = -\dfrac{y''(\sqrt{\dots}) - \frac{\dots}{\sqrt{\dots}}}{1+x^2+y^2} =$

$= \dfrac{y''(1+x^2+y^2) - x + y'^2}{(1+x^2+y^2)} = 0$

$\Leftrightarrow y''(1+x^2+y'^2) - x + y'^2 = 0$

$-\frac{\partial}{\partial x} L_{y'} = 0 \Rightarrow L_{y'} = C$

$\dfrac{y'}{\sqrt{1+x^2+y'^2}} = C$

$y'^2 = C(1+x^2+y'^2)$

$y'^2(1-C) = (1+x^2)C$

$y' = \sqrt{\dfrac{(1+x^2)C}{1-C}}$

$y = \int \sqrt{\dfrac{C}{1-C}} \underbrace{\sqrt{1+x^2}\,dx}_{\operatorname{ch} x\,dx} = \operatorname{sh} x \sqrt{\dfrac{C}{1-C}} + D$

D

$x = \operatorname{sh} t$

$dx = \operatorname{ch} t$

$y = D \int \operatorname{ch}^2 t\,dt = \dfrac{D}{2}\left(\frac{1}{2}\operatorname{sh} 2t + t\right) + E =$

$\boxed{\operatorname{ch}^2 t = \dfrac{\operatorname{ch} 2t + 1}{2}}$

$= \dfrac{D}{2}\left(x\sqrt{1+x^2} + \ln|x + \sqrt{1+x^2}|\right) + E$

$y(-1) = \dfrac{D}{2}\left(-\sqrt{2} + \ln|-1+\sqrt{2}|\right) + E = 0$

$y(1) = \dfrac{D}{2}\left(\sqrt{2} + \ln|1+\sqrt{2}|\right) + E = 1$

$\}+$

$\dfrac{D}{2} \ln(2-1) + 2E = 1 \quad \rightsquigarrow E = \frac{1}{2}$

$D(-\sqrt{2} + \ln|\sqrt{2}-1| = -1$

$D = \dfrac{-1}{-\sqrt{2} + \ln|\sqrt{2}-1|}$

$\dfrac{D}{2}\left(2\sqrt{2} + \ln\left|\dfrac{1+\sqrt{2}}{1-\sqrt{2}}\right|\right) = 1$

$\dfrac{D}{2}\left(2\sqrt{2} + \ln\dfrac{(1+\sqrt{2})^2}{1}\right) = 1$

$D = \dfrac{1}{\sqrt{2} + 2\ln(1+\sqrt{2})}$

Naj bo $L = L(y, y')$ neodvisen od x

A) Dokaži da je $y \not\equiv \tilde{C}$ kritična točka funkcionala

$$\mathcal{L}(y) = \int_a^b L(y, y')\,dx \quad y(a) = A \quad y(b) = B$$

$$\Longleftrightarrow$$

$$L - y' L_{y'} \equiv C \in \mathbb{R}$$ Beltramijeva identiteta

b) Določi kritične točke $\int_{-a}^{a} y\sqrt{1-y'^2}\,dx \quad y(-a) = y(a) = A$

$(\Leftarrow)$
$$L_y - \frac{\partial}{\partial x} L_{y'} \equiv 0$$

$$L - y' L_{y'} \equiv C \quad \Big/ \frac{\partial}{\partial x} \qquad L(x, y(x), y'(x)) = L(y(x), y'(x))$$

$$L_y \cdot y' + L_{y'} y'' - y'' L_{y'} - y' \frac{\partial}{\partial x} L_{y'} = 0$$

$$\underset{\neq 0}{y'}\left(L_y - \frac{\partial}{\partial x} L_{y'}\right) = 0 \Leftrightarrow L_y - \frac{\partial}{\partial x} L_{y'} = 0$$

b)
$$y\sqrt{1-y'^2} - y' y \frac{-y'}{\sqrt{1-y'^2}} \equiv C$$

$$y(1-y'^2) + y'^2 y = C\sqrt{1+y'^2}$$

$$y = C\sqrt{1-y'^2}$$

$$y^2 = (1+y'^2)C$$

$$y'^2 = 1 - \frac{y^2}{C}$$

$$y' = \sqrt{1 - \frac{y^2}{C}} \quad \cdots\cdots ! \quad y' = \sqrt{c^2y^2 - 1}$$

$$y = \int \frac{dx}{\sqrt{c^2y^2 - 1}\,dx} = \int dx = x + D$$

$$cy = \operatorname{sh} t$$

$$\operatorname{sh} t\,dt = cy'\,dx$$

$$= \frac{1}{c}\int dt = \frac{1}{c} t = \frac{1}{c}\operatorname{arch}(cy) = x + D$$

$$t = cx$$

$$y = \frac{1}{c}\operatorname{ch}(cx + \tilde{D})$$

Dolčeni drugi smerni odvod funkcionala

$$\mathcal{L}(y) = \int_a^b L(x, y, y')\,dx$$

$$Q\mathcal{L}_y(h) = \frac{\partial^2}{\partial t^2}\Big|_{t=0} \mathcal{L}(y+th) = \frac{\partial^2}{\partial t^2}\Big|_{t=0} \int_a^b L(x, y+th, y'+th')\,dx =$$

$$= \frac{d}{dt}\Big|_{t=0} \int_a^b h \cdot L_y(x, y+th, y'+th') + h' L_{y'}(x, y+th, y'+th')\,dx =$$

$$= \int_a^b h(hL_{yy}(x, y+th, y'+th') + h' \quad \ldots\ldots \Big|_{t=0} =$$

$$= \int_a^b h^2 L_{yy}(x,y,y') + \underbrace{2hh'}_{=(h^2)'} L_{yy'}(\ldots) + h'^2 L_{y'y'}(\ldots)\,dx$$

per partes

$$\boxed{\int_a^b h^2\left(L_{yy} - \frac{d}{dx}L_{yy'}\right) + h'^2 L_{y'y'}\,dx}$$

y_0 kritične in $Q\mathcal{L}_{y_0}(h) > 0$ ((minimum)) ali

$Q(\mathcal{L}_y(h) < 0$ ((maksimum))

$\Rightarrow$ y_0 je minimum/maksimum

V nekaj
nalog z
drugimi odvodi

$$\mathcal{A} = \{x \longmapsto (\underbrace{y(x), z(x)}^{\vec{y}}) \,;\, \vec{y}(a) = \vec{A}\,,\ \vec{y}(b) = \vec{B}\} \subset \mathcal{C}^2([a,b], \mathbb{R}^2)$$

$$\mathcal{L}(y,z) = \int_a^b L(y, y, z, y', z')\,dx$$

Dopustne variacije (h, k)

$h(a) = 0$

$h(b) = 0$

$k(a) = k(b) = 0$

EL pogoj je v tem kontekstu

- $L_y - \frac{\partial}{\partial x} L_{y'} = 0$
- $L_z - \frac{\partial}{\partial x} L_{z'} = 0$

Poišči kritične točke

$$\mathcal{L}(y,z) = \int_0^{\frac{\pi}{2}} (y'^2 + z'^2 - 2yz)\,dx = 0$$

$y(0) = 0$
$y(\frac{\pi}{2}) = 1$
$z(0) = 0$
$z(\frac{\pi}{2}) = 1$

$L_y = -2z$
$L_{y'} = 2y'$
$L_z = -2y$
$L_{z'} = 2z'$

$$-2z - \frac{\partial}{\partial x} 2y' = 0$$
$$-2y - \frac{\partial}{\partial x} 2z' = 0$$

$$-2z - 2y'' = 0$$
$$-2y - 2z'' = 0$$

$z + y'' = 0$
$y + z'' = 0$

$z = -y''$
$z'' = -y''''$

$y'''' = y$

$\lambda^4 - 1 = 0 \quad (\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i)$

$\rightsquigarrow y = Ae^x + Be^{-x} + C\cos x + D\sin x$

$y' = Ae^x - Be^{-x} - C\sin x + D\cos x$

$z = -y'' = -Ae^x - Be^{-x} + C\cos x + D\sin x$

$y(0) = A + B + C = 0$
$y(\frac{\pi}{2}) = Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} + D = 1$

$z(0) = -A - B + C = 0$
$z(\frac{\pi}{2}) = -Ae^{\frac{\pi}{2}} - Be^{-\frac{\pi}{2}} + D = 1$

$\Rightarrow C = 0$
$A + B = 0 \rightsquigarrow A = -B$

$2D = 1 \Rightarrow D = \frac{1}{2}$

$A(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}) = \frac{1}{2}$

$A = \frac{1}{2} \frac{1}{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}$

??

og dano:

$A = B = C = 0 \quad D = 1$

$z(x) = \sin x$
$y(x) = \sin x$