Lecture 36: Numerical Analysis (UMA011)

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Chapter 6th

Numerical Quadrature:

Numerical Quadrature:

Numerical Quadrature is a basic method for numerically approximating the value of a definite integral $\int_a^b f(x)dx$.

It uses $\sum_{i=0}^{n} a_i f(x_i)$ to approximate $\int_a^b f(x)dx$,

$$i.e. \int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i).$$

$$= a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) - - - + a_n f(x_n)$$

where $a_i \to$ multipliers ; $x_i \to$ nodes

Numerical Quadrature:

General formula to approximate integral

Divide the interval $[a, b]$ in to a set of $(n+1)$ distinct nodes $\{x_0, x_1, x_2, \cdots, x_n\}$.

Approximate $f(x)$ by Lagrange's interpolating polynomials which is used to approximate $f(x)$.

Thus, we can write

$$f(x) = P_n(x) + e_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where $\xi = \xi(x) \in [a, b]$ and $l_i(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$

$$P_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + --- l_n(x) f(x_n)$$

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum_{i=0}^{n} l_i(x)\, f(x_i)\right) dx + \int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x-x_i)\,dx$$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)\,dx}_{=a_i} + \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi(x)) \prod_{i=0}^{n} (x-x_i)\,dx$$

$$= \sum_{i=0}^{n} a_i f(x_i) + \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi(x)) \prod_{i=0}^{n} (x-x_i)\,dx$$

$$= \sum_{i=0}^{n} a_i f(x_i) + E(f) \quad (\text{say})$$

Numerical Quadrature:

Quadrature formula:

Therefore, $\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i)$, where $a_i = \int_a^b l_i(x)dx$,
for each $i = 0, 1, 2, \cdots, n$
with error given by

$$E(f) = \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x - x_i) f^{(n+1)}(\xi(x))dx.$$

Numerical Quadrature:

Quadrature formula:

We derive formulas by using one and two degree interpolating polynomials with equally spaced nodes. This gives:

1. Trapezoidal Rule
2. Simpson's Rule or Simpson's $\frac{1}{3}$rd rule.

Trapezoidal Rule:

To derive the Trapezoidal rule for approximating $\int_a^b f(x)dx$, let $x_0 = a, x_1 = b, h = b - a$ and use linear Lagrange's interpolating polynomial:

$$P_1(x) = \sum_{i=0}^{1} l_i(x) f(x_i) = l_0(x)\, f(x_0) + l_1(x) f(x_1)$$

$$\Rightarrow f(x) = P_1(x) + e_1(x) \rightarrow \text{error}$$

$$\Rightarrow f(x) = \sum_{i=0}^{1} l_i(x) f(x_i) + \frac{f''(\xi)}{2!}(x - x_0)(x - x_1), \ \xi \in (a, b).$$

where

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Numerical Quadrature:

Trapezoidal Rule:

$$\Rightarrow \int_a^b f(x)dx = \int_{a=x_0}^{b=x_1} \left(\overbrace{\frac{x-x_1}{x_0-x_1}}^{l_0(x)} f(x_0) + \overbrace{\frac{x-x_0}{x_1-x_0}}^{l_1(x)} f(x_1) \right) dx$$

$$+ \int_{a=x_0}^{b=x_1} \frac{f''(\xi)}{2!}(x-x_0)(x-x_1)dx$$

$$= \sum_{i=0}^{1} a_i(x) f(x_i) + E(f), \text{ (say).}$$

where

$$a_i(x) = \int_a^b l_i(x)\,dx$$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{1} a_i \quad f(x_i) + E(f)$$

$$= \sum_{i=0}^{1} a_i f(x_i) + E(f)$$

$$a_i = \int_a^b l_i(x)\,dx$$

$$a_0 = \int_{a=x_0}^{b=x_1} l_0(x)\,dx = \int_{x_0}^{x_1} \left(\frac{x-x_1}{x_0-x_1}\right) dx = \frac{-1}{h} \int_{x_0}^{x_1} (x-x_1)\,dx$$

$$= \frac{-1}{h} \left[\frac{x^2}{2} - x x_1\right]_{x_0}^{x_1}$$

$$= \frac{-1}{h}\left[\frac{x_1^2}{2} - x_1^2 - \frac{x_0^2}{2} + x_0 x_1\right]$$

$$= \frac{-1}{h}\left[-\frac{x_1^2}{2} - \frac{x_0^2}{2} + x_0 x_1\right] = \frac{-1}{h}\left[\frac{-x_1^2 - x_0^2 + 2x_0x_1}{2}\right]$$

$$= \frac{1}{h}\left[\frac{(x_1-x_0)^2}{2}\right] = \frac{h^2}{2h} = \frac{h}{2}$$

$$\Rightarrow \quad a_0 = \frac{h}{2}$$

$$\text{llly} \quad a_1 = \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0}\,dx = \frac{h}{2}$$

$$\Rightarrow \quad \int_a^b f(x)\,dx \approx a_0 f(x_0) + a_1 f(x_1) = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

$$= \frac{h}{2}\left[f(a) + f(b)\right]$$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)\right], \quad h = b-a.$$

for e.g. 1.

$$\int_1^2 x\,dx = \left(\frac{x^2}{2}\right)_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} = \text{Exact value.}$$

by Trap.

$$\int_1^2 x\,dx \simeq \frac{h}{2}\left[f(1)+f(2)\right] = \frac{1}{2}\left[f(1)+f(2)\right] = \frac{1}{2}\left[1+2\right] = \frac{3}{2}$$

2

$$\int_1^2 x^2\,dx = \left(\frac{x^3}{3}\right)_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$= \text{exact}$

≈ 2.3333

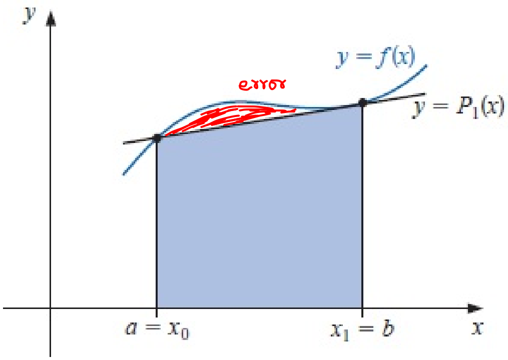
by Trap.

$$= \frac{1}{2}\left[f(1)+f(2)\right] = \frac{1}{2}\left[1+4\right]$$

$= 5/2 = 2.5$

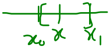
Numerical Quadrature:

Trapezoidal Rule:



Error in Trap. Rule :-

$$E(f) = \int_{a=x_0}^{b=x_1} \frac{f''(\xi(x))}{2!} \underbrace{\overset{+ve}{(x-x_0)}\,\overset{-ve}{(x-x_1)}}_{g(x)}\, dx$$



Weighted Mean value Theorem let $f(x)$ is continuous on $[a,b]$ and $g(x)$ is integrable on $[a,b]$ and does not change sign in $[a,b]$, then

$$\underbrace{\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx}$$

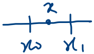
for $c \in (a,b)$

Here, we use $f(x) = f''(\xi(x))$, $g(x) = (x-x_0)(x-x_1)$

By using W.M.V.T., for $c \in (a,b)$, we get

$$E(f) = \int_{x_0}^{x_1} \frac{f''(\xi(x))}{2!} (x-x_0)(x-x_1)\, dx$$

$$E(f) = \frac{f''(c)}{2!}\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx$$



Put

$x = x_0 + ih$

$dx = h\,di$

If $x = x_0$
then $i = 0$

if $x = x_1$
then $x_1 = x_0 + ih$

$x_1 - x_0 = ih$

$\frac{h}{h} = i \Rightarrow i = 1$

$$= \frac{f''(c)}{2!}\int_0^1 (ih)(i-1)h\;h\,di$$

$$= \frac{f''(c)}{2!}h^3\int_0^1 i(i-1)\,di$$

$$= \frac{f''(c)}{2!}h^3\int_0^1 i^2 - i\;di$$

$$= \frac{f''(c)}{2!}h^3\left[\frac{i^3}{3} - \frac{i^2}{2}\right]_0^1 = \frac{h^3 f''(c)}{2!}\left[\frac{1}{3}-\frac{1}{2}\right] = \frac{-h^3 f''(c)}{2!\;6} = -\frac{h^3 f''(c)}{12}$$

Numerical Quadrature:

Exercise:

1. Derive Trapezoidal rule.
2. Show that the error in approximating $\int_a^b f(x)dx$ by trapezoidal's rule is $-\frac{h^3}{12}f''(c)$, where $c \in (a, b)$ and $h = b - a$.
3. Approximate the integral $I = \int\limits_{-0.25}^{0.25} (\cos x)^2 dx$ using the trapezoidal and compare with exact value.