CHAPTER 3 (4 LECTURES)
DIRECT METHODS FOR SOLVING LINEAR SYSTEMS

1. Introduction

System of simultaneous linear equations are associated with many problems in engineering and science, as well as with applications to the social sciences and quantitative study of business and economic problems. These problems occur in wide variety of disciplines, directly in real world problems as well as in the solution process for other problems.
The principal objective of this Chapter is to discuss the numerical aspects of solving linear system of equations having the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n. \end{cases} \tag{1.1}$$

This is a linear system of n equation in n unknowns $x_1, x_2, \cdots, x_n$. This system can simply be written in the matrix equation form

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{1.2}$$

This equations has a unique solution $x = A^{-1}b$, when the coefficient matrix A is non-singular. Unless otherwise stated, we shall assume that this is the case under discussion. If A^{-1} is already available, then $x = A^{-1}b$ provides a good method of computing the solution x.
If A^{-1} is not available, then in general A^{-1} should not be computed solely for the purpose of obtaining x. More efficient numerical procedures will be developed in this chapter. We study broadly two categories Direct and Iterative methods. We start with direct method to solve the linear system in this chapter.

2. Gaussian Elimination

Direct methods, which are technique that give a solution in a fixed number of steps, subject only to round-off errors, are considered in this chapter. Gaussian elimination is the principal tool in the direct solution of system (1.2). The method is named after Carl Friedrich Gauss (1777-1855).
To solve larger system of linear equation, we consider a following $n \times n$ system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \qquad (E_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \qquad (E_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n &= b_i \qquad (E_i) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \qquad (E_n). \end{aligned}$$

Here E_i denote the i-th row of the coefficients matrix $A = [a_{ij}]$, $i, j = 1, 2, \cdots, n$.
Firstly we write the Augmented matrix $[A, b]$ (coefficient and right hand side together) as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}.$$

Let $a_{11} \neq 0$ and eliminate x_1 from E_2, $E_3, \cdots$, E_n.
Define multipliers $m_{i1} = \dfrac{a_{i1}}{a_{11}}$, for each rows $i = 2, 3, \cdots, n$.
We write each entry in E_i as $E_i - m_{i1}E_1$, $i = 1, 2, \cdots, n$. Also b_i as $b_i - m_{i1}b_1$. This will delete x_1 from these rows.
We repeat this procedure and follow a sequential procedure for columns $j = 2, 3, \cdots, n-1$ and perform the following operations provided $a_{jj} \neq 0$:

$$E_i \longrightarrow E_i - (a_{ij}/a_{jj})E_j, \quad \text{for each } i = j+1, j+2, \cdots, n.$$

This eliminates x_i in each row below the i-th for all values of $i = 1, 2, \cdots, n-1$.
The resulting matrix is upper triangular and has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{bmatrix}$$

Solving the n-th equation for x_n gives

$$x_n = \frac{b_n}{a_{nn}}.$$

Solving the $(n-1)$st equation for x_{n-1} and using the known value for x_n yields (back substitution)

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}.$$

Continuing this process, we obtain

$$x_i = \frac{b_i - a_{i,i+1}x_{i+1} - a_{i,i+2}x_{i+2} - \cdots - a_{in}x_n}{a_{ii}} = \frac{b_i - \sum\limits_{j=i+1}^{n} a_{ij}x_j}{a_{ii}},$$

for each $j = n-1, n-2, \cdots, 2, 1$.

Example 1. *Solve the following systems using the simple Gaussian elimination method*

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 1 \\ x_1 - 3x_2 + 2x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 3. \end{aligned}$$

Sol. We label each rows as E_1, E_2 and E_3. The augmented matrix form of the given system is

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & -3 & 2 & 2 \\ 2 & -1 & 1 & 3 \end{bmatrix}.$$

Since $a_{11} = 3 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second and third rows the appropriate multipliers of the first row. In this case the multiples are given

$$m_{21} = \frac{1}{3}, \quad m_{31} = \frac{2}{3}.$$

Thus we write E_2 as $E_2 - \frac{1}{3}E_1$ and E_3 as $E_3 - \frac{2}{3}E_1$. Hence

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & -11//3 & 7/3 & 5/3 \\ 0 & -7/3 & 5/3 & 7/3 \end{bmatrix}.$$

As $a_{22} = -11/3 \neq 0$, therefore, we eliminate entry in a_{32} position by taking the multiple $m_{32} = 7/11$ and we write E_3 as $E_3 - \frac{7}{11}$, to get

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & -11//3 & 7/3 & 5/3 \\ 0 & 0 & 2/11 & 14/11 \end{bmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular and therefore, the given system has a unique solution.
Now using backward substitution to give

$$\begin{aligned} 2/11x_3 &= 14/11, & x_3 &= 7 \\ -11/3x_2 &= 5/3 - 7/3x_3 = -44/3, & x_2 &= 4 \\ 3x_1 &= 1 - 2x_2 + x_3 = 0, & x_1 &= 0. \end{aligned}$$

Partial Pivoting: In the elimination process, we divide with a_{ii} at each stage and assume that $|a_{ii}| \neq 0$. These elements are known as pivot element. If at any stage of elimination, one of the pivot becomes small (or zero) then we bring other element as pivot by interchanging the rows. This process is called Gauss elimination with partial pivoting.

Example 2. *Solve the system of equations using Gauss elimination. This system has exact solution (known from other sources!)* $x_1 = 2.6, \ x_2 = -3.8, \ x_3 = -5.0.$

$$\begin{aligned} 6x_1 + 2x_2 + 2x_3 &= -2 \\ 2x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 0. \end{aligned}$$

Sol. Let us use a floating-point representation with 4−digits and all operations will be rounded. We label each rows as E_1, E_2 and E_3. Augmented matrix (coefficients matrix with right hand side) is given by

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 2.000 & 0.6667 & 0.3333 & 1.000 \\ 1.000 & 2.000 & -1.000 & 0.0 \end{bmatrix}$$

Multipliers are $m_{21} = \dfrac{2}{6} = 0.3333$ and $m_{31} = \dfrac{1}{6} = 0.1667$.
We write E_2 as $E_2 - 0.3333E_1$ and E_3 as $E_3 - 0.1667E_1$.

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.0 & 0.0001000 & -0.3333 & 1.667 \\ 0.0 & 1.667 & -1.333 & 0.3334 \end{bmatrix}$$

Multiplier is $m_{32} = \dfrac{1.667}{0.0001} = 16670$ and we write E_3 as $E_3 - 16670E_2$.

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.0 & 0.0001000 & -0.3333 & 1.667 \\ 0.0 & 0.0 & 5555 & -27790 \end{bmatrix}$$

Using back substitution, we obtain

$$x_3 = -5.003$$
$$x_2 = 0.0$$
$$x_1 = 1.335.$$

We observe that computed solution is not compatible with the exact solution.
The difficulty is in a_{22}. This coefficient is very small (almost zero). This means that the coefficient in this position had essentially infinite relative error and this was carried through into computation involving this coefficient. To avoid this, we interchange second and third rows and then continue the

elimination.
In this case (after interchanging) multipliers is $m_{32} = 0.00005999$ and we write new E_3 as $E_3 - 000005999E_2$.

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.0 & 1.667 & -1.337 & 0.3334 \\ 0.0 & 0.0 & -0.3332 & 1.667 \end{bmatrix}.$$

Using back substitution, we obtain

$$x_3 = -5.003$$
$$x_2 = -3.801$$
$$x_1 = 2.602.$$

We see that after partial pivoting, we get the desired solution.

Example 3. *Given the linear system*

$$\begin{aligned} x_1 - x_2 + \alpha x_3 &= -2, \\ -x_1 + 2x_2 - \alpha x_3 &= 3, \\ \alpha x_1 + x_2 + x_3 &= 2. \end{aligned}$$

(1) *Find value(s) of α for which the system has no solutions.*
(2) *Find value(s) of α for which the system has an infinite number of solutions.*
(3) *Assuming a unique solution exists for a given α, find the solution.*

Sol. Augmented matrix is given by

$$\begin{bmatrix} 1 & -1 & \alpha & -2 \\ -1 & 2 & -\alpha & 3 \\ \alpha & 1 & 1 & 2 \end{bmatrix}$$

Multipliers are $m_{21} = -1$ and $m_{31} = \alpha$. Performing $E_2 \to E_2 + E_1$ and $E_3 \to E_3 - \alpha E_1$ to obtain

$$\begin{bmatrix} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 1+\alpha & 1-\alpha^2 & 2(1+\alpha) \end{bmatrix}$$

Multiplier is $m_{32} = 1 + \alpha$ and we perform $E_3 \to E_3 - m_{32}E_2$.

$$\begin{bmatrix} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1-\alpha^2 & 1+\alpha \end{bmatrix}$$

(1) If $\alpha = 1$, then the last row of the reduced augmented matrix says that $0.x_3 = 2$, the system has no solution.
(2) If $\alpha = -1$, then we see that the system has infinitely many solutions.
(3) If $\alpha \neq 1$, then the system has a unique solution.

$$x_3 = \frac{1}{1-\alpha}, \; x_2 = 1, \; x_1 = -\frac{1}{1-\alpha}.$$

Remark 2.1. *Unique solution, no solution, or infinite number of solutions.*

(1) If we have a leading one in every column, then we will have a unique solution.
(2) If we have a row of zeros equal to a non-zero number in right side, then the system has no solution.
(3) If we don't have a leading one in every column in a homogeneous system, i.e. a system where all the equations equal zero, or a row of zeros, then system have infinite number of solutions.

Example 4. *Solve the system by Gauss elimination*

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &= 1 \\ 3x_1 + 4x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= -1. \end{aligned}$$

Sol. We write augmented matrix and solve the system

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 1 \\ 3 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & -1 \\ 1 & 2 & 3 & 4 & -1 \end{bmatrix}$$

Multipliers are $m_{21} = \dfrac{3}{4}$, $m_{31} = \dfrac{1}{2}$, and $m_{41} = \dfrac{1}{4}$.
Replace E_2 with $E_2 - m_{21}E_1$, E_3 with $E_3 - m_{31}E_1$ and E_4 with $E_4 - m_{41}E_1$.

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 1 \\ 0 & 7/4 & 3/2 & 5/4 & 1/4 \\ 0 & 3/2 & 3 & 5/2 & -3/2 \\ 0 & 5/4 & 5/2 & 15/4 & -5/4 \end{bmatrix}$$

Multipliers are $m_{32} = \dfrac{6}{7}$ and $m_{42} = \dfrac{5}{7}$.
Replace E_3 with $E_3 - m_{32}E_2$ and E_4 with $E_4 - m_{42}E_2$, we obtain

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 1 \\ 0 & 7/4 & 3/2 & 5/4 & 1/4 \\ 0 & 0 & 12/7 & 10/7 & -12/7 \\ 0 & 0 & 10/7 & 20/7 & -10/7 \end{bmatrix}$$

Multiplier is $m_{43} = \dfrac{5}{6}$ and we replace E_4 with $E_4 - m_{43}E_3$.

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 1 \\ 0 & 7/4 & 3/2 & 5/4 & 1/4 \\ 0 & 0 & 12/7 & 10/7 & -12/7 \\ 0 & 0 & 0 & 5/3 & 0 \end{bmatrix}$$

Using back substitution successively for $x_4,\ x_3,\ x_2,\ x_1$, we obtain $x_4 = 0$, $x_3 = -1$, $x_2 = 1$, $x_1 = 0$.

Complete Pivoting: In the first stage of elimination, we search the largest element in magnitude from the entire matrix and bring it at the position of first pivot. We repeat the same process at every step of elimination. This process require interchange of both rows and columns.

Scaled Partial Pivoting: In this approach, the algorithm select the largest relative entries as the pivot elements at each stage of elimination. At the beginning, a scale factor must be computed for each equation in the system. We define

$$s_i = \max_{1 \le j \le n} |a_{ij}| \quad (1 \le i \le n)$$

These numbers are recored in the scaled vector $s = [s_1, s_2, \cdots, s_n]$. Note that the scale vector does not change throughout the procedure.
In starting the forward elimination process, we do not arbitrarily use the first equation as the pivot equation. Instead, we use the equation for which the ratio $\dfrac{|a_{i,1}|}{s_i}$ is greatest. We repeat the process by taking same scaling factors.

Example 5. *Solve the system*

$$\begin{aligned} 2.11x_1 - 4.21x_2 + 0.921x_3 &= 2.01 \\ 4.01x_1 + 10.2x_2 - 1.12x_3 &= -3.09 \\ 1.09x_1 + 0.987x_2 + 0.832x_3 &= 4.21 \end{aligned}$$

by using scaled partial pivoting.

Sol. The augmented matrix is

$$\begin{bmatrix} 2.11 & -4.21 & 0.921 & 2.01 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 1.09 & 0.987 & 0.832 & 4.21. \end{bmatrix}$$

The scale factors are $s_1 = 4.21, s_2 = 10.2, \&\ s_3 = 1.09$. We need to pick the largest $(2.11/4.21 = 0.501, 4.01/10.2 = 0.393, 1.09/1.09 = 1)$, which is the third entry, and interchange row 1 and row 3 and interchange s_1 and s_3 to get

$$\begin{bmatrix} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & 0.921 & 2.01. \end{bmatrix}$$

Performing $E_2 \to E_2 - 3.68E_1$, $E_3 \to E_3 - 1.94E_1$, we obtain

$$\begin{bmatrix} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & 6.57 & -4.18 & -18.6 \\ 0 & -6.12 & -0.689 & -6.16. \end{bmatrix}$$

Now comparing $(6.57/10.2 = 0.6444, 6.12/4.21 = 1.45)$, the second ratio is largest so we need to interchange row 2 and row 3 and interchange scale factor accordingly.

$$\begin{bmatrix} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.689 & -6.16 \\ 0 & 6.57 & -4.18 & -18.6. \end{bmatrix}$$

Performing $E_3 \to E_3 + 1.07E_2$, we get

$$\begin{bmatrix} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.689 & -6.16 \\ 0 & 0 & -4.92 & -25.2. \end{bmatrix}$$

Backward substitution gives $x_3 = 5.12, x_2 = 0.43, x_1 = -0.436$.

Example 6. *Solve the system*

$$\begin{aligned} 3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 &= -34 \\ 6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26 \end{aligned}$$

by hand using scaled partial pivoting. Justify all row interchanges and write out the transformed matrix after you finish working on each column.

Sol. The augmented matrix is

$$\begin{bmatrix} 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \\ 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \end{bmatrix}$$

and the scale factors are $s_1 = 13, s_2 = 18, s_3 = 6, \&\ s_4 = 12$. We need to pick the largest $(3/13, 6/18, 6/6, 12/12)$, which is the third entry, and interchange row 1 and row 3 and interchange s_1 and s_3 to get

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 16 \\ -6 & 4 & 1 & -18 & -34 \\ 3 & -13 & 9 & 3 & -19 \\ 12 & -8 & 6 & 10 & 26 \end{bmatrix}$$

with $s_1 = 6, s_2 = 18, s_3 = 13, s_4 = 12$. Performing $E_2 \to E_2 - (-6/6)E_1$, $E_3 \to E_3 - (3/6)E_1$, and $E_4 \to E_4 - (12/6)E_1$, we obtain

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 16 \\ 0 & 2 & 3 & -14 & -18 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & -4 & 2 & 2 & -6. \end{bmatrix}$$

Comparing $(|a_{22}|/s_2 = 2/18, |a_{32}|/s_3 = 12/13, |a_{42}|/s_4 = 4/12)$, the largest is the third entry so we need to interchange row 2 and row 3 and interchange s_2 and s_3 to get

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \\ 0 & -4 & 2 & 2 & -6 \end{bmatrix}$$

with $s_1 = 6, s_2 = 13, s_3 = 18, s_4 = 12$. Performing $E_3 \to E_3 - (2/12)E_2$ and $E_4 \to E_4 - (-4/12)E_2$, we get

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 0 & 13/3 & -83/6 & -45/2 \\ 0 & 0 & -2/3 & 5/3 & 3 \end{bmatrix}$$

Comparing $(|a_{33}|/s_3 = (13/3)/18, |a_{43}|/s_4 = (2/3)/12)$, the largest is the first entry so we do not interchange rows. Performing $E_4 \to E_4 - (-2/13)E_3$, we get the final reduced matrix

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 0 & 13/3 & -83/6 & -45/2 \\ 0 & 0 & 0 & -6/13 & -6/13 \end{bmatrix}$$

Backward substitution gives $x_1 = 3, x_2 = 1, x_3 = -2, x_4 = 1$.

2.1. **Operation Counts.** We count the number of operations required to solve the system $Ax = b$. Both the amount of time required to complete the calculations and the subsequent round-off error depend on the number of floating-point arithmetic operations needed to solve a routine problem. In general, the amount of time required to perform a multiplication or division on a computer is approximately the same and is considerably greater than that required to perform an addition or subtraction. The actual differences in execution time, however, depend on the particular computing system.

To demonstrate the counting operations for a given method, we will count the operations required to solve a typical linear system of n equations in n unknowns using Gauss elimination Algorithm. We will keep the count of the additions/subtractions separate from the count of the multiplications/divisions because of the time differential.

First step is to calculate multipliers $m_{ij} = \dfrac{a_{ij}}{a_{jj}}$. Then the replacement of the equation E_i by $(E_i - m_{ij}E_j)$ requires that m_{ij} be multiplied by each term in E_j and then each term of the resulting equation is subtracted from the corresponding term in E_i.

The following table states the operations count from going from A to upper triangular matrix U at each step $1, 2, \cdots, n-1$.

Step	Number of divisions	Number of multiplications	Number of additions/subtractions
1	$(n-1)$	$(n-1)^2$	$(n-1)^2$
2	$(n-2)$	$(n-2)^2$	$(n-2)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	2	4	4
$n-1$	1	1	1
Total:	$\dfrac{n(n-1)}{2}$	$\dfrac{n(n-1)(2n-1)}{6}$	$\dfrac{n(n-1)(2n-1)}{6}$

Therefore total number of additions/subtractions from A to U are $\dfrac{n(n-1)(2n-1)}{6}$ (**I**).

Total number of multiplications/divisions are $\dfrac{n(n-1)}{2} + \dfrac{n(n-1)(2n-1)}{6} = \dfrac{n(n^2-1)}{3}$ (**II**).

Now we count the number of additions/subtractions and the number of multiplications/divisions for right hand side vector b. We write each b_i as $b_i - m_{ij}b_j$. We have:

Total number of additions/subtractions $(n-1)+(n-2)+\cdots+2+1 = \dfrac{n(n-1)}{2}$ (**III**).

Total number of multiplications/divisions $(n-1)+(n-2)+\cdots+2+1 = \dfrac{n(n-1)}{2}$ (**IV**).

Lastly we count the number of additions/subtractions and multiplications/divisions for finding the solutions from the back-substitution method. Recall that

$$x_n = \frac{b_n}{a_{nn}}.$$

For each $i = n-1, n-2, \cdots, 2, 1$, we have

$$x_i = \frac{b_i - \sum\limits_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}.$$

Therefore total number of additions/subtractions $0+1+\cdots+(n-1) = \dfrac{n(n-1)}{2}$ (**V**).

Total number of multiplications/divisions $1+2+\cdots+n = \dfrac{n(n+1)}{2}$ (**VI**).

Thus the total number of operations to obtain the solution of a system of n linear equations in n variables using Gaussian elimination is:

Additions/Subtractions (I+III+V):

$$\frac{n(n-1)(2n+5)}{6}.$$

Multiplications/Divisions (II+IV+VI):

$$\frac{n(n^2+3n-1)}{3}.$$

For large n, the total number of multiplications and divisions is approximately $n^3/3$, as is the total number of additions and subtractions. Thus the amount of computation and the time required increases with n in proportion to n^3, as shown in Table.

n	Multiplications/Divisions	Additions/Subtractions
3	17	11
10	430	375
50	44,150	42,875
100	343,300	338,250

3. The LU Factorization

When we use matrix multiplication, another meaning can be given to the Gauss elimination. The matrix A can be factored into the product of the two triangular matrices.

Let $AX = b$ is the system to be solved, A is $n \times n$ coefficient matrix. The linear system can be reduced to the upper triangular system $UX = g$ with

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Here $u_{ij} = a_{ij}$. Introduce an auxiliary lower triangular matrix L based on the multipliers m_{ij} as following

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \\ m_{n,1} & 0 & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

Theorem 3.1. *Let A be a non-singular matrix and let L and U be defined as above. If U is produced without pivoting then*

$$LU = A.$$

This is called LU factorization of A.

We can use Gaussian elimination to solve a system by LU decomposition. Suppose that A has been factored into the triangular form $A = LU$, where L is lower triangular and U is upper triangular. Then we can solve for x more easily by using a two-step process.
First we let $y = Ux$ and solve the lower triangular system $Ly = b$ for y. Once y is known, the upper triangular system $Ux = y$ provide the solution x. We can check that total number of operations are same as Gauss elimination.

Example 7. *We require to solve the following system of linear equations using LU decomposition.*

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 3 \\ 3x_1 + 8x_2 + 14x_3 &= 13 \\ 2x_1 + 6x_2 + 13x_3 &= 4. \end{aligned}$$

Find the matrices L and U using Gauss elimination and using those values of L and U, solve the system of equations.

Sol. We first apply the Gaussian elimination on the matrix A and collect the multipliers m_{21}, m_{31}, and m_{32}.
We have

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Multipliers are $m_{21} = 3$, $m_{31} = 2$.
$E_2 \to E_2 - 3E_1$ and $E_3 \to E_3 - 2E_1$.

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Multiplier is $m_{32} = 2/2 = 1$ and we perform $E_3 \to E_3 - E_2$.

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

We observe that $m_{21} = 3$, $m_{31} = 2$, and $m_{32} = 1$. Therefore,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = LU$$

Therefore,

$$Ax = b \implies LUx = b.$$

Assuming $Ux = y$, we obtain,

$$Ly = b$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}.$$

Using forward substitution, we obtain $y_1 = 3$, $y_2 = 4$, and $y_3 = -6$. Now

$$Ux = y \implies \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}.$$

Now, using the backward substitution process, we obtain the final solution as $x_3 = -2$, $x_2 = 4$, and $x_1 = 3$.

Example 8. *(a) Determine the LU factorization for matrix A in the linear system $Ax = b$, where*

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \textit{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}. \tag{3.1}$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 + 3x_4 &= 8 \\ 2x_1 + x_2 - x_3 + x_4 &= 7 \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7 \end{aligned}$$

.

Sol. (a) We take the coefficient matrix and apply Gauss elimination.
Multipliers are $m_{21} = 2$, $m_{31} = 3$, and $m_{41} = -1$.
Sequence of operations $E_2 \to E_2 - 2E_1,\ E_3 \to E_3 - 3E_1,\ E_4 \to E_4 - (-1)E_1$.

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{bmatrix}.$$

Multipliers are $m_{32} = 4$ and $m_{42} = -3$.
$E_3 \to E_3 - 4E_2,\ E_4 \to E_4 - (-3)E_2$.

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}.$$

The multipliers m_{ij} and the upper triangular matrix produce the following factorization

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.$$

(b)

$$Ax = (LU)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

We first introduce the substitution $y = Ux$. Then $b = L(Ux) = Ly$. That is,

$$Ly = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

This system is solved for y by a simple forward-substitution process:

$$y_1 = 8, \quad y_2 = -9, \quad y_3 = 26, \quad y_4 = -26.$$

We then solve $Ux = y$ for x, the solution of the original system; that is,,

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}$$

Using backward substitution we obtain $x_4 = 2,\ x_3 = 0,\ x_2 = -1,\ x_1 = 3$.

Example 9. *Show that the LU factorization algorithm requires*

a. $\frac{1}{3}n^3 - \frac{1}{3}n$ *multiplications/divisions and* $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ *additions/subtractions.*

b. *Show that solving $Ly = b$, where L is a lower-triangular matrix with $l_{ii} = 1$ for all i, requires*

$$\frac{1}{2}n^2 - \frac{1}{2}n \quad \text{multiplications/divisions and} \quad \frac{1}{2}n^2 - \frac{1}{2}n \quad \text{additions/subtractions.}$$

c. *Show that solving $Ax = b$ by first factoring A into $A = LU$ and then solving $Ly = b$ and $Ux = y$ requires the same number of operations as the Gaussian elimination algorithm.*

Sol.

a. We have already counted the mathematical operation in detail in Gauss elimination. Here we provide the same for LU factorization.

We found total number of additions/subtractions from A to U are $\frac{n(n-1)(2n-1)}{6} = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ and total number of multiplications/divisions are $\frac{n(n-1)}{2} + \frac{n(n-1)(2n-1)}{6} = \frac{n(n^2-1)}{3} = \frac{1}{3}n^3 - \frac{1}{3}n.$

These counts remains same to factorize the matrix A in to L and U.

b. Solving $Ly = b$, where L is a lower-triangular matrix with $l_{ii} = 1$ for all i, requires total number of additions/subtractions $0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}$.

Total number of multiplications/divisions $0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$.

Please note that these operations can be counted in similar manner as we discussed for back substitution. As l_{ii} is always 1, so this will reduce one division at each step. These counts are same as for b.

c. Finally the counts used in b are same as required to solve $Ly = b$. Therefore, total counts remains same for LU decomposition and simple Gauss elimination.

Exercises

(1) Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = -1,\ x_2 = 1,\ x_3 = 3$.)

(a)

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11. \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1 \\ x_1 + 4x_2 + 2x_3 &= 9. \end{aligned}$$

(2) Using the four-digit arithmetic solve the following system of equations by Gaussian elimination with partial pivoting

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000. \end{aligned}$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!

(3) Use the Gaussian elimination algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

(a)

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 &= 3. \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3. \end{aligned}$$

(4) Use Gaussian elimination with scaled pivoting and three-digit chopping arithmetic to solve the following linear system and compare the approximations to the actual solution $[0, 10, 1/7]^T$.

$$\begin{aligned} 3.03x_1 - 12.1x_2 + 14x_3 &= -119 \\ -3.03x_1 + 12.1x_2 - 7x_3 &= 120 \\ 6.11x_1 - 14.2x_2 + 21x_3 &= -139. \end{aligned}$$

(5) Suppose that

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 6x_2 + 8x_3 &= 5 \\ 6x_1 + \alpha x_2 + 10x_3 &= 5, \end{aligned}$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

(a) $\alpha = 6$.
(b) $\alpha = 9$.
(c) $\alpha = -3$.

(6) Use the LU factorization to solve the following linear system:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

(7) We require to solve the following system of linear equations using LU decomposition.

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ x_1 + 2x_2 - 2x_3 &= 2 \\ -2x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Find the matrices L and U using Gauss elimination. Using those values of L and U, solve the given system of equations.

Appendix A. Algorithms

Algorithm (Gauss Elimination)
Input: number of unknowns and equations n; matrix $A = [a_{ij}]$, where $1 \le i \le n$ and $1 \le j \le n$ and column vector $b = [b_i]$, where $1 \le i \le n$.
Output: solution $x_1, x_2, \cdots, x_n$ or message that the linear system has no unique solution.
Step 1: For $j = 1, \cdots, n-1$ do Steps 2-4. (Elimination process.)
Step 2: Let p be the smallest integer with $j \le p \le n$ and $a_{pj} \neq 0$.
If no integer p can be found
then OUTPUT ('no unique solution exists');
STOP.
Step 3: If $p \neq j$ then perform $(E_p) \leftrightarrow (E_j)$.
Step 4: For $i = j+1, \cdots, n$ do Steps 5 and 6.
Step 5: Set $m_{ij} = a_{ij}/a_{jj}$.
Step 6: Perform $(E_i - m_{ij}E_j) \to (E_i),\ b_i - m_{ij}b_j \to b_i$;
Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
STOP.
Step 8: Set $x_n = b_n/a_{nn}$.(Start backward substitution.)
Step 9: For $i = n-1, \cdots, 1$ set $x_i = [b_i - \sum\limits_{j=i+1}^{n} a_{ij}x_j]/a_{ii}$.
Step 10: OUTPUT $(x_1, \cdots, x_n)$; (Procedure completed successfully.)
STOP.
Algorithm (LU Factorization)
We write $A = LU$ and procedure is similar to Gauss elimination.

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