3

Differential Equations

3.1 DEFINITION

An equation which involves differential co-efficient is called a differential equation.

For example,

1. $\dfrac{dy}{dx} = \dfrac{1+x^2}{1-y^2}$ **2.** $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - 8y = 0$ **3.** $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = k\dfrac{d^2y}{dx^2}$

4. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu,$ **5.** $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial z}{\partial y}$

There are two types of differential equations :

(1) Ordinary Differential Equation

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

(2) Partial Differential Equation

A differential equation involving partial derivatives with respect to more than one independent variable is called a partial differential equation.

3.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

The *order* of a differential equation is the order of the highest differential co-efficient present in the equation. Consider

1. $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{c} = E\sin wt.$ **2.** $\cos x\dfrac{d^2y}{dx^2} + \sin x\left(\dfrac{dy}{dx}\right)^2 + 8y = \tan x$

3. $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 = \left(\dfrac{d^2y}{dx^2}\right)^2$

The order of the above equations is 2.

The degree of a differential equation is the degree of the highest derivative after removing the radical sign and fraction.

The *degree* of the equation (1) and (2) is 1. The degree of the equation (3) is 2.

3.3 FORMATION OF DIFFERENTIAL EQUATIONS

The differential equations can be formed by differentiating the ordinary equation and eliminating the arbitrary constants.

Example 1. *Form the differential equation by eliminating arbitrary constants, in the following cases and also write down the order of the differential equations obtained.*

(a) $y = Ax + A^2$ (b) $y = A\cos x + B\sin x$ (c) $y^2 = Ax^2 + Bx + C.$

(*R.G.P.V. Bhopal, June 2008*)

Solution. (*a*) $y = Ax + A^2$ ... (1)

On differentiation $\dfrac{dy}{dx} = A$

Putting the value of A in (1), we get $y = x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$ **Ans.**

On eliminating one constant A we get the differential equation of order 1.

(*b*) $y = A\cos x + B\sin x$

On differentiation $\dfrac{dy}{dx} = -A\sin x + B\cos x$

Again differentiating

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x \Rightarrow \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0 \quad \textbf{Ans.}$$

This is differential equation of order 2 obtained by eliminating two constants A and B.

(*c*) $y^2 = Ax^2 + Bx + C$

On differentiation $2y\dfrac{dy}{dx} = 2Ax + B$

Again differentiating $2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 2A$

On differentiating again $y\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx}\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = 0 \Rightarrow y\dfrac{d^3y}{dx^3} + 3\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = 0$ **Ans.**

This is the differential equation of order 3, obtained by eliminating three constants *A, B, C.*

EXERCISE 3.1

1. Write the order and the degree of the following differential equations.

(*i*) $\dfrac{d^2y}{dx^2} + a^2x = 0;$ (*ii*) $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \dfrac{d^2y}{dx^2};$ (*iii*) $x^2\left(\dfrac{d^2y}{dx^2}\right)^3 + y\left(\dfrac{dy}{dx}\right)^4 + y^4 = 0.$

Ans. (*i*) 2,1 (*ii*) 2,2 (*iii*) 2,3

2. Give an example of each of the following type of differential equations.

(*i*) A linear-differential equation of second order and first degree **Ans.** *Q,* 1 (*i*)

(*ii*) A non-linear differential equation of second order and second degree **Ans.** *Q*, 1 (*ii*)

(*iii*) Second order and third degree. **Ans.** *Q* 1 (*iii*)

3. Obtain the differential equation of which $y^2 = 4a(x + a)$ is a solution.

Ans. $y^2\left(\dfrac{dy}{dx}\right)^2 + 2xy\dfrac{dy}{dx} - y^2 = 0$

4. Obtain the differential equation associated with the primitive $Ax^2 + By^2 = 1$.

Ans. $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

5. Find the differential equation corresponding to

$y = a\,e^{3x} + be^x.$ **Ans.** $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 0$

6. By the elimination of constants A and B, find the differential equation of which

$y = e^x (A \cos x + B \sin x)$ is a solution. **Ans.** $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

7. Find the differential equation whose solution is $y = a \cos (x = 3)$. (*A.M.I.E., Summer 2000*)

Ans. $\frac{dy}{dx} = -\tan(x+3)$

8. Show that set of function $\left\{x, \frac{1}{x}\right\}$ forms a basis of the differential equation $x^2y'' + xy' - y = 0$.

Obtain a particular solution when $y(1) = 1$, $y(1) = 2$. **Ans.** $y = \frac{3x}{2} - \frac{1}{2x}$

3.4 SOLUTION OF A DIFFERENTIAL EQUATION

In the example 1(*b*), $y = A \cos x + B \sin x$, on eliminating A and B we get the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

$y = A \cos x + B \sin x$ is called the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

The order of the differential equation $\frac{d^2y}{dx^2} + y = 0$ is two and the solution $y = A \cos x + B \sin x$ contains two arbitrary constants. The number of arbitrary constants in the solution is equal to the order of the differential equation.

An equation containing dependent variable (y) and independent variable (x) and free from derivative, which satisfies the differential equation, is called the solution (primative) of the differential equation.

3.5 DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

We will discuss the standard methods of solving the differential equations of the following types:

(*i*) Equations solvable by separation of the variables. (*ii*) Homogeneous equations.
(*iii*) Linear equations of the first order. (*iv*) Exact differential equations.

3.6 VARIABLES SEPARABLE

If a differential equation can be written in the form

$$f(y)\,dy = \phi(x)\,dx$$

We say that variables are separable, y on left hand side and x on right hand side.
We get the solution by integrating both sides.

Working Rule:

Step 1. Separate the variables as $f(y)\,dy = \phi(x)\,dx$

Step 2. Integrate both sides as $\int f(y)\,dy = \int \phi(x)\,dx$

Step 3. Add an arbitrary constant C on R.H.S.

Example 2. *Solve* : $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ (*UP, II 2008, U.P.B. Pharm (C.O.) 2005*)

Solution. We have, $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Separating the variables, we get

$$(\sin y + y \cos y)\, dy = \{x\,(2 \log x + 1)\}\, dx$$

Integrating both the sides, we get $\int (\sin y + y\cos y)dy = \int \{x(2\log x + 1)\}\, dx + C$

$$-\cos y + y\sin y - \int (1)\cdot \sin y\, dy = 2\int \log x \cdot x\, dx + \int x\, dx + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = 2\left[\log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x \cdot \frac{x^2}{2} - \int x\, dx + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x \cdot \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2 \log x + C$$ **Ans.**

Example 3. *Solve the differential equation.*

$$x^4\frac{dy}{dx} + x^3 y = -\sec(x\,y).$$ (*A.M.I.E.T.E., Winter 2003*)

Solution. $x^4\dfrac{dy}{dx} + x^3 y = -\sec(x\,y) \qquad \Rightarrow \qquad x^3\left(x\dfrac{dy}{dx} + y\right) = -\sec xy$

Put $\quad v = xy,\ \dfrac{dv}{dx} = x\dfrac{dy}{dx} + y \qquad \Rightarrow \quad x^3\dfrac{dv}{dx} = -\sec v$

$\Rightarrow \quad \dfrac{dv}{\sec v} = -\dfrac{dx}{x^3} \qquad \Rightarrow \quad \int \cos v\, dv = -\int \dfrac{dx}{x^3} + c$

$\Rightarrow \ \sin v = \dfrac{1}{2x^2} + c \qquad \Rightarrow \ \sin xy = \dfrac{1}{2x^2} + c$ **Ans.**

Example 4. *Solve* : $\qquad \cos\,(x + y)dy = dx$

Solution. $\qquad \cos\,(x + y)\, dy = dx \qquad \Rightarrow \qquad \dfrac{dy}{dx} = \sec(x+y)$

On putting $\qquad x + y = z$

So that $\qquad 1 + \dfrac{dy}{dx} = \dfrac{dz}{dx} \qquad \Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

$$\frac{dz}{dx} - 1 = \sec z \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \sec z$$

Separating the variables, we get

$$\frac{dz}{1+\sec z} = dx$$

On integrating,

$$\int \frac{\cos z}{\cos z + 1}dz = \int dx \qquad \Rightarrow \qquad \int \left[1 - \frac{1}{\cos z + 1}\right]dz = x + C$$

$$\int \left(1 - \frac{1}{2\cos^2\frac{z}{2} - 1 + 1}\right)dz = x + C$$

$$\int \left(1 - \frac{1}{2}\sec^2\frac{z}{2}\right)dz = x + C \qquad \Rightarrow \qquad z - \tan\frac{z}{2} = x + C$$

$$x + y - \tan\frac{x+y}{2} = x + C$$

$$y - \tan\frac{x+y}{2} = C \qquad \textbf{Ans.}$$

Example 5. *Solve the equation.*

$(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y\, dy = 0$ (*U.P. II Semester, Summer 2005*)

Solution. We have

$$(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y\, dy = 0$$

Re-arranging (1), we get $\dfrac{x}{y}\dfrac{dx}{dy} = \dfrac{3x^2 + 2y^2 - 8}{2x^2 + 3y^2 - 7}$

Applying componendo and dividendo rule, we get

$$\frac{x\,dx + y\,dy}{x\,dx - y\,dy} = \frac{5x^2 + 5y^2 - 15}{x^2 - y^2 - 1} \quad \Rightarrow \quad \frac{x\,dx + y\,dy}{x^2 + y^2 - 3} = 5\left(\frac{x\,dx - y\,dy}{x^2 - y^2 - 1}\right)$$

Multiplying by 2 both the sides, we get

$$\Rightarrow \quad \left(\frac{2x\,dx + 2y\,dy}{x^2 + y^2 - 3}\right) = 5\left(\frac{2x\,dx - 2y\,dy}{x^2 - y^2 - 1}\right)$$

Integrating both sides, we get

$$\log(x^2 + y^2 - 3) = 5\log(x^2 - y^2 - 1) + \log C$$

$$\Rightarrow \quad x^2 + y^2 - 3 = C\,(x^2 - y^2 - 1)^5 \qquad \textbf{Ans.}$$

where C is arbitrary constant of integration.

EXERCISE 3.2

Solve the following differential equations :

1. $\dfrac{dx}{x} = \tan y \cdot dy$ **Ans.** $x \cos y = C$
2. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **Ans.** $\sin^{-1} y = \sin^{-1} x + C$
3. $y(1+x^2)^{1/2}\,dy + x\sqrt{1+y^2}\,dx = 0$ **Ans.** $\sqrt{1+y^2} + \sqrt{1+x^2} = C$
4. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ **Ans.** $\tan x \tan y = C$
5. $(1 + x^2)\, dy - x\, y\, dx = 0$ **Ans.** $y^2 = C\,(1 + x^2)$
6. $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$ **Ans.** $(e^y + 1)\sin x = C$
7. $3\, e^x \tan y\, dx + (1 - e^x)\, \sec^2 y\, dy = 0$ **Ans.** $(1 - e^x)^3 = C \tan y$
8. $(e^y + 2)\sin x\, dx - e^y \cos x\, dy = 0$ **Ans.** $(e^y + 2)\cos x = C$
9. $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ **Ans.** $e^y = e^x + \dfrac{x^3}{3} + C$
10. $\dfrac{dy}{dx} = 1 + \tan(y - x)$ [Put $y - x = z$] **Ans.** $\sin(y - x) = e^{x+c}$
11. $(4x + y)^2 \dfrac{dx}{dy} = 1$ **Ans.** $\tan^{-1}\dfrac{4x+y}{2} = 2x + C$
12. $\dfrac{dy}{dx} = (4x + y + 1)^2$ [**Hint.** Put $4x + y + 1 = z$] **Ans.** $\tan^{-1}\dfrac{4x+y+1}{2} = 2x + C$

3.7 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$

is called a homogeneous equation if each term of $f(x, y)$ and $\phi(x, y)$ is of the same degree *i.e.*,

$$\frac{dy}{dx}=\frac{3xy+y^2}{3x^2+xy}$$

In such case we put $\boxed{y = vx. \text{ and } \frac{dy}{dx}=v+x\frac{dv}{dx}}$

The reduced equation involves v and x only. This new differential equation can be solved by *variables separable* method.

Working Rule

Step 1. Put $y = vx$ so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$ **Step 2.** Separate the variables.

Step 3. Integrate both the sides. **Step 4.** Put $v=\frac{y}{x}$ and simplify.

Example 6. *Solve the following differential equation*

$$(2xy + x^2)\, y = 3y^2 + 2xy \qquad (A.M.I.E.T.E.\ Dec.\ 2006)$$

Solution. We have, $(2xy + x^2)\frac{dy}{dx}=3y^2+2xy \Rightarrow \frac{dy}{dx}=\frac{3y^2+2xy}{2xy+x^2}$

Put $y = vx$ so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$

On substituting, the given equation becomes $v+x\frac{dv}{dx}=\frac{3v^2x^2+2vx^2}{2vx^2+x^2}=\frac{3v^2+2v}{2v+1}$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{3v^2+2v-2v^2-v}{2v+1} \qquad \Rightarrow \quad x\frac{dv}{dx}=\frac{v^2+v}{2v+1} \quad \Rightarrow \quad \left(\frac{2v+1}{v^2+v}\right)dv=\frac{dx}{x}$$

$$\Rightarrow \quad \int\left(\frac{2v+1}{v^2+v}\right)dv=\int\frac{dx}{x} \qquad \Rightarrow \quad \log(v^2+v)\log x+\log c$$

$$\Rightarrow \quad v^2+v=cx \qquad \Rightarrow \quad \frac{y^2}{x^2}+\frac{y}{x}=cx$$

$$\Rightarrow \quad y^2+xy=cx^3$$

Example 7. *Solve the equation* :

$$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x}$$

Solution.
$$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x} \qquad \dots (1)$$

Put $y = vx$ in (1) so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=v+x\sin v$$

$$\Rightarrow \quad x\frac{dv}{dx}=x\sin v \qquad \Rightarrow \quad \frac{dv}{dx}=\sin v$$

Separating the variable, we get

$$\Rightarrow \quad \frac{dv}{\sin v}=dx \qquad \Rightarrow \quad \int \operatorname{cosec} v\, dv=\int dx+C$$

$$\log\tan\frac{v}{2}=x+C \qquad \Rightarrow \quad \log\tan\frac{y}{2x}=x+C \qquad \textbf{Ans.}$$

EXERCISE 3.3

Solve the following differential equations:

1. $(y^2 - xy)\,dx + x^2 dy = 0$ **Ans.** $\dfrac{x}{y} = \log x + C$

2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$ (*AMIETE, June 2009*) **Ans.** $x^2 + y^2 = ax$

3. $x(y-x)\dfrac{dy}{dx} = y(y+x).$ **Ans.** $\dfrac{y}{x} - \log xy = a$

4. $x\,(x - y)\,dy + y^2\,dx = 0$ (*U.P. B. Pharm (C.O.) 2005*) **Ans.** $y = x \log C y$

5. $\dfrac{dy}{dx} + \dfrac{x-2y}{2x-y} = 0$ **Ans.** $y - x = C\,(x + y)^3$

6. $\dfrac{dy}{dx} = \tan\dfrac{y}{x} + \dfrac{y}{x}$ **Ans.** $\sin\dfrac{y}{x} = C\,x$

7. $\dfrac{dy}{dx} = \dfrac{3xy + y^2}{3x^2}$ **Ans.** $3x + y \log x + Cy = 0$

8. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y^2}{2xy}$ **Ans.** $4y^2 - x^2 = \dfrac{C}{x^2}$

9. $(x^2 + y^2)\,dy = xy\,dx$ **Ans.** $-\dfrac{x^2}{2y^2} + \log y = C$

10. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $\dfrac{-x^3}{3y^3} + \log y = C$

11. $(y^2 + 2xy)\,dx + (2\,x^2 + 3xy)\,dy = 0$ (*AMIETE, Summer 2004*) **Ans.** $xy^2\,(x + y) = C$

12. $(2xy^2 - x^3)\,dy + (y^3 - 2yx^2)\,dx = 0$ **Ans.** $y^2\,(y^2 - x^2) = Cx^{-2}$

13. $(x^3 - 3\,xy^2)\,dx + (y^3 - 3\,x^2y)\,dy = 0,\ y\,(0) = 1$ **Ans.** $x^4 - 6x^2\,y^2 + y^4 = 1$

14. $2\,xy^2\,dy - (x^3 + 2y^3)\,dx = 0$ **Ans.** $2y^3 = 3x^3 \log x + 3x^3 + C$

15. $x\sin\dfrac{y}{x}\,dy = \left(y\sin\dfrac{y}{x} - x\right)dx$ **Ans.** $\cos\dfrac{y}{x} = \log x + C$

16. $\left\{x\cos\dfrac{y}{x} + y\sin\dfrac{y}{x}\right\}y - \left\{y\sin\dfrac{y}{x} - x\cos\dfrac{y}{x}\right\}x\dfrac{dy}{dx} = 0$ **Ans.** $xy\cos\dfrac{y}{x} = a$

17. $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$ **Ans.** $y + \sqrt{x^2 + y^2} = Cx^2$

18. $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$ (*AMIETE, Summer 2004*) **Ans.** $\log\dfrac{y}{x} = Cx$

19. $xy\log\dfrac{x}{y}\,dx + \left(y^2 - x^2\log\dfrac{x}{y}\right)dy = 0$ given that $y\,(1) = 0$

Ans. $\dfrac{x^2}{2y^2}\log\dfrac{x}{y} - \dfrac{x^2}{4y^2} + \log y = 1 - \dfrac{3}{4e^2}$

20. $(1 + e^{\frac{x}{y}})\,dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$ (*AMIETE, June 2009*) **Ans.** $e^{\frac{x}{y}} + \dfrac{x}{y} = e^{-y} + C$

3.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

Case I. $\dfrac{a}{A} + \dfrac{b}{B}$

The equations of the form

$$\boxed{\boldsymbol{\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}}}$$

can be reduced to the homogeneous form by the substitution if $\dfrac{a}{A} + \dfrac{b}{B}$

$$x = X + h, \quad y = Y + k \qquad (h, k \text{ being constants})$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dY}{dX}$$

The given differential equation reduces to

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C} = \frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C}$$

Choose h, k so that $\quad ah + bk + c = 0$

$$Ah + Kk + C = 0$$

Then the given equation becomes homogeneous $\dfrac{dY}{dX} = \dfrac{aX+bY}{AX+BY}$

Case II. If $\dfrac{a}{A} = \dfrac{b}{B}$ then the value of h, k will not be finite.

$$\frac{a}{A} = \frac{b}{B} = \frac{1}{m} \qquad \text{(say)}$$

$$A = am, \; B = bm$$

The given equation becomes $\dfrac{dy}{dx} = \dfrac{ax+by+c}{m(ax+by)+c}$

Now put $ax + by = z$ and apply the method of variables separable.

Example 8. *Solve* : $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$

Solution. Put $\qquad x = X + h, \qquad y = Y + k.$

The given equation reduces to

$$\therefore \qquad \frac{dY}{dX} = \frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3} \qquad \left(\frac{1}{2} \neq \frac{2}{1}\right)$$

$$= \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \qquad \text{.... (1)}$$

Now choose h and k so that $h + 2k - 3 = 0$, $2h + k - 3 = 0$

Solving these equations we get $h = k = 1$

$$\therefore \qquad \frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad \text{... (2)}$$

Put $Y = vX$, so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

The equation (2) is transformed as

$$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{1+2v}{2+v}$$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v} \qquad \Rightarrow \qquad \left(\frac{2+v}{1-v^2}\right)dv = \frac{dX}{X}$$

$$\Rightarrow \qquad \frac{1}{2}\,\frac{1}{(1+v)}dv + \frac{3}{2}\,\frac{1}{1-v}dv = \frac{dX}{X} \qquad \text{(Partial fractions)}$$

On integrating, we have

$$\frac{1}{2}\log(1+v) - \frac{3}{2}\log(1-v) = \log X + \log C$$

$$\Rightarrow \quad \log \frac{1+v}{(1-v)^3} = \log C^2 X^2 \qquad \Rightarrow \quad \frac{1+v}{(1-v)^3} = C^2 X^2$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3} = C^2 X^2 \qquad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = C^2 \quad \text{or} \quad X + Y = C^2 (X - Y)^3$$

Put $X = x - 1$ and $Y = y - 1$ $\Rightarrow$ $x + y - 2 = a (x - y)^3$ **Ans.**

Example 9. *Solve* : $(x + 2y) (dx - dy) = dx + dy$

Solution. $(x + 2y) (dx - dy) = dx + dy \quad \Rightarrow \quad (x + 2y - 1) dx - (x + 2y + 1) dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \qquad ...(1)$$

Hence $\frac{a}{A} = \frac{b}{B}$ *i.e.*, $\left(\frac{1}{1} = \frac{2}{2}\right)$ (Case of failure)

Now put $x + 2y = z$ so that $1+2\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{2} = \frac{z-1}{z+1} \qquad \Rightarrow \quad \frac{dz}{dx} = 2\frac{(z-1)}{z+1} + 1 = \frac{3z-1}{z+1}$$

$$\Rightarrow \quad \frac{z+1}{3z-1} dz = dx \qquad \Rightarrow \quad \left(\frac{1}{3} + \frac{4}{3}\frac{1}{3z-1}\right) dz = dx$$

On integrating, $\frac{z}{3} + \frac{4}{9}\log(3z-1) = x + C$

$$3z + 4 \log (3z - 1) = 9x + 9C$$

$$\Rightarrow \quad 3 (x + 2y) + 4 \log (3x + 6y - 1) = 9x + 9C$$

$$3x - 3y + a = 2 \log (3x + 6y - 1)$$ **Ans.**

EXERCISE 3.4

Solve the following differential equations :

1. $\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$ **Ans.** $(2x - y)^2 = C (x + 2y - 5)$
2. $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$ **Ans.** $\log[(y+3)^2+(x+2)^2]+2\tan^{-1}\frac{y+3}{x+2} = a$
3. $\frac{dy}{dx} = \frac{x-y-2}{x+y+6}$ **Ans.** $(y + 4)^2 + 2 (x + 2) (y + 4) - (x + 2)^2 = a^2$
4. $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ (*AMIETE, Dec. 2009*) **Ans.** $-(y - 3)^2 + 2(x + 1) (y - 3) + (x + 1)^2 = a$
5. $\frac{dy}{dx} = \frac{2x-5y+3}{2x+4y-6}$ **Ans.** $(x - 4y + 3) (2x + y - 3) = a$
6. $(2x + y + 1) dx + (4x + 2y - 1) dy = 0$ **Ans.** $2 (2x + y) + \log (2x + y - 1) = 3x + C$
7. $(x - y - 2) dx - (2x - 2y - 3) dy = 0$ **Ans.** $\log (x - y - 1) = x - 2y + C$ (*U.P. B. Pharm (C.O.) 2005*)
8. $(6x - 4y + 1) dy - (3x - 2y + 1) dx = 0$ (*A.M.I.E.T. E., Dec. 2006*) **Ans.** $4x - 8y - \log (12x - xy + 1) = c$

9. $\frac{dy}{dx} = -\frac{3y-2x+7}{7y-3x+3}$ (*A.M.I.E.T.E., Summer 2004*) **Ans.** $(x+y-1)^5 (x-y-1)^2 = 1$

10. $\frac{dy}{dx} = \frac{2y-x-4}{y-3x+3}$ (*AMIETE, Dec. 2010*)

Ans. $X^2 - 5XY + Y^2 = c\left[\frac{2Y+(-5+\sqrt{21})\,X}{2Y-(5+\sqrt{21})\,X}\right]^{\frac{1}{\sqrt{21}}}$, $X = x-2$, $Y = y-3$

3.9 LINEAR DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\boxed{\frac{dy}{dx} + Py = Q} \quad \text{... (1)}$$

is called a linear differential equation, where P and Q, are functions of x (but not of y) or constants.

In such case, multiply both sides of (1) by $e^{\int P dx}$

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q e^{\int P dx} \quad \text{... (2)}$$

The left hand side of (2) is

$$\frac{d}{dx}\left[y.e^{\int P dx}\right]$$

(2) becomes

$$\frac{d}{dx}\left[y.e^{\int P dx}\right] = Q.e^{\int P dx}$$

Integrating both sides, we get

$$y.e^{\int P dx} = \int Q.e^{\int P dx} dx + C$$

This is the required solution.

Note. $e^{\int P dx}$ is called the integrating factor.

Solution is $\boxed{y \times [I.F.] = \int Q\,[I.F.]\,dx + C}$

Working Rule

Step 1. Convert the given equation to the standard form of linear differential equation

i.e. $$\frac{dy}{dx} + Py = Q$$

Step 2. Find the integrating factor i.e. I.F. = $e^{\int P dx}$

Step 3. Then the solution is $y(I.F.) = \int Q\ (I.F.) dx + C$

Example 10. *Solve*: $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$ (*A.M.I.E.T.E., Summer 2002*)

Solution. $$\frac{dy}{dx} - \frac{y}{x+1} = e^x (x+1)$$

Integrating factor $= e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$

The solution is $$y.\frac{1}{x+1} = \int e^x.(x+1).\frac{1}{x+1} dx = \int e^x dx$$

$$\frac{y}{x+1} = e^x + C$$ **Ans.**

Example 11. *Solve a differential equation*

$$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x.$$ (*Nagpur University, Summer 2008*)

Solution. We have $(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x}y = \frac{x^5 - 2x^3 + x}{x^3 - x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x}y = x^2 - 1$$

$$\text{I.F.} = e^{\int -\frac{3x^2-1}{x^3-x}dx} = e^{-\log(x^3-x)} = e^{\log(x^3-x)^{-1}} = \frac{1}{x^3 - x}$$

Its solution is

$$y(\text{I.F.}) = \int Q(I.F.)\,dx + C \qquad \Rightarrow \quad y\left(\frac{1}{x^3 - x}\right) = \int \frac{x^2 - 1}{x^3 - x}dx + C$$

$$\Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{x^2 - 1}{x(x^2 - 1)}dx + C \qquad \Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{1}{x}dx + C$$

$$\Rightarrow \quad \frac{y}{x^3 - x} = \log x + C \qquad \Rightarrow \quad y = (x^3 - x)\log x + (x^3 - x)\,C$$ **Ans.**

Example 12. *Solve* $\sin x\frac{dy}{dx} + 2y = \tan^3\left(\frac{x}{2}\right)$ (*Nagpur University, Summer 2004*)

Solution. Given equation : $\sin x\frac{dy}{dx} + 2y = \tan^3\frac{x}{2} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{2}{\sin x}y = \frac{\tan^3\frac{x}{2}}{\sin x}$

This is linear form of $\frac{dy}{dx} + Py = Q$

$$\therefore \qquad P = \frac{2}{\sin x} \quad \text{and} \quad Q = \frac{\tan^3\frac{x}{2}}{\sin x}$$

$$\therefore \qquad \text{I.F.} = e^{\int P dx} = e^{\int \frac{2}{\sin x}dx} = e^{2\int \text{cosec}\,x\,dx} = e^{2\log\tan\frac{x}{2}} = \tan^2\frac{x}{2}$$

$\therefore$ Solution is $\quad y.(\text{I.F.}) = \int \text{I.F.}(Q\,dx) + C$

$$y\tan^2\frac{x}{2} = \int \tan^2\frac{x}{2}\cdot\frac{\tan^3\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}} + C = \frac{1}{2}\int \frac{\tan^4\frac{x}{2}}{\cos^2\frac{x}{2}}dx + C$$

$$= \frac{1}{2}\int \tan^4\frac{x}{2}.\sec^2\frac{x}{2}\,dx + C \qquad \text{... (1)}$$

Putting $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$ on R.H.S. (1), we get

$$y.\tan^2\frac{x}{2} = \frac{1}{2}\int t^4(2dt) + C \qquad \Rightarrow \quad y\tan^2\frac{x}{2} = \frac{t^5}{5} + C$$

$$y\tan^2\frac{x}{2} = \frac{\tan^5\frac{x}{2}}{5} + C$$ **Ans.**

EXERCISE 3.5

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$ **Ans.** $xy = \frac{x^5}{5} - \frac{3x^2}{2} + C$

2. $(2y - 3x)\, dx + x\, dy = 0$ **Ans.** $y x^2 = x^3 + C$

3. $\dfrac{dy}{dx} + y \cot x = \cos x$ **Ans.** $y \sin x = \dfrac{\sin^2 x}{2} + C$

4. $\dfrac{dy}{dx} + y \sec x = \tan x$ **Ans.** $y = \dfrac{C - x}{\sec x + \tan x} + 1$

5. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$ **Ans.** $y = \tan x - 1 + Ce^{-\tan x}$

6. $(x+a)\dfrac{dy}{dx} - 3y = (x+a)^5$ **Ans.** $2y = (x + a)^5 + 2C\,(x + a)^3$

7. $x \cos x \dfrac{dy}{dx} + y(x \sin x + \cos x) = 1$ **Ans.** $x\, y = \sin x + C \cos x$

8. $x \log x \dfrac{dy}{dx} + y = 2 \log x$ **Ans.** $y \log x = (\log x)^2 + C$

9. $x\dfrac{dy}{dx} + 2y = x^2 \log x$ **Ans.** $yx^2 = \dfrac{x^4}{4}\log x - \dfrac{x^4}{16} + C$

10. $dr + (2r \cot\theta + \sin 2\theta)\, d\theta = 0$ **Ans.** $r \sin^2\theta = \dfrac{-\sin^4\theta}{2} + C$

11. $\dfrac{dy}{dx} + y \cos x = \dfrac{1}{2}\sin 2x$ **Ans.** $y = \sin x - 1 + Ce^{-\sin x}$

12. $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$ **Ans.** $y = \sqrt{1-x^2} + C(1-x^2)$

13. $\sec x \dfrac{dy}{dx} = y + \sin x$ (*A.M.I.E.T.E., Dec 2005*) **Ans.** $y = -\sin x - 1 + ce^{\sin x}$

14. $y' + y \tan x = \cos x,\ y(0) = 0$ (*A.M.I.E.T.E., June 2006*) **Ans.** $y = x \cos x$

15. Solve $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$ (*AMIETE, Dec. 2009*) **Ans.** $x = -\tan^{-1} y - 1 + ce^{\tan^{-1} y}$

16. Find the value of α so that e^2 is an integrating factor of differential equation $x(1 - y)\, dx - dy = 0$. (*A.M.I.E.T.E., Summer 2005*) **Ans.** $\alpha = \dfrac{1}{2}$

17. Slove the differential equation $\cot 3x \dfrac{dy}{dx} - 3y = \cos 3x + \sin 3x,\ 0 < x < \dfrac{\pi}{2}$.

(*AMIETE, Dec. 2009*) **Ans.** $y \cos 3x = \dfrac{1}{12}\left[6x - \sin 6x - \cos 6x\right]$

18. The value of α so that $e^{\alpha y^2}$ is an integrating factor of the differential equation $\left(e^{\frac{-y^2}{2}} - xy\right)dy - dx = 0$ is (*A.M.I.E.T.E. Dec., 2005*)

(*a*) -1 (*b*) 1 (*c*) $\dfrac{1}{2}$ (*d*) $-\dfrac{1}{2}$ **Ans.** (*c*)

19. The solution of the differential equation $(y + x)^2 \dfrac{dy}{dx} = a^2$ is given by

(*a*) $y + x = a \tan\left(\dfrac{y-c}{a}\right)$ (*b*) $y - x = \tan\left(\dfrac{y-c}{a}\right)$

(*c*) $y - x = a \tan(y - c)$ (*d*) $a\,(y - x) = \tan\left(y - \dfrac{c}{a}\right)$ **Ans.** (*a*)

(*AMIETE, June 2010*)

3.10 EQUATIONS REDUCIBLE TO THE LINEAR FORM (BERNOULLI EQUATION)

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad ...(1)$$

where **P** and **Q** are constants or functions of x can be reduced to the linear form on dividing by y^n and substituting $\frac{1}{y^{n-1}} = z$

On dividing bothsides of (1) by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}} P = Q \qquad ...(2)$$

Put $\frac{1}{y^{n-1}} = z$, so that $\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad \frac{1}{y^n}\frac{dy}{dx} = \frac{dz}{1-n}$

$\therefore$ (2) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$ or $\frac{dz}{dx} + P(1-n)z = Q(1-n)$

which is a linear equation and can be solved easily by the previous method discussed in article 3.8 on page 144.

Example 13. *Solve* $x^2 dy + y(x+y)\,dx = 0$ (*U.P. II Semester Summer 2006*)

Solution. We have, $x^2\,dy + y\,(x+y)\,dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

Put $-\frac{1}{y} = z$ so that $\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

The given equation reduces to a linear differential equation in z.

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

$$\textbf{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log 1/x} = \frac{1}{x}.$$

Hence the solution is

$$z.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\,dx + C \quad \Rightarrow \quad \frac{z}{x} = \int -x^{-3}\,dx + C$$

$$\Rightarrow \quad -\frac{1}{xy} = -\frac{x^{-2}}{-2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{1}{2x^2} - C \qquad \textbf{Ans.}$$

Example 14. *Solve*: $x\frac{dy}{dx} + y\log y = xy\;e^x$ (*A.M.I.E., Summer 2000*)

Solution. $x\frac{dy}{dx} + y\log y = xye^x$

Dividing by xy, we get

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x \qquad ...(1)$$

Put $\log y = z$, so that $\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes, $\frac{dz}{dx} + \frac{z}{x} = e^x$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Solution is $$z\,x = \int x\,e^x\,d\,x + C$$

$$z\,x = x\,e^x - e^x + C$$

$\Rightarrow$ $$x \log y = x\,e^x - e^x + C$$ **Ans.**

Example 15. *Solve*: $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y.$ (*Nagpur University, Summer 2000*)

Solution. $$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$$

$\Rightarrow$ $$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\,e^x \qquad \text{...(1)}$$

Put $\sin y = z$, so that $\cos y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

(1) becomes $$\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x$$

$$I.F. = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)} = e^{\log \frac{1}{1+x}} = \frac{1}{1+x}$$

Solution is $$z.\frac{1}{1+x} = \int (1+x)e^x.\frac{1}{1+x}dx + C = \int e^x dx + C$$

$$\frac{\sin y}{1+x} = e^x + C$$ **Ans.**

Example 16. *Solve*: $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$ (*Nagpur University, Summer 2000*)

Solution. $$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$$

$$\sec y \tan y \frac{dy}{dx} + \sec y\, \tan x = \cos^2 x$$

Writing $z = \sec y$, so that $\dfrac{dz}{dx} = \sec y \tan y \dfrac{dy}{dx}$

The equation becomes $\dfrac{dz}{dx} + z \tan x = \cos^2 x$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution of the equation is

$$z \sec x = \int \cos^2 x \sec x\, dx + C$$

$$\sec y \sec x = \int \cos x\, dx + C = \sin x + C$$

$$\sec y = (\sin x + C) \cos x$$ **Ans.**

Example 17. $$x\left[\frac{dx}{dy} + y\right] = 1 - y$$ (*Nagpur University, Summer 2004*)

Solution. $$x\left(\frac{dy}{dx} + y\right) = (1-y)$$

$\Rightarrow$ $$\frac{dy}{dx} + y = \frac{1}{x} - \frac{y}{x} \qquad \Rightarrow \qquad \frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}$$

which is in linear form of $\frac{dy}{dx} + Py = Q.$

$$\therefore \qquad P = \left(1 + \frac{1}{x}\right), \qquad Q = \frac{1}{x}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\log x} = e^x . e^{\log x} = e^x . x = x e^x$$

Its solution is

$$y(\text{I.F.}) = \int \text{I.F.}(Q\,dx) + C$$

$$y(x.e^x) = \int (x.e^x) \times \frac{1}{x} dx + C \Rightarrow y(x.e^x) = \int e^x dx + C$$

$$y\,(x\,.\,e^x) = e^x + C$$

$$\therefore \qquad y = \frac{1}{x} + \frac{C}{x} e^{-x}$$ **Ans.**

Example 18. *Solve the differential equation.*

$$y \log y \, dx + (x - \log y)\, dy = 0$$ (*Uttarakhand II Semester, June 2007*)

Solution. We have,

$$y \log y \, dx + (x - \log y)\, dy = 0$$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{-x + \log y}{y \log y} \qquad \Rightarrow \qquad \frac{dx}{dy} = \frac{-x}{y \log y} + \frac{\log y}{y \log y}$$

$$\Rightarrow \qquad \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log(\log y)} = \log y$$

Its solution is $\quad x.\log y = \int \frac{1}{y} (\log y)\, dy$

$$x.\log y = \frac{(\log y)^2}{2} + C$$ **Ans.**

Example 19. *Solve*: $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy.$

(*AMIETE, June 2010, 2004, R.G.P.V., Bhopal, April 2010, June 2008, U.P. (B. Pharm) 2005*)

Solution. $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2} \qquad \Rightarrow \qquad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

This is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Its solution is $\quad x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1+y^2} dy + C$

Put $\quad \tan^{-1} y = t$ on R.H.S., so that $\frac{1}{1+y^2} dy = dt$

$$x.e^{\tan^{-1} y} = \int e^t . t\, dt + C = t.e^t - e^t + C = e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) + C$$

$$x = \left(\tan^{-1} y - 1\right) + C e^{-\tan^{-1} y}$$ **Ans.**

Example 20. *Solve* : $r\sin\theta - \dfrac{dr}{d\theta}\cos\theta = r^2$ (*Nagpur University, Summer 2005*)

Solution. The given equation can be written as $-\dfrac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$ ... (1)

Dividing (1) by $r^2\cos\theta$, we get $-r^{-2}\dfrac{dr}{d\theta} + r^{-1}\tan\theta = \sec\theta$ ... (2)

Putting $r^{-1} = v$ so that $-r^{-2}\dfrac{dr}{d\theta} = \dfrac{dv}{d\theta}$ in (2), we get

$$\frac{dv}{d\theta} + v\tan\theta = \sec\theta$$

$$\text{I.F.} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

Solution is $v\sec\theta = \int \sec\theta,\ \sec\theta + C \quad \Rightarrow \quad v\sec\theta = \int \sec^2\theta\, d\theta + C$

$$\frac{\sec\theta}{r} = \tan\theta + C \quad \Rightarrow \quad r^{-1} = (\sin\theta + C\cos\theta)$$

$\therefore$ $$r = \frac{1}{\sin\theta + C\cos\theta}$$ **Ans.**

EXERCISE 3.6

Solve the following differential equations:

1. $\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y} = 2xe^{-x}$ **Ans.** $e^x + x^2y + Cy = 0$

2. $3\dfrac{dy}{dx} + 3\dfrac{y}{x} = 2x^4y^4$ **Ans.** $\dfrac{1}{y^3} = x^5 + Cx^3$

3. $\dfrac{dy}{dx} = y\tan x - y^2\sec x$ **Ans.** $\sec x = (\tan x + C)\, y$

4. $\dfrac{dy}{dx} = 2y\tan x + y^2\tan^2 x$, if $y = 1$ at $x = 0$ **Ans.** $\dfrac{1}{y}\sec^2 x = -\dfrac{\tan^3 x}{3} + 1$

5. $\dfrac{dy}{dx} + \tan x\tan y = \cos x\sec y$ **Ans.** $\sin y\sec x = x + C$

6. $dy + y\tan x\,.\,dx = y^2\sec x\,.\,dx$ **Ans.** $y\,(x + C) + \cos x = 0$

7. $(x^2y^2 + xy)\,y\,dx + (x^2y^2 - 1)\,x\,dy = 0$ **Ans.** $x\,y = \log C\,y$

8. $(x^2 + y^2 + x)\,dx + xy\,dy = 0$ **Ans.** $x^2y^2 = -\dfrac{x^4}{2} - \dfrac{2x^3}{3} + C$

9. $\dfrac{dy}{dx} + y = 3e^x y^3$ **Ans.** $\dfrac{1}{y^2} = 6e^x + Ce^{2x}$

10. $(x - y^2)\,dx + 2\,x\,y\,dy = 0$ **Ans.** $\dfrac{y^2}{x} + \log x = C$

11. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$ **Ans.** $e^{x+y} = \dfrac{e^{2x}}{2} + C$

12. $x^2y - x^3\dfrac{dy}{dx} = y^4\cos x$ **Ans.** $x^3 = y^3(3\sin x - C)$

13. $3\dfrac{dy}{dx} + \dfrac{2}{x+1}.y = \dfrac{x^2}{y^2}$ **Ans.** $y^3(x+1)^2 = \dfrac{x^5}{5} + \dfrac{x^4}{2} + \dfrac{x^3}{3} + C$

14. $\cos x\dfrac{dy}{dx} + 4y\sin x = 4\sqrt{y}\sec x$ **Ans.** $\sqrt{y}\sec^2 x = 2\left[\tan x + \dfrac{\tan^3 x}{3}\right] + C$