FLUID KINEMATICS

- Fluid kinematics is the study of fluids in motion without considering the forces responsible for the fluid motion.

METHODS OF DESCRIBING FLUID MOTION

- Flow of a real fluid is very complex.
- The parameters describing the flow characteristics (velocity, pressure and density) are not constant in a particular set of circumstances i.e. every particle of fluid in motion has an instantaneous value of fluid flow characteristics.
  - ➢ may change with respect to space and time.
  - ➢ can be measured either by a moving co-ordinate system or a fixed co-ordinate system.

In moving co-ordinate system, a co-ordinate system is attached to a particular part of a moving object.

- This method of flow study is known as Lagrangian method.
- The method tracks the position vector and velocity vector of individual particles.
  - ❖ Complex - very difficult to keep track of all the fluid particles.

In fixed co-ordinate system, a co-ordinate system fixed in space is chosen to study the motion of fluid particles.

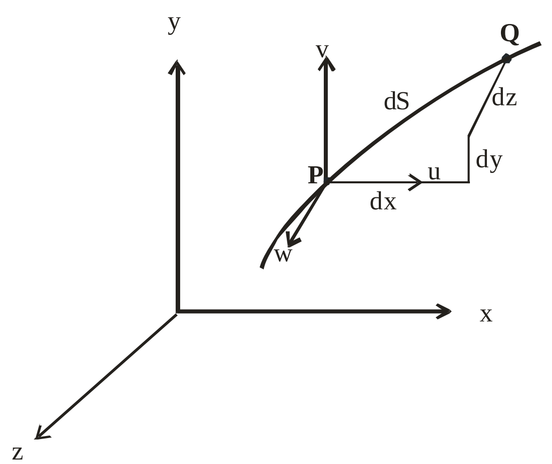
- This method of flow study is known as Eulerian method.
- The method defines a **C∀** through which fluid flows in and out
- We don't really care what happens to individual particles.
  - ❖ More practicable and is generally adopted in flow analysis.

ILLUSTRATION: An observer sitting (i) on the bank of a stream and (ii) in a boat moving in the stream - observes a floating object.

- Resembles which type of flow study method in the two cases?

VELOCITY OF A FLUID PARTICLE (EULERIAN METHOD)

- Consider a fluid particle moving in space having directions **x-, y-** and **z-**



- Let ***P*** is the position of the fluid particle at any time **t.**
- In time **dt**, let the particle moves to **Q** and the displacement is **dS**

$$\therefore V = \underset{dt \to 0}{Lt} \frac{dS}{dt}$$

If **u**, **v** and **w** are the components of velocity along **x-**, **y-** and **z-** directions, respectively and **dx**, **dy** and **dz** are the components of displacement **dS** in the respective directions, then

$$u = \underset{dt \to 0}{Lt} \frac{dx}{dt}; \qquad v = \underset{dt \to 0}{Lt} \frac{dy}{dt} \qquad \text{and} \qquad w = \underset{dt \to 0}{Lt} \frac{dz}{dt}$$

- In vector notation, $\vec{V} = f(\vec{r}, t)$
- $\vec{V} = (ui + vj + wk)$ & $\vec{r} = (xi + yj + zk)$
- Resultant velocity, $V = \sqrt{u^2 + v^2 + w^2}$

<u>ACCELERATION OF A FLUID PARTICLE</u>

- Change of velocity with respect to time.
- Acceleration components along **x**-, **y**- and **z**- directions

$$a_x = \underset{dt \to 0}{Lt} \frac{du}{dt}; \quad a_y = \underset{dt \to 0}{Lt} \frac{dv}{dt} \quad \text{and} \quad a_z = \underset{dt \to 0}{Lt} \frac{dw}{dt}$$

- As velocity of a fluid particle is a function of both position and time.
- Acceleration is also a function of both position and time
- Velocity of a fluid particle along **x-**direction can be written as;

$$u = f_1 (x, y, z, t)$$

$\therefore$ Total or substantial derivative of **u** can be written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial t} dt$$

- Divide both sides by **dt** and taking limits as **dt** $\to$ **0,** to get

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

- Similarly, acceleration components along **y-** and **z-** directions can be written as:

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \quad \text{and } a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

- Terms $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ represent change of velocity with respect to time at a particular point in the flow, known as <u>local or temporal accelerations</u>.

- Remaining terms represent change of velocity due to the change of position of particle, known as convective or spatial accelerations.

Observation:

For the velocity field in a fluid medium,

$$\overline{V} = (3x + 2y)\hat{i} + (2z + 3x^2)\hat{j} + (2t - 3z)\hat{k}$$

- What are **u**, **v** and **w** ?
- Can you find resultant '**V**' and '**a**' at (**1**, **2**, **3**) and time **t = 5**?

TYPES OR CLASSIFICATION OF FLUID FLOWS

- Steady and Unsteady flow
- Uniform and Non-uniform flow
- Laminar and Turbulent Flow
- One, Two and Three Dimensional Flows
- Rotational and Irrotational Flows

Steady and Unsteady flow (Temporal Variations)

Steady flow is defined as the flow in which fluid flow characteristics (velocity, pressure, density etc.) at a point do not change with respect to **time.**

- For velocity, steady flow may be expressed as $\left(\frac{\partial V}{\partial t}\right)_{(x_0, y_0, z_0)} = 0$
- (x_0, y_0, z_0) is a fixed point in the flow field.

Examples: Flow of liquid though a pipe at a constant rate, Flow from a tank where a constant level is maintained etc.

Unsteady flow is the flow in which any one or all the fluid flow characteristics at a point change with respect to time (functions of time).

- Much more difficult to analyse.
- Mathematically, $\left(\frac{\partial V}{\partial t}\right)_{(x_0, y_0, z_0)} \neq 0$

Examples: Flow in a pipe at a varying rate *i.e.* valve provided in the pipe is gradually opened or closed, Flow of liquid from a tank with changing level etc.

Observation: Change in the direction of flow also makes the flow unsteady. Flow of liquid in a pipe bend is an example of unsteady flow.

Uniform and Non-uniform flow (Positional Variations)

Uniform flow is the flow in which fluid flow characteristics at any time do not change with respect to **position.**

- ❖ For velocity, uniform flow may be expressed as: $\left(\frac{\partial V}{\partial S}\right)_{(t = \text{cons tan t})} = 0$

  - ➢ **S** is the distance measured from some fixed point in the path of flow.

Example: Flow in a channel having constant depth.

Non-uniform flow can be expressed as: $\left(\frac{\partial V}{\partial S}\right)_{(t = \text{cons tan t})} \neq 0$

Example: Flow in a channel having varying depths in the direction of flow

Observation: Steadiness refers to no change with time and uniformity refers to no change in position.

Laminar and Turbulent Flow

Flow is said to be **laminar** when various fluid particles move in layers or laminae, with one layer of fluid smoothly sliding over an adjacent layer. Also called streamline or viscous flow. Flow at low velocities and for highly viscous fluids is likely to be laminar.

Examples: Flow of blood in veins, flow in capillary tubes, movement of sap in trees etc.

Flow is said to be **turbulent** when various fluid particles move in zig-zag or irregular manner. This results in rapid and continuous mixing of fluid particles leading to momentum exchange between the fluid particles.

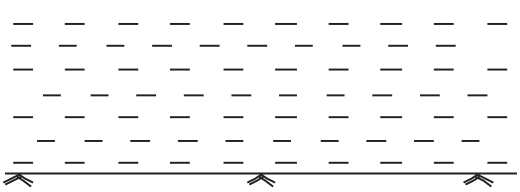
- Eddies or vortices of different shapes and sizes are present, which are responsible for high-energy losses.

❖ **TF** occurs more frequently in nature as compared to **LF.**

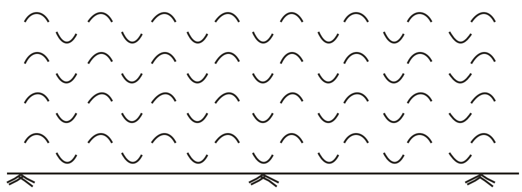
- Flow at high velocities and for low viscous fluids is likely to be turbulent

Examples: Flow in channels, canals, rivers, atmosphere etc.

- Laminar and turbulent flows are depicted below:



(a) Laminar flow



(b) **Turbulent flo**w

To determine the nature of flow (laminar or turbulent), Reynolds number of flow (**Re**) is calculated

- **Re** is defined as the ratio inertia force to viscous force.
- Given by, $$Re = \frac{\rho VL}{\mu} = \frac{VL}{\nu}$$
  - ➢ **L** is the characteristic dimension; **V** is the mean velocity of flow, **μ** is the dynamic viscosity and **ν** is the kinematic viscosity.

- For flow through pipes, **L = D, D** is the diameter of pipe
- For flow in channels, **L** = $\mathbf{R_h}$, $\mathbf{R_h}$ is known as the hydraulic radius or hydraulic mean depth
- $\mathbf{R_h}$ is the ratio of cross-sectional area of flow to the wetted perimeter of channel ($\mathbf{R_h = A/P}$)
- What is the hydraulic mean depth of flow in pipe of diameter **D**?

Type of System	Laminar flow	Turbulent flow
Pipes	Re < 2000	Re > 4000
Channels	Re < 500	Re > 2000
Between these two values, flow is in transitional state.		

One, Two and Three Dimensional Flows

- ❖ In general, the fluid flow is **3D** as the fluid flow characteristics vary in all the three coordinate directions.
- In many cases, we can simplify the analysis by assuming that characteristics vary in only two or even one direction.
- <u>In 1D flow</u>, fluid flow characteristics vary along one-direction only

Examples: Ideal flow between parallel plates, ideal flow in a pipe of constant diameter etc.

- ➢ Velocity of flow at any cross-section is constant in**1D** flow.
- In <u>2D flow</u>, the flow characteristics vary along two-directions.

Examples: Real flow between parallel plates, real flow in a pipe of constant diameter etc.

- ❖ Velocity at any cross-section is zero at the ends and maximum at the centre.
- <u>In 3D flow</u>, fluid flow characteristics vary along all the three directions.

Examples: Flow in a river, flow within the fluid machines, flow in pipes of varying diameter etc.

Observation: Mathematically, express velocity **V** for **3D** unsteady and steady flow.

Rotational and Irrotational Flows

- If moving fluid particles rotate about their mass centres in direction perpendicular to the plane of motion, flow is said to be <u>rotational</u>.
- ❖ Rotation is measured in a direction perpendicular to the plane of motion.

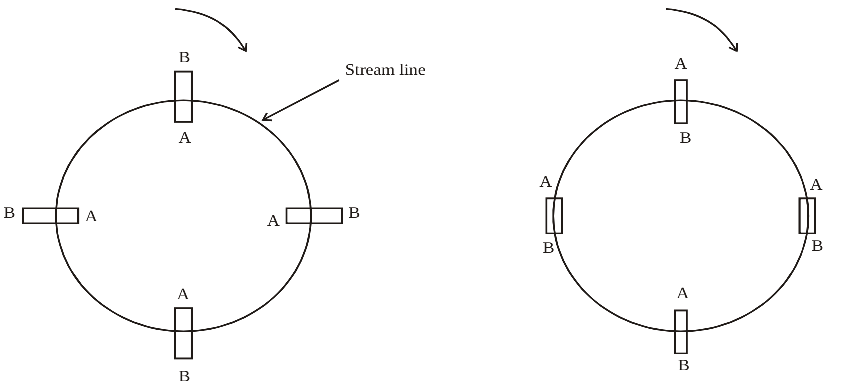
Examples: Flow of liquid in a rotating cylinder, flow of liquid inside the impeller of centrifugal pump, flow of water in the runner of turbine etc.

- <u>Flow is irrotational</u>, when fluid particles do not rotate about their mass centres.

Examples: Flow above the drain hole of a wash basin, whirlpool in a river, flow of liquid in centrifugal pump casing after it has left the impeller etc.

- Rotational and irrotational flows are illustrated in **Figs. (a) and (b).**
- A fluid particle represented by a rectangular element **AB** is rotating in a circular path/stream line.

❖



(a) Rotational flow (b) Irrotational flow

Rotation of a fluid element

- In **Figure (a),** the fluid particle rotates about its axis while moving along a circular path (streamline) and thus constitutes the rotational flow.
- In **Figure (b),** the same fluid particle does not rotate about its axis but is simply changing its position and thus constituting an irrotational flow.

<u>Observation:</u>

What is the expression for acceleration of **1D** (i) unsteady and non-uniform flow (ii) steady and uniform flow.

(i) For **1D** unsteady and non-uniform flow, total acceleration is given by;

$$a = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

(ii) For steady flow, local acceleration is zero and for uniform flow, convective acceleration is zero *i.e.*

❖ For steady-uniform flow, there is no acceleration in the direction of flow.

DESCRIPTION OF FLOW PATTERN (Fluid Flow Terminology)

- When fluid flows, its various particles move along certain lines depending upon the characteristics of flow.
- The pattern of flow may be described by means of streamlines, stream tubes, path lines and streak lines.

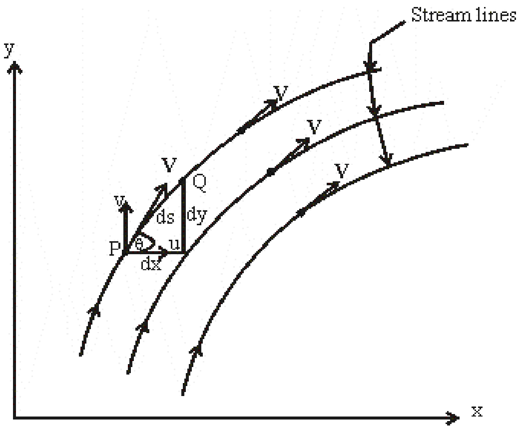
❖ The most common ways are streamlines and stream tubes.

Streamline

- is an imaginary curve in the flow field such that velocity of flow at every point on this curve is tangent to the streamline.

❖ Streamlines are equivalent to an instantaneous snapshot of fluid motion and represent flow pattern in a plane flow (**2D** flow).

- **Figure** shows some of the streamlines of a flow pattern in **xy**-plane.



- A streamline passing through point **P** is tangent to **V** vector at **P**.
- In time **dt**, let the fluid particle moves to **Q** and displacement is **dS**.
- If **u** and **v** are the components of **V; dx, dy** are the components of displacement **dS,** along **x-** and ***y-*** directions, respectively and **θ** is the angle made by **V** with horizontal, then

$$\tan\theta = \frac{dy}{dx} = \frac{v}{u} \quad \Rightarrow \frac{dx}{u} = \frac{dy}{v}$$

∴ Differential equation of streamlines in **xy**-plane is: $\frac{dx}{u} = \frac{dy}{v}$

- For **3D** flow, equation may be written as : $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

Characteristics of streamlines:

(i) Flow is only along the streamline and not across it (This is because velocity of flow at each point is tangent to the streamline).

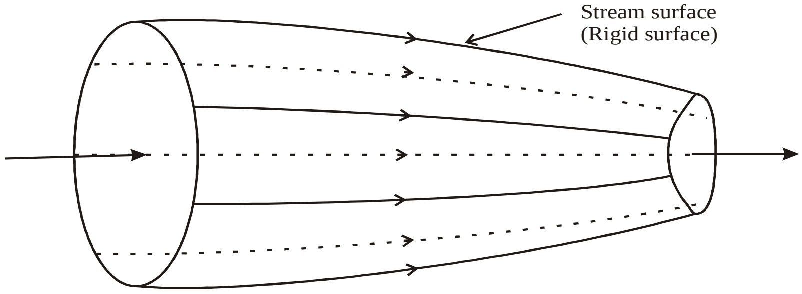
❖ A streamline can therefore be considered as a rigid boundary and the fluid lying between two streamlines can be considered in isolation.

(ii) A streamline can neither intersect itself nor can two streamlines cross each other.

(iii) For steady flow, the flow pattern remains constant but for unsteady flow, flow pattern changes with time. In unsteady flow, direction of velocity of flow may change with time).

Stream tube

- It is the collection of number of streamlines which form a passage through which fluid flows.
  Example: Flow in a pipe



- Surface of stream tube is called the stream surface.

- There is no flow across the stream surface and it behaves as a rigid surface (Stream tube is bounded on all the sides by streamlines and there is no flow perpendicular to streamlines).
- The fluid may enter or leave a stream tube only at its ends.
- **Observation**: The concept of stream tube is useful in analysing several fluid flow cases as the entire flow field may be divided into a number of stream tubes.

Path line:

- A path line represents the trajectory of a fluid particle over a period of time (or in other words, a path line shows the direction of velocity of the same fluid particle at successive instants of time
- (A streamline shows the direction of velocity of a number of fluid particles at the same time).
- **Figure** shows path line of a fluid particle as it moves from initial time $\mathbf{t_1}$ to final time $\mathbf{t_2}$.

Path line of a fluid particle

- **Observation:** A path line can intersect itself at different times.

Streak Line

- A streak line is the line traced by fluid particles passing through a fixed point in the flow field.
- **Examples:** A line formed by continuously injecting dye into the flowing fluid, a line formed by smoke particles injected into the atmosphere from chimney etc.

Basic Principles of Fluid Mechanics

- Three: Principle of conservation of; mass, energy and momentum.

Principle of conservation of mass

- Mass can neither be created nor destroyed (no nuclear reaction).
- Using this principle, **continuity equation** is derived.

Principle of conservation of energy

- Energy can neither be created nor destroyed.
- Energy equation or Bernoulli's equation is derived.

Principle of conservation of momentum

- It states that the impulse of the resultant force acting on the fluid mass in any direction is equal to the change of momentum of flow in the same direction.
- Using this principle, momentum equation is derived.

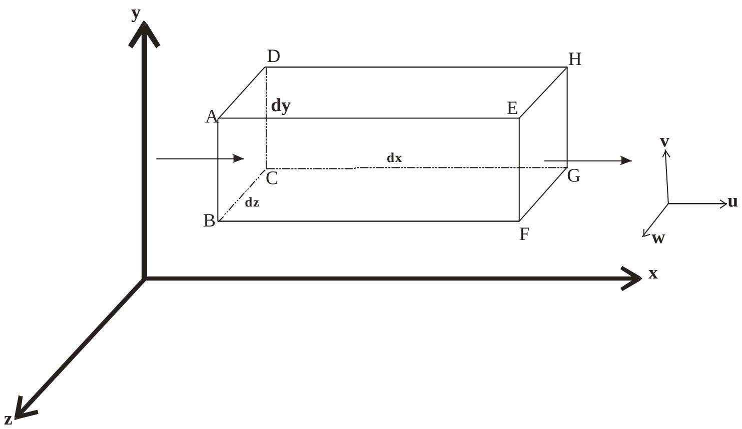
Continuity equation

- General expression for continuity eq. may be obtained by considering inflow and outflow from a fixed region (**C∀**)

- Rate of increase of fluid mass in fixed region= (Mass inflow rate – Mass outflow rate)
- This concept is used to derive equation of continuity.
- ❖ For steady flow, rate of increase of fluid mass within the fixed region is equal to zero i.e. for steady flow   Mass inflow rate = Mass outflow rate
- Depending upon the flow system, continuity equation can be derived in Cartesian coordinates, cylindrical polar coordinates and spherical coordinates.

Continuity Equation in Cartesian Coordinates

- Consider a control volume in the shape of a rectangular paralleopipe.



- Eq. is derived by considering <u>inflow and outflow</u> through three-pair of faces perpendicular to the three co-ordinate axes.
- Let **u**, **v** and **w** are the components of velocity along the three co-ordinate axes, respectively and ρ is the mass density of fluid.

<u>To start with, consider unsteady and compressible flow through the **C∀**</u>

Flow along x-direction

- Mass of fluid entering **C∀** per unit time through face **ABCD =ρu (dy dz)**
- ➢ Also, called mass inflow rate or mass influx.
- Mass of fluid leaving **C∀** per unit time through face **EFGH** = $\left[\rho u dydz + \frac{\partial}{\partial x}(\rho u dydz)dx\right]$
- ➢ Also called mass outflow rate or mass efflux.

∴ Net mass accumulated per unit time in the **C∀**= (Mass inflow rate – mass outflow rate)

$$= -\frac{\partial}{\partial x}(\rho u dydz)dx \qquad = -\frac{\partial}{\partial x}(\rho u)dxdydz$$

Flow along y- and z- directions

- Similarly, the net mass accumulated per unit time in **C∀** through inflow and outflow through the other two ⊥ pair of faces along y- and z- directions, respectively can be written as:
  $-\frac{\partial}{\partial y}(\rho v)dxdydz$ and $-\frac{\partial}{\partial z}(\rho w)dxdydz$

∴ Total mass accumulated per unit time in **CV =** $-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]dxdydz$

- Applying the principle of conservation of mass, any accumulation of fluid mass per unit time is equal to the rate of increase of fluid mass in the **C∀**
- Mass of fluid in the **C∀**is (**ρdxdydz**) and its rate of increase is $\frac{\partial}{\partial t}(\rho dxdydz)$.
- Equating the two expressions and simplifying, to get,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

❖ General form of continuity eq. in Cartesian coordinates

- The above equation may be expanded and written as:

$$\frac{\partial \rho}{\partial t}+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

- For steady flow, **(∂ρ/∂t) = 0** and for incompressible fluids, ρ does not change with **x-**, **y-** and **z-**directions.

  ∴ For steady and incompressible fluids flow, **CE** is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

❖ <u>Standard form of continuity eq. in Cartesian coordinates.</u>

Observations

(i) For **2D** flow in **xy**-plane, **w = 0**

(ii) For **1D** flow along **x-**direction, **v = 0, w = 0,** $\therefore \frac{\partial u}{\partial x}=0$, Integrating, to get

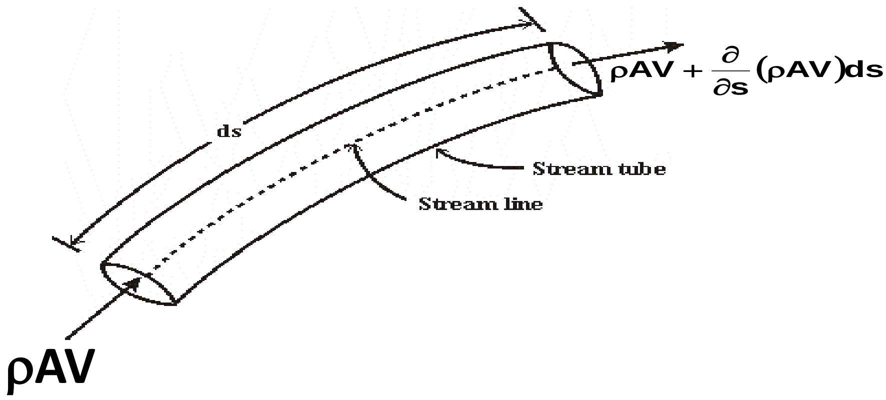
u = constant

∴ In **1D** flow, velocity of flow is constant.

(iii) CE for **2D** flow can be derived directly by taking **C∀** in the shape of a rectangle lying in **xy-**plane. Similarly, for **1D** flow, **C∀** can be taken in the shape of a stream tube.

<u>1D Continuity Equation</u>

Consider a stream tube of length **ds**, cross-sectional area **A** and mean velocity of flow **V**



- Mass influx or mass inflow rate = ρ**AV**
- Mass efflux = $\rho AV + \frac{\partial}{\partial s}(\rho AV)ds$

  $\therefore$ Net mass accumulated per unit time in stream tube = $-\frac{\partial}{\partial s}(\rho AV)ds$
- Rate of increase of fluid mass in stream tube = $\frac{\partial}{\partial t}(\rho Ads)$
- Applying the principle of conservation of mass, to write

$$-\frac{\partial}{\partial s}(\rho AV)ds = \frac{\partial}{\partial t}(\rho Ads)$$

- Divide both sides by **ds** and taking limit so as to reduce the stream tube to a point
- Continuity equation for **1D** flow is

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho AV) = 0$$

- For steady flow, $\partial(\rho A)/\partial t = 0$; $\quad \therefore \partial(\rho AV)/\partial s = 0$
- Integrating, to get $\quad \rho AV = \text{Constant}$
- For a varying duct, one can write $\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \rho_3 A_3 V_3$
- For incompressible fluids, $\rho_1 = \rho_2 = \rho_3 = \rho$ **(say)**

$$\therefore \; A_1V_1 = A_2V_2 = A_3V_3 \text{ Or } \quad AV = \text{Constant}$$

❖ (**AV**) represents the volume of fluid flowing through any section per unit time or volume rate of flow which is known as discharge, generally denoted by **Q.**

$$\therefore Q = \text{Cross sectional area of flow (A)} \times \text{Average velocity of flow (V)}$$

Observation:

- **Q** is the volumetric rate of flow whereas (ρ**Q**) is the mass rate of flow.

Units of discharge:

- cm^3/s, m^3/s, litres per second (lps), **cumec**, cusec
- 1 cumec = 1 m^3/s, 1 cusec = 1 ft^3/s
- 1 m^3 = 1000 L; 1 L = 1000 cm^3, 1 ml = 1 cm^3

Observations:

(i) Average velocity of flow across any section may be obtained by using

$$V = \frac{1}{A}\int u\,dA$$

(ii) Continuity eq. for steady and incompressible fluids flow in cylindrical polar coordinates (**r, θ, z**) is:

$$\frac{1}{r}\frac{\partial}{\partial r}(u_r r) + \frac{1}{r}\frac{\partial u_t}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

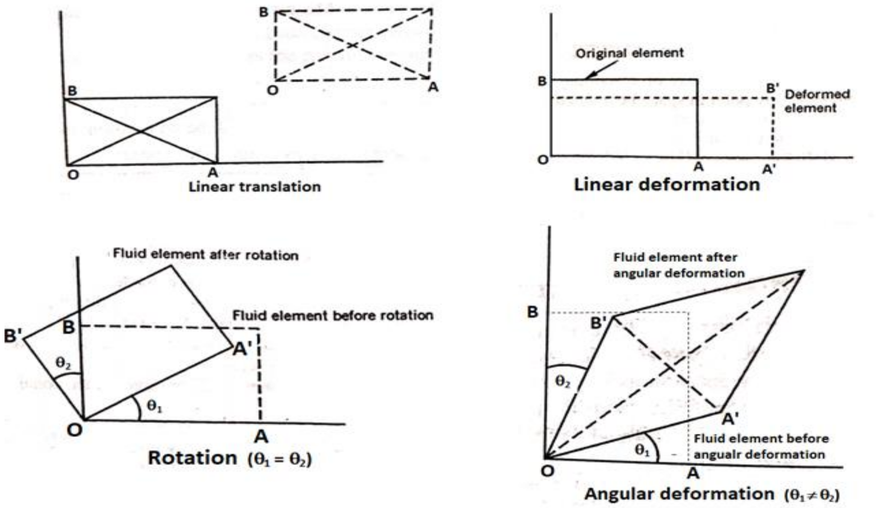
➢ $\mathbf{u_r}$ is the velocity along radial direction, $\mathbf{u_t}$ is the velocity along tangential direction and $\mathbf{u_z}$ is the velocity along **z-**direction.

DISPLACEMENTS OF A FLUID PARTICLE

- During fluid motion, the fluid particles can undergo following four types of displacements viz. linear translation, linear deformation, rotation and angular deformation

❖ It may be noted that the displacement of a fluid particle at any point may expressed in terms of displacements of two mutually perpendicular linear elements enclosing that point.

- As shown in **Figs., OA** and **OB** are the two linear elements/sides drawn through point **O** in **xy**-plane.



Types of displacements

- In **linear translation,** the dimensions of fluid elements do not change and there is a bodily movement of the fluid elements only.
- During **linear deformation,** the fluid undergoes a change in its dimensions (shape) but directions of the two elements remain the same.
- During **rotation**, the two elements of fluid rotate in the same direction and in the same amount ($\theta_1 = \theta_2$) with respect to their previous positions.

➢ Rotation is also called rotation without shear strain i.e. irrotation.

- During **angular deformation**, the two elements/sides of fluid rotate in opposite directions but with unequal amounts ($\theta_1 \neq \theta_2$) with respect to their previous positions.

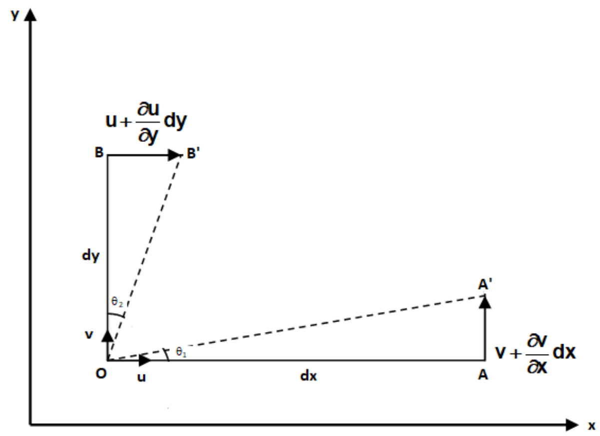
- Angular deformation is also called rotation with shear strain or angular velocity.

Observations:

(i) Rotation of fluid particle can only be produced if shear stresses (caused by viscous action) are present in the fluid flow. If shear stresses are of negligible magnitude, only translation of fluid occurs and the flow may be regarded as irrotational.

(ii) In actual flow problems, the fluid particle may undergo all four types of displacements simultaneously.

MATHEMATICAL EXPRESSION FOR ANGULAR DEFORMATION/VELOCITY (ROTATION WITH SHEAR STRAIN) OF A FLUID PARTICLE



- Let **OA = dx** and **OB = dy**.
- Velocity components at **O** are **u** and **v** along **x-** and **y-** directions, respectively.
- Velocities are increased at **A** and **B** to $\left(v+\frac{\partial v}{\partial x}dx\right)$ and $\left(u+\frac{\partial u}{\partial y}dy\right)$ respectively along **y -** and **x -**directions, respectively as shown in **Figure** [Formed by the expansion of Taylor series].
- Due to the different velocities at **O**, **A** and **B**, in small time interval **dt**, the fluid element **OA** moves to **OA′** and the fluid element **OB** moves to **OB′** respectively, relative to **O**.
- If θ_1 is the angle made by side **OA′** with **OA** and θ_2 is the angle made by side **OB′** with **OB**, then

$$\theta_1 = \frac{AA'}{OA} = \frac{\left[\left(v+\frac{\partial v}{\partial x}dx\right)-v\right]dt}{dx} = \frac{\partial v}{\partial x}dt$$ (Considering anticlockwise rotation as +ve).

$$\theta_2 = \frac{BB'}{OB} = \frac{\left[-\left(u+\frac{\partial u}{\partial y}dy\right)+u\right]dt}{dy} = -\frac{\partial u}{\partial y}dt$$

- Angular deformation *i.e.* angular velocity of a fluid particle is expressed in terms of average of the rates of rotation of elements **OA** and **OB** and further, since rotation is measured in a direction perpendicular to the plane of motion and thus angular velocity of a fluid particle **O** about **z-**axis is given by,

$$w_z = \frac{1}{2}\left(\frac{\theta_1 + \theta_2}{dt}\right)$$

$$\therefore w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- Similarly, angular velocity of a fluid particle about **x**-axis and **y**-axis are, respectively given by

$$w_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad w_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

- In vector form, rotation of a fluid particle can be expressed as:

$$\vec{w} = \left(w_x i + w_y j + w_z k\right) = \frac{1}{2}(\nabla \times V)$$

  - Cross product of ∇ (del) with any quantity (here **V**) is known as curl of that quantity

$$\therefore \nabla \times V = \text{curl } V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

- If at every point in the flowing fluid, all the rotation components are zero, then the flow is irrotational i.e. for irrotational flow; $w_x = 0$, $w_y = 0$ and $w_z = 0$ **or** $\nabla \times V = 0$

Observations:

(i) Rate of shear strain at point **O** in **xy-**plane is equal to the average of difference of the rates of rotation of elements **OA** and **OB** and is given by:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\theta_1 - \theta_2}{dt}\right) \quad \Rightarrow \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Similarly, shear strain rates at point **O** in **yz**- and **xz**- planes are given by:

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

(ii) Direct strain rate (linear strain rate i.e. dilatency) at point *O* along x-direction is given by:

$$\varepsilon_{xx} = \frac{\left(\dfrac{OA' - OA}{OA}\right)}{dt} = \frac{\left[\left(u + \dfrac{\partial u}{\partial x}dx\right) - u\right]dt}{dxdt} = \frac{\partial u}{\partial x}$$

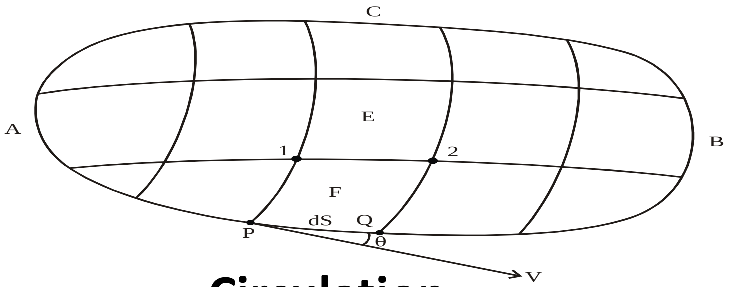
- Similarly,

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \text{ and } \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

CIRCULATION AND VORTICITY

- Flow around a closed curve is called circulation.

- Consider a closed curve **ABC** within the fluid medium as shown in **Figure.**



- Divide this curve into small meshes.
- If **PQ** is a small element of one of the meshes at circumference having length **dS**, **V** is the velocity at point **P** (tangential to the curve) and **θ** is the angle between (velocity vector) and **dS**, then
- Mathematically, circulation around **PQ** is equal to the product of velocity component along **PQ** and the length **dS** and is denoted by the symbol **Γ** (Greek capital gamma) i.e.

$$\Gamma_{PQ} = V\cos\theta \times dS$$

$\therefore$ Total circulation around the curve is equal to the sum of circulations around small meshes into which this curve has been subdivided i.e.

$$\Gamma_{ABC} = \int V\cos\theta \times dS$$

❖ In vector form, circulation is expressed as the dot product of velocity vector and length vector i.e.

$$\Gamma_{ABC} = \int \vec{V}.\overrightarrow{dS} = \int (ui + vj + wk).(idx + jdy + kdz) = \int (udx + vdy + wdz)$$

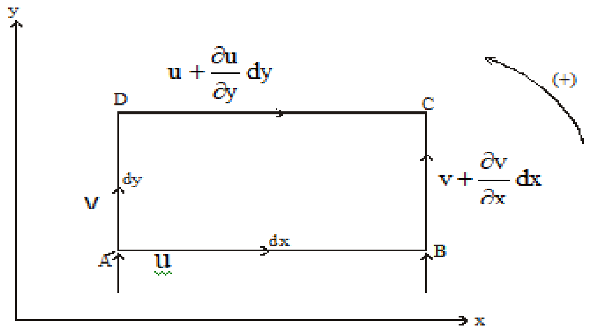
Observation: Consider two adjacent meshes **E** & **F** and let their common boundary is represented by **1-2**.

- Circulations along the common boundary of these two meshes have equal magnitude but opposite directions and hence cancel each other.
- This is also true for other meshes and thus the final result consists of circulation around the outer periphery of the closed curve only.

✓ Determination of circulation around any arbitrary closed curve is generally complicated. However, the principle may be applied easily to certain closed curves having regular shapes such as rectangles, triangles, circles etc.

Let us calculate circulation around a rectangular flow field:

- Consider a rectangular flow field **ABCD** of size **dx** and **dy** along **x-** and **y-** directions, respectively. The velocities along each side of the rectangle are shown in **Figure**.



Circulation around a rectangle

$\therefore$ Circulation around **ABCD**, $\Gamma_{ABCD} = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA}$

$\therefore \Gamma_{ABCD} = udx + \left(v + \frac{\partial v}{\partial x}dx\right)dy - \left(u + \frac{\partial u}{\partial y}dy\right)dx - vdy$ (Considering anticlockwise circulation +ve)

$$\Rightarrow \Gamma_{ABCD} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

- Now, vorticity denoted by ξ (Greek 'Geeta') at a point is defined as the circulation per unit area enclosed by the curve at that point i.e.

$$\xi = \frac{\Gamma}{dxdy} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- Also, angular velocity of a fluid particle about **z**-axis is given by, $w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$\therefore \xi = 2w_z$$

i.e. vorticity is twice the rotation component about an axis perpendicular to the plane in which the area is lying and therefore this eq. represents a component of vorticity in the **z**-direction *i.e.* ξ_z **=2w**$_z$

  - Similarly, ξ_y **= 2w**$_y$ and ξ_x **= 2w**$_x$

Observation:

  - Circulation around a circle of radius **R** is given by, $\Gamma = \int_0^{2\pi} V \times R d\theta = (2\pi R)V$

POTENTIAL FUNCTION AND STREAM FUNCTION

POTENTIAL FUNCTION

  - It is defined as a scalar function of space and time such that its derivative with respect to any direction gives velocity component in that direction.
  - denoted by ϕ (phi) and is also called velocity potential function.
  - Mathematically, for **3D** steady flow, potential function $\phi = f(x, y, z)$

$$\therefore -\frac{\partial \phi}{\partial x} = u, \; -\frac{\partial \phi}{\partial y} = v \text{ and } -\frac{\partial \phi}{\partial z} = w \qquad (1)$$

- **-ve** sign signifies that ϕ decreases with an increase in **x**, **y** and **z**.

Observations:

- ϕ is analogous to electric potential which decreases in the direction of flow of current.
- ϕ is constant along a line, known as equipotential line.
- ϕ exists for irrotational flow.

Characteristics of Potential Function

(i) For **3D**, steady and incompressible flow, continuity equation is: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

- Substituting the values of **u**, **v** and **w** from Eq. (1), to get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0 \qquad \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- This is Laplace's equation.

$\therefore$ If any potential function satisfies Laplace's equation, it is a case of steady incompressible fluid flow.

(ii) Angular velocity in xy-plane is given by: $w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

- Substituting the values of **u** and **v** from Eq. (1), to get $w_z = \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right)$
- If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$

$$\therefore w_z = 0$$

- Similarly, it can be shown that $w_y = 0$ and $w_x = 0$.
- Thus, potential function ϕ satisfies the condition of irrotational flow.

STREAM FUNCTION

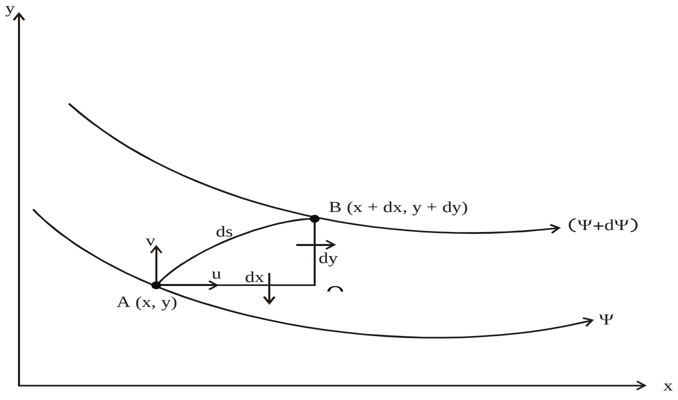
- It is defined as a scalar function of space and time such that its derivative with respect to any direction gives the velocity component at right angle to this direction.
- denoted by symbol **Ψ** (psi).
- For **2D** steady flow in **xy**- plane, $\psi = f(x, y)$

$$\therefore \frac{\partial \psi}{\partial x} = -v, \quad \text{and} \quad \frac{\partial \psi}{\partial y} = u \qquad (1)$$

- Stream function is constant along a stream line.

<u>*Derivation of Eq. (1)*</u>:

- Consider two streamlines having stream functions **Ψ** and (**Ψ+dΨ**).



Stream function

- Let **A(x, y)** and **B(x+dx, y+dy)** are the two points lying on the two streamlines, respectively as shown in **Figure**.
- Join these points by an arbitrary curve **AB** of length **dS**.
- Let **u** and **v** are the velocity components at point **A** along **x-** and **y-** directions respectively.

∴ From continuity consideration, Flow across **AB** = Flow across **AO** + Flow across **BO**

$$= -v\,(dx \times 1) + u\,(dy \times 1)$$

(**v** is acting in the downward direction which is opposite to the assumed direction)

- Also, flow across AB = **(Ψ + dΨ) - Ψ = dΨ**

- Equating Eqs., to get $$d\psi = -v\,dx + u\,dy$$

- As, **Ψ = f(x, y)** ,

∴ Total derivative of ψ, $$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

- Comparing the two Eqs. of ψ term wise, to get $$\frac{\partial \psi}{\partial x} = -v, \quad \text{and} \quad \frac{\partial \psi}{\partial y} = u$$

- Hence the result.

Characteristics of stream function

(i) Continuity equation for **2D**, steady and incompressible flow is: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

- Substituting values of **u** and **v** from Eq. (1), to get

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0 \qquad \Rightarrow \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

- If **Ψ** is a continuous function, then $$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

$\therefore$ Stream function satisfies continuity equation.

(ii) Angular velocity in **xy**-plane is given by: $w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

- Substituting values of **u** and **v** from Eq. (1), to get

$$w_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right] = -\frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

- However, if $w_z = 0$ then , $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$; This is Laplace's equation.

$\therefore$ If stream function satisfies Laplace's eq., flow is irrotational.

(iii) For **2D** flow in **xy-**plane, equation of a streamline is given by;

$$\frac{dx}{u} = \frac{dy}{v} \qquad \Rightarrow \ \mathbf{udy - vdx = 0}$$

- As, $\Psi = f(x, y)$ $\quad \therefore d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \quad \Rightarrow \quad d\psi = -vdx + udy$

$\therefore$ **d**ψ = 0 or ψ = Constant *i.e.* stream function is constant along a streamline.

- However, stream function varies from one streamline to another. Each streamline of flow pattern can be represented as $\psi_1 = C_1, \psi_2 = C_2$ **etc.**, C_1, C_2,---- are constants.

❖ Streamlines are thus lines of constant stream function.

Observation:

- To sum up, from the above discussion of potential function and stream function, the following points may be noted:

Potential function (ϕ)	Stream function (Ψ)
ϕ satisfies continuity equation.	Ψ also satisfies continuity equation.
ϕ satisfies the condition of irrotationality of flow.	If $w_z = 0$, only then the flow is irrotational.
For irrotational flow, both ϕ and Ψ satisfies Laplace's equation.	

CAUCHY'S RIEMANN'S EQUATIONS:

- For irrotational flow, both ϕ and Ψ satisfy Laplace equation and therefore they are interchangeable.
- Since, $\frac{\partial \psi}{\partial x} = -v, \ \frac{\partial \psi}{\partial y} = u$ **and** $-\frac{\partial \phi}{\partial x} = u, \ -\frac{\partial \phi}{\partial y} = v$
- These equations are known as Cauchy's Riemann's equations and can be used to find ϕ if Ψ is given and vice-versa.

ORTHOGONALITY OF STREAM LINES AND EQUIPOTENTIAL LINES

- Since, stream function, $\psi = f(x, y)$

- $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$ $\qquad\qquad$ $\therefore d\psi = -vdx + udy$
- Also, Ψ is constant along a streamline *i.e.* **$d\Psi = 0$**

$$\therefore 0 = -vdx + udy \qquad\qquad \Rightarrow \frac{dy}{dx} = \frac{v}{u}$$

- i.e. slope of a stream line (say **m_1**) = **v/u**
- Also, potential function, **$\phi = f(x, y)$**

$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \qquad\qquad \Rightarrow d\phi = -udx - vdy$$

- Also, ϕ is constant along an equipotential line (equipotential line is a line having constant potential function) *i.e.* **$d\phi = 0$**

$$\therefore 0 = -udx - vdy \qquad\qquad \Rightarrow \frac{dy}{dx} = -\frac{u}{v}$$

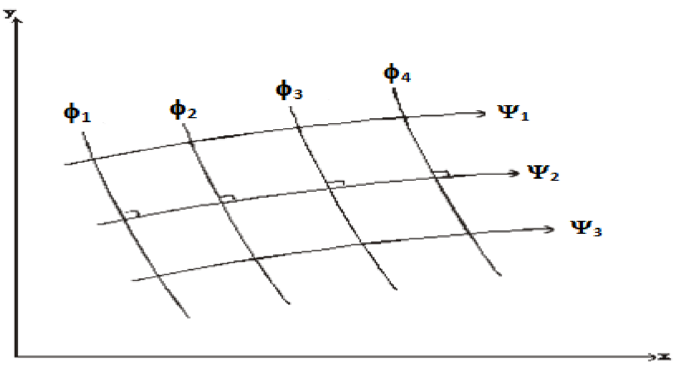
i.e. slope of an equipotential line (say **m_2**) = **- (u/v)**

∴ Product of slopes of a streamline and equipotential line *i.e.* **$m_1 \times m_2 = -1$**

- Thus, stream lines and equipotential lines intersect each other orthogonally at all points of intersection.
- This property of stream lines and equipotential lines is used in drawing a flow net.

FLOW NET

- A flow net is a graphical representation obtained by drawing a series of stream lines and equipotential lines.
- A flow net may be drawn for two-dimensional, steady and irrotational flow *i.e.* potential flow.
- **Figure**, shows a flow net obtained by drawing a number of streamlines corresponding to $\psi_1, \psi_2, \psi_3, \ldots$ and equipotential lines corresponding to $\phi_1, \phi_2, \phi_3, \ldots$.



Flow net

- A flow net can be plotted by either analytical or graphical methods.

- The analytical method involves solution of Laplace equations with given boundary conditions.
- The analytical method cannot be applied to all the cases as it may not be possible to obtain the solution of Laplace equations.
- The graphical method involves solution of Laplace equations graphically.
- Hele-Shaw and electrical analogy methods are the two graphical methods used for drawing flow nets.

Uses of Flow Net:

- The main uses of flow net analysis are:

(i) Determination of the quantity of seepage through porous media e.g. canal banks, earth dams etc.

(ii) Designing efficient boundary shapes for which the flow does not separate from the boundary surface or flow separation is minimum. This is due to the fact that a flow net provides valuable information of the flow pattern around object of any shape.

Observation: Although the flow net analysis is based on ideal fluid flow concept, it may also be applied to real fluid flow situations within certain limit. For example, in the region away from the surface, the real fluid behaves more or less like an ideal fluid.

Problems:

Q1:The velocity components in a **2D** flow field for an incompressible fluid are: $u = (y^3 + 6x - 3x^2y)$ and $(v = 3xy^2 - 6y - x^3)$

Check whether the flow is (i) continuous or non-continuous (ii) rotational or irrotational. If flow is irrotational, find the potential function and stream function.

Solution: The given velocity components are:

$$u = (y^3 + 6x - 3x^2y) \quad \text{and} \quad (v = 3xy^2 - 6y - x^3)$$

$$\therefore \frac{\partial u}{\partial x} = (6 - 6xy) \quad \text{and} \quad \frac{\partial v}{\partial y} = (6xy - 6)$$

(i) Continuity equation for steady **2D** flow is: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow (6 - 6xy + 6xy - 6) = 0$

- As continuity eq. is satisfied and therefore the flow is continuous.

(ii) For **2D** flow in **xy**-plane, the rotation component about **z**-axis is:

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\therefore \frac{\partial u}{\partial y} = (3y^2 - 3x^2) \quad \text{and} \quad \frac{\partial v}{\partial x} = (3y^2 - 3x^2)$$

$$\therefore w_z = \frac{1}{2}(3y^2 - 3x^2 - 3y^2 - 3x^2) = 0$$

- As $\omega_z = 0$ and therefore the flow is irrotational and potential function ϕ and stream function Ψ exist.

To find ϕ

- $\because -\frac{\partial\phi}{\partial x} = u$ $\quad\therefore \frac{\partial\phi}{\partial x} = -y^3 - 6x + 3x^2 y,$ Integrating, to get

$$\phi = -y^3 x - 3x^2 + x^3 y + f(y) \qquad \text{(a)}$$

- **f (y)** is a constant of integration, which can be a function of **y** only.
- Also, $-\frac{\partial\phi}{\partial y} = v$ $\quad\Rightarrow \frac{\partial\phi}{\partial y} = -3xy^2 + 6y + x^3$
- Integrating, to get

$$\phi = -xy^3 + 3y^2 + x^3 y + f(x) \qquad \text{(b)}$$

- **f (x)** is a constant of integration, which can be a function of **x** only.
- Equating Eqs. (**a**) and (**b**) term wise, to write

$$f(x) = -3x^2 \text{ and } f(y) = 3y^2$$

- Substituting in Eq. **(a)** or Eq. **(b)**, the eq. of potential function is:

$$\phi = -y^3 x - 3x^2 + x^3 y + 3y^2$$

To find Ψ

$\because \frac{\partial\psi}{\partial x} = -v$ $\quad\therefore \frac{\partial\psi}{\partial x} = -3xy^2 + 6y + x^3,$ Integrating, to get

$$\psi = -\frac{3}{2}x^2 y^2 + 6xy + \frac{x^4}{4} + f(y) \qquad \text{(c)}$$

- **f(y)** is a constant of integration, which can be a function of **y** only.
- Also, $\frac{\partial\psi}{\partial y} = u$ $\quad\Rightarrow \frac{\partial\Psi}{\partial y} = y^3 + 6x - 3x^2 y$
- Integrating, to get

$$\psi = \frac{y^4}{4} + 6xy - \frac{3}{2}x^2 y^2 + f(x) \qquad \text{(d)}$$

- **f(x)** is a constant of integration, which can be a function of **x** only.
- Equating Eqs. **(c)** and **(d)** term and term wise, to write

$$f(x) = \frac{x^4}{4} \text{ and } f(y) = \frac{y^4}{4}$$

- Substituting in Eq. **(c)** or Eq. **(d)**, the equation of stream function is

$$\psi = \frac{x^4}{4} + \frac{y^4}{4} + 6xy - \frac{3}{2}x^2 y^2$$

Observation:

- **f(x) or f(y)** in the above equations for ϕ **or** Ψ can also be determined as:

✓ Let us determine **f(y)** in Eq. **(a)**

- Eq. **(a)** is:

$$\phi = -y^3x - 3x^2 + x^3y + f(y) \qquad \text{(a)}$$

- **As,** $-\frac{\partial \phi}{\partial y} = v \qquad \Rightarrow \frac{\partial \phi}{\partial y} = -3xy^2 + 6y + x^3$
- Differentiating Eq. **(a)** with respect to **y**, to get

$$\frac{\partial \phi}{\partial y} = -x(3y^2) - 0 + x^3 + f'(y)$$

- Equating the two equations, to get

$$f'(y) = 6y$$

- Integrating

$$f(y) = 3y^2$$

- Substituting in Eq. **(a)**, the eq. of potential function is:

$$\phi = -y^3x - 3x^2 + x^3y + 3y^2$$

- Same expression.

Q2: Given **u = xy** and **v = 2yz**. Check whether these components represent **2D** or **3D** flow. If **3D**, determine the third component of velocity.

Solution:

u = xy and **v = 2yz**

$$\therefore \frac{\partial u}{\partial x} = y \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 2z \qquad \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$$

$\therefore$ Given velocity components represent **3D** flow

- For **3D** flow, continuity equation is: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$$\therefore y + 2z + \frac{\partial w}{\partial z} = 0 \qquad \Rightarrow \frac{\partial w}{\partial z} = -(y + 2z)$$

- Integrating, to get

$$w = -(yz + z^2) + f(x, y)$$

➢ **f(x, y)** is a constant of integration and can be a function of **x** & **y**.