Lecture 32: Numerical Analysis (UMA011)

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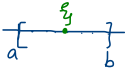
Newton Divided Difference Interpolation:

Result

Suppose that $f \in C^n[a,b]$ and $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a,b]$. Then a number ξ exists in (a,b) with

$$f[x_0, x_1, x_2, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

unknown.



$f(x) = 0$

Proof: Generalized Rolle's theorem :- let $f \in C^n[a,b]$ and $f(x)$ has $(n+1)$ distinct zeros in (a,b) then $\exists$ a no. ξ in (a,b) s.t. $f^{(n)}(\xi) = 0$.

Define $g(x) = f(x) - P_n(x)$ — (1)

where $P_n(x)$ is nth- degree Newton divided-difference interpolating polynomial

ie $P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

$+ \cdots f[x_0, x_1, x_2 \cdots x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$

Since $f \in C^n[a,b]$, $P_n(x) \in C^\infty[a,b]$, then $g \in C^n[a,b]$

Also, Put $x = x_0, x_1, x_2 \cdots x_n$ in eqn (1)

$g(x_0) = f(x_0) - P_n(x_0) = 0$, $g(x_1) = 0$, $g(x_2) = 0 \cdots g(x_n) = 0$

$\Rightarrow \quad g(x_i) = 0 \quad \forall \quad a \le i \le n.$

$\Rightarrow$ g has $(n+1)$ distinct zeros in (a,b)

By generalized Rolle's thm, $\exists$ a no. ξ in (a,b)

s.t. $\quad g^n(\xi) = 0$

Diff. eqn (1) n times

$$\left(g^{(n)}(x)\right)_{x=\xi} = \left(f^{(n)}(x)\right)_{x=\xi} - \left(P_n^{(n)}(x)\right)_{x=\xi}$$

$$0 = f^{(n)}(\xi) - n!\, f[x_0, x_1, \cdots x_n]$$

$$\because \begin{cases} P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ P_n^{(n)}(x) = a_n \, n! \end{cases}$$

$$\Rightarrow \quad f^{(n)}(\xi) = n! \, f[x_0, x_1, \ldots, x_n]$$

$$\Rightarrow \quad f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}, \quad \xi \in (a, b)$$

Newton Divided Difference Interpolation:

Example:

Let $f(x) = e^x$, show that $f[x_0, x_1, \ldots, x_m] > 0$ for all values of m and all distinct equally spaced nodes $\{x_0 < x_1 < \cdots < x_m\}$.

$f[x_0] > 0$

$f[x_0, x_1] > 0$

$f[x_0, x_1, x_2] > 0$

Solution:

Let $x_0, x_1, --x_m$ be $(m+1)$ distinct pts and h be the spacing betn them.

So, the pts are $x_0, x_0+h, x_0+2h --- x_0+mh$

Now, $f \in C^m[x_0, x_0+mh]$, then $\exists$ a no. ξ in (x_0, x_0+mh)

s.t $f[x_0, x_1, --- x_m] = \dfrac{f^{(m)}(\xi)}{m!} = \dfrac{e^{\xi}}{m!} > 0$

Exercise:

1. Let $f(x) = x^3$, compute $f[x_0, x_1, x_2, x_3]$ and $f[x_0, x_1, x_2, x_3, x_4]$ for the distinct nodes x_i, $1 \leq i \leq 4$.

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

As $f \in C^3[x_0, x_3]$ and x_0, x_1, x_2, x_3 and x_4 are distinct, then $\exists\, \xi \in (x_0, x_3)$ s.t. $f[x_0, x_1, x_2, x_3] = \dfrac{f^{(3)}(\xi)}{3!} = \dfrac{6}{3!} = 1$

$f[x_0, x_1, x_2, x_3, x_4] = \dfrac{f^{(4)}(\xi)}{4!} = 0$