

Lecture 27: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Power method:

Power method:

It is an iterative method which is used to determine the dominant eigenvalue i.e the eigenvalue with largest magnitude.

$A_{3 \times 3}$
 $1, -3, -5 \rightarrow$ e. values
largest in magnitude ✓

Let $\lambda \in \mathbb{R}$, A be any $n \times n$ matrix.

$$|A - \lambda I| = 0$$

$$p(\lambda) = 0$$

Roots of $p(\lambda)$ are eigenvalues i.e $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$

$$(A - \lambda_1 I) \check{x}_1 = 0$$

$$|A - \lambda_1 I| = 0$$

$$Bx = 0$$

$|B| \neq 0$, unique solⁿ, $x = 0$
 $|B| = 0$, it has infinite solⁿ

$$Ax = b$$

$$(A - \lambda_1 I) X_1 = 0$$

$$AX_1 - \lambda_1 I X_1 = 0$$

$$AX_1 = \lambda_1 X_1$$

$$\overset{\curvearrowright}{A}X = \overset{\curvearrowright}{\lambda}X$$

$$A\overset{\curvearrowright}{X} = \textcircled{\lambda}\overset{\curvearrowright}{X}$$

Power method:

Procedure of Power method:

Given matrix $A_{n \times n}$ and initial vector $x^0_{n \times 1}$

$$Ax^0_{n \times 1} = (y_1)_{n \times 1} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 \begin{bmatrix} y_1/\lambda_1 \\ y_2/\lambda_1 \\ \vdots \\ y_n/\lambda_1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}$$

$$Ax^{(0)} = \lambda_1 x^{(1)}$$

$$Ax^{(1)} = y_2 = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \lambda_2 \begin{bmatrix} y_1/\lambda_2 \\ 1 \\ y_n/\lambda_2 \end{bmatrix} = \lambda_2 x^{(2)}$$

$$5 \begin{bmatrix} 1/5 \\ 3/5 \\ 1 \\ 2/5 \\ 0 \end{bmatrix}$$

$$A x^{(2)} = y_3 = \lambda_3 x^{(3)}$$

$$\begin{array}{ccc} - & - & \cdot \\ - & & - \end{array}$$

$$A x^{(k)} = \underbrace{\lambda_{k+1}}_{\checkmark} x^{(k+1)} \checkmark$$

Stopping criteria

$$\text{if } |\lambda_{k+1} - \lambda_k| < \text{tol (given)}$$

$$\text{and } \|x^{(k+1)} - x^{(k)}\|_{\infty} < \text{tol (given)}$$

Power method:

Example:

Determine the largest eigenvalue and the corresponding eigenvector of the following matrix correct to one decimal using the power method with $x^{(0)} = (1, 0, 0)^t$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

let $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

Apply power method with $x^{(0)} = (1, 0, 0)^t$

10⁻¹

$$A X^{(0)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0.5 \\ -1 \\ 0 \end{bmatrix} = \lambda_1 X^{(1)}$$

$$A X^{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -5 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 0.3 \\ -1 \\ 0.2 \end{bmatrix} = \lambda_2 X^{(2)}$$

$$\|X^{(2)} - X^{(1)}\|_1 \approx 1.0$$

$$A X^{(2)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.3 \\ -1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 1.3 \\ -5 \\ 1.4 \end{bmatrix} = 5 \begin{bmatrix} 0.26 \\ -1 \\ 0.28 \end{bmatrix} = \lambda_3 X^{(3)}$$

$$\|X^{(3)} - X^{(2)}\|_1 \approx 1.0$$

$$A X^{(3)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.26 \\ -1 \\ 0.28 \end{bmatrix} = \begin{bmatrix} 1.26 \\ -5.08 \\ 1.56 \end{bmatrix} = 5.08 \begin{bmatrix} 0.248 \\ -1 \\ 0.307 \end{bmatrix} = \lambda_4 X^{(4)}$$

$$A x^{(4)} = \begin{bmatrix} 1.248 \\ -5.11 \\ 1.614 \end{bmatrix} = 5.11 \begin{bmatrix} 0.244 \\ -1 \\ 0.316 \end{bmatrix}$$

$$\|x^{(4)} - x^{(3)}\|_2 \leq 10^{-1}$$

$$4 \quad |\lambda_5 - \lambda_4| \leq 10^{-1}$$

Approximate e. value is 5.11 and corresp. e. vector is

$$\begin{bmatrix} 0.244 \\ -1 \\ 0.316 \end{bmatrix}$$

Ans.

Power method:

Inverse power method:

$$\begin{cases} \checkmark \\ AX = \lambda X \\ A^{-1}AX = \lambda A^{-1}X \\ IX = \lambda A^{-1}X \\ \frac{1}{\lambda}X = A^{-1}X \\ \Rightarrow A^{-1}X = \frac{1}{\lambda}X \end{cases}$$

To find the smallest eigen value of A ; $|A| \neq 0$
let eigenvalues of matrix $A_{n \times n}$ are $\lambda_1, \lambda_2 \dots \lambda_n$

then eigenvalues of matrix A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots \frac{1}{\lambda_n}$

Apply Power method on A^{-1} , then we
get the largest eigen value of A^{-1}

for e.g.

$$A \rightarrow \overset{\text{eigenvalues}}{\textcircled{3}} 4, 5$$

$$A^{-1} \rightarrow \sqrt{\frac{1}{3}}, \frac{1}{4}, \frac{1}{5}$$

$\sqrt{\frac{1}{3}}$ is the largest e. value of A^{-1}

\Rightarrow 3 is the smallest eigenvalue of A .

Inverse power method:

Example:

Perform first four iterations of inverse power method to approximate the smallest eigenvalue of the matrix by using an initial vector $x^{(0)} = (1, -1, 2)^t$

$$\text{let } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Solution ∴ find A^{-1}

$$A^{-1} = \begin{bmatrix} 0.75 & -0.25 & -0.25 \\ -0.25 & 0.75 & -0.25 \\ -0.25 & -0.25 & 0.75 \end{bmatrix}$$

Apply Power method on A^T with $x^{(0)} = (1, -1, 2)^t$

$$A^T x^{(0)} = \begin{bmatrix} 0.75 & -0.25 & -0.25 \\ -0.25 & 0.75 & -0.25 \\ -0.25 & -0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1.5 \\ 1.5 \end{bmatrix} = 1.5 \begin{bmatrix} 0.3333 \\ -1 \\ 1 \end{bmatrix} = \lambda_1 x^{(1)}$$

$$A^T x^{(1)} = \begin{bmatrix} 0.25 \\ -1.0833 \\ 1.5 \end{bmatrix} = 1.5 \begin{bmatrix} 0.1667 \\ -0.722 \\ 1 \end{bmatrix} = \lambda_2 x^{(2)}$$

$$A^T x^{(2)} = \begin{bmatrix} 0.0556 \\ -0.8333 \\ 0.8889 \end{bmatrix} = 0.8889 \begin{bmatrix} 0.0625 \\ -0.9375 \\ 1 \end{bmatrix} = \lambda_3 x^{(3)}$$

$$A^T x^{(3)} = \begin{bmatrix} 0.03125 \\ -0.96875 \\ 0.96875 \end{bmatrix} = 0.96875 \begin{bmatrix} 0.03226 \\ -1 \\ 1 \end{bmatrix} = \lambda_4 x^{(4)}$$

$\lambda_4 = 0.96875$ is the largest eigenvalue of A^T

$\Rightarrow \frac{1}{\lambda_4} = 1.03226$ is the smallest eigenvalue of A

and $x^{(4)} = \begin{bmatrix} 0.03226 \\ -1 \end{bmatrix}$ is the e-vector corresponding to $\frac{1}{\lambda_4}$

To check; eigenvalue and eigenvectors are correct or not

$$|A - \lambda I| = 0$$

$$(A - \lambda I) x = 0$$

Error bounds in solutions of system of linear equations:

Exercise:

- 1 Use power method and inverse power method to approximate the most dominant and smallest eigenvalues of the matrix until a tolerance of 10^{-1} is achieved with $x^{(0)} = (1, 1, 1)^t$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Exact eigen values are
 $2, 2-\sqrt{2}$
 $2+\sqrt{2}$ ✓