

(NDE)
Navadna diferencialna enačba je enačba oblike

$$f(x, y, y') = 0 \quad \text{iščemo } y(x) \text{ da velja}$$

$$f(x, y(x), y'(x)) \quad \text{za } \forall x \in D_y$$

(To je v implicitni obliki)

Mi se bomo večinoma ukvarjali s temke je podano v eksplisitni obliki, torej ko je $y' = f(x, y)$

Enačba reda $n \in \mathbb{N}$ je oblike $G(x, y, y', \dots, y^{(n)}) = 0$

Enačba je **avtonomna**, če funkcija G ni odvisna od x . Sicer je neavtonomna.

7.10

1. Za dano družino funkcij poišči pripadajočo DE

$$\left. \begin{array}{l} a) y = ce^x \\ b) y^2 = cx \\ c) y = c(x-c) \end{array} \right\} \begin{array}{l} f(x, y, c) = 0 \\ c \in \mathbb{R} \end{array}$$

$$y' = ce^x \Rightarrow y = y'$$

$$y^2 = cx$$

$$2yy' = c \Rightarrow y^2 = 2yy'x$$

$$y = 2y'x \quad \text{za } y \neq 0$$

$$y' = c$$

$$y = y'(x - c)$$

2) ugotovi rešitev naslednjih DE

$$a) y'' = -y \Rightarrow \{ \alpha \cos x + \beta \sin x \}$$

$$b) y + xy' = \cos x$$

$$c) xy' = ny$$

$$b) (xy)' = \cos x$$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

$$c) xy' = ny$$

$$xy' + y = ny + x$$

$$(xy)' = (n+1)y$$

$$y = x^n$$

Metode izoklin

$$y_c = y = y(x, c) \quad x, c \in \mathbb{R}$$
$$y \in \mathcal{C}^1$$

Izoklina je krivulja vzdolž katere ima vsaka členica družine y enak odvod po x ($y'_c(x)$)

$$I_\alpha = \{ (x, y); y = y_c(x) \text{ potem je } y'_c(x) = \alpha \}$$

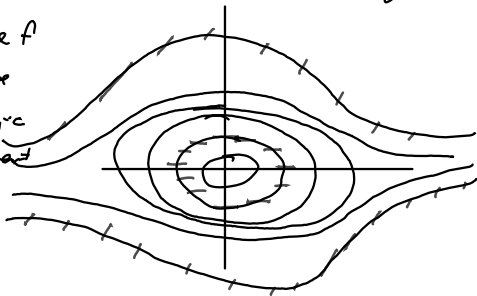
če imamo DE oblike $y' = f(x, y)$ in pripadajočo družino rešitev $y_c(x)$, potem so izokline ravno nivojnice $\{ f = \alpha \}$
tj. $y'_c(x) = f(x, y_c(x)) = \alpha$

Postopek:

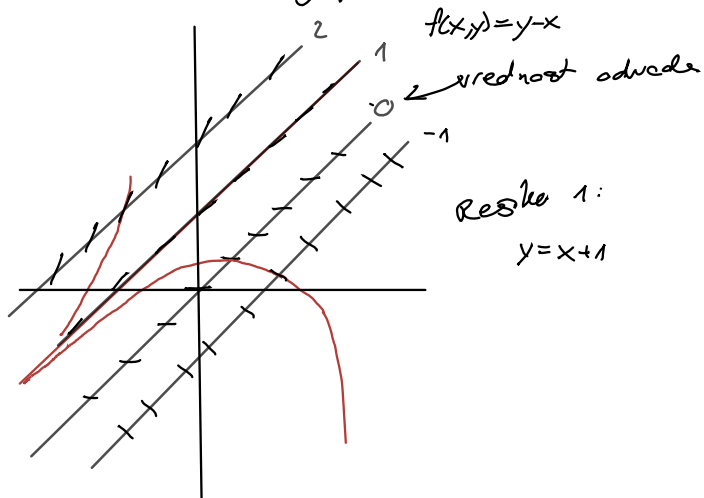
1. skiciramo funkcije f
2. vzdolž vsake nivojnice narišemo nekaj drobcev ki imajo smer in kosčičast enake nivojnici na vrednosti f na nivojnici

- 3) narišemo krivuljo ki so v presečiščih nivojnic tangente neravnino

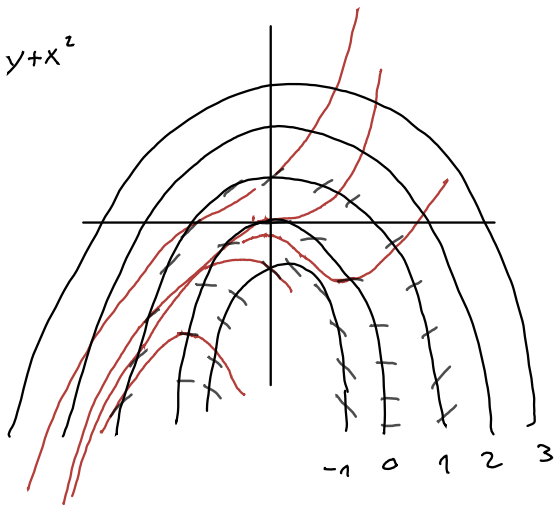
(črna so nivojnice)
m; iščemo funkcijo



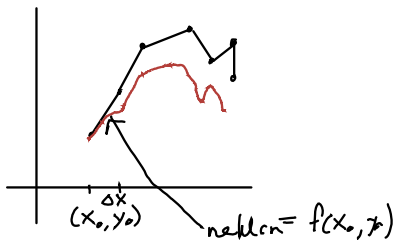
3. Približno skiciraj potek enačbe $y' = y - x$



$$y' = y + x^2$$



Eulerjeva metoda



$$x_1 = x_0 + \Delta x$$

$$y_1 = y_0 + \Delta x \cdot f(x_0, y_0)$$

$$x_2 = x_1 + \Delta x$$

$$y_2 = y_1 + \Delta x \cdot f(x_1, y_1)$$

Izberemo si Δx

in iterativno
definiramo ~~x_{n+1}~~

$$x_{n+1} = x_n + \Delta x$$

$$= x_0 + (n+1) \Delta x$$

$$y_{n+1} = y_n + \Delta x \cdot f(x_n, y_n)$$

Zakaj imamo smisel?

Če privzamemo da je
rešitev zvezo odvedljiv,
velja, da je

$$y(x + \Delta x) \approx y(x) + y'(x) \Delta x + o(\Delta x)$$

5) Z Eulerjevo metodo lokalno rešitev DŽ

$y' = f(x)$ kjer je f zvezna pri pogojem $y(0) = 0$

↙ nekateri
 $y(A) = ?$

$$\Delta x = \frac{A}{m} \quad \text{za nek } m \in \mathbb{N}$$

$$A = x_m$$

$$x_n = 0 + \Delta x n$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x \neq y_0 =$$

$$= y_{n-1} + f(x_{n-1}) \Delta x + y_0$$

$$= (f(x_{n-2}) + f(x_{n-1})) \Delta x + y_0$$

$$y_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

Riemannova
vsota

za

$$\int_0^A f(x) dx$$

To je očitno rešitev

$$y_m = \sum_{i=0}^{m-1} f(x_i) \Delta x$$

Vaja (DN):

Najdi rešitev za $y' = 2y$ z Eulerjevo metodo

$$\Delta x = \frac{A}{n}$$

$$\begin{aligned} y_n &= y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x = y_{n-1} + 2y_{n-1} \frac{A}{n} = \\ &= y_{n-1} \left(1 + \frac{2A}{n} \right) \end{aligned}$$

$$y_n = y_{n-2} \left(1 + \frac{2A}{n} \right)^2 = y_1 \left(1 + \frac{2A}{n} \right)^{n-1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{2A}{n} \right)^{n-1} &= \lim_{u \rightarrow \infty} \frac{\left(1 + \frac{1}{u} \right)^u}{\left(1 + \frac{1}{u} \right)}^{2A} = \underline{\underline{e^{2A}}} \\ \frac{1}{u} &= \frac{2A}{n} \Rightarrow n = 2Au \end{aligned}$$

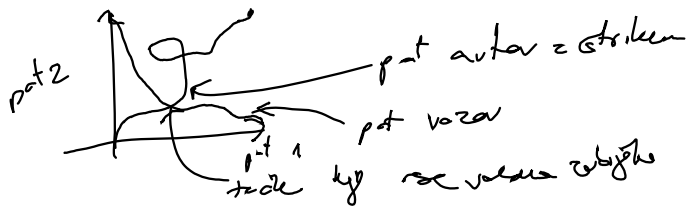
$$y = e^{2x}$$

Fazni prostor je prostor vseh možnih stanj sistema

$$y' = f(x, y) \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

↳ fazni prostor je \mathbb{R}^3

6) iz Ljubljane v Meribor vodita dve cesti po katerih lahko iz Ljubljane do meribora pripeljeta avtomobila ki sta eden na drugega privezane z vrjo dolžine $< 2l$, ne da bi jo pretrgela. Ali se lahko vozava krožne dolžine radija l , ki vozita vsek v svojo smer srečata, ne da bi trčila



$$1) \quad y' = \frac{x^2}{y} = \frac{dy}{dx}$$

$$x^2 dx = y dy$$

$$\frac{1}{3} x^3 + C = y^2$$

$$y = \pm \sqrt{\frac{2}{3} x^3 + C}$$

$$b) \quad 2x^2 y y' + y^2 = 2$$

$$2x^2 = \frac{2-y^2}{y y'} = \frac{2-y^2}{y \frac{dy}{dx}} = \frac{2-y^2}{y dy} dx$$

$$\frac{dx}{2x^2} = \frac{y dy}{2-y^2} \quad \begin{array}{l} u = 2-y^2 \\ du = -2y dy \end{array}$$

$$-\frac{1}{2} \frac{1}{x} = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} (\ln u + C)$$

$$\frac{1}{x} = \ln(2-y^2) + C$$

$$C e^{\frac{1}{x}} = 2-y^2$$

$$y = \pm \sqrt{2 - C e^{\frac{1}{x}}}$$

$$c) (1+x^2)y' = y$$

||

$$(1+x^2) \frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x^2}$$

$$\ln y = \arctan x + C$$

$$y = ce^{\arctan x}$$

$$y \equiv 0$$

$$c') (1+x^3)y' = y$$

$$y_1 \equiv 0$$

$$2) \frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)}$$

$$B = -A$$

$$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

$$x^2: B+A=0$$

$$x: B+C-A=0$$

$$1: A+C=1$$

$$C=1-\frac{A}{2}$$

$$2B+C=0$$

$$\Rightarrow C = -2B = 2A$$

$$3A=1$$

$$\frac{dy}{y} = \frac{1}{3} \frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1} dx$$

$$\int \frac{x-2}{x^2-x+1} dx = \int \frac{(x-\frac{1}{2})^2 + \frac{3}{4}}{(x-\frac{1}{2})^2 + \frac{3}{4}}$$

2)

$$2) y' = \tan(2x+3y-1)$$

$$y' = \tan z$$

$$z = 2x + 3y - 1$$

$$z' = 2 + 3y'$$

$$y' = \frac{z' - 2}{3}$$

$$\tan z = \frac{z' - 2}{3}$$

$$z' = 3 \tan z + 2$$

$$1. \text{ res\`a } : z' = 0 \Rightarrow z = \arctan\left(-\frac{2}{3}\right)$$

$$\Downarrow y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan(-\frac{2}{3}) - 2x + 1}{3}$$

$$\frac{dz}{dx} = 3 \tan z + 2$$

$$\frac{dz}{3 \tan z + 2} = dx$$

$$x + C = \int \frac{du}{(3u+2)(1-u^2)}$$

$$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2}$$

$$u = \tan z$$

$$du = \frac{1}{\cos^2 z} dz \Rightarrow dz = \frac{du}{1-u^2}$$

$$u^2: B-A=0 \quad A=B$$

$$u: 3C + 2 + 2B = 0$$

$$1: 2C + A = 1 \Rightarrow A = 1 - 2C$$

$$= \int \left(\frac{-3}{3u+2} + \frac{-3u+2}{1-u^2} \right) du =$$

$$3C + 2 - 4C = 0$$

$$C = 2 \Rightarrow A = -3 = B$$

$$= -\ln\left(u + \frac{2}{3}\right) +$$

$$+ \int \frac{1}{2} \frac{1}{u-1} + \frac{5}{2} \frac{1}{u+1} =$$

$$= -\ln\left(u + \frac{2}{3}\right) + \frac{1}{2} \ln(u-1) + \frac{5}{2} \ln(u+1)$$

$$\frac{A}{1-u} + \frac{B}{1+u} =$$

$$u: A-B = -3$$

$$1: A+B = 2$$

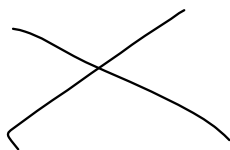
$$A = B - 3$$

$$2B - 3 = 2$$

$$2B = 5$$

$$B = \frac{5}{2}$$

$$A = -\frac{1}{2}$$



$$y''y - y' = xy'' - (y')^2$$

$$y''(y-x) = y'(1-y')$$

$$y''(y-x) + y'(y'-1) = 0 \quad y=0$$

$$a = y'$$

$$b = y-x \quad b' = y'-1$$

$$a'b + ab' = 0$$

$$(ab)' = 0$$

$$ab = C \in \mathbb{R}$$

$$y'(y-x) = C$$

$$y' = \frac{C}{y-x}$$

$$z = y-x$$

$$z' = y'-1 = \frac{C}{z} - 1$$

~~$$z = \int \left(\frac{C}{z} - 1 \right) dz = C \ln z - z + D$$~~

$$\frac{dz}{dx} = \frac{C}{z} - 1$$

$$\frac{dz}{\frac{C}{z} - 1} = dx \quad x+D = \int \frac{z}{C-z} dz = -\int \left(\frac{C}{t} - 1 \right) dt =$$

$$t = C-z \quad dt = -dz$$

$$= -C \ln(C-z) + C-z$$

$$x+D = -C \ln(C-y+x) + C - y+x$$

$$y = D e^C e^{\ln(C-y+x)+1}$$

Homogene machen

$$F(x, y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xz = y \Rightarrow y' = z + xz' = F(1, z)$$
$$z' = \frac{F(1, z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

$$A) y^2 + x^2 y' = xy y'$$

$$z = \frac{y}{x}$$

$$y' (xy - x^2) = y^2$$
$$y' = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z - 1}$$

$$y = zx$$

$$y' = z + z'x = \frac{z^2}{z - 1}$$

$$z'x = \frac{z^2 - z^2 + z}{z - 1} = \frac{z}{z - 1}$$

$$\frac{dz}{dx} x = \frac{z}{z - 1} \Rightarrow \frac{z - 1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{y}{x}$$

$$b) y = xy' - \sqrt{x^2 + y^2}$$

$$y) = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = z + \sqrt{1 + z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = z + z'x = z + \sqrt{1 + z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1 + z^2}}{x}$$

$$z = \operatorname{sh} u \quad dz = \cosh u \, du$$

$$\operatorname{arsh} \ln x + D = \int \frac{1}{\sqrt{1 + z^2}} dz = \int \frac{\cosh u}{\cosh u} du =$$

$$y = x \operatorname{ch}(\operatorname{arsh} \frac{y}{x} + D)$$

$$= -\operatorname{arcsch} \frac{y}{x} \\ \text{"} \\ \operatorname{arch} \frac{y}{x}$$

u)

$$ma = F = mg - kv^2$$

$$m\dot{v} = mg - kv^2$$

$$\dot{v} = g - \frac{k}{m}v^2 = g(1 - \frac{k}{mg}v^2) \stackrel{\alpha}{=} g(1 - \alpha v^2)$$

$$\frac{dv}{dt} = g(1 - \alpha v^2)$$

$$\frac{dv}{1 - \alpha v^2} = g dt \quad \int$$

$$g t =$$

$$\frac{A}{1 - \sqrt{\alpha}v} + \frac{B}{1 + \sqrt{\alpha}v} :$$

$$v: \sqrt{\alpha}A - \sqrt{\alpha}B = 0$$

$$A = B$$

$$1: A + B = 1$$

$$g t = \frac{1}{\sqrt{\alpha}2} (\ln(1 + \sqrt{\alpha}v) - \ln(\sqrt{\alpha}v - 1)) =$$

$$\Rightarrow A = \frac{1}{2}$$

$$B = \frac{1}{2}$$

$$\frac{1}{2\alpha} \ln\left(\frac{1 + \sqrt{\alpha}v}{1 - \sqrt{\alpha}v}\right) = g t + C$$

$$v_0 = 0 \Rightarrow$$

$$\cancel{1 + \sqrt{\alpha}v} = \cancel{1 - \sqrt{\alpha}v} + C$$

$$\Rightarrow 0 = C$$

$$\frac{1 + \sqrt{\alpha}v}{1 - \sqrt{\alpha}v} = \underbrace{e^{2\alpha g t}}_D$$

$$1 + \sqrt{\alpha}v = D - \sqrt{\alpha}v$$

$$\sqrt{\alpha}(1 - \sqrt{\alpha}D) = D - 1$$

$$v = \frac{D - 1}{\sqrt{\alpha}(1 - \sqrt{\alpha}D)}$$

V merkenje

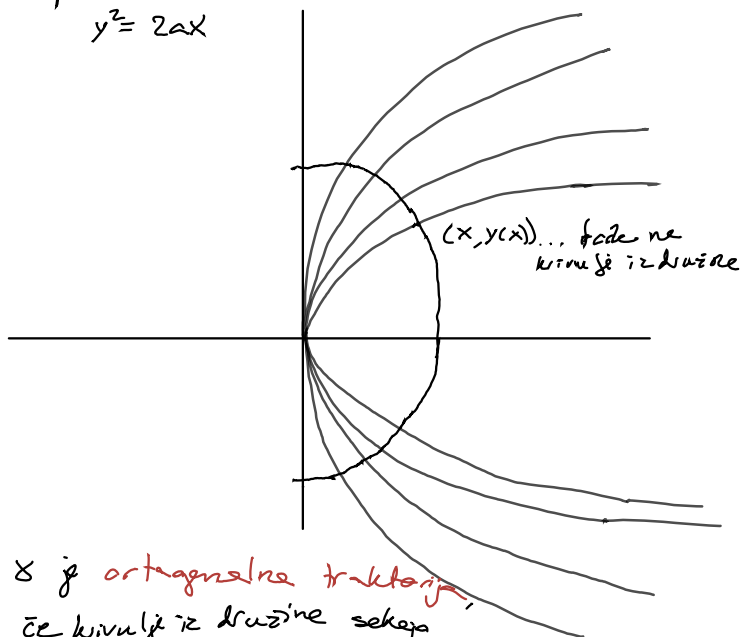
+ zandude 20 min

21.10

2) Poišči družino

ortogonalnih trajektorij na družino
parabol

$$y^2 = 2ax$$



γ je **ortogonalna trajektorija**,

če krivulje iz družine seka

γ pod pravim kotom

$$2yy' = 2a \Rightarrow y^2 = 2yy'x$$

$$V(x_0, y_0) \text{ velja } x_0 \cdot 2y_0 y' = y_0^2$$

$$y' = \frac{y_0}{2x_0} \Rightarrow \text{znatni koeficient trajektorije } -\frac{2x_0}{y_0}$$

$$y' = f(x, y) \text{ družina krivulj} \Rightarrow -\frac{1}{f(x, y)} = f(x, y) \text{ družina ort. t.}$$

$$y^2 = 2yy'x \Rightarrow y^2 = -\frac{2yx}{y'} \quad \text{iščemo } y$$

1. možnost $y=0$ ✓

2. možnost

$$y' = -\frac{2x}{y} = \frac{dy}{dx}$$

$$2x dx = y dy \quad / \int$$

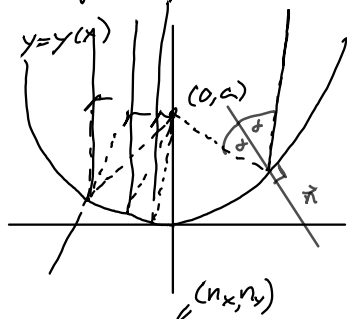
$$-\frac{2}{2} x^2 = \frac{1}{2} y^2 + C$$

$$x^2 + \frac{1}{2} y^2 + C = 0$$

$$x^2 + \frac{1}{2} y^2 = -C \quad \text{elipse}$$

3) Pri izdelavi žarotnega mera biti

odbojn površino ki odseva žarotno tako oblike, da vse svetlobne žarke odboji v isto smer predpostavimo da je odbojna površina rotacijske plošče graf f_n



$$(-x, -y(x)+a) \cdot \vec{n} = -x n_x - y n_y - a n_y = \cos \alpha \cdot \sqrt{x^2 + (y+a)^2} \sqrt{n_x^2 + n_y^2}$$

$$\vec{n} = (1, -\frac{1}{y'})$$

$$\cos \alpha \sqrt{x^2 + (y+a)^2} \sqrt{1 + \frac{1}{y'^2}} = -x + \frac{y}{y'} - \frac{a}{y'}$$

$$\cos \alpha = \frac{-xy' - y - a}{\sqrt{x^2 + (y+a)^2} \sqrt{1 + y'^2}}$$

$$\vec{n} \cdot (1, 0, 0) = \cos \alpha |\vec{n}| \cdot 1$$

$$n_y dx = \cos \alpha \sqrt{1 + \frac{1}{y'^2}}$$

$$n_y dx = \frac{1}{\sqrt{x^2 + (y+a)^2}} (-xy' - y - a) = \frac{1}{y'}$$

$$-xy' + y - a = \sqrt{x^2 + (y+a)^2}$$

???

$$z = y - a$$

$$z' = y'$$

$$x z' + z = \sqrt{x^2 + z^2} + z$$

??

$$z' = \frac{\sqrt{x^2 + z^2}}{x} = \sqrt{\frac{x^2 + z^2}{x^2}} = \sqrt{1 + \frac{z^2}{x^2}} = \sqrt{1 + \frac{z^2}{x^2}}$$

$$u = \frac{z}{x}$$

$$z' = x u'$$

$$u + x u' = \sqrt{1 + u^2} + u$$

$$\frac{u'}{\sqrt{1 + u^2}} = \frac{1}{x}$$

$$v = \sqrt{1 + u^2}$$

$$dv = \frac{u du}{\sqrt{1 + u^2}} = \frac{1 - v^2}{\sqrt{1 + u^2}} du$$

$$\ln x = \int \frac{1 - v^2}{v} dv = \ln v - \frac{1}{2} v^2 + C$$

???

$$y = a + \frac{c}{2} x^2 - \frac{1}{2a}$$

Cauchy nekdy

$$y' = f(y) \quad y(x_0) = y_0$$

a) gndi. \exists resi Dť za podmĩnĩ zĩdĩnĩ pĩmĩ

b) f Lipsicova klesĩ kĩjĩ resĩkĩ zĩ a endĩm

$$\frac{dy}{dx} = f(y)$$

$$\frac{dy}{f(y)} = dx \int$$

$$F(y) = x + C$$

$$\int \frac{dy}{f(y)}$$

$$y'(x) = f(y(x))$$

$$y(x) = y(x_0) + \int_{x_0}^x y'(t) dt$$

$$\int_{x_0}^x \frac{y'(t) dt}{f(y(t))} = \int_{x_0}^x 1 dt = x - x_0$$

daS vse na enu stran in integrujĩ

$$u = y$$

$$du = y' dt$$

$$\int_{y_0}^{y(x)} \frac{du}{f(u)} = F(y(x)) - F(y_0) = x - x_0$$

$$F(y(x)) = x - x_0 + F(y_0)$$

$$y(x) = F^{-1}(x - x_0 + F(y_0))$$

$$y(x_0) = F^{-1}(F(y_0)) = y_0$$

$$y'(x) = \frac{1}{F'(F^{-1}(x - x_0 + F(y_0)))} = \frac{1}{f(y(x))}$$

$$= f(y(x)) \quad ???$$

$$F'(F^{-1}(x)) = \frac{1}{(F^{-1})'(x)}$$

b) endičnost

$$\int_{y_0}^y \frac{dy}{f(y)} = x - x_0$$

$$|f(x) - f(y)| \leq C|x - y|$$

$$\nearrow |f(y)| \leq C|x - y|$$

rećmo
da $f(x) = 0$

$$\int_{y_0}^y \frac{dy}{f(y)} \geq \frac{dy}{C|y - y_0|} = \infty$$

Rećmo da y reši DE da de

$f(y_n) = 0$ in y ni konstanten

$$x_n > x_0 : f(y(x_n)) > 0$$

Potom je y blizu x_n za koje

$$\int_{x_n}^x \frac{y'(x) dx}{f(y(x))} = x - x_n$$

$$\parallel$$

$$\int_{y_n}^y \frac{dy}{f(y)}$$

izaberemo si $x_2 \in [x_0, x_n]$, da valja

$f(y(x_2)) = 0$ in $f(y(x)) > 0$ za $\forall x \in (x_2, x_n]$

$$\lim_{x \rightarrow x_2} \int_{x_n}^x \frac{y'(x)}{f(y(x))} = \lim_{x \rightarrow x_2} x - x_n = x_2 - x_n$$

$$= \lim_{y \rightarrow y_2} \int_{y_n}^y \frac{dy}{f(y)} \geq \lim_{y \rightarrow y_2} \int_{y_n}^y \frac{dy}{C(y - y_0)} = \infty$$

X

$$y' = \frac{f(y)}{g(x)} \quad f, g \text{ brez ničel}$$

a) pokaži da vsake rešitve da skeno krivulje polja; a $v(x, y) = (g(x), f(y))$

b) Dokaži za poljken rešitvi po gaj točko uveliku eno rešitve

Tokovnice ali integralna krivulje je krivulje, ki imajo vsaki točki odred enake vektorne polje

$$F(y) - F(y_0) = \int_{y_0}^y \frac{dy}{f(y)} = t - t_0 \quad x = G^{-1}(t - t_0 + G(x_0))$$

$$G(x) - G(x_0) = \int_{x_0}^x \frac{dx}{g(x)} = t - t_0 \Rightarrow$$

$$\gamma = (x(t), y(t))$$

$$\dot{\gamma}(t) = V(x(t), y(t))$$

$$\dot{x}(t) = g(x)$$

$$\dot{y}(t) = f(y)$$

$$\frac{\partial}{\partial t} y(x(t)) =$$

$$= \frac{\partial y}{\partial t}(x(t)) \dot{x} =$$

$$= \frac{f(y(t))}{g(x(t))} \frac{\partial x}{\partial t}$$

$$\Rightarrow \dot{x} = g(x) \quad ???$$

✓

Lin. dif. enažse (LDE)

$$y' + a(x)y = b(x) \quad a, b \in \mathcal{C}([x, \beta])$$

i) $b \equiv 0 \Rightarrow$ LDE homogena \Rightarrow ločljive spremenljivke

$$y = y_0 e^{-\int a(x) dx} = C e^{-\int a(x) dx}$$

ii) $b \neq 0 \Rightarrow$ LDE nehomogena \Rightarrow variacija konstante

$$y = e^{-\int a(x) dx} \int b(x) e^{\int a(x) dx} dx + C e^{-\int a(x) dx}$$

•) LDE homogena: y_1, y_2 rešitvi $\Rightarrow y_1 + \lambda y_2$ tudi rešitev

če vemo homogena rešitev \bar{y} so vse druge rešitve oblike $C \cdot \bar{y}$ za $C \in \mathbb{R}$

••) LDE nehomogena

Partikularno rešitev najdemo z variacijo konstante

Homogeni del: $y' + a y = 0 \Rightarrow y_h$ rešitev

Partikularni del: $y_p = C(y) y_h$

Splasnjen rešitev: $\tilde{y} = y_p + D y_h$

Odvajamo in vstavimo v DE:

$$y_p' = C'(y) y_h + C y_h' \Rightarrow C' y_h + C y_h' + a C y_h = b$$

$$C' y_h = b \Rightarrow C(x) = \int \frac{b}{y_h} dx$$

7) Pois o problema é linear:

$$a) xy' + 2y = (3x+2)e^{3x} \quad / : x$$

$$y' + y \frac{2}{x} = \underbrace{\frac{3x+2}{x}}_b e^{3x}$$

$$x=0 \Rightarrow 2y=2 \Rightarrow y=1$$

$$x=0 \Rightarrow$$

$$\text{homogênea: } y' + y \frac{2}{x} = 0$$

$$\frac{dy}{dx} + y \frac{2}{x} = 0 \quad / \frac{dx}{y}$$

$$\frac{dy}{y} = -\frac{2dx}{x} \quad / \int$$

$$\ln y = \ln x^{-2} + C$$

$$y = \frac{1}{x^2} \cdot D$$

particular no:

$$y_p = D(x)x^{-2}$$

$$y'_p = D'(x)x^{-2} - 2D(x)x^{-3}$$

$$D'(x)x^{-2} - \cancel{2D(x)x^{-3}} + \cancel{2D(x)x^{-3}} = \frac{(3x+2)e^{3x}}{x}$$

$$D'(x) = (3x^2 + 2x)e^x$$

$$\int p(x)e^{\lambda x} dx = q(x)e^{\lambda x} + C$$

$$p, q \in \mathbb{R}[x], \text{ st } p = \text{st } q$$

$$((Ax^2 + Bx + C)e^{3x})' = (3x^2 + 2x)e^{3x}$$

$$(3Ax^2 + 3Bx + 3C + 2Ax + B)e^{3x} \leadsto$$

$$3A = 3 \quad \Rightarrow A = 1$$

$$3B + 2A = 2 \quad \Rightarrow B = 0$$

$$3C + B = 0 \quad \Rightarrow C = 0$$

$$D(x) = x^2 e^{3x}$$

$$\leadsto y_p = x^2 e^{3x} \cdot x^{-2} = e^{3x}$$

$$y_s = y_h + y_p = D \frac{1}{x^2} + e^{3x}$$

$$b) y'' = \sin x + y'$$

$$z = y'$$

$$a(x) = -1 \quad b(x) = \sin x$$

$$z' = \sin x + z$$

$$b \equiv 0 \Rightarrow z' - z = 0$$

$$z' - z = \sin x$$

$$z' = z$$

$$1) z \equiv 0 \Rightarrow y = C \in \mathbb{R}$$

$$2) \int \frac{dz}{z} = dx \Rightarrow \ln z = x + C$$

$$\Rightarrow z = Ce^x \Rightarrow y = Ce^x + D$$

$$b \neq 0 \Rightarrow z_h = e^x$$

$$z_p = C(x)z_h = C(x)e^x$$

$$z = z_p + D y_h = C(x)e^x + De^x$$

$$z' = C'(x)e^x + C(x)e^x + De^x$$

$$z' - z = C'(x)e^x + C(x)(e^x - e^x) = \sin x$$

$$C'(x) = \frac{\sin x}{e^x} = e^{-x} \sin x$$

$$\text{nastavek: } C(x) = Ae^{-x} \sin x + Be^{-x} \cos x$$

$$C(x) = e^{-x}(A \sin x + B \cos x)$$

$$C'(x) = -e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)$$

$$= e^{-x}(\sin x(-A-B) + \cos x(-B+A)) = e^{-x} \sin x$$

$$A+B = -1$$

$$A-B = 0 \Rightarrow A=B \Rightarrow 2A = -1 \Rightarrow A=B = -\frac{1}{2}$$

$$\Rightarrow C(x) = -\frac{1}{2}e^{-x}(\sin x + \cos x) =$$

$$z = -\frac{1}{2}(\sin x + \cos x) + De^x = y'$$

$$y = -\frac{1}{2} \int (\sin x + \cos x) dx = \frac{1}{2} \cos x - \frac{1}{2} \sin x + De^x + C$$

$$2) \quad y' + 2xy = 2x^3 y^3$$

1) $y \equiv 0$ je rešitev

$$2) \quad y \neq 0 \Rightarrow \frac{y'}{y^3} + \frac{2x}{y^2} = 2x^3$$

$$z = \frac{y'}{y^3} \Rightarrow \int dz = \int y^{-3} dy \Rightarrow z = -\frac{1}{2}y^{-2} = -\frac{1}{2y^2}$$

Bernullijeva DE

$$y' + ay = by^a \quad a \in \{0, 1\}$$

$$\text{Nastavek: } z = y^{1-a}$$

$$\frac{2x}{y^2} = \frac{4x}{2y^2} = -4xz$$

$$\text{enačba: } z' - 4xz = 2x^3 \quad a(x) = -4x \quad b(x) = 2x^3$$

$$\text{homogeno: } z' - 4xz = 0$$

$$\int \frac{dz}{z} = \int 4x dx \Rightarrow \ln z = 2x^2 + C \Rightarrow z = Ce^{2x^2}$$

$$\text{partikularno: } z_h = e^{2x^2} \quad z_p = C(x)e^{2x^2}$$

$$z = z_p + z_h$$

$$\leadsto \text{enačba: } C'(x)e^{2x^2} + 4xe^{2x^2} \cdot C - 4x Ce^{2x^2} = 2x^3$$

$$C'(x) = e^{-2x^2} \cdot 2x^3$$

Nastavek:

$$C(x) = \int 2x^3 e^{-2x^2} dx = \int x e^u du = (Au + B)e^u$$

$$u = -2x^2$$

$$du = -4x dx$$

$$= -\frac{1}{2} \int e^u u du = \frac{1}{2} (Au + B)e^u$$

$$((Au + B)e^u)' = e^u (Au + B + A)$$

$$A + B = 0 \quad A = 1 \Rightarrow B = -1$$

$$C(x) = -\frac{1}{2} (1u - 1)e^u \leadsto$$

$$C(x) = e^{-2x^2} (x^2 - \frac{1}{2}) + D$$

$$z = x^2 - \frac{1}{2} + D e^{2x^2} \leadsto y = \frac{1}{\sqrt{x^2 - \frac{1}{2} + D e^{2x^2}}}$$

3. nalog

$$xy' + y = y^2 \ln x$$

1) $y = 0 \Rightarrow$ je rešitev

2) $y \neq 0 \Rightarrow$
 $x \neq 0 \Rightarrow$

$$\frac{y'}{y^2} + \frac{1}{xy} = \ln x$$

$$z' = \frac{y'}{y^2} \Rightarrow z = -\frac{1}{y}$$

$$\boxed{z' - \frac{z}{x} = \ln x} \quad \begin{matrix} a = -1 \\ b = \ln x \end{matrix}$$

homogeno:

$$z' - \frac{z}{x} = 0$$

$$\frac{dz}{z} = \frac{dx}{x} \Rightarrow$$

$$\ln z = \ln x + C$$

$$z = e^{\ln x} \cdot e^C \Rightarrow z = Cx$$

$$z_h = x$$

$$\Rightarrow C'x + C - C = \ln x$$

$$C' = \frac{\ln x}{x} \quad C = \int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + D$$

$$u = \ln x \quad du = \frac{dx}{x} \quad C = \frac{1}{2} (\ln x)^2 + D$$

$$z = \frac{1}{2} (\ln x)^2 x + Dx = -\frac{1}{y}$$

$$y = -\frac{2}{(\ln x)^2 x + Dx}$$

4 naloga

f omejena zvezna

$$y' + y = f(x)$$

a) Ta ima natanko eno rešitev DE

b) f periodična s periodo $w \Rightarrow \bar{y}$ periodična s periodo w

a) $y' + y = f(x)$

homogeno:

$$\frac{dy}{dx} = -y \leadsto \frac{dy}{y} = -dx \leadsto y = e^{-x}$$

partikularna: $y_p = C(x)e^{-x}$

$$C'(x)e^{-x} = f(x) \quad C(x) = \int_0^x f(t)e^t dt = F(x) + D$$

$$\bar{y} = F(x)e^{-x} + Ce^{-x} = e^{-x}(F(x) + C) = e^{-x} \int_0^x f(t)e^t dt$$

~~$$\begin{aligned} \text{b) } \bar{y}(x+w) &= e^{-x-w} \int_0^{x+w} f(t)e^t dt = e^{-x}e^{-w} \int_0^x f(t)e^t dt + \\ &\quad e^{-w}e^{-x} \int_x^{x+w} f(t)e^t dt = \\ &= e^{-w} \bar{y} + e^{-w}e^{-x} \int_x^{x+w} f(t)e^t dt = \\ &= e^{-w} \bar{y}(x) + e^{-x} \bar{y}(w) - e^{-w} \bar{y}(x) = e^{-x} \bar{y}(w) = e^{-w} \bar{y}(x) \end{aligned}$$~~

~~$$\begin{aligned} \bar{y}(x+w) &= e^{-x-w} \int_{-\infty}^{x+w} f(t)e^t dt = e^{-x-w} \int_{-\infty}^w f(t)e^t dt \\ &\quad + e^{-x-w} \int_w^{x+w} f(t)e^t dt = \\ &= e^{-x-w} \left(\int_{-\infty}^w f(t)e^t dt + \int_w^x f(t)e^t dt \right) = \\ &= e^{-x} \bar{y}(w) + e^{-w} \bar{y}(x) \end{aligned}$$~~

???

$$b) \bar{y}(x) e^{-x} \int_{-\infty}^x f(t) e^t dt = \int_{-\infty}^x f(t) e^{t-x} dt = \int_{-\infty}^0 f(u+x) e^u du$$

$$u = t-x$$

$$\Rightarrow t = u+x$$

$$\bar{y}(w+x) = \int_{-\infty}^0 f(u+x+w) e^u du = \int_{-\infty}^0 f(u+x) e^u du = \bar{y}(x)$$

5 nalaga

$$3y' + y^2 + \frac{2}{x^2} = 0$$

Uagnemo rešitev: $y = \frac{1}{x}$ $y' = -\frac{1}{x^2}$

$$-\frac{3}{x^2} + \frac{1}{x^2} + \frac{2}{x^2} = 0 \quad \checkmark$$

Splazna rešitev: $y = \frac{1}{x} + u(x)$
 $y' = -\frac{1}{x^2} + u'(x)$

$$-\frac{3}{x^2} + 3u'(x) + \frac{1}{x^2} + \frac{2u(x)}{x} + u^2(x) + \frac{2}{x^2} = 0$$

$$3u'(x) + \frac{2u(x)}{x} + u^2(x) = 0 \quad / : u^2(x)$$

$$\frac{3u'(x)}{u^2(x)} + \frac{2}{xu(x)} + 1 = 0$$

$$z' = \frac{3u'(x)}{u^2(x)} \quad z = 3 \int u^{-2} dx = -3u^{-1} = -\frac{3}{u}$$

$$\Rightarrow u = -\frac{3}{z}$$

$$z' - \frac{2}{3} \frac{z}{x} = -1$$

$$a(x) = \frac{2}{3x}$$

$$b(x) = -1$$

homogeni del:

$$\frac{dz}{z} = \frac{2}{3} \frac{dx}{x}$$

$$\ln z = \ln x^{\frac{2}{3}} + C \Rightarrow z = \frac{2}{3} Cx$$

partikularni:

$$C'(x) \frac{2}{3} x = -1$$

$$C(x) = -\frac{3}{2} \int \frac{1}{x} dx = \ln x^{-\frac{3}{2}}$$

$$z = x \ln x + \frac{2}{3} Cx$$

$$u = -\frac{3}{x \ln x + \frac{2}{3} Cx} \Rightarrow y = \frac{1}{x} - \frac{3}{x \ln x + \frac{2}{3} Cx}$$

Ricattijeva enačba

$$y' = ay^2 + by + c \quad a, b, c \text{ funkcije } x$$

1) Uganemo rešitev y_p

2) Splošno rešitev poiščemo z nastavekom

$$y = y_p + u(x)$$

3) Vstavimo nastavek z začetno enačbo

4) Dobimo Bernullijevo enačbo

$$z) \quad y = y_p + \frac{1}{u^2(x)} \Rightarrow LDE$$

1. $(1-x^2)y' = 1-y^2$

Riccatijeva enačba

$$y' = ay^2 + by + c \quad ; \quad a, b, c \text{ func. od } x$$

1) Uganimo rešitev y_p

2) splošna rešitev kot $y = y_p + u(x)$

3) Dobimo Bernoulijeva enačba

$$y = y_p + \frac{1}{u(x)^2} \Rightarrow \text{CDE}$$

$$y' = \frac{1-y^2}{1-x^2}$$

1. $y_p = x : 1 = \frac{1-x^2}{1-x^2} \quad \checkmark$

2. $y = v + y_p = v + x$ v je neznana funkcija

$$y' = v' + 1 = \frac{1 + (v+1)^2}{1-x^2} = \dots = 1 + \frac{-2vx - v^2}{1-x^2}$$

$$v' = \frac{-2vx - v^2}{1-x^2} = -\left(\frac{1}{1-x^2} v^2 + \frac{2x}{1-x^2} v\right)$$

$$\frac{v'}{v^2} = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} \frac{1}{v}$$

$$z = \frac{1}{v} \quad z' = \frac{v'}{v^2}$$

$$z' = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} z$$

homogeni del:

$$z' = -\frac{2x}{1-x^2} z$$

$$\int \frac{dz}{z} = -\int \frac{2x}{1-x^2} dx$$

$$\ln|z| = -\ln|1-x^2| + C$$

$$z_h = \frac{C}{x^2-1}$$

$$\frac{C'(x)}{x^2-1} + \frac{C}{x^2-1} = \dots = -\frac{1}{x^2-1}$$

$$C'(x) = -1 \quad C = -x$$

$$z = z_h + z_p = \frac{C-x}{x^2-1}$$

$$v = \frac{x^2-1}{C-x}$$

$$y = \dots = \frac{1-Cx}{x-C}$$

Prvi integral enačbe $y' = f(x, y)$ je funkcija, ki je konstantna vzdolž vsake rešitve

Če imamo splošno rešitev $y = \varphi(x, C)$ in je $\varphi_C \neq 0$ ↓ odvod po C
 $\Rightarrow C = C(x, y)$ je prvi integral

1) DE zapišemo v obliki $Pdx + Qdy = 0$ (*)

Opomba: (*) je ekvivalentna DE $P + Qy' = 0$

$$\langle (P, Q), (1, y') \rangle = 0$$



če velja $Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y$

→ pravimo da je enačba ~~ekvivalentna~~ eksaktna.
Njen prvi (eksaktni integral) poiščemo z integracijo $u = \int$
→ $((P, Q) \text{ je potencialno} \Leftrightarrow (P, Q) = \nabla u)$

C poiščemo z odvajanjem

$$u_y = \frac{\partial}{\partial y} \int P dx + C'(y) = Q$$

2) Poišči kak nekonstanten prvi integral

$$a) x\sqrt{x^2+y^2}dx + (y\sqrt{x^2+y^2} + x)dy = 0$$

$$b) 3x^2(1+\ln y)dx - (2y - \frac{x^3}{y})dy = 0$$

a)

$$P = x\sqrt{x^2+y^2} + y$$

$$Q = y\sqrt{x^2+y^2} + x$$

Ali je eksaktna?

$$P_x = \sqrt{x^2+y^2} + \frac{xy}{\sqrt{x^2+y^2}}$$

$$Q_y = -1$$

$$u = \int (x\sqrt{x^2+y^2} + \frac{y}{\sqrt{x^2+y^2}}) dx + C(y) =$$

$$t = x^2 + y^2$$

$$dt = 2x dx$$

$$= \frac{1}{2} \int \frac{1}{t^{\frac{1}{2}}} dt + xy + C(y) = \frac{1}{3} (x^2+y^2)^{\frac{3}{2}} + xy + C(y)$$

$$u_y = \frac{y(x^2+y^2)^{\frac{1}{2}}}{1} + C'(y) = y\sqrt{x^2+y^2} + C'(y)$$

$$C'(y) = 0 \Rightarrow C \text{ je konstanta}$$

$$u = \frac{1}{3} (x^2+y^2)^{\frac{3}{2}} + xy \quad \text{potemno da je 0}$$

↑ prvi integral

splošno rešitev dobimo če: $u(x,y) = C$

izrazimo y $y = y(x)$

$$u(x, y(x)) = C$$

1

Da je $Pdx + Qdy = 0$ (*) mora veljati

$$y' = -\frac{P}{Q} = -\frac{\mu P}{\mu Q} \Rightarrow \mu Pdx + \mu Qdy = 0 \quad (**)$$

Ti dve sta ekvivalentni za poljubno rešitev
če * ni eksaktan mora biti ** eksaktan
faktor μ se menjuje integrirajoči množitelj

V praksi ga skušamo uganiti z nastevki:

$$\mu(x), \mu(y), \mu(x+y)$$

$$\mu(x^n \pm y^m), \mu(x^2 + y^2) \dots$$

3) Podana je DE

$$Pdx + Qdy = 0$$

a) Dokaži $\mu = \mu(y) \Leftrightarrow \frac{Q_x - P_y}{P} = g(y)$

$\Rightarrow \mu Pdx + \mu Qdy = 0$ in μ integracijski faktor
DE pa je eksaktna

$$(\mu P)_y = (\mu Q)_x$$

$$\mu_y P + \mu P_y = \mu Q_x$$

$$\frac{\mu(Q_x - P_y)}{P} = \mu'(y) \quad /: \mu$$

$$\frac{Q_x - P_y}{P} = \frac{\mu'(y)}{\mu(y)} \quad \int = \text{divimo samo po } y$$

$$\Leftarrow \frac{Q_x - P_y}{P} = g(y) \quad \text{recimo da je}$$

$$\ln \mu = \int \frac{Q_x - P_y}{P} dy \Rightarrow \mu = e^{\int \frac{Q_x - P_y}{P} dy}$$

Potem je $(\mu P)_y = \frac{Q_x - P_y}{P} P + \dots$ itd itd
pride vredn

$$b) \underbrace{(xy^2 - y^3)}_P dx + \underbrace{(1 - xy^2)}_Q dy = 0$$

$$Q_x = -y^2 \quad P_y = 2xy - 3y^2$$

$$\frac{Q_x - P_y}{P} = \frac{-y^2 - (2xy - 3y^2)}{xy^2 - y^3} = \frac{2(y - x)}{y(x - y)} = -\frac{2}{y}$$

$$\mu = e^{-\int \frac{2}{y} dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$$

$$\mu P_x + \mu Q_y = 0 \quad \text{is true}$$

$$u = \int \mu P dx + C(y) =$$

$$= \int (x - y) dx + C(y) = \frac{1}{2} x^2 - xy + C(y).$$

$$u_y = -x + C'(y) = \frac{1 - xy^2}{y^2} = \frac{1}{y^2} - x$$

$$C'(y) = \frac{1}{y^2} \Rightarrow C = -\frac{1}{y}$$

$$u = \frac{x^2}{2} - xy - \frac{1}{y}$$

μ

$$u) \quad y(x^2 + y^2 + 1) - x(x^2 + y^2 - 1) dx = 0$$

$$\mu = \mu(xy)$$

$$P_y = x^2 + 3y^2 + 1$$

$$\mu P dx + \mu Q dy = 0$$

$$Q_x = -3x^2 - y^2 + 1 \quad \mu = \mu(xy)$$

$$(\mu P)_y = x\mu'P + \mu P_y$$

$$(\mu Q)_x = y\mu'Q + \mu Q_x \quad \left. \begin{array}{l} (\mu P)_y \\ (\mu Q)_x \end{array} \right\} + \mu'(xP - yQ) + \mu(P_y - Q_x) = 0$$

$$\frac{\mu'}{\mu} = \frac{-P_y + Q_x}{xP - yQ} = \frac{-4x^2 - 4y^2}{xy(2x^2 + 2y^2)} = \frac{-2}{xy}$$

$$z = xy \quad \frac{\mu'(z)}{\mu(z)} = \frac{-2}{z} \int$$

$$\frac{d\mu}{\mu(z)} = -\frac{2dz}{z} \int$$

$$|\mu(z)| = |z|^{-2}$$

$$\mu(z) = \frac{1}{z^2} = \frac{1}{(xy)^2}$$

$$u = \int \mu P dx + Q dy = \int \frac{y(x^2 + y^2 + 1)}{x^2 y^2} dx + C(y)$$

$$= \int \frac{1}{y} + \frac{y}{x^2} + \frac{1}{x^2 y} = \frac{x}{y} - \frac{y}{x} - \frac{1}{xy} + C(y)$$

$$u_y = -\frac{x}{y^2} - \frac{1}{x} + \frac{1}{y^2 x} + C'(y) = \dots$$

4)

$$y dx - x dy = 2x^3 \tan\left(\frac{y}{x}\right) dx$$

$$z = \frac{y}{x} \quad dz = \frac{dy}{x} - \frac{y}{x^2} dx$$

$$dF = F_x dx + F_y dy$$

$$-x^2 dz = -x dy + y dx$$

$$-dz = 2x \tan(z) dx$$

$$-\frac{dz}{\tan(z)} = 2x dx$$

$$-\int \frac{\cos(z)}{\sin(z)} dz = \int 2x dx$$

$$-\ln|\sin(z)| = x^2 + C$$

$$-\ln \sin \frac{y}{x} = x^2 + C$$

$$C = -x^2 - \ln\left|\sin \frac{y}{x}\right| = u(x, y) = \text{priv. integra}$$

Parametrično reševanje implicitno podanih DE

Do zdaj so enačbe bile oblike $y' = F(x, y)$

Lažje pa je v $F(x, y, y') = 0$

Poanta: $F(x, y, p) = 0$ podaja ploščo v \mathbb{R}^3
(če $F \in C^1$ in $\nabla F \neq 0$)

Vsaka rešitev bo potekala po neki krivulji:

$\gamma(t) = (x(t), y(t), p(t))$ da je $F(\gamma(t)) = 0$

Da γ res podaja rešitev DE v parametrični obliki,
mora veljati

$$\dot{\gamma}(t) = p(t) \cdot \dot{x}(t)$$

Preprost primer:

$$F(x,y) = 0$$

↓

$F(x,p)$ daje krivuljo v ravnini:

To krivuljo parametriziramo in poiščemo manjši
predpis za y komponento

6) Reži de

$$a) (y')^2 - 2y^2 = 1$$

$$b) x^2 + 1 + (y')^2 - 2y' = 1$$

$$p^2 - 2y^2 = 1$$

$$p = \operatorname{ch} t$$

$$y = \frac{1}{\sqrt{2}} \operatorname{sh}(t)$$

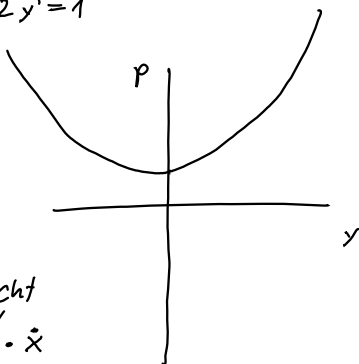
$$\dot{y} = \frac{\operatorname{ch}(t)}{\sqrt{2}} = \overset{\operatorname{ch} t}{p} \cdot \dot{x}$$

$$\dot{x} = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{2}{\sqrt{2}} t + C \quad t = \sqrt{2} x$$

kurve je $(\frac{\sqrt{2}}{2} t + C, \frac{\sqrt{2}}{2} \operatorname{sh} t)$

$$y = \frac{\operatorname{sh}(\sqrt{2}(x - C))}{\sqrt{2}}$$

3C druga rešila $\frac{\operatorname{sh}(\sqrt{2}(C-x))}{\sqrt{2}}$ druga kurva



$$b) x^2 + 1 + (y')^2 - 2y = 1$$

$$x^2 + (p-1)^2 = 1$$

$$x = \cos t$$

$$p = 1 + \sin t$$

$$\dot{y} = p \dot{x} = (1 + \sin t)(-\sin t) = -\sin^2 t - \sin t$$

$$y = \int \dot{y} dt = \int \frac{1 - \cos 2t}{2} dt + \cos t =$$

$$= \frac{1}{2} \cancel{t} \frac{1}{2} \sin 2t + \underbrace{\cos t}_x$$

$$y = \frac{1}{2} \arccos x - x \sin(\arccos x) + x$$

9.2.1.1: p-neri

a):

$$p^2 - 2y^2 = 1$$

$$\text{er } \bar{F}_p = 0 \quad F_p = 2p = 0 \Leftrightarrow p = 0$$

$\Rightarrow y$ konstante

$$y^2 = -\frac{1}{2} \quad \text{keine reelle Lösung}$$

b)

Od zednjic: Parametrične reševanje DE 11.11
oblike $F(x, y, y') = 0$

Splosni postopek

1. $p = y'$ neodvisne spremenljivke

2. izračunamo $DF(x, y, p)$

3. s pomočjo nastanka $dy = p dx$ dobimo

diferencialno enačbo $x = x(p)$ ali $y = y(p)$.

Manjšo komponento dobimo iz prve enačbe

Včasih ima DE tudi singularne rešitve

1) $G(x, y, C) = 0$ in enakosti $\frac{\partial G}{\partial C} = 0$ eliminiramo C

2) parametriziramo množico $\{F = F_p = 0\}$ in preverimo
ali reši DE \uparrow odvod po p

1. nalaga: Poišči splošno in morebitne singularne rešitve

$$x^2(y')^3 - xy' + y = 0$$

$$y = xy' - x^2(y')^3 \quad p = y' \quad dy = p dx$$

$$y = xp - x^2 p^3$$

$$dy = p dx + x dp - 2xp^3 dx - 3x^2 p^2 dp$$

$$\cancel{p dx} = \cancel{p dx} - 2xp^3 dx + (x - 3x^2 p^2) dp$$

$$2xp^3 dx = (x - 3x^2 p^2) dp$$

$$x = 0 \Rightarrow y = 0$$

$$x \neq 0 \Rightarrow$$

$$2p^3 dx = (1 - 3xp^2) dp$$

$$2p^3 x'(p) = 1 - 3xp^2 \quad p = 0 \Rightarrow y = 0$$

$$\dot{x} = \frac{1}{2p^3} - \frac{3}{2} \frac{x}{p^2}$$

$$\dot{x} + \frac{3}{2} \frac{x}{p^2} = \frac{1}{2p^3}$$

homogeni del:

$$\frac{dx}{x} = -\frac{3}{2} \frac{dp}{p}$$

$$|x| = C |p|^{-\frac{3}{2}}$$

partikularni:

$$C = C(p)$$

$$\dot{x} = C(p) p^{-\frac{3}{2}} = \frac{1}{2} p^{-3}$$

$$\int_0^r \frac{1}{2} p^{\frac{3}{2}-3} = \frac{1}{2} \int_0^r p^{-\frac{3}{2}} dx = -p^{-\frac{1}{2}}$$

$$C(p) = -p^{-\frac{1}{2}}$$

$$x = C p^{-\frac{3}{2}} - p^{-2}$$

$$y = xy' - x^2(y')^3 = C p^{-\frac{1}{2}} - p^{-1} - (C^2 p^{-3} - 2C p^{-\frac{3}{2}-2} + p^{-4}) p^3$$

$$= C p^{-\frac{1}{2}} - p^{-1} - C^2 + 2C p^{-\frac{1}{2}} - p^{-1} =$$

$$= 3C p^{-\frac{1}{2}} - 2p^{-1} - C^2$$

Dobili smo rešitev v parametrični obliki $(x(p), y(p))$

$$\text{kjer je } p = \frac{\dot{y}(p)}{x'(p)} = y'(x)$$

Se singularne rešitve

$$x = -p^{-2} + D p^{-\frac{3}{2}}$$

$$y = -2p^{-1} + 3D p^{-\frac{1}{2}} - D^2$$

$$F(x, y, p) = y - xp + x^2 p^3 = 0$$

$$\text{torej } F_p = \cancel{y} + 3x^2 p^2 = 0 \Rightarrow x(1 + 3xp^2)$$

$$x=0 \Rightarrow y=0 \text{ ena točka ni aprotin } p$$

$$1 + 3xp^2 = 0 \Rightarrow p^2 = -\frac{1}{3x} \Rightarrow p = \pm \sqrt{-\frac{1}{3x}}$$

$x < 0$

$$y - x\left(\pm \sqrt{-\frac{1}{3x}}\right) + x^2 \frac{1}{3x} \left(\pm \sqrt{-\frac{1}{3x}}\right) = 0$$

$$\text{torej } y = \pm \sqrt{-\frac{1}{3x}} \left(x - \frac{1}{3}x\right) = \pm \sqrt{-\frac{1}{3x}} \cdot \frac{2}{3}x = \pm \frac{2}{3} \sqrt{-\frac{x}{3}}$$

$$y' = \pm \frac{2}{3} \cdot \frac{1}{2\sqrt{-\frac{x}{3}}} = \pm \frac{1}{3} \sqrt{-\frac{3}{x}}$$

Ali resimo še

$$\pm \frac{2}{3} \sqrt{-\frac{x}{3}} - x \left(\pm \frac{1}{3} \sqrt{-\frac{3}{x}}\right) + x^2 \left(\pm \frac{1}{3} \cdot \frac{3}{x} \cdot \sqrt{-\frac{3}{x}}\right) = 0$$

$$= \pm \frac{2}{3} \sqrt{-\frac{x}{3}} \mp \frac{1}{3} \sqrt{3x} \pm \frac{3}{93} \sqrt{3x} \neq 0$$

$$= \pm \frac{2}{3} \sqrt{-\frac{x}{3}} \mp \frac{78}{93} \sqrt{3x} = \sqrt{x} \left(\pm \frac{2}{3\sqrt{3}} \mp \frac{78\sqrt{3}}{93}\right) =$$

$$= \sqrt{3x} \left(\pm \frac{2}{9} \dots\right)$$

2. način: primerjamo y' in p

$$\pm \frac{1}{3} \sqrt{-\frac{3}{x}} = \pm \sqrt{-\frac{1}{3x}}$$

$$\pm \frac{\sqrt{3}}{3\sqrt{3}} = \pm \frac{1}{\sqrt{3}} \quad \times$$

2. nalyse

Dalo oi resitve

$$(y')^2 - xy' - y + \frac{x^2}{2} = 0$$

$$y = (y')^2 - xy' + \frac{x^2}{2}$$

$$y = p^2 - xp + \frac{x^2}{2}$$

$$dy = 2p dp - x dp - p dx + x dx \quad dy = p dx$$

$$2p dx - x dx = 2p dp - x dp$$

$$(2p - x) dx = (2p - x) dp$$

$$1. 2p - x = 0$$

$$2p = x \Rightarrow p = \frac{x}{2}$$

$$y = p^2 - 2p^2 + 2p^2 = p^2$$

$$2. 2p - x \neq 0$$

$$dx = dp$$

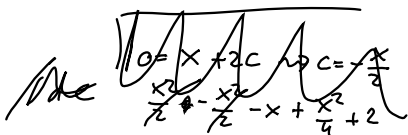
$$p'(x) = 1$$

$$p(x) = x + c$$

$$\begin{aligned} y &= 1 - x + \frac{x^2 + 2xc + c^2}{2} = \\ &= \frac{x^2 + 2x(c-1) + c^2 + 2}{2} = \\ &= \frac{x^2}{2} + x(c-1) + c^2 + 2 \end{aligned}$$

??

$$y = \frac{x^2}{2} + xc + c^2$$



singulærne:

$$0 = x + 2c$$

$$c = -\frac{x}{2}$$

$$\Rightarrow y = \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{4} = \frac{x^2}{4}$$

Clairautova enačba

$$y = xy' + \psi(y')$$

Videl: bomo, da ogrinjajo najdemo že sproti

3. naloga

$$a) y = xy' + \frac{1}{y'}$$

$$b) y = xy' + (y')^2 + 1$$

$$a) y = xy' + \frac{1}{y'} \quad \psi(t) = \frac{1}{t} \quad y' = p$$

$$y = px + \frac{1}{p}$$

$$dy = p dx + x dp - \frac{1}{p^2} dp \quad p dx = dy$$

$$\frac{1}{p^2} dp = x dp$$

$$0 = \left(\frac{1}{p^2} - x\right) dp$$

Clairautova enačba se vedno reducira na obliko $f(x, p) dp = 0$

$$1) dp = 0 \Rightarrow p = c$$

$$y = xc + \frac{1}{c}$$

$$2) \frac{1}{p^2} = x \Rightarrow p = \pm \sqrt{\frac{1}{x}}$$

$$y = \pm \sqrt{x} + \pm \sqrt{x} = \pm 2\sqrt{x}$$

$$y' = \pm \frac{1}{\sqrt{x}}$$

$dp = 0 \dots$ splošna rešitev

$f(x, p) = 0 \dots$ ogrinjajo

$$b) y = xp + p^2 + 1$$

$$dy = xdp + p dx + 2p dp \quad dy = p dx$$

$$(x + 2p) dp = 0$$

$$dp = 0 \Rightarrow p = C \Rightarrow y = xC + C^2 + 1$$

$$x = -2p \Rightarrow p = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x^2 + \frac{1}{4}x^2 + 1 = -\frac{1}{4}x^2 + 1$$

$$y' = -\frac{1}{2}x$$

Opomba o singularnih rešitvah Clairotovih enačb:

$$y = xy' + \psi(y')$$

$$F(x, y, p) = xp + \psi(p) - y = 0$$

$$F_p = x + \psi'(p) = 0$$

Splazna rešitev

$$0 = G(x, y, c) = xc + \psi(c) - y = F(x, y, c) = 0$$

Torej odvajanje v p ti vedno da ~~ta~~ ogrinjajo
pri Clairotovi enačbi

Lagrangeva enacba

$$y = x \varphi(y') + \psi(y')$$

4. Reži

$$y = \frac{xy'}{2} + \frac{2}{y'} \quad \varphi(y') = \frac{y'}{2}$$

$$dy = \frac{x}{2} dp + \frac{p}{2} dx - \frac{1}{p^2} dp \quad dy = dp \cdot p dx$$

$$\frac{1}{2} p dx = \frac{x}{2} dp - \frac{1}{p^2} dp$$

$$p dx = \left(x - \frac{1}{p^2}\right) dp$$

$$\dot{x} = \frac{dx}{dp} = \frac{x}{p} - \frac{1}{p^3} \quad \text{LDE za } x = x(p)$$

$$\dot{x} - \frac{x}{p} = -\frac{1}{p^3} \quad \text{homogeno:}$$

$$\frac{dx}{x} = \frac{1}{p} dp$$

parti:

$$|x| = C |p| \quad x = C \cdot p$$

$$C'(p) \cdot p = -\frac{1}{p^3} \Rightarrow C'(p) = -\frac{1}{p^4}$$

$$C = 3 \cdot 4 \cdot p^{-3}$$

$$\frac{x}{p} = 12 \cdot p^{-3} \cdot p + C \cdot p \Rightarrow \frac{x}{p} = \frac{12}{p^2} + C \cdot p$$

$$y = \frac{1}{2} \left(\frac{12}{p^2} + C p^2 \right) + \frac{2}{p} = \frac{6}{p^2} + \frac{1}{2} C p^2 + \frac{2}{p} = \frac{8}{3} + \frac{C}{2} p$$

Singularna rešitev: $0 = \frac{x}{2} - \frac{2}{p^2} = F_p$

$$\frac{x}{p^2} = \frac{x}{2}$$

$$p^2 = \frac{4}{x}$$

$$\Rightarrow y = \frac{2}{p^2} \cdot p + \frac{2}{p} = \frac{4}{p}$$

Ali je res $dy = p dx$?

$$\dot{y} = -\frac{4}{p^2}$$

Ali velja $\dot{y} = p \cdot \dot{x}$

$$\dot{x} = -\frac{8}{p^3}$$

$$\frac{4}{p^2} \cdot p \cdot \left(-\frac{8}{p^3}\right) = -\frac{8}{p^2} \neq -\frac{4}{p^2}$$

\Rightarrow To ni singularna rešitev

5. Resi $y = \frac{x(y)^2}{2} + \frac{2}{y}$.



;

Povzetek (o reševanju impl. DE)

$F(x, y, y') = 0 \quad \forall F \neq 0$ F dovolj gladka

$F(x, y, p)$ je ploskev

$$P(x_0, y_0, p_0) \in \mathbb{R}^3$$

$$1) F_p(P) \neq 0 \Rightarrow p = p(x, y)$$

$$F(x, y, p(x, y)) = 0 \text{ za } x, y \text{ blizu } x_0, y_0$$

$$p(x_0, y_0) = p_0$$

če je $x \mapsto y(x)$ rešitev začetnega problema $F(x, y, y')$
 $y(x_0) = y_0$

$$\text{Potem je } y' = p(x, y)$$

tako dobimo splošno rešitev

$$2) F_p(P) = 0 \Rightarrow$$

dobimo množico $\{F = F_p = 0\} \leftarrow$ unija točk in krivulj

če je P_0 izolirana točka:

na neki okolici U od P_0 velja

$$\forall F \neq 0 \text{ za } P \in U - \{P_0\}$$

dobimo splošno rešitev, vendar se lahko zgodi, da več čenic splošne rešitve poteka skozi P_0

če je $\{F = F_p = 0\}$ krivulja, jo parametiziramo

npr. kot $(x(p), y(p), p)$; če velja $p \cdot \dot{x}(p) = \dot{y}(p)$,

potem je to singularna rešitev (in hkrati

ogrinjača splošne rešitve)

Primer od zadnjic

18.11

$$y = \frac{xy^2}{2} + \frac{2}{y}$$

1) $F_p = 0$

$$F(x, y, p) = \frac{xp^2}{2} + \frac{2}{p} - y$$

$$F_p = 0 \Rightarrow xp - \frac{2}{p^2} = 0 \Rightarrow x = \frac{2}{p^3}$$

$$y = \frac{xp^2}{2} + \frac{2}{p} = \frac{2}{p^3} \cdot \frac{p^2}{2} + \frac{2}{p} = \frac{3}{p}$$

$$\dot{y} = -\frac{3}{p^2}$$

$$p\dot{x} = p \left(-\frac{6}{p^3}\right) = -\frac{6}{p^2} \neq \dot{y} \Rightarrow G \text{ ni ogrinjaca}$$

oz ni sing. res

2) $dy = p dx$

$$dy = xp dp + \frac{p^2}{2} dx - \frac{2}{p^2} dp$$

$$\left(\frac{p^2}{2} - p\right) dx + \left(xp - \frac{2}{p^2}\right) dp = 0$$

Ker nas zanimajo rešitve v obliki grafar funkcij $x \mapsto y(x)$ lahko predpostavimo, da $dx \neq 0$

\Rightarrow formalno lahko delimo z dx ; dobimo DE za

$$p = p(x).$$

Običajno je lažje rešavati DE za $x = x(p)$

želimo deliti z dp , da dobimo $\frac{dx}{dp} = \dot{x}(p)$

Ali je lahko $dp = 0$? $\Rightarrow p \equiv C = y \Rightarrow y(x) = Cx + D$ ustrezen D

$$\text{Sicer} \Rightarrow \left(\frac{p^2}{2} - p\right) \frac{dp}{dx} + xp - \frac{2}{p^2} = 0$$

$$\text{Opazimo: } dp = 0 \Rightarrow \frac{p^2}{2} - p = 0$$

$$dp \neq 0 \Rightarrow \text{dobimo DE, če } \frac{p^2}{2} - p \neq 0$$

$$\text{če } \frac{p^2}{2} - p = 0 \Rightarrow p = 0 \vee p = 2$$

\downarrow
ni definirana

$$\rightarrow y = 2x + 1$$

Ali je rešitev singularna?

$$F_p = xp - \frac{2}{p^2}$$

Premica $y = 2x + 1$ ustreza k. v. u. l. j. i

$$x \mapsto (x, y(x), y'(x)) =$$

$$(x, 2x + 1, 2)$$

$$F_p = 2x - \frac{1}{2} \Rightarrow$$

Povsod razen v $x = \frac{1}{4}$ je ta izraz $\neq 0$

Ta premica torej pripada splošni rešitvi

$x = \frac{1}{4}$ pa je singularna točka

lahko

\swarrow
omo goiskali:
zadnjic

Pavzetek pavzeta

- 1) $F_p = 0$, parametriziramo/eliminiramo p iz enačb
 $F = F_p = 0$, preverimo če je resitev
↳ singularna
- 2) Splošno rešitev iščemo s pomočjo nastanka $dy = p dx$

Naj bo $y = \Phi(x, c)$ splošna rešitev DE $F(x, y, y') = 0$
in 4 ogrinjača. Pokaži da skozi vsako na grafu 4
poteka neskončno mnogo rešitev

DN

Reš DE

$$y = xp^2 - 2p^3$$

$$y = (xy')^2 - 2(y')^3$$

$$dy = p dx = p^2 dx + 2p x dx - 6p^2 dp$$

$$dx(p^2 + p) + 6p^2 dp = 0$$

$$\uparrow dp = 0 \Rightarrow p = c, y = C^2 x - 2C^3$$

$$y' = C^2 = p = c$$

$$C^2 = c \Rightarrow c = 0 \vee c = 1$$

$$\downarrow \\ y = c \\ \text{sing.}$$

$$\rightarrow y = x - 2 \\ \text{ni singularna}$$

$$F_p = F = 0$$

$$-2xp + 6p^2 = 0$$

$$-2p(x - 3p) = 0$$

$$p = 0 \Rightarrow$$

$$x = 3p \Rightarrow$$

$$F = 0 \Rightarrow 3p^3 - 2p^3 = 0 \\ p^3 = 0$$

$$\dot{x}(p) = 3$$

$$\dot{y}(p) = 3p^2$$

$$p \dot{x} \neq \dot{y} \Rightarrow \text{to ni rešitev}$$

$$dp \neq 0 \Rightarrow$$

$$\dot{x}(p - p^2) + 6p^2 - 2px = 0 \quad / : p(1-p) \quad \leftarrow \text{tako že dobavljamo,}$$

koje to 0

$$\dot{x} + x \frac{-2p}{p-p^2} = \frac{6p^2}{p^2-p}$$

homogeni del:

$$\frac{dx}{x} = \frac{-2p dp}{p-p^2} = \frac{-2}{1-p} dp$$

$$\ln|x| = -2 \ln|1-p|$$

$$x = D(1-p)^{-2}$$

partikularni:

$$D'(1-p)^{-2} = \frac{6p}{p-1}$$

$$D' = 6p(p-1) = 6p^2 - 6p$$

$$D = 2p^3 - 3p^2 + C$$

$$\text{Splošno: } x = \frac{2p^3 - 3p^2 + C}{(1-p)^2}$$

Enačbe višjega reda

$$F(x, y^{(k)}, \dots, y^{(n)}) = 0 \quad 0 \leq k \leq n$$

Lahko znižamo red z nastavitvijo: $z = y^{(k)}$

$$\hookrightarrow F(x, z, \dots, z^{(n-k)}) = 0$$

$$xy''' = y''$$

$$z = y''$$

$$xz' = z$$

$$\frac{dz}{z} = C \frac{dx}{x} \leadsto |x| = |z| \Rightarrow x = |z| = |y''|$$

$$z = C \cdot x = y''$$

↖ ± skrajšamo v konstanto

$$\begin{aligned} \int \int (Cx \, dx) \, dx &= \int C \frac{1}{2} x^2 \, dx = \int Cx^2 \, dx = C \frac{1}{3} x^3 + Dx + E = \\ &= Cx^3 + Dx + E \end{aligned}$$

②

$$F(y, y', \dots, y^{(k)}) = 0 \quad \hookrightarrow F \text{ neodvisen od } x$$

$$z(y) = y'(y) \quad \text{zod } y \text{ je neodvisen od } y$$

Interpretiramo kot:

$$z(y) = y'(x(y)) \quad z(y(x)) = y'(x)$$

$y \mapsto x(y)$ inverz residue y
(obstaja ker $y'(x) \neq 0$)

Reš enačbo

$$y y'' = y'(y' + 1)$$

$$y' = 0 \Rightarrow y y'' = 0$$

Možnost ko ne moremo operirati
naštevek

$$1) y = 0$$

$$2) y'' = 0 \Rightarrow$$

$$y' = 0 = 0 \Rightarrow y = C$$

$$\Rightarrow y = C \in \mathbb{R}$$

$$z = y'(x(y))$$

$$z(y(x)) = y'(x)$$

$$\frac{dz}{dy} \cdot y' = y''(x)$$

$$\dot{z} y' = y''$$

$$\dot{z} \cdot z = y''$$

Nesemo v enačbo

$$y \cdot \dot{z} \cdot z = z(z+1)$$

$$\uparrow z = 0$$

$$2) z \neq 0 \Rightarrow \text{lahko delimo}$$

$$2) \quad y \dot{z} = z + 1$$

$$\text{hom: } \frac{dz}{dy} \cdot y = z \quad z = C y$$

partikularna:

$$y \cdot C' y = 1$$

$$C' = \frac{1}{y^2} \leadsto C = -\frac{1}{y} + D$$

$$z = -1 + D y$$

$$\frac{y'}{-1 + D y} = 1$$

$$x = \int \frac{dy}{-1 + D y} = \frac{1}{D} \ln(-1 + D y) + C$$

$$D x + C = \ln(D y - 1)$$

$$\frac{C e^{D x} + 1}{D} = y$$

$$D = 0 \Rightarrow$$

$$\frac{y'}{-1} = 1$$

$$y' = -1$$

$$y = z - x$$

$$F(x, y, y', \dots) = 0$$

F homogena v y, y', \dots

$$F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y', \dots) \quad \text{za nek } k \text{ i za } \forall \lambda \in \mathbb{R}$$

$$\dots \text{ uvedemo } z(x) = \frac{y'}{y}$$

$$yy'' - x^2 y^2 + (y')^2$$

$$z = \frac{y'}{y} \quad yz = y' \quad y'' = y'z + z'y = z^2 y + z'y$$

$$y(yz^2 + z'y) = x^2 y^2 + (yz)^2$$

$$y=0 \Rightarrow \text{je rešitev}$$

$$y \neq 0 \Rightarrow$$

$$z^2 + z' = x^2 + z^2$$

$$z = \frac{1}{3}x^3 + C = \frac{y'}{y} \Rightarrow \ln|y| = \frac{x^4}{12} + Cx + D$$

$$y = D e^{\frac{x^4}{12} + Cx}$$

Ekstremni izreki

$f: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ zvezna in Lipschitzova v y

$$|f(x, y_0) - f(x, y_1)| < L|y_0 - y_1| \quad \text{za } L \geq 0$$

$\Rightarrow \exists!$ rešitev $y' = f(x, y)$, $y(x_0) = y_0$ $\forall (x_0, y_0) \in D$, ki je

definirana na intervalu $(x_0 - \alpha, x_0 + \alpha)$, kjer je

$\alpha = \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$, kjer sta $a, b > 0$ takšna, da je

$$\underbrace{[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]}_{C_{a,b}} \subset D \quad \text{in je } \text{~~to maksimum~~}$$

$$M = \max_{x, y \in C_{a,b}} |f(x, y)|$$

območje: $f \in C^1[a, b] \Rightarrow f$ je Lipschitzova

Podana je Cauchyjeva naloga

$$y' = y + \cos y \quad y(0) = 0$$

a) Utemelji da obstaja natančno ena rešitev in določi

Taylorjev polinom stopnje 3 v $x=0$

b) določi α

c) Dokazi, da se rešitev razširi na \mathbb{R}

d) Dokazi da velja $\lim_{x \rightarrow \infty} y(x) = \infty$

a) Fj je zveza na \mathbb{R}^2

Ali je Lipschitzeva? Jeker je $v C^1$

\Rightarrow Rešitev obstaja in je enolično določena

$$y'' = y' - \sin(y)y' = (y + \cos y)(1 - \sin(y))$$

$$\begin{aligned} y''' &= (y' - y' \sin(y))(1 - \sin(y)) + (y + \cos y)(1 - \cos(y)y') = \\ &= (y + \cos y)(1 - \sin(y))^2 + (y + \cos(y))(1 - y \cos y + \cos^2 y) \end{aligned}$$

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + o(x^3) = \\ &= x + \frac{1}{2} x^2 + o(x^3) \end{aligned}$$

$$b) \quad L = \max_{C(a,b)} \left| \frac{\partial f}{\partial y} \right| \quad f(x,y) = y + \cos y$$

$$\left| \frac{df}{dy} \right| = |1 - \sin(y)| \leq 2 \quad L = 2$$

$$\alpha = \min(a, \frac{b}{M}, \frac{1}{2})$$

$$\max_{C(a,b)} f(\cdot, y) = \max_{[-b,b]} f(\cdot, y) = M$$

$$|f(x,y)| = |y + \cos y| \leq |y| + |\cos y| \leq b + 1$$

$$\frac{b}{M} \geq \frac{b}{b+1}$$

$$\alpha = \min \left\{ a, \frac{b}{b+1}, \frac{1}{2} \right\}$$

je izabrano a in b dovolj velika, je

$$\text{lahko } \alpha > \frac{1}{2}$$

\Rightarrow primer ne $[-\frac{1}{2}, \frac{1}{2}]$

c) Pogoj iz b) velja za vsako točko.

Torej če si postopoma izbiramo točke vedno bolj na robu, ki je manjša, lahko razširimo intervale na cel \mathbb{R}

razen namesto b) opredelimo b) + b)

Sklep: če lahko α omejeno nevedel neodvisno
od začetnega pogoja, potem se rešitev
razširi na \mathbb{R}

V posebnem to velja: če sta f in f_y omejena

$$d) \lim_{y \rightarrow 0} y(x) = \infty$$

y narašča pozitivno od 1

$$y' > 0 \text{ velja za } t > 1$$

$$1 - \sin t \geq 0 \text{ za } t > 0 \Rightarrow \text{funkcija narašča}$$

Dokazujemo da je $y(x) > 0 \forall x > 0$

$$\text{Reimamo } y(\tilde{x}) = 0 \text{ za nek } \tilde{x} > 0$$

$$y'(\tilde{x}) = 0 + \cos(0) = 1$$

Torej je mogla biti prej funkcija negativna

$$Z = \frac{1}{2} x, > 0, \quad x(x) = 0$$

$$y_0 = 1 \quad y = f(x)$$

$$\Rightarrow f(x) > 0$$

$$\text{za } x \in (0, \epsilon) \text{ za } \epsilon > 0$$



$$\text{Za } \forall \epsilon: x_1 \in (0, \tilde{x})$$

$$f'(x_1) = 0$$

$$f'(x_1) = f(x_1) + \cos(f(x_1)) = 0$$

$$\Rightarrow \cos f(x_1) < 0$$

$$f(x_1) > \frac{\pi}{2}$$

$$f'(x_1) > \frac{\pi}{2} + \cos(\lambda_1) > 0 \quad *$$

$$\Rightarrow f(x) > 0 \text{ za } \forall x > 0$$

$$\Rightarrow f'(0) \text{ jo naraščajoča}$$

Ekzistenčni izrek

$f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ zveza in Lipschitzova

$\Rightarrow \exists!$ rešitev $y = f(x, y)$ $y(x_0) = y_0$ za $\forall (x_0, y_0)$,
k je definirana na intervalu $(x_0 - \alpha, x_0 + \alpha)$,
kjer je $\alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$; $M = \max |f|$, L Lipsch.,
 $a, b > 0$. $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset D$

$f \in C^1[a, b] \Rightarrow f$ je Lipschitzova

$$F(x, y, y') = 0$$

F homogena v y, y' ($F(x, \lambda y, \lambda y') = \lambda^k F(x, y, y')$)

\Rightarrow uvedemo $z(x) = \frac{y'}{y}$

Enačba višjega reda $F(x, y^{(k)}, \dots, y^{(n)}) \Rightarrow z = y^{(k)}$

Reševanje impl. DE

$$F(x, y, y') = 0 \quad \nabla F \neq 0$$

1) $F(x, y, p) = 0$ je ploskev

$$P = (x_0, y_0, z_0)$$

2.1) $F_p(P) \neq 0 \Rightarrow p = p(x, y)$

$$F(x, y, p(x, y)) = 0 \text{ v okolici } P$$

$$p(x_0, y_0) = p_0$$

$$y(x) \text{ rešitev} \Rightarrow y' = p(x, y)$$

Dobimo **splošno rešitev**

2.2) $F_p(P) = 0 \Rightarrow \{F = F_p = 0\}$ je unija točk in krivulj

P_0 izdirlana točka $\Rightarrow \exists$ okolica P . $P \in U - \{P_0\}$ v $F \neq 0$

Dobimo **splošno rešitev**

$\{F = F_p = 0\}$ je krivulja \Rightarrow parametriziramo
npr. $(x(p), y(p), p)$.

$$p \cdot \dot{x}(p) = \dot{y}(p) \Rightarrow \text{Singularen rešitev}$$


Ogrinjača splošne rešitve

Clairautova enačba

$$y = xy' + 4(y') \Rightarrow y' = p \Rightarrow p dx = dy$$

$$\text{Dobimo: } f(x, p) dp = 0$$

$$dp = 0 \dots \text{singularna rešitev}$$

$$f(x, p) = 0 \dots \text{ogrinjača}$$

Odvajanje po p , ti vedno da ogrinjačo pri tej enačbi

Prvi integral enačbe $y' = f(x, y)$ je funkcija, ki je konst.

vzdolž vsake rešitve

$$y = \varphi(x, C) \wedge \varphi_C \neq 0 \Rightarrow C = C(x, y)$$

$$1) \text{ DE zapišemo ko } P dx + Q dy = 0$$

$$Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y \Rightarrow (P, Q) \text{ je potencialno}$$

$$Q = \frac{\partial}{\partial y} \int P dx + C'(x, y)$$

Riccatijeva enačba

$$y' = a^2 y + b y + c$$

1) Uganemo rešitev y_p

2) Splošna rešitev je $y = y_p + u(x)$

3) Dobimo Bernullijev enačbo $y = y_p + \frac{1}{u(x)} \Rightarrow \text{LDE}$

Bernullijeva enačba

$$y' + a y = b y^\alpha \quad \alpha \notin \{0, 1\} \Rightarrow z = y^{1-\alpha}$$

25.11

Od zadatje naprej $y' = y + \cos y$ $y(0) = 0$

Rešitev f je definirana za $\forall x \in \mathbb{R}$

$y > 0$ za $\forall x \geq 0$

$$Z = \{x > 0, f(x) = 0\}$$

$$f'(0) = f(0) + \cos(0) = 0 + \cos 0 = 1$$

$$\Rightarrow \text{za nek } \varepsilon > 0 \text{ je } f(x) > 0 \quad \forall x \in (0, \varepsilon)$$

$$\text{oznaino } x_0 = \inf Z > 0$$

$$f \text{ zvezna} \Rightarrow f(x_0) = 0$$

$$\text{Rolle: } \exists x_1 \in (0, x_0), f'(x_1) = 0 \wedge f(x) > 0 \quad \forall x \in (0, x_1)$$

$$f \text{ res} \text{ CN} \Rightarrow f'(x_1) = f(x_1) + \cos(f(x_1))$$

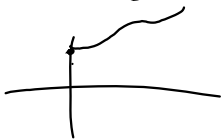
$$\underset{0}{\vee} \Rightarrow f(x_1) > \frac{\pi}{2}$$

$$\Rightarrow f(x_1) - \cos(f(x_1)) > 0$$

$$\Rightarrow Z = \emptyset$$

✗

$$g(t) = t + \cos t \quad g(0) = 1 \quad g'(t) = 1 - \sin t \geq 0 \quad \forall t$$



$$g(t) \geq 1, \quad \forall t \geq 0$$

$$f'(x) = g(f(x)) \geq 1 \quad \forall x \geq 0$$

$$f(x) = f(0) + \int_0^x f'(t) dt \geq \int_0^x 1 dt = x \xrightarrow{x \rightarrow \infty} \infty$$

1) Dane je Cauchyjeva naloga

$$y' = \frac{y}{1+x^2+y^2} \quad y(0)=1$$

Dokazi da obstaja ! rešitev, ki je definirana na \mathbb{R} .

Ali ima y ničlo?

$f(x,y)$ mora biti zveza in Lipschitzova v y

$$f(x,y) = \frac{y}{1+x^2+y^2} \quad \text{zveza zaradi elementarnosti}$$

Naredimo pravokotnik $[a, a] \times [-b, b]$

Ta m je Lipschitzova, ker je skrajno omejen

\Rightarrow obstaja in je enolično določena na $[-\alpha, \alpha]$

$$\text{kjer je } \alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$$

a, b poljubno velika

$$M = \max_{[a,b]} |f(x,y)| = \max_{x^2=0} \frac{y}{1+y^2} = \left\{ \pm \frac{1}{2} \right\} = \frac{1}{2}$$
$$\hookrightarrow \frac{(1+y^2) - 2y^2}{(1+y^2)^2} = \frac{1-y^2}{1+y^2}$$

$$L = \max \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{1+x^2+y^2-2y^2}{(1+x^2+y^2)^2} = \frac{1+x^2-y^2}{(1+x^2+y^2)^2}$$

L je tudi omejena

Dobili smo α , ki je isti za vse točke in neodvisen od začetne pogoja, tako da lahko z obstojem enim rešenjem rešimo največ točk

$$b) \quad y' = 0 \Leftrightarrow y = 0$$

Recimo da je $f(x_0) = 0$

$$z \equiv 0 \quad z' = f(x,y) \text{ res } z(x_0) = 0$$

f tudi res to nalogo $\Rightarrow f = z$ (zaradi enoličnosti)

\rightarrow ,ker je $y(0) = 1$

Trditev: Za α lahko vzamemo $\alpha = \min\{\alpha, \frac{1}{n}\}$
L mae pa obstajati na tem pravokotniku

2)

$f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ zvezna

$$y'(x) = 1 + \frac{y(x)^2}{2} \sin f(x) \quad ; x \in [0, 2] \\ y(0) = 0$$

Dokaži da je

$$\max_{x \in [0, 2]} y(x) \geq 1$$

Pretpostavimo da $\max y(x) < 1 \Rightarrow$

$$y'(x) \geq 1 + \frac{y(x)^2}{2} \geq \frac{1}{2}, \forall y \in [0, 1]$$

$y \geq 0$

$x_0 = \inf \{x; y(x_0) = 0\}$

Pretpostavimo da $\exists x_0 > 0$ da je $y(x_0) = 0$

Potem po Rollovemu izreku, $\exists x_1 \in (0, x_0)$ da je $y'(x_1) = 0$

$$y'(x_1) = 1 + \frac{(y(x_1))^2}{2} \sin(f(x_1))$$

$$0 \leq y(x_1) \leq 1$$

$$y'(x_1) \geq 1 - \frac{(y(x_1))^2}{2} \geq \frac{1}{2}$$

$\Rightarrow y > 0$ za $\forall x \in \mathbb{R}_{\geq 0} \cap [0, 2]$ i kar je odred u o pozitivna

$$\Rightarrow y'(x) \geq 1 - \frac{1}{2} \geq \frac{1}{2}$$

$$\text{Potem je } y(2) = y(0) + \int_0^2 y'(t) dt \geq 0 + 2 \cdot \frac{1}{2} = 1 \quad *$$

$$\Rightarrow \max y \geq 1$$

3.

$$y' = f(y) \quad f \in C^1(\mathbb{R})$$

Pokaži da je \forall periodična rešitev konstantna

Recimo da $\exists \omega > 0$. $y(x + \omega) = y(x) \quad \forall x$

$$\int_0^\omega y' dx = \int_0^\omega f(y) dx = c$$

y periodična $\Rightarrow f(y)$ periodična

če je $f(c) = 0$, potem je $y \equiv c$ rešitev

$f(y)$ je odvod $\Rightarrow f$ ima ničlo (ker je y periodična)

$\Rightarrow \exists x_0 \in \mathbb{R}$. $f(y(x_0)) = 0$ $y \equiv y(x_0)$ je ena rešitev
sistema in zaradi enoličnosti je to edina

4.

$$y' = y + e^y \quad y(0) = 1$$

a) Dokaži, da je definicijski interval rešitve, kot ga določi obstojni interval omejen

b) Dokaži, da je f naraščajoča v \forall točki $x \geq 0$, kjer je definirana

c) Dokaži, da ima φ ničlo

d) $\exists w. 0 < w < \infty. \lim_{x \rightarrow w} \varphi(x) = \pm \infty$

a) $y + e^y$ je vedno odvedljiva \Rightarrow Fendlicna rešitev

na $[\alpha, \beta]$ $\alpha = \min\{a, \frac{b}{M}, \frac{1}{\Delta}\}$

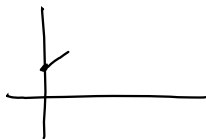
$$y \in [1-b, 1+b] \Rightarrow y + e^y \leq 1+b + e^{1+b}$$

$$M = 1+b + e^{1+b} \Rightarrow \frac{b}{M} = \frac{b}{1+b + e^{1+b}} \leq C \quad \forall b > 0$$

omejena ta funkcija je

b) $y' = y + e^y$

$$y' \geq 0 \quad y(0) = 1$$



Rešimo da je $y + e^y = 0$

$$e^y = -y$$

\forall
0

če je $y=0$ pomeni, da je $y' = 1$ kar pomeni, da v vsaki točki narašča, kar pa ne more biti, ker rešen za eno ničlo

c) f ma nido

Rečno da nima nido. pden je $f(x) > 0 \forall x \in \mathbb{R}$,

pden je pa f naraščajoa

n odvod je večji od 1. pden pnda nima

skhati x as ali pa nido ne ulejo ne mereno
razširiti če je pd, ne more biti nagnen
tečj so pachten, pden je pa odvod

< 0 ✗

$$f(x) = f(0) + \int_0^x f'(t) dt \leq f(0) + x \xrightarrow{x \rightarrow \infty} -\infty$$

d) Dokazujemo:

$$\exists \omega > 0. \lim_{x \rightarrow \infty} f(x) = \infty$$

Dokazimo da je inverz omejen. (inverz obstaja ker je f
strogo naraščajoa na $x \geq 0$)

$$\frac{1}{z'} = y + e^y \Rightarrow z' = \frac{1}{y + e^y}$$

$$z(y) = \int_0^y \frac{1}{t + e^t} dt \leq \int_0^y \frac{1}{e^t} dt = e^{-1} - e^{-y} \leq M$$

dostaja tukaj

↑
 $y \geq 0$

Naj bo $f \in C^1(\mathbb{R}^2)$, zadošča predpostavkam eksistenčnega izreka. Recimo, da se naše rešitve y ne da razširiti cez interval (α, ω) ; $-\infty < \alpha < x_0 < \omega < \infty$, potem velja $\lim_{x \rightarrow \omega} y(x) = \pm \infty$ in $\lim_{x \rightarrow \alpha} y(x) = \pm \infty$

5)

$$y' = \frac{1}{x^2 + y^2} ; y(0) > 0$$

a) Pokaži, da $\exists!$ φ rešitev, ki je definirana na \mathbb{R}

b) pokaži $\lim_{x \rightarrow \infty} \varphi(x) < \infty$

$f(x, y) = \frac{1}{x^2 + y^2}$ je omejena razen v točki 0

$f_y(x, y) = \frac{2y}{(x^2 + y^2)^2}$ je omejen razen na točki 0

Dokazimo da je rešitev omejene nevsakega konarnega intervala

v 0 je rešitev pozitivna in narašča

$$\varphi(x) = \varphi(0) + \int_0^x \frac{1}{t^2 + \varphi(t)^2} dt = \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2 + (\varphi(t))^2} dt \leq$$

$$\leq \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2} dt = \varphi(x_0) - \frac{1}{x} + \frac{1}{x_0} \leq M$$

↑ omejeno

\Rightarrow Po trditvi: je $\omega = \infty$

Popravimo f tako, da nima več pda v (0,0) in velja

$f = \tilde{f}$ izven neke majhne okolišice (0,0) ki je gost φ ne seka

v ~~tem~~ ∞ smer je simetrična $\varphi(-x)$

6)

Pokažimo da je $\varphi(x)$ naraščajoča

in omejena \Rightarrow ima limito

Vektorsko polje na \mathbb{R}^n je preslikava $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Takovnica polja V je krivulja $\gamma: (a, b) \rightarrow \mathbb{R}^n$, ki
 \downarrow
zadostuje, da je $\dot{\gamma} = V(\gamma(t))$

$$\gamma(t) = (x_1 \dots x_n)$$

$$\dot{x}_1 = V_1(\gamma(t))$$

$$\vdots$$
$$\dot{x}_n = V_n(\gamma(t))$$

Do zdaj: obravnavali smo primere za $n=0$, vendar v
povečini oblike $y'(x) = f(x, y(x))$

\downarrow
 x

\downarrow
 y

Newtonovi sistemi

Enačbi $y' = f(x, y)$ priredimo sistem oz. avtonomno
vektorsko polje

$$\dot{x} = 1$$

$$y' = f(x, y)$$

2.12

V 1. uro

Misl: mdeje dekej panembno

3. neka
 $f \in C^\infty(\mathbb{R})$

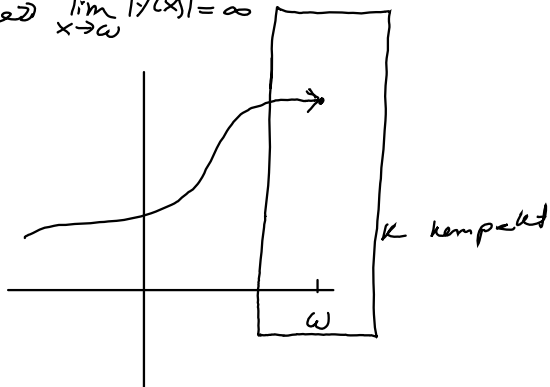
y rešitev

$$y' = f(x, y)$$

$y(x_0) = y_0$, tako da se ne razširi čez

$$(\alpha, \omega) \quad \omega < \infty$$

Dokazi $\lim_{x \rightarrow \omega} |y(x)| = \infty$



Graf rešitve je vektorsko polje $V(1, f(x, y))$. Izberemo kompaktni pravokotnik $[\omega - \delta, \omega + \delta] \times [-l, l]$

Izrek nam pove: če je x dovolj blizu ω , je tokanica bodisi pod $-l$ in nad l

Sklep: če x dovolj blizu ω je $|y(x)| > l$ za $\forall l$
 $\Rightarrow \lim_{x \rightarrow \omega} |y(x)| = \infty$

$$y' = \frac{1}{x^2 + y^2}$$

$$y(0) > 0$$

y je definirana na \mathbb{R}

Odvod je omejen, ker je $y' > 0 \forall x \in \mathbb{R}$, torej je y naraščajoča, večji kot je x^2 , večji je y , vedno manjši je odvod. \Rightarrow

$$y' = \frac{1}{x^2 + y^2} \leq \frac{1}{y^2} \leq \frac{1}{y_0^2} \quad \text{za } x \geq 0$$

$$\leadsto y(x) \leq \frac{1}{y_0} x + y_0 \Rightarrow \text{če je } \omega \text{ končen,}$$

$y(x)$ ne zapusti vsakega kompakta

za $x < 0$. y vedno narašča

$$y' = \frac{1}{x^2 + y^2} \leq \frac{1}{x^2} \rightarrow 0 \Rightarrow y' \rightarrow 0$$

$\Rightarrow y'$ je omejen

N4.

a) Izrazi polje $(1,0)$ in $(0,1)$, ki sta padeni v polarnih koordinatah (r, φ)

b) Določi tok polja $V(x,y) = a(x^2+y^2)(y-x)$, kjer je $a \in C^\infty(\mathbb{R})$

$$a) \quad \dot{r} = 1 \Rightarrow r(t) = r_0 + t$$

$$\dot{\varphi} = 0 \Rightarrow \varphi(t) = \varphi_0$$

$$x = r \cos \varphi \leadsto x(t) = r_0 \cos \varphi_0 + t \cdot \cos \varphi_0$$

$$y = r \sin \varphi \leadsto y(t) = r_0 \sin \varphi_0 + t \sin \varphi_0$$

$$\dot{x}(t) = \cos \varphi_0 = \cos \varphi(t)$$

$$\dot{y} = \sin \varphi_0 = \sin \varphi(t) \quad \text{''}$$

$$\left. \begin{aligned} \dot{x} &= \frac{x}{\sqrt{x^2+y^2}} \\ \dot{y} &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned} \right\} W(x,y)$$

$$\dot{r} = 0 \leadsto r = r_0$$

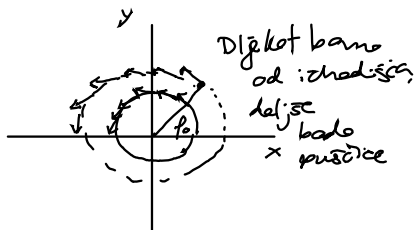
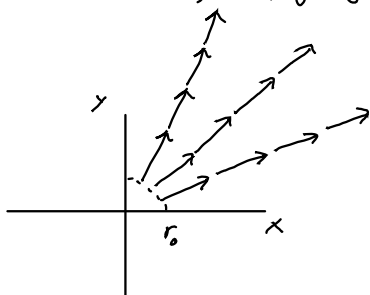
$$\dot{\varphi} = 1 \leadsto \varphi = \varphi_0 + t$$

$$\dot{x} = r \cos \varphi = r_0 \cos(\varphi_0 + t)$$

$$y = r \sin \varphi = r_0 \sin(\varphi_0 + t)$$

$$\dot{x} = -r_0 \sin(\varphi_0 + t) = -y$$

$$\dot{y} = r_0 \cos(\varphi_0 + t) = x$$



$$Q(x,y) = (-y, x)$$

b)

$$a(x^2+y^2) \cdot (y, -x) \stackrel{\substack{\uparrow \\ \text{v polarnih} \\ \text{koordinatah}}}{=} a(r^2)(-(0, 1)) = ar^2(-Q) \quad \dot{r} = 0$$

$$\dot{r} = a(r^2) \cdot 0 = 0 \quad \leadsto r = r_0$$

$$\dot{\varphi} = a(r^2) \cdot 1 = a(r^2) \quad \leadsto \varphi(t) = \int a(r^2) dt$$

$$(x, y) = F(r, \varphi)$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_F \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = J_F \cdot \underset{\substack{\uparrow \\ \text{nekopije}}}{P(r, \varphi)} = \varphi_0 + t a(r^2)$$

$$x(t) = r(t) \cos \varphi(t) = r_0 \cos(\varphi_0 - a(r_0^2)t)$$

$$y(t) = r(t) \sin \varphi(t) = r_0 \sin(\varphi_0 - a(r_0^2)t)$$

$$x(t) = \underbrace{r_0 \cos \varphi_0}_{x_0} \cos(a(r_0^2)t) + \underbrace{r_0 \sin \varphi_0}_{y_0} \sin(a(r_0^2)t) =$$

$$= x_0 \cos(a(x_0^2 + y_0^2)t) + y_0 \sin(a(x_0^2 + y_0^2)t)$$

DN $\dot{r} = \sin r$

$$\dot{\varphi} = \cos r$$

$$\vec{X} = A(t) \times G(t)$$

$$A(t) = \text{const}$$

$$\hookrightarrow e^{A \cdot t}$$

$$e^A = 1 + A + \frac{A^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Dado $A \in \mathbb{R}^{2 \times 2}$

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R}$

c) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

a)
$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^k}{k!} & 0 \\ 0 & \frac{(2t)^k}{k!} \end{bmatrix} =$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

b) Jordan normal form

$$A = \lambda I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

~~$A^k = (\lambda I + N)^k$~~ \leftarrow kommutira $\text{linear combination}$

$$e^{At} = e^{(\lambda I + N)t} = e^{\lambda I t} e^{N t} =$$

$$= e^{\lambda t} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} N^k \right) =$$

$$= e^{\lambda t} \left(\frac{t^2}{2} N^2 + \frac{t}{1} N + \frac{t^2}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) =$$

$$e^{\lambda t} \begin{bmatrix} 1 + \frac{t^2}{2} & t & \frac{t^2}{2} \\ 0 & 1 + t & t \\ 0 & 0 & 1 \end{bmatrix}$$

c) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ Učinkovito: to je matrika s kompleksnim

Ansatz:

$$e^{At} = e^{(a+bi)t} = e^{at} e^{ibt} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

$$A = aI + bJ$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$J^2 = -I$$

$$J^3 = -J \quad J^4 = I$$

$$e^{At} = e^{(aI+bJ)t} = e^{at} e^{bJt} =$$

$$= e^{at} \left(I + \frac{J^2 t^2}{2!} + \frac{J^4 t^4}{4!} + \frac{J^6 t^6}{6!} + \dots \right) =$$

$$= e^{at} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (bt)^{2k} I + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (bt)^{2k+1} J \right) =$$

$$= e^{at} (\cos bt I + \sin bt J) = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

Sistemi NDE s konstantnimi koeficienti

$$\dot{\vec{x}} = A\vec{x} \quad A \in M_{n \times n}$$

Vseke rešitev je oblike $\vec{x}(t) = \Phi(t)\vec{x}(0)$, kjer je

Φ fundamentalna rešitev (rešitvena matrika) sistema

$$\dot{\Phi} = A\Phi \quad \Phi(0) = id$$

Kako poiščemo Φ ?

$$\begin{aligned} \psi &= e^{At} & \psi(0) &= I \\ \dot{\psi} &= Ae^{At} = A\psi(t) \end{aligned} \quad \checkmark$$

$$\Phi = \psi$$

$A = PJP \dots$ jordanova A matrika

$$e^{At} = e^{(PJP^{-1})t} = \sum \frac{P J^k P^{-1} t^k}{k!} = P e^{Jt} P^{-1}$$

Predpostavljajmo, da ima A realne lastne vrednosti

Poisci fundamentalna resjku sistema

$$\dot{X} = X - Y + 4Z$$

...

we feb!

Poizdi Amdaneteln ~

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 3 & 4 \\ -2 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 3-\lambda & 1 & 2 \\ 3 & 3-\lambda & 4 \\ -2 & -1 & -1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) =$$

$$(3-\lambda)(\lambda^2 + \lambda(1-3) + 4) - 1(-3\lambda - 3 + 8) + 2(-3 + 6 - 2)$$

$$= (3-\lambda)(\lambda-1)^2 + (-\lambda+1) =$$

$$= (\lambda-1)(-\lambda^2 + \lambda(3+1) - 3 - 1) =$$

$$= -(\lambda-1)(\lambda^2 - 4\lambda + 4) = -(\lambda-1)(\lambda-2)^2$$

$$J = \begin{bmatrix} 1 & & 0 \\ & 2 & 0 \\ & & 2 \end{bmatrix} \quad P = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1:$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 4 \\ -2 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 2 \\ & & \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ & & \end{bmatrix}$$

$$x=0$$

$$y=-2z$$

$$\lambda_2 = 2$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ & & \end{bmatrix}$$

$$\ker(A-2I):$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$x+z=0$$

$$\ker(A-2I) = \text{Lin} \left\{ \begin{bmatrix} x \\ y \\ -x \end{bmatrix} \right\} =$$

$$\text{Lin} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\leftarrow \text{je } v \text{ jedru, ni } \mu \text{ lastu, } v \text{ lastu}$$

$$\vec{v}_1 = (A-2I)\vec{v}_2$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^t & & \\ & e^{2t} + e^{2t} & \\ & & e^{2t} \end{bmatrix}$$

Nelinearni: linearni sistem. NDE s konst. koef.

$$\dot{\vec{v}} = A\vec{v} + f(t) \quad // \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\vec{v}_h = e^{At} \vec{v}_0 \quad \vec{v}_p = e^{At} \cdot w(t)$$

$$\begin{aligned} \vec{v}_p &= A \underbrace{e^{At}}_{\parallel A\vec{v}_p} w + e^{At} \cdot \dot{w} = A\vec{v}_p + f \\ e^{At} \cdot \dot{w} &= f \\ \dot{w} &= e^{-At} \cdot f \end{aligned}$$

$\Phi(t)$ fundamentalni rešetor $\Rightarrow \Phi$ obratljiva za vse t , kjer je definirana
 $\Phi(t) = \Phi(-t)$

$$w(t) = \int_0^t e^{-A\xi} f(\xi) d\xi$$

$$v_p = e^{At} \cdot w(t) = \int_0^t e^{A(t-\xi)} f(\xi) d\xi$$

Poiščite splošno rešitev sistema

$$\dot{x} = x - y + \frac{1}{\cos t}$$

$$\dot{y} = 2x - y$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = -(1-\lambda)(-1-\lambda) + 2 = \lambda^2 - 1 + 2 = \lambda^2 + 1$$

$$\lambda = \pm i$$

vedno nove koordinate $u = x$

$$v = y - x$$

$$\dot{v} = \dot{y} - \dot{x}$$

$$y = v + u$$

$$\dot{y} = \dot{v} + \dot{u}$$

$$\dot{u} = u - v - u + \frac{1}{\cos t} = -v + \frac{1}{\cos t}$$

$$\dot{v} - \frac{1}{\cos t} = 2u - v - u =$$

$$\dot{v} = u + \frac{1}{\cos t}$$

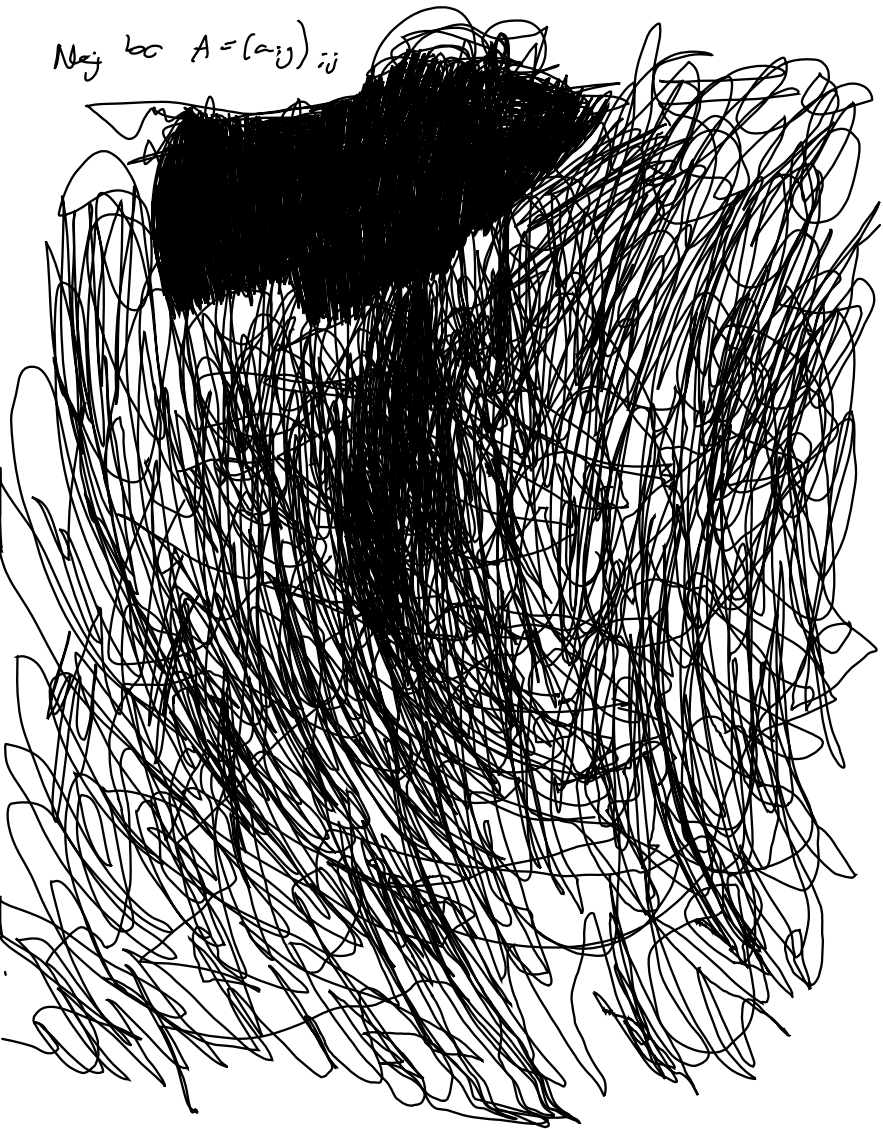
$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad e^{Bt} = e^{it} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$w = \int_a^t e^{-iz} \begin{bmatrix} \frac{1}{\cos z} \\ -\frac{1}{\cos z} \end{bmatrix} dz = e^{-Bt} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$w = \begin{bmatrix} \int_{-t}^t (1 - \tan z) dz \\ \int_{-t}^t (-\tan z - 1) dz \end{bmatrix} = \begin{bmatrix} t + \ln |\cos t| \\ -t - \ln |\cos t| \end{bmatrix}$$

$$V(t) = e^{Bt} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + e^{Bt} w(t)$$

May be $A = (a_{ij})_{ij}$



he stands

$$x_j(t) = e^{\lambda_1 \dots t} \cdot p_j(t)$$

$st(p_j) \leq \text{alg.}$
vedu laste vlnovky
ako λ_j

Vide jo kuhl

12.12

Spomnimo se

$$\vec{x} = A \cdot \vec{x} + f(z)$$

$$\Phi = P e^{J^+} P^{-1}$$

$$x_n = \Phi x_0$$

$$x_r = \int_a^+ \Phi(t) \Phi(s)^{-1} f(s) ds$$

N: Določiti splošno rešitev sistema

$$\dot{x} = y - z + t$$

$$\dot{y} = x - z + 1$$

$$\dot{z} = 2x + 2y - 3z + t + 1$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$$

$$f(t) = \begin{bmatrix} t \\ 1 \\ t+1 \end{bmatrix}$$

$$\det(A - \lambda I) =$$

$$\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -\lambda(3\lambda + \lambda^2 + 2) - 1(-3-\lambda+2) - 1(2+2\lambda)$$

$$= -\lambda((\lambda+1)(\lambda+2)) + (\lambda+1) - 2(\lambda+1) =$$

$$= (\lambda+1)(\lambda^2 + 2\lambda - 1) = \dots \text{ ups} = -(\lambda+1)^3$$

$$A+I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \ker(A+I) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \text{Lin} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\ker(A+I)^2 = \mathbb{R}^{2 \times 3}$$

(ker se dimenzija zmanjša na vsaki 1)

$$\Rightarrow \text{dopolnimo do baze: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

drugi stolpec

$$(A+I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \ker(A+I)$$

↑ tretji stolpec

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{ali pa} \quad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

oba sta vredni

(ker sta oba lin. neod. od $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$)

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 \end{array} \right]$$

$$\sim \dots \quad P^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad e^{Jt} = \begin{bmatrix} e^{-t} & & \\ & e^{-t} & te^{-t} \\ & & e^{-t} \end{bmatrix}$$

$$\Phi(t) = e^{At} = P e^{Jt} P^{-1}$$

$$\Phi^{-1}(s) = P e^{-Js} P^{-1}$$

$$x_p = P e^{Jt} P^{-1} \int_0^t P e^{-Js} P^{-1} f(s) ds =$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} 1 & 1 & -s \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} ds$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} 1 & 1 & -s \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s-1 \\ 1 \\ 0 \end{bmatrix} ds =$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} s-1 \\ 1 \\ 0 \end{bmatrix} ds = P e^{Jt} \begin{bmatrix} se^s - te^s \\ e^s \\ 0 \end{bmatrix} \Big|_0^t =$$

$$\int s e^s ds = s e^s - \int e^s ds = s e^s - e^s$$

$$u = s \quad du = e^s$$

$$du = 1 \quad v = e^s$$

$$= P e^{Jt} \left(e^t \begin{bmatrix} t-2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) = x_p(t)$$

splošna rešitev:

$$X(t) = \Phi(t) x_0 + x_p(t)$$

Linearna enačbe reda n z konstantnimi koeficienti

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

Sistem

$$x_1 = y$$

$$x_2 = y'$$

$$\vdots$$

$$x_n = y^{(n-1)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ -\frac{a_1}{a_0} x_{n-1} - \frac{a_2}{a_0} x_{n-2} \dots + \frac{a_n}{a_0} x_1 \end{bmatrix}$$
$$= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Dobimo sistem oblike $\dot{\vec{v}} = A\vec{v}$

Komponente rešitev so oblike: $\sum p(t) e^{\lambda t}$