Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x\,dx$$

$FV(f)$ konvergira k f, če je f zvezna v x, če pa ni, pa konvergira k $\frac{f(x^-) + f(x^+)}{2}$

$$\boxed{\begin{aligned} \int x \cos(nx)\,dx &= \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C \\ \int x \sin(nx)\,dx &= -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C \end{aligned}}$$

f soda $\Rightarrow b_n = 0$

f liha $\Rightarrow a_n = 0$

$$\int x^2 \cos(nx)\,dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx)\,dx = \frac{-x^2}{n} \cos x + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

(2)

$f(x) = |x|$ razvij v Fourierovo vrsto in seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx \underset{\text{sodost}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(nx) \overset{\text{lihost}}{=} 0$$

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2}\cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi}\cdot\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{1}\frac{1}{(2k+1)^2} = -\frac{\pi}{2}\cdot\frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodatno

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \overset{?}{=} S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

$S \dots$ lih.

$S' \dots$ ostalo

$$= S + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) =$$

$$= S + \frac{1}{4}\Sigma = \Sigma$$

$$S = \frac{3}{4}\Sigma$$

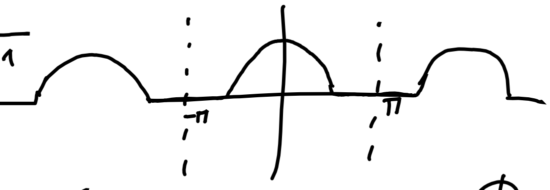
$$\Sigma = \frac{4}{3}S = \frac{\pi^2}{6}$$

(3)

$f(x) = \max(\cos x, 0)$

$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$

$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(x) = \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

ničelnost $f(x)$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos(nx) \cancel{+2\int_{\frac{\pi}{2}} \cos(nx)\,dx}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right) =$$

$n \neq 1$

$$\frac{1}{\pi}\left[\frac{1}{n+1}\sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1}\sin\left((n-1)\frac{\pi}{2}\right)\right]$$

$n = 4k$: $\frac{1}{\pi}\frac{1}{n+1}\cdot 1 + \frac{1}{n-1}(-1)\frac{1}{\pi} = \frac{-2}{n^2-1}\cdot\frac{1}{\pi}$ $\quad \frac{-2}{16k^2-1}\cdot\frac{1}{\pi}$

n lih: 0

$n = 4k+2$: $\frac{1}{\pi}\frac{1}{4k+3}\sin\frac{3\pi}{2} + \frac{1}{\pi}\frac{1}{4k+1}\sin\frac{\pi}{2} =$

$$= \frac{1}{\pi}\left(-\frac{1}{4k+3} + \frac{1}{4k+1}\right)$$

$$= \frac{1}{\pi}\begin{cases} 0 & ; n \text{ liho} \\ \frac{(-1)^{m+1}\cdot 2}{(2n)^2-1} & ; n = 2m \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{1}{\pi} + \frac{\sin(2x)}{2}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}$$

$b_n = 0$ ker f soda

$$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\cdot 2}{(2m)^2-1}\cos(2mx)$$

$$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}S_1 \Rightarrow$$

$$S_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)\cdot\frac{\pi}{2}(-1) = \frac{1}{2} - \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^2-1}\cdot\underbrace{\cos(m\pi)}_{=(-1)^m} =$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\underbrace{\sum_{m=1}^{\infty}\frac{1}{4m^2-1}}_{=S_2}$$

$$S_2 = -\frac{1}{\pi}\cdot\frac{\pi}{2} = -\frac{1}{2}$$

④

$f(x) = x^2 \qquad f: [0, \pi] \longrightarrow \mathbb{R}$

a) Razvij v kosinusno FV in skiciraj graf

b) razvij f v sinusno FV in skiciraj njen graf

1) če f razširimo do sode funkcije

$f_s : [-\pi, \pi] \longrightarrow \mathbb{R}$

$x < 0 \Rightarrow f_s(x) = f(-x)$

$$FV_{\cos}(f)(x) = FV(f)(x)$$

2) $f_l : [-\pi, \pi] \longrightarrow \mathbb{R} \qquad f_l(x) = -f(-x)$

$$FV_{\sin}(f)(x) = FV(f_l)(x)$$

1) $f_s(x) = x^2$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

Poglejmo:

$$\int x^2 \cdot e^{inx} =$$

$$\boxed{\int x e^{inx} = -\frac{i \cdot x \cdot e^{ix}}{n} + \frac{e^{inx}}{n^2}}$$

$u = x^2 \qquad dv = e^{inx}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{in} e^{inx}$

$$= x^2 \cdot \frac{e^{inx}}{in} - \int 2x \frac{e^{inx}}{in}\,dx \overset{\text{od prej}}{=}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \frac{2}{in}\left(-\frac{ixe^{inx}}{n} + \frac{e^{inx}}{n^2}\right) + C$$

$$= \underset{\substack{\| \\ \cos nx + i\sin nx}}{e^{inx}}\left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3}\right)$$

$$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)$$

$$\int x^2\sin(nx)\,dx = \frac{-x^2}{n}\cos x + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)$$

$$a_n = \frac{1}{\pi}\left(\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) + \frac{2}{n^3}\sin(nx)\right)\Big|_{-\pi}^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right) = \frac{4(-1)^n}{n^2}$$

$$FV\cos(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{n^2} \cos(x$$

b) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\text{soda}}{f_p} (x) \sin(x) = \frac{2}{\pi} \underbrace{\int_0^{\pi} f_p(x) \sin(nx)}$

$= f(x)$ na $[0, \pi] = x^2$

$$= \frac{2}{\pi} \left[-\frac{x^2}{n} \cos(nx) + \frac{2}{n^2} \sin nx + \frac{2}{n^3} (\cos(nx)) \right] \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left(-\frac{\pi^2}{n} (-1)^n + \frac{2}{n^3} (-1)^n - \frac{2}{n^3} \right)$$

$$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left((-1)^{n+1} \frac{\pi^2}{2} + \frac{2(1-(-1)^n)}{n^3} \right) \sin(nx)$$

$f(x) = x(\pi - x)$ [illegible]

⑥

$f(x) = \sin^3 x$ razvij v FV

Pred premislek:

$f(x) = \sin 2x$ je že FV

$b_2 = 1$, ostali so 0

$$f(x) = \sin^2 x = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right) =$$

$$= \frac{1}{2}\left(\cos(0) - \cos(2x)\right) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2}\sin x - \frac{1}{2}\cos 2x \sin x =$$

$$= \frac{1}{2}\sin x - \frac{1}{2}\cdot\frac{1}{2}\left(\sin(3x) - \sin(x)\right) =$$

$$= \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

$\forall \underbrace{p}_{\text{polinom}}(\sin x, \cos x)$ ima končno furierovo vrsto

$p \in \mathbb{R}[x, y]$