Course: UMA 035 (Optimization Techniques)

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Optimal solution a LPP

A feasible solution of a LPP is said to be an optimal solution of the considered LPP if corresponding to the considered feasible solution, the value of the objective function of the considered LPP is

- **Maximum in case of a maximization problem**
- **Minimum in case of a minimization problem**

Methods for finding an optimal solution of a LPP

- **Algebraic method**
- **Graphical method (only for one and two variables)**
- **Simplex method**
- **Big-M method**
- **Two Phase method**
- **Dual Simplex method**

Algebraic method

Step 1 Find all the basic feasible solutions of the considered LPP.

Step 2 Find the value of the objective function of the considered LPP corresponding to each basic feasible solution.

Step 3 If the considered LPP is of maximization then find the maximum of all the values which are obtained in Step 2.

If the considered LPP is of minimization then find the minimum of all the values which are obtained in Step 2.

Step 4 All the basic feasible solutions, corresponding to which maximum/minimum occur, will be optimal solution for the considered LPP.

Step 5 If there exist more than one optimal solutions, say q optimal solutions, $(x_1, y_1, \ldots, z_1), (x_2, y_2, \ldots, z_2), \ldots, (x_q, y_q, \ldots, z_q)$.

First Optimal Solution	**Second Optimal Solution**	**…**	**qth Optimal Solution**
x_1	x_2		x_q
y_1	y_2		y_q
. . .	. . .		. . .
z_1	z_2		z_q

Then, there will exist infinite number of optimal solutions which can be obtained by varying $a_1, a_2, \ldots, a_q$ in the following optimal solution.

$a_1x_1 + a_2x_2 + \ldots + a_qx_q$
$a_1y_1 + a_2y_2 + \ldots + a_qy_q$
.   .   .
$a_1z_1 + a_2z_2 + \ldots + a_qz_q$

where,

- $0 <= a_1, a_2, \ldots, a_q <= 1$
- $a_1 + a_2 + \ldots, + a_q = 1$

Example:

Using algebraic method, find all the optimal solutions of the following LPP.

Maximize $(3x_1 - 2x_2)$

Subject to

$x_1 - 4x_2 <= 5,$

$2x_1 - 8x_2 >= 10,$

$x_1 >= 2,\ x_2 >= -8.$

Solution: As discussed in the last lecture,

- $y_1=0$ and $y_2=0$, $S_1= -29$ and $S_2= 58$ is a non-degenerate basic infeasible solution. (Case 1)

- $y_1=0$ and $y_2=29/4$, $S_1=0$ and $S_2= 0$ is a Degenerate basic feasible solution. (Case 2)

- $y_1=0$ and $y_2=29/4$, $S_1=0$ and $S_2= 0$ is a Degenerate basic feasible solution. (Case 3)

- $y_1 = -29$ and $y_2=0$, $S_1=0$ and $S_2= 0$ is a Degenerate basic infeasible solution. (Case 4)

- $y_1= -29$ and $y_2=0$, $S_1=0$ and $S_2= 0$ is a Degenerate basic infeasible solution. (Case 5)

- $y_1= a$ and $y_2= (29+a)/4$, $S_1=0$ and $S_2= 0$ is non-basic solution. (Case 6)

The basic feasible solutions are

- $y_1=0$ and $y_2=29/4$, $S_1=0$ and $S_2= 0$ is a Degenerate basic feasible solution. (Case 2)

- $y_1=0$ and $y_2=29/4$, $S_1=0$ and $S_2= 0$ is a Degenerate basic feasible solution. (Case 3)

Both are same

Unique basic feasible solution so it is optimal.

$y_1=0$ and $y_2=29/4$, $S_1=0$ and $S_2= 0$

Value of objective function corresponding to this basic feasible solution ($3y_1 - 2y_2+22$) is $3*0- 2*(29/4)+22=7.5$

Optimal solution for transformed problem is $y_1=0$ and $y_2=29/4$, $S_1=0$ and $S_2= 0$

$x_1 = y_1+2 =2$

$x_2 = y_2-8 =29/4 -8= -0.75$

Optimal solution for given problem is $x_1=2$ and $x_2 = -0.75$

Example:

Using algebraic method, find all the optimal solutions of the following LPP.

Maximize ($3x_1 - 2x_2+x_3$)

Subject to

$x_1 -4x_2 +x_3 <= 5$,

$x_1>=0$, $x_2>= 0$, $x_3>= 0$.

Solution:

Convert in Standard form

Maximize ($3x_1 - 2x_2+x_3$)

Subject to

$x_1 -4x_2 +x_3 +S_1 = 5$,

$x_1>=0$, $x_2>= 0$, $S_1>=0$

Number of equations = m= 1

Number of variables = n = 4

Extra variables = n–m=4–1=3

Need to assume three variables as 0 at a time.

Number of Cases = $^{n}C_{m}$= $^{4}C_{1}$=4

Case 1: x_1= x_2= x_3=0

x_1 –4x_2 +x_3 +S_1 = 5 implies 0–0+0+S_1=5

First solution: x_1= x_2= x_3=0, S_1=5

Unique solution (Basic solution)

Feasible solution

Basic feasible solution

Case 2: x_1= x_2= S_1=0

x_1 –4x_2 +x_3 +S_1 = 5 implies 0–0+ x_3+0=5

First solution: x_1= x_2= S_1=0, x_3=5

Unique solution (Basic solution)

Feasible solution

Basic feasible solution

Case 3: x_2= x_3= S_1=0

$x_1 - 4x_2 + x_3 + S_1 = 5$ implies $x_1 - 0 + 0 + 0 = 5$

First solution: $x_2 = x_3 = S_1 = 0,\ x_1 = 5$

Unique solution (Basic solution)

Feasible solution

Basic feasible solution

Case 4: $x_1 = x_3 = S_1 = 0$

$x_1 - 4x_2 + x_3 + S_1 = 5$ implies $0 - 4x_2 + 0 + 0 = 5$

First solution: $x_1 = x_3 = S_1 = 0,\ x_2 = -5/4$

Unique solution (Basic solution)

Infeasible solution

Basic infeasible solution

Three basic feasible solutions

- **$x_1 = x_2 = x_3 = 0,\ S_1 = 5$**
- **$x_1 = x_2 = S_1 = 0,\ x_3 = 5$**
- **$x_2 = x_3 = S_1 = 0,\ x_1 = 5$**

Value of objective function $3x_1 - 2x_2 + x_3$ corresponding to the

- **First basic feasible solution is 3*0 – 2*0+0 = 0**
- **Second basic feasible solution is 3*0 – 2*0+5 = 5**
- **Third basic feasible solution is 3*5 – 2*0+0 = 15**

Since, the problem is of maximization. So, we need to find maximum {0, 5, 15}.

Maximum value is 15 which is corresponding to the basic feasible solution

$x_2 = x_3 = S_1 = 0$, $x_1 = 5$

The optimal solution is $x_2 = x_3 = S_1 = 0$, $x_1 = 5$

The optimal value is 15

Remark: If the problem is

Minimize $(3x_1 - 2x_2 + x_3)$

Subject to

$x_1 - 4x_2 + x_3 <= 5$,

$x_1 >= 0$, $x_2 >= 0$, $x_3 >= 0$.

Then

The optimal solution is $x_2 = x_3 = x_1 = 0$, $S_1 = 5$

The optimal value is 0

Example:

Using algebraic method, find all the optimal solutions of the following LPP.

Maximize $(2x_1 - 8x_2 + 2x_3)$

Subject to

$x_1 - 4x_2 + x_3 <= 5,$

$x_1 >= 0, x_2 >= 0, x_3 >= 0.$

Solution:

Convert in Standard form

Maximize $(2x_1 - 8x_2 + 2x_3)$

Subject to

$x_1 - 4x_2 + x_3 + S_1 = 5,$

$x_1 >= 0, x_2 >= 0, S_1 >= 0$

Number of equations = m= 1

Number of variables = n = 4

Extra variables = n–m=4–1=3

Need to assume three variables as 0 at a time.

Number of Cases = $^{n}C_{m} = {}^{4}C_{1} = 4$

Case 1: $x_1 = x_2 = x_3 = 0$

$x_1 - 4x_2 + x_3 + S_1 = 5$ **implies** $0 - 0 + 0 + S_1 = 5$

First solution: $x_1 = x_2 = x_3 = 0$, $S_1 = 5$

Unique solution (Basic solution)

Feasible solution

Basic feasible solution

Case 2: $x_1 = x_2 = S_1 = 0$

$x_1 - 4x_2 + x_3 + S_1 = 5$ **implies** $0 - 0 + x_3 + 0 = 5$

First solution: $x_1 = x_2 = S_1 = 0$, $x_3 = 5$

Unique solution (Basic solution)

Feasible solution

Basic feasible solution

Case 3: $x_2 = x_3 = S_1 = 0$

$x_1 - 4x_2 + x_3 + S_1 = 5$ **implies** $x_1 - 0 + 0 + 0 = 5$

First solution: $x_2 = x_3 = S_1 = 0$, $x_1 = 5$

Unique solution (Basic solution)

Feasible solution

Basic feasible solution

Case 4: $x_1 = x_3 = S_1 = 0$

$x_1 - 4x_2 + x_3 + S_1 = 5$ **implies** $0 - 4x_2 + 0 + 0 = 5$

First solution: $x_1 = x_3 = S_1 = 0$, $x_2 = -5/4$

Unique solution (Basic solution)

Infeasible solution

Basic infeasible solution

Three basic feasible solutions

- $x_1 = x_2 = x_3 = 0, S_1 = 5$
- $x_1 = x_2 = S_1 = 0, x_3 = 5$
- $x_2 = x_3 = S_1 = 0, x_1 = 5$

Value of objective function $2x_1 - 8x_2 + 2x_3$ corresponding to the

- **First basic feasible solution is 2*0 – 8*0+2*0 = 0**
- **Second basic feasible solution is 2*0 – 8*0+2*5 = 10**
- **Third basic feasible solution is 2*5 – 8*0+2*0 = 10**

Since, the problem is of maximization. So, we need to find maximum {0, 10, 10}.

Maximum value is 10 which is corresponding to the basic feasible solutions

$x_2 = x_3 = S_1 = 0, x_1 = 5$ **and** $x_1 = x_2 = S_1 = 0, x_3 = 5$

Since, there are two optimal solutions. So, there will exist infinite number of optimal solutions.

First solution	Second solution	Alternate optimal solutions
$x_1=5$	$x_1=0$	$x_1^{new} = a_1*5 + a_2*0 = a_1*5$
$x_2=0$	$x_2=0$	$x_2^{new} = a_1*0 + a_2*0 = 0$
$x_3=0$	$x_3=5$	$x_3^{new} = a_1*0 + a_2*5 = a_2*5$

$S_1=0$	$S_1=0$	$S_1^{new} = a_1*0+ a_2*0 = 0$

where,

$0<=a_1<=1$, $0<=a_2<=1$ and $a_1+ a_2 =1$

Some alternate optimal solutions

	$a_1=0.5$, $a_2=0.5$	$a_1=0.3$, $a_2=0.7$
$x_1^{new} = a_1*5+ a_2*0 = a_1*5$	$x_1^{new} = 2.5$	$x_1^{new} = 1.5$
$x_2^{new} = a_1*0+ a_2*0 = 0$	$x_2^{new} = 0$	$x_2^{new} = 0$
$x_3^{new} = a_1*0+ a_2*5 = a_2*5$	$x_3^{new} = 2.5$	$x_3^{new} = 3.5$
$S_1^{new} = a_1*0+ a_2*0 = 0$	$S_1^{new} = 0$	$S_1^{new} = 0$