School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

UMA007 : Numerical Analysis
Assignment 5
Iterative Techniques in Matrix Algebra

1. Find l_∞ and l_2 norms of the vectors.

(a) $x = (3, -4, 0, \frac{3}{2})^t$.

(b) $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.

2. The following linear system $Ax = b$ have x as the actual solution and $\bar{x}$ as an approximate solution. Compute $\|x - \bar{x}\|_\infty$ and $\|A\bar{x} - b\|_\infty$. Also compute $\|A\|_\infty$.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 3x_2 + 4x_3 &= -1 \\ 3x_1 + 4x_2 + 6x_3 &= 2, \\ x &= (0, -7, 5)^t \\ \bar{x} &= (-0.2, -7.5, 5.4)^t. \end{aligned}$$

3. Find the first two iterations of Jacobi and Gauss-Seidel using $x^{(0)} = 0$.

$$\begin{aligned} 4.63x_1 - 1.21x_2 + 3.22x_3 &= 2.22 \\ -3.07x_1 + 5.48x_2 + 2.11x_3 &= -3.17 \\ 1.26x_1 + 3.11x_2 + 4.57x_3 &= 5.11. \end{aligned}$$

4. The linear system

$$\begin{aligned} x_1 - x_3 &= 0.2 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425 \\ x_1 - \frac{1}{2}x_2 + x_3 &= 2 \end{aligned}$$

has the solution $(0.9, -0.8, 0.7)^T$.

(a) Is the coefficient matrix strictly diagonally dominant?

(b) Compute the spectral radius of the Gauss-Seidel iteration matrix.

(c) Perform four iterations of the Gauss-Seidel iterative method to approximate the solution.

(d) What happens in part (c) when the first equation in the system is changed to $x_1 - 2x_3 = 0.2$?

5. Check whether you can apply the Jacobi and Gauss-Seidel iterative techniques to solve the following linear system.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= -1 \\ 3x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 1. \end{aligned}$$

6. Find the first two iterations of the SOR method with $\omega = 1.1$ for the following linear system, using $x^{(0)} = 0$.

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1. \end{aligned}$$

7. Compute the condition numbers of the following matrices relative to $\|.\|_\infty$.

(a) $\begin{bmatrix} 3.9 & 1.6 \\ 6.8 & 2.9 \end{bmatrix}$

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(b) $\begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}$.

8. Use Gaussian elimination and three-digit rounding arithmetic to approximate the solutions to the following linear systems. Then use one iteration of iterative refinement to improve the approximation, and compare the approximations to the actual solutions.

(a)

$$\begin{aligned} 0.03x_1 + 58.9x_2 &= 59.2 \\ 5.31x_1 - 6.10x_2 &= 47.0. \end{aligned}$$

Actual solution $(10, 1)^t$.

(b)

$$\begin{aligned} 3.3330x_1 + 15920x_2 + 10.333x_3 &= 7953 \\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 0.965 \\ -1.5611x_1 + 5.1792x_2 - 1.6855x_3 &= 2.714. \end{aligned}$$

Actual solution $(1, 0.5, -1)^t$.

9. The linear system $Ax = b$ given by

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$$

has solution $(1, 1)^t$. Use seven-digit rounding arithmetic to find the solution of the perturbed system

$$\begin{bmatrix} 1 & 2 \\ 1.000011 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.00001 \\ 3.00003 \end{bmatrix}$$

Is matrix A ill-conditioned? What does this say about the linear system?

10. Determine the largest eigenvalue and the corresponding eigenvector of the following matrix correct to three decimals using the power method with $x^{(0)} = (-1, 2, 1)^t$ using the power method.

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

11. Use the inverse power method to approximate the most dominant eigenvalue of the matrix until a tolerance of 10^{-2} is achieved with $x^{(0)} = (1, -1, 2)^t$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

12. Find the eigenvalue of matrix nearest to 3

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

using inverse power method. Solve the resulting system of equations using LU factorization.