

# Relations

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# Contents

- Relations and Its Introduction
- Representation of Relations:
  - Using Matrices
  - Using Diagraph
- Properties of Relations
- Inverse and Complementary Relations
- Combining Relations
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- Equivalence Relations
- Equivalence Classes
- Equivalence Relations and Partitions
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- **Partial Ordering and Partially Ordered Set**
- Lexicographic Ordering
- Hasse diagram
- Topological Sorting
- Lattices
- Special Types of Lattices

# Partial Ordering

- A relation  $R$  on a set  $P$  is called a **partial ordering**, or **partial order**, if it is:
  - Reflexive
  - Antisymmetric
  - Transitive
- A set  $P$  together with a partial order relation  $R$ , defined on it, is called a **partially ordered set**, or **poset**, and is denoted by  $(P, R)$ . Members of  $P$  are called *elements* of the poset.

# Examples

1. Consider the set of integers. Is the relation “less than or equal” ( $\leq$ ), a partial ordering on the given set of Integers?
  - Reflexive?
  - Antisymmetric?
  - Transitive?

Yes

## Examples (Cont..)

2. Consider the set of integers. Is the relation “divisibility” ( $|$ ), a partial ordering on the given set of Integers?

- ❑ Reflexive?
- ❑ Antisymmetric?
- ❑ Transitive?

No

$(\mathbb{Z}^+, |)$  is a POSET.

## Examples (Cont..)

3. Show that the inclusion relation ( $\subseteq$ ) is a partial ordering on the power set of a set  $S$ .
- Reflexive?
  - Antisymmetric?
  - Transitive?

# Comparability

- The elements  $a$  and  $b$  of a poset  $(P, \preceq)$  are *comparable* if either  $a \preceq b$  or  $b \preceq a$ .
- When  $a$  and  $b$  are elements of  $P$  so that neither  $a \preceq b$  nor  $b \preceq a$ , then  $a$  and  $b$  are called *incomparable*.

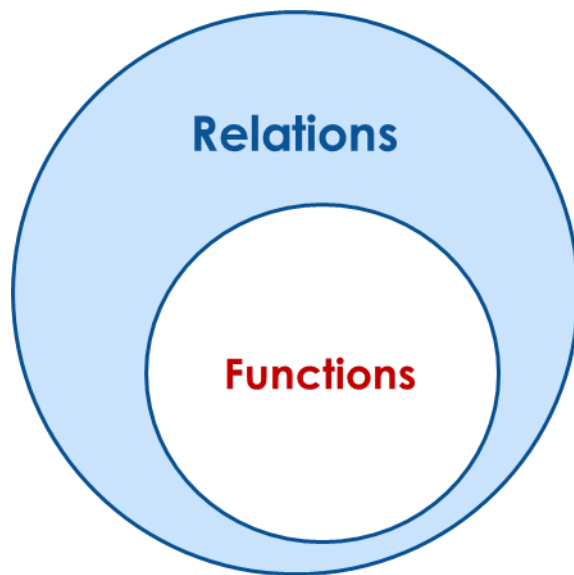
The symbol  $\preceq$  is used to denote the relation in any poset.

# Comparability (Cont..)

- If  $(P, \preceq)$  is a poset and every two elements of  $P$  are comparable,  $P$  is called a **totally ordered** or **linearly ordered set**, and  $\preceq$  is called a **total order** or a **linear order**.
- A totally ordered set is also called a *chain*.
- $(P, \preceq)$  is a **well-ordered set** if it is a poset such that  $\preceq$  is a total ordering and every nonempty subset of  $P$  has a least element.

Thank  
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# Lexicographic Ordering

- Given two posets  $(A_1, \preceq_1)$  and  $(A_2, \preceq_2)$ , the *lexicographic ordering* on  $A_1 \times A_2$  is defined by specifying that  $(a_1, a_2)$  is less than  $(b_1, b_2)$ , that is,
 
$$(a_1, a_2) < (b_1, b_2),$$
 either if  $a_1 <_1 b_1$  or if  $a_1 = b_1$  and  $a_2 <_2 b_2$ .
- This definition can be easily extended to a lexicographic ordering on strings.

# Examples

1. Consider strings of lowercase English letters.

A lexicographic ordering can be defined using the ordering of the letters in the alphabet.

This is the same ordering as that used in dictionaries.

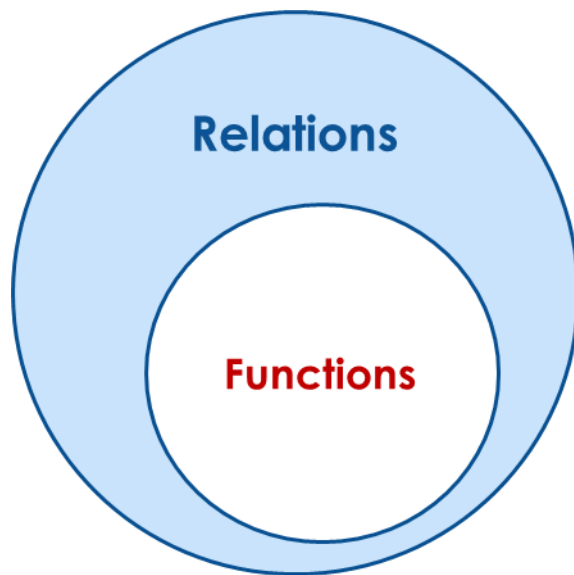
- *discreet* < *discrete*, because these strings differ in the seventh position and  $e < t$ .
- *discreet* < *discretion*, because the first six letters agree, but the strings differ in the seventh position and  $e < t$ .

## Examples (Cont..)

2. Determine whether  $(3, 5) < (4, 8)$ , whether  $(3, 8) < (4, 5)$  and whether  $(4, 9) < (4, 11)$  in the poset  $(\mathbb{Z} \times \mathbb{Z}, \preceq)$ , where  $\preceq$  is the lexicographic ordering constructed from the usual  $\leq$  relation on  $\mathbb{Z}$ .

Thank  
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# Hasse Diagram

- A **Hasse diagram** is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

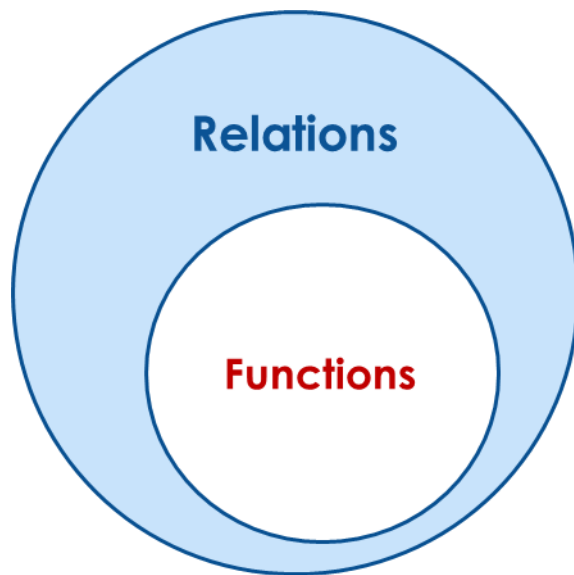
# Procedure for drawing a Hasse Diagram

- To represent a finite poset  $(S, \preceq)$  using a Hasse diagram, start with the directed graph of the relation:
  - ❑ Remove the loops  $(a, a)$  present at every vertex due to the reflexive property.
  - ❑ Remove all edges  $(x, y)$  for which there is an element  $z \in S$  such that  $x \prec z$  and  $z \prec y$ . These are the edges that must be present due to the transitive property.
  - ❑ Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

# Examples

Thank  
you!!!





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# Topological Sorting

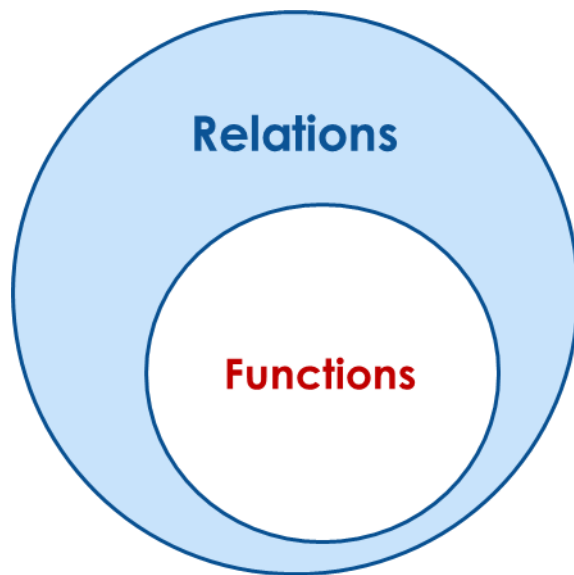
- If  $A$  is a poset with partial order  $\preceq$ , we sometimes need to find a linear order  $<$  for the set  $A$  in the sense that if  $a \preceq b$  then  $a < b$ .
- The process of constructing a linear order is called Topological Sorting.

**Linear Order corresponding to partial ordering**

# Examples

Thank  
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# Extremal Elements of POSET

- Maximal Element
- Minimal element

# Maximal and Minimal Elements

- Let  $(A, \preceq)$  be a POSET.
- An element  $M \in A$  is called a **maximal element** of  $A$  if there is no element  $x$  in  $A$  such that  $M < x$ .
- An element  $m \in A$  is called a **minimal element** of  $A$  if there is no element  $x$  in  $A$  such that  $x < m$ .

# Examples

1. Let  $A$  be the poset of all non -ve real numbers with the usual partial order  $\leq$ .

Minimal Element?

0

Maximal Element?

No maximal element

2. Let us consider the poset  $(\mathbb{Z}, \leq)$ .

No Minimal Element

No Maximal Element

# Greatest Element and Least Element

- An element  $a \in A$  is called a greatest element of  $A$  if  $x \leq a, \forall x \in A$ .
- An element  $a \in A$  is called a least element of  $A$  if  $a \leq x, \forall x \in A$ .

# Examples

1. Let  $A$  be the POSET of all non -ve real numbers with the usual partial order  $\leq$ .

Least Element?

0

Greatest Element?

No greatest element

2. Let us consider the POSET  $(P(S), \subseteq)$ .

Least Element?

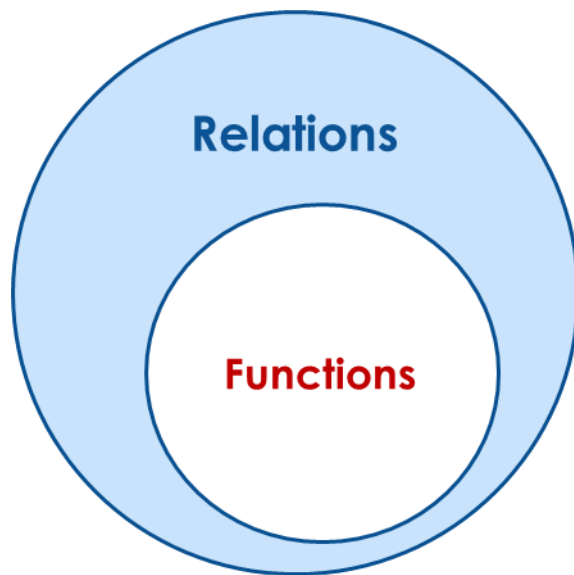
$\emptyset$

Greatest Element?

Set  $S$

Thank  
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# Upper Bound and Lower Bound

- Let  $A$  be a POSET and  $B \subseteq A$ .
- An element  $a \in A$  is called an upper bound of  $B$  if
$$b \leq a, \forall b \in B$$
- An element  $a \in A$  is called a lower bound of  $B$  if
$$a \leq b, \forall b \in B$$

# Greatest Lower Bound (GLB)

- Let  $A$  be a POSET and  $B \subseteq A$ .
- An element  $a \in A$  is called a **Greatest Lower Bound (GLB)** of  $B$  if  $a$  is a lower bound of  $B$  and  $a' \leq a$ , whenever  $a'$  is a lower bound of  $B$ .
- Thus,  $a = \text{GLB}(B)$ , if  $a \leq b, \forall b \in B$  and if whenever  $a' \in A$  is also a lower bound of  $B$  ( $a' \leq b, \forall b \in B$ ) then  $a' \leq a$ .

# Least Upper Bound (LUB)

- Let  $A$  be a POSET and  $B \subseteq A$ .
- An element  $a \in A$  is called a **Least Upper Bound (LUB)** of  $B$  if  $a$  is an upper bound of  $B$  and  $a \leq a'$ , whenever  $a'$  is an upper bound of  $B$ .
- Thus,  $a = \text{LUB}(B)$ , if  $b \leq a, \forall b \in B$  and if whenever  $a' \in A$  is also an upper bound of  $B$  ( $b \leq a', \forall b \in B$ ) then  $a \leq a'$ .

# Examples

- Let  $(A, \preceq)$  be a POSET on  $A = \{a, b, c, d, e, f, g, h\}$ .  
Hasse Diagram is shown Below:

Find all **upper and lower bounds** of the following subsets of  $A$ :

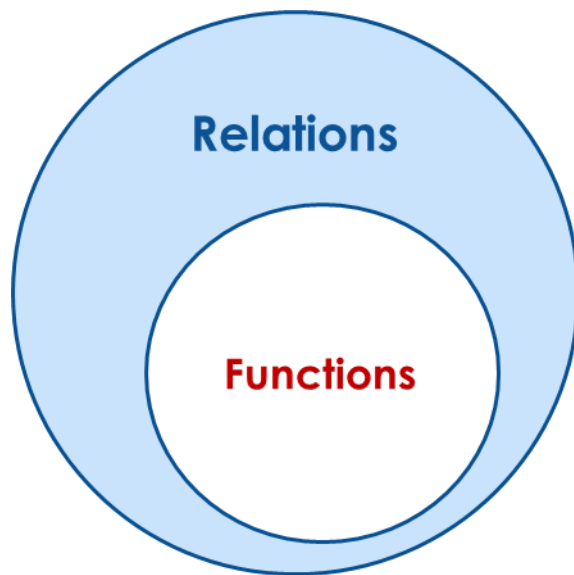
a)  $B_1 = \{a, b\}$

b)  $B_2 = \{c, d, e\}$

Also find **Least Upper Bound (LUB)** and **Greatest Lower Bound (GLB)** for above subsets of  $A$ .

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## Lattice

- A lattice is a POSET  $(L, \leq)$  in which every subset  $\{a, b\}$  consisting of 2 elements has a Least Upper Bound (LUB) and a Greatest Lower Bound (GLB).
  - $LUB(\{a, b\}) = a \vee b$  (Join of  $a$  and  $b$ )
  - $GLB(\{a, b\}) = a \wedge b$  (Meet of  $a$  and  $b$ )

## Examples

1. Let us consider the POSET  $(P(S), \subseteq)$ .

□ Is this a lattice?

**Yes**

- Let  $A$  and  $B$  are 2 elements of  $P(S)$ .
- Then the join of  $A$  and  $B$  is their union  $A \cup B$ ,
- and the meet of  $A$  and  $B$  is their  $A \cap B$ .
- Hence,  $L$  is lattice.

# Examples

2. Let us consider the POSET  $(\mathbb{Z}^+, \leq)$ , where  $a \leq b$  iff  $a/b$ .

□ Is this a lattice?

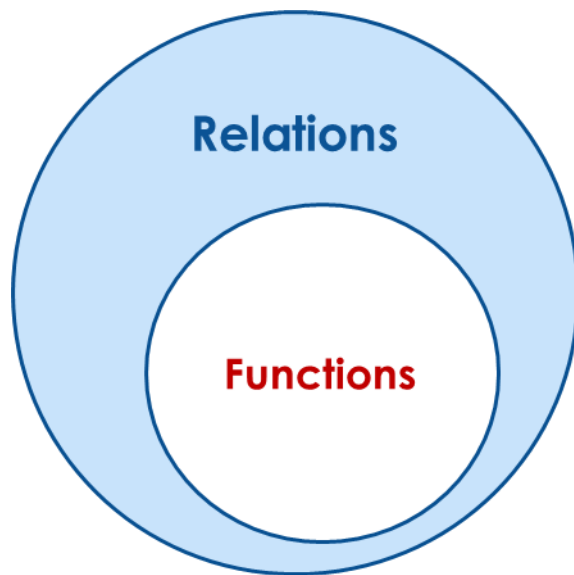
**Yes**

$$a \vee b = LCM(a, b)$$

$$a \wedge b = GCD(a, b)$$

Thank  
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# Special Types of Lattices

- Bounded Lattice
- Distributive Lattice
- Complemented Lattice
- Boolean Lattice

## Bounded Lattice

- A **lattice**  $L$  is said to be **bounded** if it has a greatest element  $I$  and a least element  $0$ .

### Examples

1. Let us consider lattice  $\mathbb{Z}^+$  under the partial order of divisibility.

Is it a bounded lattice?

No

## Examples (Cont..)

2. Let us consider lattice  $Z$  under the partial order of  $\leq$ .

Is it a bounded lattice?

No

3. Let us consider the lattice  $P(S)$  of all subsets of a set  $S$  with partial order subset.

Is it a bounded lattice?

Yes

**Theorem:** If  $L$  is a finite lattice then  $L$  is bounded.

# Distributive Lattice

- A **lattice**  $L$  is called distributive if for any elements  $a, b$  and  $c$  in  $L$ , we have the following distributive laws:
  - 1)  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  - 2)  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- If  $L$  is not distributive, then  $L$  is called nondistributive lattice.

# Examples

1. Let us consider Lattice  $P(S)$  with partial order of subset.

Is it Distributive?

**Yes**

2. Let us consider lattice  $Z^+$  under the partial order of  $\leq$ .

Is it Distributive?

**Yes**

# Complemented Lattice

- Let  $L$  be a bounded lattice with greatest element  $I$  and least element  $0$ , and let  $a \in L$ .
- An element  $a' \in L$  is called a complement of  $a$  if

$$a \vee a' = I$$

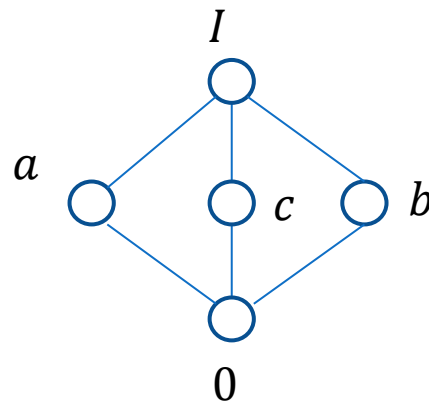
$$\text{and, } a \wedge a' = 0$$

$$0' = I \text{ and } I' = 0$$

**A lattice  $L$  is called complemented if it is bounded and if every element in  $L$  has a complement.**

# Examples

1. Lattice  $L = P(S)$  with subset partial order is **Complemented Lattice**.
2. Let us consider following Lattice:



Is it a complimented Lattice?

**Yes**

# Boolean Lattice

- A **lattice**  $L$  is called **Boolean Lattice** if it is
  - Bounded
  - Distributive
  - Complemented

## Example

- ❖ Lattice  $L = P(S)$  with subset partial order is a Boolean Lattice.

Thank  
you!!!

