Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$$

$FV(f)$ konvergira k f, če je f zvezna v x,
če pa ni, pa konvergira k $\frac{f(x^-)+f(x^+)}{2}$

$$\boxed{\begin{aligned}\int x\cos(nx)\,dx &= \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C\\ \int x\sin(nx)\,dx &= -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C\end{aligned}}$$

f soda $\Rightarrow b_n = 0$
f liha $\Rightarrow a_n = 0$

$$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$$

$$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$$

(2)

$f(x) = |x|$ razvij v Fourierovo vrsto in seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\,dx \underset{\text{sodost}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(nx) \overset{\text{lihost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2}\cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi}\cdot\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{1}\frac{1}{(2k+1)^2} = -\frac{\pi}{2}\cdot\frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad =? \quad S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

$S \dots$ lih.

$S' \dots$ ostalo

$$= S + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) =$$

$$= S + \frac{1}{4}\Sigma = \Sigma$$

$$S = \frac{3}{4}\Sigma$$

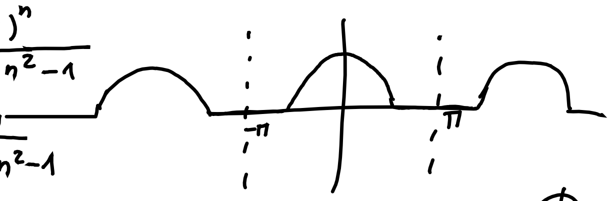
$$\Sigma = \frac{4}{3}S = \frac{\pi^2}{6}$$

③

$f(x) = \max(\cos x, 0)$

$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$

$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$



$a_0 = 2\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(x) = \frac{1}{\pi}\sin(\frac{\pi}{2}) = \frac{1}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

ničelnost $f(x)$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos(nx) \cancel{+2\int_{\frac{\pi}{2}} \cos(nx)\,dx}$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos((n+1)x) + \cos((n-1)x)) =$

$n \neq 1$

$\frac{1}{\pi}\left[\frac{1}{n+1}\sin((n+1)\frac{\pi}{2}) + \frac{1}{n-1}\sin((n-1)\frac{\pi}{2})\right]$

$n = 4k: \frac{1}{\pi}\frac{1}{n+1}\cdot 1 + \frac{1}{n-1}(-1)\frac{1}{\pi} = \frac{-2}{n^2-1}\cdot\frac{1}{\pi} \quad \frac{-2}{16k^2-1}\cdot\frac{1}{\pi}$

n lih: 0

$n = 4k+2: \frac{1}{\pi}\frac{1}{4k+3}\sin\frac{3\pi}{2} + \frac{1}{\pi}\frac{1}{4k+1}\sin\frac{\pi}{2} =$

$= \frac{1}{\pi}\left(-\frac{1}{4k+3} + \frac{1}{4k+1}\right)$

$= \frac{1}{\pi}\begin{cases} 0 & ;\ n \text{ liho} \\ \frac{(-1)^{m+1}\cdot 2}{(2n)^2-1} & ;\ n = 2m \end{cases}$

$a_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{1}{\pi} + \frac{\sin(2x)}{2}\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}$

$b_n = 0$ ker f soda

$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\cdot 2}{(2m)^2-1}\cos(2mx)$

$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}S_1 \Rightarrow$

$S_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)\cdot\frac{\pi}{2}(-1) = \frac{1}{2} - \frac{\pi}{4}$

$f(\frac{\pi}{2}) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2-1}\cdot\underbrace{\cos(m\pi)}_{(-1)^m} =$

$= \frac{1}{\pi} - \frac{2}{\pi}\underbrace{\sum_{m=1}^{\infty} \frac{1}{4m^2-1}}_{S_2}$

$S_2 = -\frac{1}{\pi}\cdot\frac{\pi}{2} = -\frac{1}{2}$

④

$f(x)=x^2 \qquad f:[0,\pi]\longrightarrow \mathbb{R}$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v sinusno FV in skiciraj njen graf

1) če f razširimo do sode funkcije

$f_s:[-\pi,\pi]\longrightarrow \mathbb{R}$

$x<0 \Rightarrow f_s(x)=f(-x)$

$FV_{cos}(f)(x)=FV(f)(x)$

2) $f_l:[-\pi,\pi]\longrightarrow \mathbb{R} \qquad f_l(x)=-f(-x)$

$FV_{sin}(f)(x)=FV(f_l)(x)$

1) $f_s(x)=x^2$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{1}{\pi}\int_0^{\pi}x^2\,dx=\frac{\pi^2}{3}$$

$b_n=0$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos(nx)\,dx$$

Poglejmo:

$$\int x^2 e^{inx} =$$

$$\boxed{\int xe^{inx}=-\frac{i\cdot x\cdot e^{ix}}{n}+\frac{e^{inx}}{n^2}}$$

$u=x^2 \qquad dv=e^{inx}dx$

$du=2x\,dx \qquad v=\frac{1}{in}e^{inx}$

$$=x^2\cdot\frac{e^{inx}}{in}-\int 2x\,\frac{e^{inx}}{in}\,dx \overset{\text{od prej}}{=}$$

$$=x^2\cdot\frac{e^{inx}}{in}-\frac{2}{in}\left(\frac{ixe^{inx}}{n}+\frac{e^{inx}}{n^2}\right)+C$$

$$=e^{inx}\left(\frac{ix^2}{-n}+\frac{2x}{n^2}+\frac{2i}{n^3}\right)$$

$e^{inx}=\cos nx+i\sin nx$

$$\int x^2\cos(nx)\,dx=\frac{x^2}{n}\sin(nx)+\frac{2x}{n^2}\cos(nx)-\frac{2}{n^3}\sin(nx)$$

$$\int x^2\sin(nx)\,dx=\frac{-x^2}{n}\cos x+\frac{2x}{n^2}\sin(nx)+\frac{2}{n^3}\cos(nx)$$

$$a_n=\frac{1}{\pi}\left(\frac{x^2}{n}\sin(nx)+\frac{2x}{n^2}\cos(nx)+\frac{2}{n^3}\sin(nx)\right)\Big|_{-\pi}^{\pi}=$$

$$=\frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right)=\frac{4(-1)^n}{n^2}$$

$$FV\cos(f)(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(x$$

b) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\text{soda}}{f_p(x)\sin(x)} = \frac{2}{\pi} \int_0^{\pi} \underbrace{f_p(x)\sin(nx)}$

$= f(x)$ na $[0,\pi] = x^2$

$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2}{n^2}\sin nx + \frac{2}{n^3}(\cos(nx))\right]\Big|_0^{\pi}$

$= \frac{2}{\pi}\left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right)$

$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left((-1)^{n+1}\frac{\pi^2}{2} + \frac{2(1-(-1)^n)}{n^3}\right)\sin(nx)$

$f(x) = x(\pi - x)$ [illegible]

(6)

$f(x) = \sin^3 x$ razvij v FV

Pred premislek:

$f(x) = \sin 2x$ je že FV

$b_2 = 1$, ostali so 0

$$f(x) = \sin^2 x = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right) =$$

$$= \frac{1}{2}\left(\cos(0) - \cos(2x)\right) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2}\sin x - \frac{1}{2}\cos 2x \sin x =$$

$$= \frac{1}{2}\sin x - \frac{1}{2}\cdot\frac{1}{2}\left(\sin(3x) - \sin(x)\right) =$$

$$= \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

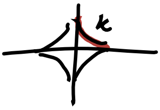
polinom

$\forall p(\sin x, \cos x)$ ima končno furierove vrsto

$p \in \mathbb{R}[x,y]$

$a>0$ Izračunaj težišče homogenega

loka astroide



$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

$$x_T=\frac{\int_K x\,dm}{m(K)}=\frac{\int_K x\,\rho\,ds}{\int_K \rho\,ds}\quad = l(K)$$

$\vec{r}(t)$ ... parametrizacija

$ds=|\dot{\vec{r}}(t)|\,dt=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt$

u ... skalarno polje

$$\int_K u\,ds=\int_a^b u(\vec{r}(t))\,|\dot{\vec{r}}(t)|\,dt$$

$x=a\cos^3 t$
$y=a\sin^3 t$

$t\in[0,\frac{\pi}{2}]$ ← ker je vse pozitivno (gledamo pravilok)

$\dot{\vec{r}}(t)=(-3a\cos^2 t\cdot\sin t,\ 3a\sin^2 t\cos t)$

$|\dot{\vec{r}}|=3a\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}=$
$=3a\cos t\sin t$

$$l(K)=\int_0^{\frac{\pi}{2}}|\dot{\vec{r}}(t)|\,dt=\int_0^{\frac{\pi}{2}}3a\cos t\sin t\,dt=$$

$u=\sin t\,dt$
$du=\cos t\,dt$

$$=3a\int_0^1 u\,du=\frac{3}{2}a$$

$$\int_K x\,ds=\int_0^{\frac{\pi}{2}}a\cos^3 t\;3a\cos t\sin t\,dt=$$

$\cos t=u \qquad du=-\sin t$

$$=3a^2\int_0^1 u^4\,du=\frac{3}{5}a^2$$

$y_T=\frac{3}{5}a^2$

$a>0 \quad \alpha\in[0,2\pi]$

$K = S(0,a)\cap\Pi$

$\Pi: y = x\tan\alpha$

$I: \int_K (y-z) + (z-x)dy + (x-y)dz$

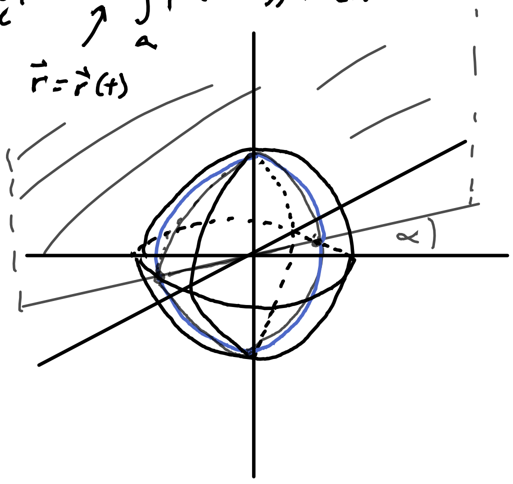
$\vec f(x,y,z) = (y-z,\ z-x,\ x-y)$ vektorsko polje

$I = \int_K \vec f\, d\vec r$

$\vec r = (x,y,z) \Rightarrow d\vec r$

$\int_K \vec f\, d\vec r = \int_a^b \vec f(\vec r(t))\, \dot{\vec r}(t)\, dt$

$\vec r = \vec r(t)$



Parametrizacija K

$x = a\cos\alpha\cdot\cos\vartheta$

$y = a\sin\alpha\cos\vartheta$

$z = a\sin\vartheta$

$r(\vartheta) = (a\cos\vartheta\cos\alpha,\ a\cos\vartheta\sin\alpha,\ a\sin\vartheta)$

$\vartheta\in[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\dot{\vec r}(\vartheta) = (-a\sin\vartheta\cos\alpha,\ -a\sin\vartheta\sin\alpha,\ a\cos\vartheta)$

$\vec f(\vec r(\vartheta)) = a(\cos\vartheta\sin\alpha - \sin\vartheta,\ \sin\vartheta - \cos\vartheta\cos\alpha,$
$\cos\vartheta\sin\alpha - \cos\vartheta\cos\alpha)$

$\vec f\cdot\dot{\vec r} = a^2(-\cos\alpha\cos\cancel{\vartheta\sin\alpha\sin\vartheta} + \sin^2\vartheta\cos\alpha\ \cancel{\cos}$

$-\sin^2\vartheta\sin\alpha + \cos\vartheta\cancel{\cos\alpha\sin\vartheta}\sin\alpha +$

$+\cos^2\vartheta\sin\alpha - \cos^2\vartheta\cos\alpha) =$

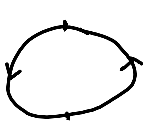
$= a^2(\cos 2\vartheta\sin\alpha - \cos 2\vartheta\cos\alpha)$ ✗

Ne

$= a^2(\cos\alpha - \sin\alpha$

$\int_K \vec f\, d\vec r = \int_0^{2\pi} a^2(\cos\alpha - \sin\alpha)\, d\vartheta =$

$2\pi a^2(\cos\alpha - \sin\alpha)$

↑ pri orientaciji 

$a, b, c \in \mathbb{R}$

$$I = \int_S a\,dy\,dz + b\,dx\,dz + c\,dx\,dy$$

$S = \{(x,y,z);\ x^2+y^2+z^2 = 1,\ x,y,z \geq 0\}$

$$I = \int \vec{f}\, d\vec{S}$$

$\vec{f} = (a,b,c)$

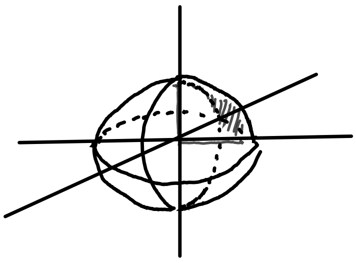
enotske normale

$\int_S (\vec{f}\cdot\vec{N})\,dS$ orientacija ploskve je usmerjena s smerjo normale

$$\int_S \vec{f}\,d\vec{S} = \int_D \vec{f}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv$$

$r = \vec{r}(u,v)$

smer se ujema s predpisano orientacijo



sferične koordinate

$x = \cos\vartheta \cos\varphi \qquad \vartheta \in [0, \frac{\pi}{2}]$

$y = \cos\vartheta \sin\varphi \qquad \varphi \in [0, \frac{\pi}{2}]$

$z = \sin\vartheta$

$\vec{r}(\vartheta,\varphi) = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, \sin\vartheta)$

$\vec{r}_\vartheta = (-\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\vartheta)$

$\vec{r}_\varphi = (-\cos\vartheta\sin\varphi, \cos\vartheta\cos\varphi, 0)$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta\cos^2\varphi - \cos\vartheta\sin\vartheta\sin^2\varphi) =$$

$$= (-\cos^2\vartheta\cos\varphi, \cos^2\vartheta\sin\varphi, -\sin\vartheta\cos\vartheta) =$$

$$= -\cos\vartheta\,\underbrace{(\cos\vartheta\cos\varphi, -\cos\vartheta\sin\varphi, \sin\vartheta)}$$

$$\int_S \vec{f}\,d\vec{S} = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (-a\cos^2\vartheta\cos\varphi + b\cos^2\vartheta\sin\varphi - c\sin\vartheta\cos\vartheta)\,d\vartheta$$

$$= \iint_0^{\frac{\pi}{2}} \left((-a\cos\varphi + b\sin\varphi)\cos^2\vartheta - \frac{c}{2}\sin 2\vartheta\right) d\vartheta =$$

Prevedemo na B funkcijo

$$= \int_0^{\frac{\pi}{2}} \left((-a\cos\varphi + b\sin\varphi)\frac{\pi}{4} - \frac{c}{2}\right) =$$

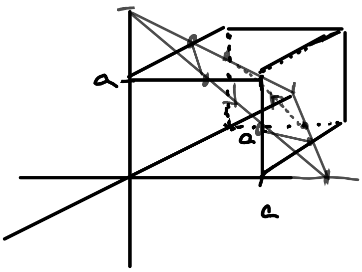
$$= \left(-\frac{\pi a}{4}\sin\varphi + \frac{\pi}{4}b\cos\varphi - \frac{c}{2}\varphi\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4}a - \frac{\pi}{4}b - \frac{\pi}{4}c = -\frac{\pi}{4}(a+b+c)$$

$a > 0$

$$\vec{f}(x,y,z) = (y^2 - z^2,\ z^2 - x^2,\ x^2 - y^2)$$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z = \frac{3a}{2}$



kivulja 6 kotnik

cirkulacija.... integral vektorskega polja vzdolž sklenjene krivulje

K_1:

$x = x$

$y = 0$

$z = \frac{3a}{2} - x$

$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$

$$\int_{K_1} \vec{f} dr$$

$\dot{\vec{r}}(x) = (1, 0, -1)$

$\vec{f}(r(x)) = (0 - (\frac{3}{2} - x)^2, (\frac{3}{2}a - x)^2 - x^2, x^2 - 0)$

$\vec{f} \cdot \dot{\vec{r}} = -(\frac{3}{2}a - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$

$$\int_{\frac{a}{2}}^{a} \left(-(\frac{3}{2}a - x)^2 - x^2\right) dx =$$

$$-\frac{1}{3}(\frac{3}{2}a - x)^3 \Big|_{\frac{a}{2}}^{a} - \frac{1}{3}x^3 \Big|_{\frac{a}{2}}^{a} =$$

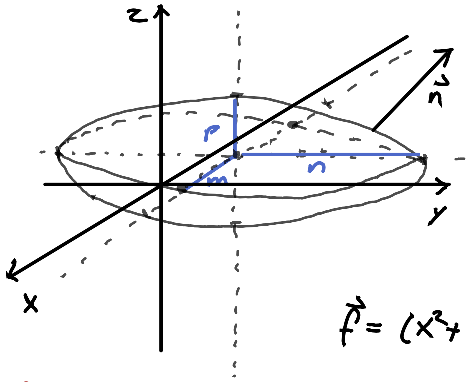
$$= -\cancel{\frac{1}{3}\frac{a^3}{8}} + \cancel{\frac{1}{3}a^3} - \cancel{\frac{1}{3}a^3} + \cancel{\frac{1}{3}\frac{a^3}{8}} = 0$$

$m, n, p > 0$
$a, b, c \in \mathbb{R}$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanja stran te ploskve



$\vec{f} = (x^2 + y^2 + z^2)$

$$\boxed{I = \int_{\substack{S \\ \| \\ \partial D}} \vec{f}\, dS = \int_D \operatorname{div} \vec{f}\, dV}$$

gaussov izrek

Normala mora biti zunanja

$$\int_D (2x + 2y + 2z)\, dV = 2x_T + 2 \cdot y_T + 2 z_T$$

$$x_T = \int_D x\, dV = x_T \cdot V(D) = a V(D)$$

enotska krogla B: $V(B) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

$\Downarrow$

$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3}\pi$$

D dobimo če B raztegnemo za faktor m v smer: x, n, p v drugih dveh

B opišemo:

$x = r\cos\vartheta\cos\varphi$
$y = r\cos\vartheta\sin\varphi$
$z = r\sin\vartheta$

$r \in [0, 1]$

D opišemo:

$x = m r\cos\varphi\cos\vartheta$
$y = n r\cos\vartheta\sin\varphi$
$z = p r\sin\vartheta$

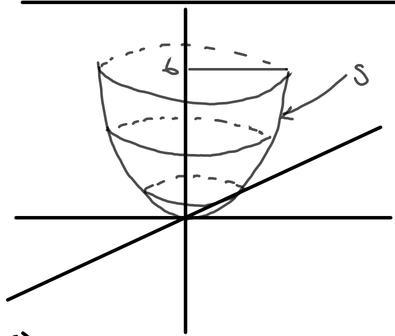
$$I = 2 \frac{4}{3}\pi m \cdot n \cdot p\,(a + b + c)$$

$\vec{f}(\vec{r}) = |\vec{r}|^2 \vec{r}$ $b > 0$

izračunaj pretok $\vec{f}$ skozi

a) rob območja $D = \{x,y,z;\ 2z \geq x^2+y^2;\ z \leq b$

b) ploskev $2z = x^2+y^2;\ z \leq b$



Pretok skozi S:
$\int_S \vec{f}\, d\vec{S}$

a)
$$\int_{\partial D} \vec{f}\, d\vec{S} = \int_D \operatorname{div} \vec{f}\, dV$$

$$\vec{f}(x,y,z) = (x^2+y^2+z^2)(x,y,z) =$$
$$(x(x^2+y^2+z^2),\ y(x^2+y^2+z^2),\ z(x^2+y^2+z^2)) =$$
$$(x^3+y^2x+z^2x,\ x^2y+y^3+z^2y,\ x^2z+y^2z+z^3)$$

$$\operatorname{div}\vec{f} = 3x^2+y^2+z^2 + x^2+3y^2+z^2 + y^2+x^2+3z^2$$
$$= 5x^2+5y^2+5z^2$$

$$\int_D \operatorname{div}\vec{f}\, dV = 5\int_D (x^2+y^2+z^2)\, dV$$

$x = r\cos\varphi$ $r^2 = x^2+y^2 = 2z$
$y = r\sin\varphi$ $z = \frac{r^2}{2}$
$z = z$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^{b} (r^2+z^2)\, r\, dz$$

$$= 5\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \frac{1}{3}z^3\right)\Big|_{\frac{r^2}{2}}^{b} dr$$

$$= 10\pi \int_0^{\sqrt{2b}} \left(b r^3 + \frac{1}{3}b^3 - \frac{r^5}{2} - \frac{1}{6}r^6\right) dr =$$

$$= 10\pi \left(\frac{1}{4}b r^4 + \frac{1}{3}b^3 r - \frac{1}{12}r^6 - \frac{1}{6\cdot 7}r^7\right)\Big|_0^{\sqrt{2b}} =$$

$$= 10\pi \left(\frac{1}{4}b^3\cdot 4 + \frac{1}{3}b^3\sqrt{2b} - \frac{1}{12}8b^3 - \frac{1}{6\cdot 7}8b^3\sqrt{2b}\right)$$

~~narobe~~ narobe

$$= 10\pi\left(b^3 + \frac{b^4}{3} - \frac{2}{3}b^3 - \frac{b^4}{12}\right) =$$
$$10\pi\left(\frac{b^3}{3} + \frac{b^4}{4}\right)$$

b) $\partial D = S \cup S_0$

$$\int_S \vec{f}\, d\vec{S} = \int_{\partial D} \vec{f}\, d\vec{S} - \int_{S_0} \vec{f}\, d\vec{S}$$

$$\int_{S_0} \vec{f}\, d\vec{S} = \int_{S_0} (\vec{f}\cdot\vec{n})\, dS = \qquad \vec{n} = (0,0,1)$$

$$= \int_{S_0} (zx^2+zy^2+z^3)\, dS = \int_{S_0} (bx^2+by^2+b^3)\, dS =$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$= b\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r(r^2+b^2)\, dr = b\left(\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr + \underbrace{\int_{S_0} b^2\, dS}_{P(S_0) = 2\pi b}\right)$$

$$= b\left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr\right) =$$

$$= b\left(2\pi b^3 + 2\pi\,\frac{1}{4}\cdot 4b^2\right) =$$

$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \int_S \vec{f}\, d\vec{S} = \frac{10}{3}\pi b^3 + \frac{10}{3}\pi b^4 - 2\pi b^4 - 2\pi b^3$$

Izračunaj $I = \int_K \frac{x\,dy - y\,dx}{x^2+y^2}$

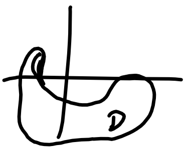
če je $K \subseteq \mathbb{R}^2$ sklenjena krivulja;

a) ki ne obkroži izhodišča

b) ki obkroži izhodišče

Omejimo se na primer: $K = \partial D$ za D odsekoma gladkim robom

a)



$$\boxed{\int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx\,dy}$$

pozitivna orientacija

greenova formula

$P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^4} = \frac{y^2-x^2}{(x^2+y^2)^4}$$

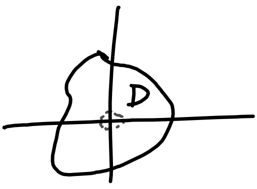
$$P_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\int_D (Q_x - P_y)\,dx\,dy = \int_D 0\,dx\,dy = 0$$

∂D ne obkroži $(0,0) \Leftrightarrow (0,0) \notin D$

$(x^2+y^2) \neq 0$

b)



Iz D izrežemo dovolj majhen krog $K(0,\varepsilon)$

$D' = D - K(0,\varepsilon)$

Uporabimo greenovo formulo na D'

$$\int_{D'} \cancel{(Q\,dx + P\,dy)} = \int_{\partial D'} (Q_x - P_y)\,dx\,dy = 0$$

$\partial D' = \partial D^+ + \partial K(0,\varepsilon)^-$

$$0 = \int_{\partial D^+} P\,dx + Q\,dy + \int_{\partial K^-} P\,dx + Q\,dy$$

Pozor orientacija obratna kot ponavadi ☺

$$\Longrightarrow \int_{\partial D^+} \dots = \int_{\partial K^+} \dots$$

$x = \varepsilon\cos\varphi \qquad y = \varepsilon\sin\varphi$

$dx = -\varepsilon\sin\varphi$

$dy = \varepsilon\cos\varphi$

$$\int_{\partial K} \dots = \frac{1}{\varepsilon}\int_0^{2\pi} \cos^2\varphi + \sin^2\varphi = 2\pi$$

$r_1 \dots r_n \in \mathbb{R}^3$ različni

$e_1 \dots e_n \in \mathbb{R}$

$$f(\vec r) = \sum_{i=1}^{n} \operatorname{grad}\left(\frac{-e_i}{4\pi |r-r_i|}\right)$$

izračunaj pretok $\vec f$ skozi zaključeno ploskev S, ki objame točke $r_1 \dots r_n$

$r_1 \dots r_n \in \operatorname{Int} D$ $\quad S = \partial D$

$$\int_{\partial D} \vec f \, d\vec S = \int_D \operatorname{div} \vec f \, dV$$

gaussov izrek vrjetno ne velja, ker ker ima f singularne točke v D

$$\operatorname{div}(\vec f(r)) = \sum_{i=1}^{n} \operatorname{div}\left(\operatorname{grad}\left(\frac{-e_i}{4\pi |r-r_i|}\right)\right)$$

Spomnimo se: $\operatorname{grad} \frac{1}{|\vec r - \vec a|} = -\frac{\vec r - \vec a}{|\vec r - \vec a|^3}$

in $\operatorname{div}\left(\operatorname{grad} \frac{1}{|\vec r - \vec a|}\right) = 0$

Iz D izrežemo krogle $K_i = K(r_i, \varepsilon)$

Tako da $\bar K_i \subset \operatorname{Int}(D)$ in $K_i \cap K_j = \emptyset$ za $i \neq j$

$D' = D - \left(\bigcup_{i=1}^{n} K_i\right)$

Na D' uporabimo gaussov izrek

$$\int_{\partial D'} \vec f(\vec r)\, dS = \int_{D'} \operatorname{div} \vec f \, dV = 0$$

$$\partial D' = \partial D^+ \cup \left(\bigcup_{j=1}^{n} \partial K_j^-\right) \Rightarrow \int_{\partial D'} = \int_{\partial D^+} \dots + \sum_{i=1}^{n} \int_{\partial K_i^-} \dots = 0$$

$$\Rightarrow \int_{\partial D^+} \dots = \sum_{i=1}^{n} \int_{\partial K_i^+} \dots$$

$$\int_{\partial K_i} \vec f \, d\vec S = \int_{K_i} \sum_{j=1}^{n} \operatorname{grad}\left(\frac{-e_j}{4\pi |r-r_j|}\right) \cancel{dS}$$

$$\sum_{j=1}^{n} \int_{\partial K_i} \left(\frac{+e_j}{4\pi} \frac{r-r_j}{|\vec r - \vec r_j|^3}\right) dS$$

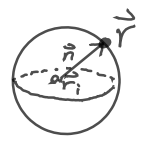
$$\int_{\partial K_j} \frac{\vec r - r_j}{|\vec r - \vec r_j|^3} dS \overset{i \neq j}{\Longrightarrow} r_j \in \bar K_i \Rightarrow$$

lahko uporabimo gaussov izrek

$\Longrightarrow \; = 0$

$$\sum_{j=1}^{n} \int_{\partial K_i} \frac{e_j}{4\pi} \frac{\vec r - \vec r_j}{|\vec r - \vec r_j|^3} d\vec S = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{\vec r - \vec r_i}{|\vec r - \vec r_i|^3} d\vec S$$

$d\vec S = \vec n \, dS$



$\vec n = \frac{\vec r - \vec r_i}{|\vec r - \vec r_i|}$

$$= \frac{e_i}{4\pi} \int_{\partial K_i} \frac{|\vec r - \vec r_i|^2}{|\vec r - \vec r_i|^4} dS = \frac{e_i}{4\pi} \int_{\partial K_i} \frac{dS}{|r - r_i|^2} =$$

na ∂K_i je $|r - r_i| = \varepsilon$

$$= \frac{e_i}{4\pi} \int \frac{dS}{\varepsilon^2} =$$

$$\frac{e_i}{4\pi\varepsilon^2} P(\partial K_i) = \frac{e_i}{4\pi\varepsilon^2} 4\pi\varepsilon^2 = e_i$$

$$\Rightarrow \int f \, d\vec S = \sum_{i=1}^{n} e_i$$

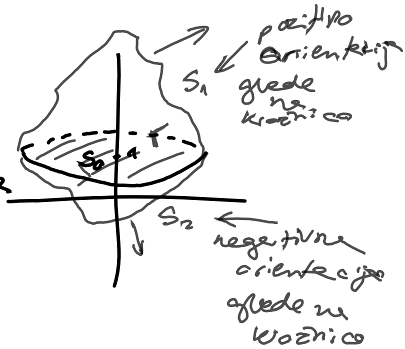
$$I = \int_S (1+x^2) f(x)\, dy\,dz - 2xy\, f(x)\, dz\,dx - 3z\, dx\,dy$$

Določi: $f \in C^1$ da I enak ~~za~~ za vse ploskve S, katerih rob je krožnica $\{(\cos t, \sin t, 1)\}$

Določi to vrednost za I

$$\vec f = ((1+x^2) f(x),\ 2xy\, f(x),\ -3z)$$

$$\Longrightarrow I = \int_S \vec f\, d\vec S$$



$S_1 \cup S_2$ je sklenjena ploskva

$S_1^+ \cup S_2^- = \partial D^+$

Gauss na D:

$$\int_{\partial D^+} \vec f\, d\vec S = \int_D \operatorname{div} \vec f\, dV$$

$$\partial D = \int_{S_1} - \int_{S_2} = 0 \Longleftarrow \operatorname{div} \vec f\, dV = 0$$

$\vec f$ je solenoidalno ($\operatorname{div} \vec f = 0$) $\Longleftrightarrow$ $f(x)$ je oblike $3\arctan x + C$

$$\int_S \vec f\, d\vec S = \int_S \big((1+x^2)(3\arctan x + C),\ -2xy(3\arctan x + C),\ -3z\big)\, d\vec S$$

$$= \int_{S_0} \vec f\, dS =$$

samo krog: $d\vec S = \vec n \cdot dS \quad \vec n = (0,0,1)$

$$= \int_{S_0} -3z\, dS = -3 \int_{S_0} z\, dS = -3 z_T\, P(S_0) =$$

$$= -3 \cdot 1 \cdot \pi$$

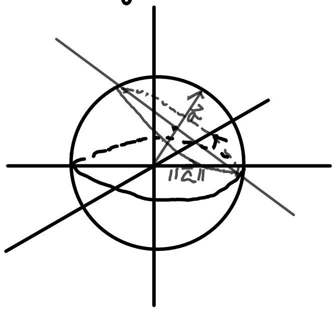
Opomba: Lahko bi dokazali, da je tadi $\operatorname{div}\vec{F}$ potreben pogoj

$\vec a \neq \vec 0$

$\vec b \in \mathbb{R}$

$\vec F(\vec r) = (\vec r - \vec a) \times (\vec r - \vec b)$

izračunaj cirkulacijo $\vec F$ vzdolž krivulje $K: |\vec r| = |\vec a| \wedge \vec r \cdot \vec a = \frac{|\vec a|^2}{2}$



enačba ravnine:

$\vec r \cdot \vec n = d$

normala

ravnina z normalo $\vec a$ in točka $\vec r = \frac{\vec a}{2}$

K je krožnica

B naj bo krog z robom K $K = \partial B$

$$\int_{\partial B} \vec f \, d\vec r = \int_B \operatorname{rot} \vec f \, d\vec S$$

Stokesov izrek

normala: B: $\vec n = \frac{\vec a}{\|\vec a\|}$

$$\operatorname{rot} \vec f = \operatorname{rot}(\underbrace{\vec r \times \vec r}_{0} - a \times r - \vec r \times \vec b + \vec a \times \vec b) =$$

$$\boxed{\text{Vemo: } \operatorname{rot}(\vec r \times \vec c) = -2\vec c}$$

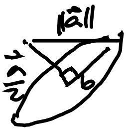
$$= -2\vec a + 2\vec b + 0 = 2(\vec b - \vec a)$$

$$I = \int_B 2(\vec b - \vec a) \, d\vec S = \frac{2}{\|\vec a\|} \int_B (\vec b - \vec a) \cdot \vec a \, dS =$$

$$= \frac{2(\vec b - \vec a) \cdot \vec b}{\|\vec a\|} \cdot P(B)$$

$P(B) = ?$

polmer:



$$b = \sqrt{\|a\|^2 - \frac{\|a\|^2}{4}} = \frac{\sqrt 3}{2} \|a\|$$

$$P(B) = \pi b^2 = \pi \cdot \frac{3}{4} \|a\|$$

$$I = 3\pi (\vec b - \vec a) \vec b \|a\|$$

ni isti $\vec b$ kot b

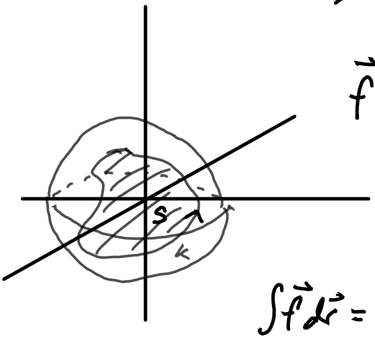
3.4

K zaključena krivulja
na sferi $x^2+y^2+z^2=1$

Izračunaj $I = \int_K \frac{dx+dy+dz}{(x^2+y^2+z^2)^2} = \int_K \vec f \, d\vec r$

$$\vec f = \left(\frac{1}{(x^2+y^2+z^2)^2}, \frac{1}{(x^2+y^2+z^2)^2}, \frac{1}{(x^2+y^2+z^2)^2} \right)$$

$\vec f(\vec r) = (1,1,1)$ za $\vec r \in \vec S$



$K = \partial S$

$$\int_K \vec f \, d\vec r = \int_S \mathrm{rot}\, \vec f \, d\vec S = \ldots$$

obstaja boljši način

$$\int_K \vec f \, d\vec r = \int_K \underbrace{(1,1,1)}_{\vec g(\vec r)} d\vec r = \int_S \mathrm{rot}(\vec g(\vec r)) \, d\vec S =$$

$$= \int_S 0 \, d\vec S = 0$$

Kaj če $K \neq \partial S$

če je polje $\vec f$ potencialno ($\vec f = \mathrm{grad}\, u$) in K krivulja z začetkom v a in koncem v b $\Rightarrow \int_K \vec f \, d\vec r = u(b) - u(a)$

Posledica: če je K sklenjena je integral vzdolž K od $\int_K \mathrm{grad}\, u \, d\vec r = \vec 0$

$\vec f$ ni potencialno, ampak $\vec g$ je potencialno (ker $\mathrm{rot}\, \vec g = 0$)

$\vec g = \mathrm{grad}(x+y+z)$ $\quad I = \int_K \vec g \, d\vec r = 0$

Opomba: $\int_K h(x^2+y^2+z^2)(dx+dy+dz)$

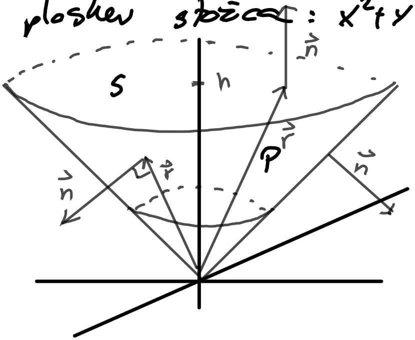
$\downarrow$ na S,

konstantno polje

$h > 0$

$\vec{f}(\vec{r}) = \vec{r}$

Izračunaj pretok $\vec{f}$ skozi plašč in osnovno ploskev stožca: $x^2+y^2 \leq z^2 \quad z \in [0, h]$



$\vec{r} \cdot \vec{n} = 0$

$$\int_P \vec{f}\, d\vec{S} = \int_P \vec{f}\,\vec{n}\, dS = \int_P \vec{r}\,\vec{n}\, dS = \int 0\, dS = 0$$

$$\int_S \vec{f}\, d\vec{S} = \int_S \vec{r}\,\vec{n}\, dS = \int_S z\, dS = z_T \cdot P(S) =$$

$$\vec{n} = (0,0,1) \qquad = h \cdot \pi h^2 = \pi h^3$$

2. način (z gaussom)

$$\int \vec{f}\, dS = \int_D \operatorname{div} \vec{f}\, dV = \int_D (1+1+1)\, dV = 3P(V) =$$

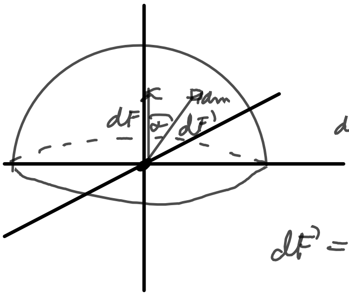
$$= 3 \cdot \frac{\pi h^2 \cdot h}{3} = \pi h^3$$

$$\int_{\partial D} = \int_S + \int_P$$

$a > 0$

Določi privlačno silo med točkastim telesom z maso m_0, ki je v izhodišče in ploskvijo $x^2+y^2+z^2=a^2$ $z \geq 0$ z maso M

homogena



$$F = \frac{m_1 m_2}{r^2} G$$

$dm = \rho dS$

$$dF' = \frac{m_0 dm}{r^2} G = \frac{m_0 \rho dS}{r^2} G$$

Rezultanta bo kazala gor torej nas zanima samo z komponenta

$dF = dF' \cos\alpha$

$x = a\cos\vartheta\cos\varphi$ $\quad \alpha = \frac{\pi}{2} - \vartheta$

$y = a\cos\vartheta\sin\varphi$

$z = a\sin\vartheta$

$$dF = \frac{\overset{\xi}{G} m_0 \rho}{r^2} \cos\left(\frac{\pi}{2} - \vartheta\right) dS$$

$$dS = a^2 \cos\vartheta \, d\varphi \, d\vartheta$$

od prej z $\sqrt{EG - F^2} = a^2 \cos\vartheta$

$$F = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \frac{G m_0 \rho}{a^2} \sin\vartheta \, a^2 \cos\vartheta \, d\vartheta =$$

$$= G m_0 \rho \, 2\pi \int_0^{\frac{\pi}{2}} \sin\vartheta\cos\vartheta \, d\vartheta =$$

$$G m_0 \rho \, \pi \int_0^{\frac{\pi}{2}} \sin 2\vartheta \, d\vartheta = \frac{\pi}{2} G m_0 \rho \cos 2\vartheta \Big|_0^{\frac{\pi}{2}} =$$

$$= \pi G m_0 \rho = \frac{G m_0 \cdot M}{2a^2}$$

$$\rho = \frac{M}{P(S)} = \frac{M}{2\pi a^2}$$

$D \subseteq \mathbb{R}^3$ gaussasto območje (območje z gladkim robom) in prostornino V

$a \in \mathbb{R}^3$

Izračunaj:

$$\int_{\partial D} (\vec{r} \times \vec{a}) \times d\vec{S} = I$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{y} \cdot \vec{z})\vec{x}$$

$$I = \int_{\partial D} (\vec{r} \cdot d\vec{S})\vec{a} - (\vec{a} \cdot d\vec{S})\vec{r} =$$

$$d\vec{S} = \vec{n} \cdot dS$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n})\vec{a} - (\vec{a} \cdot \vec{n})\vec{r})\, dS =$$

$a = (a_1, a_2, a_3)$

$r = (x, y, z)$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n})\, dS)(a_1, a_2, a_3) - (\vec{a}\vec{n}\, x, \vec{a}\vec{n}\, y, \vec{a}\vec{n}\, z)\, dS$$

$$= \left(\int_{\partial D} (\vec{r} \cdot \vec{n})\, dS\right)\vec{a} - \int (\underbrace{\vec{a}\vec{n}\, x\, dS}_{x\vec{a}\cdot\vec{n}\, dS}, \vec{a}\vec{n}\, y\, dS, \vec{a}\vec{n}\, z\, dS)$$

$$\left(\int x\vec{a} \cdot \vec{n}\, dS, \int y\vec{a}\vec{n}\, dS, z\vec{a}\vec{n}\, dS\right)$$

$$\int_{\partial D} (\vec{r}\vec{n})\, dS = \int_{\partial D} \vec{r}\, d\vec{S} = \int_D \operatorname{div} \vec{r}\, dV =$$

$$= 3\int_D dV = 3V$$

$$\int_{\partial D} x\vec{a}\vec{n}\, dS = \int_{\partial D} x\vec{a}\, d\vec{S} = \int_D \operatorname{div}(x\vec{a})\, dV = a_1 V$$

$$\operatorname{div} x\vec{a} = \nabla(xa_1, xa_2, xa_3) = a_1$$

Podobno y in

$$I = 3V\vec{a} - (a_1 V, a_2 V, a_3 V) = 2\vec{a}V$$

Dokaži da je za

$\vec{f} = (2x\cos y - y^2\sin x,\ 2y\cos x - x^2\sin y,\ 4)$

$\int\limits_K \vec{f}\,d\vec{r}$ enak za vse krivulje med $(0,0,0)$ in $(5,3,\pi)$

in izračunaj

$\int\limits_{K_0} \vec{f}\,d\vec{r}$ za $K_0: \vec{r}(t) = (\cos t, \sin t, t)$; $t \in [-2\pi, 2\pi]$

Dokazujemo: $\vec{f}$ je potencialno

(smo že naredili na začetku)

$u = x^2\cos y + y^2\cos x + 4z$

potencial za $\vec{f}$

$\int\limits_{K_0} \vec{f}\,d\vec{r} = u(b) - u(a) = u(1, 0, 2\pi) - u(1, 0, -2\pi)$

$= 4 \cdot 2\pi - (-4 \cdot 2\pi) = 16\pi$

$$I = \int_K u(x,y)(y\,dx + x\,dy)$$

$$\vec{f}(x,y) = y, x)$$

$$\vec{g} = u \cdot \vec{f} = u(x,y)\cdot y, \; u(x,y)\, x)$$

Radi bi da je $\vec{g}$ potencialno polje

$$\vec{g} = \operatorname{grad} v = (v_x, v_y$$

$$v_x = u(x,y)\cdot y \qquad v_y = u(x,y)\cdot x$$

če odvajamo
eno po x enkrat pa po y mora priti enako

$$v_{xy} = u(x,y) + u_y y \qquad v_{yx} = u_x x + u(x,y)$$

$$\rightsquigarrow \quad y u_y = x u_x$$

- Ta pogoj lahko dobimo z greenom
- vložimo v ravnine v $\mathbb{R}^3$

$$t = x \cdot y$$
$$s = \frac{x}{y}$$

$$I = \int_K y u\,dx + x u\,dy + 0\,dz$$

$\vec{h} = (yu, xu, 0)$ je potencialno $\Leftrightarrow \operatorname{rot} \vec{h} = 0$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} = y\frac{\partial}{\partial t} + \frac{1}{y}\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial y} = x\frac{\partial}{\partial t} - \frac{x}{y^2}\frac{\partial}{\partial s}$$

$$x\left(y u_t + \frac{1}{y} u_s\right) - y\left(x u_t - \frac{x}{y^2} u_s\right) = 0$$

$$t u_t + s u_s - t u_t + s u_s = 2 s u_s \qquad s > 0$$

$$u_s = 0 \Rightarrow u = u(t) \;\ldots \text{samo od } t$$

$u = h(t)$ odvisen

$$u = h(x \cdot y) \qquad \text{kjer } h \in C^1$$

$u, l \in C^2$ $\quad D \subseteq \mathbb{R}^3$ z odsekoma gladkim robom

$\vec{e} = \mathbb{R}^3$ enotski

$$\boxed{\text{Smerni odvod: } \frac{\partial u}{\partial \vec{e}} = \operatorname{grad} u \cdot \vec{e}}$$

Opomba: $\frac{\partial u}{\partial (1,0,0)} = \frac{\partial u}{\partial x} \cdot \vec{e}$

$\vec{n}$ ..... zunanja enotska normala za ∂D

a) Dokaži:

$$\int_{\partial D} u \cdot \frac{\partial v}{\partial \vec{n}} \, dS = \int_D (\operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v) \, dV$$

b) $$\int_{\partial D} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) dS = \int_D (u \Delta v - v \Delta u) \, dV$$

$$\int_{\partial D} u \frac{dv}{d\vec{n}} dS = \int_{\partial D} u \operatorname{grad} v \underbrace{\vec{n} \, dS}_{d\vec{S}} = \int_{\partial D} \underbrace{u \operatorname{grad} v}_{\vec{f}} \, d\vec{S}$$

$$\operatorname{div} \vec{f} = \operatorname{div}(u \operatorname{grad} v) = \operatorname{div}(u v_x, u v_y, u v_z) =$$

$$= (u v_x)_x + (u v_y)_y + (u v_z)_z =$$

ko razpišemo

$$= \operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v$$

b)

$$\int_{\partial D} \left(u \frac{dv}{d\vec{n}} - v \frac{du}{\partial \vec{n}}\right) dS = \int_{\partial D} u \frac{dv}{d\vec{n}} dS - \int_{\partial D} v \frac{du}{d\vec{n}} dS =$$

$$= \int_D \cancel{\operatorname{grad} u \cdot \operatorname{grad} v} + u \Delta v - \cancel{\operatorname{grad} v \cdot \operatorname{grad} u} - v \Delta u$$

$f: \mathbb{R}^3 \to \mathbb{R}$ harmonična ($\Delta f = 0$)

Dokaži da za $a>0$ velja:

$$\frac{1}{4\pi a^2}\int_{S(0,a)} f\,dS = f(0,0,0) = I(a)$$

Nasvet: Parametriziraj $S(0,a)$ in nato s pomočjo odvajanja in gaussovega izreka dokaži trditev.

$x = a\cos\vartheta\cos\varphi$ $\quad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = a\cos\vartheta\sin\varphi$ $\quad \varphi \in [0, 2\pi]$

$z = a\sin\vartheta$ $\quad \rightsquigarrow \vec{r}(\varphi,\vartheta)$

$$dS = |r_\varphi \times r_\vartheta|\,d\varphi d\vartheta = a^2\cos\vartheta\,d\varphi\,d\vartheta$$

$$I(a) = \frac{1}{4\pi \cancel{a^2}}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cancel{a^2}\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))}_{g(a,\varphi,\vartheta)}\,d\vartheta$$

$$I'(a) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g}{\partial a}\,d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\,\left(f_x(\vec{r}(\varphi,\vartheta))\cdot x_a + f_y\cdot y_a + f_z\cdot z_a\right)d\vartheta =$$

$$f(x,y,z) = f(a\cos\varphi\cos\vartheta, a\cos\varphi\sin\vartheta, a\sin\vartheta)$$

v $\vec{r}$ se skriva a

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\,(f_x \cos\varphi\cos\vartheta + f_y \cos\vartheta\sin\varphi + f_z \sin\vartheta)\,d\vartheta$$

komponente enotske normale $\vec{n}$ za $S(0,1)$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\,\operatorname{grad} f\cdot\vec{n}\,d\vartheta = \qquad dS = a^2\cos\vartheta\,d\vartheta\,d\varphi$$

dS za $a=1$

$$= \frac{1}{4\pi}\int_{S(0,1)} \operatorname{grad} f\,\vec{n}\cdot dS = \frac{1}{4\pi}\int_{S(0,1)} \operatorname{grad} f\,d\vec{S} =$$

$$= \frac{1}{4\pi}\int_{K(0,1)} \overbrace{\operatorname{div}(\operatorname{grad} f)}^{\Delta f}\,dV = 0$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

$I'(a)$ je $0 \Rightarrow I(a)$ je konstantna

$$I(a) = \frac{1}{4\pi a^2}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\,d\vartheta$$

$$\lim_{a\to 0} I(a) = \frac{1}{4\pi}\lim_{a\to 0}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\,d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{a\to 0} \cos\vartheta\, f(\vec{r}(\varphi,\vartheta))\,d\vartheta =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta\, f(0,0,0)\,d\vartheta =$$

$$= \frac{1}{4\pi} f(0,0,0)\cdot \underbrace{4\pi}_{P(S(0,1))} = f(0,0,0)$$

ker je $\cos\vartheta$ determinanta

Holomorfne funkcije

$D \overset{\text{odp.}}{\subseteq} \mathbb{C}$ območje $\quad f \in \mathcal{O}(D)$.

Katere od f_1, f_2 in f_3 so holomorfne

$$f_1(z) = \overline{f(z)}$$
$$f_2(z) = f(\bar{z})$$
$$f_3(z) = \overline{f(\bar{z})}$$

$$\boxed{\text{holomorfnost: } \forall a \in D.\ \exists \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} =: f'(a)}$$

$$f(x+iy) = u(x,y) + iv(x,y) \qquad u = \operatorname{Re} f \quad v = \operatorname{Im} f$$

$$f \in \mathcal{O}(D) \iff u_x = v_y \ \wedge\ u_y = -v_x$$

$u, v \in C^1(D)$

$$f_1(x+yi) = \underbrace{u(x,y)}_{u_1(x,y)} \underbrace{-iv(x,y)}_{v_1(x,y)}$$

$$u_{1x} = u_x \qquad u_{1y} = u_y$$
$$v_{1x} = -v_x \qquad v_{1y} = -v_y$$
$$u_{1x} = u_x = v_y = -v_{1y}$$
$$u_{1y} = -v_y$$

$$\Longrightarrow \quad v_y = -v_y, \quad v_x = -v_x \quad \text{in} \ \dots$$

???

$$\Downarrow$$

$$v_x = 0 = v_y$$
$$u_x = u_y = 0$$

u, v konstantni

f_1 ni holomorfne

$$f_2(x+yi) = u_2(x,y) + \cancel{u_2}\, v_2(x,y); \quad \left.\right\} f_2 \text{ na } \overline{D}$$

$$\| $$

$$u(x,-y) + v(x,-y)$$

$$\cancel{u_{2x} = u_x \qquad u_{2y} = u_y \cdot (-1) = -u_y}$$

$$\cancel{v_{2x} = v_x \qquad v_{2y} = -v_y}$$

$$u_{2x}(x,y) = u_x(x,-y) \qquad u_{2y}(x,y) = -u_y(x,-y)$$

$$v_{2x}(x,y) = v_x(x,-y) \qquad v_{2y}(x,y) = -v_y(x,-y)$$

$$u_{2x}(x,y) = u_x(x,-y) = v_y(x,-y) = -v_{2y}(x,y)$$

Torej f_2 ni holomorfna, razen če so ~~dve~~ u, v konstanti

$$f_3(x+yi) = u_3(x,y) + iv(x,y) \Rightarrow$$

$$u_3(x,y) = u(x,-y)$$

$$v_3(x,y) = -v(x,-y)$$

$$(u_3)_x(x,y) = u_x(x,-y) \qquad v_{3x}(x,y) = v_x(x,y)$$

$$(u_3)_y(x,y) = u_y(x,-x) \qquad v_{3y}(x,y) = +v_y(x,-y)$$

$$(u_3)_x = (v_3)_y \Leftrightarrow$$

$$u_x(x-y) = +v_y(x,-y) \Leftrightarrow u_x = v_y \quad \checkmark$$

$$u_{3y} = -v_{3x} \Leftrightarrow$$

$$-u_y(x,-y) = -v_x(x,-x) \Leftrightarrow v_y = u_x \qquad \ldots$$

je holomorfna

Dokaži da obstaj $f \in \mathcal{O}(\mathbb{C})$ za katero je $u(x,y) = x^3 - 3xy^2$ in jo določi

$$u_x = 3x^2 - 3y^2 = v_y$$
$$u_y = -6xy = -v_x$$

$$v = 3\int (x^2 - y^2)\,dy = 3x^2y - y^3 + C(x)$$
$$v = 6\int xy\,dx = 3x^2y + D(y)$$

$$v = 3x^2y - y^3 + C$$ "unija členov"

$$f(x+yi) = x^3 - 3xy^2 + i(3x^2y - y^3 + C)$$
$f(z) = z^3 + iC \leftarrow$ z ugibanjem

Brez navodila:

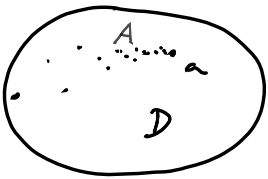
$z = x + yi \Rightarrow x = \frac{z+\bar{z}}{2} \qquad y = \frac{z-\bar{z}}{2i}$

DN: vstavi notri

Imenitna trditev:

$D \subseteq \mathbb{C}$ neko območje, $f, g \in \mathcal{O}(D)$

funkciji ki se ujemata na množici A s stekališčem v D



v A obstaja zaporedje $(a_n)_n$ $a_n \in A$ $a_n \neq a$

$\lim_{n \to \infty} a_n = a$

$\Longrightarrow f = g$

$u = x^3 - 3xy^2 \quad v = 3x^3y - y^3 + iC$

če najdemo $g \in \mathcal{O}(\mathbb{C})$ ki se z f ujema na $\mathbb{R}$, po trditvi T sledi $f = g$

$x \in \mathbb{R} \quad f(x) = f(x + i0) = u(x,0) + v(x,0) = x^3 + iC$

$\Longrightarrow f(z) = z^3 + iC$

Še imenitnejši premislek :

$u(x,y) \rightsquigarrow f = ?$

$x \in \mathbb{R}: f(x) = u(x,0) + i\, v(x,0) \overset{x \to z}{\rightsquigarrow} f(z)$:)

$\checkmark$?

$$v(x,0) = \int v_x(x,0)\,dx \underset{\text{CRS}}{=} -\int u_y(x,0)\,dx$$

$$u_y = -6xy$$

$$v(x,0) = -\int -6x \cdot 0\,dx = +C$$

$$\Longrightarrow f(x) = x^3 + iC \Longrightarrow f(z) = z^3 + iC$$

Podobno

$$u(x,0) = \int u_x(x,0)\,dx = \int v_y(x,0)\,dx$$

$$u(x,y) = e^x(\cos(ky)+\sin(ky))$$

Določi k da je $u = \mathrm{Re} f$ za $f \in \mathcal{O}(\mathbb{C})$

Če je $u = \mathrm{Re} f \Rightarrow$

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \rightsquigarrow \begin{aligned} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \end{aligned} \Rightarrow u_{xx} = -u_{yy}$$

Podobno

$v_{xx} = -v_{yy}$

$\Rightarrow \Delta u = 0$

$$u_{xx} = e^x(\cos(ky)+\sin(ky)) = u$$

$$u_y = e^x(-k\sin(ky)+k\cos(ky))$$

$$u_{yy} = k^2 e^x(-\cos(ky)-\sin(ky)) = -k^2 v$$

$$\Delta u = e^x(1-k^2)(\cos(ky)+\sin(ky)) = 0 \text{ za } \forall x,y$$

$$\Rightarrow k = \pm 1$$

$$v(x,0) = -\int u_y(x,0)\,dx = -\int e^x k\,dx = -ke^x + C$$

$$f(x) = u(x,0) + iv(x,0) = e^x - ike^x + Ci = e^x(1-ik) + iC$$

$$f(z) = e^z(1 \pm i) + iC$$