LECTURE-20

UEI407

Translation in the frequency domain

If f(t) $\leftrightarrow$ F(ω), according to the property it can be written as follows,

$$e^{j\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0).$$

Therefore, multiplication of f(t) by $e^{j\omega_0 t}$ in time domain is equivalent to shifting its Fourier transform to the right by ω_0. If f(t) is multiplied by $e^{-j\omega_0 t}$, it implies that its Fourier transform will be shifted to the left by ω_0

Therefore, $e^{-j\omega_0 t} f(t) \leftrightarrow F(\omega + \omega_0)$.

Proof:

Since $F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\,\omega t}\, dt$

$$\therefore F\left[e^{j\,\omega_0 t}\, f(t)\right] = \int_{-\infty}^{\infty} e^{j\,\omega_0 t} f(t)\, e^{-j\,\omega t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{-j\,(\omega-\omega_0)t}\, dt = F(\omega - \omega_0)$$

Similarly, $F\left[e^{-j\,\omega_0 t}\, f(t)\right] = F(\omega + \omega_0)$

Modulation Theorem

If any signal f(t) is multiplied by a cosine or sine signal ($\cos\omega_0 t$ or $\sin\omega_{0t)}$, it splits up the Fourier transform of f(t) into two parts, one shifted to the left by ω_0 and other shifted to the right by ω_0 each part having half the magnitude of $F(\omega + \omega_0)$ and $F(\omega - \omega_0)$.

If f(t) ↔ F(ω), according to this theorem,

$$f(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$$

and $f(t)\sin\omega_0 t \leftrightarrow \frac{1}{2j}[F(\omega-\omega_0)-F(\omega+\omega_0)]$

Proof:

We know that $F\left[f(t)e^{j\omega_0 t}\right] = F(\omega-\omega_0)$ and

$F\left[f(t)e^{-j\omega_0 t}\right] = F(\omega+\omega_0)$

Therefore, we can write the following:

$$F[f(t)\cos\omega_0 t] = F\left[\left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right)f(t)\right] =$$

$$F\left[\frac{1}{2}e^{j\omega_0 t}f(t)+\frac{1}{2}e^{-j\omega_0 t}f(t)\right]$$

$$= \frac{1}{2}F(\omega-\omega_0)+\frac{1}{2}F(\omega+\omega_0)$$

$$= \frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$$

Similarly,

$$F[f(t)\sin\omega_0 t] = F\left[\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)f(t)\right]$$

$$= F\left[\frac{1}{2j}e^{j\omega_0 t}f(t)-\frac{1}{2j}e^{-j\omega_0 t}f(t)\right]$$

$$= \frac{1}{2j}F(\omega-\omega_0)-\frac{1}{2j}F(\omega+\omega_0)$$

$$= \frac{j}{2}[F(\omega+\omega_0)-F(\omega-\omega_0)]$$

Symmetry or Duality Property

This property suggests that there must be dual relation between time and frequency domain. The sampling function is obtained from Fourier transform of a gate function. Therefore, the Fourier transform of sampling function in time domain will involve gate function in frequency domain.

It states that if $f(t) \leftrightarrow F(\omega)$, it can be written as $F(t) \leftrightarrow 2\pi f(-\omega)$.

Proof:

We now that $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$.

Therefore , $f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t}\, d\omega$

i.e., $2\pi\, f(-t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t}\, d\omega$

Changing ω to t and t to ω, we get

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dt$$

Therefore, the suggests that Fourier transform of F(t) is $2\pi f(-\omega)$ i.e.,

$F(t) \leftrightarrow 2\pi f(-\omega)$

Time Convolution Property

It states that convolution of two functions in time domain is equivalent to their multiplication in frequency domain. To the magnetic resonance scientist, the most important theorem concerning Fourier transforms is the convolution theorem.

If $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$, according to this property it can be written that

$$f_1(t) * f_2(t) \leftrightarrow F_1(\omega)\, F_2(\omega)$$

where the symbol * means 'convolve with'.

The convolved signal is defined by the convolution integral i.e.,

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda)\, d\lambda \quad or \quad f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_2(\lambda) f_1(t-\lambda)\, d\lambda$$

Proof:

$F[f(t)] = \int_{-\infty}^{\infty} f(t)\ e^{-j\ \omega t}\ dt$

Let $f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda) d\lambda$

Therefore,F $[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} [f_1(t) * f_2(t)]\ e^{-j\ \omega t}\ dt$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda) d\lambda \right] e^{-j\ \omega t}\ dt$$

Let $t = x + \lambda$.

Therefore $dt = dx$

F $[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\lambda) f_2(x)\ d\lambda \right] e^{-j\ \omega(x+\lambda)}\ dx$

$= \left[\int_{-\infty}^{\infty} f_1(\lambda) e^{-j\ \omega\ \lambda} d\lambda \right] \left[\int_{-\infty}^{\infty} f_2(x) e^{-j\ \omega\ x}\ dx \right] = F_1(\omega)\ F_2(\omega)]$

Frequency convolution

This property states that the multiplication of two time functions $f_1(t)$ and $f_2(t)$ in time domain is equivalent to convolving their Fourier transforms in Frequency domain divided by 2π. This is dual time convolution property. Mathematically, we can write

If $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$, according to this property it can be said that

$$f_1(t) \cdot f_2(t) \leftrightarrow \frac{F_{1(\omega)*}\ F_{2(\omega)}}{2\pi}$$

Proof:

$$F^{-1}\left[\frac{F_1(\omega)*F_2(\omega)}{2\pi}\right] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}[F_1(\omega)*F_2(\omega)]\, e^{j\omega t}\, d\omega$$

Again, $F_1(\omega)*F_2(\omega) = \int_{-\infty}^{\infty} F_1(u)F_2(\omega-u)du$

Hence,

$$F^{-1}\left[\frac{F_1(\omega)*F_2(\omega)}{2\pi}\right] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F_1(u)F_2(\omega-u)du\right] e^{j\omega t}\, d\omega = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} F_1(u)\left[\int_{-\infty}^{\infty} F_2(\omega-u)du\right] e^{j\omega t}\, d\omega$$

Let us put $\omega = u + \lambda$.

i.e., $d\omega = d\lambda$

$$F^{-1}\left[\frac{F_1(\omega)*F_2(\omega)}{2\pi}\right] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} F_1(u)\left[\int_{-\infty}^{\infty} F_2(\lambda)e^{j(\lambda+u)\, t}\, d\lambda\right] du$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u)e^{ju\, t}\, du\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\lambda)e^{j\lambda\, t}\, d\lambda\right]$$

$= f_1(t)\ f_2(t)$

Frequency differentiation

This property states that multiplication of a function f(t) in time domain by (–jt) is equivalent to the differentiation of Fourier transform in 'ω' domain. Therefore, this property is a sort of dual for time differentiation. Mathematically, we can write

if $f(t) \leftrightarrow F(\omega)$, according to this property it can be written as follows

$$(-\mathrm{j}t)f(t) \leftrightarrow \frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega} \qquad (-\mathrm{j}t)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{\mathrm{d}\omega^n}$$

Proof:

Fourier Transform of f(t) is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\mathrm{j}\omega\, t}\, \mathrm{d}t$$

Differentiating both sides with respect to ω, we get

$$\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega} = \int_{-\infty}^{\infty} [(-\mathrm{j}\ t)f(t)] e^{-\mathrm{j}\omega\, t}\, \mathrm{d}t$$

Therefore, Fourier Transform of $[(-\mathrm{j}\ t)f(t)]$ is $\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}$.

Hence we can write

$$[(-\mathrm{j}\ t)f(t)] \leftrightarrow \frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}$$

Similarly, we can prove

$$[(-\mathrm{j}\ t)^n f(t)] \leftrightarrow \frac{d^n F(\omega)}{\mathrm{d}\omega^n}$$