<u>One Sided Limits</u>

We know that the deleted neighborhood of a can be written as,

$(a-\delta, a+\delta) - \{a\} = (a-\delta, a) \cup (a, a+\delta)$

- $(a-\delta, a)$ is the left hand side of the deleted neighborhood of a.
- $(a, a+\delta)$ is the right hand side of the deleted neighborhood of a.

Definition of the left hand side limit

$f(x)$ is a real valued function and $a \epsilon R$. We say that the left limit of $f(x)$ as x approaches a is L if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$.

We write this fact as $\lim_{x \to a^-} f(x) = L$

Definition of the right hand side limit

$f(x)$ is a real valued function and $a \epsilon R$. We say that the right limit of $f(x)$ as x approaches a is L if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.

We write this fact as $\lim_{x \to a^+} f(x) = L$

Theorem 9

Let f be a real valued function. Then $\lim_{x \to a} f(x)$ exists & $\lim_{x \to a} f(x) = L$ if and only if both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exists & $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$

Exercise 2

1. $f: R \to R$ is a real valued function defined as

   $f(x) = \begin{Bmatrix} x & ; x \geq 1 \\ -x+1 & ; x < 1 \end{Bmatrix}$ Show that $\lim_{x \to 1} f(x)$ does not exists.

2. $f(x) = \frac{|x|}{x}$; $x \neq 0$ Plot the $y = f(x)$ graph. Show that $\lim_{x \to 0} f(x)$ does not exists using the definition of limits.

3. $f(x) = |x| + |x-1|$; $x \in R$ Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$. Show that $\lim_{x \to 1} f(x)$ exists.

4. Prove the following limits using the definition.

   i. $\lim_{x \to 1^+} \frac{1}{x+1} = \frac{1}{2}$

   ii. $\lim_{x \to -2^-} \frac{2x}{x-2} = 1$

   iii. $\lim_{x \to 3^-} \frac{2x-1}{x-1} = \frac{5}{2}$

   iv. $\lim_{x \to 1^+} x^3 = 1$

   v. $\lim_{x \to 1^+} \frac{x^3}{x^2+1} = \frac{1}{2}$

Limits As x tends to infinity

In here we define a definition of the existence of a finite limit of a real valued function when x tends to positive or negative infinity.

Definition

We say that the limit of $f(x)$ as x tends to ∞ is L if $\forall \varepsilon > 0, \exists a > 0$ such that $x > a \Rightarrow |f(x) - L| < \varepsilon$

When this happen we write $\lim_{x \to \infty} f(x) = L$

We will define the limit as x tends to $-\infty$ in similar manner.

Definition

We say that the limit of $f(x)$ as x tends to $-\infty$ is L if $\forall \varepsilon > 0, \exists b < 0$ such that $x < b \Rightarrow |f(x) - L| < \varepsilon$

When this happen we write $\lim_{x \to -\infty} f(x) = L$

Infinity as a limit

There are some functions which tend to a infinity when x tends to a finite value.

Definition

We say that $f(x)$ tends to ∞ as x approaches a if $\forall M > 0, \exists \delta > 0$, such that $0 < |x - a| < \delta \Rightarrow f(x) > M$. We can write this fact as $\lim_{x \to a} f(x) = \infty$

Definition

We say that $f(x)$ tends to $-\infty$ as x approaches a if $\forall N < 0, \exists \delta > 0$, such that $0 < |x-a| < \delta \Rightarrow f(x) < N$

We can write this fact as $\lim_{x \to a} f(x) = -\infty$

Infinity as a Limit When x Tends To Infinity

Definition

We say that $f(x)$ tends to ∞ as x tends to ∞ if $\forall M > 0, \exists a > 0$, such that $x > a \Rightarrow f(x) > M$

We can write this fact as $\lim_{x \to \infty} f(x) = \infty$

Definition

We say that $f(x)$ tends to ∞ as x tends to $-\infty$ if $\forall M > 0, \exists b < 0$, such that $x < b \Rightarrow f(x) > M$

We can write this fact as $\lim_{x \to -\infty} f(x) = \infty$

Definition

We say that $f(x)$ tends to $-\infty$ as x tends to ∞ if $\forall N < 0, \exists a > 0$, such that $x > a \Rightarrow f(x) < N$

We can write this fact as $\lim_{x \to \infty} f(x) = -\infty$

Definition

We say that $f(x)$ tends to $-\infty$ as x tends to $-\infty$ if $\forall N < 0, \exists b < 0$, such that $x < b \Rightarrow f(x) < N$

We can write this fact as $\lim_{x \to -\infty} f(x) = -\infty$

Exercise 3

- Try to solve following limits using definition

1. $\lim_{x\to 2} \frac{3}{(x-2)^2} = \infty$
2. $\lim_{x\to -3} \frac{1}{(x+3)^2} = \infty$
3. $\lim_{x\to \infty} \frac{x+3}{x-2} = 1$
4. $\lim_{x\to \infty} \frac{2x+5}{x+9} = 2$
5. $\lim_{x\to \infty} \frac{7}{x-2} = 0$
6. $\lim_{x\to \infty} 4x + 3 = \infty$
7. $\lim_{x\to -\infty} -3x + 1 = \infty$
8. $\lim_{x\to \infty} -5x = -\infty$
9. $\lim_{x\to -\infty} x + 1 = -\infty$
10. $\lim_{x\to \infty} x^2 + 4x + 1 = \infty$
11. $\lim_{x\to 2^+} x^2 + 4x + 1 = 13$
12. $\lim_{x\to 1^-} 2x^2 + 3 = 5$

-----End of the Tutorial-----

Dasun Madushan
B.Sc. Eng. (Hons) – 1st Class
Electronic & Telecommunication Engineering
University of Moratuwa