

(NDE)

Naravna diferencialna enačba je enačba oblike

$f(x, y, y') = 0$ izčemo $y(x)$ da velja

$f(x, y(x), y'(x)) = 0 \forall x \in D_y$

(To je v implicitni obliki)

Mi se bomo večoma ukvarjali s tem ke je podano v eksplicitni obliki, torej ko je $y = f(x, y')$

Enačba reda $n \in \mathbb{N}$ je oblike $G(x, y, y', \dots, y^{(n)}) = 0$

Enačba je **autonomna**, če funkcija G ni odvisna od x . Sicer je neautonomna.

1. Za dano druzino funk cij posci
pripradejoo DE

$$\left. \begin{array}{l} a) y = ce^x \\ b) y^2 = cx \\ c) y = c(x - c) \end{array} \right\} \begin{array}{l} f(x, y, c) = 0 \\ c \in \mathbb{R} \end{array}$$

$$y' = ce^x \Rightarrow y = y'$$

$$\begin{aligned} y^2 &= cx \\ 2yy' &= c \quad \Rightarrow \quad y^2 = 2yy'x \\ y &= 2y'x \quad \text{a} y \neq 0 \end{aligned}$$

$$y' = c$$

$$y = y'(x - c)$$

2) uegan: rezipitor nezáleží j. h DE

a) $y'' = -y \Rightarrow \{ \alpha \cos x + \beta \sin x \}$

b) $y + xy' = \cos x$

c) $xy' = ny$

b)
 $(xy)' = \cos x$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

c) $xy' = ny$

$$xy' + y = ny + x$$

$$(xy)' = (n+1)y$$

$$y = x^n$$

Metoda izoklin

$$y_c = y = y(x, c) \quad x, c \in \mathbb{R}$$
$$y \in \mathcal{C}^1$$

Izoklina je krivulja vzdolž katere ima vsaka denica družine y enak odvod po x ($y'_c(x)$)

$$I_\alpha = \{(x, y); y = y_c(x) \text{ potem je } y'_c(x) = \alpha\}$$

če imamo DE oblike $y' = f(x, y)$ in predstavimo
družino rešitev $y_c(x)$, potem so izokline ravno
nivojnice $\{f = \alpha\}$
tj. $y'_c(x) = f(x, y_c(x)) = \alpha$

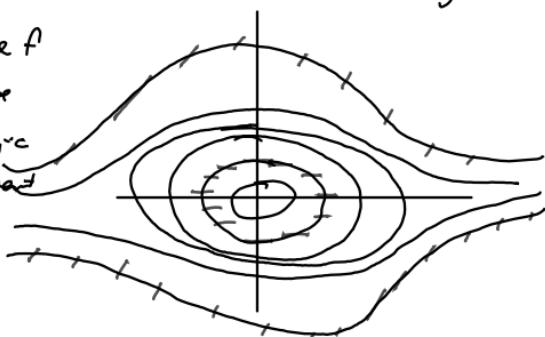
Postopek:

1) Skiciramo funkcijo f

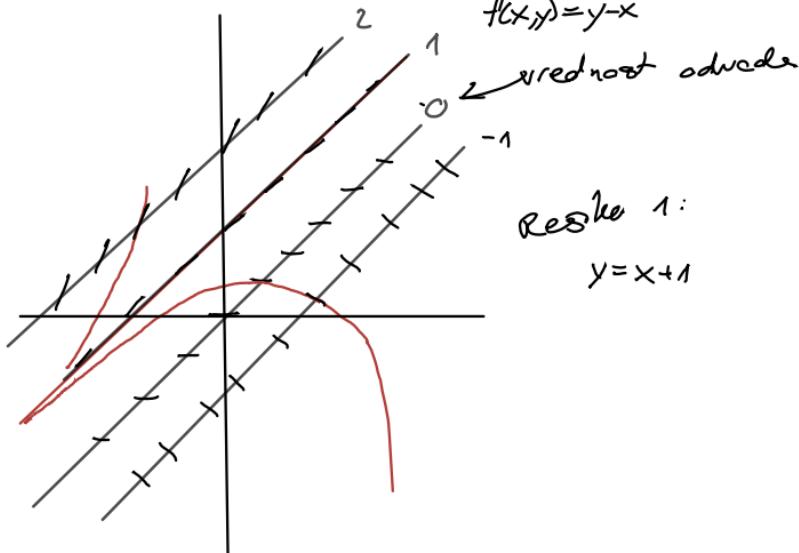
2) Vzdolž neke nivojnice
narišemo nekej bolj
vzgojne stene na koordinat
enih sistema in
uredimo f na
nivojnicu

3) narišemo krivulje ki so
v presečnih nivojnic
tangente na ravni

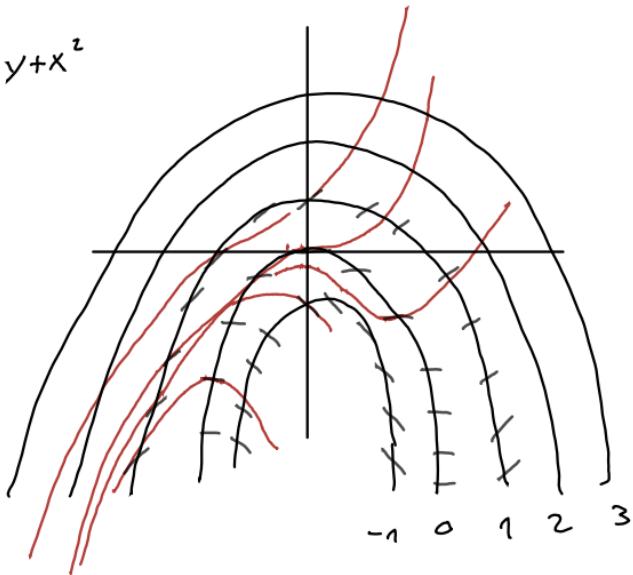
(crne so nivojnice)
mi isčemo kriboje



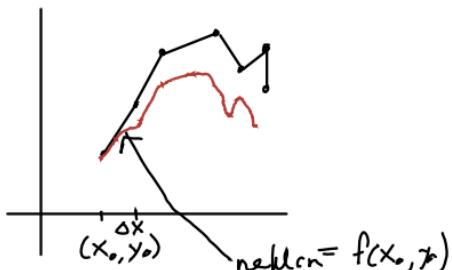
3. Približno skiciraj potek enačbe $y = y - x$



$$y^1 = y + x^2$$



Eulerjeva metoda



$$x_1 = x_0 + \Delta x$$

$$y_1 = y_0 + \Delta x \cdot f(x_0, y_0)$$

$$x_2 = x_1 + \Delta x$$

$$y_2 = y_1 + \Delta x \cdot f(x_1, y_1)$$

Izberemo si Δx

in iterativno
determinamo x_{n+1}

$$\begin{aligned}x_{n+1} &= x_n + \Delta x \\&= x_0 + (n+1) \Delta x\end{aligned}$$

$$y_{n+1} = y_n + P(x_n, y_n) \Delta x$$

Zakaj imeto greska?

ce pravilnemu da je
resitev vsebuje odredjene
velje, leje

$$y(x + \Delta x) = y(x) + y'(x) \Delta x + \text{greska}$$

5) Z Eulerjeva metoda lokal rešenja će

$y' = f(x)$ kada je + weva pri počaju $y(0) = 0$

\checkmark nekako
 $y(A) = ?$

$$\Delta x = \frac{A}{m} \quad \text{za neki } m \in \mathbb{N}$$
$$A = x_m$$
$$x_n = 0 + \Delta x n$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x \neq y_0 =$$

$$= y_{n-1} + f(x_{n-1}) \Delta x + y_0$$

$$= (f(x_{n-2}) + f(x_{n-1})) \Delta x + y_0$$

Riemannova
veza

$$y_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

za

$$\int_0^A f(x) dx$$



$$y_n = \sum_{i=0}^{m-1} f(x_i) \Delta x$$

To je očitno rešenje

Vojna (DN):

Najdi rešitev za $y' = 2y$ z Eulerjevo metodo

$$\Delta x = \frac{A}{n}$$

$$y_n = y_{n-1} + f(x_n, y_{n-1}) \Delta x = y_{n-1} + 2y_{n-1} \frac{A}{n} = \\ = y_{n-1} \left(1 + \frac{2A}{n}\right)$$

$$y_n = y_0 \left(1 + \frac{2A}{n}\right)^n = y_0 \left(1 + \frac{2A}{n}\right)^{n-1}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2A}{n}\right)^{n-1} \stackrel{\substack{| u = \frac{1}{n} \\ u \rightarrow 0}}{=} \lim_{u \rightarrow 0} \frac{\left(1 + \frac{1}{u}\right)^u}{\left(1 + \frac{1}{u}\right)} = \underline{e^{2A}}$$
$$\frac{1}{u} = \frac{2A}{n} \Rightarrow n = 2Au$$

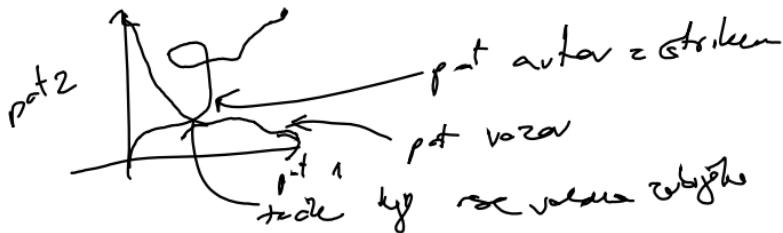
$$y = e^{2Ax}$$

Fazni prostor je prostor vseh možnih stanj sistema

$$y' = f(x, y) \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\hookrightarrow \text{fazni prostor je } \mathbb{R}^3$$

6) Iz lukljane v Maribor vodilne direkcie po katerih lahko iz lukljane do maribora prepeljejo avtomobile h ista eden nadaljeva preveren z vrvo delitve $\langle 2 \rangle$, ne da bodo pretežki. Ni se lahko vorava krovne dolice radija <math>\ell</math>, ki vodiča vseh v svojo smr grecata, ne da bi trdila



$$1) \quad y' = \frac{x^2}{y} = \frac{dy}{dx}$$

$$x^2 dx = y dy$$

$$\frac{1}{3}x^3 + C = y^2$$

$$y = \pm \sqrt{\frac{1}{3}x^3 + C}$$

b)

$$2x^2 y y' + y^2 = 2$$

$$2x^2 = \frac{2-y^2}{yy'} = \frac{2-y^2}{y \frac{dy}{dx}} = \frac{2-y^2}{y dy} dx$$

$$\frac{dx}{2x^2} = \frac{y dy}{2-y^2} \quad u=2-y^2 \\ -\frac{1}{2} \frac{1}{x} = \frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} (\ln u + C)$$

$$\frac{1}{x} = \ln(2-y^2) + C$$

$$ce^{\frac{1}{x}} = 2-y^2$$

$$y = \pm \sqrt{2-ce^{\frac{1}{x}}}$$

$$c) (1+x^2)y' = y$$

$$\frac{dy}{y} = \frac{dx}{1+x^2}$$

$$\ln y = \arctan x + C$$

$$y = ce^{\arctan x}$$

$$y=0$$

$$c) (1+x^2)y' = y$$

$$y \neq 0$$

$$2) \frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)} \quad B = -A$$

$$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

$$x^2: B+A=0$$

$$x: B+C-A=0$$

$$1: A+C=-1$$

$$C=1-\frac{1}{3}$$

$$2B+C=0$$

$$\Rightarrow C=-2B=2A$$

$$3A=1$$

$$\frac{dy}{y} = \frac{1}{3} \frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1} dx$$

$$\int \frac{x-2}{x^2-x+1} dx = \int \frac{3}{(x-\frac{1}{2})^2 + \frac{3}{4}}$$

2)

$$\text{a) } y = \tan(2x+3y-1)$$

$$y' = \tan z$$

$$z = 2x + 3y - 1$$

$$z' = 2 + 3y'$$

$$y' = \frac{z'-2}{3}$$

$$\tan z = \frac{z'-2}{3}$$

$$z' = 3 \tan z + 2$$

$$1. \text{ rezipitor: } z' = 0 \Rightarrow z = \arctan\left(-\frac{2}{3}\right)$$

$$\Downarrow \\ y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan\left(-\frac{2}{3}\right) - 2x + 1}{3}$$

$$\frac{dz}{dx} = 3 \tan z + 2$$

$$\frac{dz}{3 \tan z + 2} = dx$$

$$x + C = \int \frac{du}{(3u+2)(1-u^2)}$$

$$u = \tan z$$

$$du = \frac{1}{\cos^2 z} dz \Rightarrow dz = \frac{du}{1-u^2}$$

$$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2} \quad \begin{array}{l} u^2 \cdot B - A = 0 \\ u: 3C \cancel{+} 2B = 0 \\ 1: 2C + A = 1 \rightarrow A = 1 - 2C \end{array} \quad \begin{array}{l} A = B \\ u: 3C + 2B = 0 \\ 1: 2C + A = 1 \rightarrow A = 1 - 2C \end{array}$$

$$= \int \left(\frac{-3}{3u+2} + \frac{-3u+2}{1-u^2} \right) du = \quad \begin{array}{l} 3C + 2 - 4C = 0 \\ C = 2 \Rightarrow A = -3 = B \end{array}$$

$$= -\ln(u + \frac{2}{3}) +$$

$$+ \int \frac{1}{2} \frac{1}{u-1} + \frac{5}{2} \frac{1}{u+1} =$$

$$= -\ln(u + \frac{2}{3}) + \frac{1}{2} \ln(u-1) + \frac{5}{2} \ln(u+1)$$

$$\frac{A}{1-u} + \frac{B}{1+u} = \quad \begin{array}{l} u: A-B = -3 \\ 1: A+B = 2 \end{array}$$

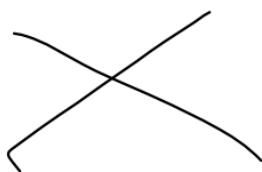
$$A = B-3$$

$$2B-3=2$$

$$2B=5$$

$$B=\frac{5}{2}$$

$$A=-\frac{1}{2}$$



$$y''y - y' = xy'' - (y')^2$$

$$y''(y-x) = y'(y'-1)$$

$$y''(y-x) + y'(y'-1) = 0 \quad y \neq 0$$

$$a = y'$$

$$b = y-x \quad b' = y'-1$$

$$a'b + ab' = 0$$

$$(ab)' = 0$$

$$ab = c \in \mathbb{R}$$

$$y'(y-x) = c$$

$$y' = \frac{c}{y-x} \quad z = y-x$$

$$\underline{z = \int \left(\frac{c}{z}-1\right) dz = c \ln z - z + D}$$

$$\frac{dz}{dx} = \frac{c}{z} - 1$$

$$\frac{dz}{\frac{c}{z}-1} = dx \quad x+D = \int \frac{z}{c-z} dz = - \int \left(\frac{c}{z}-1\right) dt =$$
$$t = c-z \quad dt = -dz \quad = -c \ln(c-z) + c-z$$

$$x+D = -c \ln(c-y+x) + C - y \cancel{+k}$$

$$y = D e^C e^{\ln(c-y+x)+1}$$

Homogene erwecke

$$F(x,y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xy = y \Rightarrow y^1 = z + xz^1 = F(1, z)$$
$$z^1 = \frac{F(1, z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

$$A) y^2 + x^2 y^1 = xy y^1 \quad z = \frac{y}{x}$$

$$y^1 (xy - x^2) = y^2$$
$$y^1 = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z-1}$$

$$y = zx$$
$$y^1 = z + z^1 x = \frac{z^2}{z-1}$$

$$z^1 x = \frac{z^2 - z^2 + z}{z-1} = \frac{z}{z-1}$$

$$\frac{dz}{dx} x = \frac{z}{z-1} \Rightarrow \frac{z-1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{x}{z}$$

$$b) \quad y = xy' - \sqrt{x^2 + y^2}$$

$$y' = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \cancel{\sqrt{1 + \frac{y^2}{x^2}}} = \cancel{z} + \sqrt{1 + z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = z + z^2 x = z + \sqrt{1+z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1+z^2}}{x}$$

$$z = \operatorname{sh} u \quad dz = -\operatorname{du}$$

$$\operatorname{sh} x \ln x + D = \int \frac{1}{\sqrt{1+z^2}} dz = \int \frac{1}{\cosh u} - \operatorname{ch} du =$$

$$y = x \operatorname{ch} \underbrace{(\ln x + D)}_{\text{say}}$$

$$- \operatorname{arcsinh} \frac{x}{z}$$

$$\operatorname{arccosh} \frac{z}{x}$$

$$m\ddot{a} = F = m \cdot g - k v^2$$

$$m\ddot{v} = mg - k v^2 \quad //$$

$$\ddot{v} = g - \frac{k}{m} v^2 = g \left(1 - \frac{k}{mg} v^2\right) = g(1 - \alpha v^2)$$

$$\frac{dv}{dt} = g(1 - \alpha v^2)$$

$$\frac{dv}{1 - \alpha v^2} = g dt \quad / \int$$

$$gt = \frac{A}{1 - \sqrt{\alpha} v} + \frac{B}{1 + \sqrt{\alpha} v} : \quad \begin{array}{l} v: \sqrt{\alpha} A - \sqrt{\alpha} B = 0 \\ A = B \end{array}$$

$$1: \quad A + B = 1$$

$$gt = \frac{1}{\sqrt{\alpha}} \left(\ln(1 + \sqrt{\alpha} v) - \ln(1 - \sqrt{\alpha} v) \right) = \Rightarrow A = \frac{1}{2}$$

$$\frac{1}{2\sqrt{\alpha}} \ln \left(\frac{1 + \sqrt{\alpha} v}{1 - \sqrt{\alpha} v} \right) = gt + C \quad \begin{array}{l} B = \frac{1}{2} \\ \end{array}$$

$$V_0 = 0 \Rightarrow \cancel{1 + \sqrt{\alpha} v} = 1 - \sqrt{\alpha} v + C$$

$$\Rightarrow 0 = C$$

$$\frac{1 + \sqrt{\alpha} v}{1 - \sqrt{\alpha} v} = \underbrace{e^{2\sqrt{\alpha} gt}}_D$$

$$1 + \sqrt{\alpha} v = D - \sqrt{\alpha} D v$$

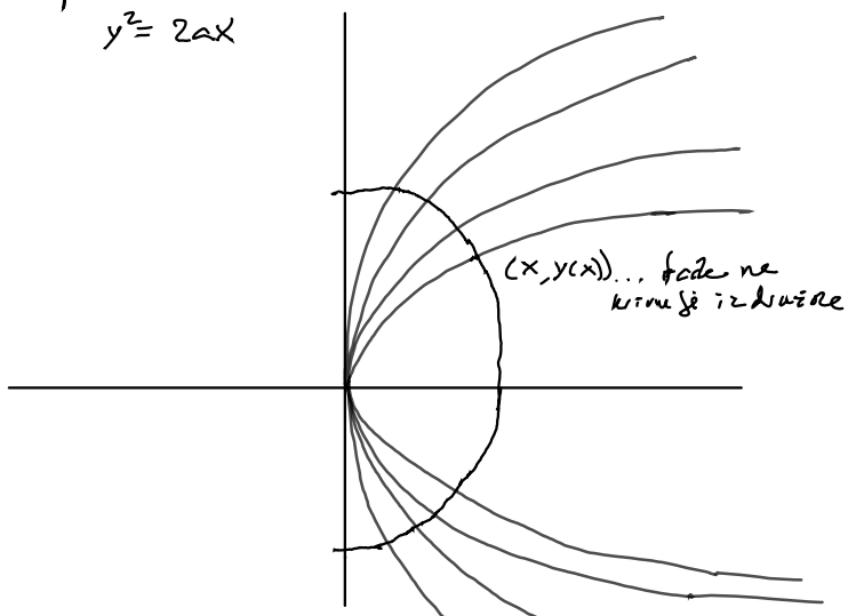
$$V \sqrt{\alpha} (1 - \sqrt{\alpha} D) = D - 1$$

$$v = \frac{D-1}{\sqrt{\alpha}(1 - \sqrt{\alpha} D)}$$

V meukje

+ zandduide 20 min 21.10

2) Poisci druzino
ortogonalnih trajektorij na druzino
parabol
 $y^2 = 2ax$



γ je ortogonalne trajektorije,

če krovje iz držine sekajo

pod pravim ptam

$$2yy' = 2a \Rightarrow y^2 = 2yy'x$$

$$\nu(x_0, y_0) \text{ fale velja } x_0 2y_0 y' = y_0^2$$

$$y' = \frac{y_0}{2x_0} \Rightarrow \text{zamerni koeficient trajektorije} - \frac{2x_0}{y_0}$$

$y' = f(x, y)$ druzina krovij $\Rightarrow -\frac{1}{y^2} = f(x, y)$ druzina
ort. t.

$$y^2 = 2yy'x \Rightarrow y^2 = \frac{2yx}{y'} \quad \text{izcemo } y$$

1. mesto $y=0$ ✓

2. mesto

$$y' = -\frac{2x}{y} = \frac{dy}{dx}$$

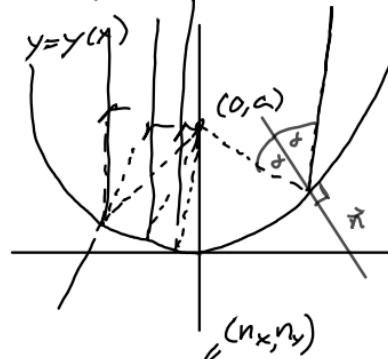
$$2xdx = ydy / \int$$

$$-\frac{2}{2} x^2 = \frac{1}{2} y^2 + C$$

$$x^2 + \frac{1}{2} y^2 + C = 0$$

$$x^2 + \frac{1}{2} y^2 = -C \quad \text{elipse}$$

3) Pr.: izdelavi zvezdico mera biti
odbojn parosine ki obseva zvezlico take
oblike, da vse svetlobne žarke obliju v
isto smer predstavimo lejodan parosin
vzajemno plaski graf f,



$$(-x, -y(x)+a) \cdot \vec{n} = -x n_x - y n_y - a n_y = \cos \alpha \sqrt{x^2 + (y+a)^2} \sqrt{n_x^2 + n_y^2}$$

$$\vec{n} = (1, -\frac{1}{y^1})$$

$$\cos \alpha \sqrt{x^2 + (y+a)^2} \sqrt{1 + \frac{1}{y^1}} = -x + \frac{-y - a}{y^1}$$

$$\cos \alpha = \frac{-x y^1 - y - a}{\sqrt{x^2 + (y+a)^2} \sqrt{1 + y^1}}$$

$$\vec{n} \cdot (1, 0, 0) = \cos \alpha |\vec{n}| \cdot 1$$

$$n_y D x = \cos \alpha \sqrt{1 + \frac{1}{y^1}}$$

$$n_y n_x = \frac{1}{\sqrt{x^2 + (y+a)^2}} (-x y^1 - y - a) = \frac{1}{y^1}$$

$$-x y^1 - y - a = \sqrt{x^2 + (y+a)^2}$$

???

$$\begin{aligned} z &= y - a \\ z^1 &= y^1 \end{aligned}$$

$$u \times z^1 + z = \sqrt{x^2 + z^2} + z$$

$$\therefore z^1 = \frac{\sqrt{x^2 + z^2}}{u} = \sqrt{\frac{x^2 + z^2}{u^2}} = \sqrt{1 + \frac{z^2}{x^2}} = \sqrt{1 + \frac{z^2}{x^2} + \frac{z^2}{x^2}}$$

$$u = \frac{z}{x}$$

$$z^1 = x u \quad u + x u' = \sqrt{1 + u^2} + u$$

$$\frac{u'}{\sqrt{1+u^2}} = \frac{1}{x}$$

$$v = \sqrt{1+u^2}$$

$$du = \frac{u du}{\sqrt{1+u^2}} = \frac{1-v^2}{\sqrt{1+v^2}} dv$$

$$\ln x = \int \frac{1-v^2}{v} dv = \ln v - \frac{1}{2} v^2 + C$$

???

$$y = a + \frac{c}{2} x^2 - \frac{1}{2c}$$

Cauchy náleží

$$y = f(y) \quad y(x_0) = y_0$$

a) gálož. \exists reál. D za podporu zádušního požadavku

b) Lipschicova vlastnost když existuje $\varepsilon > 0$ takže

$$\frac{dy}{dx} = f(y)$$

$$\exists \delta \text{ f } f(y) \neq 0$$

$$\frac{dx}{f(y)} = dx / \int$$

$$\text{Když je } \exists \delta \text{ f } f(y_0) = 0$$

$$F(y) = x + C$$

$$\text{počítáme } f(y) \equiv x_0$$

$$\int \frac{dy}{f(y)}$$

$$y'(x) = 0$$

$$f(y(x)) = f(y_0) = 0$$

$$y'(x) = f(y(x))$$

$$y(x) = y(x_0) + \int_{x_0}^x y'(t) dt$$

$$\int_{x_0}^x \frac{y'(t) dt}{f(y(t))} = \int_{x_0}^x 1 dt = x - x_0$$

dá se využít všechna stranu v intervalu

$$u = y$$

$$du = y' dt$$

$$\int_{y_0}^{y(x)} \frac{du}{f(u)} = F(y(x)) - F(y_0) = x - x_0$$

$$F(y(x)) = x - x_0 + F(y_0)$$

$$y(x) = F^{-1}(x - x_0 + F(y_0))$$

$$y(x_0) = F^{-1}(F(y_0)) = y_0$$

$$y'(x) = \frac{1}{F'(F^{-1}(x - x_0 + F(y_0)))} = \frac{1}{f(F^{-1}(x - x_0 + F(y_0)))}$$

$$\therefore = f(y(x)) \quad ???$$

$$F'(F^{-1}(x)) = \frac{1}{(F^{-1})'(x)}$$

b) endlich

$$\int_{y_0}^y \frac{dy}{f(y)} = x - x_0$$

$$|f(x) - f(y)| \leq c|x-y|$$

$$\xrightarrow{\text{rechts}} |f(y)| \leq c|x-y|$$

da $f(x_0) = 0$

$$\int_{y_0}^y \frac{dy}{f(y)} \xrightarrow{\substack{f > 0 \\ y > y_0, f(y_0) = 0}} \frac{dy}{c(y-y_0)} = \infty$$

Rechnung da y reell DE da ist
 $f(x_0) = 0$ in y ni konstanten

$$x_n > x_0 : f(y(x_n)) > 0$$

Stellen je y blz zu x_n annehmen

$$\int_{x_1}^x \frac{y'(x)dx}{f(y(x))} = x - x_1$$

$$\int_{y_n}^y \frac{dy}{f(y)}$$

zu zeigen si $x_2 \in [x_0, x_n]$, da $y(x)$

$f(y(x_2)) = 0$ in $f(y(x)) > 0 \geq \forall x \in (x_2, x_1]$

$$\lim_{x \rightarrow x_2} \int_{x_0}^x \frac{y'(x)}{f(y(x))} = \lim_{x \rightarrow x_2} x - x_1 = x_2 - x_1$$
$$= \lim_{y \rightarrow y_2} \int_{y_n}^y \frac{dy}{f(y)} \geq \lim_{y \rightarrow y_2} \int_{x_1}^y \frac{dy}{c(y-y_0)} = \infty$$

X

$$y' = \frac{f(y)}{g(x)} \quad f, g \text{ brez nih}$$

a) Podež de vsekih rešitev de teimo krvitje
polja $v(x,y) = (g(x), f(x))$

b) Določi za poljuben zacetni pogoj določo
velikost vseh rešitev

Takounice ali integralna krvitje je krovje,
ki ima vselej tudi odvod male vrednosti
poljka

$$F(y) - F(y_0) = \int_{y_0}^y \frac{dy}{f(y)} = t - t_0 \quad x = G^{-1}(t - t_0 + G(x_0))$$

$$G(x) - G(x_0) \int_{x_0}^x \frac{dx}{g(x)} = t - t_0 \rightsquigarrow$$

$$\gamma = (x(t), y(t))$$

$$\frac{\partial}{\partial t} y(x(t)) =$$

$$\dot{x}(t) = V(x(t), y(t))$$

$$= \frac{\partial y}{\partial t} (x(t)) \dot{x} =$$

$$\dot{x}(t) = g(x)$$

$$= \frac{f(y(t))}{g(x(t))} \frac{\partial x}{\partial t}$$

$$\dot{y}(t) = f(y)$$

$$\rightsquigarrow \dot{x} = g(x) \quad ???$$

Zvezek

lin. dif. enačbe (LDE)

$$y' + a(x)y = b(x) \quad a, b \in C([x_1, x_2])$$

i) $b=0 \Rightarrow$ LDE homogen \Rightarrow lastje speenjnice

$$y = y_0 e^{-\int a(x) dx} = C e^{-\int a(x) dx}$$

ii) $b \neq 0 \Rightarrow$ LDE nehomogen \Rightarrow variacija konstante

$$y = e^{-\int a(x) dx} \int b(x) e^{\int a(x) dx} dx + C e^{-\int a(x) dx}$$

•) LDE homogen: y_1, y_2 rešitev $\Rightarrow y_1 + \lambda y_2$ tudi rešitev

če vema homogeno rešitev \bar{y} so vse druge rešitve oblike $C\bar{y}$ za celi

•) LDE nehomogen

Partikularno rešitev nujemo z variacijo konstante

Homogeni del: $y' + a y = 0 \Rightarrow y_h$ rešitev

Partikularni del: $y_p = C(y) y_h$

Splastna rešitev: $\tilde{y} = y_p + D x$

Odvajamo in vstavimo v DE:

$$y_p' = C' y_h + C y_h' \Rightarrow C' y_h + C y_h' + a C y_h = b$$

$$C' y_h = b \Rightarrow C(x) = \int \frac{b}{y_h} dx$$

1) Pois oj splaynaregeln:

$$2) xy' + 2y = (3x^2+2)e^{3x} \quad / :x$$

$$y' + y \frac{2}{x} = \underbrace{\frac{3x+2}{x}}_b e^{3x}$$

$$x=0 \Rightarrow 2y = 2 \Rightarrow y=1$$

$$x=0 \Rightarrow$$

$$\text{homogen: } y' + y \frac{2}{x} = 0$$

$$\frac{dy}{dx} + y \frac{2}{x} = 0 \quad / \frac{dx}{y}$$

$$\frac{dy}{y} = -\frac{2dx}{x} \quad / \int$$

$$\ln y = \ln x^{-2} + C$$

$$y = \frac{1}{x^2} \cdot D$$

partikular no:

$$y_p = D(x)x^{-2}$$

$$y'_p = D'(x)x^{-2} - 2D(x)x^{-3}$$

$$D'(x)x^{-2} - 2D(x)x^{-3} + 2D(x)x^{-3} = \frac{(3x^2+2)e^{3x}}{x}$$

$$D'(x) = (3x^2+2x)e^x$$

$$\int p(x) e^{3x} dx = g(x) e^{3x} + C$$

$$p, g \in \mathbb{R}[x], \text{ st } p = sg$$

$$\left((Ax^2+Bx+C)e^{3x} \right)' = (3x^2+2x)e^{3x}$$

$$(3Ax^2+3Bx+3C+2Ax+B)e^{3x} \rightsquigarrow$$

$$3A=3 \Rightarrow A=1$$

$$3B+2A=2 \Rightarrow B=0$$

$$3C+B=0 \Rightarrow C=0$$

$$D(x) = x^2 e^{3x}$$

$$\rightsquigarrow y_p = x^2 e^{3x} \cdot x^{-2} = e^{3x}$$

$$y_s = y_h + y_p = D \frac{1}{x^2} + e^{3x}$$

$$b) \quad y' = \sin x + y$$

$$z = y$$

$$z' = \sin x + z$$

$$z' - z = \sin x$$

$$a(x) = -1 \quad b(x) = \sin x$$

$$b \equiv 0 \Rightarrow z' - z = 0$$

$$z' = z$$

$$1) \quad z \equiv 0 \Rightarrow y = c \in \mathbb{R}$$

$$2) \quad \int \frac{dz}{z} = dx \Rightarrow \ln z = x + C$$

$$\Rightarrow z = ce^x \Rightarrow y = ce^x + D$$

$$b \neq 0 \Rightarrow z_h = e^x$$

$$z_p = C(x)z_h = C(x)e^x$$

$$z = z_p + D y_h = C(x)e^x + D e^x$$

$$z' = C'(x)e^x + C(x)e^x + D e^x$$

$$z' - z = C'(x)e^x + C(x)(e^x - e^x) = \sin x$$

$$C'(x) = \frac{\sin x}{e^x} = e^{-x} \sin x$$

$$\text{nastavek: } C(x) = A e^{-x} \sin x + B e^{-x} \cos x$$

$$C(x) = e^{-x}(A \sin x + B \cos x)$$

$$C'(x) = -e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)$$

$$= e^{-x}(\sin x(-A - B) + \cos x(-B + A)) = e^{-x} \sin x$$

$$A + B = -1$$

$$A - B = 0 \Rightarrow A = B \Rightarrow 2A = -1 \Rightarrow A = B = -\frac{1}{2}$$

$$\Rightarrow C(x) = -\frac{1}{2} e^{-x} (\sin x + \cos x) =$$

$$z = -\frac{1}{2} (e^{-x} \sin x + e^{-x} \cos x) + D e^x = y$$

$$y = -\frac{1}{2} \int \sin x + \cos x \, dx = \frac{1}{2} \cos x - \frac{1}{2} \sin x + D e^x + C$$

2)

$$y' + 2xy = 2x^3y^3$$

1) $y \equiv 0$ je rešitev

$$2) y \neq 0 \Rightarrow \frac{y'}{y^3} + \frac{2x}{y^2} = 2x^3$$

$$z' = \frac{y'}{y^3} \Rightarrow \int dz = \int y^{-3} dy \Rightarrow z = -\frac{1}{2}y^{-2} = -\frac{1}{2y^2}$$

Bernullijeva DE

$$y' + ay = by^\alpha \quad \alpha \in \{0, 1\}$$

$$\text{Nastavek: } z = y^{1-\alpha}$$

$$\frac{2x}{y^2} = \frac{4x}{z^2} = -4xz$$

$$\text{enakba: } z' - 4xz = 2x^3 \quad a(x) = -4x \quad b(x) = 2x^3$$

$$\text{homogeno: } z' - 4xz = 0$$

$$\int \frac{dz}{z} = \int 4x dx \Rightarrow \ln z = 2x^2 + C \Rightarrow z = C e^{2x^2}$$

$$\text{partikulärna: } z_h = e^{2x^2} \quad z_p = C(x) e^{2x^2}$$

$$z = z_p + z_h$$

$$\rightsquigarrow \text{enakba: } C'(x) e^{2x^2} + 4x e^{2x^2} \cdot C - 4x C e^{2x^2} = 2x^3$$

$$C'(x) = e^{-2x^2} \cdot 2x^3$$

Nastavek:

$$C(x) = \int 2x^3 e^{-2x^2} dx = \int x e^u du = (Ax + B) e^u$$

$$u = -2x^2$$

$$du = -4x dx$$

$$= -\frac{1}{2} \int e^u u du = \frac{1}{2} (Au + B) e^u$$

$$(Au + B) e^u)' = e^u (Au + B + A)$$

$$A + B = 0 \quad A = 1 \Rightarrow B = -1$$

$$C(x) = -\frac{1}{2} (1u - 1) e^u \rightsquigarrow$$

$$C(x) = e^{-2x^2} \left(x^2 - \frac{1}{2} \right) + D$$

$$z = x^2 - \frac{1+D e^{-2x^2}}{2} = -\frac{1}{y^2} \quad \rightsquigarrow y = \frac{1}{\sqrt{x^2 - \frac{1}{2} + D e^{-2x^2}}}$$

3. relogz

$$xy' + y = y^2 \ln x$$

1) $y=0 \Rightarrow$ je rechter

2) $y \neq 0 \Rightarrow$

$$x \neq 0 \Rightarrow$$

$$\frac{y'}{y^2} + \frac{1}{xy} = \ln x$$

$$z' = \frac{y'}{y^2} \rightsquigarrow z = -\frac{1}{y}$$

$$\boxed{z' - \frac{z}{x} = \ln x} \quad \begin{matrix} a=-1 \\ b=\ln x \end{matrix}$$

homogen:

$$z' - \frac{z}{x} = 0$$

$$\frac{dz}{z} = \frac{dx}{x} \rightsquigarrow$$

$$\ln z = \ln x + C$$

$$z = e^{\ln x} \cdot e^C \rightsquigarrow z = C \cancel{e^x}$$

$$z_n = x$$

$$\rightsquigarrow C'x + C - C = \ln x$$

$$C' = \frac{\ln x}{x} \quad C = \int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + D$$

$$u = \ln x \quad du = \frac{dx}{x} \quad C = \frac{1}{2} (\ln x)^2 + D$$

$$z = \frac{1}{2} (\ln x)^2 x + Dx = -\frac{1}{y}$$

$$y = -\frac{2}{(\ln x)^2 x + Dx}$$

4 nerlage

fomejene zwzne

$$y' + y = f(x)$$

a) Ta jme netanko eno resitev DE

b) f periodica s periodo $\omega \Rightarrow \bar{y}$ periodica s periodom ω

a) $y' + y = f(x)$

homogeno:

$$\frac{dy}{dx} = -y \rightsquigarrow \frac{dy}{y} = -dx \rightsquigarrow y = e^{-x}$$

partikularne: $y_p = C(x)e^{-x}$

$$C'(x)e^{-x} = f(x) \quad C(x) = \int_0^x f(t)e^t dt = F(x) + D$$

$$\bar{y} = F(x)e^{-x} + Ce^{-x} = e^{-x}(F(x) + C) = e^{-x} \int_0^x f(t)e^t dt$$

b) $\bar{y}(x+\omega) = e^{-x-\omega} \int_0^{x+\omega} f(t)e^t dt = e^{-x}e^{-\omega} \int_0^x f(t)e^t dt +$
 ~~$e^{-\omega} e^{-x} \int_x^{x+\omega} f(t)e^t dt =$~~
 $= e^{-\omega} \bar{y}(x) + e^{-\omega} e^{-x} \int_0^x f(t)e^t dt - e^{-\omega} e^{-x} \int_0^x f(t)e^t dt =$
 $= e^{-\omega} \bar{y}(x) + e^{-x} \bar{y}(\omega) - e^{-\omega} \bar{y}(x) = e^{-x} \bar{y}(\omega) = e^{-\omega} \bar{y}(x)$

~~$\bar{y}(x+\omega) = e^{-x-\omega} \int_{-\infty}^{x+\omega} f(t)e^t dt = e^{-x-\omega} \int_{-\infty}^{\omega} f(t)e^t dt + e^{-x-\omega} \int_{\omega}^{x+\omega} f(t)e^t dt =$~~
 $= e^{-x-\omega} \left(\int_{-\infty}^{\omega} f(t)e^t dt + \int_0^x f(t)e^t dt \right) =$
 $= e^{-x-\omega} \bar{y}(\omega) + e^{-\omega} \bar{y}(x)$

???

$$b) \bar{Y}(\bar{x}) e^{-\bar{x}} \int_{-\infty}^{\bar{x}} f(t) e^t dt = \int_{-\infty}^{\bar{x}} f(t) e^{t+\bar{x}} = \int_{-\infty}^0 f(u+x) e^u du$$

$u=t+\bar{x}$
 $\Rightarrow t=u-x$

$$\bar{Y}(x+\omega) = \int_{-\infty}^0 f(u+x+\omega) e^u du = \int_{-\infty}^0 f(u+x) e^u du = \bar{Y}(x)$$

5. nelage

$$3y' + y^2 + \frac{2}{x^2} = 0$$

Ugensmemo resistor: $y = \frac{1}{x}$ $y' = -\frac{1}{x^2}$

$$-\frac{3}{x^2} + \frac{1}{x^2} + \frac{2}{x^2} = 0 \quad \checkmark$$

Splasha resistor: $y = \frac{1}{x} + u(x)$
 $y' = -\frac{1}{x^2} + u'(x)$

$$-\frac{3}{x^2} + 3u'(x) + \frac{1}{x^2} + \frac{2u(x)}{x} + u^2(x) + \frac{2}{x^2} = 0$$

$$3u'(x) + \frac{2u(x)}{x} + u^2(x) = 0 \quad /: u^2(x)$$

$$\frac{3u'(x)}{u^2(x)} + \frac{2}{xu(x)} + 1 = 0$$

$$z' = \frac{3u'(x)}{u^2(x)} \quad z = 3 \int u^{-2} dx = -3u^{-1} = -\frac{3}{u}$$

$$z' - \frac{2}{3} \frac{z}{x} = -1 \quad \Rightarrow u = -\frac{3}{z}$$

$$a(x) = \frac{2}{3x}$$

homogen del:

$$b(x) = -1$$

$$\frac{dz}{z} = \frac{2}{3} \frac{dx}{x}$$

partikularm:

$$\ln z = \ln x^{\frac{2}{3}} + C \Rightarrow z = \frac{2}{3} C x$$

$$C'(x) \frac{2}{3} x = -1$$

$$C(x) = -\frac{3}{2} \int \frac{1}{x} dx = \ln x^{-\frac{3}{2}}$$

$$z = x \ln x + \frac{2}{3} C x$$

$$u = -\frac{3}{x \ln x + \frac{2}{3} C x} \quad \Rightarrow y = \frac{1}{x} - \frac{3}{x \ln x + \frac{2}{3} C x}$$

Riccatijeva enačba

$$y' = ay^2 + by + c \quad a, b, c \text{ funkcije } x$$

1) Uganemo rešitev y_p

2) Splošno rešitev poisciemo z nastavkom

$$y = y_p + u(x)$$

3) Vstavimo nastavek z začetno enačbo

4) Dobimo Bernullijeve enačbo

$$5) \quad y = y_p + \frac{1}{u'(x)} \Rightarrow LDE$$

$$1. \quad (1-x^2)y' = 1-y^2$$

Riccatijeva enačba

$$y' = ay^2 + by + c \quad ; \quad a, b, c \text{ funkcije od } x$$

- 1) Uganimo rešitev y_p
- 2) Spostavljamo rešitev kot $y = y_p + u(x)$
- 3) Dobimo Bernolijeva enačba

$$y = y_p + \frac{1}{u(x)^2} \Rightarrow \text{ODE}$$

$$y' = \frac{1-y^2}{1-x^2}$$

$$1. \quad y_p = x : \quad 1 = \frac{1-x^2}{1-x^2} \quad \checkmark$$

$$2. \quad y = v + y_p = v+x \quad v \text{ je rezanne funkcije}$$

$$y' = v' + 1 = \frac{1+(v+1)^2}{1-x^2} = \dots = 1 + \frac{-2vx - v^2}{1-x^2}$$

$$v' = \frac{-2vx - v^2}{1-x^2} = -\left(\frac{1}{1-x^2} v^2 + \frac{2x}{1-x^2} v\right)$$

$$\frac{v'}{\sqrt{v^2}} = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} \frac{1}{\sqrt{v}}$$

$$z = \frac{1}{v} \quad z' = \frac{v'}{v^2}$$

$$z' = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} z$$

homogen: del:

$$z' = -\frac{2x}{1-x^2} z$$

$$\int \frac{dz}{z} = -\int \frac{2x}{1-x^2} dx$$

$$\ln|z| = -\ln|1-x^2| + C$$

$$z_h = \frac{C}{x^2-1}$$

~~$$z_h = \frac{C(x)}{x^2-1} + \frac{C}{x^2-1} = \frac{C}{x^2-1}$$~~

$$C'(x) = -1 \quad C = -x$$

$$z = z_h + z_p = \frac{c-x}{x^2-1}$$

$$v = \frac{x^2-1}{c-x}$$

$$y = \dots = \frac{1-cx}{x-c}$$

Prvi integral enačbe $y' = f(x,y)$ je funkcija, ki je konstantne vrednosti vsake rešitve

če imamo splašno rešitev $y = \varphi(x, c)$ in je $\varphi_c \neq 0$
 $\Rightarrow C = C(x, y)$ je prvi integral

⇒ DE zapisemo v obliki: $Pdx + Qdy = 0$ (*)

Opozoril: (*) je ekvivalentna DE $P + Qy' = 0$

$$\langle (P, Q), (1, y') \rangle = 0$$



$$\text{če velja } \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y$$

pravimo da je enačba eksaktna.
 Njen prvi (eksaktni integral) poisciemo z integracijo $u = \int$
 $\left((P, Q) \text{ je potencialne} \Leftrightarrow (P, Q) = \nabla u \right)$

C povišenom z odvajanjem

$$u_y = \frac{\partial}{\partial y} \int P dx + C(y) = Q$$

2) Poisci tek nekonstanten prvi integral

$$a) x\sqrt{x^2+y^2}dx + (x\sqrt{x^2+y^2}+xy)dx = 0$$

$$b) 3x^2(1+\ln y)dx - \left(2y - \frac{x^3}{y}\right)dy = 0$$

a)

$$P = x\sqrt{x^2+y^2} + y$$

$$Q = y\sqrt{x^2+y^2} + x$$

Ali je ekvivalentna?

$$P_x = \sqrt{x^2+y^2} + \frac{xy}{\sqrt{x^2+y^2}}$$

$$Q_y = -1$$

$$u = \int(x\sqrt{x^2+y^2} + \cancel{y})dx + C(y) =$$

$$t = x^2+y^2$$

$$dt = 2xdx$$

$$= \frac{1}{2} \int t^{\frac{1}{2}} dt + xy + C(y) = \frac{1}{3} (x^2+y^2)^{\frac{3}{2}} + xy + C(y)$$

$$U_y = \cancel{y(x^2+y^2)^{\frac{1}{2}}} + \cancel{x} + C'(y) = y\sqrt{x^2+y^2} + \cancel{x}$$

$$C'(y) = 0 \Rightarrow C \text{ je konstanta}$$

$$u = \frac{1}{3} (x^2+y^2)^{\frac{3}{2}} + xy \quad \text{izberemo da } \tilde{y} < 0$$

prvi integral

spremembo resitev dobimo $\tilde{y} = y(x)$

izrazimo y $\tilde{y} = y(x)$

$$u(x, \tilde{y}(x)) = C$$

Da je $Pdx + Qdy = 0$ ^{b)} može veljet

$$y' = -\frac{P}{Q} = -\frac{\mu P}{\mu Q} \Rightarrow \mu Pdx + \mu Qdy = 0 \quad (**)$$

Tj. da sta ekvivalentni za pravilno rešitev
če * n: eksakten mora biti \Leftrightarrow eksakten
faktor μ se meniže integracijo mnozitve

V praksi ga skušamo ugeniti z nastanki:

$$\mu(x), \mu(y), \mu(xy)$$

$$\mu(x^n \pm y^m), \mu(x^2 + y^2) \dots$$

3) Podeva je DE

$$Pdx + Qdy = 0$$

a) Dokaz $\mu = \mu(y) \Leftrightarrow \frac{Q_x - P_y}{P} = g(y)$

$\Rightarrow \mu Pdx + \mu Qdy = 0$ in μ : integracijski model
DE pa je desetka

$$(\mu P)_y = (\mu Q)_x$$

$$\mu_y P + \mu P_y = \mu Q_x$$

$$\frac{\mu(Q_x - P_y)}{P} = \mu'(y) \quad / : \mu$$

$$\frac{Q_x - P_y}{P} = \frac{\mu'(y)}{\mu(y)} \quad \text{zadnje samo } y$$

$$\Leftarrow \frac{Q_x - P_y}{P} = g(y) \quad \text{recimo da je}$$

$$\ln \mu = \int \frac{Q_x - P_y}{P} dy \quad \Rightarrow \mu = e^{\int \frac{Q_x - P_y}{P} dy}$$

Potem je $(\mu P)_y = \frac{Q_x - P_y}{P} P + \dots$ isto id
pri de vredni

$$b) \underbrace{(xy^2 - y^3)}_P dx + \underbrace{(1 - xy^2)}_Q dy = 0$$

$$Q_x = -y^2 \quad P_y = 2xy - 3y^2$$

$$\frac{Q_x - P_y}{P} = \frac{y^2 - 2xy}{xy^2 - y^3} = \frac{2(y-x)}{y(x-y)} = -\frac{2}{y}$$

$$\mu = e^{-\int \frac{2}{y} dy} = e^{\ln y^{-2}} = \cancel{y^2} \frac{1}{y^2}$$

$$\mu P dx + \mu Q dy = 0 \text{ durch}$$

$$\begin{aligned} u &= \int \mu P dx + C(y) = \\ &= \int (x-y) dx + C(y) = \frac{1}{2}x^2 + xy + C(y). \end{aligned}$$

$$u_y = -x + C'(y) - \frac{1-xy^2}{y^2} = \frac{1}{y^2} - x$$

$$C'(y) = \frac{1}{y^2} \Rightarrow C = -\frac{1}{y}$$

$$u = \frac{x^2}{2} - xy - \frac{1}{y}$$

μ

$$u) \quad y(x^2+y^2+1)-x(x^2+y^2-1) dx = 0$$

$$\mu = \mu(x, y)$$

$$P_y = x^2 + 3y^2 + 1 \quad \mu P dx + \mu Q dy = 0$$

$$Q_x = -3x^2 - y^2 + 1 \quad \mu = \mu(x, y)$$

$$\begin{aligned} (\mu P)_y &= x\mu' P + \mu P_y \\ (\mu Q)_x &= y\mu' Q + \mu Q_x \end{aligned} \quad \left. \begin{array}{l} \text{---} \\ + \end{array} \right\} \mu'(xP - yQ) + \mu(P_y - Q_x) = 0$$

$$\frac{\mu'}{\mu} = \frac{-P_y + Q_x}{xP - yQ} = \frac{-\mu x^2 - 4y^2}{xy(2x^2 + 2y^2)} = \frac{-2}{xy}$$

$$z = xy \quad \frac{\mu'(z)}{\mu(z)} = -\frac{2}{xy} z \quad / \int$$

$$\frac{d\mu}{\mu(z)} = -\frac{2dz}{z} \quad / \int$$

$$|\mu(z)| = |z|^{\frac{1}{2}}$$

$$\mu(z) = \frac{1}{\sqrt{|z|}} z^2 = \frac{1}{(xy)^2}$$

$$u = \int \mu P dx + \mu Q dy = \int \left(\frac{y(x^2+y^2+1)}{x^2y^2} \right) dx + C(y)$$

$$= \int \frac{1}{y} + \frac{y}{x^2} + \frac{1}{x^2y} = \frac{x}{y} - \frac{y}{x} - \frac{1}{xy} + C(y)$$

$$u_y = -\frac{x}{y^2} - \frac{1}{x} + \frac{1}{y^2x} + C'(y) =$$

4) $ydx - xdy = 2x^3 \tan\left(\frac{y}{x}\right)dx$

$$z = \frac{y}{x} \quad dz = \frac{dy}{x} - \frac{y}{x^2}dx \quad dF = F_x dx + F_y dy$$

$$-x^2 dz = -xdy + ydx$$

$$-dz = 2x \tan(z) dx$$

$$-\frac{dz}{\tan(z)} = 2x dx$$

$$-\int \frac{\cos(z)}{\sin(z)} dz = \int 2x dx$$

$$-\ln|\sin(z)| = x^2 + C$$

$$-\ln|\sin\frac{y}{x}| = x^2 + C$$

$$C = -x^2 - \ln|\sin\frac{y}{x}| = u(x,y) = \text{prv: integra}$$

Parametrično rezervanje implicitno podelnih DE

Do zdej so enačbe bile dolike $y' = F(x, y)$

Lahko pa je v $F(x, y, y') = 0$

Poanta: $F(x, y, p) = 0$ podaja plasko v \mathbb{R}^3
(če $F \in C^1$ in $\nabla F \neq 0$)

Vseka rešitev bo potekala po neki krivulji:

$$\gamma(t) = (x(t), y(t), p(t)) \text{ da je } F(\gamma(t)) = 0$$

Da γ res podaja rešitev DE v parametrični obliki;
mora uveljavljeti

$$\dot{\gamma}(t) = p(t) \cdot \dot{x}(t)$$

Preprost primer:

$$F(x, y) = 0$$

↳

$F(x, p)$ podaja krivuljo v ravni;

To krivuljo parametriziramo in posredno minkajadi predpis za y komponento

6) Revisão de

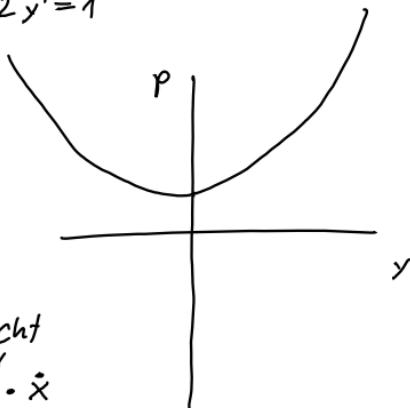
$$a)(y^1)^2 - 2y^2 = 1$$

$$b) x^2 + 1 + (y^1)^2 - 2y^1 = 1$$

$$p^2 - 2y^2 = 1$$

$$p = cht$$

$$y = \frac{1}{\sqrt{2}} \sin(t)$$



$$\ddot{y} = \frac{ch(t)}{\sqrt{z}} = \frac{ch}{p \cdot \dot{x}}$$

$$\dot{x} = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{2}{\sqrt{2}} t + C \quad t = \sqrt{2}x$$

$$k \sin \theta = \left(\frac{\sqrt{2}}{2} t + c, \frac{\sqrt{2}}{2} sht \right)$$

$$y = \frac{\sin(\sqrt{2}x + \phi)}{\sqrt{2}}$$

SC drug

regia

$$\frac{\sin(\sqrt{z}(c-x))}{\sqrt{z}}$$

Aug
wir

$$b) x^2 + 1 + (y')^2 - 2y = 1$$

$$x^2 + (p-1)^2 = 1$$

$$x = \cos t$$

$$p = 1 + \sin t$$

$$\dot{y} = p \dot{x} = (1 + \sin t)(-\sin t) = -\sin^2 t - \sin t$$

$$y = \int \dot{y} dt = \int \frac{1 - \cos 2t}{2} dt + C =$$

$$= \frac{1}{2} \cancel{\dot{1}} \cancel{\dot{2}} \sin 2t + \underbrace{\cos t}_x$$

$$y = \frac{1}{2} \arccos x - x \sin(\arccos x) + C$$

zu Lern: physik

a):

$$p^2 - 2y^2 - 1$$

$$\text{für } F_p = 0 \quad F_y = 2p = 0 \Leftrightarrow p = 0$$

$\Rightarrow y$ konstant

$$y^2 = -\frac{1}{2} \cancel{\dot{1}} \cancel{\dot{2}} \quad *: \text{reduzieren nach } y$$

b)

Od zadnjih: Parametrike reševanje DE
oblike $F(x, y, y') = 0$

11.11

Splosni postopek

1. $P = y'$ neodvisne spremenljivke

2. izračunamo $\overbrace{DF(x, y, p)}$

3. s pomočjo nastavka $dy = pdx$ dobimo

diferencialno enačbo $x = x(p)$ ali $y = y(p)$.

Menkajoča komponenta dobimo iz prvečne enačbe

Včasih ima DE tudi singularne rešitve

- 1) $G(x, y, C) = 0$ in enakosti $\frac{\partial G}{\partial C} = 0$ eliminiramo C
- 2) parametriziramo množico $\{F = F_p = 0\}$ in preverimo
ali reši DE

odvad po p

1. neloage: Poissoi splasne in merobitne sre.
resitive

$$x^2(y')^3 - xy' + y = 0$$

$$y = xy' - x^2(y')^3 \quad p = y' \quad dy = pdx$$

$$y = xp - x^2p^3$$

$$dy = pdx + xdp - 2xp^3dx - 3x^2p^2dp$$

$$pdx = pdx - 2xp^3dx + (x - 3x^2p^2)dp$$

$$2xp^3dx = (x - 3x^2p^2)dp$$

$$x=0 \Rightarrow y=0$$

$$x \neq 0 \Rightarrow$$

$$2p^3dx = (1 - 3xp^2)dp$$

$$2p^3x'(p) = 1 - 3xp^2 \quad p=0 \Rightarrow y=0$$

$$\dot{x} = \frac{1}{2p^3} - \frac{3}{2} \frac{x}{p^2}$$

$$\dot{x} + \frac{3}{2} \frac{x}{p^2} = \frac{1}{2p^3} \quad \text{homogeni del:}$$

$$\frac{dx}{x} = -\frac{3}{2} \frac{dp}{p}$$

partikularn:

$$C = C(p)$$

$$|x = C|p|^{-\frac{3}{2}}$$

$$\dot{x} = C(p)p^{-\frac{3}{2}} = \frac{1}{2}p^{-3}$$

$$\int_0^r \frac{1}{2}p^{\frac{3}{2}-3} dp = \frac{1}{2} \int_0^{-\frac{3}{2}} dx = -p^{-\frac{1}{2}}$$

$$C(p) = -p^{-\frac{1}{2}}$$

$$x = C p^{-\frac{3}{2}} - p^{-2}$$

$$y = xy' - x^2(y')^3 = Cp^{-\frac{1}{2}} - p^{-1} - (Cp^{-3} - 2Cp^{-\frac{3}{2}-2} + p^{-4})p^3$$

$$= Cp^{-\frac{1}{2}} - p^{-1} - C^2 + 2Cp^{-\frac{1}{2}} - p^{-1} =$$

$$= 3Cp^{-\frac{1}{2}} - 2p^{-1} - C^2$$

Dobili smo resitev v parametrični dolžini $(x(p), y(p))$

$$\text{kor. je } p = \frac{y(p)}{x'(p)} = y'(x)$$

Se singularne rešitev

$$x = -p^{-2} + D p^{-\frac{3}{2}}$$

$$y = -2p^{-1} + 3D p^{-\frac{1}{2}} - D^2$$

$$F(x, y, p) = y - xp + x^2 p^3 = 0$$

~~$$\text{tj. } F_p = \sqrt{3}x + 3x^2 p^2 = 0 \times (1+3xp^2)$$~~

$x=0 \Rightarrow y=0$ ena točka ni ogrinjača

$$1+3xp^2=0 \Rightarrow p^2 = +\frac{1}{3x} \Rightarrow p = \pm \sqrt{\frac{1}{3x}}$$

$\xrightarrow{x \neq 0}$

$$y - x \left(\pm \sqrt{\frac{1}{3x}} \right) + x^2 \frac{1}{3x} \left(\pm \sqrt{\frac{1}{3x}} \right) = 0$$

~~$$y = \pm \sqrt{\frac{1}{3x}} \left(x - \frac{1}{3}x \right) = \pm \sqrt{\frac{1}{3x}} \cdot \frac{2}{3}x = \pm \frac{2}{3} \sqrt{\frac{x}{3}}$$~~

$$y' = \pm \frac{2}{3} \frac{\frac{1}{3}}{2\sqrt{\frac{x}{3}}} = \pm \frac{1}{3} \sqrt{\frac{3}{x}}$$

Ali resi ene dva

$$\pm \frac{2}{3} \sqrt{\frac{x}{3}} - x \left(\pm \frac{1}{3} \sqrt{\frac{3}{x}} \right) + x^2 \left(\pm \frac{1}{3} \frac{3}{x} \cdot \sqrt{\frac{3}{x}} \right) = \text{ (76:2)}$$

$$= \pm \frac{2}{3} \sqrt{\frac{x}{3}} \mp \frac{1}{3} \sqrt{3x} \pm \frac{3}{3^3} \sqrt{3x} \neq 0$$

$$= \pm \frac{2}{3} \sqrt{\frac{x}{3}} \mp \frac{78}{9^3} \sqrt{3x} = \sqrt{x} \left(\frac{4}{3\sqrt{3}} \mp \frac{78\sqrt{3}}{9^3} \right) =$$

$$= \sqrt{3x} \left(\pm \frac{2}{9} \right) \dots$$

2. nacin primerjamo y' in p

$$\pm \frac{1}{3} \sqrt{\frac{3}{x}} = \pm \sqrt{\frac{1}{3x}}$$

$$\pm \frac{\sqrt{3}}{3\sqrt{3}} = \pm \frac{1}{\sqrt{3}} \quad \times$$

Znalezę

Dla ośi rezywne

$$(y')^2 - xy' - y + \frac{x^2}{2} = 0$$

$$y = (y')^2 - xy' + \frac{x^2}{2}$$

$$y = p^2 - xp + \frac{x^2}{2}$$

$$dy = 2p dp - x dp - pdx + x dx \quad dy = pdx$$

$$2pdx - x dx = 2p dp - x dp$$

$$(2p - x) dx = (2p - x) dp$$

$$1. 2p - x = 0$$

$$2p = x \Rightarrow p = \frac{x}{2}$$

$$y = p^2 - 2p^2 + 2p^2 = p^2$$

$$2. 2p - x \neq 0$$

$$dx = dp$$

$$y = 1 - x + \frac{x^2 + 2x + c + c^2}{2} =$$

$$p'(x) = 1$$

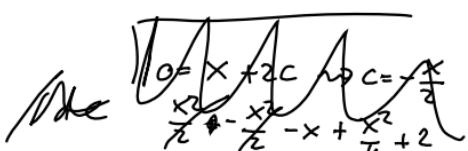
$$= \frac{x^2 + 2x(c-1) + c^2 + 2}{2} =$$

$$p(x) = x + c$$

$$= \frac{x^2}{2} + x(c-1) + c^2 + 2$$

??

$$y = \frac{x^2}{2} + xc + c^2$$



singularne:

$$0 = x + 2c$$

$$c = -\frac{x}{2}$$

$$\Rightarrow y = \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{4} = \frac{x^2}{4}$$

Clairautova enačba

$$y = xy' + \Psi(y')$$

Videli bomo, da ogrinječo najdemo že sproti

3. naloge

a) $y = xy' + \frac{1}{y'}$

b) $y = xy' + (y')^2 + 1$

a) $y = xy' + \frac{1}{y'} \quad \Psi(p) = \frac{1}{p} \quad y' = p$

$$y = px + \frac{1}{p}$$

$$dy = pdx + xdp - \frac{1}{p^2} dp \quad pdx = dx$$

$$\frac{1}{p^2} dp = xdp$$

$$0 = \left(\frac{1}{p^2} - x \right) dp$$

Clairautova enačba se vedno reducira na obliko $f(x, p) dp = 0$

1) $dp = 0 \Rightarrow p = C$

$$y = xC + \frac{1}{C}$$

$dp = 0 \dots$ splošna rešitev

$f(x, p) = 0 \dots$ ogrinječ

2) $\frac{1}{p^2} = x \Rightarrow p = \pm \sqrt{\frac{1}{x}}$

$$y = \pm \sqrt{x} + \pm \sqrt{x} = \pm 2\sqrt{x}$$

$$y' = \mp \frac{1}{\sqrt{x}}$$

$$b) y = xp + p^2 + 1$$

$$dy = xdp + pdx + 2pdp \quad dy = pdx$$

$$(x + 2p) dp = 0$$

$$dp = 0 \Rightarrow p \equiv C \Rightarrow y = xC + C^2 + 1$$

$$x = -2p \Rightarrow p = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x^2 + \frac{1}{4}x^2 + 1 = -\frac{1}{4}x^2 + 1$$

$$y' = -\frac{1}{2}x$$

Opmere o singularnih rešitevih Clairotovih enačb:

$$y = xy' + \Psi(y')$$

$$F(x, y, p) = xp + \Psi(p) - y = 0$$

$$F_p = x + \Psi'(p) = 0$$

Splasna rešitev

$$0 = G(x, y, C) = xC + \Psi(C) - y = F(x, y, C) = 0$$

Torej odvisanje v p ti vedno da ~~je~~ ogrinjačo
pri clairotovi enačbi

Lagrangeova enačba

$$y = x\varphi(y) + \psi(y)$$

4. Reši:

$$y = \frac{xy'}{2} + \frac{2}{y} \quad \varphi(y) = \frac{y}{2}$$

$$dy = \frac{x}{2}dp + \frac{p}{2}dx - \frac{1}{p^2}dp \quad dy = dp/p dx$$

$$\frac{1}{2}pdx = \frac{x}{2}dp - \frac{1}{p^2}dp$$

$$pdx = \left(x - \frac{4}{p^2}\right)dp$$

$$\dot{x} = \frac{dx}{dp} = \frac{x}{p} - \frac{4}{p^3} \quad \text{LDE za } x = x(p)$$

$$\dot{x} - \frac{x}{p} = -\frac{4}{p^3} \quad \text{homogeno:}$$

$$\frac{dx}{x} = -\frac{1}{p^3}dp$$

parti:

$$|x| = C|p| \quad x = C \cdot p$$

$$C(p) \cdot p = -\frac{4}{p^3} \Rightarrow C(p) = -\frac{4}{p^6}$$

$$C = \bar{3}4 p^{-3}$$

$$\bar{x} = 12 \cdot p^{-3} \cdot p + C \cdot p \Rightarrow \bar{x} = \frac{12p^4}{p^3} + C \cdot p$$

$$y = \frac{1}{2} \left(\frac{44p}{3p} + Cp^2 \right) + \frac{2}{p} = \frac{2}{3}p \left(\frac{2}{3} + \frac{1}{2} \right) + \frac{Cp^2}{2} = \frac{8}{3} + \frac{C}{2}p$$

Singulárne rešenie: $0 = \frac{x}{2} - \frac{2}{p^2} = F_p$

$$\frac{2}{p^2} = \frac{x}{2}$$

$$p^2 = \frac{4}{x}$$

$$\Rightarrow y = \frac{2}{p^2} \cdot p + \frac{2}{p} = \frac{4}{p}$$

Až: je res $dy = pdx$?

$$\dot{y} = -\frac{4}{p^2}$$

$$\dot{x} = -\frac{8}{p^3}$$

Až: veda $\dot{y} = p \cdot \dot{x}$

$$t \cancel{\text{je}} \quad p \cdot \left(-\frac{8}{p^3}\right) = -\frac{8}{p^2} \neq -\frac{4}{p^2}$$

→ To n: singulárne rešenie

5. Resi $y = \frac{x(y')^2}{2} + \frac{2}{x}$.



∴

Povzetek (o reševanju impl. DE)

$$F(x, y, y') = 0 \quad \nabla F \neq 0 \quad F \text{ dovolj gladka}$$

$F(x, y, p)$ je plaskov

$$P(x_0, y_0, p_0) \in \mathbb{R}^3$$

$$1) F_p(P) \neq 0 \Rightarrow p = p(x, y)$$

$$F(x, y, p(x, y)) = 0 \quad \forall x, y \text{ bližu } x_0, y_0$$

$$p(x_0, y_0) = p_0$$

če je $x \mapsto y(x)$ rešitev začetnega problema $\begin{cases} F(x, y, y') \\ y(x_0) = y_0 \end{cases}$

$$\text{Potem je } y' = p(x, y)$$

tako dobimo splošno rešitev

$$2) F_p(p) = 0 \Rightarrow$$

dobimo množico $\{F = F_p = 0\} \leftarrow$ unija točk in krivulj

če je P_0 izolirana točka:

ne niki okolicu U od P_0 velja

$$\nabla F \neq 0 \Leftrightarrow P \in U - \{P_0\}$$

dobimo splošno rešitev, vendar se lahko zgoditi, da več ženč splošne rešitve poteka

skozi P_0

če je $\{F = F_p = 0\}$ krivulja, jo parametriziramo

npr. kot $(x(p), y(p), p)$; če velja $p \cdot \dot{x}(p) = \dot{y}(p)$,

potem je to singularna rešitev (in hkrati

ograničena splošne rešitve

Primer od zadnjic

$$y = \frac{xy^2}{2} + \frac{2}{y}$$

$$1) F_p = 0$$

$$F(x, y, p) = \frac{xp^2}{2} + \frac{2}{p} - y$$

$$F_p = 0 \Rightarrow xp - \frac{2}{p} = 0 \Rightarrow x = \frac{2}{p^2}$$

$$y = \frac{xp^2}{2} + \frac{2}{p} = \frac{2}{p^3} \cdot \frac{p^2}{2} + \frac{2}{p} = \frac{3}{p}$$

$$\dot{y} = -\frac{3}{p^2}$$

$$p \dot{x} = p \left(-\frac{6}{p^3} \right) = -\frac{6}{p^2} \neq \dot{y} \Rightarrow \text{G n: agr: irnjača oz n: sing. reš}$$

$$2) dy = pdx$$

$$dy = xp dp + \frac{p^2}{2} dx - \frac{2}{p^2} dp$$

$$\left(\frac{p^2}{2} - p \right) dx + \left(xp - \frac{2}{p^2} \right) dp = 0$$

Ker nas zanimajo rešitve v obliku grafov funkcij
 $x \mapsto y(x)$ lahko predpostavimo, da $dx \neq 0$

\Rightarrow formalno lahko delimo z dx ; dobimo DE za

$$p = p(x).$$

Običajno je lažje reševati DE za $x = x(p)$

$$\text{želimo deliti z } dp, \text{ de dobimo } \frac{dx}{dp} = \dot{x}(p)$$

Ali: je lahko $dp = 0$? $\Rightarrow p \equiv C = y \Rightarrow y(x) = Cx + D$ ustrezen

$$\text{Sicer } \left(\frac{p^2}{2} - p \right) \frac{dp}{dx} + xp - \frac{2}{p^2} = 0$$

$$\text{Opazimo: } dp = 0 \Rightarrow \frac{p^2}{2} - p = 0$$

$$dp \neq 0 \Rightarrow \text{dobimo DE, če } \frac{p^2}{2} - p \neq 0$$

$$\text{če } \frac{p^2}{2} - p = 0 \Rightarrow p = 0 \vee p = 2$$

$\begin{matrix} \downarrow \\ \text{n: definirana} \end{matrix} \quad \begin{matrix} \rightarrow \\ y = 2x + 1 \end{matrix}$

Ali: je resitev singularna?

$$F_p = xp - \frac{2}{p^2}$$

Premica $y = 2x + 1$ ustreza kritični/ji
 $x \mapsto (x, y(x), y'(x)) =$

$$(x, 2x+1, 2)$$

$$F_p = 2x - \frac{1}{2} \Rightarrow$$

Povsed razen v $x = \frac{1}{4}$ je te izraz $\neq 0$
 Ta premica torej priпадa splasti: rešitvi

$x = \frac{1}{4}$ pa je singularna točka
 lahko

smo poiskali:
 zadnje

Povzetek povezave

- 1) $F_p = 0$, parametriziramo / eliminiramo p iz enačbe
 $F = F_p = 0$, pravimo če je rešitev
 \hookrightarrow singularna
- 2) Splošno rešitev isčemo s pomičjo nastavke $dy = pdx$

Naj bo $y = \Phi(x, c)$ splošna rešitev DE $F(x, y, y') = c$
in je ogrinjača. Pokaži deštevje na grafu y
poteka nekončno mnog rešitev

DN

Rozl. DE

$$y = xp^2 - 2p^3$$

$$y = (xy^2) - 2(y^3)$$

$$dy = pdx = p^2 dx + 2pxdx - 6p^2 dp$$

$$dx(p^2 + p) + 6p^2 dp = 0$$

$$\Rightarrow dp = 0 \Rightarrow p = c, y = C^2 x - 2c^3$$

$$y^3 = C^2 = p = c$$

$$C^2 = c \Rightarrow c = 0 \vee c = 1$$

$$\begin{array}{c} y = c \\ \text{sing.} \end{array}$$

$$\begin{array}{c} y = x - 2 \\ \text{ni singularne} \end{array}$$

$$F_p = F = 0$$

$$-2xp + 6p^2 = 0$$

$$p = 0 \Rightarrow$$

$$-2p(x - 3p) = 0$$

$$x = 3p \Rightarrow$$

$$F = 0 \Rightarrow 3p^3 - 2p^3 = 0$$

$$p^3 = 0$$

$$\dot{x}(p) = 3$$

$$\dot{y}(p) = 3p^2$$

$$p \neq 0 \Rightarrow \text{to n; reakce}$$

$$dp \neq 0 \Rightarrow$$

$$\dot{x}(p - p^2) + 6p^2 - 2px = 0 \quad / : p(1-p) \quad \begin{array}{l} \text{zde dle vlastnosti} \\ \text{koje to je} \end{array}$$

$$\dot{x} + x \frac{-2p}{p - p^2} = \frac{6p^2}{p^2 - p}$$

homogen: del:

$$\frac{dx}{x} = \frac{-2p dp}{p - p^2} = \frac{-2}{1-p} dp$$

$$\ln|x| = -2 \ln|1-p|$$

$$x = D(1-p)^{-2}$$

partikulární:

$$D'(1-p)^{-2} = \frac{6p}{p-1}$$

$$D' = 6p(p-1) = 6p^2 - 6p$$

$$D = 2p^3 - 3p^2 + C$$

$$\text{Splosná: } x = \frac{(2p^3 - 3p^2 + C)}{(1-p)^2}$$

Brojne višeg reda

$$F(x, y^{(n)}, \dots, y^{(n)}) = 0 \quad 0 \leq k \leq n$$

Lahko enčemo red z nastavkom: $z = y^{(k)}$

$$\hookrightarrow F(x, z, \dots, z^{(n-k)}) = 0$$

$$xy^{(1)} = y^{(1)}$$

$$z = y^{(1)}$$

$$xz' = z$$

$$\frac{dz}{z} = C \frac{dx}{x} \rightsquigarrow |x| = |z| \Rightarrow x = dz = |y^{(1)}|$$

$$\begin{aligned} z &= C \cdot x = y^{(1)} \\ &\text{↑ skriveno v konstanto} \\ \int (Cx dx) dx &= \int C \frac{1}{2} x^2 + D dx = \int C x^2 dx - C \frac{1}{3} x^3 + D x + E = \\ &= C x^3 + D x + E \end{aligned}$$

(2)

$$F(y, y', \dots, y^{(k)}) = 0 \rightarrow F \text{ neutralsen od } x$$

$$\text{"}z(y) = y^1(y)\text{"} \quad \text{zadaj je yodrok od } y$$

Interpretacina krt:

$$z(y) = y^1(x(y)) \quad z(y(x)) = y^1(x)$$

$y \mapsto x(y)$ inverz reative x

(obsteja ker $y'(x) \neq 0$)

Resi enako

$$yy'' = y^1(y'+1) \quad \leftarrow \text{Možnost kone moremo uporabiti}$$

$$y' = 0 \Rightarrow yy'' = 0 \quad 1) y = 0$$

$$2) y'' = 0 \Rightarrow$$

$$y^1 = c = 0 \Rightarrow y = c$$

$$\Rightarrow y = c \in \mathbb{R}$$

$$z = y^1(x(y))$$

$$z(y(x)) = y^1(x)$$

$$\frac{dz}{dy}, y' = y''(x) \quad \dot{z} y' = y''$$

$$\dot{z} \cdot z = y''$$

Nesene o venciško

$$y \cdot \dot{z} \cdot z = z(z+1)$$

$$\dot{z} = 0$$

2) $z \neq 0 \Rightarrow$ lahko delimo

$$2) \quad y \dot{z}' = z + 1$$

hom:

$$\frac{dz}{dy} \cdot y = z \quad z = cy$$

partikularne:

$$y \cdot c' y = 1$$

$$c' = \frac{1}{y^2} \rightsquigarrow c = -\frac{1}{y} + D$$

$$z = -1 + Dy$$

$$\frac{y'}{-1 + Dy} = 1$$

$$x = \int_{-1+Dy} \frac{dy}{y'} = \frac{1}{D} \ln(-1 + Dy) + C$$

$$Dx + C = \ln(Dy - 1)$$

$$\frac{ce^{Dx}}{D} + 1 = y$$

$$D = 0 \rightarrow$$

$$\frac{y'}{-1} = 1$$

$$y' = -1$$

$$y = z - x$$

$$F(x, y, y'), \dots) = 0$$

F homogena v y, y', \dots

$$F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y', \dots) \quad \text{za neki } k \in \mathbb{N} \quad \lambda \in \mathbb{R}$$

-- uvedemo $z(x) = \frac{y}{x}$

$$yy'' - x^2 y'^2 + (y')^2$$

$$z = \frac{y}{x} \quad yz' = y' \quad y'' = y'z + z'y = z^2 y + z'y$$

$$y(yz^2 + z'y) = x^2 y'^2 + (yz)^2$$

$$y=0 \Rightarrow \text{je rešitev}$$

$$y \neq 0 \Rightarrow$$

$$z^2 + z' = x^2 + z^2$$

$$z = \frac{1}{3}x^3 + C = \frac{y}{x} \Rightarrow \ln|y| = \frac{x^4}{12} + cx + D$$

$$y = D e^{\frac{x^4}{12} + cx}$$

Eksistencijski rezultat

$f: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ evene in Lipschitzova v y

$$|f(x, y_0) - f(x, y_1)| < L|y_0 - y_1| \quad \text{za } L \geq 0$$

$\Rightarrow \exists!$ rešitev $y = f(x, y)$, $y(x_0) = y_0 \quad \forall (x_0, y_0) \in D$, kjer je

dopolnjena na intervalu $(x_0 - \alpha, x_0 + \alpha)$, kjer je

$\alpha = \min\left\{\frac{a}{M}, \frac{b}{L}\right\}$, kjer sta $a, b > 0$ tenušna, dej

$$\underbrace{[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]}_{C_{a,b}} \subset D \quad \text{inj. in mehanizem}$$
$$M = \max_{x, y \in C_{a,b}} |f(x, y)|$$

obzorje: $f \in C^1[C_{a,b}] \Rightarrow f$ je Lipschitzova

Podane je Cauchyjeva naloge

$$y' = y + \cos y \quad y(0) = 0$$

a) Ustremi, da obstaja rešitev enačke in dokazi

Taylorjev polinom stopnje 3 v $x=0$

b) delni α

c) Dokazi, da se rešitev razteži na \mathbb{R}

d) Dokazi da velja $\lim_{x \rightarrow \infty} y(x) = \infty$

a) F je zvezna na \mathbb{R}^2

A1: je Lipschitzova? dokazi jo v C^1

\Rightarrow Rešitev obstaja in je enolična

$$y'' = y' - \sin(y)y = (y + \cos y)(1 - \sin(y))$$

$$\begin{aligned} y''' &= (y' - y \sin(y))(1 - \sin(y)) + (y + \cos y)(1 - \cos(y)y) = \\ &= (y + \cos y)(1 - \sin(y))^2 + (y + \cos y)(1 - y \cos y + \cos^2 y) \end{aligned}$$

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + o(x^3) = \\ &= x + \frac{1}{2} x^2 + o(x^3) \end{aligned}$$

$$b) L = \max_{C(a,b)} \left| \frac{\partial f}{\partial y} \right| \quad f(x,y) = y + \cos y$$

$$\left| \frac{df}{dy} \right| = |1 - \sin(y)| \leq 2 \quad L = 2$$

$$\alpha = \min\left(a, \frac{b}{M}, \frac{1}{2}\right)$$

$$\max_{C(a,b)} f \stackrel{def}{=} \max_{[a,b]} f(\cdot, y) = M$$

$$|f(x,y)| = |y + \cos y| \leq |y| + |\cos y| \leq b+1$$

$$\frac{b}{M} \geq \frac{b}{b+1}$$

$$\alpha \text{ z.m. : } n \notin \left\{ a, \frac{b}{b+1}, \frac{1}{2} \right\}$$

je istormo a in b doval vela, je

$$\text{takže } \alpha > \frac{1}{2}$$

$$\Rightarrow \text{režim na } (-\frac{1}{2}, \frac{1}{2})$$

c) Pogute izb velja za vsoto fader.

To je da si postopoma izbiramo take vedno bolj ne robavih iz manega, lahko razstirimo interval na cel R

razen nenečo 6 spodnje pisanje b+pt

Sklep: Če lahko x amejmo nezadel modvisor
ad zacetne pogoj, potem se registrira
razdir: ne R

V poslednjem to velja: če sta f in fy amejena

$$d) \lim_{y \rightarrow \infty} y(x) = \infty$$

y^* nerasčajača in večja od 1

$$y' > 0 \quad \text{vsi } x \geq t > 1$$

$$1 - \sin t \geq 0 \quad \text{za } t > 0 \Rightarrow \text{Ras nerasča}$$

Dokazano da je $y(x) > 0 \quad \forall x > 0$

Reimode $y(\tilde{x}) = 0 \quad \text{za nek } x' > 0$

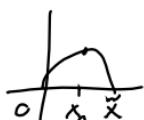
$$y'(\tilde{x}) = 0 + \cos(0) = 1$$

Torej je mogoče dati prej funkcije negativne,

$$\mathcal{Z} = \{x, x > 0, y(x) = 0\}$$

$$y_0 = 1 \quad y = \varphi(x)$$

$$\Rightarrow \varphi'(x) > 0 \quad \text{za } x \in (0, \epsilon) \quad \epsilon > 0$$



$$\text{Za: } x_1 \in (0, \tilde{x})$$

$$\varphi'(x_1) = 0$$

$$\varphi'(x_1) = \varphi(x_1) + \cos(y(x_1)) = 0$$

$$\Rightarrow \begin{matrix} \vee \\ 0 \\ \cos y(x_1) < 0 \end{matrix}$$

$$\varphi(x_1) > \frac{\pi}{2}$$

$$\varphi'(x_1) > \frac{\pi}{2} + \cos(x_1) > 0 \quad *$$

$$\Rightarrow \varphi(x) > 0 \quad \text{za } \forall x > 0$$

$$\Rightarrow \varphi'(0) \text{ je nerasčačna}$$

Eksistencija rešenja

$f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ kveze u Lipschitzove

$\Rightarrow \exists!$ rešenje $y = f(x, y)$ $y(x_0) = y_0 \in A(x_0, y_0)$,

kao je definisano na intervalu $(x_0 - \alpha, x_0 + \alpha)$,

ker je $\alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$; $M = \max |f|$, L Lipsch.,

$a, b > 0$. $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset D$

$f \in C^1[a, b]$ $\Rightarrow f$ je Lipschitzova

$$F(x, y, y', \dots) = 0$$

F homogene v y, y' ($F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y', \dots)$)

$$\Rightarrow \text{uvedemo } z(x) = \frac{y'}{y}$$

Enačba višje reda $F(x, y^{(k)}, \dots, y^{(n)}) \Rightarrow z = y^{(k)}$

Reševanje impl. DE

$$F(x, y, y') = 0 \quad \nabla F \neq 0$$

1) $F(x, y, p) = 0$ je plaskov

$$P = (x_0, y_0, z_0)$$

2.1) $F_p(P) \neq 0 \Rightarrow p = p(x, y)$

$$F(x, y, p(x, y)) = 0 \quad \forall \text{okolic} P$$

$$p(x_0, y_0) = p_0$$

$$y(x) \text{ rešitev} \Rightarrow y' = p(x, y)$$

Dobimo splasno rešitev

2.2) $F_p(P) = 0 \Rightarrow \{F = F_p = 0\}$ je unija točki in krivulji

Po izolirana točka $\Rightarrow \exists$ okolina P . $P \in U - \{P_0\}$. $\nabla F \neq 0$

Dobimo splasno rešitev

$\{F = F_p = 0\}$ je krivulja \Rightarrow parametriziramo

npr. $(x(p), y(p), p)$.

$p \cdot \dot{x}(p) = \dot{y}(p) \Rightarrow$ Singularna rešitev



Ograničena splasna rešitev

Clairautova enačba

$$y = xy' + \psi(y') \Rightarrow y' = p \Rightarrow pdx = dy$$

$$\text{Dobimo: } f(x, p) dp = 0$$

$d_p = 0 \dots$ singularna rešitev
 $f(x, p) = 0 \dots$ ogrinjača

Odvajanje po p , ti vedno kaže ogrinjačo pri tehnici:

Prvi integral enačbe $y' = f(x, y)$ je funkcija, ki je konst.

vzdelž vsake rešitve

$$y = \varphi(x, c) \wedge \varphi_c \neq 0 \Rightarrow c = C(x, y)$$

1) DE zapisemo kot $Pdx + Qdy = 0$

$$Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y \Rightarrow (P, Q) \text{ je potencijalno}$$

$$Q = \frac{\delta}{\delta y} \int P dx + C(x, y)$$

Riccatijeva enačba

$$y' = a^2y + b y + c$$

1) Ugotovimo rešitev y_p

2) Slednja rešitev je $y = y_p + u(x)$

3) Dobimo Bernullijevu enačbo $y = y_p + \frac{1}{u'(x)} \Rightarrow$ LDE

Bernullijeva enačba

$$y' + ay = by^\alpha \quad \alpha \notin \{0, 1\} \Rightarrow z = y^{1-\alpha}$$

Od zadnječ neprek
 $y' = y + \cos y \quad y(0) = 0$

Rešitev f je definirana za $\forall x \in \mathbb{R}$

$$y > 0 \iff \forall x > 0$$

$$Z = \{x > 0, \varphi(x) = 0\}$$

$$\begin{aligned} \varphi'(0) - \varphi(0) + \cos(\varphi(0)) &= 0 + \cos 0 = 1 \\ \Rightarrow \text{a nek } \varepsilon > 0 \quad \text{je } \varphi(x) > 0 \quad \forall x \in (0, \varepsilon) \end{aligned}$$

$$\text{ozn. zimo } x_0 = \inf Z > 0$$

$$\varphi \text{ zvezna} \Rightarrow \varphi(x_0) = 0$$

Rolle: $\exists x_1 \in (0, x_0), \varphi'(x_1) = 0 \wedge \varphi(x) > 0 \quad \forall x \in (0, x_0)$

$$\varphi \text{ resit CN} \Rightarrow \varphi'(x_1) = \varphi(x_1) + \cos(\varphi(x_1))$$

$$\stackrel{\vee}{0} \Rightarrow \varphi(x_1) > \frac{\pi}{2}$$

$$\Rightarrow \varphi(x_1) - \cos(\varphi(x_1)) > 0$$

$$\Rightarrow Z = \emptyset \quad \times$$

$$g(t) = t + \cos t \quad g(0) = 1 \quad g'(t) = 1 - \sin t \geq 0 \quad \forall t$$



$$g(t) \geq 1 \quad \forall t \geq 0$$

$$\varphi'(x) = g(\varphi(x)) \geq 1 \quad \forall x \geq 0$$

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(t) dt \geq \int_0^x 1 dt = x \xrightarrow{x \rightarrow \infty} \infty$$

1) Dan je Cauchyjeva naloge

$$y' = \frac{y}{1+x^2+y^2} \quad y(0)=1$$

Dokazi da obstaja ! resitev, ki je definirana na \mathbb{R} .

Ali ima y nico?

$f(x,y)$ mora biti vezne in lipschitze v y

$$f(x,y) = \frac{y}{1+x^2+y^2} \quad \text{vernost zaradi elementarnosti}$$

Nekdime pravokotnik $[a,b] \times [-b, b]$

Ta m je lipschitza, ker je vseh omejen

\Rightarrow obstaja sje enotne delocene na $[-a, a]$

$$\text{kjer je } a = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$$

a, b poljubno velike

$$M = \max_{[a,b]} |f(x,y)| \models \max_{x^2=0} \frac{y}{1+y^2} = \left\{ \begin{array}{l} \pm 1/2 = 1/2 \\ \hookrightarrow \frac{(1+y^2)-2y^2}{(1+y^2)^2} = \frac{1-y^2}{1+y^2} \end{array} \right.$$

$$L = \max \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = \frac{1+x^2+y^2-2y^2}{(1+x^2+y^2)^2} = \frac{1+x^2-y^2}{(1+x^2+y^2)^2}$$

L je tudi omejena

Dobili smo a, b ki je isto za vse funkcije neodvisen od poznejih podatkov, tako da lahko z eksistencim rešenjem rezultira na vse funkcije

5) $y'=0 \Leftrightarrow y=0$

Resimo dej je $f(x_0)=0$

$$z=0 \quad z'=f(x_0, y) \text{ resi } z(x_0)=0$$

f tudi resi to naloge $\Rightarrow f=z$ (zaradi enotnosti)

\Rightarrow ker je $f(0)=1$

Trditev: Za α lahko vzamemo $\alpha = \min\{\alpha, \frac{b}{M}\}$
L moč pa obstajat ne tem pravokotnikov

z)

$f : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ zweck

$$y'(x) = 1 + \frac{y(x)^2}{2} \sin(f(x)) \quad ; x \in [0, 2] \\ y(0) = 0$$

Dabei ist y je

$$\max_{x \in [0, 2]} y(x) \geq 1$$

Rechts ist $\max y(x) < 1 \Rightarrow$

$$y'(x) \geq 1 + \frac{y(x)^2}{2} \geq \frac{1}{2}, \text{ da } y \in [0, 1]$$

$$y \geq 0$$

$$x_0 = \inf \{x; y(x_0) = 0\}$$

Rechts ist $\exists \leftarrow x_0 > 0$ da $y(x_0) = 0$

Patem po Rollovem: zeker, $\exists x_1 \in (0, x_0)$ da $y'(x_1) = 0$

$$y'(x_1) = 1 + \frac{(y(x_1))^2}{2} \sin(f(x_1))$$

$$0 \leq y(x_1) \leq 1$$

$$y'(x_1) \geq 1 - \frac{(y(x_1))^2}{2} \geq \frac{1}{2}$$

$\Rightarrow y > 0 \quad \forall x \in \mathbb{R}_{>0} [0, 2]$ kein quadrat v 0 positiven

$$\Rightarrow y'(x) \geq 1 - \frac{1}{2} \geq \frac{1}{2}$$

$$\text{Patem je } y(2) = y(0) + \int_0^2 y'(t) dt \geq 0 + 2 \cdot \frac{1}{2} = 1 \quad \times$$

$$\Rightarrow \max y \geq 1$$

3.

$$y' = f(y) \quad f \in C^1(\mathbb{R})$$

Pokaz: da je \forall periodičke rešitve konstantne

Recimo da $\exists \omega > 0$. $y(t + \omega) = y(t)$ $\forall t$

$$\int_0^\omega y' dx = \int_0^\omega f(y) dx = C$$

y periodična $\Rightarrow f(y)$ periodična

$\text{če je } f(c) = 0$, potem je $y \equiv c$ rešitev

$f(y)$ neoddadol $\Rightarrow f$ ima nulo (ker je y periodična)

$\rightarrow \exists x_0 \in \mathbb{R}. f(y(x_0)) = y'(x_0)$ je ena rešitev
sisteme in zaradi enodolnosti je to edina

4.

$$y' = y + e^y \quad y(0) = 1$$

- a) Dokaż, że je definiująca interval rezytyw, kiedy dolne ekstenzje istnieją i są one unikalne
- b) Dokaż, że je f mierząca w & to dla $x \geq 0$, kiedy je definiowana
- c) Dokaż, że nie φ nikt
- d) $\exists w > 0 < \infty$. $\lim_{x \rightarrow w} \varphi(x) = \pm \infty$

a) $y + e^y$ je wewnątrz odwrotnie \Rightarrow funkcja rezytywna

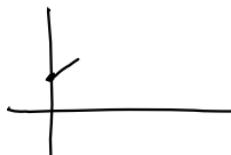
$$\text{na } [\alpha, -\infty] \quad \alpha = \min\left\{\alpha, \frac{b}{M}, \frac{V}{k}\right\}$$

$$y \in [1-b, 1+b] \Rightarrow y + e^y \leq 1+b+e^{1+b}$$

$$M = 1+b+e^{1+b} \Rightarrow \frac{b}{M} = \frac{b}{1+b+e^{1+b}} \leq c \quad \forall b > 0 \\ \text{funkcja jest rosnąca}$$

b) $y' = y + e^y$

$$y' \geq 0 \quad y(0) = 1$$



Rozważmy, że $y + e^y = 0$

$$e^y = -y$$

V
O

że je $y = 0$ p. dem jo $y' = 1$ terazem de wtedy
nicht mierząca, kiedy je na more kiedy reg. razy za
enu nikt

c) f je nizko

Rečimo da nizko nizko podan je $f(x) > 0 \forall x \in \mathbb{R}$,

podan je po naraščajoča

in odvod je vrednost 1. podan poda nizko

znači x ali po negativne velje večino
razstavitev je pod, ne more biti negativen,
torej jo poskrbi, podan je avrod

$$< 0 \quad *$$

$$f(x) = f(0) + \int_0^x f'(t) dt \leq f(0) + x \xrightarrow{x \rightarrow -\infty} -\infty$$

✓

d) Dokazujemo:

$$\exists M > 0. \lim_{x \rightarrow \infty} f(x) = \infty$$

Dokazimo da je nizko omejen. (nizko obsegajoče leje f
straga naraščajoča na $x \geq 0$)

$$\frac{1}{z^1} = y + e^y \Rightarrow z^1 = \frac{1}{y+e^y}$$

$$z^1(y) = \int_1^y \frac{1}{t e^t} dt \leq \int_1^y \frac{1}{e^t} dt = e^{-1} - e^{-y} \leq M$$

dokazuje

Naj bo $f \in C^1(\mathbb{R}^2)$, zadešča predpostavka eksistencije izreka. Recimo, da se naše rezultate ne de razsiriti cez interval (α, ω) ; $-\infty < \alpha < x_0 < \omega < \infty$.
Potem velja $\lim_{x \rightarrow \omega} y(x) = \pm \infty$ in $\lim_{x \rightarrow \alpha} y(x) = \pm \infty$

$$5) \quad y = \frac{1}{x^2+y^2} ; \quad y(a) > 0$$

- a) Paket, da $\exists! \varphi$ rešitev, ki je določena na \mathbb{R}
 b) paket $\lim_{x \rightarrow \infty} \varphi(x) < \infty$
-

$$f(x,y) = \frac{1}{x^2+y^2} \quad \text{je omejena vzdolj} \quad 0$$

$$f_y(x,y) = \frac{2y}{(x^2+y^2)^2} \quad \text{je omejen razen na dolici} \quad 0$$

Dokazimo da je rešitev omejena nevezke končna in tekujoča
 v 0 je rešitev v pozitivni natančnosti

$$\begin{aligned} \varphi(x) &= \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2 + \varphi(t)^2} dt = \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2 + (\varphi(t))^2} dt \leq \\ &\leq \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2} dt = \varphi(x_0) - \frac{1}{x} + \frac{1}{x_0} \leq M \end{aligned}$$

omejeno

\Rightarrow Po trdilih: je $\omega = \infty$

\sqrt{f}

Popravimo f tako, da nima več podnivo $(0,0)$ in velja
 $f = \tilde{f}$ izven neke majhne okolice $(0,0)$ ki je gotov ne
 sile

$\sqrt{\tilde{f}}$ pa je simetrična $y(-x)$

6)

Pokazujemo da je $f(x)$ neravščjaka

in omejena \Rightarrow ima limito

Vektorško polje na \mathbb{R}^n je preslikava $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Takovnica polja V je krivulja $\gamma(t; a, b) \xrightarrow[\psi]{} \mathbb{R}^n$, ki

začasča, da je $\dot{\gamma} = V(\gamma(t))$

$$\gamma(t) = (x_1, \dots, x_n)$$

$$\dot{x}_1 = V_1(\gamma(t))$$

⋮

$$\dot{x}_n = V_n(\gamma(t))$$

To zdej: obrazovali smo pravere za $n=0$, vendar v
pravolini oblike $y'(x) = f(x, y(x))$

$\begin{cases} \dot{x} \\ \dot{y} \end{cases}$ neautonomni sistemi;

Erečbi $y' = f(x, y)$ pravodimo sistem oz avtomatsko
vektorško polje

$$\dot{x} = 1$$

$$y = f(x, y)$$

2.12

F 1. uro

Mislíme jeď dekej pamětní

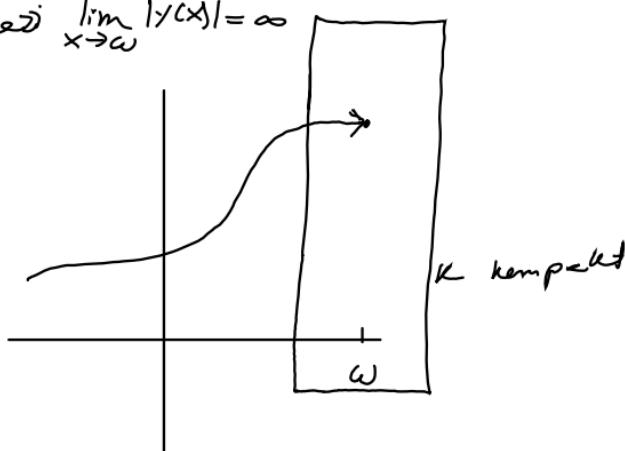
3. nebeseg
 $f \in C^\infty(\mathbb{R})$

y resitev

$$y' = f(x, y)$$

$y(x_0) = y_0$, tako da se ne raztegne cez
 (x_0, y_0) $w < \infty$

Dokaz: $\lim_{x \rightarrow w} |y(x)| = \infty$



Graf resitve je veletarsko polje $V(1, f(x, y))$. Izberemo kompakt pravokotnik $[w-\delta, w+\delta] \times [-l, l]$

Irek nem pove: če je x dovolj blizu w , je tokanica bodisi pod $-l$ in nad $+l$

Sklep: če x dovolj blizu w je $y(x) > l$ $\Rightarrow \lim_{x \rightarrow w} |y(x)| = \infty$

$$y' = \frac{1}{x^2+y^2}$$

$$y(0) > 0$$

y je definisvana na \mathbb{R}

Odvod jeomejen, ker je $y' > 0 \forall x \in \mathbb{R}$, torej je y naraščajoča, večji kot je x^2 , večji je y , vedno manjši je odvod. \Rightarrow

$$y' = \frac{1}{x^2+y^2} \leq \frac{1}{y^2} \leq \frac{1}{y_0^2} \quad \text{za } x \geq 0$$

$$\Rightarrow y(x) \leq \frac{1}{y_0} x + y_0 \Rightarrow \text{če je w končen,}$$

$y(x)$ nezapusti vsekoga kompakta

za $x < 0$. y vedno rarašča

$$y' = \frac{1}{x^2+y^2} \leq \frac{1}{x^2} \rightarrow 0 \Rightarrow y' \rightarrow 0$$

$\Rightarrow y'$ je omejen

N4.

a) Izrazi polje $(1,0)$ in $(0,1)$, ki sta padeni v polarnih koordinatih (r in φ)

b) Določi tak polje $V(x,y) = \alpha(x^2+y^2)(y-x)$, kjer je $\alpha \in C^\infty(\mathbb{R})$

$$\begin{aligned} \dot{r} &= 1 \Rightarrow r(t) = r_0 + t \\ \dot{\varphi} &= 0 \Rightarrow \varphi(t) = \varphi_0 \end{aligned}$$

$$\begin{aligned} x &= r \cos \varphi \rightsquigarrow x(t) = r_0 \cos \varphi_0 + t \cdot \cos \varphi_0 \\ y &= r \sin \varphi \rightsquigarrow y(t) = r_0 \sin \varphi_0 + t \sin \varphi_0 \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= \cos \varphi_0 = \cos \varphi(t) \\ \dot{y}(t) &= \sin \varphi_0 = \sin \varphi(t) \end{aligned}$$

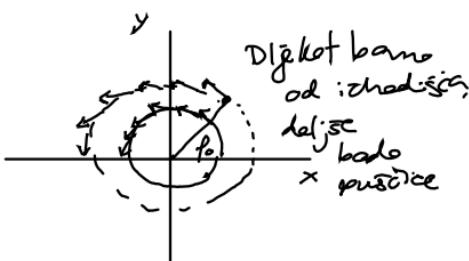
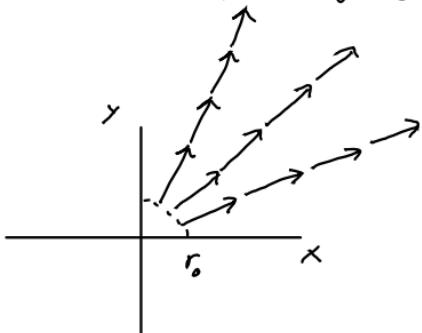
$$\left. \begin{aligned} \dot{x} &= \frac{x}{\sqrt{x^2+y^2}} \\ \dot{y} &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned} \right\} W(x,y)$$

$$\begin{aligned} \dot{r} &= 0 \rightsquigarrow r = r_0 \\ \dot{\varphi} &= 1 \rightsquigarrow \varphi = \varphi_0 + t \end{aligned}$$

$$\begin{aligned} \dot{x} &= r \cos \varphi = r_0 \cos(\varphi_0 + t) \\ y &= r \sin \varphi = r_0 \sin(\varphi_0 + t) \end{aligned}$$

$$\dot{x} = -r_0 \sin(\varphi_0 + t) = -y$$

$$\dot{y} = r_0 \cos(\varphi_0 + t) = x$$



$$Q(x,y) = (-y, x)$$

b)

$$\alpha(x^2+y^2) \cdot (y, -x) = \alpha(r^2)(-(0, 1)) = \alpha r^2(-Q)$$

v polar mit
koordinaten

$$\dot{r} = 0$$

$$\dot{r} = \alpha(r^2) \cdot 0 = 0 \Rightarrow r = r_0$$

$$\dot{\varphi} = \alpha(r^2) \cdot 1 = \alpha(r^2) \Rightarrow \varphi(t) = \int \alpha(r^2) dt \Leftarrow \\ (x, y) = F(r, \varphi)$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_F \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = J_F \cdot P(r, \varphi) = \underbrace{r_0}_{\text{neu kopplig}} + t \alpha(r^2)$$

$$x(t) = r(t) \cos \varphi(t) = r_0 \cos(\varphi_0 - \alpha(r_0^2)t) \Rightarrow$$

$$y(t) = r(t) \sin \varphi(t) = r_0 \sin(\varphi_0 - \alpha(r_0^2)t)$$

$$x(t) = \underbrace{r_0 \cos \varphi_0}_{x_0} \cos(\alpha(r_0^2)t) + \underbrace{r_0 \sin \varphi_0}_{y_0} \sin(\alpha(r_0^2)t) =$$

$$= x_0 \cos(\alpha(x_0^2+y_0^2)t) + y_0 \sin(\alpha(x_0^2+y_0^2)t)$$

$$\text{DN} \quad \dot{r} = s \cdot n \cdot r$$

$$\dot{\varphi} = \cos r$$

$$\vec{x} = A(t) \times G$$

$$A(t) = \text{const}$$

$$\hookrightarrow e^{A^t} \\ e^A = d + A + \frac{A^2}{2!} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Dodataci A^{et} za

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R}$

c) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

a) $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (A + I)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^k}{k!} & 0 \\ 0 & \frac{(2t)^k}{k!} \end{bmatrix} =$
 $A^k = \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix} \quad = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$

b) Jordanova forma

$$A = \lambda I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

~~$A^k = (\lambda I + N)^k =$~~ \leftarrow komutira linearne oblike
 $e^{At} = e^{\lambda t} I + e^{At} N = e^{\lambda t} e^{Nt} =$ $\downarrow \lambda t + \frac{t^2}{1!} + \frac{t^3}{2!} + \dots$

$$= e^{\lambda t} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} N^k \right) =$$

$$= e^{\lambda t} \left(\cancel{\frac{1}{0!} I} + \frac{t}{1!} N + \frac{t^2}{2!} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) =$$

$$e^{\lambda t} \begin{bmatrix} 1 + \frac{t^2}{2} \\ 0 & 1 + \\ 0 & 0 & 1 \end{bmatrix}$$

$$c) A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{Vyslozeni: To jeme matici s kompl. koef.}$$

$$A \approx a + bi$$

$$e^{At} = e^{(a+bi)t} = e^{at} e^{bit} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

$$A \approx aI + bJ$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$J^2 = -I$$

$$J^3 = -J$$

$$J^4 = I$$

$$e^{At} = e^{(aI+bJ)t} = e^{bt} e^{bJt} =$$

$$= e^{bt} \left(I + bJ + \frac{b^2 J^2}{2!} + \frac{b^3 J^3}{3!} + \frac{b^4 J^4}{4!} + \frac{b^5 J^5}{5!} \right) =$$

$$= e^{bt} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (bt)^{2k} I + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (bt)^{2k+1} J \right) =$$

$$= e^{bt} (\cos bt I + \sin bt J) = e^{bt} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

Sistemi NDE s konstantnimi koeficienti

$$\dot{\vec{x}} = A\vec{x} \quad A \in M_{nn}$$

Vseka rešitev je oblike $\vec{x}(t) = \vec{\Phi}(t)\vec{x}(0)$, kjer je $\vec{\Phi}$ fundamentalna rešitev (res.) matricni sistem $\dot{\vec{x}} = A\vec{x} \quad \vec{\Phi}(0) = id$

Kako poiščemo $\vec{\Phi}$?

$$\begin{aligned}\Psi &= e^{At} & \Psi(0) &= I \\ \Psi &= Ae^{At} = A\Psi(t) & \checkmark\end{aligned}$$

$$\vec{\Phi} = \Psi$$

$A = PJP^{-1}$ jordanova forma

$$e^{At} = e^{(PJP^{-1})t} = \sum \frac{PJP^{-1} + \mu}{k!} = Pe^{Jt}P^{-1}$$

Predpostavki: Lanz da ima A realne lastne vrednosti

Projekt fundamentale risiken ordnen

$$\dot{x} = x - y + u z$$

...
we fehl)

Po; \hat{x}_j sind zu bestimmen \rightarrow

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 3 & 4 \\ -2 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 3-x & 1 & 2 \\ 3 & 3-x & 4 \\ -2 & -1 & -1-x \end{bmatrix}$$

$$\det(A - xI) =$$

$$(3-x)(x^2 + x(1-3)+4) - 1(-3x-3+8) + 2(-3+6-2x)$$

$$= (3-x)(x-1)^2 + (-x+1) =$$

$$= (x-1)(-x^2 + x(3+1) - 3 - 1) =$$

$$= -(x-1)(x^2 - 4x + 4) = -(x-1)(x-2)^2$$

$$J = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{bmatrix} \quad P = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1:$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 4 \\ -2 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_2 = 2$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$x = 0 \\ y = -2z$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\ker(A - 2xI)$:

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \quad x+2=0$$

$$\ker(A - 2xI) = \text{Lin} \left\{ \begin{bmatrix} x \\ -x \\ 1 \end{bmatrix} \right\} =$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{Lin} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{v}_1 = (A - 2I) \vec{v}_2 \leftarrow v_1 \text{ je ein, n: re. bethr. vektor}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 2 & & \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^t & & \\ e^{2t} + e^{2t} & & \\ e^{2t} & & \end{bmatrix}$$

Nelne mogeni: linearni sistem NDE s konst. koef.

$$\vec{V} = A\vec{v} + f(t) \quad \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \right]$$

$$\vec{V}_0 = e^{At} \vec{v}_0 \quad \vec{V}_r = e^{At} \cdot w(t)$$

$$\vec{V}_p = \underbrace{A e^{At} w}_{\vec{v}} + e^{At} \cdot \dot{w} = A\vec{v} + f$$

$$A\vec{v} \quad e^{At} \cdot \dot{w} = f$$

$$\dot{w} = e^{-At} \cdot f$$

$\bar{\Phi}(t)$ fundamentalna resiter $\Rightarrow \bar{\Phi}$ obratnica

ve se t, kjer je definirana

$$\bar{\Phi}(t) = \bar{\Phi}(-t)$$

$$w(t) = \int_0^t e^{-A\xi} f(\xi) d\xi$$

$$v_p = e^{At} \cdot w(t) = \int_0^t e^{A(t-\xi)} f(\xi) d\xi$$

Poisci: sponzno rešitev sistema

$$\dot{x} = x - y + \frac{1}{\cos t}$$

$$\dot{y} = 2x - y$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = -(1-\lambda)(1+\lambda) + 2 = \lambda^2 - 1 + 2 = \lambda^2 + 1$$

$$\lambda = \pm i$$

novi nove koordinate $u = x$

$$v = y - x$$

$$\dot{v} = \dot{y} - \dot{x}$$

$$\dot{u} = u - v - u + \frac{1}{\cos t} = -v + \frac{1}{\cos t}$$

$$y = v + u$$

$$\dot{y} = \dot{v} + \dot{u}$$

$$\dot{v} - v + \frac{1}{\cos t} = 2u - y - u =$$

$$\dot{v} = u - \frac{1}{\cos t}$$

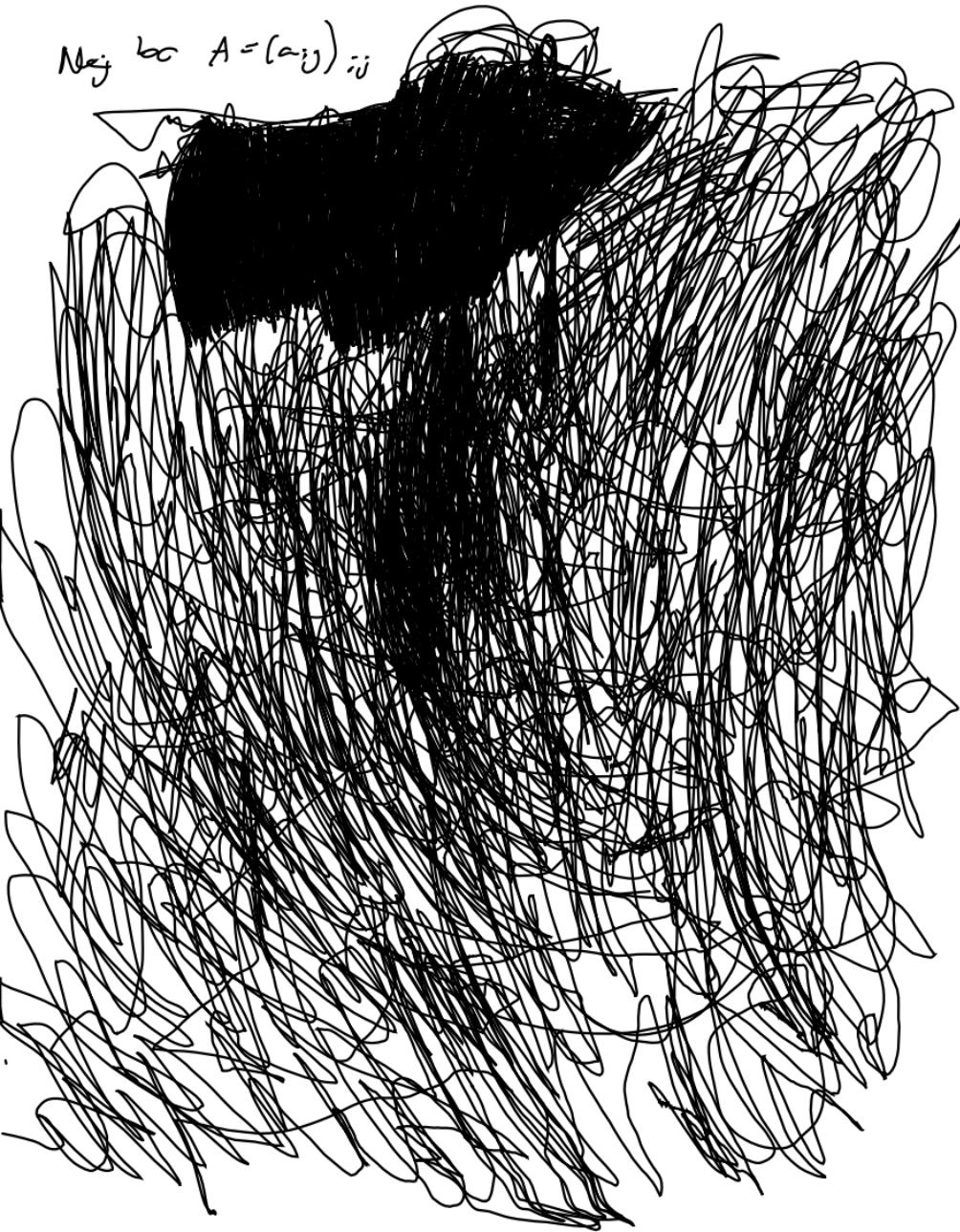
$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad e^{Bt} = e^{it} = \begin{bmatrix} \cos t - \sin t \\ \sin t \cos t \end{bmatrix}$$

$$w = \int_0^t e^{-is} \begin{bmatrix} 1 \\ -\frac{1}{\cos s} \end{bmatrix} ds = e^{-Bt} \begin{bmatrix} \cos t \sin t \\ -\sin t \cos t \end{bmatrix}$$

$$w = \begin{bmatrix} \int_0^t (1 - \tan s) ds \\ \int_0^t (-\tan s - 1) ds \end{bmatrix} = \begin{bmatrix} t + \ln |\cos t| \\ -t - \ln |\cos t| \end{bmatrix}$$

$$V(t) = e^{Bt} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + e^{Bt} w(t)$$

Neg bc $A = (a_{ij})_{ij}$



ne Sternen

$$x_j(t) = e^{\lambda_1 \dots + \lambda_j} \cdot p_j(t)$$

$s^*(p_j) \leq \text{alg.}$
 $\text{reale latte vektoren}$
 $\text{aus } \lambda_j$

Vide jo kud

12. 12

Spomnimo se

$$\vec{x} = A \cdot \vec{x} + f(z)$$

$$\bar{\Phi} = P e^{\int^t_0 P^{-1}}$$

$$x_n = \bar{\Phi} x_0$$

$$x_r = \int_{\alpha}^t \bar{\Phi}(t) \bar{\Phi}^{-1}(s) f(s) ds$$

N: Določi splašnico reziter sistema

$$\dot{x} = y - z + t$$

$$\dot{y} = x - z + 1$$

$$\dot{z} = 2x + 2y - 3z + t + 1$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$$

$$f(t) = \begin{bmatrix} t \\ 1 \\ -1 \end{bmatrix}$$

$$\det(A - I) =$$

$$\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -\lambda(3\lambda^2 + \lambda^2 + 2) - 1(-3\lambda + 2) - 1(2 + 2\lambda)$$

$$= -\lambda((\lambda+1)(\lambda+2)) + (\lambda+1) - 2(\lambda+1) =$$

$$= (\lambda+1)(\lambda^2 + 2\lambda - 1) = \dots \text{ups} = -(\lambda+1)^3$$

$$A+I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \ker(A+I) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \text{Lin} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\ker(A+I)^2 = \mathbb{R}^{2 \times 3}$$

(ker se dimenzija jedra poveča za vsaj 1)

$$\Rightarrow \text{dopolnilna baza: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

en od teh dveh je lahko prvi stolpec

$$J = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

drugi stolpec

$$(A+I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \ker(A+I)$$

treći stolpec

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{ali pa} \quad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

oba sta vredni

(ker sta oba lin. neod. od $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$)

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$\sim \dots \quad P^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad e^{Jt} = \begin{bmatrix} e^{it} & & \\ & e^{-it} - te^{it} & \\ & & e^{-it} \end{bmatrix}$$

$$\bar{\Phi}(t) = e^{At} = Pe^{Jt}P^{-1}$$

$$\bar{\Phi}^{-1}(s) = P e^{-Js} P^{-1}$$

$$x_p = Pe^{Jt} P^{-1} \int Pe^{-Js} P^{-1} f(s) ds =$$

$$= Pe^{Jt} \int_0^t e^{sJ} \begin{bmatrix} 1 & 1-s & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} ds$$

$$= Pe^{Jt} \int_0^t e^{sJ} \begin{bmatrix} 1 & s-1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ 0 \end{bmatrix} ds =$$

$$= Pe^{Jt} \int_0^t e^{sJ} \begin{bmatrix} s \\ 1 \\ 0 \end{bmatrix} ds = Pe^{Jt} \begin{bmatrix} se^s - e^s \\ e^s \\ 0 \end{bmatrix} =$$

$$\int s e^s ds = se^s - \int e^s ds = se^s - e^s$$

$$u = s \quad du = e^s$$

$$du = 1 \quad u = e^s$$

$$= Pe^{Jt} \left(e^{it} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) = X_p(t)$$

splašnica rezitu:

$$X(t) = \bar{\Phi}(t) x_0 + X_p(t)$$

Lineärne enečbe reda n z konstantnimi koeficienti

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

sistem

$$\begin{aligned}x_1 &= y \\x_2 &= y' \\&\vdots \\x_n &= y^{(n-1)}\end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ -\frac{a_n}{a_0} x_{n-1} - \frac{a_2}{a_0} x_{n-2} - \dots - \frac{a_1}{a_0} x_1 \end{bmatrix}$$
$$= A \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Dobimo sistem oblike $\vec{\dot{v}} = A \vec{v}$

Komponente rešitev so oblike: $\sum p(t) e^{\lambda t}$