Lecture 6: Numerical Analysis (UMA011)

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Error Analysis: Algorithms and Stability

Example:

Write an algorithm to calculate the expression $e^x - \cos x$ when x is near 0 and rewrite it to be stable.

$x_0 : x = 0.001$

$x_1 : e^{x_0}$ ✓

$x_2 : \cos x_0$ ✓

$x_3 : x_1 - x_2$ ✓

$$C.N = \left|\frac{x_0 f'(x_0)}{f(x_0)}\right| = \left|\frac{x_0 \cancel{e^{x_0}}}{\cancel{e^{x_0}}}\right| < 1$$

$$e^x - \cos x = \left(\cancel{1} + x + \frac{x^2}{2!} - - - -\right) - \left(\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - - -\right)$$

$$= \left(x + \frac{2x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} - - - -\right)$$

Chapter 2: Solution of root-finding problem

$f(x) = 0$

$x - 3 = 0$

$x = 3$

$x^2 - 2x + 3 = 0$

$x = \frac{2 \pm \sqrt{4-12}}{2}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Analytic

$\downarrow$

Exact

$\cos x - 2 \sin x = 0$

$e^x - \cos x = 0$

$\log x - e^x = 0$

Numerical methods

$\downarrow$

approximation

$x = 5.3297$

$x^* = 5.330$

Root-finding problem

Methods for root-finding problem:

To find a solution of an equation $f(x) = 0$, we discuss the following four methods:

1. Bisection method
2. Fixed point Iteration
3. Newton method
4. Secant method

Iterative methods

✓ x_1

✓ x_2

✓ x_3

$\langle x_n \rangle \to x \to$ Exact root

$\downarrow$

approximation

Root-finding problem

→ to find interval in which root of $f(x)=0$ lie.

Intermediate Value Theorem (IVT)

$f(x)=0$

Let $f(x)$ be a continuous function on $[a, b]$ and $f(a) * f(b) < 0$, then $\exists$ a no. $c \in (a,b)$ s.t. $f(c)=0$ ie c is root of eqn $f(x)=0$

0

$f(0) = -ve$

$f(1) = -ve$

$f(2) = -ve$

$f(3) = +ve$

$f(-1) = +ve$

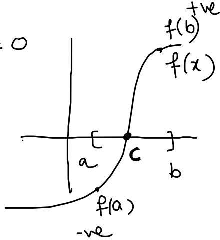
$[1, 2]$

$C = ?$

$f(1) = -ve$

$f(2) = +ve$ $-ve$

$f(3) = +ve$



Root-finding problem

Bisection method: Procedure

from IVT $[a,b]$ ✓

$f(a)$ -ve, $f(b)$ +ve (assume)

$x_1 = \frac{a+b}{2}$

$[a, x_1]$ ✗ and $[x_1, b]$ ✓

Check the sign of $f(x_1)$ = -ve (assume)

Root lie in $[x_1, b]$ ✓

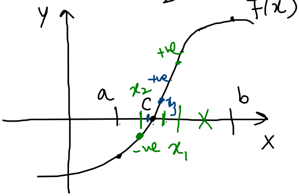
$x_2 = \frac{x_1+b}{2}$

$[x_1, x_2]$ ✓ and $[x_2, b]$ ✗

Check sign of $f(x_2)$ = +ve (assume)

Root lie in $[x_1, x_2]$

$x_3 = \frac{x_1+x_2}{2}$



10^{-2}

✓ x_1
✓ x_2
x_3
x_4 $n \to \infty$
⋮
x_n
↓
x

Root-finding problem

Bisection method: Stopping Criteria

$f(a) \approx 0$

$f(b) \approx 0$

$f(a) = -ve$

$f(b) = +ve$

$[a, b]$

$|a-b| < tol$

$[x_1, b]$

$|x_1 - b| < tol$

$|x_1 - x_2| < tol$

$|f(x_n)| \simeq tol$

$\langle x_n \rangle \longrightarrow x$

x_1

x_2

$|x_1 - x_2| \nless tol = 10^{-2}$

x_3

$|x_2 - x_3| \nless 10^{-2}$

x_4

$|x_3 - x_4| \nless 10^{-2}$

x_5

$|x_5 - x_4| < 10^{-2}$

$x_5 \to root$

$|x_n - x_{n-1}| < tol \to (given)$