

$$x = 0, 1$$

Pokaži da velja \*

$$x = \sum_{i=1}^{\infty} (2^{-k_i} + 2^{-4_i - 1})$$

b) linearni zapis za  $x$

c) zapis v IEEE formu

IEEE754

enojna natančnost  $P(2, 24, -125, 128)$

$$(-1)^{\sigma} (1+m) 2^{\tilde{e}-127}$$

$m$  dolžine 23

$\tilde{e}$  dolžine 8

$\sigma$  dolžine 1

dvajna natančnost  $P(2, 53, -1021, 1024)$

$$(-1)^{\sigma} (1+m) 2^{\tilde{e}-1023}$$

$m$  52

$\tilde{e}$  11

$\sigma$  1

$$a) X = \sum_{i=1}^{\infty} 2^{-4i} = \frac{\frac{1}{16}}{1 - \frac{1}{16}} + \frac{1}{2} \frac{\frac{1}{16}}{1 - \frac{1}{16}} = \frac{2}{2 \cdot 16} \frac{16}{16-1} = \frac{1}{15} = 0,1$$

$$b) 0,0001100110011 = 0,0\overline{0011}_{(2)}$$

$$c) \begin{aligned} &1,1\overline{0011} \cdot 2^{-4} = \\ &1 + 0,1\overline{0011} \cdot 2^{-4} \end{aligned}$$

$$0,100110\dots\dots 001101$$

$$\tilde{e} - 127 = -4 \Rightarrow e = 123 = 1111011$$

$$\begin{array}{rcl} 123 : 2 & = & 61 \quad 1 \\ 61 : 2 & = & 30 \quad 1 \\ 30 : 2 & = & 15 \quad 0 \\ 15 : 2 & = & 7 \quad 1 \\ 7 : 2 & = & 3 \quad 1 \\ 3 : 2 & = & 1 \quad 1 \\ 1 : 2 & = & 0 \quad 1 \end{array}$$

$$x = 2^{-1} + 2^{-k} + 2^{-t}$$

$$y = 2^{-1} + 2^{-k}$$

$$k = \frac{t}{2} + 1$$

$$t = 2k - 2$$

z obravnave relativne napake pdežile da  
izračunari direktno stabilen

$x^2 + y^2$  izračunamo z  
izrazom

$$x \cdot x - y \cdot y$$

$$x^2 = (2^{-1} + 2^{-k} + 2^{-t})^2 = 2^{-2} + 2^{-2k} + 2^{-2t} + 2 \cdot 2^{-1-k} + 2 \cdot 2^{-k-t} + 2 \cdot 2^{-t-1} =$$

$$\cancel{2^{-2}} + \cancel{2^{-2k}} + 2^{-4k+4} + \cancel{2^{-k}} + 2^{-k-t+1} + 2^{-t}$$

$$y^2 = (2^{-1} + 2^{-k})^2 = 2^{-2} + \cancel{2^{-2k}} + \cancel{2^{-k}} + 2^{-2k}$$

$$x^2 - y^2 = 2^{-4k+4} + 2^{-k-t+1} + 2^{-t}$$

$$= 2^{-2t} + 2^{-t} + 2^{-k-t+1}$$

$$fl(x) = 0, \overset{-2}{0}, \overset{-k}{0}, \dots, \overset{-t}{1}, \dots, \overset{-2k}{1}, \overset{-t-k+1}{0}, \dots, 1, \dots =$$

$$= 0, 0, 1, \dots, 1, \dots, 1, 1$$

$$fl(y) = 0, \overset{-k}{0}, \overset{-t}{0}, \dots, \overset{-k}{1}, \dots, \overset{-t}{0}, \overset{-k}{0}, \overset{-t}{1} = 0, 0, 1, \dots, 1, \dots, 0, 1, 0$$

niveč denar zad:

zato izberemo številce  
soda zadnja številca, ker je  
1 in megla  $2^{-k}$  zadnja  
v ~~zadnji~~ zbiru

Y

22.10

$$g(x) = -x^2 + 8x - 12 \quad x_{r+1} = g(x_r)$$

$$\lim_{r \rightarrow \infty} x_r = 4 \quad \forall x_0 \in (3, 5)$$

 $\Leftrightarrow$ 

$$\lim_{r \rightarrow \infty} |x_r - 4| = 0$$

$$= |x_{r+1} - 4|$$

$$\lim_{r \rightarrow \infty} |x_{r+1} - 4| = \lim_{r \rightarrow \infty} |-x_r^2 + 8x_r - 16| =$$

$$= \lim_{r \rightarrow \infty} |x_r - 4|^2 = \lim_{r \rightarrow \infty} |x_{r-1} - 4|^4 = \dots = \lim_{r \rightarrow \infty} |x_0 - 4|^{2^{r+1}}$$

konvergenz  $\forall |x_0 - 4| < 1 \Rightarrow x_0 \in (3, 5)$   
 p.c.t.i o

## Red konvergence

$\alpha \dots$  negibna točka

$$g'(x) = g''(x) = \dots = g^{(r-1)}(x) = 0 \quad g^{(r)} \neq 0$$

potem je **red konvergence** enak  $p$

- a) pokažite da lahko  $\sqrt{\text{pozitivnega števila } a}$   
izračunamo z iteracijo  $x_{r+1} = x_r \cdot \frac{x_r^2 + 3a}{3x_r^2 + a}$
- b) delošte red konvergence
- c) dokažite da iteracija konvergira k  $\sqrt{a}$  za  $\forall$   
začetni približek  $x_0 > 0$

1. Preverimo če je  $\sqrt{a}$  negibna točka iteracijske funkcije

$$\sqrt{a} \cdot \frac{\sqrt{a}^2 + 3a}{3\sqrt{a}^2 + a} = \sqrt{a} \cdot \frac{4a}{4a} = \sqrt{a}$$

1. negibnost  
2. privlačnost

2. preverjamo privlačnost  $g'(\sqrt{a}) < 1$

$$g(x) = \frac{x^3 + 3ax}{3x^2 + a}$$

$$g'(x) = \frac{(3x^2 + 3a)(3x^2 + a) - 6x(x^3 + 3ax)}{(3x^2 + a)^2} =$$

$$g'(\sqrt{a}) = \frac{(3a + 3a)(3a + a) - 6a^2 - 18aa}{(3a + a)^2} =$$

$$= 0 < 1$$

$$b) g''(x) = \left( \frac{3 \cancel{(3x^2 + a)} (3x^2 + a) - 6x^2 (x^2 + 3a)}{(3x^2 + a)^2} \right)' =$$

$$= 3 \frac{(2x + a)(3x^2 + a) + 6x(x^2 + a) - 4x(x^2 + 3a) - 4x^3(3x^2 + a)}{(3x^2 + a)^3}$$

$$= -3 \frac{(6x^3 + 3ax^2 + 2ax + a + 6x^3 + 6a - 4x^3 + 12xa - 4x^3)(3x^2 + a)}{(3x^2 + a)^3}$$

$$g'' = \frac{48xa(x^2 + a)}{(3x^2 + a)^3}$$

$$g''' = h_1' h_2 + h_1 h_2' = h_1'(x^2 + a) + h_1(x) 2x$$

$$g'''(\sqrt{a}) = h_1(\sqrt{a}) \cdot 0 + h_1(\sqrt{a}) 2\sqrt{a}$$

$$h_1 \sqrt{a} = \frac{48a\sqrt{a}}{3a + a}$$

$$g''' \sqrt{a} = \frac{48 \cdot 2a^2}{(4a)^3} = \frac{3}{2a} > 0 \neq 0$$

$\Rightarrow$  red konvergence  $= 3$

$$c) \quad g(x) = x \frac{x^2 + 3a}{3x^2 + a}$$

lootmo primeru ko  $x_0 < \sqrt{a}$

$$x_1 > x_0 ?$$

$$x_1 > \sqrt{a} \text{ ali } x_1 < \sqrt{a} ?$$

$$x_1 = \cancel{x_0} \frac{x_0^2 + 3a}{3x_0^2 + a} > \cancel{x_0}$$

$$x_0^2 + 3a > 3x_0^2 + a$$

$$2a > 2x_0^2 \quad \checkmark$$

$$\sqrt{a} > x_0 \quad \checkmark$$

$$\underline{\underline{x_1 < \sqrt{a}}}$$

$$x_0(x_0^2 + 3a) < \sqrt{a}(3x_0^2 + a)$$

$\Leftrightarrow$

$$\sqrt{a}(x_0^2 + 3a) < \sqrt{a}(3x_0^2 + a)$$

$$2a < 2x_0^2$$

$$\text{Sled: } x_1 < \sqrt{a}$$

tonaj  $x_i$  narašča in so omejen  
konvergenca nelemu

konvergenca k  $\sqrt{a}$  ?

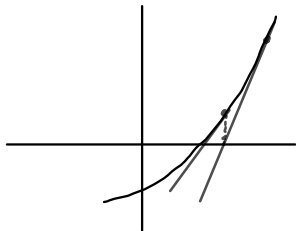
$$x_0 \in (\sqrt{a}, \infty)$$

pokužimo da imamo padejoče  
nevezdno omejeno zaporedje

## Tangentna metoda

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$x_{r+1} = g(x_r)$$



Babilonska metoda za računanje  $\sqrt{a}$   $a > 0$   
temelji na iteraciji

$$x_{r+1} = \frac{1}{2} \left( x_r + \frac{a}{x_r} \right)$$

a) Proverite, da iteracija odgovara tangentnoj metodi  
za funkciju  $f(x) = x^2 - a$

b) dobijte red konvergence

c) Dokazite da iteracija konvergira k  $\sqrt{a}$  za  $\forall x_0 > 0$

$$\begin{aligned} \text{a)} \quad g(x) &= \frac{1}{2} \left( x + \frac{a}{x} \right) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - a}{2x} = \\ &= \frac{2x^2 - x^2 + a}{2x} = \frac{1}{2} \left( x + \frac{a}{x} \right) \quad \checkmark \end{aligned}$$

$$\text{b)} \quad g'(x) = \frac{1}{2} \left( 1 - \frac{a}{x^2} \right) \rightarrow g'(\sqrt{a}) = 0$$

$$g''(x) = \frac{1}{2} \left( \frac{a}{x^3} \right) \rightarrow \frac{1}{2\sqrt{a}^3} > 0$$

red konvergence je 2



$$c) \quad g(x) = \frac{1}{2} \left( x + \frac{a}{x} \right)$$

gnera-sejoo ca  $x \in (\sqrt{a}, \infty)$

$$\sqrt{a} < x_{r+1} < x_r$$

se ca

$$\sqrt{a} < \frac{1}{2} \left( x_r + \frac{a}{x_r} \right) < x_r$$

$$2\sqrt{a}x_r < x_r^2 + a < 2x_r^2$$

$$\underbrace{\hspace{10em}}_{a < x_r^2} \checkmark$$

$$x_r^2 - 2\sqrt{a}x_r + a > 0$$

$$(x_r - \sqrt{a})^2 > 0$$

pedejo-se navedol amejeno zepvalje

$$x_0 \in (0, \sqrt{a}) \Rightarrow x_1 \in (\sqrt{a}, \infty)$$

$$x_1 = \frac{1}{2} \left( x_0 + \frac{a}{x_0} \right) \sqrt{a}$$

$$x_0^2 + a > 2x_0\sqrt{a}$$

$$x_0^2 - 2x_0\sqrt{a} + a > 0$$

$$(x_0 - \sqrt{a})^2 > 0 \quad \checkmark$$

Naj bo  $f \in \mathcal{C}^2$  a njena enostavna ničla

a) Dokazite, da metoda

$$x_{r+1} = x_r - \frac{2f(x_r) \cdot f'(x_r)}{2f'(x_r)^2 - f(x_r)f''(x_r)} \text{ ustreza tangentni}$$

metodi za funkcijo  $F(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$

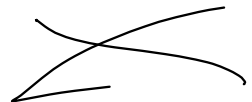
b) poenostavite metodo za  $f(x) = x^2 - a$

$$a) F'(x) = \frac{f'(x)\sqrt{|f'(x)|} - \sqrt{|f'(x)|}}{|f'(x)|} =$$

$$\sqrt{|f'(x)|} = \frac{f''(x)}{2\sqrt{|f'(x)|}} \frac{f'(x)}{|f'(x)|}$$

$$= \frac{\frac{f'(x)}{\sqrt{|f'(x)|}} \left(1 - \frac{f''(x)}{2|f'(x)|}\right)}{|f'(x)|} = \frac{\operatorname{sgn} f \left(1 - \frac{1}{2}\operatorname{sgn} f\right)}{\sqrt{|f'(x)|}} =$$

$$= \frac{\operatorname{sgn} f - \frac{1}{2}}{\sqrt{|f'(x)|}}$$

$$\frac{F(x)}{F'(x)} = \frac{f(x)}{\sqrt{|f'(x)|} \left(\operatorname{sgn} f - \frac{1}{2}\right)} = \frac{f(x)}{\operatorname{sgn} f - \frac{1}{2}}$$


1.1

$$P(2, 3, -1, 3) \pm m.b^e \quad L \leq e \leq U$$

$$P(b, t, L, U)$$

base		e: -1	1	2	3
0					
0,100		0,01	1	10	100
0,101		0,0101	1,01	10,1	101
0,110		0,011	1,1	11	110
0,111		0,0111	1,11	11,1	111

1.2

$$P(2, 9, -10, -10)$$

$$X = 47, 712$$

$$\begin{aligned} 47:2 &= 23 \text{ in } 1 \\ 23:2 &= 11 \text{ in } 1 \\ 11:2 &= 5 \text{ in } 1 \\ 5:2 &= 2 \text{ in } 1 \\ 2:2 &= 1 \text{ in } 0 \\ 1:2 &= 1 \text{ in } 1 \end{aligned}$$

$$1 + 2 + 2^2 + 2^3 + 2^5 =$$

$$\begin{array}{r} 32 \\ 8 \\ 4 \\ 3 \\ \hline 47 \end{array} \quad 10111$$

$$\begin{array}{r} 0,712 \cdot 2 \\ \hline 1,424 \end{array} \quad 1$$

$$0,101111010 \cdot 10^6$$

$$\begin{array}{r} 0,424 \cdot 2 \\ \hline 0,848 \end{array} \quad 0$$

$$\begin{array}{r} 0,848 \cdot 2 \\ \hline 1,696 \end{array} = 1$$

$$\begin{array}{r} 0,696 \cdot 2 \\ \hline 1,392 \end{array} \quad 1$$

$$\begin{array}{r} 0,392 \cdot 2 \\ \hline 0,784 \end{array} \quad 0$$

$$0,1 = \sum_{i=1}^{\infty} (2^{-4i} + 2^{-4i-1})$$

-----

$$\sum_{i=1}^{\infty} (2^{-4})^i = \frac{1}{1-2^{-4}} = \frac{2^4}{2^4-1}$$

$$\frac{1}{2} \sum_{i=1}^{\infty} (2^{-4})^i = \frac{2^3}{2^4-1}$$

$$+ = \frac{2^4+2^3}{15} = \frac{24}{15} = \frac{8}{5} = \frac{16}{10} = 1,6$$

$$\frac{3}{2} \frac{16}{15} = \frac{8}{5}$$

1

2.3

$$f(x) = x^5 - 10x + 1$$

$$f(0) = 1$$

$$f(0,2) = 0,2^5 - 2 + 1 < 0$$

ima vsej eno ničlo

se ima več, ima številno c. točko

$$f'(x) = 5x^4 - 10 = 0$$

$$x^4 = 2$$

$$x = \pm \sqrt[4]{2} \text{ ni med } 0 \text{ in } 0,2$$

b)

$$x_0 = 0$$

$$x_{r+1} = g(x_r)$$

$$g(x) = (x^5 + 1) / 10$$

$$x_{r+1} = \frac{(x_r^5 + 1)}{10}$$

$$\lim_{r \rightarrow \infty} f(x_r)$$

$$g'(x) = \frac{5x^4}{10} \quad g'(0) = 0 <$$

$$g(g(0)) = g\left(\frac{1}{10}\right) = \frac{\frac{1}{10^5} + 1}{10} = \frac{10^5 + 1}{10^6}$$

$$g(x) = -x^2 + 8x - 12$$

$$g(x) = x$$

$$-x^2 + 8x - 12 - x = 0$$

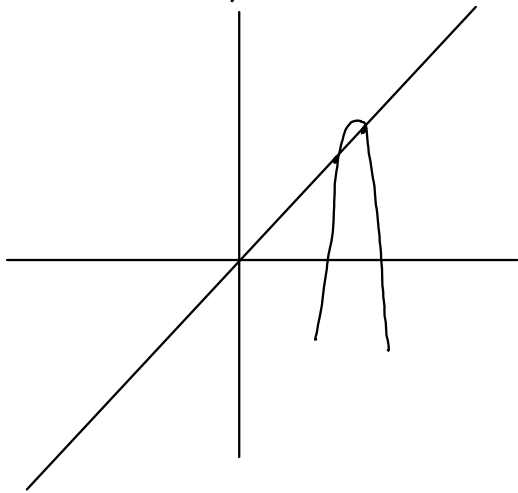
$$-x^2 + 7x - 12 = 0$$

$$-(x-3)(x-4) = 0$$

$$g'(x) = -2x + 8 \quad | -2x + 8 | < 1$$

$$g'(3) = 2 \quad \text{addge} \quad \in \left( \frac{7}{2}, \frac{9}{2} \right)$$

$$g'(4) = 0 \quad \text{remove}$$



$$b_n = \frac{10}{3} b_{n-1} - b_{n-2}$$

$$b_{n-2} = \frac{10}{3} b_{n-1} - b_n$$

2,7 naloga

$$f(x) = (x^2 - a)x = x^3 - ax$$

$$g(x) = x \frac{x^2 + 3a}{3x^2 + a} \text{ je}$$

negibne točke  $x$ :  $\sqrt{a}$ ,  $-\sqrt{a}$ ,  $0$

za  $x_0 > 0$  konvergira k  $\sqrt{a}$

$$x_r - \sqrt{a} = g(x_{r-1}) - \sqrt{a} = \frac{(x_{r-1} - \sqrt{a})^3}{3x_{r-1}^2 + a} < 0$$

$$x_0 - \sqrt{a} < 0$$

$$\Rightarrow x_{r-1} - \sqrt{a} < 0$$

di je ~~straga~~ ~~nerazpore?~~  $x_r < \sqrt{a}$

2.19 naloga

$$f(x) = x^2 - a \quad a > 0$$

Hallyjeva metoda

$$g(x) = x - \frac{2f(x)f'(x)}{2f'(x)^2 - f(x)f''(x)}$$

$$f'(x) = 2x \quad f''(x) = 2$$

$$g(x) = x - \frac{2(x^2 - a)2x}{2(2x)^2 - (x^2 - a)2} =$$

$$= x - \frac{4x^3 - 4ax}{8x^2 - 2x^2 + 2a} = \frac{2x^3 + 6ax}{6x^2 + 2a} =$$

$$= \frac{x(x^2 + 3a)}{3x^2 + a}$$



2.15 relage

$$x_{r+1} = \frac{x_{r-1}x_r + a}{x_{r-1} + x_r}$$

$$x_0, x_1 > \sqrt{a}$$

sekantna metoda

$$x_{r+1} = x_r - \frac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}$$

$$f(x) = x^2 - a$$

$$\begin{aligned} x_{r+1} &= x_r - \frac{(x_r^2 - a)(x_r - x_{r-1})}{x_r^2 - a - x_{r-1}^2 + a} = \\ &= \frac{\cancel{x_r^3} - x_{r-1}^2 x_r - \cancel{x_r^3} + ax_r + x_{r-1}x_r^2 - ax_{r-1}}{x_r^2 - x_{r-1}^2} = \\ &= \frac{x_{r-1}x_r(x_r - x_{r-1}) + a(x_r - x_{r-1})}{x_r^2 - x_{r-1}^2} = \frac{x_{r-1}x_r + a}{x_r + x_{r-1}} \end{aligned}$$

$$\underline{\underline{x_{r+1} < x_r}}$$

$$\begin{aligned} x_{r+1} &< \frac{x_{r-1}x_r + a}{x_r + x_{r-1}} \Rightarrow x_{r+1}x_r + x_{r+1}x_{r-1} < x_{r-1}x_r + a \\ &< \\ 2x_{r+1}x_{r-1} &< \cancel{x_{r-1}x_r} + a \\ &\quad \quad \quad \downarrow a < x_r \\ x_{r+1} &< \frac{x_r + a}{2} < x_r \quad \dots \end{aligned}$$

### 3.3 norm

$$N_{\infty}(A) = \max |a_{ij}|$$

1) N-norm

$$N(A)N(B) = N(AB)$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \quad \times$$

2)  $\|A\| = n N_{\infty}(A)$  is matrix norm  
positive definite

•  $\|A\| \geq 0$  : and is 0, in  $n > 0$

$$\|A\| = 0 \Leftrightarrow n = 0 \vee a_{ij} = 0 \forall a_{ij} \Rightarrow A = 0$$

• homogeneous

$$\|\alpha A\| = n N(\alpha A) = n |\alpha| N(A) = |\alpha| \|A\| \quad \checkmark$$

• triangle inequality

$$\|A+B\| = n N_{\infty}(A+B) = n \max_j |a_{ij} + b_{ij}| \leq n \max_j |a_{ij}| + n \max_j |b_{ij}|$$

• multiplicative

$$\leq \|A\| + \|B\|$$

$$\|A \cdot B\| \leq \|A\| \|B\|$$

$$\|A \cdot B\| = n \max_i \left| \sum_{j=1}^n a_{ij} b_{ij} \right| \leq n \sum_{j=1}^n |a_{ij}| |b_{ij}|$$

$$n \max_i |a_{ij}| \sum_{j=1}^n |b_{ij}| \leq n \max_i |a_{ij}| n \max_j |b_{ij}| \leq \|A\| \|B\|$$

3.8 relage

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |k_{ij}|^2}$$

$$\|A\|_2 = \max \sqrt{\lambda_i(A^H A)}$$

laste vektorest od  $A^H A$

$$1) \frac{1}{\sqrt{n}} \|A_F\| \leq \|A\|_2 \leq \|A\|_F$$

$$\det(A^H A - \lambda I) = 0$$

3.10

$$\|A\|_2 \leq \|A\|_F$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^H = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$A^H A = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

Vespera

## LU razcep

• brez pivotiranja (zadnjič ☹)

• delno pivotiranje  $PA=LU$

$$Ax=b$$

$$LUx=Pb$$

$$1) Ly=Pb$$

$$2) Ux=y$$

• kompletno pivotiranje  $Ly=Pb$  2)  $Uz=y$  3)  $Q^{-1}x=z$

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 2 & 1 & -4 & 2 \\ 3 & -2 & 3 & -1 \\ -1 & 3 & -1 & 1 \end{bmatrix}$$

izračunaj LU razcep in  $\det(A)$

$$\begin{array}{c} \begin{array}{c} \downarrow \\ \begin{bmatrix} 3 & -2 & 3 & -1 \\ 2 & 1 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} 1 - (2)(\frac{2}{3}) \\ \begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & \frac{7}{3} & -4 & \frac{5}{3} \\ -\frac{1}{3} & \frac{7}{3} & 0 & \frac{2}{3} \end{bmatrix} \end{array} \end{array}$$

$$P \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

$$1 - (-2)(\frac{2}{3})$$

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & 1 & 2 & -1 \\ -\frac{1}{3} & 1 & 6 & -2 \end{bmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{bmatrix} -3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$-1 - (-\frac{2}{3})$$

$$U = \begin{bmatrix} -3 & -2 & 3 & -1 \\ & \frac{7}{3} & -6 & \frac{8}{3} \\ & & 6 & -2 \\ & & & -\frac{1}{3} \end{bmatrix} \quad L = \begin{bmatrix} 1 & & & \\ \frac{2}{3} & 1 & & \\ -\frac{1}{3} & 1 & 1 & \\ \frac{2}{3} & 1 & \frac{1}{3} & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det(A) = \frac{\det(U) \det(L)}{\det(P)} = \det(U) = (-7)(-2) = 14$$

$$\pm 1 = (-1)^{\text{št. zam. vrstic}}$$

$$\frac{1}{2}$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix}$$

WS komplektieren pivotingen + system  $Ax=b$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ -3 & 2 & 1 \\ 3 & 3 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 4 & -1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & \frac{1}{4} & 1 & \end{bmatrix} \quad U = \begin{bmatrix} -6 & 2 & 4 \\ & 4 & -1 \\ & & -\frac{3}{4} \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

zamenjevanje vrstic  
zamenjevanje stolpcer

$$Ly = Pb$$

$$\begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ \frac{1}{2} & \frac{1}{4} & 1 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$$y_1 = 14$$

$$-\frac{1}{2}x_1 + x_2 = 0 \rightarrow x_2 = 7$$

$$y_3 = -\frac{3}{4}$$

$$Uz = y$$

$$x = Qz$$

a) Izračunajte LU razcep brez pivotairanja  
 $A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$  kaj opazite

b) Naj bo  $\begin{bmatrix} a_{11} & b_1 \\ & a_{12} & \\ & & \ddots & \\ & & & a_{nn} & b_n \end{bmatrix}$  splošna 3-diagonalna matrika

zapišite algoritem za razcep te tridialne matrike in preštejte št. operacij  
 koliko operacij potrebujemo za reševanje sistema

$$AX = Z$$

a)  $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & -\frac{2}{7} & \frac{11}{7} & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & -\frac{2}{7} & \frac{11}{7} & 6 \\ 0 & 0 & \frac{2}{7} & -\frac{20}{7} \end{bmatrix}$   $G \cdot 7 = 42$   
 $L = I \quad U = A_{uz}$   
 $2 - \frac{42}{11} = \frac{-20}{11}$

b) for i in range (1, n):  
 $= L[i][i-1] = A[i][i-1] / A[i-1][i-1]$   
 $U[i][i] = A[i][i] - L[i][i-1] \cdot A[i-1][i]$

$$u_1 = a_1$$

for i = (1:n)

$$l_i = \frac{c_i}{u_i}$$

$$u_{i+1} = a_{i+1} - l_i \cdot b_i$$

end

n-1 deljenja z l

n-1 množenj

n-1 odštevanj

3 · (n-1) operacij

$$AX = Z$$

$$LY = Z: \quad y_1 = z_1 \quad y_2 = l_{12} y_1 + z_2 \quad \dots \quad y_i = z_i - l_{i-1,i} y_{i-1}$$

2 · (n-1) operacij

$$UX = Z: \quad x_n = u_n^{-1} z_n$$

$$\cancel{b_{n-1} x_n} + u_{n-1} x_{n-1} = z_n$$

$$x_{n-1} = \frac{z_n - b_{n-1} x_n}{u_{n-1}} \quad x_i = \frac{z_{i+1} - b_i x_{i+1}}{u_i}$$

$$= 3 \cdot (n-1) + 1 = 3n-2 \text{ operacij}$$

skupaj 5n-4 operacij

Opisite postopek za reševanje sistema linearnih enačb

$$\begin{bmatrix} U & -I \\ B & L \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ kjer je } B=LU \text{ nesingularna}$$

Prestave števil operacij

$$\begin{bmatrix} Ux - y \\ Bx + Ly \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$Bx + Ly = L(Ux + Ly) = L(Ux + Iy)$$

$$y = Ux - a \quad Ux = y + a$$

$$\begin{aligned} Bx + Ly &= L(Ux + Ux - a) = L(2Ux - a) = \\ &= L(\underbrace{2y + a}) = b \end{aligned}$$

1.  $Lz = b$   $n^2$  operacij

2.  $y = \frac{z - a}{2}$   $2n$  operacij

3.  $Ux = a + y$   $n + n^2 + n =$   
abstr. sum

skupaj:  $2n + 4n$



13.11

Y menjhele  
ene wo

## Razcep Choleskega

$A \in \mathbb{R}^{n \times n}$  simetrična, pozitivno definitna  
( $\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$ )

$$A = V \cdot V^T \quad V = \begin{matrix} \text{nesingularna s} \\ \text{pozitivnimi diag. elementi} \end{matrix}$$

for  $j = 1:n$

$$V_{j,j} = \sqrt{a_{j,j} - \sum_{k=1}^{j-1} V_{j,k}^2}$$

for  $j = i+1:n$

$$V_{j,j} = \frac{1}{V_{j,i}} \left( a_{j,i} - \sum_{k=1}^{j-1} V_{j,k} V_{i,k} \right)$$

a) Dobiš faktor Choleskega (v)

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 8 & -2 & 8 \\ -2 & -2 & 14 & -11 \\ 3 & 8 & -11 & 15 \end{bmatrix}$$

b) Naj bo

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

i) za kakšno  $\alpha$  je to poz. def.

ii)  ~~$\alpha = 23$~~  23 razlik s teden

$$Ax = \begin{bmatrix} 6 \\ 15 \\ 2 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ -2 & 1 & 3 & \\ 3 & 1 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

$$V = \begin{bmatrix} 2 & & & \\ 3 & 3 & & \\ 1 & -1 & 1 & \\ -2 & 3 & 1 & \sqrt{\alpha - 14} \end{bmatrix}$$

$$V_{ji} := \frac{1}{V_{ii}} \left( a_{ji} - \sum_{k=1}^{j-1} V_{jk} V_{ik} \right)$$

$$V_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} V_{ik}^2}$$

$$V_{21} = \frac{1}{1} (2 - 0)$$

$$V_{22} = \sqrt{18 - 2^2} = 4$$

$$V_{32} = \frac{1}{2} (-2 - (-4)) = 1$$

$$V_{42} = \frac{1}{2} (8 - (3 \cdot 2)) = 1$$

$$\alpha > 14$$

26.11

Dani sta matriki  $A$  in  $B \in \mathbb{R}^{n \times n}$ ,  $B$  je pozitivno definitna. Sestavite učinkovit postopek za izračun sledi od  $A^T B^{-1} A$  in preštejte š. operacij

Na kubi:

razsvetlje sistemov nelinearnih enačb

$$\begin{aligned} f_1(x_1 \dots x_n) &= 0 \\ f_2(x_1 \dots x_n) &= 0 \\ &\vdots \\ f_n(x_1 \dots x_n) &= 0 \end{aligned} \quad \leadsto F(x_1 \dots x_n) = \begin{bmatrix} f_1(x_1 \dots x_n) \\ \vdots \\ f_n(x_1 \dots x_n) \end{bmatrix}$$

$$F(\underline{x}) = 0$$

Jakobijeva iteracija

$$G(\underline{x}) = \begin{bmatrix} g_1(\underline{x}) \\ \vdots \\ g_n(\underline{x}) \end{bmatrix} \quad \underline{x}^{(r+1)} = G(\underline{x}^{(r)})$$

Seidlove iteracije

$$x_i^{(r+1)} = g_i(x_1^{(r+1)}, \dots, x_{i-1}^{(r+1)}, x_i^{(r)}, \dots, x_n^{(r)})$$

- $\alpha$  je privlačna rešitvena točka, če obstaja kakšna metrična norma, da velja  $\|J_G(\alpha)\| < 1$   
( $G$  mora biti odvedljiva v  $\alpha$ )

Potem  $\exists$  okolica  $\Omega \ni \alpha$ .  $\forall \underline{x}^{(0)} \in \Omega$  zaporedje konvergira k  $\alpha$

Dan je sistem enačb

$$\begin{aligned}x &= \sin\left(\frac{2x-y}{4}\right) \\ y &= \cos\left(\frac{x+2y}{4}\right)\end{aligned}$$

- a) pri začetnem približku  $x^0 = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$  naredite dva koraka Jakobijeve in se idlerne iteracije
- b) Pokazite, da iteraciji konvergira za vsi začetni približki

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{4}\right) \\ \cos\left(\frac{x+2y}{4}\right) \end{bmatrix}$$

$$x^1 = G\left(\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sin\left(\frac{4\pi}{4}\right) \\ \cos\left(\frac{2\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x^2 = G\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Se nadalja:

$$x^1 = G\left[\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x^2 = G\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} \sin\frac{1}{4} \\ \cos\left(\frac{-\sin\frac{1}{4}+2}{4}\right) \end{bmatrix}$$

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{u}\right) \\ \cos\left(\frac{x+2y}{u}\right) \end{bmatrix}$$

$$J_G(x,y) = \begin{bmatrix} \cos\left(\frac{2x-y}{u}\right) \cdot \frac{1}{2} - \cos\left(\frac{2x-y}{u}\right) \frac{1}{u} \\ \underbrace{-\sin\left(\frac{x+2y}{u}\right) \cdot \frac{1}{u}}_{|| < 1} - \sin\left(\frac{x+2y}{u}\right) \cdot \frac{1}{2} \end{bmatrix}$$

Vremeno na norme

vrste po abs vrednost vistic:

$$|J_G(x,y)|_{\infty} = \max \left\{ \left| \cos\left(\frac{2x-y}{u}\right) \frac{1}{2} \right| + \left| \cos\left(\frac{2x-y}{u}\right) \right|, \dots \right\} \leq$$

$$\leq \max \left\{ \frac{1}{2} + \frac{1}{u}, \frac{1}{2} + \frac{1}{u} \right\} < 1$$

To je za vsa  $x, y$  kjer obsega pomen  
kerle den so vsi pomeni

za seide:

$$G(x,y) = \begin{bmatrix} \sin\left(\frac{2x-y}{u}\right) \\ \cos\left(\frac{\sin\left(\frac{2x-y}{u}\right) + 2y}{u}\right) \end{bmatrix}$$

$$J_G = \begin{bmatrix} \frac{1}{2} \cos\left(\frac{2x-y}{u}\right) - \frac{1}{u} \cos\left(\frac{2x-y}{u}\right) \\ -\frac{1}{2} \cos\left(\frac{2x-y}{u}\right) \sin\left(\frac{\sin\left(\frac{2x-y}{u}\right) + 2y}{u}\right), \dots \end{bmatrix}$$

spet preverimo || na norme in vidimo  
je manjša od 1

## Newton's method

$$x^{(r+1)} = x^{(r)} - J_F(x^{(r)})^{-1} F(x^{(r)})$$

2)

$$J_F(x^{(r)})(x^{(r+1)} - x^{(r)}) = -F(x^{(r)})$$

$$1) J_F(x^{(r)}) \Delta x^{(r)} = -F(x^{(r)})$$

$$2) x^{(r+1)} = \Delta x^{(r)} + x^{(r)}$$



Dan je sistem  $x^2 + y^2 = 4$   
 $x^2 - y^2 = 1$

a) Naredite 2. korak newtonove metoda  
 pri  $x^{(0)} = (2, 1)$

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad F = \begin{bmatrix} x^2 + y^2 - 4 \\ x^2 - y^2 - 1 \end{bmatrix}$$

$$JF = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$J_F(x^{(0)}) \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 4\bar{x} + 2\bar{y} \\ 4\bar{x} - 2\bar{y} \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & | & -1 \\ 4 & -2 & | & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & | & -1 \\ 0 & -4 & | & -1 \end{bmatrix} \rightsquigarrow y = \frac{1}{4}$$

$$4x + 2y = -1$$

$$4x = -\frac{3}{2}$$

$$x = -\frac{3}{8}$$

3.12

f 1. netlog

## Predloženi sistemi

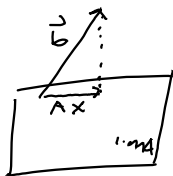
$$Ax = b \quad A \in \mathbb{R}^{m \times n}, \text{ m} \geq n \quad b \in \mathbb{R}^m$$

Iščemo  $x$ , ki minimizira  $\|Ax - b\|_2$

A polnege range  $\Rightarrow$  enolična rešitev

Normalni sistem

$$Ax \in \text{im}(A)$$



$$\Rightarrow Ax - b \perp \text{im}(A)$$

$$\Leftrightarrow$$

$$(Ax - b) \perp A^T \quad \leftarrow \text{postolpek}$$

$$\Leftrightarrow$$

$$A^T(Ax - b) = 0$$

$$\Leftrightarrow$$

$$A^T A x = A^T b$$

$A^T A$  je simetrična, pozitivno definitna  
 $\Rightarrow$  lahko uporabimo razcep  
Choleskega

$$1) A^T A = V V^T$$

$$2) A^T A x = A^T b$$

$$V V^T x = A^T b$$

$$\underbrace{\quad}_y$$

$$V y = A^T b$$

f podana v točkah

$$f(-1) = \frac{11}{4} \quad \text{Poiščite parabolo, ki se}$$

$$f(0) = \frac{7}{4} \quad \text{po metodi najmanjših}$$

$$f(1) = \frac{1}{4} \quad \text{kvadratov najboljše prilaga}$$

$$f(2) = \frac{13}{4} \quad \text{funkciji f. Izpeljite normalen}$$

in ga rešite z razcepom Cholesky

$$p(x) = ax^2 + bx + c$$

$$p(-1) = a - b + c = \frac{11}{4}$$

$$p(0) = c = \frac{7}{4}$$

$$p(1) = a + b + c = \frac{1}{4}$$

$$p(2) = 4a + 2b + c = \frac{13}{4}$$

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ \frac{7}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

Cholesky  $V = \begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \frac{8-3}{\sqrt{5}} & \sqrt{18-9-5} \end{bmatrix} =$

$= \begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix} \quad V y = A^T b$

$$A^T b = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 7 \\ 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= 4 \\ 4 + \sqrt{5}y_2 &= 4 \Rightarrow y_2 = 0 \\ 12 + 2y_3 &= 16 \Rightarrow y_3 = 2 \end{aligned}$$

$$V^T x = y$$

$$\begin{bmatrix} 2 & 1 & 3 \\ & \sqrt{5} & \sqrt{5} \\ & & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} x_3 &= 1 \\ \sqrt{5}(x_2 + 1) &= 0 \Rightarrow x_2 = -1 \\ 2x_1 - 1 + 3 &= 4 \\ x_1 &= 1 \end{aligned}$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} a &= 1 \\ b &= -1 \\ c &= 1 \end{aligned}$$

$$\underline{p(x) = x^2 - x + 1}$$

## QR razcep

$$A = QR \quad A \in \mathbb{R}^{m \times n} \quad m > n \quad Q \in \mathbb{R}^{m \times n} \quad R \in \mathbb{R}^{n \times n}$$

$\nearrow$  ortonormirani stolpci       $\nwarrow$  zgornje trikotna

$$Q^T Q = I$$

$$Q Q^T \neq I \text{ ker vrstice niso ortonormirane}$$

## • Modificiran Gram-Schmidov postopek

$$k = 1, \dots, n$$

$$g_k = a_k$$

$$i = 1, \dots, k-1$$

$$r_{ik} = g_i^T g_k$$

$$g_k = g_k - r_{ik} g_i$$

$$r_{kk} = \|g_k\|_2$$

$$q_k = \frac{g_k}{r_{kk}}$$

S pomočjo MGS (modificiran gram Schmidt)  
izračunajte QR razcep za matriko

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}$$

$$g_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \|g_1\| = \sqrt{2}$$

$$g_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{matrix} r_{12} = 0 \\ r_{22} = \sqrt{2} \end{matrix} \quad \begin{matrix} g_2 = g_2 - 0 \\ g_2 = \frac{g_2}{\sqrt{2}} \end{matrix}$$

$$g_3 = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \end{bmatrix} \quad r_{13} = \frac{-1}{\sqrt{2}} - \frac{3}{\sqrt{2}} = -2\sqrt{2}$$

$$g_3 = \begin{bmatrix} -2 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$r_{23} = 0 \quad r_{33} = \sqrt{12} = 2\sqrt{3}$$

$$g_3 = g_3 \quad \leadsto$$

$$g_3 = \frac{1}{2\sqrt{3}} g_3$$

## Modificiran Gram-Schmidt

$$k = 1 : n$$

$$z_k = a_k$$

$$j = 1, \dots, k-1$$

$$r_{jk} = z_j^T z_k$$

$$z_k = z_k - r_{jk} z_j$$

$$r_{kk} = \|z_k\|_2$$

$$z_k = \frac{1}{r_{kk}} z_k$$

Pri reševanju sistema z MGS naredimo QR  
razcep na razširjeni matriki

$$[A; b] = [Q; z_{n+1}] \begin{bmatrix} R & z \\ c & s \end{bmatrix} \quad \left( \begin{array}{l} Ax = b \\ \min_x \|Ax - b\|_2 \end{array} \right)$$

Rešjeno  $Rx = z$   $\min_x \|Ax - b\| = s$

Od zadnjic

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \quad R = \begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} \\ & \sqrt{2} & 0 \\ & & 2 \end{bmatrix}$$

b) Preko QR razcepa rešite predložen sistem  
 $Ax = \begin{bmatrix} 4 \\ 6 \\ -1 \\ 2 \end{bmatrix}$  Po metodi najmanjših kvadratov.

koliko je  $\min \|Ax - b\|_2$

$$[A; b] = \begin{bmatrix} 1 & 0 & -1 & 4 \\ 1 & 0 & -3 & 6 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \quad z_4 = \begin{bmatrix} 6 \\ 6 \\ -1 \\ 2 \end{bmatrix}$$

$j=1$

$$r_{14} = \frac{4}{\sqrt{2}} + \frac{6}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

$j=2$ :

$$r_{24} = -\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -\frac{3}{\sqrt{2}} = -\frac{\sqrt{2} \cdot 3}{2}$$

$$z_4 = z_4 - 5\sqrt{2} z_1$$

$$z_4 = z_4 + \frac{3}{\sqrt{2}} z_2 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$z_4 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$j=3$

$$r_{34} = \frac{1}{2}$$

$$z_3 = z_4 + \frac{1}{2} z_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$r_{44} = \sqrt{\frac{9}{16} \cdot 2} = \sqrt{\frac{3}{8}} = \frac{3}{2\sqrt{2}}$$

$$z_4 = \frac{1}{r_{44}} z_4 = \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Rx = z$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ & \sqrt{2} & 0 \\ & & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} \\ -\frac{3}{2}\sqrt{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$x_3 = -\frac{1}{4}$$

$$x = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{1}{4} \end{bmatrix}$$

$$x_2 = -\frac{3}{2}$$

$$\sqrt{2} x_1 - 2\sqrt{2} x_3 = 5\sqrt{2}$$

$$x_1 = \frac{1}{2} + 5 = \frac{3}{2}$$

$$\min_x \|Ax - b\|_2 = 2$$



Y

10.12

1.12

## Razširjeni QR razcep

$$A = \tilde{Q} \tilde{R}$$

$\tilde{Q} : m \times m$  ortogonalna  $\tilde{Q}^T \tilde{Q} = \tilde{Q} \tilde{Q}^T = I$

$\tilde{R} : m \times n$  zgornja trapezna

$$A = \tilde{Q} \tilde{R} = [Q \ Q_1] \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Reševanje sistema:  $\swarrow$  množenje z  $\tilde{Q}$  ne spremeni z-nem

$$\|Ax - b\|_2 = \|\tilde{Q} \tilde{R}x - b\|_2 = \|\tilde{Q}^T (\tilde{Q} \tilde{R}x - b)\|_2 =$$

$$= \|\tilde{R}x - Q^T b\| = \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} x - \begin{bmatrix} Q^T \\ Q_1^T \end{bmatrix} b \right\|$$

$x$ , ki to minimizira dobimo iz  $Rx = Q^T b$

$$\text{minimum} = \|Q_1^T b\|_2$$

## Givensove rotacije

vhodni podatki:  $A, b$

izhod:  $\tilde{R} \tilde{Q}^T b$

for  $j=1:n$

for  $k=j+1:m$

if  $a_{kj} \neq 0$ :

$$R_{ik}^T = I$$

$$r = \sqrt{a_{jj}^2 + a_{kj}^2}$$

$$c = \frac{a_{jj}}{r}$$

$$s = \frac{a_{kj}}{r}$$

$$(R_{ik}^T)_{jj} = c$$

$$(R_{ik}^T)_{ik} = s$$

$$R_{ik}^T([j, k], [j, k]) = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$A = R_{ik}^T A$$

$$b = R_{ik}^T b$$

izhod:  $A (= \tilde{R})$ ,  $b (= \tilde{Q}^T b)$

Naloga: s pomočjo givenskih rotacij  
reši predložen sistem. koliko je min  $\|Ax - b\|$ ?

$$\begin{bmatrix} 3 \\ -4 \\ 12 \\ -84 \end{bmatrix} x = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 85 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix}$$

$$i=1 \quad k=2$$

$$\begin{aligned} r &= 5 \\ c &= \frac{3}{5} \\ s &= -\frac{4}{5} \end{aligned} \quad R = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & & \\ & \frac{4}{5} & \frac{3}{5} & \\ & & 1 & 1 \end{bmatrix} \quad \begin{matrix} RA \uparrow \\ RB \uparrow \end{matrix}$$

$$\frac{144}{25} \\ 169$$

$$i=1 \quad k=3$$

$$r = \sqrt{5^2 + 12^2} = 13$$

$$c = \frac{5}{13} \quad s = \frac{12}{13}$$

$$R = \begin{bmatrix} \frac{5}{13} & 0 & \frac{12}{13} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} RA \uparrow \\ RB \uparrow \end{matrix}$$

$$i=1 \quad k=4$$

$$r = \sqrt{13^2 + 84^2} = 85$$

$$c = \frac{13}{85} \quad s = -\frac{84}{85}$$

$$R = \begin{bmatrix} \frac{13}{85} & & -\frac{84}{85} & \\ & 1 & & \\ \frac{84}{85} & & 1 & \\ & & & \frac{13}{85} \end{bmatrix}$$

$$A = \begin{bmatrix} 85 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix}$$

$$Q = I$$

$$\rightsquigarrow R = 85$$

$$Q_1^T b = \begin{bmatrix} 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix}$$

$$\text{minimum: } \sqrt{25 + \frac{35^2}{13^2} + \frac{84^2}{13^2}} \\ = \sqrt{74}$$