Lecture 31: Numerical Analysis (UMA011)

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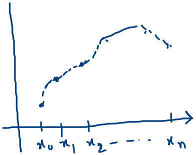
Newton Divided Difference Interpolation:

$x_0, x_1, x_2 \,--\, x_n$

$f(x_0), f(x_1) \,---\, f(x_n)$



Derivation:

The divided differences of f with respect to $x_0, x_1, \cdots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \quad — (1)$$

for appropriate constants $a_0, a_1, \cdots, a_n$.

To find a_0, Put $x = x_0$ in eq^n (1)

$$P_n(x_0) = a_0 + 0 + 0 = a_0 \Rightarrow a_0 = f(x_0)$$

To find a_1, Put $x = x_1$ in eq^n (1)

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) + 0 + - \quad 0$$

$$f(x_1) = a_0 + a_1(x_1 - x_0)$$

$$a_1 = \frac{f(x_1) - a_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

To write these co-efficients in terms of divided difference, we first define these differences :-

Newton Divided Difference Interpolation:

Divided Differences

$f[x_0] = f(x_0)$

The zeroth divided difference of the function f with respect to x_i, denoted $f[xi]$, is simply the value of f at x_i : $f[x_i] = f(x_i)$.

x_i

$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

The first divided difference of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and defined as: $f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$.

The second divided difference, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$.

$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

The kth divided difference, $f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k}]$, is defined as

$f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+k}] - f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$.

$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_6}$

So, we get

$$a_0 = f[x_0]$$

$$\Rightarrow \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1]$$

To find a_2, put $x = x_2$ in eqn 1

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\frac{f(x_2) - f(x_0) - \frac{(f(x_1) - f(x_0))}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = a_2$$

$$= \frac{x_1 f(x_2) - x_0 f(x_2) - x_1 f(x_0) + \cancel{x_0 f(x_0)} - x_2 f(x_1) + x_2 f(x_0) + x_0 f(x_1) - \cancel{x_0 f(x_0)} - x_1 f(x_1) + x_1 f(x_1)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{x_1 \left(f(x_2) - f(x_1)\right) - x_0 \left(f(x_2) - f(x_1)\right) + x_1 \left(f(x_1) - f(x_0)\right) - x_2 \left(f(x_1) - f(x_0)\right)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{\left(f(x_2) - f(x_1)\right)(x_1 - x_0) - \left(f(x_1) - f(x_0)\right)(x_2 - x_1)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2] \checkmark$$

$$\vdots \qquad \vdots$$

$$a_n = f[x_0, x_1, x_2 \cdots x_n] \checkmark$$

$$\Rightarrow \quad P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+ \cdots \quad f[x_0, x_1, x_2 \cdots x_n](x - x_0) \cdots (x - x_{n-1})$$

Newton D.D interpolating polynomial

To interpolate at $x = p$.

$$P_n(p) \simeq f(p)$$

Table of D.D

x_i	$f(x_i) = f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	$f(x_0)$			
		$\frac{f(x_1)-f(x_0)}{x_1-x_0} = f[x_0,x_1]$		
x_1	$f(x_1)$		$\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = f[x_0,x_1,x_2]$	
		$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f[x_1,x_2]$		$f[x_0,\dots,x_3]$
x_2	$f(x_2)$		$\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
				$f[x_1,\dots,x_4]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$f(x_n)$	$\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}} = f[x_{n-1},x_n]$		

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \dots$$

$$+ \dots f[x_0,x_1,x_2 \dots x_n](x-x_0)\dots(x-x_{n-1})$$

Newton Divided Difference Interpolation:

Example:

Complete the divided difference table for the following data:

x	1	1.3	1.6	1.9	2.2
$f(x)$	0.7652	0.6201	0.4554	0.2818	0.1103

and construct the interpolating polynomial that uses all this data and hence find the value of $f(x)$ at $x = 1.5$.

Solution:

Table of D.D.

i	x_i	zero D.D $f(x_i)$	first D.D $f[x_i, x_{i+1}]$	2^{nd} D.D $f[x_i, x_{i+1}, x_{i+2}]$	3^{rd} D.D $f[x_i, \dots, x_{i+3}]$	4^{th} D.D $f[x_i, \dots, x_{i+4}]$
0	1	0.7652	$\frac{0.6201 - 0.7652}{1.3 - 1} = -0.4837$	-0.1087	0.0659	0.001825
1	1.3	0.6201	$\frac{0.4554 - 0.6201}{1.6 - 1.3} = -0.5489$	-0.0494	0.0681	
2	1.6	0.4554	$\frac{0.2818 - 0.4554}{1.9 - 1.6} = -0.5786$	0.0118		
3	1.9	0.2818	$\frac{0.1102 - 0.2818}{2.2 - 1.9} = -0.5715$			
4	2.2	0.1102				

$$P_4(x) = 0.7652 + (-0.4837)(x-1) + (-0.1087)(x-1)(x-1.3) + 0.0659 * (x-1)(x-1.3)(x-1.6) + (0.001825)(x-1)(x-1.3)(x-1.6)(x-1.9)$$

Put $x = 1.5$

$P_4(1.5) = 0.51182$ Ans

Newton Divided Difference Interpolation:

Exercise:

1. Using Newton's divided difference interpolation, construct interpolating polynomials of degree one, two, and three for the following data. Approximate $f(0.43)$ using the polynomial.

   $$f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.4816.$$

2. Show that the Newton polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4