Course: UMA 035 (Optimization Techniques)

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Prove that the set S ={$X_1 - X_2$: $X_1 \in S_1$ and $X_2 \in S_2$} is a convex set, where S_1 and S_2 are two disjoint non-empty convex subsets of a convex set.

Proof:

Let $X_1 - X_2$ and $Y_1 - Y_2$ be two elements of the set S.

Since, $X_1 - X_2$ and $Y_1 - Y_2$ belongs to the set S. So, these elements will satisfy the following property of the set S.

$X_1 \in S_1$ and $X_2 \in S_2$

$Y_1 \in S_1$ and $Y_2 \in S_2$

The convex linear combination of $X_1 - X_2$ and $Y_1 - Y_2$ is $a_1 (X_1 - X_2) + a_2 (Y_1 - Y_2)$

where,

$a_1 >= 0$,

$a_2 >= 0$,

$a_1 + a_2 = 1$.

Now,

$a_1 (X_1 - X_2) + a_2 (Y_1 - Y_2)$

$= (a_1 X_1 + a_2 Y_1) - (a_1 X_2 + a_2 Y_2) \in S$

as

$a_1 X_1 + a_2 Y_1 \in S_1$ (Since, X_1 and $Y_1 \in S_1$ and S_1 is a convex set)

and

$a_1 X_2 + a_2 Y_2 \in S_2$ (Since, X_2 and $Y_2 \in S_2$ and S_2 is a convex set)

Prove algebraically that the set $S=\{(x_1,x_2): (x_1)^2+(x_2)^2<=r^2\}$ is a convex set.

Proof:

Let (x_1,x_2) and (y_1,y_2) be two elements of the set S.

Since, (x_1,x_2) and (y_1,y_2) belongs to the set S. So, these elements will satisfy the following property of the set S.

$(x_1)^2+(x_2)^2 <= r^2$

$(y_1)^2+(y_2)^2 <= r^2$

The convex linear combination of (x_1,x_2) and (y_1,y_2) is $a_1\ (x_1,x_2)+ a_2\ (y_1,y_2)=$

$(a_1x_1+ a_2y_1, a_1x_2+ a_2y_2)$

where,

$a_1>=0,$

$a_2>=0,$

$a_1+ a_2=1.$

Now,

$(a_1x_1+ a_2y_1)^2+ (a_1x_2+ a_2y_2)^2$

$= (a_1)^2(x_1)^2 + (a_2)^2(y_1)^2+2(a_1)(a_2)(x_1)(y_1)+ (a_1)^2(x_2)^2 + (a_2)^2(y_2)^2+2(a_1)(a_2)(x_2)(y_2)$

$=(a_1)^2((x_1)^2 +(x_2)^2)+ (a_2)^2((y_1)^2+(y_2)^2)+2(a_1)(a_2)((x_1)(y_1)+(x_2)(y_2))$

$<=(a_1)^2(r^2)+ (a_2)^2(r^2)+ 2(a_1)(a_2)((x_1)(y_1)+(x_2)(y_2))$ (since, $(x_1)^2+(x_2)^2 <= r^2$,

$(y_1)^2+(y_2)^2 <= r^2)$

$<=(a_1)^2(r^2)+ (a_2)^2(r^2)+ 2(a_1)(a_2)(\sqrt{((x_1^2+x_2^2))(y_1^2+y_2^2)})$

(since, $(x_1)(y_1)+(x_2)(y_2)<=\sqrt{((x_1^2+x_2^2))(y_1^2+y_2^2)}$)

$<=(a_1)^2(r^2)+ (a_2)^2(r^2)+ 2(a_1)(a_2)(\sqrt{(r^2)(r^2)})$

$<=(a_1)^2(r^2)+ (a_2)^2(r^2)+ 2(a_1)(a_2)(r^2)$

$<=(r^2) ((a_1)^2+ (a_2)^2+ 2(a_1)(a_2))$

$<=(r^2) (a_1+ a_2)^2$

$<=(r^2) (1)^2$

$<=r^2$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination a_1 (x_1,x_2)+ a_2 (y_1,y_2).

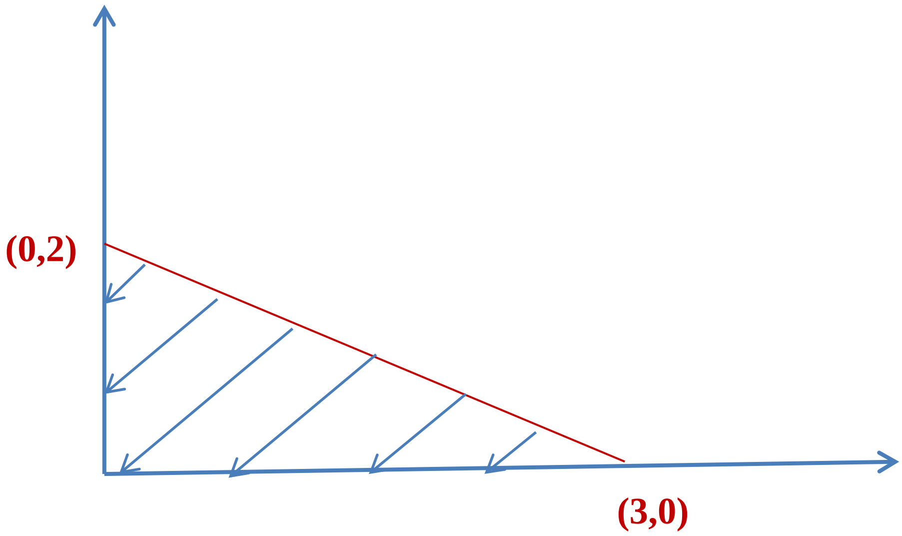
Hence, the set S is a convex set.

Remark: To show that a result is valid there is a need to prove that result

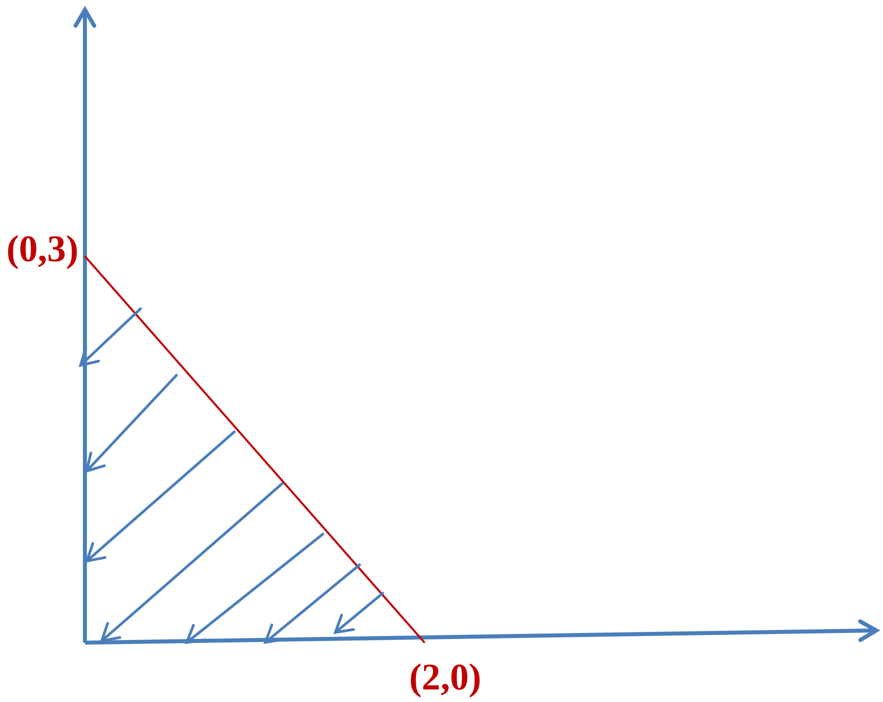
Remark: To show that a result is not valid it is sufficient to provide an example for which the result is not valid

Give an example to show that the union of two convex sets is not necessarily a convex set.

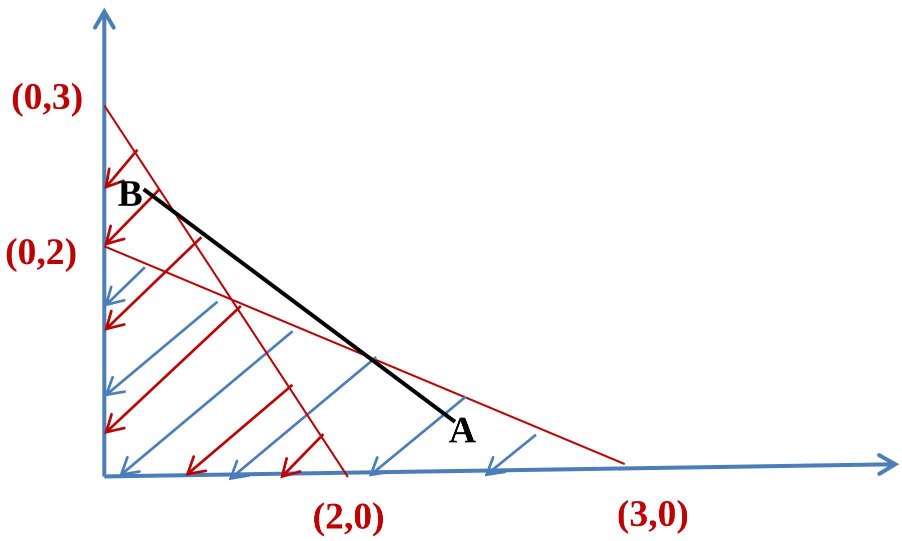
Let $S_1=\{(x_1, x_2): 2x_1+3x_2<=6\}$. It is obvious from the below graph that the set S_1 is a convex set.



Let $S_2=\{(x_1, x_2): 3x_1+2x_2<=6\}$. It is obvious from the below graph that the set S_2 is a convex set.



While, it is obvious from the below graph of $S_1 \cup S_2=\{(x_1, x_2): 2x_1+3x_2<=6$ or $3x_1+2x_2<=6\}$ that $S_1 \cup S_2$ is not a convex set as it is possible to find two points A and B in the region such that some portion of the line segment AB lies outside the region.



Check that (2, 3) can be written as convex linear combination of three points (1,0), (1,2) and (3,4) or not.

Let

$(2,3) = a_1(1,0)+ a_2(1,2)+ a_3(3,4)$, where, $a_1>=0$, $a_2>=0$, $a_3>=0$ and $a_1+ a_2+ a_3=1$.

$(2,3)=(a_1+a_2+3a_3 , 2a_2+4a_3)$

$2= a_1+a_2+3a_3$

$3= 2a_2+4a_3$

Solving

$a_1+a_2+3a_3=2$

$2a_2+4a_3 = 3$

and

$a_1+ a_2+ a_3=1$

$a_3=1/2$

$a_2=1/2$

$a_1=0$

Since, all the conditions $a_1>=0$, $a_2>=0$, $a_3>=0$ are satisfying. So, (2, 3) can be written as convex linear combination of three points (1,0), (1,2) and (3,4)

Check that (1/2, 3) can be written as convex linear combination of three points (1,0), (1,2) and (3,4) or not.

Let

$(1/2,3) = a_1(1,0)+ a_2(1,2)+ a_3(3,4)$, where, $a_1>=0$, $a_2>=0$, $a_3>=0$ and $a_1+ a_2+ a_3=1$.

$(1/2,3)=(a_1+a_2+3a_3 , 2a_2+4a_3)$

$1/2= a_1+a_2+3a_3$

$3= 2a_2+4a_3$

Solving

$a_1+a_2+3a_3=1/2$

$2a_2+4a_3 = 3$

and

$a_1+ a_2+ a_3=1$

$a_3=-1/4$

$a_2=2$

$a_1=-3/4$

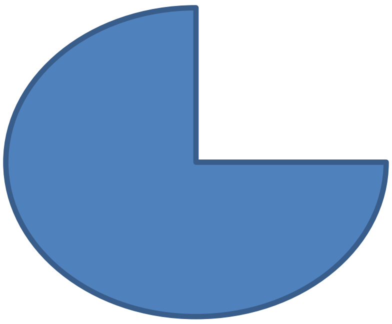
Since, the conditions $a_1>=0$, $a_2>=0$, $a_3>=0$ are not satisfying. So, (1/2, 3) can not be written as convex linear combination of three points (1,0), (1,2) and (3,4)

Remark: A set formed by the intersection of finite number of closed half spaces is called as polyhedron or polyhedral. Furthermore, if the intersection is non-empty and bounded. Then, it is called a polytope.

Remark: Let S be a nonempty set. Then, the set of all the possible convex linear combinations of the points of the set S is called convex hull of S. It is denoted by [S].

OR

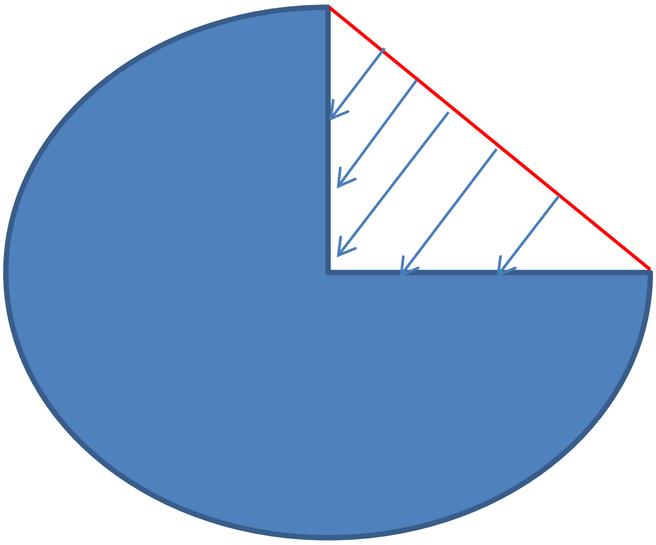
Let S be a nonempty set. Then, the convex hull [S] is the smallest convex set containing S.



The above set is convex or not.

Ans: **No**

Which is the smallest convex set containing the above set or which is convex hull for the above set?



Remark: **The convex linear combination of n numbers $a_1, a_2, \ldots, a_n$ will always be greater than or equal to minimum $\{ a_1, a_2, \ldots, a_n \}$ and less than or equal to maximum $\{ a_1, a_2, \ldots, a_n \}$**

State and prove fundamental theorem of LPP

Statement:

If the given LPP has an optimal solution then at least one vertex of the feasible region is optimal.

Proof

Let the value of the objective function of a LPP having "n" vertices x_1, x_2, …, x_n be maximum at x^*

Case (i): If x^* is a vertex then the proof is complete.

Case (ii): If x^* is not a vertex then x^* can be written as a convex linear combination of the vertices x_1, x_2, …, x_n

i.e.,

$x^*=a_1x_1+ a_2x_2+\ldots+ a_nx_n$ (where, $a_1>=0$, $a_2>=0$,…, $a_n>=0$ and $a_1+ a_2+\ldots+ a_n=1$)

$cx^*=c(a_1x_1+ a_2x_2+\ldots+ a_nx_n)$

$cx^*= a_1 (c\, x_1)+ a_2(cx_2)+\ldots+ a_n(cx_n)$

$cx^*<=$ maximum $\{c\, x_1, c\, x_2 ,\ldots, c\, x_n\}$

$cx^*<= c\, x_p$ (where, cx_p = maximum $\{c\, x_1, c\, x_2 ,\ldots, c\, x_n\}$)

Furthermore as, the value of the objective function is maximum at x^*. So, the value of the objective function at x^* will be either equal or greater than the value of the objective function at x_p i.e.,

$cx_p <= cx^*$

Using $cx^* <= c\,x_p$ and $cx_p <= cx^*$

$cx^* = c\,x_p$

This implies that the value of the objective function at an arbitrary point x_p is equal to the value of the objective function at x^* and we had considered that the value of the objective function at x^* is maximum.

Therefore, the point x_p (vertex) is also an optimal solution.

OR

Let the value of the objective function of a LPP having "n" vertices x_1, x_2, …, x_n be minimum at x^*

Case (i): If x^* is a vertex then the proof is complete.

Case (ii): If x^* is not a vertex then x^* can be written as a convex linear combination of the vertices x_1, x_2, …, x_n

i.e.,

$x^*=a_1x_1+ a_2x_2+\ldots+ a_nx_n$ (where, $a_1>=0$, $a_2>=0,\ldots, a_n>=0$ and $a_1+ a_2+\ldots+ a_n=1$)

$cx^*=c(a_1x_1+ a_2x_2+\ldots+ a_nx_n)$

$cx^*= a_1 (c x_1)+ a_2(cx_2)+\ldots+ a_n(cx_n)$

$cx^*>=$ minimum $\{c x_1, c x_2 ,\ldots, c x_n\}$

$cx^*>= c x_p$ (where, cx_p = minimum $\{c x_1, c x_2 ,\ldots, c x_n\}$)

Furthermore as, the value of the objective function is minimum at x^*. So, the value of the objective function at x^* will be either equal or less than the value of the objective function at x_p i.e.,

$cx_p >= cx^*$

Using $cx^*>= c x_p$ and $cx_p >= cx^*$

$cx^*= c x_p$

This implies that the value of the objective function at an arbitrary point x_p is equal to the value of the objective function at x^* and we had considered that the value of the objective function at x^* is minimum.

Therefore, the point x_p (vertex) is also an optimal solution.