Unit Parabolic Function

The unit parabolic function is defined by

$$a(t) = \begin{cases} \dfrac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

or, $$a(t) = \frac{t^2}{2}u(t)$$

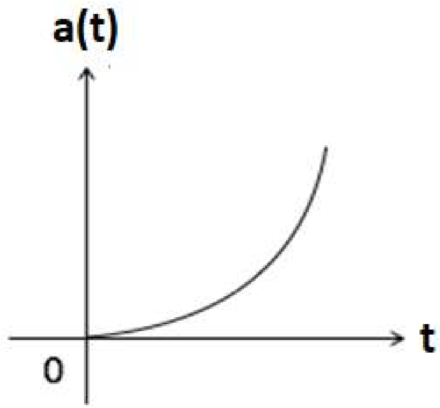


Figure 1

Figure 1 shows the parabolic function.

If the integration of unit ramp function is taken, the unit parabolic function is obtained.

$$a(t) = \int r(t)dt = \int t dt = \frac{t^2}{2} \qquad \text{(for } t \geq 0\text{)}$$

Also, $r(t) = \int u(t)dt = t \quad \text{(for } t \geq 0\text{)}$

$$\therefore a(t) = \int \left[\int u(t)dt \right] dt$$

$$\text{Step function} \xrightarrow{\textit{Integration}} \text{Ramp Function} \xrightarrow{\textit{Integration}} \text{Parabolic Function}$$

The derivative of the unit parabolic function gives the unit ramp function. We have

$$r(t) = \frac{da(t)}{dt} = t$$

Also, $u(t) = \dfrac{dr(t)}{dt} = \dfrac{d^2 a(t)}{dt^2} = 1$

$$\text{Parabolic function} \xrightarrow{\textit{Differentiation}} \text{Ramp Function} \xrightarrow{\textit{Differentiation}} \text{Step Function}$$

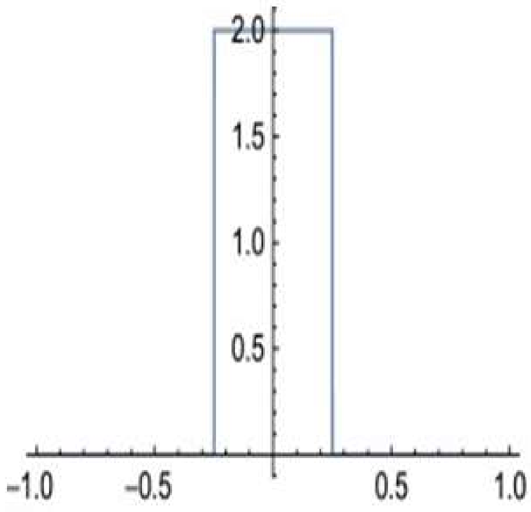
Unit Impulse Function

The unit impulse function $\delta(t)$ that imparts an impulse of magnitude one at $t = t_0$ but it is zero for all other values of t other than $t = t_0$. The unit impulse function $\delta(t)$, which is also known as Dirac delta function, has a very important role in system analysis. The properties of this function are

$$\delta(t) = \begin{cases} 0 & \mathrm{t} \neq 0 \\ \infty & \mathrm{t} = 0 \end{cases}$$

and $$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$

Figure 2 shows the unit impulse function which is defined as the limit of a suitably chosen conventional pulse function. This function has unity area over an infinitesimal time interval. This has been shown in Figure 3.



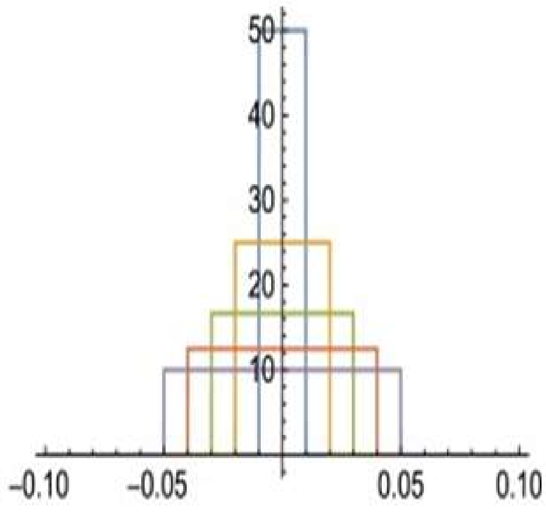


Figure 2

Figure 3

Figure 4 shows the delta function.

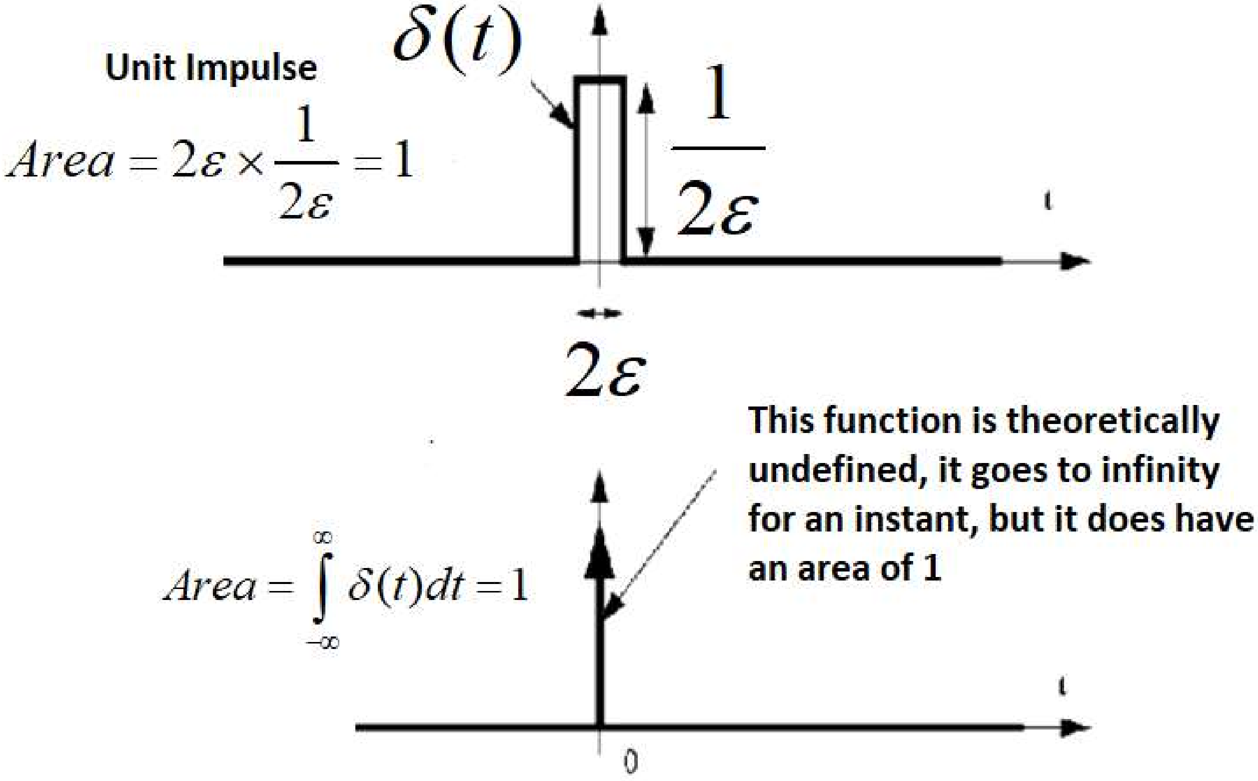


Figure 4

Figure 5 also shows the delta function.

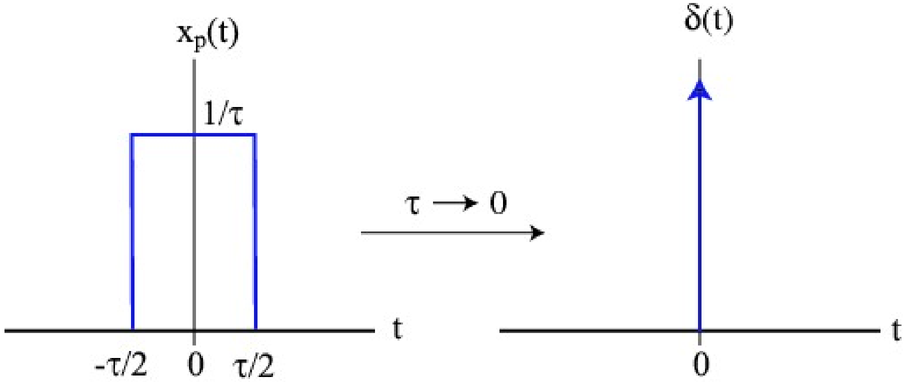


Figure 5

Figure 6 shows $\delta(t)$ function and Figure 7 shows $\delta(t\text{-}t_0)$ function.

$$\delta(t) = \begin{cases} 0 & for \quad t \neq 0 \\ \infty & for \quad t = 0 \end{cases}$$

$$Also \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\delta(t) = \delta(-t)$$

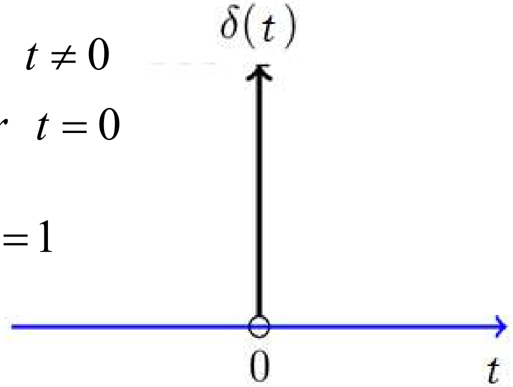
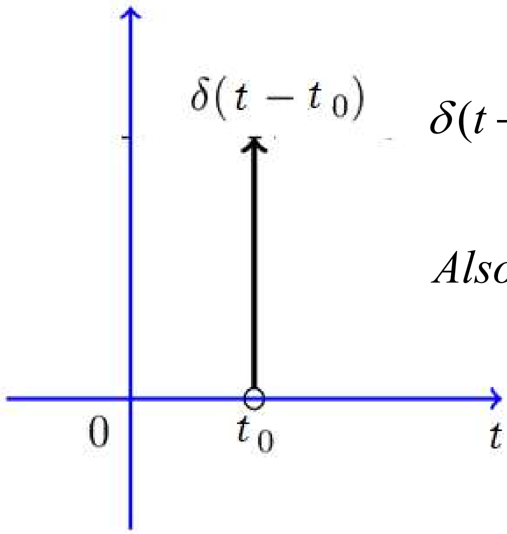


Figure 6

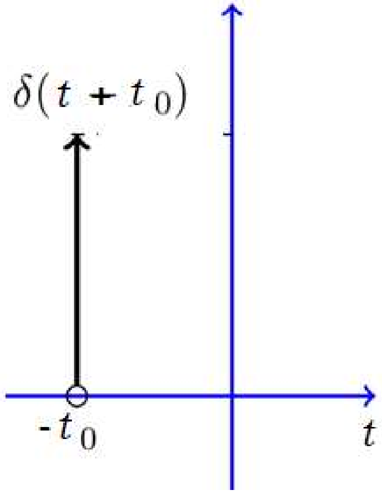


$$\delta(t - t_0) = \begin{cases} 0 & for \quad t \neq t_0 \\ \infty & for \quad t = t_0 \end{cases}$$

$$Also \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1$$

Figure 7

Figure 8 shows $\delta(t+t_0)$ function.



$$\delta(t+t_0) = \begin{cases} 0 \;\; for \;\; t \neq -t_0 \\ \infty \;\; for \;\; t = -t_0 \end{cases}$$

$$Also \int_{-\infty}^{\infty} \delta(t+t_0)dt = 1$$

Figure 8

Delta Function is symmetrical:

Figure 9 shows that delta function is symmetrical.

$$\delta(-(t-t_0)) = \delta(-t+t_0) = \delta(t_0-t)$$

$$\delta(t-t_0) = \delta(t_0-t)$$

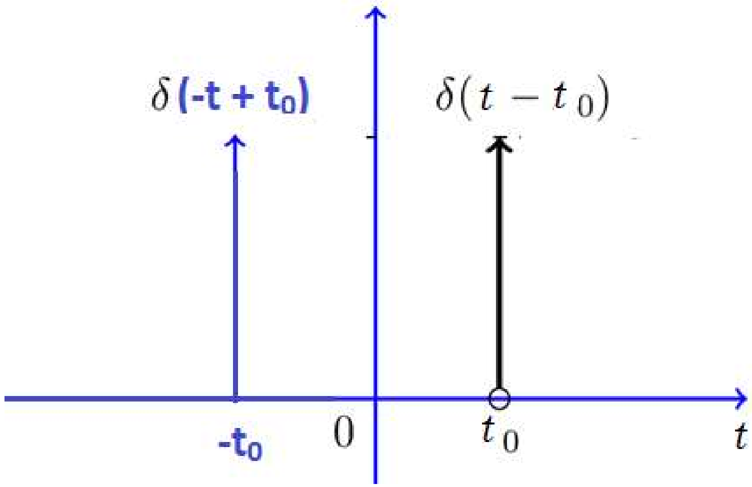


Figure 9

If $\phi(t)$ is a regular function which is continuous a $t = 0$, we have

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)dt = \phi(0)$$

The function $\delta(t)$ can also be defined as

$$\int_{\alpha}^{\beta} \phi(t)\delta(t)dt = \begin{cases} \phi(0) & \alpha < 0 < \beta \\ 0 & \alpha < \beta < 0 \text{ or } 0 < \alpha < \beta \\ \textit{undefined} & \alpha = 0 \text{ or } \beta = 0 \end{cases}$$

The above two equations should not be taken as ordinary Riemann integral. These are symbolic expressions. $\phi(t)$ considered above is known as testing function whereas $\delta(t)$ is known as a generalized function. A delayed delta function $\delta(t - t_0)$ is expressed by

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t - t_0)dt = \phi(t_0)$$

The properties of unit impulse function are discussed below.

1. $\int_{-\infty}^{+\infty} \phi(\mathrm{t})\delta(\mathrm{t})dt = \phi(0)$

A function $\phi(t)$ is considered here which is continuous at $t = 0$ having a value of $\phi(0)$ at $t = 0$. The product of $\phi(t)$ and $\delta(t)$ is $\phi(t)\ \delta(t)$. The impulse exists at $t = 0$. We have

$\phi(t)\delta(t) = \phi(0)\delta(t)$. We can write

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)dt = \int_{-\infty}^{+\infty} \phi(0)\delta(t)dt = \phi(0)\int_{-\infty}^{+\infty} \delta(t)dt = \phi(0)$$

$$\therefore \int_{-\infty}^{+\infty} \phi(t)\delta(t)dt = \phi(0)$$

2. $\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$

The signal is $\phi(t)$ is taken which is continuous at $t = t_0$. The value of $\phi(t)$ at $t = t_0$ is $\phi(t_0)$. The impulse exists at $t = t_0$. We have

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$$

3. $$\int_{-\infty}^{+\infty} \phi(t)\delta(t-t_0)\,\mathrm{dt} = \phi(t_0)$$

Here $$\int_{-\infty}^{+\infty} \phi(t)\delta(t-t_0)\,\mathrm{dt} = \phi(t_0)\int_{-\infty}^{+\infty} \delta(t-t_0)\; dt = \phi(t_0)$$

4. $$\delta(\mathrm{at}) = \frac{1}{a}\delta(t)$$

The following integral is considered.

$$\int_{-\infty}^{+\infty} \phi(t)\delta(at)\,\mathrm{dt} \quad \text{for a} > 0$$

Putting at = β in the above integral, we have

$$\int_{-\infty}^{+\infty} \phi(t)\delta(at)\,\mathrm{dt} = \frac{1}{a}\int_{-\infty}^{+\infty} \phi\left(\frac{\beta}{a}\right)\delta(\beta)\,\mathrm{d}\beta \quad \left[\because \beta = at, t = \frac{\beta}{a} \text{ } i.e., \text{ } dt = \frac{d\beta}{a}\right]$$

From property 1, $\int_{-\infty}^{+\infty} \phi(t)\delta(t)dt = \phi(0)$

And also $\int_{-\infty}^{+\infty} \delta(t)dt = 1$

$$\therefore \int_{-\infty}^{+\infty} \phi(t)\delta(at)dt = \frac{1}{a}\phi(0)$$

If 'a' is negative (a < 0),

$$\int_{-\infty}^{+\infty} \phi(t)\delta(-at)\,\mathrm{dt} = \frac{1}{a}\int_{-\infty}^{+\infty} \phi(\frac{\beta}{a})\delta(-\beta)\,\mathrm{d}\,\beta \quad \left[\because \beta = at, t = \frac{\beta}{a} \; i.e., \; dt = \frac{d\beta}{a}\right]$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} \phi(\frac{\beta}{a})\delta(\beta)\,\mathrm{d}\,\beta \quad [\because \delta(-\beta) = \delta(\beta)]$$

$$= \frac{\phi(0)}{a}$$

In general, $\displaystyle\int_{-\infty}^{+\infty} \phi(t)\delta(at)dt = \frac{1}{|a|}\phi(0)$

Since $\int_{-\infty}^{+\infty} \phi(t)\delta(t)\,\mathrm{dt} = \phi(0)$

$$\therefore \int_{-\infty}^{+\infty} \phi(t)\delta(at)\,\mathrm{dt} = \frac{1}{|a|}\int_{-\infty}^{+\infty} \phi(t)\delta(t)\,\mathrm{dt}$$

$Therefore,\ it\ must\ hold\ good\ \delta(at) = \frac{1}{|a|}\delta(t)$

$where\ 'a'\ is\ a\ non-zero\ scalar.$

$Also, \delta(a(t-t_0)) = \frac{1}{|a|}\delta(t-t_0)$

$where\ 'a'\ is\ a\ non-zero\ scalar.$

5. $\int_{-\infty}^{+\infty} \phi(\lambda)\delta(t-\lambda)dt = \phi(\lambda)$

Let us consider property 3 i.e., $\int_{-\infty}^{+\infty} \phi(t)\delta(t-t_0)\,\mathrm{dt} = \phi(t_0)$

Replace t by λ in above equation, $\int_{-\infty}^{+\infty} \phi(\lambda)\delta(\lambda-t_0)\,\mathrm{dt} = \phi(t_0)$

Replacing t_0 by t in above equation, $\int_{-\infty}^{+\infty} \phi(\lambda)\delta(\lambda-t)\,\mathrm{dt} = \phi(t)$

Using symmetry property of impulse function,

$$\delta(\lambda-t) = \delta(t-\lambda)$$

$$\therefore \int_{-\infty}^{+\infty} \phi(\lambda)\delta(t-\lambda)\,\mathrm{d}\lambda = \phi(t)$$