Let $\{a_n\}$ be a sequence then the sum of the terms of the sequence is called a series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

If the number of terms in a series is finite the series is called a finite series. If the number of terms in a series is infinite the series is called an infinite series.

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Convergent, Divergent and Oscillatory Series

Let $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be a series and S_n be the sum of the first n terms of the series.

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$, Then $\{S_n\}$ will be a sequence.

- If S_n tends to a finite number when $n \to \infty$, then the series is said to be a convergent series.
- If S_n tends to infinity when $n \to \infty$, then the series is said to be a divergent series.
- If S_n does not tends to a unique limit when $n \to \infty$, then the series is said to be an oscillatory series.

Positive Term Series

If all terms after few negative terms in an infinite series are positive, such a series is a positive term series.

Definition

Let $\sum a_n$ be a series, If $\sum |a_n|$ is convergent, we say that $\sum a_n$ is absolutely convergent.

- If $\sum a_n$ is absolutely convergent then it is convergent.

Definition

If $\sum a_n$ is a convergent series, but not absolutely convergent, then we say that it is conditionally convergent.

1. **Alternating Series Test**

   Let $\{a_n\}$ be a decreasing sequence of positive terms such that $\lim_{n\to\infty} a_n = 0$ and if $b_n = (-1)^n a_n$ or $b_n = (-1)^{n+1} a_n$ then,

   $\sum b_n$ is convergent.

Theorem

Let $\sum a_n$ be a series, $\sum a_n$ is convergent $\Longrightarrow \lim_{n\to\infty} a_n = 0$

2. **Cauchy's Fundamental Test for Divergence**

   $\lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n$ is divergent

3. **Integral Test**

   Suppose $f(x)$ is a positive continuous and decreasing function defined on $[1,\infty)$, then

   $\sum_{n=1}^{\infty} f(n)$ is convergent $\Leftrightarrow \int_1^{\infty} f(x)\,dx < \infty$

<u>**p-Series**</u>

$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \infty$ is called the $p-$ series.

- When $p > 1$, the series is convergent.
- When $p \leq 1$, the series is divergent.

4. **Comparison Test**

   I. **Direct Comparison Test**

   Let two positive term series, $\sum u_n$ and $\sum v_n$,

   If $\exists n_0 \in \mathbb{Z}^+ \ s.t.\ 0 \leq u_n \leq v_n$; $\forall n > n_0$, then

   - $\sum v_n$ is convergent $\Longrightarrow \sum u_n$ is convergent
   - $\sum u_n$ is divergent $\Longrightarrow \sum v_n$ is divergent

II. **Limit Comparison Test**

Let two positive term series, $\sum u_n$ and $\sum v_n$,

If $\lim_{n\to\infty} \frac{u_n}{v_n} = \boldsymbol{finite\ number}$, then both series converge or diverge.

5. Ratio Test

If $\sum u_n$ is a positive term series such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$ then,

- If $k < 1$, series is convergent
- If $k > 1$, series is divergent
- If $k = 1$, test fails

Exercise

1. Prove that the geometric series $1 + r + r^2 + r^3 + \cdots \infty$ is
   a. Convergent if $|r| < 1$
   b. Divergent if $r \geq 1$
   c. Oscillatory if $r \leq -1$
2. Prove that $\sum \frac{(-1)^n}{n}$ is convergent.
3. Prove that $\sum (-1)^n \left(\frac{n}{n+2}\right)$ is not convergent.
4. Prove that $\sum \frac{1}{n}$ is divergent.
5. Prove that $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots \infty$ is convergent.
6. Prove that $\sum \frac{1}{n(n+1)}$ is convergent.
7. Prove that $\sum \frac{n}{n+1}$ is divergent.
8. Prove that $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Exercise

Examine the convergence or divergence of the following series.

1. $2 + \frac{3.1}{2.4} + \frac{4}{3.4^2} + \frac{5}{4.4^3} + \cdots$
2. $1 + \frac{1.2}{1.3} + \frac{1.2.3}{1.3.5} + \frac{1.2.3.4}{1.3.5.7} + \cdots$
3. $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$
4. $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \cdots$
5. $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots$
6. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$

7. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\cdots$

8. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots$

9. $\frac{1}{3}+\frac{2!}{3^2}+\frac{3!}{3^3}+\cdots$

10. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$

11. $\sum_{n=1}^{\infty}\frac{2n^3+5}{4n^5+1}$

12. $\sum_{n=1}^{\infty}\frac{a^n}{x^n+n^a}$

13. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

14. $\sum_{n=1}^{\infty}\frac{n^2}{e^n}$

15. $\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$

16. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

17. $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n.3^n}$

18. $\sum_{n=1}^{\infty} ne^{-n^2}$

19. $\sum_{n=1}^{\infty}\frac{\ln n}{n}$

20. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n+1}{n}\right)^n$

21. $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n^2+2n+3}$

22. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\left(\frac{\pi}{n}\right)}{n^2}$

23. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

24. $\sum_{n=1}^{\infty}\frac{n^3}{2^n}$

-----*End of the Tutorial*-----

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