Lecture 2: Numerical Analysis (UMA011)

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Error Analysis: Chopping and Rounding Errors

42.965
$= (0.4296) \times 10^2$

Maximum error bound in rounding

Let x be any real number we want to represent in a computer. Let $fl(x)$ be the representation by rounding with $n-$digits of x then what is largest possible values of $\frac{|x - fl(x)|}{|x|}$? $\frac{|x - fl(x)|}{|x|} \le \frac{\frac{1}{2} \times 10^{e-n}}{|x|}$?

$\left|\frac{x - fl(x)}{x}\right| \le L$?

$x \to$ real
$fl(x) \to$ app.

$A.E = |x - fl(x)|$

$R.E = \frac{A.E.}{|x|}$

Proof: Let x be an exact real no.

then $x = (0.a_1 a_2 a_3 \cdots a_n a_{n+1} \cdots)_{10} \times 10^e$ $\quad 0 \le a_i \le 9$

$x = \left(\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots\right) \times 10^e$

$x = \left(\sum_{i=1}^{\infty} \frac{a_i}{10^i}\right)_{10} \times 10^e$

$fl(x)$ be the app. to x by rounding with n-digits

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Proof(continued):

$$fl(x) = \begin{cases} (0.a_1 a_2 \cdots a_n)_{10} \times 10^e, & 0 \le a_{n+1} < 5 \\ (0.a_1 a_2 \cdots a_n + 1)_{10} \times 10^e, & 5 \le a_{n+1} \le 9 \end{cases}$$

$$= \begin{cases} (0.a_1 a_2 \cdots a_n)_{10} \times 10^e, & 0 \le a_{n+1} < 5 \\ \left[(0.a_1 \cdots a_n)_{10} + (0.000 \cdots \underset{n^{th}\ \text{place}}{1})_{10}\right] \times 10^e, & 5 \le a_{n+1} \le 9 \end{cases}$$

$$= \begin{cases} \left(\sum_{i=1}^{n} \frac{a_i}{10^i}\right) \times 10^e, & 0 \le a_{n+1} < 5 \\ \left(\sum_{i=1}^{n} \frac{a_i}{10^i} + \frac{1}{10^n}\right) \times 10^e, & 5 \le a_{n+1} \le 9 \end{cases} \checkmark$$

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Proof(continued):

$$|x - fl(x)| = \begin{cases} \left(\sum_{i=n+1}^{\infty} \frac{a_i}{10^i}\right) \times 10^e, & 0 \le a_{n+1} < 5 \\ \left(\sum_{i=n+1}^{\infty} \frac{a_i}{10^i} - \frac{1}{10^n}\right) \times 10^e, & 5 \le a_{n+1} \le 9 \end{cases}$$

$0 \le a_i \le 9$

Case I $\quad 0 \le a_{n+1} < 5 \Rightarrow a_{n+1} \le 4$

$$|x - fl(x)| = \left(\sum_{i=n+1}^{\infty} \frac{a_i}{10^i}\right) \times 10^e = \left(\frac{a_{n+1}}{10^{n+1}} + \sum_{i=n+2}^{\infty} \frac{a_i}{10^i}\right) \times 10^e$$

$$\le \left(\frac{4}{10^{n+1}} + \sum_{i=n+2}^{\infty} \frac{9}{10^i}\right) \times 10^e$$

$$|x - fl(x)| \le \left(\frac{4}{10^{n+1}} + 9 \sum_{i=n+2}^{\infty} \frac{1}{10^i} \right) \times 10^e$$

G.P. series

$a = \frac{1}{10^{n+2}}$ sum $= \frac{a}{1-r}$

$r = \frac{1}{10}$

$$= \left(\frac{4}{10^{n+1}} + 9 \times \frac{\frac{1}{10^{n+2}}}{1 - \frac{1}{10}} \right) \times 10^e$$

$$= \left(\frac{4}{10^{n+1}} + \cancel{9} \times \frac{\cancel{10}}{\cancel{9}} \times \frac{1}{10^{n+\cancel{2}}} \right) \times 10^e$$

$$= \left(\frac{4}{10^{n+1}} + \frac{1}{10^{n+1}} \right) \times 10^e = \frac{5}{10^{n+1}} \times 10^e$$

$$= \frac{5}{10} \times 10^{e-n} = \frac{1}{2} \times 10^{e-n}$$

NOW

$$x = (0.a_1 a_2 \cdots a_n a_{n+1} \cdots)_{10} \times 10^e$$

$0 \le a_i \le 9$, $i = 2, \cdots \infty$

$0 < a_1 \le 9 \Rightarrow 1 \le a_1 \le 9$

$$|x| = |(0.a_1 a_2 \cdots\cdots)_{10}| \times 10^e$$

$$\geq |(0.1000 \cdots\cdots)_{10}| \times 10^e = |(0.1)_{10}| \times 10^e$$

$$\Rightarrow \frac{1}{|x|} \le \frac{1}{(0.1)_{10} \times 10^e}$$

$$R.E = \frac{|x - fl(x)|}{|x|} \le \frac{\frac{1}{2} \times \cancel{10^{e}}^{-n}}{(0.1) \times \cancel{10^e}} = \frac{1}{2} \times 10^{1-n} = \frac{1}{2} \times 10^{1-n}$$

$$R.E \le \frac{1}{2} \times 10^{1-n}$$

Case II $\quad 5 \le a_{n+1} \le 9 \qquad -5 \ge -a_{n+1} \ge -9$

$a \ge b$

$$|x - fl(x)| = \left| \sum_{i=n+1}^{\infty} \frac{a_i}{10^i} - \frac{1}{10^n} \right| \times 10^e = \left| \frac{1}{10^n} - \sum_{i=n+1}^{\infty} \frac{a_i}{10^i} \right| \times 10^e$$

$\because |a-b| = |b-a|$

$a-b-c \le a-b$

$$= \left| \frac{1}{10^n} - \frac{a_{n+1}}{10^{n+1}} - \sum_{i=n+2}^{\infty} \frac{a_i}{10^i} \right| \times 10^e$$

$0 \le a_i \le 9$

$|\cos x| \le 1$

$|\cos x| \le 2$

$a \le b$

$-a \ge -b$

$0 \ge -a_i \ge -9$

$$\le \left| \frac{1}{10^n} - \frac{a_{n+1}}{10^{n+1}} \right| \times 10^e \le \left| \frac{1}{10^n} + \frac{-5}{10^{n+1}} \right| \times 10^e$$

$b \qquad a$

$$\le \left| \frac{10-5}{10^{n+1}} \right| \times 10^e = \frac{5}{10^{n+1}} \times 10^e = \frac{1}{2} \times 10^{e-n}$$

$$|x| \leq (0.1) \times 10^{e}$$

$$\boxed{\frac{|x - fl(x)|}{|x|} \leq \frac{1}{2} \times 10^{1-n}}$$