

CHAPTER 7 (4 LECTURES)

INITIAL-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Differential equations are used to model problems in science and engineering that involve the change of some variable with respect to another. Most of these problems require the solution of an initial-value problem, that is, the solution to a differential equation that satisfies a given initial condition. In common real-life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to modify the problem by simplifying the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original problem. The other approach, which we will examine in this chapter, uses methods for approximating the solution of the original problem. This is the approach that is most commonly taken because the approximation methods give more accurate results and realistic error information.

In this chapter, we discuss the numerical methods for solving the ordinary differential equations of initial-value problems (IVP) of the form

$$\frac{dy}{dt} = f(t, y), \quad t \in \mathbb{R}, \quad y(t_0) = y_0 \quad (1.1)$$

where y is a function of t , f is function of t and y , and t_0 is called the initial value. The numerical values of $y(t)$ on an interval containing t_0 are to be determined.

We divide the domain $[a, b]$ into subintervals

$$a = t_0 < t_1 < \cdots < t_N = b.$$

These points are called mesh points or grid points. Let equal spacing is h . The uniform mesh points are given by $t_i = t_0 + ih$, $i = 0, 1, 2, \dots$. The set of points y_0, y_1, \dots, y_N are the numerical solution of the initial-value problem (IVP).

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Definition 2.1. A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on some domain if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in domain. The constant L is called a Lipschitz constant for f .

Example 1. Let $f(t, x) = y^2 e^{-t^2} \sin t$ be defined on

$$D = \{(t, y) \in \mathbb{R}^2 : 0 \leq y \leq 2\}.$$

Show that f satisfies Lipschitz condition.

Sol. Let $(t, y_1), (t, y_2) \in D$.

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= |y_1^2 e^{-t^2} \sin t - y_2^2 e^{-t^2} \sin t| \\ &= |e^{-t^2} \sin t| |y_1 + y_2| |y_1 - y_2| \\ &\leq (1)(4) |y_1 - y_2| = 4 |y_1 - y_2|. \end{aligned}$$

Thus we may take $L = 4$ and f satisfies a Lipschitz condition in D with Lipschitz constant 4.

Example 2. Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) | 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\}$.

Sol. For each pair of points (t, y_1) and (t, y_2) in D , we have

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= ||t| y_1 - |t| y_2| \\ &\leq |t| |y_1 - y_2| \\ &\leq 2|y_1 - y_2|. \end{aligned}$$

Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant $L = 2$.

Theorem 2.2. If $f(t, y)$ is continuous in $a \leq t \leq b$, $-\infty \leq y \leq \infty$, and

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for some positive constant L (which means f is Lipschitz continuous in y), then the IVP (1.1) has a unique solution in the interval $[a, b]$.

Example 3. Show that there is a unique solution to the initial-value problem

$$y' = 1 + t \sin(ty), \quad 0 \leq t \leq 2, \quad y(0) = 0.$$

Sol. Take two points (t, y_1) and (t, y_2) in the domain, we have

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= |(1 + t \sin(ty_1)) - (1 + t \sin(ty_2))| \\ &= |t| |\sin(ty_1) - \sin(ty_2)|. \end{aligned}$$

Holding t constant and applying the Mean Value Theorem for $f(t, y) = \sin(ty)$, we get

$$\begin{aligned} |\sin(ty_1) - \sin(ty_2)| &= |t \cos(t\xi)| |y_1 - y_2|, \quad \xi \in (y_1, y_2) \\ &= |t| |\cos(t\xi)| |y_1 - y_2|. \end{aligned}$$

Therefore

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= t^2 |\cos(t\xi)| |y_1 - y_2|, \\ &\leq 2^2 \cdot 1 \cdot |y_1 - y_2| \\ &= 4|y_1 - y_2|. \end{aligned}$$

Hence f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$.

Additionally, $f(t, y)$ is continuous when $0 \leq t \leq 2$ and $-\infty \leq y \leq \infty$, so Existence Theorem implies that a unique solution exists to this initial-value problem.

2.1. Picard's method. This method is also known as method of successive approximations. We consider the following IVP

$$\frac{dy}{dt} = f(t, y), \quad t \in \mathbb{R}, \quad y(t_0) = y_0$$

Let $f(t, y)$ to be a continuous function on the given domain. The initial value problem is equivalent to following integral equation,

$$y(t) = y(0) + \int_{t_0}^t f(s, y(s)) ds.$$

Writing $y(0) = y_0$ and we can compute the solution $y(t)$ at any time t by integrating above equation. Note that $y(t)$ also appears in integral in $f(t, y(t))$. Therefore we take any approximation of $y(t)$ to start the procedure.

The successive approximations for solutions are given by

$$y_0(t) = y_0, \quad y_{k+1}(t) = y_0 + \int_{t_0}^t f(s, y_k(s)) ds, \quad k = 0, 1, 2, \dots$$

Example 4. Consider the initial value problem

$$\frac{dy}{dt} = 1 + ty, \quad y(0) = 1.$$

Show that $f(t, y) = 1 + ty$ satisfies a Lipschitz condition for region $0 \leq t \leq 2$. Also find first three approximations of the solution using Picard's method.

Sol. Consider the initial value problem (IVP)

$$\frac{dy}{dt} = 1 + ty = f(t, y), \quad y(0) = 1.$$

We have

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= |1 + ty_1 - 1 - ty_2| \\ &= |t| |y_1 - y_2| \\ &\leq (2)|y_1 - y_2| = 2|y_1 - y_2|. \end{aligned}$$

Thus given f satisfies a Lipschitz condition in given region with Lipschitz constant $L = 2$. The integral equation corresponding to this IVP is

$$y(t) = 1 + \int_0^t (1 + sy(s))ds.$$

The successive approximations are given by

$$y_0(t) = 1, \quad y_{k+1}(t) = 1 + \int_0^t (1 + sy_k(s))ds, \quad k = 0, 1, 2, \dots.$$

Thus

$$\begin{aligned} y_1(t) &= 1 + \int_0^t (1 + s) ds = 1 + t + \frac{t^2}{2}, \\ y_2(t) &= 1 + \int_0^t \left[1 + s \left(1 + s + \frac{s^2}{2} \right) \right] ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}, \\ y_3(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8} + \frac{t^5}{15} + \frac{t^6}{48}. \end{aligned}$$

2.2. Taylor's Series method. Consider the one dimensional initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

where f is a function of two variables t and y and (t_0, y_0) is a known point on the solution curve. If the existence of all higher order partial derivatives is assumed for y at some point $t = t_i$, then by Taylor series the value of y at any neighboring point $t_i + h$ can be written as

$$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{3!} y'''(t_i) + \dots + O(h^{p+1}).$$

Since at t_i , y_i is known, y' at x_i can be found by computing $f(t_i, y_i)$.

Similarly higher derivatives of y at t_i can be computed by making use of the relation $y' = f(t, y)$. Hence the value of y at any neighboring point $t_i + h$ can be obtained by summing the above infinite series. If the series has been terminated after the p -th derivative term then the approximated formula is called the Taylor series approximation to y of order p and the error is of order $p + 1$.

Example 5. Given the IVP $y' = x^2y - 1$, $y(0) = 1$. By Taylor series method of order 4 with step size 0.1. Find y at $x = 0.1$ and $x = 0.2$.

Sol. Given IVP

$$\begin{aligned} y' &= x^2y - 1, \quad y'' = 2xy + x^2y', \quad y''' = 2y + 4xy' + x^2y'', \quad y^{(4)} = 6y' + 6xy'' + x^2y'''. \\ \therefore y'(0) &= -1, \quad y''(0) = 0, \quad y^{(3)}(0) = 2, \quad y^{(4)}(0) = -6. \end{aligned}$$

The fourth-order Taylor's formula is given by

$$y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{3!} y'''(x_i) + \frac{h^4}{4!} y^{(4)}(x_i) + O(h^5)$$

Therefore

$$y(0.1) = 1 + (0.1)(-1) + 0 + (0.1)^3(2)/6 - (0.1)^4(-6)/24 = 0.900033$$

Similarly

$$y(0.2) = 0.80227.$$

3. NUMERICAL METHODS FOR IVP

We consider the following IVP

$$\frac{dy}{dt} = f(t, y), \quad t \in \mathbb{R} \quad (3.1)$$

$$y(t_0) = y_0. \quad (3.2)$$

Its integral form is the following equation

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s))ds.$$

3.1. Euler's Method: The Euler method is named after Swiss mathematician Leonhard Euler (1707-1783). This is the one of the simplest method to solve the IVP. Consider the IVP given in Eqs(3.1-3.2).

We can approximate the derivative $\frac{dy}{dt}$ as following by assuming that all nodes t_i are equally spaced with spacing h and $t_{i+1} = t_i + h$.

Now by the definition of derivative

$$y'(t_0) \approx \frac{y(t_0 + h) - y(t_0)}{h}.$$

Apply this approximation to the given IVP at point $t = t_0$ gives

$$y'(t_0) = f(t_0, y_0).$$

Therefore

$$\frac{1}{h}[y(t_0 + h) - y(t_0)] = f(t_0, y_0)$$

$$\implies y(t_0 + h) - y(t_0) = hf(t_0, y_0)$$

which gives

$$y(t_1) = y(t_0 + h) = y(t_0) + hf(t_0, y_0).$$

In general, we write

$$t_{i+1} = t_i + h$$

$$y_{i+1} = y_i + hf(t_i, y_i)$$

where $y_i = y(t_i)$. This procedure is called Euler's method.

Alternatively we can derive this method from a Taylor's series. We write

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \dots$$

If we truncate the series at $y'(t_i)$, we obtain

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hy'(t_i) \\ \implies y(t_{i+1}) &= y(t_i) + hf(t_i, y(t_i)) \\ \implies y_{i+1} &= y_i + hf(t_i, y_i). \end{aligned}$$

If truncation error has term h^{p+1} , then order of the numerical method is p . Therefore, Euler's method is a first-order method.

3.2. The Improved or Modified Euler's method. We write the integral form of $y(t)$ as

$$\frac{dy}{dt} = f(t, y) \iff y(t) = y(t_0) + \int_{t_0}^t f(t, y(t)) dt.$$

Approximate integral using the trapezium rule:

$$y(t_1) = y(t_0) + \frac{h}{2}[f(t_0, y(t_0)) + f(t_0 + h, y(t_0 + h))], \quad t_1 = t_0 + h.$$

Use Euler's method to approximate $y(t_1) \approx y(t_0) + hf(t_0, y(t_0))$ in above equation gives

$$y(t_1) = y(t_0) + \frac{h}{2}[f(t_0, y(t_0)) + f(t_0 + h, y(t_0) + hf(t_0, y(t_0)))].$$

Hence the modified Euler's scheme can be written as

$$\begin{aligned} K_1 &= hf(t_0, y_0) \\ K_2 &= hf(t_1, y_0 + K_1) \\ y_1 &= y_0 + \frac{K_1 + K_2}{2}. \end{aligned}$$

In general, the modified Euler's scheme is given by

$$\begin{aligned} t_{i+1} &= t_i + h \\ K_1 &= hf(t_i, y_i) \\ K_2 &= hf(t_{i+1}, y_i + K_1) \\ y_{i+1} &= y_i + \frac{K_1 + K_2}{2}. \end{aligned}$$

Example 6.

$$y' + 2y = 2 - e^{-4t}, y(0) = 1$$

By taking step size 0.1, find y at $t = 0.1$ and 0.2 by Euler method.

Sol.

$$y' = -2y + 2 - e^{-4t} = f(t, y), y(0) = 1$$

$$f(0, 1) = -2(1) + 2 - 1 = -1$$

By Euler's method with step size $h = 0.1$,

$$\begin{aligned} t_1 &= t_0 + h = 0 + 0.1 = 0.1 \\ y_1 &= y_0 + hf(0, 1) = 1 + 0.1(-1) = 0.9 \\ \therefore y_1 &= y(0.1) = 0.9. \end{aligned}$$

$$\begin{aligned} t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \\ y_2 &= y_1 + hf(0.1, 0.9) = 0.9 + 0.1(-2 \times 0.9 + 2 - e^{-4(0.1)}) \\ &= 0.9 + 0.1(-0.47032) = 0.852967 \\ \therefore y_2 &= y(0.2) = 0.852967. \end{aligned}$$

Example 7. For the IVP $y' = t + \sqrt{y}$, $y(0) = 1$. Calculate y in the interval $[0, 0.6]$ with $h = 0.2$ by using modified Euler's method.

Sol.

$$\begin{aligned} y' &= t + \sqrt{y} = f(t, y), t_0 = 0, y_0 = 1, h = 0.2, t_1 = 0.2 \\ K_1 &= hf(t_0, y_0) = 0.2(1) = 0.2 \\ K_2 &= hf(t_1, y_0 + K_1) = hf(0.2, 1.2) = 0.2591 \\ y_1 &= y(0.2) = y_0 + \frac{K_1 + K_2}{2} = 1.22955. \end{aligned}$$

Similarly we can compute solutions at other points.

Example 8. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \leq t \leq 2, \quad y(1) = 0$$

- (i) Use Euler's method with $h = 0.1$ to approximate the solution in the interval $[1, 1.6]$.
- (ii) Use the answers generated in part (i) and linear interpolation to approximate y at $t = 1.04$ and $t = 1.55$.

Sol. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t = f(t, y)$$

$$t_0 = 1.0, \quad y(t_0) = 0.0, \quad h = 0.1.$$

By Euler's method, approximation of solutions at different time-level are given by

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hf(t_i, y(t_i)). \\ \therefore y(t_1) = y(1.1) &= y(0) + hf(1, 0) = 0.0 + 0.1 \left[\frac{2}{1.0} 0.0 + 1.0^2 e^{1.0} \right] = 0.271828. \\ t_1 &= 1.1 \\ y(t_2) = y(1.2) &= 0.271828 + 0.1 \left[\frac{2}{1.1} 0.271828 + (1.1)^2 e^{1.1} \right] = 0.684756 \\ t_2 &= 1.2 \\ y(t_3) = y(1.3) &= 0.684756 + 0.1 \left[\frac{2}{1.2} 0.684756 + (1.2)^2 e^{1.2} \right] = 1.27698. \\ t_3 &= 1.3 \end{aligned}$$

Similarly

$$\begin{aligned} t_4 &= 1.4 \\ y(t_4) = y(1.4) &= 2.09355 \\ t_5 &= 1.5 \\ y(t_5) = y(1.5) &= 3.18745 \\ t_6 &= 1.6 \\ y(t_6) = y(1.6) &= 4.62082. \end{aligned}$$

Now using linear interpolation, approximate y can be found in the following way.

$$\begin{aligned} y(1.04) &= \frac{1.04 - 1.1}{1 - 1.1} y(1.0) + \frac{1.04 - 1}{1.1 - 1} y(1.1) = 0.10873120. \\ y(1.55) &= \frac{1.55 - 1.6}{1.5 - 1.6} y(1.5) + \frac{1.55 - 1.5}{1.6 - 1.5} y(1.6) = 3.90413500. \end{aligned}$$

3.3. Runge-Kutta Methods: This is the one of the most important method to solve the IVP. These techniques were developed around 1900 by the German mathematicians C. Runge and M. W. Kutta. If we apply Taylor's Theorem directly then we require that the function have higher-order derivatives. The class of Runge-Kutta methods does not involve higher-order derivatives which is the advantage of this class.

Euler's method is an example of the Runge-Kutta method of first-order and modified Euler's method is an example of second-order Runge-Kutta method.

Third-order Runge-Kutta methods: Like-wise modified Euler's, using Simpson's rule to approximate the integral, we obtain the following Runge-Kutta method of order three.

$$\begin{aligned} t_{i+1} &= t_i + h \\ K_1 &= hf(t_i, y_i) \\ K_2 &= hf(t_i + h/2, y_i + K_1/2) \\ K_3 &= hf(t_i + h, y_i - K_1 + 2K_2) \\ y_{i+1} &= y_i + \frac{1}{6}(K_1 + 4K_2 + K_3). \end{aligned}$$

There are different Runge-Kutta method of order three. Most commonly used method is Heun's method, given by

$$\begin{aligned} t_{i+1} &= t_i + h \\ y_{i+1} &= y_i + \frac{h}{4} \left[f(t_i, y_i) + 3f \left(t_i + \frac{2h}{3}, y_i + \frac{2h}{3}f(t_i, y_i) \right) \right]. \end{aligned}$$

Runge-Kutta methods of order three are not generally used. The most common Runge-Kutta method in use is of order four, is given by the following.

Fourth-order Runge-Kutta method:

$$\begin{aligned} t_{i+1} &= t_i + h \\ K_1 &= hf(t_i, y_i) \\ K_2 &= hf(t_i + h/2, y_i + K_1/2) \\ K_3 &= hf(t_i + h/2, y_i + K_2/2) \\ K_4 &= hf(t_i + h, y_i + K_3) \\ y_{i+1} &= y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) + O(h^5). \end{aligned}$$

Local truncation error in the Runge-Kutta method is the error that arises in each step because of the truncated Taylor series. This error is inevitable. The fourth-order Runge-Kutta involves a local truncation error of $O(h^5)$.

Example 9. Using Runge-Kutta fourth-order, solve $\frac{dy}{dt} = \frac{y^2 - t^2}{y^2 + t^2}$, $y(0) = 1$ at $t = 0.2$ and 0.4 with $h = 0.2$.

Sol.

$$\begin{aligned} f(t, y) &= \frac{y^2 - t^2}{y^2 + t^2}, t_0 = 0, y_0 = 1, h = 0.2 \\ K_1 &= hf(t_0, y_0) = 0.2f(0, 1) = 0.200 \\ K_2 &= hf\left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2f(0.1, 1.1) = 0.19672 \\ K_3 &= hf\left(t_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2f(0.1, 1.09836) = 0.1967 \\ K_4 &= hf(t_0 + h, y_0 + K_3) = 0.2f(0.2, 1.1967) = 0.1891 \\ y_1 &= y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + 0.19599 = 1.196 \\ \therefore y(0.2) &= 1.196. \end{aligned}$$

Now

$$\begin{aligned} t_1 &= t_0 + h = 0.2 \\ K_1 &= hf(t_1, y_1) = 0.1891 \\ K_2 &= hf\left(t_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.2f(0.3, 1.2906) = 0.1795 \\ K_3 &= hf\left(t_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2f(0.3, 1.2858) = 0.1793 \\ K_4 &= hf(t_1 + h, y_1 + K_3) = 0.2f(0.4, 1.3753) = 0.1688 \\ y_2 &= y(0.4) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.196 + 0.1792 = 1.3752. \end{aligned}$$

4. NUMERICAL SOLUTION OF SYSTEM AND SECOND-ORDER EQUATIONS

We can apply the Euler and Runge-Kutta methods to find the numerical solution of system of differential equations. Second-order equations can be changed in to system of differential equations. The application of numerical methods are explained in the following examples.

Example 10. Solve the following system

$$\begin{aligned}\frac{dx}{dt} &= 3x - 2y \\ \frac{dy}{dt} &= 5x - 4y \\ x(0) &= 3, y(0) = 6.\end{aligned}$$

Find solution by Euler's method at 0.1 and 0.2 by taking time increment 0.1.

Sol. Given $t_0 = 0$, $x_0 = 3$, $y_0 = 6$, $h = 0.1$.

Write $f(t, x, y) = 3x - 2y$, $g(t, x, y) = 5x - 4y$.

By Euler's method

$$\begin{aligned}x_1 &= x(0.1) = x_0 + hf(t_0, x_0, y_0) = 3 + 0.1(3 \times 3 - 2 \times 6) = 2.7. \\ y_1 &= y(0.1) = y_0 + hg(t_0, x_0, y_0) = 6 + 0.1(5 \times 3 - 4 \times 6) = 5.1.\end{aligned}$$

Similarly

$$\begin{aligned}x_2 &= x(0.2) = x_1 + hf(t_1, x_1, y_1) = 2.7 + 0.1(3 \times 2.7 - 2 \times 5.1) = 2.49. \\ y_2 &= y(0.2) = y_1 + hg(t_1, x_1, y_1) = 5.1 + 0.1(5 \times 2.7 - 4 \times 5.1) = 4.41.\end{aligned}$$

Example 11. Solve the following system

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy$$

for $x = 0.3$ by using fourth-order Runge-Kutta method. Given $y(0) = 0$, $z(0) = 1$.

Sol. Given

$$\begin{aligned}\frac{dy}{dx} &= 1 + xz = f(x, y, z), \quad \frac{dz}{dx} = -xy = g(x, y, z) \\ x_0 &= 0, y_0 = 0, z_0 = 1, h = 0.3 \\ K_1 &= hf(x_0, y_0, z_0) = 0.3f(0, 0, 1) = 0.3 \\ L_1 &= hg(x_0, y_0, z_0) = 0.3g(0, 0, 1) = 0 \\ K_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right) = 0.3f(0.15, 0.15, 1) = 0.346 \\ L_2 &= hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right) = -0.00675 \\ K_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right) = 0.34385 \\ L_3 &= hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right) = -0.007762 \\ K_4 &= hf(x_0 + h, y_0 + K_3, z_0 + L_3) = 0.3893 \\ L_4 &= hg(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.03104.\end{aligned}$$

Hence

$$\begin{aligned}y_1 &= y(0.3) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.34483 \\ z_1 &= z(0.3) = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) = 0.9899.\end{aligned}$$

Example 12. A simple model to account for the way in which two different animal species sometimes interact is the predator-prey model. If $u(t)$ is the number of individuals in the predator species and $v(t)$ is the number of individuals in the prey species, then under suitable simplifying assumptions, the model is given by

$$\begin{aligned}\frac{du}{dt} &= 0.25u - 0.01uv, \quad u(0) = 80 \\ \frac{dv}{dt} &= 0.01uv - v, \quad v(0) = 30.\end{aligned}$$

Use the fourth-order Runge-Kutta method with step-size $h = 1$ to approximate the solution at $t = 1$.

Sol. We have

$$\begin{aligned}\frac{du}{dt} &= 0.25u - 0.01uv = f(t, u, v) \\ \frac{dv}{dt} &= 0.25u - 0.01uv = g(t, u, v), \\ u_0 &= 80, v_0 = 30, t_0 = 0, h = 1, t_1 = t_0 + h = 1.\end{aligned}$$

In order to apply Runge-Kutta method of order four, we calculate the following values:

$$\begin{aligned}K_1 &= hf(t_0, u_0, v_0) = -4 \\ L_1 &= hg(t_0, u_0, v_0) = -6 \\ K_2 &= hf\left(t_0 + \frac{h}{2}, u_0 + \frac{K_1}{2}, v_0 + \frac{L_1}{2}\right) = -1.56. \\ L_2 &= hg\left(t_0 + \frac{h}{2}, u_0 + \frac{K_1}{2}, v_0 + \frac{L_1}{2}\right) = -5.94 \\ K_3 &= hf\left(t_0 + \frac{h}{2}, u_0 + \frac{K_2}{2}, v_0 + \frac{L_2}{2}\right) = -1.6082 \\ L_3 &= hg\left(t_0 + \frac{h}{2}, u_0 + \frac{K_2}{2}, v_0 + \frac{L_2}{2}\right) = -5.6168 \\ K_4 &= hf(t_0 + h, u_0 + K_3, v_0 + L_3) = 0.4835 \\ L_4 &= hg(x_0 + h, u_0 + K_3, v_0 + L_3) = -5.2688.\end{aligned}$$

Therefore

$$\begin{aligned}u(1) \approx u_1 &= u_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 78.3579. \\ v(1) \approx v_1 &= v_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) = 24.2696.\end{aligned}$$

Example 13. Solve by using fourth-order Runge-Kutta method for $x = 0.2$ with $h = 0.2$:

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2, \quad y(0) = 1, \quad y'(0) = 0.$$

Sol. Let

$$\frac{dy}{dx} = z = f(x, y, z).$$

Therefore

$$\frac{dz}{dx} = xz^2 - y^2 = g(x, y, z).$$

Now

$$\begin{aligned}x_0 &= 0, y_0 = 1, z_0 = 0, h = 0.2 \\ K_1 &= hf(x_0, y_0, z_0) = 0.0 \\ L_1 &= hg(x_0, y_0, z_0) = -0.2 \\ K_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right) = -0.02 \\ L_2 &= hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right) = -0.1998 \\ K_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right) = -0.02 \\ L_3 &= hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right) = -0.1958 \\ K_4 &= hf(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.0392 \\ L_4 &= hg(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.1905.\end{aligned}$$

Hence

$$y_1 = y(0.2) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.9801.$$

$$z_1 = y'(0.3) = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) = -0.1970.$$

EXERCISES

- (1) Show that each of the following initial-value problems (IVP) has a unique solution, and find the solution.

(a) $y' = y \cos t, 0 \leq t \leq 1, y(0) = 1.$

(b) $y' = \frac{2}{t}y + t^2 e^t, 1 \leq t \leq 2, y(1) = 0.$

- (2) Generate $y_0(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$ for the initial-value problem using Picard's method

$$y' = -y + t + 1, \quad 0 \leq t \leq 1, \quad y(0) = 1.$$

- (3) Use Taylor's method of order two and four to approximate the solution for the following initial-value problem.

$$y' = y/t - (y/t)^2, \quad 1 \leq t \leq 1.2, \quad y(1) = 1, \quad h = 0.1.$$

- (4) Use Euler's method to approximate the solutions for each of the following initial-value problems.

(a) $y' = te^{3t} - 2y, 0 \leq t \leq 1, y(0) = 0, h = 0.5$

(b) $y' = 1 + (t - y)^2, 2 \leq t \leq 3, y(2) = 1, h = 0.5.$

- (5) Show that the following initial-value problem has a unique solution.

$$y' = t^{-2} (\sin 2t - 2ty), \quad 1 \leq t \leq 2, \quad y(1) = 2.$$

Find $y(1.1)$ and $y(1.2)$ with step-size $h = 0.1$ using modified Euler's method.

- (6) Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \leq t \leq 2, \quad y(1) = -1,$$

with exact solution $y(t) = -\frac{1}{t}$.

- (a) Use modified Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of y .

- (b) Use the answers generated in part (a) and linear interpolation to approximate the following values of y , and compare them to the actual values.

- i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$.

- (7) Use the modified Euler's method to approximate the solution to the following initial-value problem

$$y' = te^{3t} - 2y, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad h = 0.5.$$

- (8) A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8$ m/s is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8$ m/s² and $k = 0.002$ kg/m. The differential equation for the velocity v is given by

$$mv' = -mg - kv|v|.$$

- (a) Find the velocity after 0.1, 0.2, ..., 1.0s.

- (b) To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.

- (9) Using Runge-Kutta fourth-order method to solve the IVP in $[0, 1]$ for

$$\frac{dy}{dx} = y - x^2 + 1, \quad y(0) = 0.5$$

with mesh length $h = 0.5$.

- (10) Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2 \sqrt{2g} \frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft/s², and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)$ ft³. Use the Runge-Kutta method of order four to find the following.

- (a) The water level after 10 min with $h = 20$ s.
- (b) When the tank will be empty, to within 1 min.

- (11) The following system represent a much simplified model of nerve cells

$$\begin{aligned}\frac{dx}{dt} &= x + y - x^3, \quad x(0) = 0.5 \\ \frac{dy}{dt} &= -\frac{x}{2}, \quad y(0) = 0.1\end{aligned}$$

where $x(t)$ represents voltage across the boundary of nerve cell and $y(t)$ is the permeability of the cell wall at time t . Solve this system using Runge-Kutta fourth-order method to generate the profile up to $t = 0.2$ with step size 0.2.

- (12) The motion of a swinging pendulum is described by the following second-order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0, \quad \theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0,$$

where θ be the angle with vertical at time t , length of the pendulum $L = 2$ ft, and $g = 32.17$ ft/s². With $h = 0.1$ s, find the angle θ at $t = 0.1$ using Runge-Kutta fourth order method.

- (13) Use Runge-Kutta method of order four to solve

$$t^2 y'' - 2ty' + 2y = t^3 \ln t, \quad 1 \leq t \leq 1.2, \quad y(1) = 1, \quad y'(1) = 0$$

with $h = 0.1$.

APPENDIX A. ALGORITHMS

Algorithm for second-order Runge-Kutta method:

for $i = 0, 1, 2, \dots$ do

$$\begin{aligned}t_{i+1} &= t_i + h = t_0 + (i + 1)h \\ K_1 &= hf(t_i, y_i) \\ K_2 &= hf(t_{i+1}, y_i + K_1) \\ y_{i+1} &= y_i + \frac{1}{2}(K_1 + K_2).\end{aligned}$$

end for

Algorithm for fourth-order Runge-Kutta method:

for $i = 0, 1, 2, \dots$ do

$$\begin{aligned}t_{i+1} &= t_i + h \\ K_1 &= hf(t_i, y_i) \\ K_2 &= hf(t_i + h/2, y_i + K_1/2) \\ K_3 &= hf(t_i + h/2, y_i + K_2/2) \\ K_4 &= hf(t_{i+1}, y_i + K_3) \\ y_{i+1} &= y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4).\end{aligned}$$

end for

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