# Lecture 8: Numerical Analysis (UMA011)

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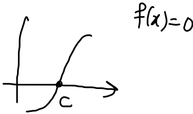
## Fixed point iteration

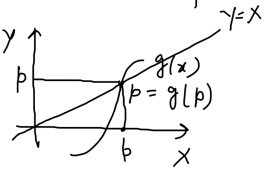
**Fixed point:**

A fixed point for a function $g(x)$ is a number at which the value of function does not change, when function is applied.




$p$ is a fixed pt. for a function $g(x)$ if

$g(p)=p$




Graphically, where $g(x)$ cuts $y=x$ line.

## Fixed point iteration: Fixed point

**Example**

Determine any fixed point of the function $g(x) = x^2 - 2$.

If $p$ is a fixed pt. for $g(x) = x^2 - 2$ ie

$$g(p) = p$$

$$p^2 - 2 = p$$

$$p^2 - p - 2 = 0$$

$$p = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2, -1$$

## Fixed point iteration → gives us fixed pt.

### Connection between fixed point problems and root finding problems:

(F.P.P.) (R.F.P.)

$f(x) = 0$

$g(x) = x$

?

1) Given a root finding problem $f(x)=0$, let $p$ be the root of this equation, we can define a function $g(x)$ with a fixed pt at $p$.

$$g(x) = x - f(x) \quad \text{or} \quad x + f(x)$$

$$g(p) = p - f(p) = p - 0 = p$$

$$g(x) = x + c\,f(x), \quad c \in \mathbb{R}.$$

$$\Rightarrow g(p) = p + c\,f(p) = p$$

R.F.P.

$x^2 - x - 2 = 0$

$x^2 - 2 = x$

$x = \sqrt{x+2}$

$x + x^2 - x - 2 = x$

F.P.P.

2) If the function $g(x)$ has a fixed pt at $p$ ie $g(p)=p$, then the function defined by

$g(x)=x$
$f(x)=0$

$$f(x) = g(x)-x \quad \text{or} \quad x-g(x)$$

has a root at point $p$

$$\because \quad f(p) = g(p)-p = p-p=0$$

## Fixed point iteration

**Fixed point forms:** $f(x) = 0$

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. Write all the possible ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.

Solution

$$x = x + x^3 + 4x^2 - 10 = g_1(x) \checkmark$$

$$x = x - (x^3 + 4x^2 - 10) = g_2(x)$$

$$x = x + c(x^3 + 4x^2 - 10) = g_3(x), \quad c \in \mathbb{R}.$$

from $f(x)=0$, $x^3 = 10 - 4x^2$

$$x = (10 - 4x^2)^{1/3} = g_4(x)$$

$4x^2 = 10 - x^3$

$$x = \frac{\sqrt{10 - x^3}}{2} = g_5(x)$$

$$x + 4 - \frac{10}{x^2} = 0$$

$$x = \frac{10}{x^2} - 4 = g_6(x)$$

$$x^2(x+4) = 10$$

$$x = \sqrt{\frac{10}{x+4}} = g_7(x)$$

$$x^2 + 4x = \frac{10}{x} \Rightarrow \quad x = \frac{1}{4}\left(\frac{10}{x} - x^2\right) = g_9(x)$$

$$x = \sqrt{\frac{10}{x} - 4x} = g_8(x)$$

Procedure of F.P.I.

1st step $\rightarrow$ $g(x) = x$ $[a, b]$

To find $g(x)$

let $p_0 \in [a, b]$

$\hookrightarrow$ initial guess

$$p_1 = g(p_0) \neq p_0$$

$$p_2 = g(p_1) \neq p_1$$

$$p_3 = g(p_2) \neq p_2$$

$$\vdots$$

$$p_n = g(p_{n-1})$$

$\hookrightarrow$ $p$ (exact fixed pt.)

$|p_n - p_{n-1}| < tol$ (given)

$p_{n+1} = g(p_n) \rightarrow p$

## Fixed point iteration

$C[a,b] \to$ class of continuous function on $[a,b]$

### Convergence conditions satisfied by $g(x)$:

(i) (existence) If $g \in C[a,b]$ and $g(x) \in [a,b]$, $\forall x \in [a,b]$, then $g(x)$ has at least one fixed point in $[a,b]$.

$[a,b]$

(ii) (uniqueness) If, in addition, $g'(x)$ exists in $(a,b)$ and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a,b)$, then there is exactly one fixed point in $[a,b]$.

$\Rightarrow |g'(x)| < 1$

(iii) (convergence) If conditions of (i) and (ii) are satisfied, then for any number $p_0 \in [a,b]$, the sequence defined by $p_n = g(p_{n-1})$, $n \geq 1$ converges to the unique fixed point $p$ in $[a,b]$.

## Root finding problem

**Exercise:**

1. The equation $x^3 - 7x + 2 = 0$ has a unique root in $[0, 1]$. Write all the possible ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.