Solids and Structures
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Axially Loaded Indeterminate Structures

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Statically Indeterminate Axially Loaded Members

Statically Determinate and Statically Indeterminate Structures

- In many simple structures and mechanical systems constructed with axially loaded members, it is possible to determine the reactions at supports and the internal forces in the individual members by drawing the free body diagram and solving equilibrium equations of Statics.
- Such type of structures and systems are classified as ***Statically Determinate***.
- When the equations of equilibrium alone are not sufficient for the determination of axial forces in the members and the reactions at the supports, such type of systems are called ***Statically Indeterminate Structures***.
- Additional equations are required to supplement the equations of statics to determine the unknown forces.
- Usually, these equations are obtained from deformation conditions of the system and are known as **Compatibility Equations.**

General procedure to solve such problems can be organized into five steps:

Problem 1: A rigid horizontal bar *AB* hinged at *A* and supported by a 1.2 *m* long steel rod and a 2.4 *m* long bronze rod, both rigidly fixed at the upper ends (as shown in *Figure*). A load of 48 *kN* is applied at a point that is 3.2 *m* from the hinge point A. The areas of cross section of the steel and bronze rods are 850 mm^2 and 650 mm^2 respectively. **Find the stress in each rod.**

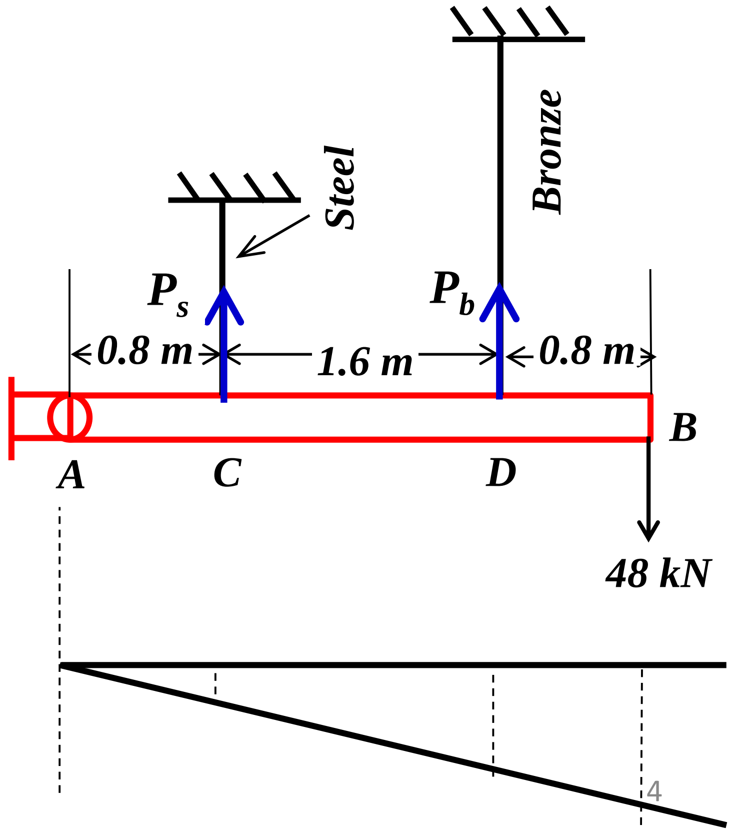
Take: E_s = 205 *GPa*; E_b = 82 *GPa*

1. Equilibrium Equations: The equations are expressed in terms of unknown axial forces are derived for the structure.

   Take moment about point *A*

$$P_s \times 800 + P_b \times 2400 = 48000 \times 3200$$

$$P_s + 3P_b = 19200 \qquad (a)$$

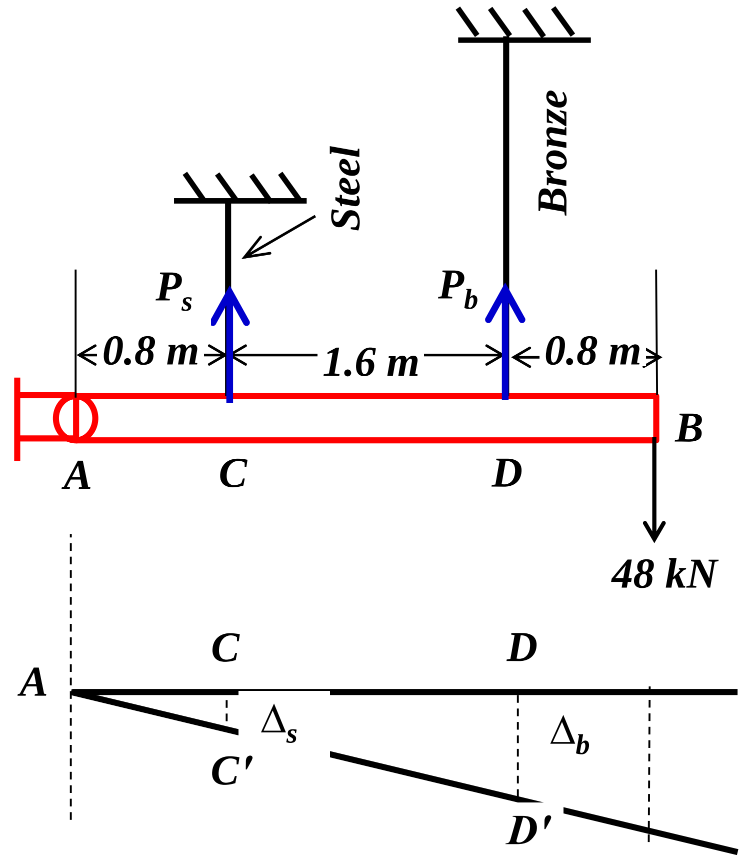


Statically Indeterminate Axially loaded Member

2. Geometry of Deformation: The geometry of the specific structure is evaluated to determine how the deformations of the axial members are related.

Consider two similar triangles ACC' and ADD'

$$\frac{\Delta_s}{800} = \frac{\Delta_b}{2400} \Rightarrow \Delta_s = \frac{1}{3}\Delta_b \qquad (b)$$



3. Force Deformation Relationship: The relation between the internal forces in an axial member and corresponding its elongation.

$$\Delta_s = \frac{P_s L_s}{A_s E_s} \; and \; \Delta_b = \frac{P_b L_b}{A_b E_b} \qquad (c)$$

4. Compatibility Equation: The force – displacement relations are put into geometry deformation equation to obtain an equation that is based on the structure's geometry.

   Now, substitute the value of Δ_s and Δ_b from *eq. (c)* into *eq. (b)*

$$\Delta_s = \frac{1}{3}\Delta_b \Rightarrow \frac{P_s L_s}{A_s E_s} = \frac{1}{3}\frac{P_b L_b}{A_b E_b} \qquad (d)$$

5. Solve the Equations: The equilibrium *eq. (a)* and compatibility *eq. (d)* are solved simultaneously to compute the unknown axial force.

$$\frac{P_s \times 1200}{850 \times 205000} = \frac{1}{3} \times \frac{P_b \times 2400}{650 \times 82000} \Rightarrow P_s = 2.179 P_b \qquad (e)$$

   On solving *eq. (a)* and *eq. (e)* simultaneously, we get P_b = *37073 N* and P_s = *80781 N*

$$\sigma_s = \frac{P_s}{A_s} = \frac{80781}{850} = 95.04\,MPa; \sigma_b = \frac{P_b}{A_b} = \frac{37073}{650} = 57.04\,MPa$$

Statically Indeterminate Axially loaded Members

Problem 2: Three equally spaced rods in the same vertical plane support a rigid bar *AB*. Two outer rods are of brass, each 600 *mm* long and of 25 *mm* in diameter. The central rod is of steel that is 800 *mm* long and 30 *mm* in diameter. Determine the forces in the rods due to an applied load of 120 *kN* through the midpoint of the bar. The bar remains horizontal after the application of load.

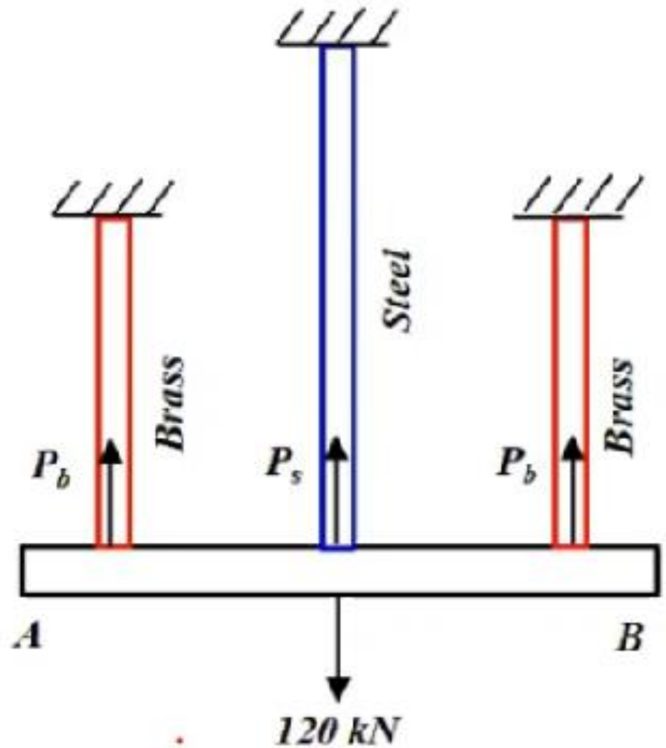
Take $E_s/E_b = 2$.

Steel Rod:

$l_s = 800\ mm$; $d_s = 30\ mm$; $A_s = (3.14/4)\times(d_s)^2$;

Bronze Rod:

$l_b = 600\ mm$; $d_b = 25\ mm$; $A_b = (3.14/4)\times(d_b)^2$;



1. Equilibrium Equation: The equations are expressed in terms of unknown axial forces are derived for the structure.

$$P_s + 2P_b = 120000 \qquad (a)$$

Numerical Problem

2. Geometry of Deformation: The geometry of the specific structure is evaluated to determine how the deformations of the axial members are related.

$$\Delta_s = \Delta_b \qquad \text{(b)}$$

3. Force Deformation Relationship: The relation between the internal forces in an axial member and corresponding its elongation.

$$\Delta_s = \frac{P_s L_s}{A_s E_s} \; and \; \Delta_b = \frac{P_b L_b}{A_b E_b} \qquad \text{(c)}$$

4. Compatibility Equation: The force – displacement relations are put into geometry deformation equation to obtain an equation that is based on the structure's geometry.

Now, substitute the value of Δ_s and Δ_b from *eq. (c)* into *eq. (b)*

$$\Delta_s = \Delta_b \Rightarrow \frac{P_s L_s}{A_s E_s} = \frac{P_b L_b}{A_b E_b} \qquad \text{(d)}$$

Numerical Problem

5. Solve the Equations: The equilibrium *eq. (a)* and compatibility *eq. (d)* are solved simultaneously to compute the unknown axial force.

$$\frac{P_s \times 800}{(3.14/4)\times(30)^2 \times 2E_b} = \frac{P_b \times 600}{(3.14/4)\times(25)^2 \times E_b} \Rightarrow P_b = 0.463 \times P_s \qquad \text{(e)}$$

On substituting the value of P_b from *eq. (e)* into *eq. (a)*

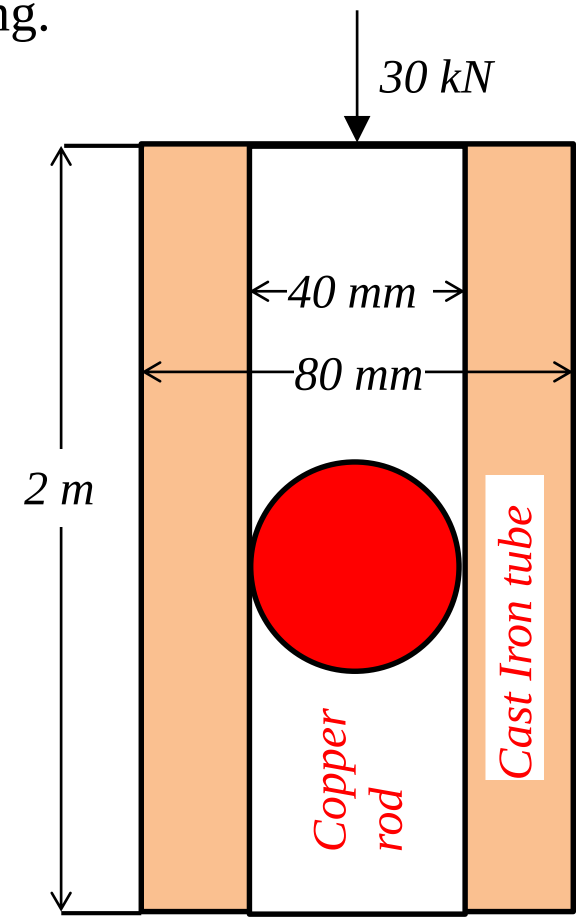
$$P_s + 2\times(0.463)\times P_s = 120000$$

we get P_b = *28.84 kN* and P_s = *62.3 kN*

Numerical Problem

Problem 3: A copper rod of 40 *mm* diameter is surrounded tightly by a cast-iron tube of 80 *mm* external diameter, the ends being firmly fastened together. When put to a compressive load of 30 *kN*, what load will be shared by each? Also determine the amount by which the compound bar shortens if it is 2 *m* long.

Take $E_{c.i} = 175\ GN/m^2$ and $E_c = 75\ GN/m^2$.



Copper Rod:

$l_{cu} = 2000\ mm$; ; $d_{cu} = 40\ mm$; $A_{cu} = (3.14/4)\times(40)^2$;

Cast Iron Rod:

$l_{ci} = 2000\ mm$ $(d_{ci})_o = 80\ mm$; $(d_{ci})_i = 40\ mm$

$A_{ci} = (3.14/4)\times((80)^2-(40)^2)$;

1. Equilibrium Equation: The equations are expressed in terms of unknown axial forces are derived for the structure.

$$P_{cu} + P_{ci} = 30000 \qquad \text{(a)}$$

Numerical Problem

2. Geometry of Deformation: The geometry of the specific structure is evaluated to determine how the deformations of the axial members are related.

$$\Delta_{cu} = \Delta_{ci} \qquad \text{(b)}$$

3. Force Deformation Relationship: The relation between the internal forces in an axial member and corresponding its elongation.

$$\Delta_{cu} = \frac{P_{cu} L_{cu}}{A_{cu} E_{cu}} \text{ and } \Delta_{ci} = \frac{P_{ci} L_{ci}}{A_{ci} E_{ci}} \qquad \text{(c)}$$

4. Compatibility Equation: The force – displacement relations are put into geometry deformation equation to obtain an equation that is based on the structure's geometry.

   Now, substitute the value of Δ_{cu} and Δ_{ci} from *eq. (c)* into *eq. (b)*

$$\Delta_{cu} = \Delta_{ci} \Rightarrow \frac{P_{cu} L_{cu}}{A_{cu} E_{cu}} = \frac{P_{ci} L_{ci}}{A_{ci} E_{ci}} \qquad (d)$$

Numerical Problem

5. Solve the Equations: The equilibrium *eq. (a)* and compatibility *eq. (d)* are solved simultaneously to compute the unknown axial force.

$$\frac{P_{cu}\times 2000}{(3.14/4)\times(40^2)\times 75000}=\frac{P_{ci}\times 2000}{(3.14/4)\times(80^2-40^2)\times 175000}$$

$$\Rightarrow P_{ci}=6.999\,P_{cu}$$

$$6.999P_{cu}+P_{cu}=30000\Rightarrow 7.999P_{cu}=30000$$

$$P_{cu}=\frac{30000}{7.999}=3.75\,kN\quad\Rightarrow P_{ci}=6.999\,P_{cu}\Rightarrow P_{ci}=26.24\,kN$$

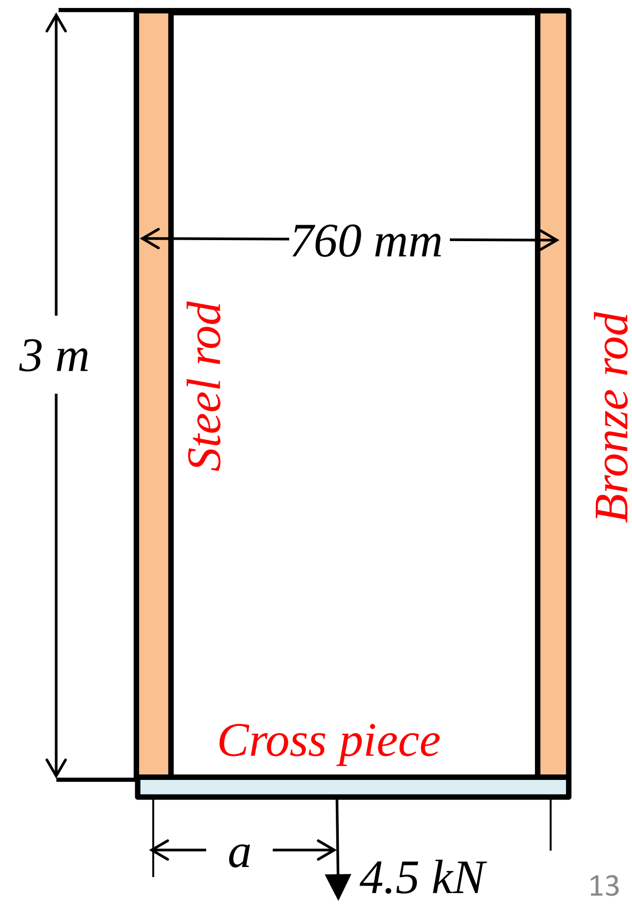
Amount by which bar shortens

$$\Delta_{cu}=\frac{P_{cu}L_{cu}}{A_{cu}E_{cu}}=\frac{3.75\times 10^3\times 2000}{(3.14/4)\times(40)^2\times 75000}=0.796\,mm$$

Numerical Problem

Problem 4: Two vertical rods, one of steel and other of bronze, are rigidly fastened at upper ends at a horizontal distance of 760 *mm* apart. Each rod is 3 *m* long and 25 *mm* in diameter. A horizontal cross-piece connects the lower ends of the bars. Where should a load of 4.5 *kN* be placed on the cross-piece so that it remains horizontal after being loaded? Determine the stress in each rod.

Take $E_s = 210\ GN/m^2$ and $E_b = 112.5\ GN/m^2$



Steel rod:

$L_s = 3000\ mm$; $d_s = 25\ mm$; $A_s = (3.14/4)\times(25)^2$;

Bronze rod:

$L_b = 3000\ mm$; $d_b = 25\ mm$; $A_b = (3.14/4)\times(25)^2$;

1. Equilibrium Equation: The equations are expressed in terms of unknown axial forces are derived for the structure.

$$P_s + P_b = 4500 \qquad \text{(a)}$$

Numerical Problem

2. Geometry of Deformation: The geometry of the specific structure is evaluated to determine how the deformations of the axial members are related.

$$\Delta_s = \Delta_b \qquad (b)$$

3. Force Deformation Relationship: The relation between the internal forces in an axial member and corresponding its elongation.

$$\Delta_s = \frac{P_s L_s}{A_s E_s} \text{ and } \Delta_b = \frac{P_b L_b}{A_b E_b} \qquad (c)$$

4. Compatibility Equation: The force – displacement relations are put into geometry deformation equation to obtain an equation that is based on the structure's geometry.

Now, substitute the value of Δ_s and Δ_b from *eq. (c)* into *eq. (b)*

$$\Delta_s = \Delta_b \Rightarrow \frac{P_s L_s}{A_s E_s} = \frac{P_b L_b}{A_b E_b} \qquad (d)$$

Numerical Problem

5. Solve the Equations: The equilibrium *eq. (a)* and compatibility *eq. (d)* are solved simultaneously to compute the unknown axial force.

$$\frac{P_s \times 3000}{(3.14/4)\times(25^2)\times 210000} = \frac{P_b \times 3000}{(3.14/4)\times(25^2)\times 112500}$$

$$\Rightarrow P_s = 1.87\, P_b$$

$$(1.87 \times P_b) + P_b = 4500 \Rightarrow 2.87 P_b = 4500 \Rightarrow P_b = 1566\, N$$

$$P_s = 1.87 \times 1566 = 2928 N$$

$$\sigma_s = \frac{P_s}{(3.14/4)\times 25^2} = \frac{2928}{490.62} = 5.96\, N/mm^2;$$

$$\sigma_b = \frac{P_b}{A_b} = \frac{1566}{(3.14/4)\times 25^2} = 3.19\, N/mm^2$$

Numerical Problem

Let a be the distance from the steel rod where the load P should be placed so that cross section remains horizontal after being loaded;

$$P_b \times 760 = 4500 \times a$$

$$\Rightarrow a = \frac{1566 \times 760}{4500} = 265\,mm$$

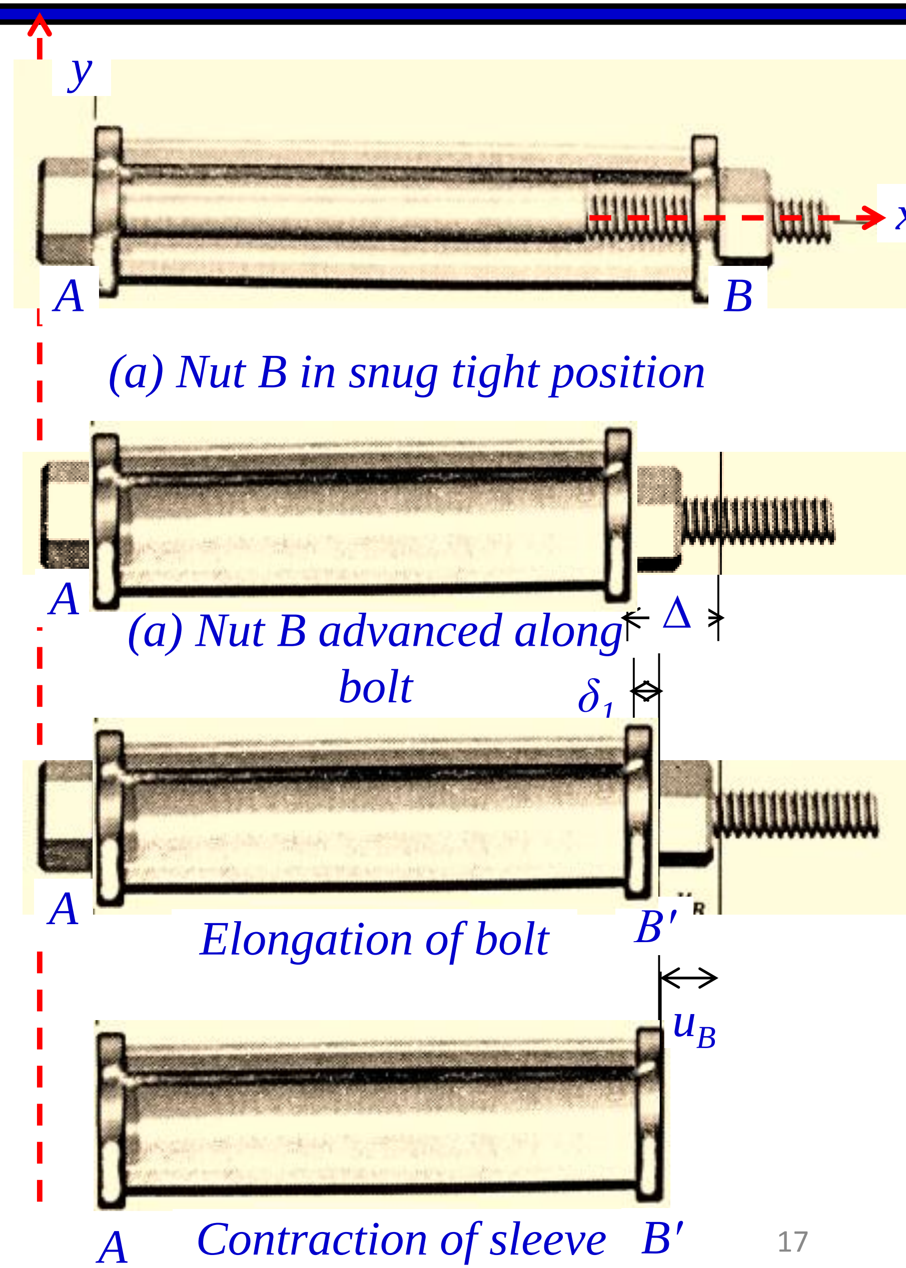
Bolt and Sleeve (Bolt and Nut Problem)

From the snug-tight position, the nut at *B* is rotated, which causes it to advance along the bolt towards *A*. The displacement of nut from its initial position at *B* denoted by symbol Δ.

Moving the nut towards *A* shortens the bolt, but it also compresses the sleeve. The resistance of the sleeve to this contraction creates a tension force in the bolt. The bolt elongates in response to tension force in it. The bolt deformation is denoted by δ_1.

The displacement u_B of the nut from its snug-tight position at *B* to its final position B′ can be expressed as:

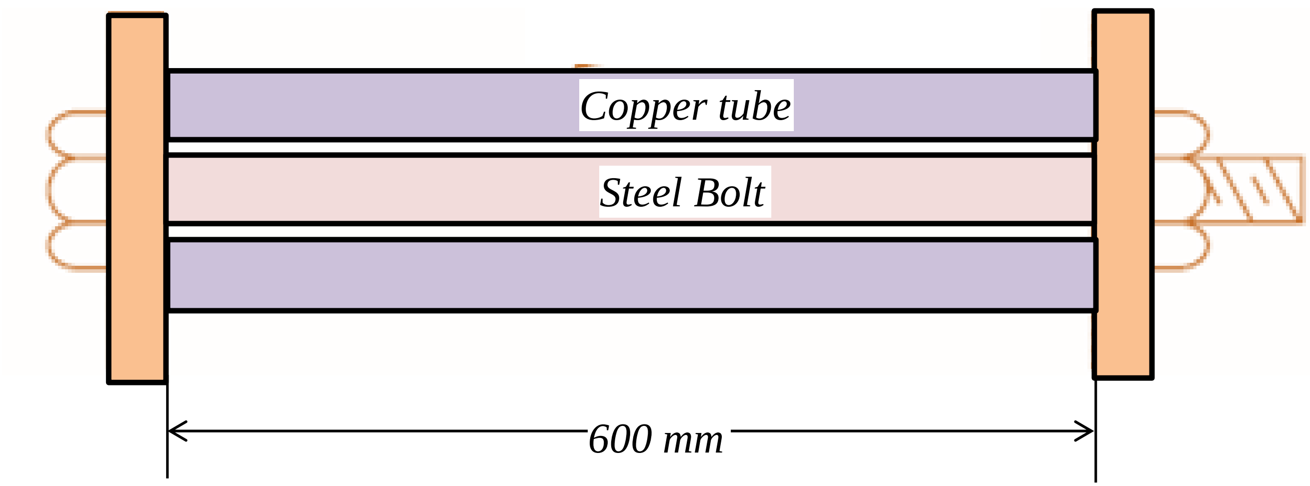
$$u_B = -\Delta + \delta_1$$



Numerical Problem

Problem 6: A steel bolt of 20 mm diameter passes centrally through a copper tube of internal diameter 28 mm and external diameter 40 mm. The length of whole assembly is 600 mm. After tight fitting of the assembly, the nut is over tightened by quarter of a turn. What are the stresses introduced in the bolt and tube, if pitch of nut is 2 mm?

Take $E_s = 2 \times 10^5$ N/mm^2 and $E_c = 1.2 \times 10^5$ N/mm^2



Numerical Problem

Solution: Let the force shared by bolt be P_s and that by tube be P_c. Since there is no external force, static equilibrium condition gives:

$$P_s + P_c = 0$$

Thus, the two forces are equal in magnitude but opposite in nature. Obviously bolt is in tension and tube is in compression.

Let the magnitude of force be P. Due to quarter turn of the nut, the nut advances by (¼) × pitch = (¼) × 2 = 0.5 mm

During this process bolt is extended and copper tube is shortened due to force P developed. Let Δ_s be extension of bolt and Δ_c shortening of copper tube. Final position of assembly be Δ, then

$$\Delta_s + \Delta_c = \Delta$$

$$\frac{P_s L_s}{A_s E_s} + \frac{P_c L_c}{A_c E_c} = 0.5$$

$$\frac{P_s \times 600}{(3.14/4) \times 20^2 \times 200000} + \frac{P_c \times 600}{(3.14/4) \times \left(40^2 - 28^2\right) \times 120000} = 0.5$$

Numerical Problem

$$P = 28816.8\, N$$

Stresses developed in steel bolt (tensile in nature)

$$\sigma_s = \frac{P_s}{A_s} = \frac{28816.8}{\frac{\pi}{4} \times 20^2} = 91.72\, N / mm^2$$

Stresses developed in copper tube (compressive in nature)

$$\sigma_c = \frac{P_c}{A_c} = \frac{28816.8}{\frac{\pi}{4} \times \left(40^2 - 28^2\right)} = 44.96\, N / mm^2$$

σ_c = 44.96 N/mm^2 (Compression)

σ_s = 91.72 N/mm^2 (Tension)

The Flexibility method and Stiffness method are used to analyze the statically indeterminate structures.

Flexibility Method:

Consider prismatic bar *AB* is attached at both ends to rigid supports and is axially loaded by the force *P* at an intermediate point *C* (as shown in Figure a).

The reactions R_A and R_B will develop at the ends of the bar. These reactions cannot be found statically alone,

$$R_A + R_B = P$$

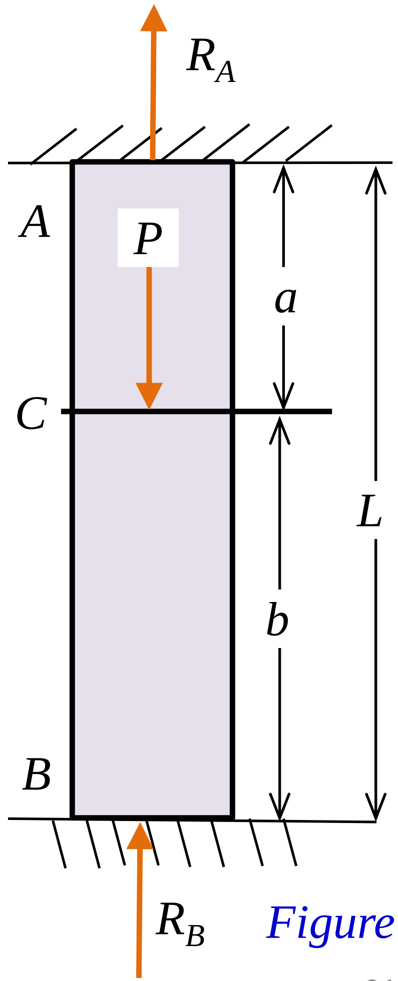


Figure a

Statically Indeterminate Structures

i) For analysis, firstly one of the unknown reactions is selected as redundant and then released from the structure by cutting through the bar and removing the support.

ii) When the unknown reaction R_A is removed from the structure, the effect is to release the support at end *A*, thereby producing the statically determinate and stable structure as shown in figure (b).

The structure that remains after releasing redundant is called the released structure, or the primary structure.

iii) **Total displacement due to applied forces and redundant is zero at 'A'.**

Displacement of point *A* in the released structure due to applied loads (P) acting downwards is given by

$$\delta_P = \frac{Pb}{EA}$$

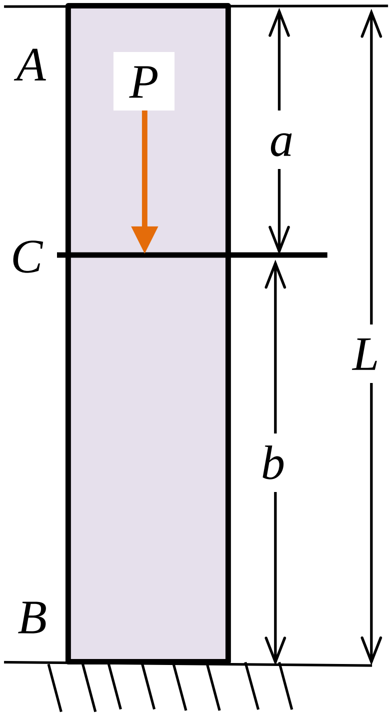


Figure b

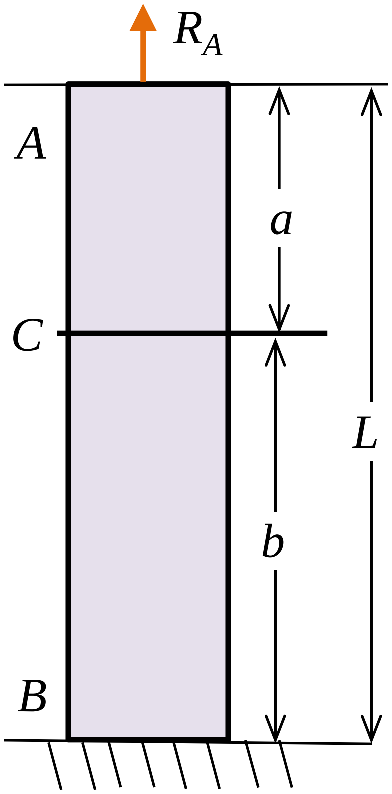


Figure c

Effect of the redundant R_A on the displacement of point A.

R_A is unknown quantity, and now visualized as the load acting on the released structure.

The upward displacement of point A due to R_A is

$$\delta_R = \frac{R_A L}{EA}$$

The final displacement δ of point A due to both P and R_A acting simultaneously is found by combining δ_P and δ_R. Thus, take downward displacement as positive,

$\delta = \delta_P - \delta_R$

But the actual displacement δ of point A is equal to zero.

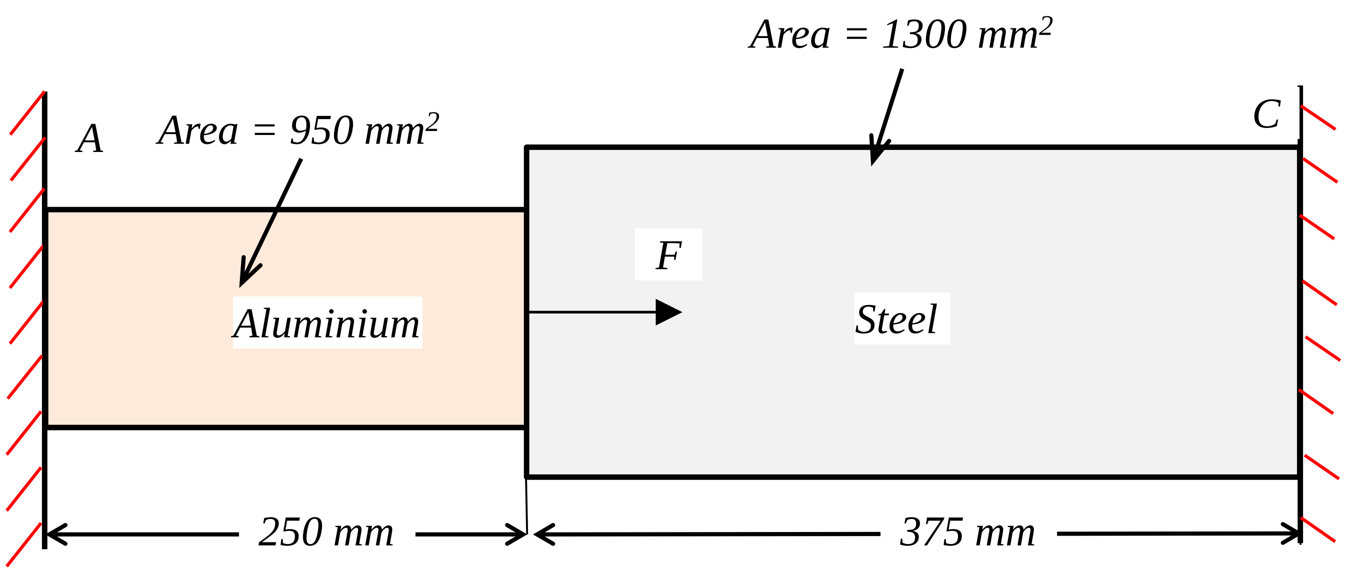
$$\frac{R_A L}{EA} = \frac{Pb}{EA} \Rightarrow R_A = \frac{Pb}{L}$$

Thus, the redundant reaction has been calculated from an equation related to the displacements of the bar. Now, we can find the R_B from equilibrium.

Numerical Problem

Problem 7: A composite bar as shown in the figure is firmly attached to unyielding supports at the ends and is subjected to the axial load F. if the aluminium is stressed to 70 MPa, what is the stress in the steel?

Take E_s = 207 GN/m^2; E_{al} = 68 GN/m^2



Numerical Problem

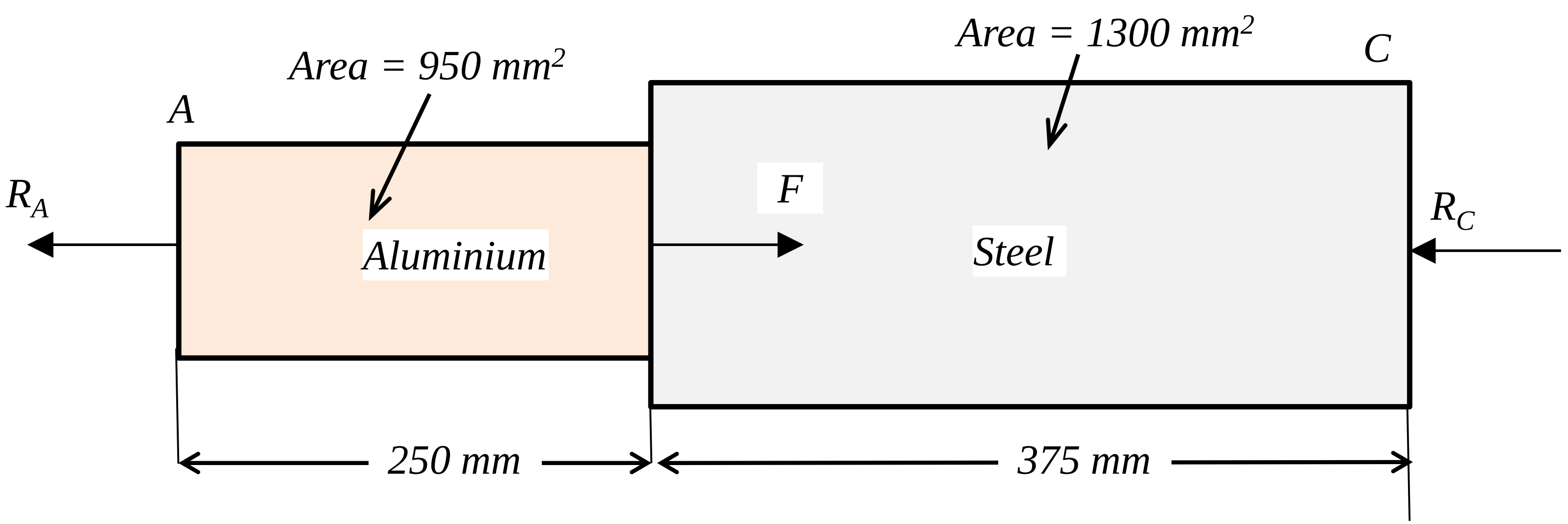
$$R_A + R_C = F, \quad \text{let } R_C \text{ is reduntant}$$

$$\delta_{C,F} = \frac{F \times 250}{950 \times 68000} = 3.87 \times 10^{-6}\, F \;\; mm \qquad (a)$$

$$\delta_{C,R_C} = \frac{R_C \times 250}{950 \times 68000} + \frac{R_C \times 375}{1300 \times 207000} = 5.26 \times 10^{-6}\, R_C \; mm \qquad (b)$$

$$\text{Compare } eq.(a) \text{ and } eq.(b)$$

$$3.87 \times 10^{-6}\, F = 5.26 \times 10^{-6}\, R_C \Rightarrow R_C = 0.736\, F \qquad (c)$$



Numerical Problem

$$R_A + R_C = F \Rightarrow R_A = F - 0.736F = 0.264F \qquad (d)$$

Since, aluminium is stressed to 70 MPa

$$\sigma_{al} = \frac{R_A}{950} \Rightarrow 70 = \frac{R_A}{950} \Rightarrow R_A = 66.5\,kN \qquad (e)$$

Now, substitute the value of R_A from *eq. (e)* into *eq. (b)*

$$F = \frac{R_A}{0.264} = \frac{66.5}{0.264} = 251.9\,kN \qquad (f)$$

Now, substitute the value of F from *eq. (f)* into *eq. (c)*

$$R_C = 0.736F = 0.736 \times 251.9 = 185.40\,kN$$

Stress in steel

$$\sigma_s = \frac{R_C}{1300} = \frac{185.40 \times 10^3}{1300} = 142.6\,MPa$$

Numerical Problem

Problem 5: A rigid bar *ABCD* pinned at B and connected to two vertical rods, is shown in Figure. Assume that the bar was initial horizontal and the rods were stress free. Determine the stress in each rod after the load $P = 500\ kN$ is applied.

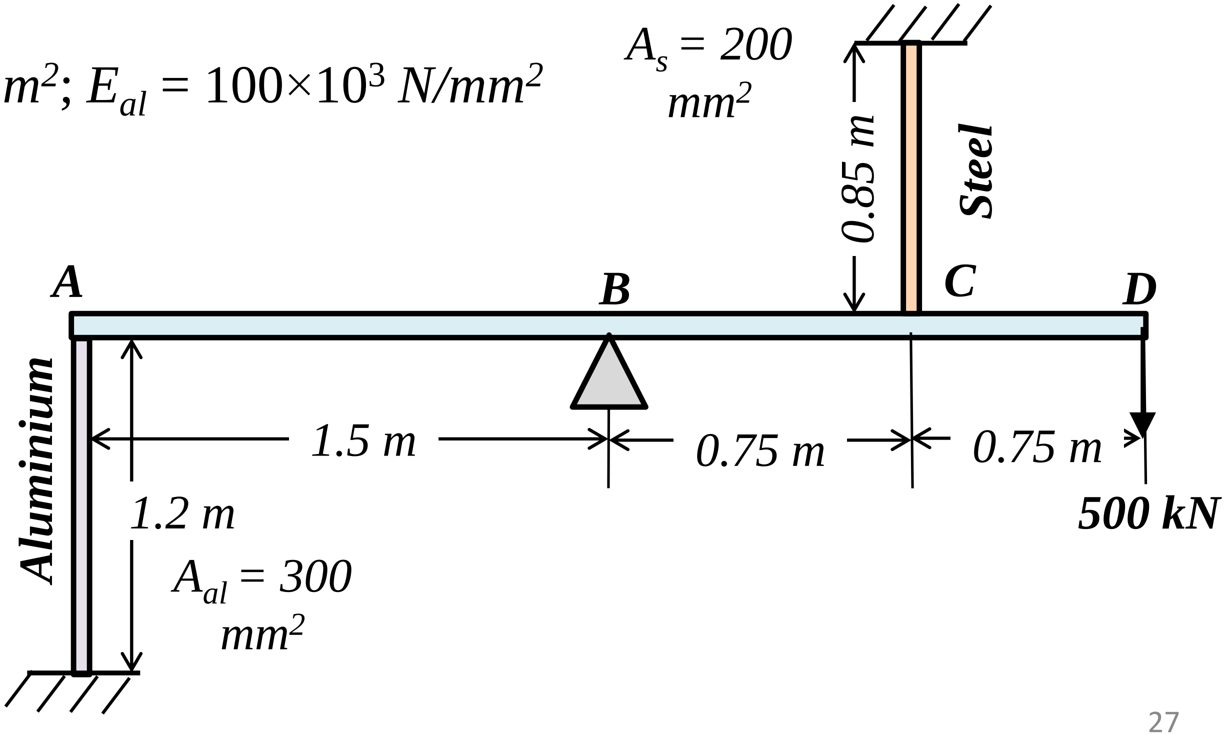
Take $E_s = 200\ GN/m^2$ and $E_{al} = 100\ GN/m^2$.

Steel rod:

$L_s = 850\ mm$; $A_s = 200\ mm^2$; $E_s = 200 \times 10^3\ N/mm^2$

Aluminium rod:

$L_{al} = 1200\ mm$; $A_{al} = 300\ mm^2$; $E_{al} = 100 \times 10^3\ N/mm^2$



Numerical Problem

1. Equilibrium Equation: The equations are expressed in terms of unknown axial forces are derived for the structure.

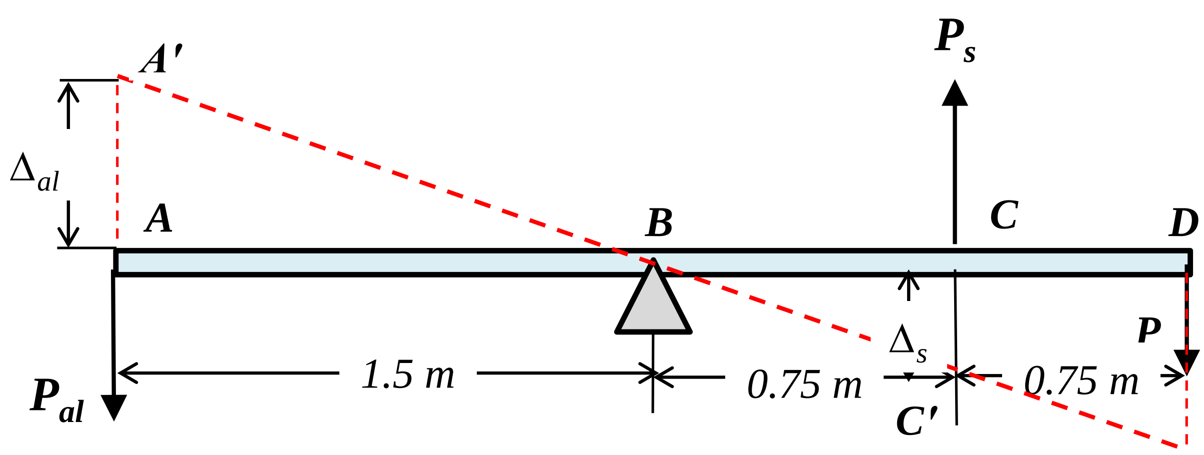
Take moment about point *B*.

$$P_{al} \times 1500 + P_s \times 750 = 1500 \times P \Rightarrow 2P_{al} + P_s = P \quad (a)$$

2. Geometry of Deformation: The geometry of the specific structure is evaluated to determine how the deformations of the axial members are related.

consider similar triangles *AA′B* and *CC′B*

$$\frac{\Delta_s}{750} = \frac{\Delta_{al}}{1500} \Rightarrow \Delta_{al} = 2\Delta_s \quad (b)$$



Numerical Problem

3. Force Deformation Relationship: The relation between the internal forces in an axial member and corresponding its elongation.

$$\Delta_s = \frac{P_s L_s}{A_s E_s} \; and \; \Delta_{al} = \frac{P_{al} L_{al}}{A_{al} E_{al}} \qquad \text{(c)}$$

4. Compatibility Equation: The force – displacement relations are put into geometry deformation equation to obtain an equation that is based on the structure's geometry.

Now, substitute the value of Δ_s and Δ_{al} from *eq. (c)* into *eq. (b)*

$$\Delta_{al} = 2\Delta_s \Rightarrow \frac{P_{al} L_{al}}{A_{al} E_{al}} = 2 \times \frac{P_s L_s}{A_s E_s} \qquad \text{(d)}$$

5. Solve the Equations: The equilibrium *eq. (a)* and compatibility *eq. (d)* are solved simultaneously to compute the unknown axial force.

Numerical Problem

$$\frac{P_{al}\times 1200}{300\times 100\times 10^3}=2\times\frac{P_s\times 850}{200\times 200\times 10^3}\Rightarrow P_{al}=1.06P_s$$

$$2\times 1.06\times P_s+P_s=P\Rightarrow 3.12P_s=1000$$

$$P_s=320.5\,N$$

$$P_{al}=1.06\,P_s=1.06\times 320.5\,N=339.7\,N$$

Stress in steel rod:

$$\sigma_s=\frac{P_s}{A_s}=\frac{320.5}{200}=1.6\,N/mm^2=1.6\,MN/m^2$$

Stress in aluminium rod:

$$\sigma_{al}=\frac{P_{al}}{A_{al}}=\frac{339.7}{300}=1.13\,N/mm^2=1.13\,MN/m^2$$