Engineering Mathematics

Real Analysis **Part 2**

__Some results related to absolute value and inequalities__

1. $\forall x \in \mathbb{R}$; $-|x| \leq x \leq |x|$
2. For any $a > 0$; $|x| \leq a \Longrightarrow -a \leq x \leq a$
3. $\forall x, y \in \mathbb{R}$; $|x + y| \leq |x| + |y|$ - Triangle inequality
4. $\forall x, y \in \mathbb{R}$; $|x - y| \geq |x| - |y|$ - Reverse triangle inequality

Exercise 2

1. Prove that $(1,2)$ is bounded.
2. Prove that $(-\infty, 4)$ is bounded above.
3. Prove that $[1, \infty)$ is not bounded above using the contradiction method.
4. Prove that $A = [-1,3]$ is bounded above and $Sup(A) = 3$
5. There exists two sets A and B defined on $\mathbb{R}$ and bounded. If another set C is defined as $C = \{x + y \mid x \in A, y \in B\}$ then prove that,
   i. $Sup(C) = Sup(A) + Sup(B)$
   ii. $Inf(C) = Inf(A) + Inf(B)$

6. Prove that if A be a set of real numbers which is bounded,
   i. Let $p > 0$, $Sup(pA) = p.Sup(A)$, where $pA = \{px \mid x \in A\}$
   ii. Let $p < 0$, $Sup(pA) = p.Inf(A)$, where $pA = \{px \mid x \in A\}$

7. $f: (0,2] \to \mathbb{R}$, $f(x) = x^2$ Prove that $f(x)$ is bounded & find the $Sup[f(x)]$.

8. Let A be a non-empty set of real numbers and let f, g are functions defined on A which are bounded. If $f(x) \leq g(x)$; $\forall x \in A$
   Prove that $Sup[f(A)] \leq Sup[g(A)]$

9. Let A be a non-empty set of real numbers and let f, g are functions defined on A which are bounded. If $f(x) \leq g(y)$; $\forall x, y \in A$
   Prove that $Sup[f(A)] \leq Inf[g(A)]$

10. Let A be a non-empty set of real numbers and let f, g are functions defined on A which are bounded. Prove that

    $Sup\{f(x) + g(x) \mid x \in A\} \leq Sup\{f(A)\} + Sup\{g(A)\}$

    $Sup\{f(x) + g(x) \mid x \in A\} \leq Sup\{f(A)\} + Sup\{g(A)\}$

11. Let A be a non-empty set of real numbers and let f, g are functions defined on A which are bounded. Prove that

    $Sup\{f(x) + g(y) \mid x, y \in A\} = Sup\{f(A)\} + Sup\{g(A)\}$

    $Sup\{f(x) - g(y) \mid x, y \in A\} = Sup\{f(A)\} - Inf\{g(A)\}$

12. Prove that if A is bounded set

    i. $\forall \epsilon > 0 , \exists a \in A$ such that $Sup(A) - \epsilon \leq a$

    ii. $\forall \epsilon > 0 , \exists b \in A$ such that $Inf(A) + \epsilon \geq b$

13. Prove that $\forall \varepsilon > 0 , \exists n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ (Archimedean Property)
14. Show that the set of positive integers is not bounded above.
15. If $A = \left\{\frac{1}{2} , \frac{2}{3} , \frac{3}{4} , \frac{4}{5} , \dots \dots \dots\right\}$ Prove that $Sup(A) = 1$
16. Prove that there is a rational between any two real numbers.
17. Find the $Sup(A)$ if $A = \{x \mid 3x^2 - 17x + 10 < 0 , x \in \mathbb{R}\}$
18. If $A = \left\{\frac{1}{n} ; n \in \mathbb{Z}^+\right\}$ then show that A is bounded. Find the $Sup(A)$ and $Inf(A)$.
19. If $A = \{x \mid x^2 < 2 , x \in \mathbb{Q}\}$ then show that A is bounded above and find the $Sup(A)$
20. If $A = \{e^{-x} \mid x \in [0, \infty)\}$ Find $Sup(A)$
21. If A is a set with some real numbers which are greater than 1. $B = \left\{\frac{1}{x} \mid x \in A\right\}$

    Prove that $Sup(B) = \frac{1}{Inf(A)}$

22. If $A = \left\{n + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$ Prove that A is not bounded above but bounded below and $Inf(A) = 0$

-----End of the Tutorial-----

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