Sequence is a function which is defined on the set of positive integers.
$f: Z^+ \to R$, $f(n) = a_n$
So the set, $\{a_1 , a_2 , a_3 , \dots\}$ is called a sequence.

- <u>Definition 1</u>

Let $\{ a_n\}$ be a sequence. A number L is said to be the limit of $\{a_n\}$ if $\forall\ \varepsilon > 0$, $\exists\ n_0 \in Z_+$ Such that $n > n_0 \Rightarrow |a_n - L| < \varepsilon$. When this happens, we write $\lim\limits_{n\to\infty} a_n = L$

- <u>Definition 2</u>

A sequence $\{a_n\}$ said to be convergent if $\exists\ L \in R$ such that $\lim\limits_{n\to\infty} a_n = L$. When a sequence is not convergent, it is said to be divergent.

- <u>Definition 3</u>

Let $\{a_n\}$ be a sequence,

i) If $a_n \le a_{n+1}$: $\forall n$, then the sequence is said to be increasing.
ii) If $a_n \ge a_{n+1}$: $\forall n$, then the sequence is said to be decreasing.
iii) If a sequence is either increasing or decreasing, then it is said to be a monotonic sequence.

- <u>Definition 4</u>

Let $\{a_n\}$ be a sequence,

i) If $\exists A \in R$ such that $a_n \le A\ \ \forall n$, then $\{a_n\}$ is said to be bounded above.
ii) If $\exists B \in R$ such that $a_n \ge B\ \ \forall n$, then $\{a_n\}$ is said to be bounded below.
iii) If there are $A, B \in R$ such that $B \le a_n \le A\ \ \forall n$, then $\{a_n\}$ is said to be bounded.

➢ <u>Theorem 1</u>

A convergent sequence has a unique limit.

➢ <u>Theorem 2</u>

Let $\{a_n\}$, $\{b_n\}$ be two sequences such that $\lim\limits_{n\to\infty} a_n = L$ & $\lim\limits_{n\to\infty} b_n = M$ then,

i) $\lim\limits_{n\to\infty} a_n + b_n = L + M$
ii) $\lim\limits_{n\to\infty} ra_n = rL \quad \forall r \in R$
iii) $\lim\limits_{n\to\infty} a_n b_n = LM$
iv) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$; $when\ M \neq 0$

➢ <u>Theorem 3</u>

Every convergent sequence is bounded.

➢ <u>Theorem 4</u>

If $\{a_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} a_n = L$ where $L = sup\{a_n |\, n \in Z^+\}$

➢ <u>Theorem 5</u>

If $\{a_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} a_n = M$ where $M = inf\{a_n |\, n \in Z^+\}$

❖ Cauchy Sequence

- <u>Definition 5</u>

A sequence $\{a_n\}$ is said to be a Cauchy sequence if and only if $\forall \varepsilon > 0\,, \exists n_0 \in Z_+$ such that $m, n > n_0 \Rightarrow |a_m - a_n| < \varepsilon$

➢ <u>Theorem 6</u>

Every Cauchy sequence is bounded.

➢ <u>Theorem 7</u>

Let $\{a_n\}$ be a sequence. Then $\{a_n\}$ is convergent if and only if $\{a_n\}$ is a Cauchy sequence.

- **Archimedean Property**

  $\forall a \in R\,, \exists n \in Z^+$ s.t. $n > a$

Exercise

1) Prove that following sequences are convergent using definition
   i) $a_n = \frac{1}{n}$
   ii) $a_n = \frac{n}{n+1}$
   iii) $a_n = \frac{n^2}{n^2+1}$
   iv) $a_n = \frac{n+4}{n+3}$

2) Prove that $\{(-1)^n\}$ is not convergent.
3) Prove that $\{n\}$ is not convergent.
4) A sequence $\{a_n\}$ defined by the recurrence relation $a_{n+1} = \frac{(a_n)^2+3}{4}$ and the initial term $a_1 = 0$. Show that this sequence is
   i) Bounded above by 1
   ii) Increasing
   iii) Convergent

   And what is the limit of this sequence?

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