Lecture 11: Numerical Analysis (UMA011)

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Fixed point iteration

Example of FPI:

Find the root of an equation $x^3 + 4x^2 - 10 = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.

Solution:

$x^2(x+4) = 10$ $\qquad [1,2]$

$g(x) = x = \sqrt{\frac{10}{x+4}}$ ✓

$g(x)$ is cont. function on $[1,2]$

$g(1) = \sqrt{\frac{10}{5}} = \sqrt{2} \in [1,2]$

$g(2) = \sqrt{\frac{10}{6}} = \sqrt{\frac{5}{3}} \in [1,2]$

$$g'(x) = \sqrt{10}\ \frac{-1}{2}(x+4)^{-3/2} = \frac{-\sqrt{10}}{2(x+4)^{3/2}} < 0 \quad \forall\, x \in [1,2]$$

$\Rightarrow$ $g(x)$ is decreasing on $x \in [1,2]$

$\Rightarrow$ Max value of $g(x)$ at $[1,2]$ is

$g(1) = \sqrt{2} \in [1,2]$

& Min. value of $g(x)$ at $[1,2]$ is $g(2) \in [1,2]$

$\Rightarrow$ $g(x) \in [1,2] \quad \forall\, x \in [1,2]$

$$|g'(x)| = \left| \frac{-\sqrt{10}}{2\,(x+4)^{3/2}} \right|$$

$$G(x) = g'(x) = \frac{-\sqrt{10}}{2\,(x+4)^{3/2}}$$

$$G'(x) = g''(x) = \frac{-\sqrt{10}}{2}\left(\frac{-3}{2}\right)(x+4)^{-5/2} = \frac{3\sqrt{10}}{4\,(x+4)^{5/2}} > 0$$

$\forall\, x \in [1,2]$

$$|g'(2)| = \left| \frac{-\sqrt{10}}{2\,(2+4)^{3/2}} \right|$$

$$g'(1) = \left| \frac{-\sqrt{10}}{2\,(5)^{3/2}} \right|$$

$\Rightarrow$ $g'(x)$ is an increasing f'n on $[1,2]$

$\Rightarrow$ Max value of $g'(x)$ is $g'(2) < 1$

Min value - - - - $g'(1) < 1$

Fixed point iteration

Example of FPI:

Find the root of an equation $x^3 - 7x + 2 = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.

Solution:

$f(x) = x^3 - 7x + 2$

$f(0) = +ve$

$f(1) = 1 - 7 + 2 = -ve$

$\Rightarrow$ $f(x) = 0$ has a root in $[0, 1]$

1.) $x = x + x^3 - 7x + 2$

$= g_1(x) = x^3 - 6x + 2$

$g_1(0) = 2 \notin [0, 1]$

2.) $x = x - x^3 + 7x - 2$

$= g_2(x) = -x^3 + 8x - 2$

$g_2(0) = -2 \notin [0, 1]$

$$\Rightarrow \quad |g'(x)| < 1 \quad \forall\, x \in [1,2]$$

$$g(x) = \sqrt{\frac{10}{x+4}} \quad \text{on} \quad x \in [1,2]$$

$$p_0 = 1.5$$

$$p_1 = g(p_0) = \sqrt{\frac{10}{1.5+4}} = \sqrt{\frac{10}{5.5}} = 1.3484 \qquad |1.5 - 1.3484| \nless 10^{-2}$$

$$p_2 = g(p_1) = \sqrt{\frac{10}{1.3484+4}} = 1.3674 \qquad |1.3484 - 1.3674| \nless 10^{-2}$$

$$p_3 = g(p_2) = \sqrt{\frac{10}{1.3674+4}} = 1.3650 \checkmark \qquad |1.3650 - 1.3674| < 10^{-2}$$

3)

$$x^3 = 7x - 2$$

$$x = (7x-2)^{1/3} = g_3(x)$$

$$g_3(0) = (-2)^{1/3} \notin [0,1]$$

$p_0 = 0.5$

p_1

p_2

p_3

$\downarrow$

p

may or may not converge

$$g_4(x) = \frac{2}{7-x^2}$$

$$g_4'(x) = 2\,(-1)\,(7-x^2)^{-2}(-2x)$$

$$= \frac{4x}{(7-x^2)^2} > 0 \quad \forall\, x \in [0,1]$$

$\Rightarrow g_4(x)$ is an increasing f^n on $[0,1]$ $\Rightarrow \begin{cases} \text{Min value of } g_4 \text{ at } [0,1] \\ \quad \text{is } g_4(0) \in [0,1] \\ \text{Max value is } g_4(1) \in [0,1] \end{cases}$

4)

$$x(x^2-7) = -2$$

$$x = \frac{2}{7-x^2} = g_4(x)$$

$$g_4 \in C[0,1]$$

$$g_4(0) = \frac{2}{7} \in [0,1]$$

$$g_4(1) = \frac{2}{6} = \frac{1}{3} \in [0,1]$$

Now,

$$|g_4'(x)| = \left|\frac{4x}{(7-x^2)^2}\right|$$

$$g_4''(x) = 4x(-2)(7-x^2)^{-3}(-2x) + \frac{4}{(7-x^2)^2}$$

$$= \frac{16x^2}{(7-x^2)^3} + \frac{4}{(7-x^2)^2} > 0 \quad \forall\, x \in [0,1]$$

$\Rightarrow$ $g_4'(x)$ is an increasing f^n on $[0,1]$

Min value of $g_4'(x)$ is at 0 ie $g_4'(0) = 0$

Max value " " at 1 ie $g_4'(1) = \frac{4}{36} = \frac{1}{9}$

$|g_4'(0)| < 1$ and $|g_4'(1)| < 1$

$\Rightarrow \quad |g_4'(x)| < 1 \quad \forall\, x \in [0,1]$

$g(x) = \dfrac{2}{7-x^2}$ on $x \in [0,1]$

$x_{n+1} = g(x_n) = \dfrac{2}{7-x_n^2}$

$p_0 = 0.5$

$p_1 = g(0.5) = \dfrac{2}{7-(0.5)^2} = 0.2960$

$|0.2960 - 0.2893| \not< 10^{-2}$

$p_2 = g(0.296) = \dfrac{2}{7-(0.296)^2} = 0.2893$

$|0.2893 - 0.2891| < 10^{-2}$

$p_3 = g(p_2) = \dfrac{2}{7-(0.2893)^2} = 0.2891$ ✓

There is an another $g(x)$ which satisfies the convergence conditions :-

$$7x = x^3 + 2$$

$$x = \frac{x^3+2}{7}$$

$$g(x) = \frac{x^3+2}{7} \quad \text{on} \quad [0,1].$$

Fixed point iteration

$p_{n+1} = g(p_n)$

$x_{n+1} = g(x_n)$

Example:

The iterates $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ converge to $p = 1$ for some constant c. Find the value or bound for c for which convergence occurs.

Solution: $x_{n+1} = 2 - (1+c)x_n + cx_n^3 = g(x_n)$

$\Rightarrow g(x) = 2 - (1+c)x + cx^3$

Given that $x_{n+1} = g(x_n)$ converges to $p=1$

$\Rightarrow$ $g(x)$ satisfies the convergence conditions on nbd of 1 ie $(1-\delta, 1+\delta)$, $\delta > 0$ (say)

$|g'(x)| < 1 \quad \forall x \in (1-\delta, 1+\delta)$

$$g'(x) = -(1+c) + 3cx^2$$

Now, $|g'(x)| < 1 \quad \forall x \in [1-\delta, 1+\delta)$

$\Rightarrow |g'(1)| < 1$

$|-(1+c) + 3c| < 1$

$|-1+2c) < 1$

$-1 < -1+2c < 1$

$0 < 2c < 2 \Rightarrow 0 < c < 1$ Ans.

Fixed point iteration

Root in bet^n.
Either or
$[2,3]$ $[2.5, 3]$

Exercise:

1. Find the root of an equation $x^3 - 2x^2 - 5 = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.
2. Let A be a given positive constant and g(x) = $2x - Ax^2$:
   (a) Show that $1/A$ is a fixed point for $g(x)$. ie $g(\frac{1}{A}) = \frac{1}{A}$ ✓
   (b) Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Hint :-

$|g'(x)| < 1$

$|2 - 2Ax| < 1$

$\Rightarrow \quad x \in \left(\frac{1}{2A}, \frac{3}{2A}\right) \rightarrow$ interval containing $\frac{1}{A}$.