*For reference purpose only

SET THEORY

- Ms. Rajanpreet Kaur Chahal
Assistant Professor, CSE Dept
Thapar Institute of Engineering & Technology
rajanpreet@thapar.edu

Topics covered

- Definition of set
- Representation of set
- Equal sets, empty set, singleton set
- Venn diagrams
- Subsets, proper subsets
- Cardinality of a set
- Power set

Definition of a Set

- A *set* is defined as an unordered collection of objects. These objects are known as *elements* or *members* of a set.
- Let S be a set consisting of elements a, b and c. Then S can be written as:

  $S = \{a,b,c\}$

- We can also say that $a \in S, b \in S \ and\ c \in S$.
- Also, $d \notin S$ signifies that set S does not contain element d.

Representation of a Set

- There are two ways to represent a set:
  - Roster Method: All members of the set are listed between braces.
    - For example, $A = \{2,4,6,8,10\}$.

  - Set Builder Notation: Specify the property/properties that the elements of the set must satisfy/possess.
    - For example, $A = \{x | x \text{ is an even positive integer less than or equal to } 10\}$ or $A = \{x \in Z^+ | x \text{ is even and } x \leq 10\}$.

Commonly used sets

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

Equal Sets

- Two sets A and B are equal if and only if they have the same elements. This can be written as $A = B$.
- In other words, two sets A and B are equal if and only if $\forall x\,(x \in A \leftrightarrow x \in B)$.
  - $\forall x\,(x \in A \leftrightarrow x \in B)$ is equivalent to saying that $\forall x\,(x \in A \rightarrow x \in B)$ and $\forall x\,(x \in B \rightarrow x \in A)$
- For example, the sets $\{a,b,c\}$ and $\{c,b,a\}$ are equal because both contain the same elements. Note that the order of elements does not matter. Also, it does not matter if an element is listed more than once*. Hence, $\{a,a,b,b,b,c,c,c,c,c\}$ is same as the set $\{a,b,c\}$.

*except in case of multi-set.

Equal Sets

- Simply stated, to show that two sets *A* and *B* are equal, show that all the elements of set *A* are contained in set *B*, and all the elements of set *B* are contained in set *A*.
- For ex, set M = {2,4,6,8,10} and set N = {1,2,3,4,5,6,7,8,9,10}. Now, all the elements of set M belong to set N but all the elements of set N are not included in set M ($1 \in N$ but $1 \notin M$. Similarly, 3,5,7,9 belong to set N but not to set M). Hence, set M and N are not equal.

Empty Set

- It is a set that has no elements.
- Also known as null set.
- Represented as ∅ or { }.
- A set of elements having a certain property may turn out to be a null set. For example, a set consisting of positive integers greater than their squares is an empty set.

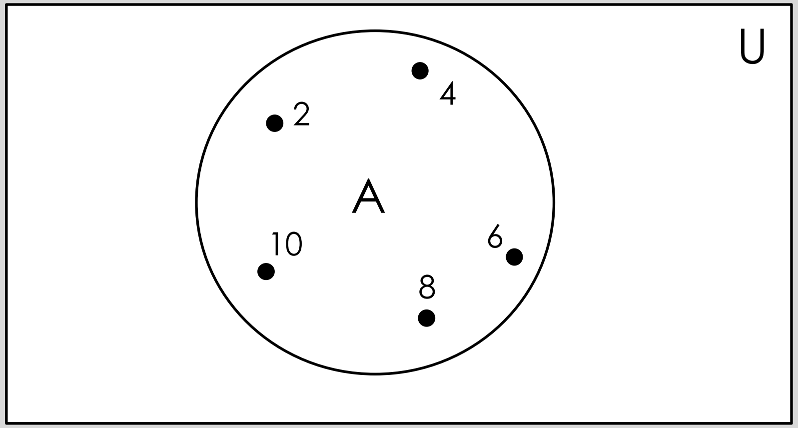
Singleton Set

- It is a set consisting of a single element.
- For example, $M = \{\ x \in N\ |x^3 = 125\ \}$ is a singleton set with 5 as its only element.

**Imp: $\{\emptyset\}$ is not the same as $\emptyset$. $\{\emptyset\}$ is a singleton set containing one element $\emptyset$, whereas $\emptyset$ is an empty set which does not contain any element.

Venn Diagrams

- Venn diagrams are used to represent sets graphically.
- The universal set U consists of all the elements under consideration. It is represented using a rectangle.
- Inside this rectangle, sets are represented using circles. Points can be used within the circles to represent particular elements of the set.

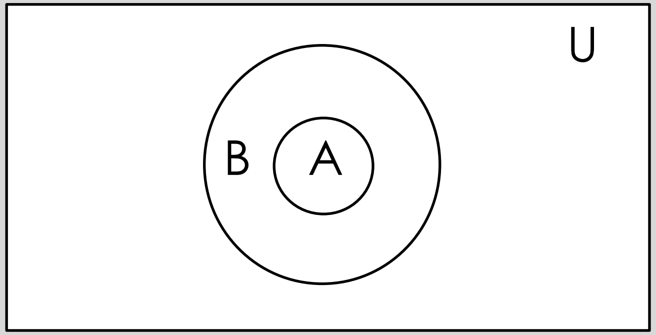


For ex, in the given diagram, U is the universal set consisting of all the positive integers; and

$$A = \{x \in Z^{+} \mid x \text{ is even and } x \leq 10\}.$$

Subsets

- Let A and B be two sets. We say that A is a subset of B **if and only if every element of *A* is also an element of *B***. This is written as $A \subseteq B$.
- In other words, A is a subset of B if and only if $\forall x\ (x \in \mathrm{A} \rightarrow x \in \mathrm{B})$ is true.
- To show that A is not a subset of B, find one element x such that $x \in \mathrm{A}$ but $x \notin B$.
- For ex, consider set $A = \{x \in Z^+ | x \text{ is even and } x \leq 10\}$ and $B = \{x \in Z^+ | x \leq 10\}$. We can say that $A \subseteq B$.
- For every set S, *i)* $\emptyset \subseteq S$ *ii)* $S \subseteq \mathrm{S}$.



Venn diagram showing A is a subset of B

Subsets

- Find all the subsets of the set $S = \{a,b,c\}$.

  Ans: The subsets of set S are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{b,c\}$, $\{a,c\}$, $\{a,b,c\}$.

- To show that two sets A and B are equal, we need to show that $A \subseteq B$ and $B \subseteq A$. In other words, $A = B$ if and only if $\forall x\,(x \in A \rightarrow x \in B)$ and $\forall x\,(x \in B \rightarrow x \in A)$ or if and only if

$$\forall x\,(x \in A \leftrightarrow x \in B)$$

- It is also possible for sets to have other sets as its elements.
  - For ex, $X = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and $Y = \{\, x \mid x \text{ is a subset of set } \{1,2\}\}$.
  - In the above example, $X = Y$.
  - Also, $\{1\} \in X$ but $1 \notin X$.

Set $C = \{\emptyset, \{1\}, \{2\}\}$ is not the same as $D = \{\emptyset, 1, 2\}$ because C contains sets $\{1\}$ and $\{2\}$ as its elements whereas D contains values 1 and 2 as its elements.

Subsets

- **Proper Subset:** Given two sets A and B, we say that A is a proper subset of B, denoted as $A \subset B$, if A is a subset of B but $A \neq B$. In other words, $A \subset B$ if and only if

$$\forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$$

- **Number of subsets of a set:** If a set S has n distinct elements, then it will have 2^n subsets. For ex, set S = {a,b,c} has 3 elements. Hence, number of subsets of S will be $2^3 = 8$.

Subsets

- Determine whether $O = \{1,3,5,7,9\}$ is a subset of $B = \{1,2,3,4,5,6,7,8,9,10\}$.

  Ans: Since every element of O is contained in set B, hence, O is a subset of B, i.e. $O \subseteq B$.

- Determine whether sets O and B are equal.

  Ans: For O and B to be equal, $O \subseteq B$ and $B \subseteq O$.

  $O \subseteq B$ as proved above. But B is not a subset of O because $2 \in B$ but $2 \notin O$ (Similarly, 4,6,8,10 belong to B but not to O).

  Hence, sets O and B are **not** equal.

Subsets

- Determine whether $O = \{1,3,5,7,9\}$ is a **proper** subset of $B = \{1,2,3,4,5,6,7,8,9,10\}$.

  Ans: O will be a proper subset of B if $O \subseteq B$ but $O \neq B$. Hence, O is a proper subset of B, i.e., $O \subset B$.

- ***Caveat: i) Empty set is a subset of every set.***

  ii) Every set is a subset of itself.

Well-defined sets

- A set is said to be well-defined if there is no ambiguity/confusion regarding whether an object is a member of the set or not, i.e., it is clear which objects will be members of the set and which will be not.
- For ex, set $C = \{black, white, green, purple\}$ is a well-defined set because it is clear which elements belong to the set.
- $D = \{x \mid x \text{ is a vowel}\}$ is also a well-defined set.
- On the other hand, $P = \{x|x \text{ is a tall boy in the class}\}$ is not a well-defined set since it is not clear what tall exactly means. Is a height of 5'7" considered tall ?

Cardinality of a Set/Size of a Set

- Let *S* be a set. If *S* contains *n* distinct elements where *n* is a non-negative integer, then we say that cardinality of set *S* is *n*. Also in this case, *S* is a finite set.
- Cardinality of set *S* can be written as |S| or *n(S)*.
- For ex, consider set $A = \{x \in Z^+ \mid x \text{ is even and } x \leq 10\}$. Here, |A|= 5.
- For null set, |∅| = 0.
- A set which is not finite is known as an infinite set. For ex, set of natural numbers.
- For infinite sets, their cardinality/size is ∞.

Power Set

- Let S be a set. The power set of S, denoted by $P(S)$, is defined as the set of all subsets of S.
- What is the power set of set $S = \{a,b,c\}$?

  Ans: $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

  Note: Null set and the set itself are always elements of the power set of a set.
- Find the power set of the null set.

  Ans: Null set has exactly one subset, i.e., itself. Hence $P(\emptyset) = \{\emptyset\}$.
- Find the power set of the set $\{\emptyset\}$.

  Ans: $P(\{\emptyset\}) = \{\emptyset,\{\emptyset\}\}$
- **If a set has n elements, then its power set has 2^n elements.**

Equivalent Sets

- Two sets A and B are equivalent if they have the same number of elements or same cardinality.
- For ex, $A = \{1,2,3\}$ and $B = \{a,b,c\}$ are equivalent because $n(A) = 3$ and $n(B) = 3$.
- However, $A = \{1,2,3\}$ and $C = \{1,2,3,4\}$ are not equivalent because $n(A) = 3$ and $n(C) = 4$.

Equal sets and equivalent sets

- Equal sets have **same elements** whereas equivalent sets **have same number of elements.**
- For ex, $A = \{1,2,3\}$ and $D = \{3,1,2\}$ are equal because both the sets contain same elements (order does not matter!). Sets A and D are also equivalent because $n(A) = 3$ and $n(D) = 3$.
- However, $A = \{1,2,3\}$ and $B = \{a,b,c\}$ are equivalent because $n(A) = 3$ and $n(B) = 3$ but they are not equal because they contain different elements.
- Also, $A = \{1,2,3\}$ and $C = \{1,2,3,4\}$ are neither equal nor equivalent because $n(A) = 3$ and $n(C) = 4$; and they have different elements.

Note: i) Equal sets are always equivalent.

ii) Equivalent sets may or may not be equal.

Exercise…

- Determine whether following pairs of sets are equal or equivalent or both:

  i) $A = \{x : x \in N \text{ and } x \leq 5\}$, $B = \{x : x \in W \text{ and } 1 \leq x \leq 5\}$ **Equal and equivalent**

  ii) $C = \{x$: *x is the colour in a rainbow*$\}$, $D = \{x$: *x is a day of the week*$\}$
  Equivalent but not equal

  iii) $D = \{2,3,4,5\}$, $E = \{5,3,4,2\}$ **Equal and equivalent**

  iv) $P = \{$*The set of letters in the word "ensure"*$\}$, $Q = \{$*The set of letters in the word "assure"*$\}$ **Equivalent but not equal**

Exercise…

- Determine whether the following pairs of sets are equal.

  i) $\{\{1\}\}, \{1,\{1\}\}$ **No!!**

  ii) $\emptyset, \{\emptyset\}$ **No!!**

  iii) $\{1,3,3,3,5,5,5,5,5\}, \{5,3,1\}$* **Yes!**

- Determine whether the following statements are true or false.

  i) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **True!**

  ii) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$ **False!!**

*exception in case of multi-set

Exercise…

- What is the cardinality of the following sets?

  i) $\{\{a\}\}$ **Ans: 1**

  ii) $\{\emptyset, \{\emptyset\}\}$ **Ans: 2**

  iii) $\{a,\{a\},\{a,\{a\}\}\}$ **Ans: 3**

- If a and *b* are distinct elements, then how many elements are there in the following sets?

  i) $P(\{a,b,\{a,b\}\})$ **Ans: 8**

  ii) $P(\{\emptyset,a,\{a\},\{\{a\}\}\})$ **Ans: 16**

  iii) $P(P(\emptyset))$ **Ans: 2**

Exercise…

- Find the number of subsets of $S = \{1,2,3,4\}$.

  Ans: Since S has 4 elements, therefore, number of subsets will be $2^4 = 16$.

- Find all the subsets of $S = \{1,2,3,4\}$.

  ***Ans:* $\emptyset$, {1}, {2}, {3}, {4}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4} , {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}, {1,2,3,4}.**

Exercise…

- Identify the well-defined sets from the following:

  i) All the colors in the rainbow. **Well-defined**

  ii) All the hardworking teachers in a school. **Not well-defined**

  iii) All the points that lie on a straight line. **Well-defined**

  iv) All the honest members in the family. **Not well-defined**

  v) All the letters in the word APPLE. **Well-defined**

  vi) All the prime numbers less than 50. **Well-defined**

Exercise…

- Identify finite and infinite sets from the following:

  i) $A = \{x : x \in N \text{ and } x \text{ is odd}\}$ **Infinite**

  ii) $B = \{x : x \in N \text{ and } x \text{ is a multiple of } 5\}$ **Infinite**

  iii) $C = \{x : x \in Z \text{ and } x^2 = 36\}$ **Finite**

  iv) $D = \{x : x \in R \text{ and } 5x - 8 = 0\}$ **Finite**

  v) $E = \{\text{The set of persons living in a home}\}$ **Finite**

References

- Rosen H. K., Discrete Mathematics and its Applications, McGraw Hill (2011) 7th ed.

Questions ?