Definition of Continuity of a function at a point

The function $f(x)$ is continuous at $x = a$ if and only if we can guarantee $f(x)$ to be close to the value $f(a)$ by restricting x to be close to a. To rephrase, we say f(x) is continuous at $x = a$ if, given any positive tolerance $\varepsilon > 0$ we choose for $f(x)$ as an approximation for $f(a)$, we can then find a positive tolerance $\delta > 0$ for x as an approximation for a so that δ −tolerance in x allows at most ε tolerance in $f(x)$. The definition below is very technical, but through reflection and exposure to examples, one eventually sees that this is exactly what is required.

The function $f(x)$ is continuous at the point $x = a$ if and only if

$$(\forall\epsilon > 0)\ (\exists\delta > 0)\ (\forall x)\ s.t. |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

➢ <u>*Continuity Theorems*</u>

- Suppose that $f(x)$ and $g(x)$ are continuous at $x = a$.
  Then so is $f(x) + g(x)$.
- If $f(x)$ is continuous at $x = a$, then so is $cf(x)$ for any constant c.
- Suppose that $f(x)$ and $g(x)$ are continuous at $x = a$.
  Then so is $f(x)g(x)$.
- Polynomial functions are continuous at every $a \in R$.
- $(g(x)$ continuous at $x = a) \wedge (g(a) \neq 0) \Rightarrow \frac{1}{g(x)}$ is continuous at $x = a$.
- If $f(x)$ and $g(x)$ are continuous at $x = a$, and $g(a) \neq 0$, then $f(x)/g(x)$ is also continuous at $x = a$.
- If $f(x) = \frac{p(x)}{q(x)}$; where p and q are polynomials, then $f(x)$ is continuous at every $a \in R$ except where $q(a) = 0$, at which points $f(x)$ is undefined and therefore discontinuous.

➢ <u>*One − Sided Continuity*</u>

Definition

We call $f(x)$ left-continuous at $x = a$ if and only if
$(\forall\epsilon > 0)\ (\exists\delta > 0)\ (\forall x)\ s.t.\ x \in (a - \delta, a] \Rightarrow |f(x) - f(a)| < \epsilon$

We call $f(x)$ right-continuous at $x = a$ if and only if
$(\forall\epsilon > 0)\ (\exists\delta > 0)\ (\forall x)\ s.t.\ x \in [a, a + \delta) \Rightarrow |f(x) - f(a)| < \epsilon$

- Try to solve the following sums.

1. Prove that $f(x) = 3x + 5$ is continuous at $x = 4$
2. $f(x) = x^2 + 5x + 8 \quad ; x \geq 1$
   $\quad\;\; = 13x + 1 \quad\quad ; x < 1$
   Prove that $f(x)$ is continuous at $x = 1$
3. Prove that $f(x) = e^x$ is continuous at $x = 0$
4. Prove that $f(x) = \frac{x+1}{x-1}$ is not continuous at $x = 1$
5. Prove that $f(x) = |x|$ is continuous at $x = 0$

From above definition then we can go to define the continuity of a function on a interval. So if $f; A \rightarrow R$, f is a function then we can say $f(x)$ is continuous on A iff

$$(\forall a \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)\ s.t. |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

- Prove that $f(x) = x^2$ is continuous on R.
- Prove that $f(x) = \frac{x}{2x+3}$ is continuous on [1,3].
- Prove that $f(x) = x^3 + 3x$ is continuous on [0,1].

➢ <u>*Continuity Vs Uniformly Continuity*</u>

Mainly there are 2 types of continuity.
If $f(x)$; $f: A \rightarrow R$ be a function

- We say that f is continuous on A if and only if
  $(\forall a \in A)(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in A)\ s.t.\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

- Hence f is not continuous on A if and only if
  $(\exists a \in A)(\exists \varepsilon > 0)\ (\forall \delta > 0)\ (\exists x \in A)\ s.t. |x - a| < \delta\ and\ |f(x) - f(a)| \geq \varepsilon$

- We say that f is uniformly continuous on A if and only if
  $(\forall \varepsilon > 0)\ (\exists \delta > 0)(\forall a \in A)\ (\forall x \in A)\ s.t. |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

- Hence f is not uniformly continuous on A if and only if
  $(\exists \varepsilon > 0)\ (\forall \delta > 0)(\exists a \in A)\ (\exists x \in A)\ s.t.\ |x - a| < \delta\ and\ |f(x) - f(a)| \geq \varepsilon$

The main difference of this two definitions is the defined δ. In continuous definition it says δ depends on both a, ε.
But the other definition says δ depends on only the ε. Not on a.

So,
➢ If $f(x)$ is uniformly continuous then it is continuous.
➢ But we can't say if $f(x)$ is continuous then it is uniformly continuous.
➢ But if $f(x)$ is continuous on closed bounded interval such that $[a, b]$ then $f(x)$ will be uniformly continuous on that interval.

➢ *Lipschitz* <u>*Continuity*</u>

Definition

A function $f : A \to R$ is Lipschitz continuous on A, if

$$\exists\, M > 0, \forall\, x, y \in A\ s.t. |f(x) - f(y)| < M|x - y|$$

- If $f(x)$ is lipschitz continuous on A which is a sub set of R, then $f(x)$ is uniformly continuous on A.
- But there are some function which are uniformly continuous but not have Lipchitz continuous on some specific intervals.

1. Prove that $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$.
2. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[1,2]$.
3. Prove that if a function is Lipchitz continuous on an interval then it is uniformly continuous.
   Using $|\sin x| < |x|$; $\forall x \in R$, Prove that $f(x) = \sin x$ is continuous at any $a \in R$.

Intermediate Value Theorem

Suppose f is continuous on $[a.b]$ and $f(a) < u < f(b)$ or $f(b) < u < f(a)$ then,
There exists $c \in (a, b)$ such that $f(c) = u$.

- ❖ If f is continuous on $[a, b]$ then f is bounded on $[a, b]$.
- ❖ If f is continuous on $[a, b]$ then f has a maximum and a minimum on $[a, b]$.

Differentiability

Let $f(x)$ be a real valued function defined on A.
Then we say that f is differentiable at a if and only if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \in R$
We call this limit as the derivative of the function at a. and we write it as $f'(a)$.

$f'_+(a) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ is called the right derivative of the function at a.
$f'_-(a)\ \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$ is called the left derivative of the function at a.

If the function is differentiable at $x = a$ if and only if $f'_+(a) = f'_-(a) = f'(a) \in R$.

Theorem

If a function differentiable at $x = a$, then it is continuous at $x = a$.

Definition

If f has a local maximum at a if and only if

$$\exists\delta > 0 , \forall x \; |x - a| < \delta \Rightarrow f(x) \geq f(a)$$

If f has a local minimum at a if and only if

$$\exists\delta > 0 , \forall x \; |x - a| < \delta \Rightarrow f(x) \leq f(a)$$

Commonly maximum or minimum is named as an extreme point.

Roller's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$ then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(a)-f(b)}{a-b}$

Problems

1. Using intermediate value theorem prove that $x + e^x = 0$ has a root between $[-1,0]$.
2. Prove that $f(x) = |x|$ is not differentiable at $x = 0$.
3. Using mean value theorem show that $|\sin x - \sin y| \leq |x - y|$; $\forall x, y \in R$

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