Lecture 40: Numerical Analysis (UMA011)

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Recall

<u>Quadrature</u> formulas

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

$x_i \downarrow$ equally spaced nodes

If n is given $\Rightarrow$ $x_i \to$ given

then $h = \frac{b-a}{n}$

In Comp. Trap.

$$\int_a^b f(x)\,dx \approx a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + \cdots a_n f(x_n)$$

$$a_0 = \frac{h}{2},\quad a_n = \frac{h}{2},\quad a_1 = a_2 = \cdots a_{n-1} = h$$

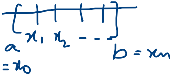
$$\Rightarrow \int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + f(x_n) + 2\left(f(x_1) \cdots f(x_n)\right)\right]$$

$\parallel^{ly}$ in comp. Simpson's, we can find a_i & x_i are equally spaced.

Numerical Quadrature:

Quadrature formulas:

1. The quadrature formula is called Newton-Cotes formula if all points are equally spaced.
2. All the Newton-Cotes formulas use values of the function at equally-spaced points
3. It can significantly decrease the accuracy of the approximation.



Gaussian Quadrature:

Gaussian quadrature chooses the best points for evaluation rather than equally spaced. So, Gaussian quadrature is more accurate.

In the numerical integration method $\int_a^b f(x)dx \approx \sum_{i=1}^{n} \lambda_i f(x_i)$, if both nodes and multipliers are unknown then method is called Gaussian quadrature.

λ_i, $\lambda_1, \lambda_2 - - - \lambda_n$ 2n unknowns.

x_i, $x_1, x_2 - - - x_n$

Numerical Quadrature:

Gaussian Quadrature:

The coefficients $\lambda_1, \lambda_2, \cdots, \lambda_n$ in the approximation formula are arbitrary, and the nodes $x_1, x_2, \cdots, x_n$ are restricted only by the fact that they must lie in $[a, b]$. This gives us $2n$ unknowns to choose.

Numerical Quadrature:

Gaussian Quadrature:

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We can obtain these unknowns by making the method exact for the class of polynomials of degree at most $2n - 1$ which gives $2n$ equations in these $2n$ unknowns.

$f(x) = 1, x, x^2 - - - - x^{2n-1}$

Numerical Quadrature:

Legendre polynomials:

we use Legendre polynomials, a collection $\{P_0(x), P_1(x), \cdots, P_n(x), \cdots\}$ with properties:

1. For each n, $P_n(x)$ is a monic polynomial of degree n.
2. $\int_{-1}^{1} P(x)P_n(x)dx = 0$, whenever $P(x)$ is a polynomial of degree less than n.
3. The first few polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - \frac{1}{3}$, $P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.
4. The roots of these polynomials are distinct, lie in the interval $(-1, 1)$ have a symmetry with respect to the origin, and, most importantly, are the correct choice for determining the parameters that give us the nodes and coefficients for our quadrature method.

Monic poly.

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots - a_1 x + a_0$, $a_n = 1$

highest degree co-eff is 1

i.e. $a_n = 1$

Roots of these poly.

linear $x = 0$

quadratic $x = \pm \frac{1}{\sqrt{3}}$

cubic $x\left(x^2 - \frac{3}{5}\right) = 0$

$x = 0, \pm\sqrt{\frac{3}{5}}$

Numerical Quadrature:

Gauss-Legendre Integration Methods::

The Gaussian quadrature formulas are derived for the interval $[-1, 1]$, and any interval $[a, b]$, can be transformed to $[-1, 1]$, by taking the transformation $x = \frac{b-a}{2} t + \frac{b+a}{2}$.

when $x = a$,

$$a = \frac{b-a}{2} t + \frac{b+a}{2}$$

$$a - \left(\frac{b+a}{2}\right) = \frac{b-a}{2} t$$

$$\frac{a-b}{2} = \frac{b-a}{2} t$$

$$t = -1$$

$\|^{ly}$ for $x = b$, $t = 1$

Numerical Quadrature:

Gauss-Legendre one-point formula:

For $n = 1$, the formula is given by $\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1)$.

The formula has two unknowns λ_1 and x_1. Make the method exact for $f(x) = 1, x$, we obtain

For $f(x) = 1$, we have $\int_{-1}^{1} 1\, dx = \lambda_1 .1 \Rightarrow \lambda_1 = 2$ — 1

For $f(x) = x$, we have $\int_{-1}^{1} x\, dx = \lambda_1 .x_1 \Rightarrow 2x_1 = 0.$ $\Rightarrow x_1 = 0$

Therefore, one point formula is given by $\int_{-1}^{1} f(x)dx = 2\, f(0)$.

Gauss-Legendre two-point formula:

For $n = 2$, the formula is given by
$\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1) + \lambda_2 f(x_2)$.

The formula has four unknowns $\lambda_1, \lambda_2, x_1$ and x_2. Make the method exact for $f(x) = 1, x, x^2, x^3$, we obtain

For $f(x) = 1$, $$\int_{-1}^{1} 1dx = \lambda_1 .1 + \lambda_2 .1 \Rightarrow \lambda_1 + \lambda_2 = 2. \tag{1}$$

For $f(x) = x$, $$\int_{-1}^{1} xdx = \lambda_1 .x_1 + \lambda_2 .x_2 \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 = 0. \tag{2}$$

For $f(x) = x^2$, $$\int_{-1}^{1} x^2dx = \lambda_1 .x_1^2 + \lambda_2 .x_2^2 \Rightarrow \lambda_1 x_1^2 + \lambda_2 x_2^2 = \frac{2}{3}. \tag{3}$$

Numerical Quadrature:

For $f(x) = x^3$, $\int_{-1}^{1} x^3 dx = \lambda_1 . x_1^3 + \lambda_2 . x_2^3 \Rightarrow \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0.$ (4)

Sub. (2) $* x_1^2$ from (4), we get

$$\cancel{\lambda_1 x_1^3} + \lambda_2 x_2^3 - \cancel{\lambda_1 x_1^3} - \lambda_2 x_2 x_1^2 = 0$$

$$\lambda_2 x_2 (x_2^2 - x_1^2) = 0$$

$$\lambda_2 x_2 (x_2 - x_1)(x_2 + x_1) = 0$$

either $\lambda_2 = 0$

if $\lambda_2 = 0$, then formula reduces to one-pt. formula

or $x_2 = 0$

from eqn (2) $\lambda_1 x_1 = 0$

either $\lambda_1 = 0$ or $x_1 = 0 \Rightarrow$

not possible

$$\int_{-1}^{1} f(x)\,dx = \lambda_1 f(0) + \lambda_2 f(0)$$

$= (\lambda_1 + \lambda_2) f(0) \rightarrow$ reduces to 1-pt. formula.

or $\quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$, from eqn 2 $(\lambda_1 + \lambda_2) x_1 = 0$

Again $x_1 \neq 0$, So $\lambda_1 + \lambda_2 = 0$

But from 1, $\lambda_1 + \lambda_2 = 2$

$$\lambda_1 + \lambda_2 \neq 0$$

$$\Rightarrow \quad x_1 \neq x_2$$

or.

$$x_1 + x_2 = 0$$

$$x_2 = -x_1$$

Put $x_2 = -x_1$ in 3

$$\lambda_1 x_1^2 + \lambda_2 (x_1^2) = \frac{2}{3}$$

$$(\lambda_1 + \lambda_2) x_1^2 = 2/3$$

from eqn 1 $\quad \lambda_1 + \lambda_2 = 2$

$$\Rightarrow \quad 2x_1^2 = \frac{2}{3}$$

$$x_1^2 = \frac{1}{3}$$

$$x_1 = \pm \frac{1}{\sqrt{3}}$$

$$\Rightarrow x_2 = \mp \frac{1}{\sqrt{3}}$$

from eqn 2 $\quad \lambda_1 x_1 + \lambda_2 x_2 = 0$

$$\frac{\lambda_1}{\sqrt{3}} - \frac{\lambda_2}{\sqrt{3}} = 0$$

$$\lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2$$

Numerical Quadrature:

from eqn 1 $\quad \lambda_1 + \lambda_2 = 2$

$$\lambda_1 + \lambda_1 = 2$$

$$2\lambda_1 = 2$$

$$\lambda_1 = 1$$

$\lambda_1 = 1, \; \lambda_2 = 1, \quad x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = \frac{-1}{\sqrt{3}}$

Therefore, two point formula is given by

$$\int_{-1}^{1} f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right).$$

Numerical Quadrature:

Gauss-Legendre three-point formula:

For $n = 3$, the formula is given by

$$\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3).$$

The method has six unknowns, make it exact for
$f(x) = 1, x, x^2, x^3, x^4, x^5$.
After solving the 6 equations in 6 unknowns, we get

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}\left(5f\left(\frac{\sqrt{3}}{\sqrt{5}}\right) + 8f(0) + 5f\left(\frac{-\sqrt{3}}{\sqrt{5}}\right)\right).$$

Numerical Quadrature:

Example:

Approximate the integral $\int_0^{\pi/4} (\cos x)^2\, dx$ using Gauss-Legendre 1, 2 and 3 point formula. Also compare with the exact value.

$x = \frac{b-a}{2} t + \frac{b+a}{2}$

$= \frac{\pi}{8} t + \frac{\pi}{8}$

Solution: Exact value $\int_0^{\pi/4} (\cos x)^2\, dx = \underline{0.6427}$

To apply Gauss-Legendre formulas, change interval $[0, \pi/4]$ to $[-1, 1]$ by taking $x = \frac{\pi}{8} t + \frac{\pi}{8} = \frac{\pi}{8}(t+1)$

$dx = \frac{\pi}{8} dt$

$$\Rightarrow \int_{-1}^{1} \left(\cos\left(\frac{\pi}{8}(t+1)\right)\right)^2 \frac{\pi}{8}\, dt$$

$$= \frac{\pi}{8}\int_{-1}^{1} \cos^2\left(\frac{\pi}{8}(t+1)\right) dt = \int_{-1}^{1} f(t)\, dt, \quad f(t) = \frac{\pi}{8}\cos^2\left(\frac{\pi}{8}(t+1)\right)$$

By one-pt. formula.

$$\int_{-1}^{1} f(t)\,dt = 2f(0) = 2 * \cos^2 \frac{\pi}{8} = \underline{0.3927}$$

By two pt. formula. $\int_{-1}^{1} f(t)\,dt = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right) = \underline{0.6423}$

By three pt. formula $\int_{-1}^{1} f(t)\,dt = \frac{1}{9}\left(5f\left(\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(-\sqrt{\frac{3}{5}}\right)\right) = \underline{0.6427}$

Numerical Quadrature:

Exercise:

1. Evaluate the integral

$$\int_{-1}^{1} e^{-x^2} \cos x \, dx$$

by using the Gauss-Legendre one, two and three point formulas.

2. Evaluate

$$I = \int_0^1 \frac{\sin x \, dx}{2 + x}$$

by subdividing the interval $[0, 1]$ into two equal parts and then by using Gauss-Legendre two point formula.

Hint for Que 2 $\quad I = \int_0^{1/2} \frac{\sin x}{2+x} dx + \int_{1/2}^{1} \frac{\sin x}{2+x} dx$