Engineering Mathematics

Real Analysis **Functions** **Part 3**

1. Ordered Pairs

A set with two elements which are ordered is called an ordered pair. An ordered pair is denoted by (a, b) where a is called the first element and b is the second.

Definition - If (a, b) is an ordered pair then it can be defined as $(a, b) = \{\{a\}, \{a, b\}\}$.
According to the above definition we can prove that if $(a, b) = (c, d)$ then $a = b$ & $c = d$.

Definition - Given two sets A & B, the set of all ordered pairs (a, b) such that $a \in A$ & $b \in B$ is called the Cartesian product of A & B.
It is denoted by $A \times B$

2. Relations

Definition - Any set of ordered pairs is called a relation.

If A & B are non-empty sets then a relation $S: A \rightarrow B$ is a non-empty subset of $A \times B$.
A is called the *domain* of S. B is called the *codomain* of S.
$\{y \mid (x, y) \in S\}$ is called the *range* of S or $ranS$.
$\{x \mid (x, y) \in S\}$ is called the $pre - range$ of S or $preranS$.

❖ One Many Relations

If S is one many relation $\Leftrightarrow \exists x \in A,\ \exists y_1, y_2 \in B\ \ s.t.\ (x, y_1)\ and\ (x, y_2) \in S\ \&\ y_1 \neq y_2$

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❖ Many Many Relations

If S is both one-many and many-one relation then S is called many many relation.

❖ One One Relation (Injection)

If S is not one-many and not many-one relation then S is called one one relation.

- **Onto Relation (Surjection)**

  If $S: A \to B$ relation in onto $\Longleftrightarrow ranS = B$

  If S is onto $\Leftrightarrow \forall y \in B,\ \exists x \in A\ \ s.t.\ \ (x, y) \in S$

  - ✓ If a relation is both one to one & onto then it is called **bijection.**

- **Inverse Relation**

  Definition - $S^{-1}: B \to A$ defined by $S^{-1} = \{(y, x) \mid (x, y) \in S\}$ is called the inverse relation of S.

3. Functions

Definition – Let A & B are non-empty sets. Then a function $f: A \to B$ is a relation which is not one many and $A \equiv prerange\ of\ f$

Now it can be written $(x, y) \in f$ as $f(x) = y$

- ✓ If function f is not many one also then it is called a $\boldsymbol{one\ one\ function}$ or $\boldsymbol{one\ to\ one\ function}$.
- ✓ If the range of f, $\boldsymbol{ranf = B}$ then the function is called an $\boldsymbol{onto\ function}$.
- ✓ If a function is both $\boldsymbol{one\ to\ one}$ and $\boldsymbol{onto}$ then it is called a $\boldsymbol{bijection\ function}$.

- **Inverse Function**

  The inverse function f^{-1} is the inverse relation $f^{-1}: B \to A$ which is also a relation.

  - ✓ For this f should be bijection.

> ➢ If A & B are subsets of $\mathbb{R}$, then $f: A \to B$ is called a real valued function.

- **Composite Functions**

  Let $f: A \to B$ & $g: B \to C$ are both functions then $g\ o\ f: A \to C$ is the composite function. $(g\ o\ f)(x) = g[f(x)]$

1. <u>Increasing Function</u>

   If $x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$, then $f(x)$ is an increasing function.

2. <u>Strictly Increasing Function</u>

   If $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$, then $f(x)$ is a strictly increasing function.

3. <u>Decreasing Function</u>

   If $x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$, then $f(x)$ is a decreasing function.

4. <u>Strictly Decreasing Function</u>

   If $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$, then $f(x)$ is a strictly decreasing function.

❖ **Function Notations**

1. $(f+g)(x) = f(x) + g(x)$
2. $(rf)(x) = r.f(x)$
3. $(fg)(x) = f(x).g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

❖ **Function Types**

1. Polynomial Functions
2. Rational Functions
3. Exponential Functions
4. Logarithm Functions
5. Trigonometric Functions (Circular Functions)
6. Inverse Trigonometric Function
7. Hyperbolic Functions

Exercise

1. If $S = \{(1,2)\,, (4,3)\,, (2,5)\,, (3,2)\,, (5,7)\}$. Show that S is many one relation.
2. $f: \mathbb{R} \to \mathbb{R}\ , f(x) = 3x - 2$ be a function then,
   i. Prove that f is a bijection.
   ii. Prove that f is strictly increasing.
   iii. Find the formula for f^{-1}
3. If $f: \mathbb{R} \to \mathbb{R}\,, f(x) = x^2$ then,

i. Prove that $f(x)$ is not a one to one function.

ii. Prove that $f(x)$ is not an onto function.

4. Let $f: A \to B$ be a function. Prove that

   f is a bijection $\Leftrightarrow f^{-1}: B \to A$ exists as a bijection.

5. Show that if $f: A \to B$ is a bijection then,

   i. $(f^{-1}\ o\ f)(x) = x\ ;\ \forall x \in A$

   ii. $(f\ o\ f^{-1})(x) = x\ ;\ \forall x \in A$

6. If $f: \mathbb{R}_0^+ \to \mathbb{R}\, , f(x) = x^2 - 2x + 2$ then prove that f^{-1} function is not exists.
7. If $f: \mathbb{R}_0^+ \to \mathbb{R}_0^+\, , f(x) = |x - 2|$ then prove that f is not a one to one function.
8. If $f: (-1, \infty) \to \mathbb{R}\, , f(x) = \frac{x}{x+1}$ be a function, then prove that

   i. f is one to one

   ii. f is strictly increasing

   iii. f is not an onto function.

9. Let $f: \mathbb{R} \to \mathbb{R}\, , f(xy) = f(x).f(y)\ ;\ \forall x, y \in \mathbb{R}$ and then prove that exactly one of the following facts occurs.

   i. $f(x) = 0\ ;\ \forall x \in \mathbb{R}$

   ii. $f(1) = 1$

10. Let $f: \mathbb{R} \to \mathbb{R}\, , f(x + y) = f(x) + f(y)\ ;\ \forall x, y \in \mathbb{R}$ and then prove the followings.

    i. $f(0) = 0$

    ii. $f(mx) = m.f(x)\ ;\ \forall m \in \mathbb{Z}\, , x \in \mathbb{R}$

    iii. $f(rx) = r.f(x)\ ;\ \forall r \in \mathbb{Q}\, , x \in \mathbb{R}$

-----*End of the Tutorial*-----

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