Lecture 30: Numerical Analysis (UMA011)

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Lagrange Interpolating polynomials:

Result (error term):

Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots, x_n$, and hence in (a, b), exists with

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n). \quad (1)$$

Exact App. Error term.

where $P_n(x)$ is n–th degree Lagrange's interpolating polynomial.

$[x_0, x_n]$

$\subseteq [a, b]$

ξ

$a = x_0$ $b = x_n$

$P_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$

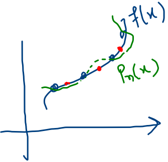
$f(x) \to$ exact

$P_n(x) \to$ App.

$|f(x) - P_n(x)|$

= error function

Proof: Generalized Rolle's theorem :- If $f \in C^n[a,b]$ and f has zeros at $(n+1)$ distinct numbers, then there exists a no. ξ in (a,b) for which $f^{(n)}(\xi) = 0$.



If we have $x = x_k \ \forall \ k = 0, 1, 2 \cdots n$ in (1)

then $f(x_k) = P_n(x_k) \ \forall \ k = 0, 1, 2 \cdots n$

and for any $\xi(x)$

Now, if $x \neq x_k$, then we define a function g for t in $[a,b]$ by

$$\begin{cases} g(t) = f(t) - P_n(t) - \left[f(x) - P_n(x)\right] \dfrac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)} \\[2ex] \quad\;\;\, = f(t) - P_n(t) - \left(f(x) - P_n(x)\right) \displaystyle\prod_{i=0}^{n} \dfrac{(t-x_i)}{(x-x_i)} \end{cases}$$

Since $f \in C^{n+1}[a,b]$ and $P_n \in C^{\infty}[a,b]$ then $g \in C^{n+1}[a,b]$

for $t = x_0$

$$g(x_0) = f(x_0) - P_n(x_0) - \left(f(x) - P_n(x)\right)(0)$$

$$= f(x_0) - P_n(x_0) = 0 \checkmark$$

$||^{ly}$ $g(x_1) = 0 \checkmark$, $g(x_2) = 0$ - - - $g(x_n) = 0$

$\Rightarrow$ $g(x_k) = 0 \quad \forall \; k = 0, 1, 2 \; --- \; n$.

Lagrange Interpolating polynomials:

Proof (continued): $\Rightarrow$ g has $(n+1)$ zeros in $[a,b]$

Moreover, if we take $t = x$ in $\cancel{*}$, then

$$g(x) = f(x) - P_n(x) - \left(f(x) - P_n(x)\right) \quad (1)$$

$$= 0$$

$\Rightarrow$ $g \in C^{n+1}[a,b]$ and g has $(n+2)$ zeros in (a,b)

then by Generalized Rolle's thm, $\exists$ a no. ξ in (a,b) s.t. $g^{(n+1)}(\xi) = 0$

Differentiate $g(t)$ $(n+1)$ times.

$$\left(g^{(n+1)}(t)\right)_{t=\xi} = \left(f^{(n+1)}(t)\right)_{t=\xi} - \left(P_n^{(n+1)}(t)\right)_{t=\xi}$$

$$- \frac{\left(f(x) - P_n(x)\right)}{\prod_{i=0}^{n}(x-x_i)} \left\{ \frac{d^{n+1}}{dt^{n+1}} (t-x_0)(t-x_1) \cdots (t-x_n) \right\}_{t=\xi}$$

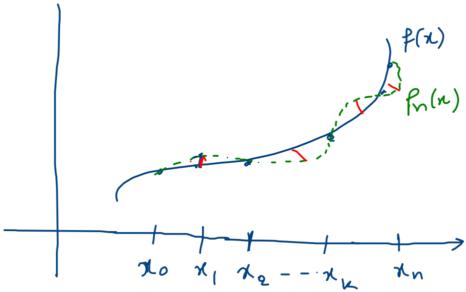
$$g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - 0 - \frac{\left(f(x) - P_n(x)\right)}{\prod_{i=0}^{n}(x-x_i)} \frac{d^{n+1}}{dt^{n+1}} \left(t^{n+1} + (\,)\,t^{n} + (\,)\,t^{n-1} - \cdots \right)$$

$$0 = f^{(n+1)}(\xi) - \left(f(x) - P_n(x)\right)(n+1)! \Big/ \prod_{i=0}^{n}(x-x_i)$$

$$0 = f^{(n+1)}(\xi) - \frac{\left(f(x) - P_n(x)\right)(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

$$\left(f(x) - P_n(x)\right)(n+1)! = f^{(n+1)}(\xi)\,(x-x_0)(x-x_1)\cdots(x-x_n)$$

$$f(x) = P_n(x) + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(\xi)\,(x-x_0)(x-x_1)\cdots(x-x_n)}_{\text{error term.}}$$



max. error in given interal.

Lagrange Interpolating polynomials:

Example:

Use the error formula to find the error bound for the polynomial which is used to approximate $f(x) = \frac{1}{x}$ on $[2, 4]$ with the nodes $x_0 = 2, x_1 = 2.75$, and $x_2 = 4$.

Proof: Error formula for (n+1) pts in [a, b] is given by

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Error formula for 3 pts. in [2, 4] is given by

$$\frac{f^{(3)}(\xi)}{3!}(x-2)(x-2.75)(x-4)$$

Max error bound in $[2,4]$ is $= M$ (say)

$$\max_{2 \le x \le 4} \left(\max_{2<\xi(x)<4} \frac{|f'''(\xi(x))|}{3!} \right) |(x-2)(x-2.75)(x-4)|$$

Now, $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = -\frac{6}{x^4}$

$$M = \max_{2<\xi(x)<4} \left| \frac{-6}{(\xi(x))^4} \right| * \frac{1}{3!} = \frac{\cancel{6}}{2^4} * \frac{1}{\cancel{3!}}$$

$$= \frac{1}{2^4} = \frac{1}{16}$$

$\max_{2 \le x \le 4} |g(x)|$ where $g(x) = (x-2)(x-2.75)(x-4)$

$g(x) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$

$g'(x) = 3x^2 - \frac{70}{4}x + \frac{49}{2} = 0$

$g'(x) = \frac{1}{2}(3x-7)(2x-7) = 0$

$x = \frac{7}{3}, \frac{7}{2}$

Now, we have values of g at $x = 7/3, 7/2$

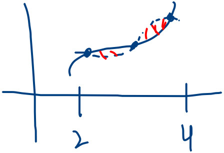
$g(7/3) = \frac{25}{108}$

$g(7/2) = -\frac{9}{16}$ ✓

$\max_{2 \le x \le 4} |g(x)| = \frac{9}{16}$ ✓

max. error bound is

$$\leq \frac{1}{16} * \frac{9}{16} = \frac{9}{256}$$ ✓



$[0,1]$

Example:

Determine the spacing h in a table of equally spaced values of the function $f(x) = e^x$ between 0 and 1, so that interpolation with a linear polynomial will yield an accuracy of 10^{-6}.

$x_0 \quad x_0+h$

$0 \qquad 1$

$h = ?$

Proof: Let the equally spaced values be x_0, x_0+h.

max error bound to linear interpolate the function $f(x)$ in $[0,1]$ $\leq 10^{-6}$

$$\max_{0<x<1} \left| \max_{0<\xi(x)<1} \left| \frac{f^{(2)}(\xi)}{2!} \right| (x-x_0)(x-x_0-h) \right| \leq 10^{-6}$$

$$M = \max_{0<\xi(x)<1} f^{(2)}(\xi(x))/2! = \max_{0<\xi(x)<1} \frac{e^{\xi(x)}}{2!} \leq \frac{e^1}{2!} = \frac{e}{2}$$

$$\max_{0<x<1} |g(x)|$$

where $g(x) = (x - x_0)\,(x - x_0 - h)$

$$g'(x) = (x - x_0) + (x - x_0 - h) = 0$$

$$2x - 2x_0 = h$$

$$x = x_0 + h/2$$

$$g(x_0 + h/2) = \frac{h}{2}\left(-h/2\right) = -h^2/4$$

$\Rightarrow$ max error bd. $\quad \frac{e}{2} * \left|-\frac{h^2}{4}\right| \leq 10^{-6} \Rightarrow h \leq 1.72 * 10^{-3}$

Ans.

Exercise:

1. Determine the spacing h in a table of equally spaced values of the function $f(x) = \sqrt{x}$ between 1 and 2, so that interpolation with a quadratic polynomial will yield an accuracy of 5×10^{-4}.
2. Find a bound for the absolute error on the interval $[x_0, x_n]$.
   a. $f(x) = \sin x,\ x_0 = 2.0,\ x_1 = 2.4,\ x_2 = 2.6,\ n = 2.$
   b. $f(x) = e^{2x} \cos 3x,\ x_0 = 0,\ x_1 = 0.3,\ x_2 = 0.6,\ n = 2.$