: Lagrange Interpolating Polynomial
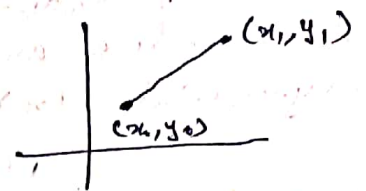
Line joining (x_0, y_0) and (x_1, y_1) is

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$\Rightarrow y = y_0 + \frac{(x-x_0)}{(x_1-x_0)} y_1 - \frac{(x-x_0)}{(x_1-x_0)} y_0$$

$$= \left[1 - \frac{(x-x_0)}{x_1-x_0}\right] y_0 + \left(\frac{x-x_0}{x_1-x_0}\right) y_1$$

$$= \left(\frac{x_1-x_0-x+x_0}{x_1-x_0}\right) y_0 + \left(\frac{x-x_0}{x_1-x_0}\right) y_1$$

$$= \left(\frac{x_1-x}{x_1-x_0}\right) y_0 + \left(\frac{x-x_0}{x_1-x_0}\right) y_1 = \left(\frac{x-x_1}{x_0-x_1}\right) y_0 + \left(\frac{x-x_0}{x_1-x_0}\right) y_1$$

$$= L_0(x) y_0 + L_1(x) y_1 = L_0(x) f(x_0) + L_1(x) f(x_1)$$

$\therefore P(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$ be the unique polynomial of degree at most one that passes through (x_0, y_0) and (x_1, y_1) where $L_0(x_0) = 1$, $L_0(x_1) = 0$, $L_1(x_0) = 0$, $L_1(x_1) = 1$ so that $P(x_0) = f(x_0) = y_0$ and $P(x_1) = f(x_1) = y_1$

Generalizing the concept, Consider apply the construction of a polynomial of degree at most n that passes through the $n+1$ points:

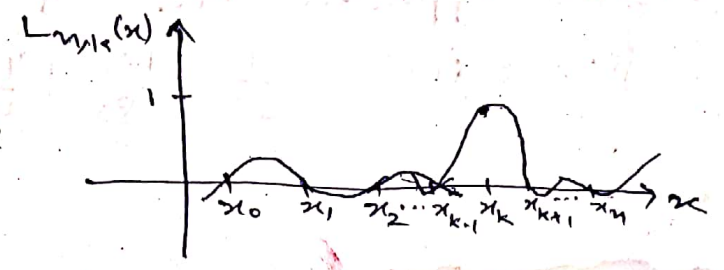$$(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$$

First construct, for each $k = 0, 1, 2, \cdots, n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ for $i \neq k$ and

$$L_{n,k}(x_k) = 1.$$
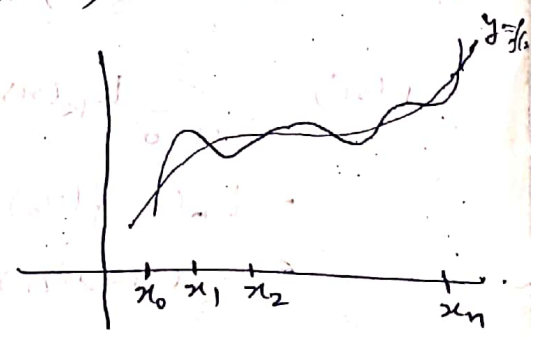
$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Once, $L_{n,k}(x)$ is known, we can define the nth Lagrange interpolating polynomial.

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Ex $\not\!p$(a) Use the numbers $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

(b) Use the polynomial to $\not\!f\!i$ approximate $f(3) = \frac{1}{3}$.

Sol. Three points $x_0, x_1, x_2 \Rightarrow$ polynomial of at most degree 2.

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75)$$

$$f(x_0) = f(2) = \frac{1}{2}, \quad f(x_1) = f(2.75) = \frac{1}{2.75} = \frac{100}{275} = \frac{4}{11}, \quad f(x_2) = f(4) = \frac{1}{4}$$

$$\therefore P(x) = \sum_{k=0}^{2} L_k(x) f(x_k)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

$$= \frac{2}{3}(x-2.75)(x-4)\left(\frac{1}{2}\right) + \frac{16}{15}(x-2)(x-4)\left(\frac{4}{11}\right) + \frac{2}{5}(x-2)(x-2.75)\left(\frac{1}{4}\right)$$

$$= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

(b) An approximation to $f(3) = \frac{1}{3}$

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955$$

Divided Difference:

Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots x_n$.

* Although $P_n(x)$ is unique, there are alternate algebraic representations that are useful in certain situations.

Two points $x_0, x_1 \Rightarrow \overset{y_0}{f(x_0)}, \overset{y_1}{f(x_1)}$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \Rightarrow y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$\Rightarrow P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= a_0 + a_1(x - x_0) \quad \text{where } a_0 = f(x_0) \text{ and} \qquad \boxed{f[x_0]}$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \rightarrow \boxed{f[x_0, x_1]}$$

-lly $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)\ldots(x - x_{n-1})$

for appropriate constants $a_0, a_1, \ldots, a_n$.

* For $x = x_0$, $P_n(x_0) = a_0 = f(x_0)$

For $x = x_1$, $P_n(x_1) = a_0 + a_1(x_1 - x_0) \Rightarrow f(x_0) + a_1(x_1 - x_0) = f(x_1)$

$$\Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Zeroth Divided difference, of the function f w.r.t x_i:

$$f[x_i] = f(x_i), \qquad i = 0, 1, 2, \ldots, n$$

First Divided difference of f w.r.t x_i and x_{i+1}:

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Second Divided difference of f w.r.t x_i, x_{i+1} and x_{i+2}:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Kth Divided difference of f w.r.t $x_i, _{i+1}, x_{i+2}, \ldots, x_{i+k}$ is

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

* $x_0, x_1, \ldots, x_n$, So nth divided difference w.r.t $x_0, x_1, \ldots, x_n$

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

$$\therefore P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$= f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \cdots +$$
$$\qquad f[x_0, x_1, \ldots, x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$= f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x-x_0)(x-x_1)\cdots(x-x_{k-1})$$

* Here the value $f[x_0, x_1, \ldots, x_k]$ is independent of the order of the numbers $x_0, x_1, \ldots, x_k$.

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
x_0	$f(x_0)$	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f(x_1)$	$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$	$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f(x_2)$	$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$	$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f(x_3)$	$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$	$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f(x_4)$	$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$	$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
x_5	$f(x_5)$			

Newton's Divided Difference Formula

$x_0, x_1, \ldots, x_n \leftarrow (n+1)$ points

To find the divided difference coefficients of the interpolating polynomial $P_n(x)$ at $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$.

Input: $x_0, x_1, \ldots, x_n$ and $f(x_0), f(x_1), \ldots, f(x_n)$
$\qquad\qquad\qquad\qquad\quad F_{0,0} \quad F_{1,0} \qquad\qquad F_{n,0}$

Output: $F_{0,0}, F_{1,1}, F_{2,2}, \ldots, F_{n,n}$ so that

$$P_n(x) = F_{0,0} + \sum_{i=0}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j)$$

Exp

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Obtain the divided difference table for the given data and Construct the interpolating polynomial that uses all this data.

$$f[x_0] = f(x_0) = 0.7651977$$

x	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
1.0	0.7651977				
		$f[x_0, x_1] = \dfrac{f[x_1] + f[x_0]}{x_1 - x_0}$ $= -0.4837057$			
1.3	0.6200860		$f[x_0, x_1, x_2] = -0.1087339$		
		$f[x_1, x_2] = -0.5489460$		$f[x_0, x_1, x_2, x_3] = 0.0658784$	
1.6	0.4554022		$f[x_1, x_2, x_3] = -0.0494433$		$f[x_0, x_1, x_2, x_3, x_4] = 0.0018251$
		$f[x_2, x_3] = -0.5786120$		$f[x_1, x_2, x_3, x_4] = 0.0680685$	
1.9	0.2818186		$f[x_2, x_3, x_4] = -0.0118183$		
		$f[x_3, x_4] = -0.5715210$			
2.2	0.1103623				

$$P_4(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)$$
$$(x - x_1)(x - x_2) + f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$= (0.7651977) + (-0.4837057)(x - 1) + (-0.1087339)(x - 1)(x - 1.3) +$$
$$(0.0658784)(x - 1)(x - 1.3)(x - 1.6) + (0.0018251)(x - 1)(x - 1.3)(x - 1.6)(x - 1.9)$$

Now $P_4(1.5) = 0.5118200$

$x_0, f(x_0)$ $(x_1, f(x_1))$. When $f'(x)$ exists

by Mean Value theorem, there exists

c_y between x_0 and x_1 such that

$$f'(c_y) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \Rightarrow \quad f[x_0, x_1] = f'(c_y), \text{ for some } c_y \text{ b/w } x_0 \text{ and } x_1$$
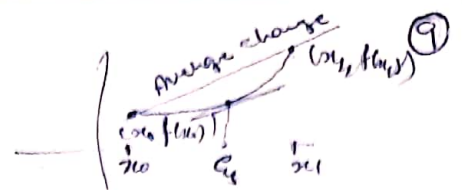
Generalize this result in following theorem

Theorem: Suppose that $f \in C^n[a,b]$ and $x_0, x_1, x_2, \ldots, x_n$ are distinct numbers in $[a,b]$. Then a number c_y exists in (a,b) with

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(c_y)}{n!}$$

4.1. Newton interpolation for equally spaced points. Newton's divided-difference formula can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing. Let $n+1$ points $x_0, x_1, \cdots, x_n$ are arranged consecutively with equal spacing h.

Let

$$h = \frac{x_n - x_0}{n} = x_{i+1} - x_i, \; i = 0, 1, \cdots, n$$

Then each $x_i = x_0 + ih, \; i = 0, 1, \cdots, n$.

For any $x \in [a, b]$, we can write $x = x_0 + sh, \; s \in \mathbb{R}$.

Then $x - x_i = (s - i)h$.

Now Newton interpolating polynomial is given by

$$
\begin{aligned}
P_n(x) &= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \, (x - x_0) \cdots (x - x_{k-1}) \\
&= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \, (s - 0)h \, (s - 1)h \, \cdots (s - k + 1)h \\
&= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \, s(s - 1) \cdots (s - k + 1) \, h^k \\
&= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \, k! \binom{s}{k} h^k
\end{aligned}
$$

where the binomial formula

$$\binom{s}{k} = \frac{s(s - 1) \cdots (s - k + 1)}{k!}.$$

This formula is called the Newton forward divided difference formula.

Now we introduce the forward difference operator

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i).$$

$$\Delta^k f(x_i) = \Delta^{k-1} \Delta f(x_i) = \Delta^{k-1}[f(x_{i+1}) - f(x_i)], \; i = 0, 1, \cdots, n - 1$$

Using the Δ notation, we can write

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{h}\Delta f(x_1) - \frac{1}{h}\Delta f(x_0)}{2h} = \frac{1}{2!h^2} \Delta^2 f(x_0)$$

In general

$$f[x_0, x_1, \cdots, x_k] = \frac{1}{k!h^k} \Delta^k f(x_0).$$

Therefore

$$P_n(x) = P_n(x_0 + sh) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0).$$

This is the Newton forward divided difference interpolation.

If the interpolation nodes are arranged recursively as $x_n, x_{n-1}, \cdots, x_0$, a formula for the interpolating polynomial is similar to previous result. In this case, Newton divided difference formula can be written as

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}] \, (x - x_n) \cdots (x - x_{n-k+1}).$$

If nodes are equally spaced with spacing

$$h = \frac{x_n - x_0}{n}, \; x_i = x_n - (n - i)h, \; i = n, n - 1, \cdots, 0.$$

Let $x = x_n + \cdot sh$.

Therefore

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}] \, (x - x_n) \cdots (x - x_{n-k+1})$$

$$= f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}] \, (s)h \, (s+1)h \cdots (s+k-1)h$$

$$= f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}] \, (-1)^k \binom{-s}{k} h^k k!$$

where the binomial formula is extended to include all real values s

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}.$$

This formula is called the Newton backward divided-difference formula. Like-wise the forward difference operator, we introduce the backward-difference operator by symbol ∇ (nabla) and

$$\nabla f(x_i) = f(x_i) - f(x_{i-1}).$$

$$\nabla^k f(x_i) = \nabla^{k-1} \nabla f(x_i) = \nabla^{k-1}[f(x_i) - f(x_{i-1})]$$

then

$$\cdot f[x_n, x_{n-1}] = \frac{1}{h} \nabla f(x_n)$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{f[x_n, x_{n-1}] - f[x_{n-1}, x_{n-2}]}{x_n - x_{n-2}} = \frac{\frac{1}{h}\nabla f(x_n) - \frac{1}{h}\nabla f(x_{n-1})}{2h} = \frac{1}{2!h^2}\nabla^2 f(x_n).$$

In general

$$f[x_n, x_{n-1}, x_{n-2} \cdots, x_{n-k}] = \frac{1}{k!h^k}\nabla^k f(x_n).$$

Therefore by using the backward-difference operator, the Newton backward divided-difference formula can be written as

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} \binom{-s}{k}(-1)^k \, \nabla^k f(x_n).$$

This is the Newton backward difference interpolation formula.

Example 9. *Using the following table for* $\tan x$, *approximate its value at 0.71 using Newton interpolation.*

x_i	0.70	72	0.74	0.76	0.78
$\tan x_i$	0.84229	0.87707	0.91309	0.95045	0.98926

Sol. As the point $x = 0.71$ lies in the beginning, we will use Newton forward interpolation. The forward difference table is:

x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$
0.70	0.84229	0.03478	0.00124	0.0001	0.00001
0.72	0.87707	0.03602	0.00134	0.00011	
0.74	0.91309	0.03736	0.00145		
0.76	0.95045	0.03881			
0.78	0.98926				

Here $x_0 = 0.70$, $h = 0.02$, $x = 0.71 = x_0 + sh$ gives $s = 0.5$. The Newton forward difference polynomial is given by

$$P_3(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f(x_0).$$

Substituting the values from table (first entries of each column starting from second), we obtain

$$P_3(0.71) = \tan(0.71) = 0.8596.$$

Algorithm: (Divided-Difference Algorithm) Given n distinct interpolation points $x_0, x_1, \cdots, x_n$, and the values of a function $f(x)$ at these points, the following algorithm computes the coefficients $c_j = f[x_0, x_1, \cdots, x_j]$ of the Newton interpolating polynomial.

```
for i = 0, 1, ⋯, n do
    aᵢ = f(xᵢ)
end
for i = 1, 2, ⋯, n do
    for j = n, n − 1, ⋯, i do
        aⱼ = (aⱼ − aⱼ₋₁)/(xⱼ − xⱼ₋ᵢ)
    end
end
```

It follows that the interpolating polynomial $P_n(x)$ can be obtained using the Newton divided-difference formula as follows:

$$P_n(x) = \sum_{j=0}^{n} f[x_0, \cdots, x_j] \prod_{i=0}^{j-1} (x - x_i).$$

Example 1. *Given the following four data points. Find a polynomial in Lagrange form to interpolate*

x_i	0	1	3	5
y_i	1	2	6	7

the data.

Sol. The Lagrange functions are given by

$$l_0(x) = \frac{(x-1)(x-3)(x-5)}{(0-1)(0-3)(0-5)} = -\frac{1}{15}(x-1)(x-3)(x-5).$$

$$l_1(x) = \frac{(x-0)(x-3)(x-5)}{(1-0)(1-3)(1-5)} = \frac{1}{8}(x-0)(x-3)(x-5).$$

$$l_2(x) = \frac{(x-0)(x-1)(x-5)}{(3-0)(3-1)(3-5)} = -\frac{1}{12}(x)(x-1)(x-5).$$

$$l_3(x) = \frac{(x-0)(x-1)(x-3)}{(5-0)(5-1)(5-3)} = \frac{1}{40}(x)(x-1)(x-3).$$

The interpolating polynomial in the Lagrange form is

$$P_3(x) = l_0(x) + 2l_1(x) + 6l_2(x) + 7l_3(x).$$

Example 2. *Let $f(x) = \sqrt{x - x^2}$ and $P_2(x)$ be the interpolation polynomial on $x_0 = 0$, x_1 and $x_2 = 1$. Find the largest value of x_1 in $(0,1)$ for which $f(0.5) - P_2(0.5) = -0.25$.*

Sol. If $f(x) = \sqrt{x - x^2}$ then our nodes are $[x_0, x_1, x_2] = [0, x_1, 1]$ and $f(x_0) = 0$, $f(x_1) = \sqrt{x_1 - x_1^2}$ and $f(x_2) = 0$. Therefore

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - x_1)(x - 1)}{x_1},$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x - 1)}{x_1(x_1 - 1)},$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - 1)}{(1 - x_1)}.$$

$$\therefore P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$= \frac{(x-x_1)(x-1)}{x_1}.0 + \frac{x(x-1)}{x_1(x_1-1)}.\sqrt{x_1 - x_1^2} + \frac{x(x-1)}{(1-x_1)}.0$$

$$= -\frac{x(x-1)}{\sqrt{x_1(1-x_1)}}.$$

If we now consider $f(x) - P_2(x)$, then

$$f(x) - P_2(x) = \sqrt{x - x^2} + \frac{x(x-1)}{\sqrt{x_1(1-x_1)}}.$$

Hence $f(0.5) - P_2(0.5) = -0.25$ implies

$$\sqrt{0.5 - 0.5^2} + \frac{0.5(0.5-1)}{\sqrt{x_1(1-x_1)}} = -0.25$$

Solving for x_1 gives

$$x_1^2 - x_1 = -1/9$$

or

$$(x_1 - 1/2)^2 = 5/36$$

which gives $x_1 = \frac{1}{2} - \sqrt{\frac{5}{36}}$ or $x_1 = \frac{1}{2} + \sqrt{\frac{5}{36}}$.
The largest of these is therefore

$$x_1 = \frac{1}{2} + \sqrt{\frac{5}{36}} \approx 0.8727.$$

2.3. Error Analysis for Polynomial Interpolation. We are given $x_0, x_1, \cdots, x_n$, and the corresponding function values $f(x_0), f(x_1), \cdots, f(x_n)$, but the we don't know the expression for the function. Let $P_n(x)$ be the polynomial of order $\leq n$ that passes through the $n+1$ points $(x_0, f(x_0))$, $(x_1, f(x_1)), \cdots, (x_n, f(x_n))$. Question: What is the error between $f(x)$ and $P_n(x)$ even we don't know $f(x)$ in advance?

Definition 2.3 (Truncation error). *The polynomial $P_n(x)$ coincides with $f(x)$ at all nodal points and may deviates at other points in the interval. This deviation is called the truncation error and we write*

$$E_n(f;x) = f(x) - P_n(x).$$

Theorem 2.4. *Suppose that $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a,b]$ and $f \in C^{n+1}[a,b]$. Let $P_n(x)$ be the unique polynomial of degree $\leq n$ that passes through $n+1$ nodal points then prove that*

$$\forall x \in [a,b], \ \exists \xi = \xi(x) \in (a,b)$$

such that

$$E_n(f;x) = f(x) - P_n(x) = \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

Proof. Let $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a,b]$ and $f \in C^{n+1}[a,b]$. Let $P_n(x)$ be the unique polynomial of degree $\leq n$ that passes through $n+1$ nodal points. The truncation error in interpolation is given by

$$E_n(f;x) = f(x) - P_n(x).$$

$$E_n(f;x_i) = 0, \ i = 0, 1, \cdots, n.$$

Now for any t in the domain, define

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t-x_0)\cdots(t-x_n)}{(x-x_0)\cdots(x-x_n)} \tag{2.2}$$

Now $g(t) = 0$ at $t = x, x_0, x_1, \ldots, x_n$. Therefore $g(t)$ satisfy the conditions of Rolle's Theorem which states that between $n+2$ zeros of a function, there is at least one zero of $(n+1)$th derivative of the function. Hence there exists a point ξ such that

$$g^{(n+1)}(\xi) = 0$$

where ξ is some point such that

$$\min(x_0, x_1, \cdots, x_n, x) < \zeta < \max(x_0, x_1, \cdots, x_n, x).$$

Now differentiate (2.2) $(n+1)$ times, we get

$$g^{(n+1)}(t) = f^{(n+1)}(t) - P^{(n+1)}(t) - [f(x) - P(x)]\frac{(n+1)!}{(x - x_0)\cdots(x - x_n)}$$

$$= f^{(n+1)}(t) - [f(x) - P(x)]\frac{(n+1)!}{(x - x_0)\cdots(x - x_n)}$$

Here $P^{(n+1)}(t) = 0$ as P_n is a n-th degree polynomial.

Now $g^{(n+1)}(\xi) = 0$ and then solving for $f(x) - P(x)$, we obtain

$$f(x) - P(x) = \frac{(x - x_0)\cdots(x - x_n)}{(n+1)!}f^{(n+1)}(\xi)$$

Truncation error is given by

$$E_n(f;x) = f(x) - P(x) = \frac{(x - x_0)\cdots(x - x_n)}{(n+1)!}f^{(n+1)}(\xi).$$

∎

Corollary 2.5. *If* $|f^{(n+1)}(\xi)| \leq M$ *then we can obtain a bound of the error*

$$|E_n(f;x)| \leq \frac{M}{(n+1)!}\max_{x\in[a,b]}|(x - x_0)\cdots(x - x_n)|.$$

Example 3. *Suppose* $f(x) = \sin x$ *is approximated by an interpolating polynomial* $P(x)$ *of degree 9 in* $[0,1]$. *Estimate* $|f(x) - P(x)|$, *for all* $x \in [0,1]$.

Sol. $f(x) = \sin x$, $n = 9$.

Now

$$|f^{(10)}(\xi)| \leq 1.$$

$$|x - x_i| \leq 1, \implies |\prod_{i=0}^{n}(x - x_i)| \leq 1, \forall x \in [0,1].$$

Hence

$$|f(x) - P(x)| = \frac{1}{10!}|f^{(10)}(\xi)|\,|\prod_{i=0}^{n}(x - x_i)| \leq \frac{1}{10!}.$$

Example 4. *Denoting the interpolating polynomial* $f(x)$ *on the distinct points* $x_0, \cdots, x_n$ *by* $\sum_{k=0}^{n}l_k(x)f(x_k)$, *find an expression for* $\sum_{k=0}^{n}l_k(0)x_k^{n+1}$.

Sol. Lagrange interpolating polynomial is

$$f(x) = \sum_{k=0}^{n}l_k(x)f(x_k) + \frac{(x - x_0)\cdots(x - x_n)}{(n+1)!}f^{(n+1)}(\xi).$$

Let $f(x) = x^{n+1}$,

$$x^{n+1} = \sum_{k=0}^{n}l_k(x)x_k^{n+1} + \frac{(x - x_0)\cdots(x - x_n)}{(n+1)!}(n+1)!$$

$$\implies x^{n+1} = \sum_{k=0}^{n}l_k(x)x_k^{n+1} + (x - x_0)\cdots(x - x_n).$$

Now put $x = 0$ to obtain

$$\sum_{k=0}^{n}l_k(0)x_k^{n+1} = (-1)^n x_0 x_1 \cdots x_n.$$

The next example illustrates how the error formula can be used to prepare a table of data that will ensure a specified interpolation error within a specified bound.

Example 5. *Suppose a table is to be prepared for the function $f(x) = e^x$, for x in $[0,1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x–values, the step size, is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0,1]$?*

Sol. Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated, x be in $[0,1]$, and suppose i satisfies $x_i \leq x \leq x_{i+1}$.

The error in linear interpolation is

$$|f(x) - P(x)| = \left| \frac{1}{2} f^2(\xi)(x - x_i)(x - x_{i+1}) \right| = \frac{|f^2(\xi)|}{2} |(x - x_i)||(x - x_{i+1})|.$$

The step size is h, so $x_i = ih$, $x_{i+1} = (i+1)h$, and

$$|f(x) - p(x)| \leq \frac{1}{2} |f^2(\xi)| \, |(x - ih)(x - (i+1)h|.$$

Hence

$$|f(x) - p(x)| \leq \frac{1}{2} \max_{\xi \in [0,1]} e^\xi \max_{x_i \leq x \leq x_{i+1}} |(x - ih)(x - (i+1)h|$$

$$\leq \frac{e}{2} \max_{x_i \leq x \leq x_{i+1}} |(x - ih)(x - (i+1)h|.$$

Consider the function $g(x) = (x - ih)(x - (i+1)h)$, for $ih \leq x \leq (i+1)h$. Because

$$g'(x) = (x - (i+1)h) + (x - ih) = 2\left(x - ih - \frac{h}{2}\right),$$

the only critical point for g is at $x = ih + h/2$, with $g(ih + h/2) = (h/2)^2 = h^2/4$. Since $g(ih) = 0$ and $g((i+1)h) = 0$, the maximum value of $|g'(x)|$ in $[ih, (i+1)h]$ must occur at the critical point which implies that

$$|f(x) - p(x)| \leq \frac{e}{2} \max_{x_i \leq x \leq x_{i+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}.$$

Consequently, to ensure that the the error in linear interpolation is bounded by 10^{-6}, it is sufficient for h to be chosen so that

$$\frac{eh^2}{8} \leq 10^{-6}.$$

This implies that $h < 1.72 \times 10^{-3}$.
Because $n = (1 - 0)/h$ must be an integer, a reasonable choice for the step size is $h = 0.001$.

Example 6. *Determine the step size h that can be used in the tabulation of a function $f(x)$, $a \leq x \leq b$, at equally spaced nodal points so that the truncation error of the quadratic interpolation is less than ε.*

Sol. Let x_{i-1}, x_i, x_{i+1} are three eqispaced points with spacing h. The truncation error of the quadratic interpolation is given by

$$|f(x) - P(x)| \leq \frac{M}{3!} \max_{a \leq x \leq b} |(x - x_{i-1},)(x - x_i)(x - x_{i+1})|$$

where $M = \max\limits_{a \leq x \leq b} |f^{(3)}(x)|$.

To simplify the calculation, let

$$x - x_i = th$$
$$\therefore x - x_{i-1} = x - (x_i - h) = (t + 1)h$$
$$\text{and } x - x_{i+1} = x - (x_i + h) = (t - 1)h.$$

$$\therefore |(x - x_{i-1},)(x - x_i)(x - x_{i+1})| = h^3 |t(t+1)(t-1)| = g(t)(\text{say})$$

Now $g(t)$ attains its extreme values if

$$\frac{dg}{dt} = 0.$$

which gives $t = 1 \pm \frac{1}{\sqrt{3}}$. At end points of the interval g becomes zero.

For both values of $t = 1 \pm \frac{1}{\sqrt{3}}$, we obtain $\max\limits_{x_{i-1} \leq x \leq x_{i+1}} |g(t)| = \frac{2}{3\sqrt{3}}$.

Truncation error

$$|f(x) - P_2(x)| < \varepsilon$$

$$\implies \frac{h^3}{9\sqrt{3}} M < \varepsilon$$

$$\implies h < \left[\frac{9\sqrt{3}\varepsilon}{M}\right]^{1/3}$$

Algorithm (Lagrange Interpolation)

- Read the degree n of the polynomial $P_n(x)$
- Read the values of $x[i]$ and $f[i]$, $i = 0, 1, \ldots, n$
- Read the point of interpolation p
- Calculate the Lagrange's fundamental polynomials $l_i(x)$ using the following loop:

```
for i=0 to n
l[i] = 1.0
for j=0 to n
if j ≠ i
```

$$l[i] = \frac{p - x[j]}{x[i] - x[j]} l(i)$$

```
end j
end i
```

- Calculate the approximate value of the function at $x = p$ using the following loop:

```
sum=0.0
for i=0 to n
sum = sum + l[i] * f[i]
end i
```

- Print sum.

Example 8. *Given the following four data points. Find a polynomial in Newton form to interpolate*

x_i	0	1	3	5
y_i	1	2	6	7

the data (the same exercise was done by Lagrange interpolation).

Sol. To write the Newton form, we draw divided difference table as following

x_i	y_i	first d.d.	second d.d.	third d.d.
0	1	1	1/3	$-17/120$
1	2	2	$-3/8$	
3	6	1/2		
5	7			

$$P_3(x) = f(x_0) + (x - 0)f[0, 1] + (x - 0)(x - 1)f[0, 1, 3] + (x - 0)(x - 1)(x - 3)f[0, 1, 3, 5]$$

$$P_3(x) = 1 + x + 1/3x(x - 1) - 17/120x(x - 1)(x - 3).$$

$P(2) = 1 + 2 + \dfrac{2(1)}{3} - \dfrac{2(1)(-1)17}{120}$

$= 1 + 2 + 0.6667 + 0.2833$

$= 3.95003$

Note that x_i can be re-ordered but must be distinct. When the order of some x_i are changed, one obtain the same polynomial but in different form.

Remark 4.1. *If more data points are added to the interpolation problem, we have to recalculate all the cardinal numbers in Lagrange form but in Newton form we need not to recalculate which is the great advantage of Newton form.*

Theorem 4.2. *Let $f \in C^n[a, b]$ and $x_0, \cdots, x_n$ are distinct numbers in $[a, b]$. Then there exists ξ such that*

$$f[x_0, x_1, x_2, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$