Navadna diferencialna enačba (NDE) je enačba oblike

$$f(x,y,y')=0 \quad \text{iščemo } y(x) \text{ da velja}$$

$$f(x,y(x),y'(x)) \text{ za } \forall x \in D_y$$

(To je v implicitni obliki)

Mi se bomo večinoma ukvarjali s temi, ki je podano v eksplicitni obliki, torej ko je $y'=f(x,y)$

Enačba reda $n \in \mathbb{N}$ je oblike $G(x,y,y',\dots,y^{(n)})=0$

Enačba je avtonomna, če funkcija G ni odvisna od x. Sicer je neavtonomna.

1. Za dano družino funkcij poišči pripadajočo DE

a) $y = ce^x$

b) $y^2 = cx$

c) $y = c(x-c)$

$\left.\right\}$ $f(x,y,c)=0$, $c \in \mathbb{R}$

$y' = ce^x \Rightarrow y = y'$

$y^2 = cx$

$2yy' = c \implies y^2 = 2yy'x$

$y = 2y'x$ za $y \neq 0$

$y' = c$

$y = y'(x-c)$

2) ugani rešitev naslednjih DE

a) $y'' = -y \Rightarrow \{\alpha \cos x + \beta \sin x\}$

b) $y + xy' = \cos x$

c) $xy' = ny$

b)

$$(xy)' = \cos x$$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

c) $xy' = ny$

$$xy' + y = ny + y$$

$$(xy)' = (n+1)y$$

$$y = x^n$$

Metoda izoklin

$y_c = y = y(x, C) \quad x, C \in \mathbb{R}$

$y \in \mathcal{C}^1$

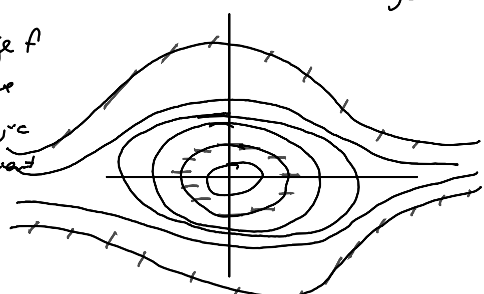
Izoklina je krivulja vzdolž katere ima vsaka članica družine y enak odvod po x ($y_c'(x)$)

$I_\alpha = \{(x,y); y = y_c(x)$ potem je $y_c'(x) = \alpha$

Če imamo DE oblike $y' = f(x,y)$ in pripadajočo družino rešitev $y_c(x)$, potem so izokline ravno nivojnice $\{f = \alpha\}$

tj. $y_c'(x) = f(x, y_c(x)) = \alpha$

(črno so nivojnice)
mi iščemo funkcijo

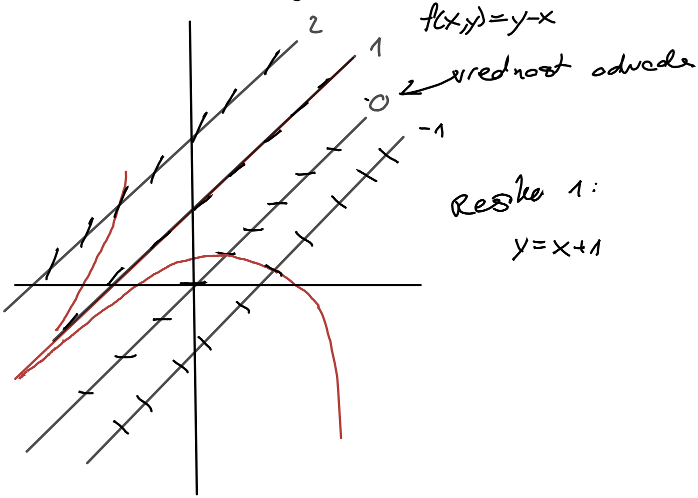


Postopek:

1. Skiciramo funkcije f
2) vzdolž vsake nivojnice narišemo nekaj koljic ki imajo smerni koeficient enak ~~nivojnica na~~ vrednosti f na nivojnici
3) narišemo krivuljo ki so v presečiščih nivojnic tangentne na ravnic

3. Približno skiciraj potek enačbe $y'=y-x$

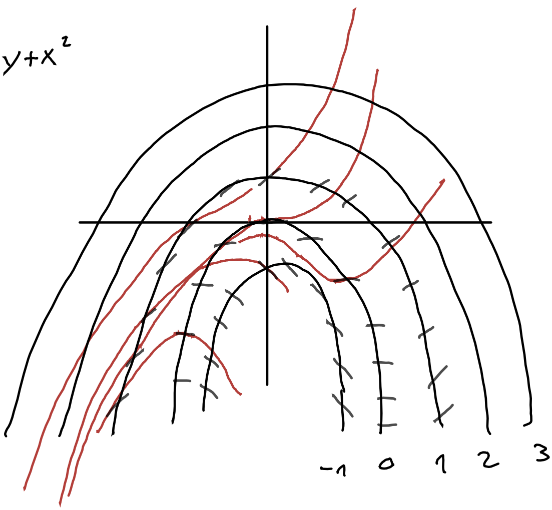
$f(x,y)=y-x$



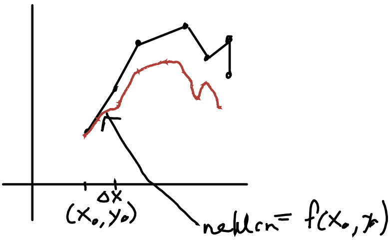
Rešitev 1:

$y=x+1$

$y'=y+x^2$



Eulerjeva metoda



$x_1 = x_0 + \Delta x$

$y_1 = y_0 + \Delta x\, f(x_0, y_0)$

$x_2 = x_1 + \Delta x$

$y_2 = y_1 + \Delta f(x, y_0)$

Izberemo si Δx in iterativno definiramo

$$x_{n+1} = x_n + \Delta x = x_0 + (n+1)\Delta x$$

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

Zakaj ima to smisel? Če privzamemo da je rešitev zvezno odvedljiva, velja, da je

$$y(x+\Delta x) = y(x) + y'(x)\Delta x + o(\Delta x)$$

5) Z Eulerjeva metodo dobi rešitev DZ

$y' = f(x)$ kjer je f zvezna pri pogoju $y(0)=0$

neka točka

$y(A) = ?$

$\Delta x = \frac{A}{m}$ za nek $m \in \mathbb{N}$

$A = x_m$

$x_n = 0 + \Delta x_n n$

$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x \cancel{+ y_0} =$

$= y_{n-1} + f(x_{n-1}) \Delta x \cancel{} + y_0$

$= \left(f(x_{n-2}) + f(x_{n-1})\right) \Delta x + \underset{0}{\overset{\parallel}{y_0}}$

Riemanova vsota

$$y_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

za $\int_0^A f(x)\,dx$

$$y_m = \sum_{i=0}^{m-1} f(x_i) \Delta x$$

To je očitno rešitev

Vaja (DN):

Najdi rešitev za $y'=2y$ z eulerjevo metodo

$$\Delta x = \frac{A}{n}$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\Delta x = y_{n-1} + 2y_{n-1}\frac{A}{n} =$$

$$= y_{n-1}\left(1+\frac{2A}{n}\right)$$

$$y_n = y_{n-2}\left(1+\frac{2A}{n}\right)^2 = y_1\left(1+\frac{2A}{n}\right)^{n-1}$$

$$\lim_{n\to\infty}\left(1+\frac{2A}{n}\right)^{n-1} = \lim_{u\to\infty}\frac{\left(\left(1+\frac{1}{u}\right)^u\right)^{2A}}{\left(1+\frac{1}{u}\right)} = \underline{\underline{e^{2A}}}$$

$$\frac{1}{u} = \frac{2A}{n} \Rightarrow n = 2Au$$

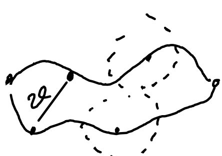
$$y = e^{2x}$$

Fazni prostor je prostor vseh možnih stanj sistema

$y' = f(x,y)$ $\quad f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

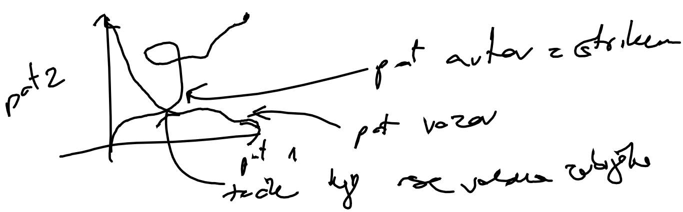
↳ fazni prostor je $\mathbb{R}^3$

6) iz Ljubljane v Maribor vodita dve cesti po katerih lahko iz Ljubljane do Maribora pripeljete avtomobila tista eden na drugega privezana z vrvjo dolžine $<2l$, ne da bi jo pretrgala. Ali se lahko vozova krožne dolžine radia l, ki vozita vsak v svojo smer srečata, ne da bi trčila



$a \in S$

$[0, 1] \times [0, 1]$



1) 14.10

a) $y' = \frac{x^2}{y} = \frac{dy}{dx}$

$$x^2 dx = y dy$$

$$\frac{1}{3}x^3 + c = y^2$$

$$y = \pm\sqrt{\frac{1}{3}x^3 + c}$$

b)

$$2x^2 y y' + y^2 = 2$$

$$2x^2 = \frac{2-y^2}{yy'} = \frac{2-y^2}{y\frac{dy}{dx}} = \frac{2-y^2}{y\,dy}dx$$

$$\frac{dx}{2x^2} = \frac{y\,dy}{2-y^2}$$

$u = 2-y^2$

$du = -2y\,dy$

$$-\frac{1}{2}\frac{1}{x} = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}(\ln u + c)$$

$$\frac{1}{x} = \ln(2-y^2) + c$$

$$ce^{\frac{1}{x}} = 2-y^2$$

$$y = \pm\sqrt{2-ce^{\frac{1}{x}}}$$

c) $(1+x^2)y' = y$

$$\parallel$$

$$(1+x^2)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x^2}$$

$$\ln y = \arctan x + c$$

$$y = ce^{\arctan x}$$

$$y \equiv 0$$

c') $(1+x^3)y' = y$

$$y_1 \equiv 0$$

2) $$\frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)}$$

$$B = -A$$

$$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

x^2: $B + A = 0$

x: $B + C - A = 0$

1: $A + C = 1$

$C = 1 - \frac{1}{3}$

$2B + C = 0$

$\Rightarrow C = -2B = 2A$

$3A = 1$

$$\frac{dy}{y} = \frac{1}{3}\frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1}dx$$

$$\int \frac{x-2}{x^2-x+1}dx = \int \frac{}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

2)

a) $y' = \tan(2x+3y-1)$

$$y' = \tan z$$
$$z = 2x+3y-1$$
$$z' = 2+3y'$$
$$y' = \frac{z'-2}{3}$$
$$\tan z = \frac{z'-2}{3}$$
$$z' = 3\tan z + 2$$

1. rešitev : $z' = 0 \Rightarrow z = \arctan(-\frac{2}{3})$

$\Downarrow$

$y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan(-\frac{2}{3}) - 2x + 1}{3}$

$$\frac{dz}{dx} = 3\tan z + 2$$
$$\frac{dz}{3\tan z + 2} = dx$$

$u = \tan z$

$du = \frac{1}{\cos^2 \cancel{z}} \cancel{dz} \Rightarrow dz = \frac{du}{1-u^2}$

$$x + C = \int \frac{du}{(3u+2)(1-u^2)}$$

$$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2}$$

$u^2: B - A = 0 \quad A = B$

$u: 3\cancel{C}\cancel{-2} + 2B = 0$

$1: 2C + A = 1 \Rightarrow A = 1 - 2C$

$3C + 2 - 4C = 0$

$C = 2 \Rightarrow A = -3 = B$

$$= \int\left(\frac{-3}{3u+2} + \frac{-3u+2}{1-u^2}\right) du =$$

$$= -\ln\left(u + \frac{2}{3}\right) +$$

$$+ \int \frac{1}{2}\frac{1}{u-1} + \frac{5}{2}\frac{1}{u+1} =$$

$\frac{A}{1-u} + \frac{B}{1+u} =$

$u: A - B = -3$

$1: A + B = 2$

$A = B - 3$

$2B - 3 = 2$

$2B = 5$

$B = \frac{5}{2}$

$A = -\frac{1}{2}$

$$= -\ln\left(u + \frac{2}{3}\right) + \frac{1}{2}\ln(u-1) + \frac{5}{2}\ln(u+1)$$

$y''y - y' = xy'' - (y')^2$

$y''(y-x) = y'(1-y')$

$y''(y-x) + y'(y'-1) = 0$ $\qquad y \equiv 0$

$a = y'$

$b = y - x \qquad b' = y' - 1$

$a'b + ab' = 0$

$(ab)' = 0$

$ab = c \in \mathbb{R}$

$y'(y-x) = c$

$y' = \frac{c}{y-x} \qquad z = y - x$

$z' = y' - 1 = \frac{c}{z} - 1$

$\cancel{z = \int\left(\frac{c}{z} - 1\right)dz = c\ln z - z + D}$

$\frac{dz}{dx} = \frac{c}{z} - 1$

$\frac{dz}{\frac{c}{z} - 1} = dx \qquad x + D = \int \frac{z}{c-z}\,dz = -\int\left(\frac{c}{t} - 1\right)dt =$

$t = c - z$

$dt = -dz \qquad = -c\ln(c-z) + c - z$

$z = y - x$

$\cancel{x} + D = -c\ln(c - y + x) + c - y + \cancel{x}$

$y = De^{c}e^{\ln(c-y+x)+1}$

Homogene enačbe

$$F(x,y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xz = y \Rightarrow y' = z + xz' = F(1,z)$$

$$z' = \frac{F(1,z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

A) $y^2 + x^2 y' = xyy'$ $\qquad z = \frac{y}{x}$

$$y'(xy - x^2) = y^2$$

$$y' = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z-1}$$

$$y = zx$$

$$y' = z + z'x = \frac{z^2}{z-1}$$

$$z'x = \frac{z^2 - z^2 + z}{z-1} = \frac{z}{z-1}$$

$$\frac{dz}{dx} x = \frac{z}{z-1} \Rightarrow \frac{z-1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{y}{x}$$

b) $y = xy' - \sqrt{x^2+y^2}$

$$y' = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} + \overset{\operatorname{sgn} x}{}\sqrt{1+\frac{y^2}{x^2}} = z + \overset{\operatorname{sgn} x}{}\sqrt{1+z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = \cancel{z} + z'x = \cancel{z} + \sqrt{1+z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1+z^2}}{x} \qquad z = \operatorname{sh} u \quad dz = -\operatorname{ch} u\,du$$

$$\operatorname{sgn} x \ln x + D = \int \frac{1}{\sqrt{1+z^2}}\,dz = \int \frac{-\operatorname{ch} du}{\operatorname{ch} u} =$$

$$-\operatorname{arcsh}\frac{y}{x}$$

$$''$$

$$\operatorname{arch}\frac{y}{x}$$

$$y = x \operatorname{ch}(\operatorname{sgn} x \ln x + D)$$

a)

$$ma = F = m \cdot g - kv^2$$

$$m\dot{v} = mg - kv^2$$

$$\dot{v} = g - \frac{k}{m}v^2 = g\left(1 - \underbrace{\frac{k}{mg}}_{\alpha}v^2\right) = g(1 - \alpha v^2)$$

$$\frac{dv}{dt} = g(1 - \alpha v^2)$$

$$\frac{dv}{1-\alpha v^2} = g\,dt \quad / \int$$

$$gt =$$

$$\frac{A}{1-\sqrt{\alpha}v} + \frac{B}{1+\sqrt{\alpha}v} :$$

$v: \sqrt{\alpha}A - \sqrt{\alpha}B = 0$

$A = B$

$1: \quad A + B = 1$

$\Rightarrow A = \frac{1}{2}$

$B = \frac{1}{2}$

$$gt = \frac{1}{\sqrt{\alpha}2}\Big(\ln(1+\sqrt{\alpha}v) - \ln(\sqrt{\alpha}v - 1)\Big) =$$

$$\frac{1}{2\sqrt{\alpha}}\ln\left(\frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v}\right) = gt + C$$

$$V_0 = 0 \Rightarrow \quad \cancel{1+\sqrt{\alpha}}v = \cancel{1}-\sqrt{\cancel{\alpha}}v + C$$

$$\Rightarrow 0 = C$$

$$\frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v} = \underbrace{e^{2\sqrt{\alpha}gt}}_{D}$$

$$1 + \sqrt{\alpha}v = D - \sqrt{\alpha}Dv$$

$$v\sqrt{\alpha}(1 - \sqrt{\alpha}D) = D - 1$$

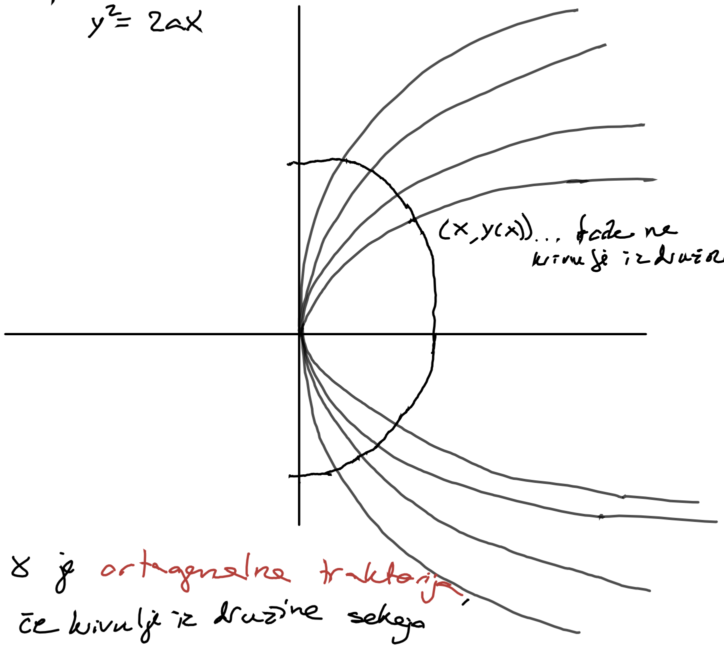
$$v = \frac{D-1}{\sqrt{\alpha}(1-\sqrt{\alpha}D)}$$

V merkenje

+ zamrude 20 min   21.10

2) Poišči družino ortagonalnih trajektorij na družino parabol

$y^2 = 2ax$



γ je ortagonalna trajektorija, če krivulje iz družine sekajo γ pod pravim kotom

$2yy' = 2a \rightsquigarrow y^2 = 2yy'x$

v (x_0, y_0) velja $x_0 2 y_0 y' = y_0^2$

$y' = \frac{y_0}{2x_0}$ smerni koeficient trajektorije $-\frac{2x_0}{y_0}$

$y' = f(x,y)$ družina krivulj $\Rightarrow -\frac{1}{y'} = f(x,y)$ družina ort. t.

$y^2 = 2yy'x \rightsquigarrow y^2 = -\frac{2yx}{y'}$ iščemo y

1 možnost $y \equiv 0$ ✓

2. možnost

$y' = -\frac{2x}{y} = \frac{dy}{dx}$

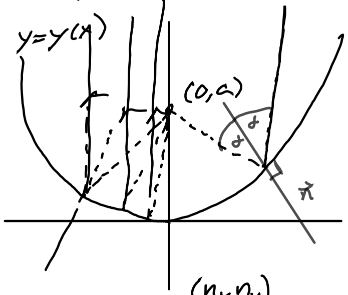
$2x\,dx = y\,dy \;/\int$

$-\frac{2}{2}x^2 = \frac{1}{2}y^2 + C$

$x^2 + \frac{1}{2}y^2 + C = 0$

$x^2 + \frac{1}{2}y^2 = -C$ elipse

3) Pri izdelavi žarometa mora biti odbojna površina ki obdaja žarnico take oblike, da vse svetlobne žarke odbije v isto smer. Predpostavimo, da je odbojna površina rotacijske ploskve grafa f_n



(n_x, n_y)

$$(-x,-y(x)+a)\cdot\vec{n} = -xn_x - yn_y - an_y = \cos\alpha\sqrt{x^2+(y+a)^2}\sqrt{n_x^2+n_y^2}$$

$$\vec{n} = \left(1, -\frac{1}{y'}\right)$$

$$\cos\alpha\sqrt{x^2+(y+a)^2}\sqrt{1+\frac{1}{y'^2}} = -x + \frac{-y-a}{y'}$$

$$\cos\alpha = \frac{-xy'-y-a}{\sqrt{x^2+(y+a)^2}\sqrt{1+y'^2}}$$

$$\vec{n}\cdot(0,1) = \cos\alpha\,|\vec{n}|\cdot 1$$

$$n_y\,\cancel{n_x} = \cos\alpha\sqrt{1+\frac{1}{y'^2}}$$

$$n_y\,\cancel{n_x} = \frac{\cancel{\frac{1}{y'}}(-xy'-y-a)}{\sqrt{x^2+(y+a)^2}} = \frac{1}{\cancel{y'}}$$

???

$$-xy'+y-a = \sqrt{x^2+(y+a)^2}$$

$z = y-a$

$z' = y'$

$$\cancel{+}xz'+z = \sqrt{x^2+z^2}+z$$

??

$$z' = \frac{\sqrt{x^2+z^2}}{\cancel{m}x} = \sqrt{\frac{x^2+z^2}{x^2}} = \sqrt{1+\frac{z^2}{x^2}}+\frac{z}{x}$$

$u = \frac{z}{x}$

$z' = xu'$

$$\cancel{u}+xu' = \sqrt{1+u^2}+\cancel{u}$$

$$\frac{u'}{\sqrt{1+u^2}} = \frac{1}{x}$$

$v = \sqrt{1+u^2}$

$$dv = \frac{u\,du}{\sqrt{1+u^2}} = \frac{1-v^2}{\sqrt{1+u^2}}\,du$$

$$\ln x = \int\frac{1-v^2}{v}\,dv = \ln v - \frac{1}{2}v^2 + C$$

???

$$y = a + \frac{c}{2}x^2 - \frac{1}{2c}$$

Cauchy naloge

$y' = f(y) \qquad y(x_0) = y_0$

a) 9nlo?. ∃ reši DE za poljubni začetni pogoj

b) f Lipšicova kdaj reševanje iz a enolična

$$\frac{dy}{dx} = f(y)$$

če $f(y) \neq 0$

$$\frac{dy}{f(y)} = dx \quad / \int$$

Kaj pa če je $f(y_0) = 0$

Poskusimo $f(y) \equiv x_0$

$$F(y) = x + C$$

$y'(x) = 0$

$$\int \frac{d\cancel{y}}{f(\cancel{y})}$$

$f(y(x)) = f(y_0) = 0$

$$y'(x) = f(y(x))$$

$$y(x) = y(x_0) + \int_{x_0}^{x} y'(t)\,dt$$

daš vse na eno stran in integriraš

$$\underbrace{\int \frac{y'(t)\,dt}{f(y(t))}} = \int_{x_0}^{x} 1\,dt = x - x_0$$

$u = y$

$du = y'\,dt$

$$\int_{y_0}^{y(x)} \frac{du}{f(u)} = F(y(x)) - F(y_0) = x - x_0$$

$$F(y(x)) = x - x_0 + F(y_0)$$

$$y(x) = F^{-1}(x - x_0 + F(y_0))$$

$$y(x_0) = F^{-1}(F(y_0)) = y_0$$

$$y'(x) = \frac{1}{F'(F^{-1}(x - x_0 + F(y_0)))} = f(\cancel{x - x_0 + F(y_0)})$$

$$= f(y(x)) \quad ???$$

$$F'(F^{-1}(x)) = \frac{1}{(F^{-1})'(x)}$$

b) enoličnost

$$\int_{y_0}^{y} \frac{dy}{f(y)} = x - x_0$$

$$|f(x) - f(y)| \leq C|x-y|$$

recimo da $f(x)=0$ $\longrightarrow$ $|f(y)| \leq C|x-y|$

$f \geq 0$ $y > y_0$, $f(y_0) = 0$

$$\int_{y_0}^{y} \frac{dy}{f(y)} \geq \frac{dy}{C|y-y_0|} = \infty$$

Recimo da y reši DE da je

$f(y_1) = 0$ in y ni konstanten

$x_1 > x_0 : f(y(x_1)) > 0$ BŠZ

potem je y blizu x_1 zakoreta

$$\int_{x_1}^{x} \frac{y'(x)dx}{f(y(x))} = x - x_1$$

$$\| $$

$$\int_{y_1}^{y} \frac{dy}{f(y)}$$

izberemo si $x_2 \in [x_0, x_1]$, da velja

$f(y(x_2)) = 0$ in $f(y(x)) > 0$ za $\forall x \in (x_2, x_1]$

$$\lim_{x \to x_2} \int_{x_1}^{x} \frac{y'(x)}{f(y(x))} = \lim_{x \to x_2} x - x_1 = x_2 - x_1$$

$$= \lim_{y \to y_2} \int_{y_1}^{y} \frac{dy}{f(y)} \geq \lim_{y \to y_2} \int_{y_1}^{y} \frac{dy}{C(y-y_0)} = \infty$$

$\times$

$y' = \frac{f(y)}{g(x)}$ f,g brez ničel

a) Pokaži da vsaka rešitev da tokovno krivuljo polja $v(x,y) = (g(x), f(y))$

b) Dokaži za poljuben začetni pogoj dobimo natanko eno rešitev

Tokovnica ali integralna krivulja je krivulja, ki ima v vseh točkah odvod enak vektorskemu polju

$$F(y)-F(y_0) = \int_{y_0}^{y} \frac{dy}{f(y)} = t - t_0 \qquad x = G^{-1}(t-t_0 + G(x_0))$$

$$G(x)-G(x_0) = \int_{x_0}^{x} \frac{dx}{g(x)} = t - t_0 \rightsquigarrow$$

$\gamma = (x(t), y(t))$

$\dot{\gamma}(t) = V(x(t), y(t))$

$\dot{x}(t) = g(x)$

$\dot{y}(t) = f(y)$

$\frac{\partial}{\partial t} y(x(t)) =$

$= \frac{\partial y}{\partial t}(x(t))\, \dot{x} =$

$= \frac{f(y(t))}{g(x(t))} \frac{\partial x}{\partial t}$

$\rightsquigarrow \dot{x} = g(x)$???