Engineering Mathematics

Real Analysis **Functions** **Part 4**

1. **Neighborhood**

Definition 1 – Let $a \in \mathbb{R}$. A neighborhood of a is an open interval $(a-\delta, a+\delta)$ for any $\delta > 0$.

Definition 2 – A deleted neighborhood of a is $(a-\delta, a+\delta) - \{a\}$.

2. **Limit of a Function**

Let $f(x)$ be a real valued function defined on $\mathbb{R}$. If the value of $f(x)$ gets closer to the value L, as x gets closer and closer to a, then we say that the limit of $f(x)$ as x approaches a is L.

$$\lim_{x\to a} f(x) = L$$

In mathematics there is a formal definition to the existence of a limit of a function.

Definition of limit

Let $f(x)$ be a real valued function defined on an open interval containing a possibly (but not necessarily) excluding a itself. We say that the limit of $f(x)$ as x approaches a is L if $\forall \varepsilon > 0, \exists \delta > 0$ such that $|f(x) - L| < \varepsilon \; \forall x \in (a-\delta, a+\delta) - \{a\}$.

Note that an equivalent statement to this would be
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$

$$\boxed{\lim_{x\to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon}$$

There are some theorems related to limits.

Theorem 1

For any $a \in \mathbb{R}$, $\lim_{x\to a} x^n = a^n$ where n is a positive integer.

Theorem 2

Let $\lim_{x\to a} f(x) = L$ and $r \in \mathbb{R}$, then $\lim_{x\to a} rf(x) = rL$

Theorem 3

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a} f(x) + g(x) = L + M$

Theorem 4

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a} f(x).g(x) = LM$

Theorem 5

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

Theorem 6

Let $\lim_{x\to a} f(x) = L$ and $L \neq 0$, then $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}$

Theorem 7

Let $\lim_{x\to a} f_1(x) = L_1, \lim_{x\to a} f_2(x) = L_2, \dots \dots, \lim_{x\to a} f_n(x) = L_n$ then

$\lim_{x\to a} f_1(x) + f_2(x) + \cdots + f_n(x) = L_1 + L_2 + \cdots + L_n$

Theorem 8

Let $f(x)$ be a polynomial, then $\lim_{x\to a} f(x) = f(a)$

Exercise

Prove by definition.

1. $\lim_{x\to 1} 5 = 5$
2. $\lim_{x\to 1} 3x + 1 = 4$
3. $\lim_{x\to 2} 7x - 2 = 12$
4. $\lim_{x\to 1} x^2 + x + 1 = 3$
5. $\lim_{x\to -1} 2x^2 - x - 2 = 1$
6. $\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$
7. $\lim_{x\to 2} \frac{1}{3x-1} = \frac{1}{5}$
8. $\lim_{x\to 1} \frac{x}{x+3} = \frac{1}{4}$
9. $\lim_{x\to 2} \frac{2x-1}{x+1} = 1$
10. $\lim_{x\to 1} \frac{1}{2x-1} = 1$
11. $\lim_{x\to 2} x^3 = 8$
12. $\lim_{x\to 1} \frac{2x+1}{x^2+2x+2} = \frac{3}{5}$
13. $\lim_{x\to -1} \frac{3x+4}{x^2+x+1} = 1$

-----End of the Tutorial-----