

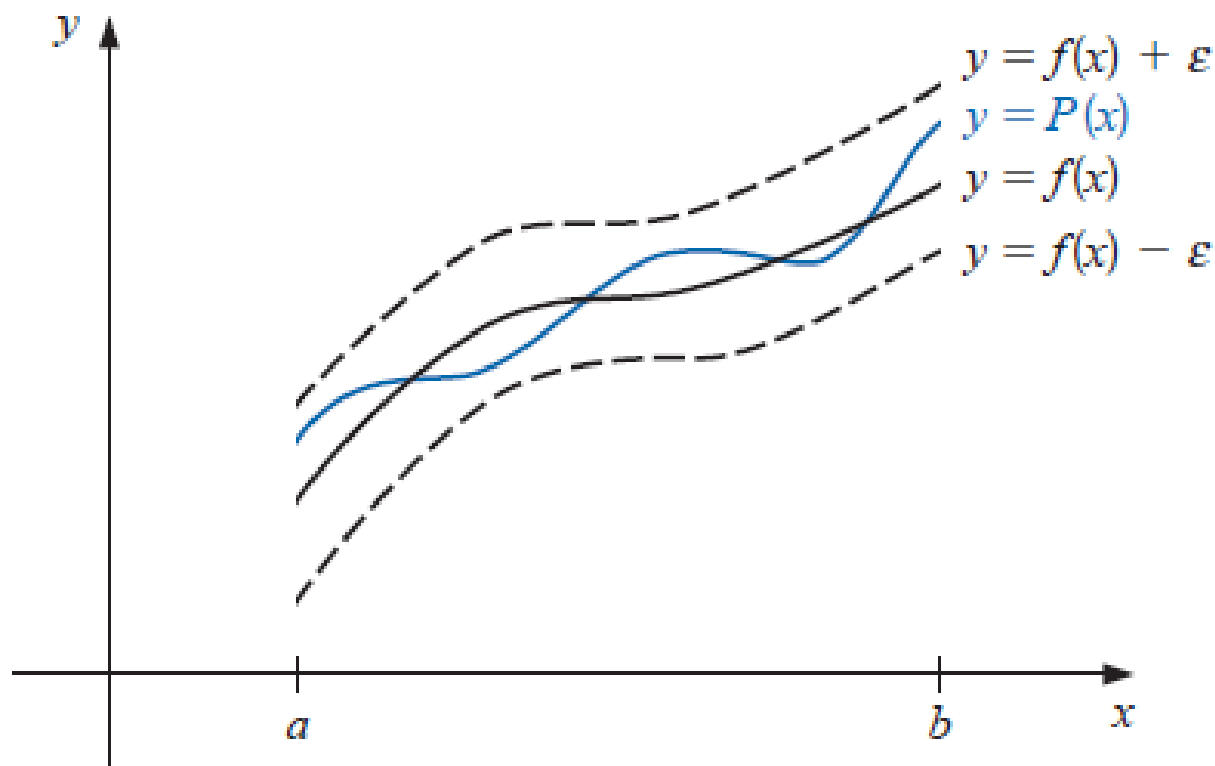
Interpolation and Polynomial Approximation

Dr. Jolly Puri
Assistant Professor,
School of Mathematics,
TIET, Patiala, Punjab, India

(Weierstrass Approximation Theorem)

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

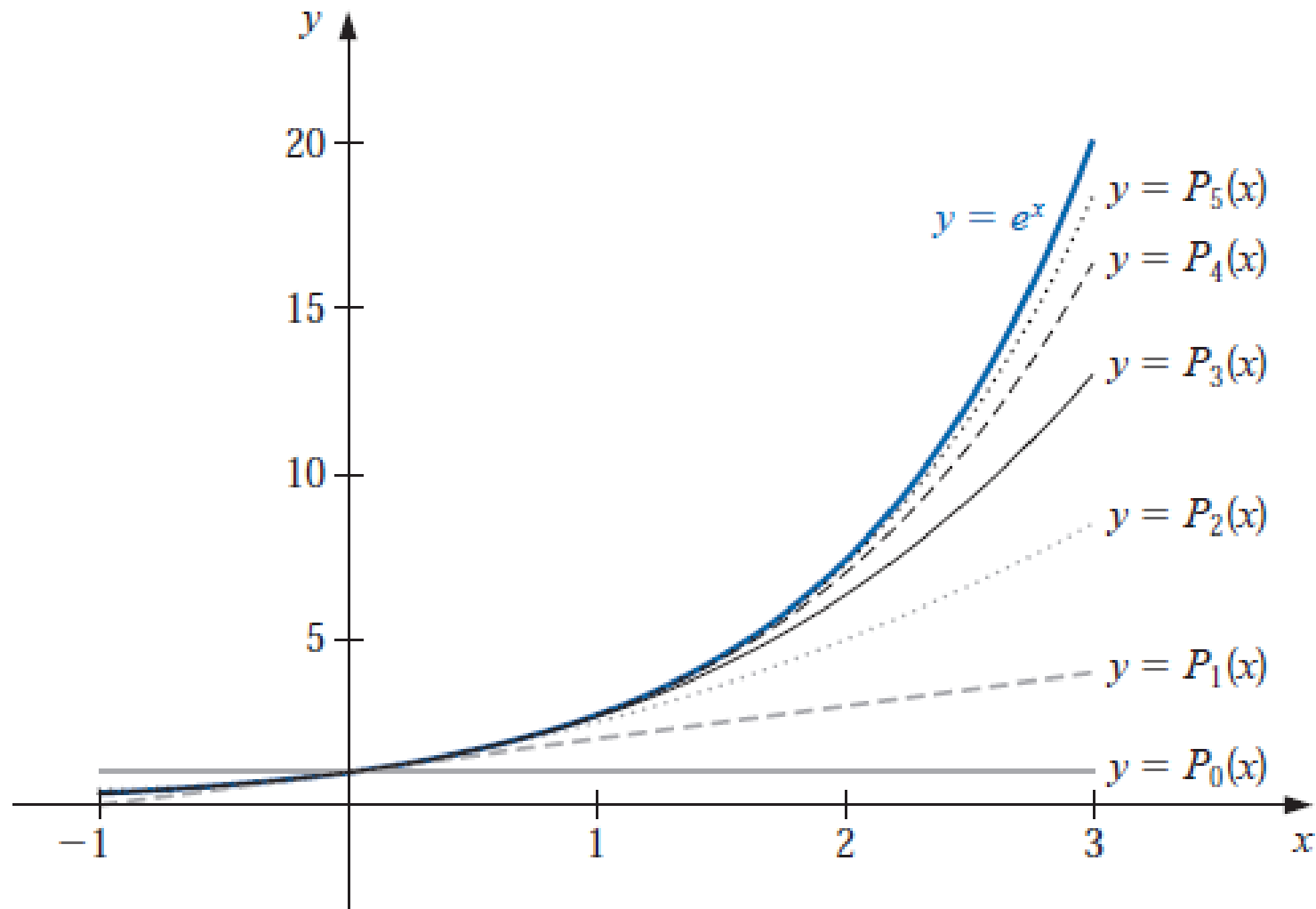
$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b].$$



Why polynomials?

- ✓ Since the derivative and indefinite integral of a polynomial are easy to determine and are also polynomials. Therefore, polynomials are often used for approximating continuous functions.
- ✓ The Taylor polynomials are well known polynomials that are used to approximate continuous functions. However, the Taylor polynomials agree as closely as possible with a given function at a specific point, and concentrate their accuracy near that point.
- ✓ A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.
- ✓ **Example 1:** Calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$. Since the derivatives of $f(x)$ are all e^x , which evaluated at $x_0 = 0$ gives 1. The Taylor polynomials are

$$P_0(x) = 1, \quad P_1(x) = 1 + x, \quad P_2(x) = 1 + x + \frac{x^2}{2}, \quad P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$$
$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}, \quad \text{and} \quad P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$



Better approximations are obtained for $f(x) = e^x$ if higher-degree Taylor polynomials are used, this is not true for all functions.

Example 2: Calculate Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = (-1)^2 2 \cdot x^{-3},$$

$$f^{(k)}(x) = (-1)^k k! x^{-k-1},$$

The **Taylor polynomials** for $f(x) = 1/x$ is given by

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(1)}{k!} (x-1)^k = \sum_{k=0}^n (-1)^k (x-1)^k.$$

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

When we approximate $f(3) = 1/3$ by $P_n(3)$ for larger values of n , the approximations become increasingly inaccurate.

- ✓ For the Taylor polynomials, all the information used in the approximation is concentrated at the point x_0 .
- ✓ So these polynomials will generally give inaccurate approximations as we move away from x_0 .
- ✓ This limits Taylor polynomial approximation to the situation in which approximations are needed only at numbers close to x_0 .

Lagrange Interpolation