Theory of Machines

Module : Linkage Mechanisms

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Lecture Contents and Learning Outcomes

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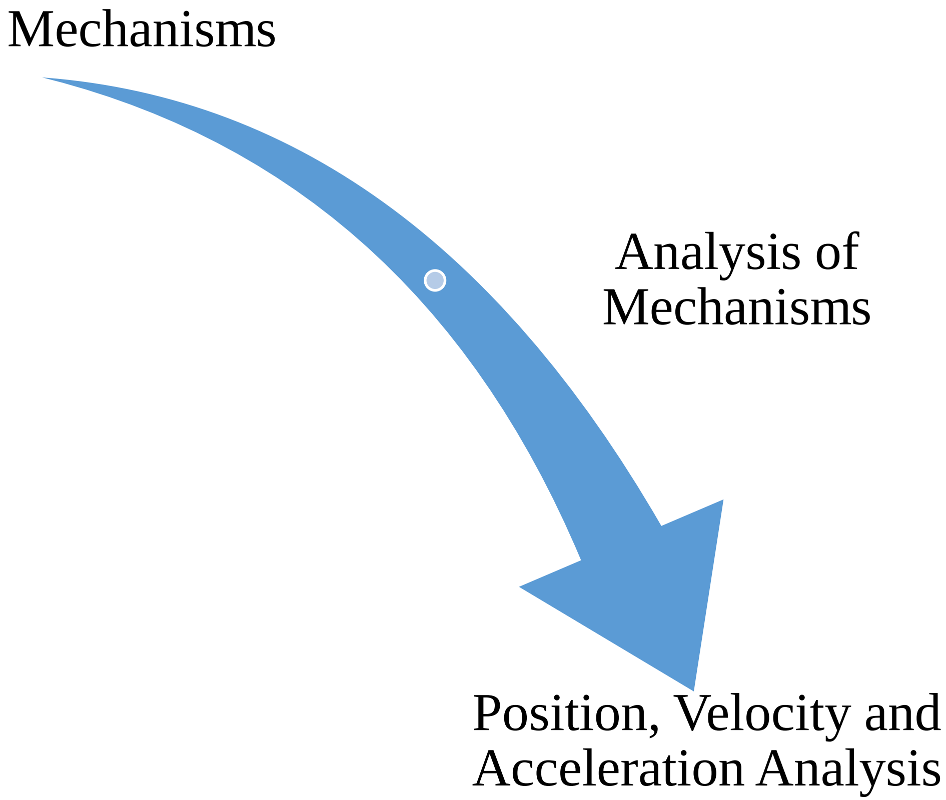
- Links
- Kinematic Pairs
- Degree of Freedom
- Inversions
- Mechanisms
- Transmission Angle and Mechanical Advantage
- Vector and Matrix Methods for position, velocity and acceleration analysis

Learning Outcomes



References

1. S S Ratan “Theory of Machines” 3rd Edition, Tata Macgraw Hill Publications

2. J. J. Uicker, G. R. Pennock, and J. E. Shigley “Theory of Machines and Mechanisms” Oxford Press (2009)

3. Neil Sclater, Nicholas P. Chironis “Mechanisms and Mechanical Devices Sourcebook” 4th Edition, McGraw Hill Publications

4. R S Khurmi “A text Book of Theory of Machines” S Chand Publications

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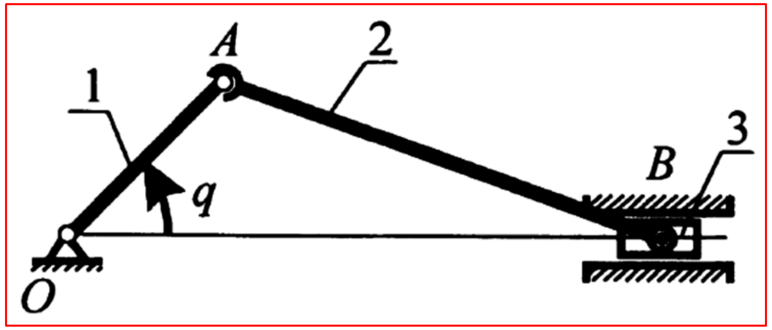
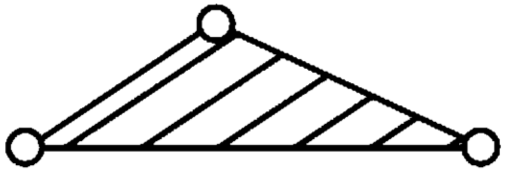
Links

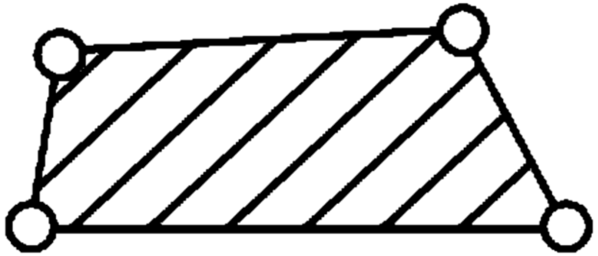
- A resistant body or a group of resistant bodies with rigid connections preventing their relative motion is known as link.
- It may also be defined as a member or a combination of members of a mechanism, connecting other members and having motion relative to them.
- A link may consists of one or more resistant bodies.
- A link may also be known as kinematic link or element.
- Types of link: Binary, Ternary and Quaternary



Binary



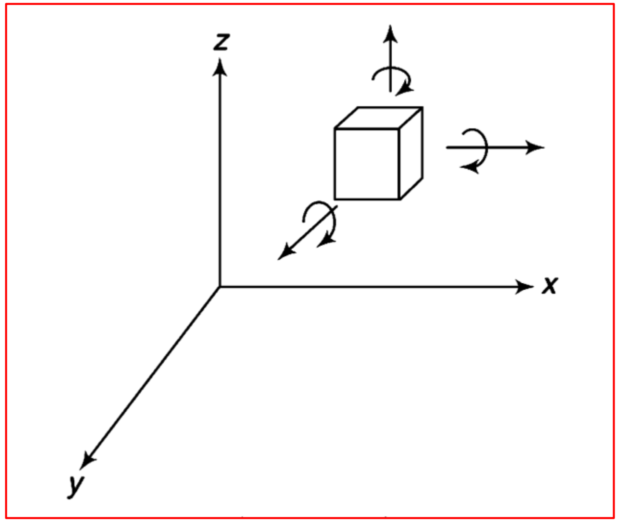
Ternary



Quaternary

Degrees of Freedom

- Unconstrained rigid body moving in a space:
  1. *Translational motion along any three mutually perpendicular axes x, y and z*
  2. *Rotational motion about these axes*



- Thus a rigid body posses a six degrees of freedom.
- Connection of a link with another posses a certain constrains over relative motion
- The restrains never be zero (joint is disconnected) or six (joint becomes solid)
- *Degrees of freedom* of a pair can be defines as the number of independent relative motions, both translational and rotational a pair can have.

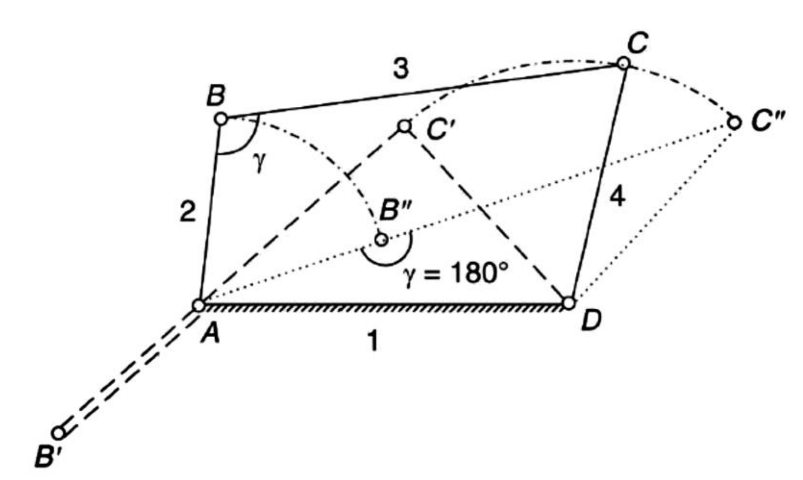
Degrees of Freedom = 6 – Number of restrains

Mechanical Advantage

- Definition: The ratio of the output force or torque to the input force or torque at any instant.
- Assumption: Friction and inertia forces are ignored
- Input torque T_2 is applied at link 2 to drive the output link 4 with a resisting torque T_4, then

Power input = Power output

$$\boxed{T_2\omega_2 = T_4\omega_4} \Rightarrow \boxed{\text{MA} = \frac{T_4}{T_2} = \frac{\omega_2}{\omega_4}}$$



Adopted from SS Ratan, "Theory of Machines"

- *Thus it is reciprocal of velocity ratio*
- At extreme positions ($AB'C'D$) and ($AB''C''D$), the velocity ω4 of the output link DC (rocker) is zero, it makes mechanical advantage equal to infinity. A small input torque can overcome large output torque load.
- The extreme positions are called as *toggle* positions.

Transmission Angle

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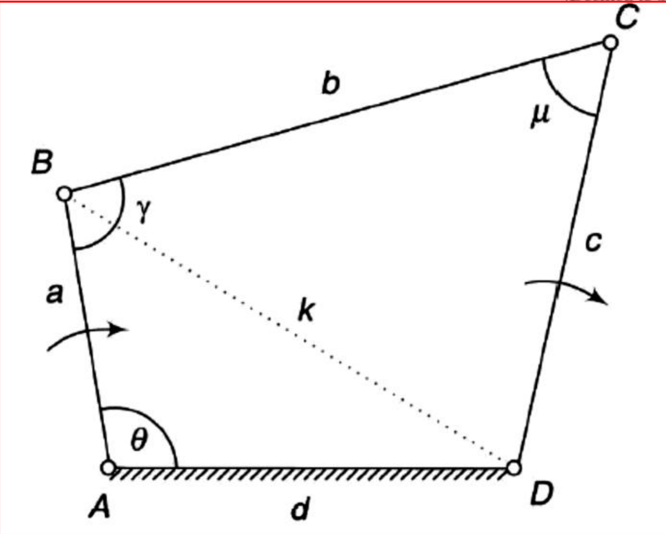
- Definition: The angle μ between output link(rocker) and coupler is known as transmission angle.
- Link *AB* is the input link and the force applied to the output link *DC* is transmitted through coupler *BC*.
- For a particular value of force in the coupler rod, the torque transmitted to the output link is maximum when $\mu = 90^\circ$



Adopted from SS Ratan, "Theory of Machines"

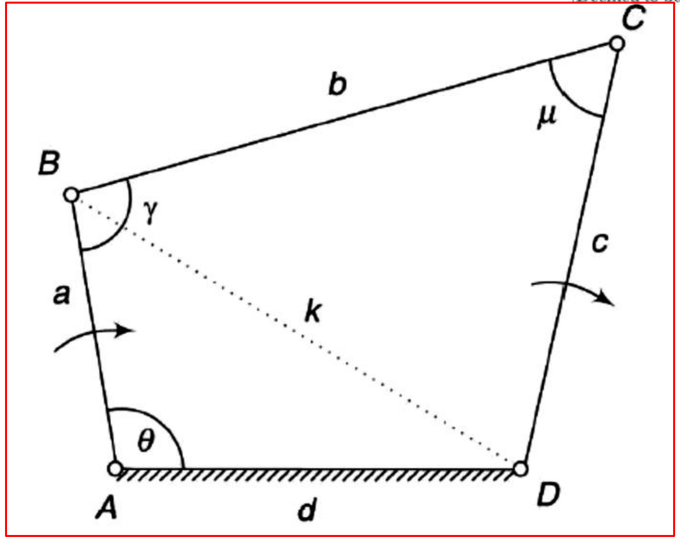
- If μ deviates significantly from 90°, the torque on the output link decreases and mechanism will get jam.
- If link *BC* and *CD* becomes coincident, the transmission angle is zero and the mechanism would lock or jam.
- Hence μ is usually kept more than 45°.
- The best mechanisms, therefore, have a transmission angle that does not deviate much from 90°

Transmission Angle contd..

- Applying cosine rule to triangles ABD and BCD

$$a^2 + d^2 - 2ad\ cos\theta = k^2 \quad (1)$$

$$b^2 + c^2 - 2bc\ cos\mu = k^2 \quad (2)$$



Adopted from SS Ratan, “Theory of Machines”

- From equations (1) and (2)

$$a^2 + d^2 - 2ad\ cos\theta = b^2 + c^2 - 2bc\ cos\mu$$

$$a^2 + d^2 - b^2 - c^2 - 2ad\ cos\theta + 2bc\ cos\mu = 0$$

- Maximum or minimum values of the transmission angle can be found by putting $d\mu/d\theta$ equal to zero.

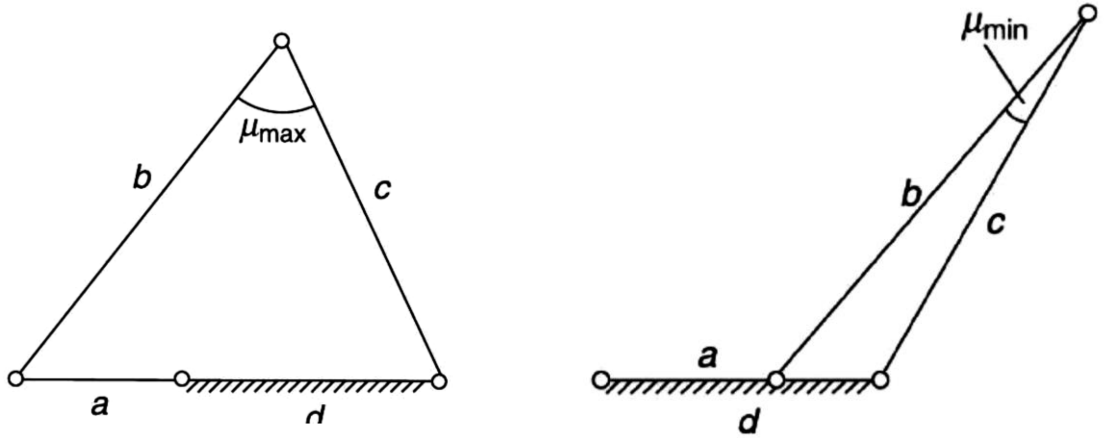
- Differentiating above equation with respect to θ:

$$2ad\ sin\theta - 2bc\ sin\mu . d\mu/d\theta = 0$$

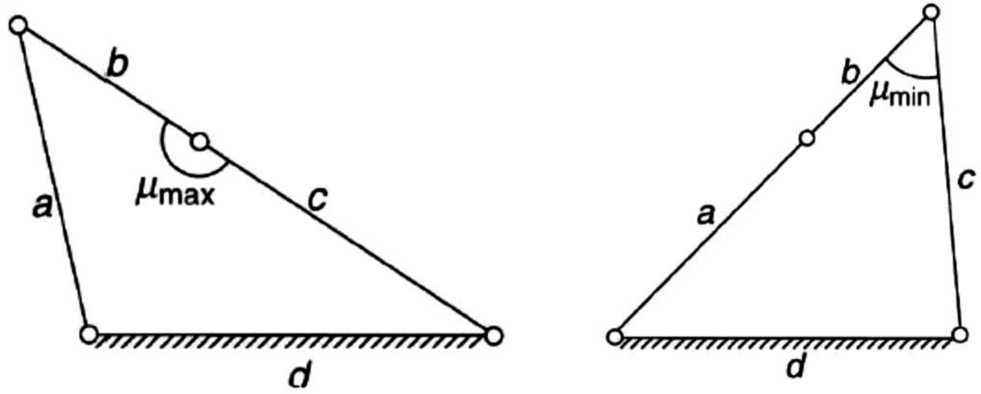
$$\frac{d\mu}{d\theta} = \frac{ad\ sin\theta}{bc\ sin\mu}$$

Transmission Angle contd..

- If $d\mu/d\theta$ is to be zero, the term $ad\sin\theta$ has to be zero, which means θ is either zero or 180°.
- It can be seen that μ is maximum when θ is 180° and minimum when θ is 0°.



Crank rocker Mechanism



Double-rocker Mechanism

Adopted from SS Ratan, "Theory of Machines"

Vector Approach: Position and Velocity Analysis

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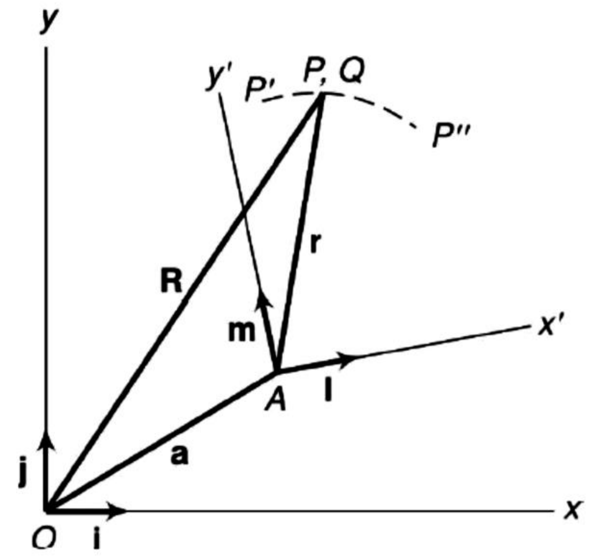
- Let *xyz*: Fixed co-ordinate system, *x′y′z′*: Moving co-ordinate system, *OAQ*: Plane moving body

- Let *i*, *j*, *k* : unit vectors for absolute co-ordinate system, *l*, *m*, *n*: unit vectors for moving co-ordinate system, ω: Angular velocity of rotation of moving system, *R*: vector relative to fixed system, *r*: vector relative to moving system, *P′PP′′*: Path on which *P* moves relative to moving co-ordinate system.



Adopted from SS Ratan, "Theory of Machines"

- At any instant, the position of *P* relative to the fixed system is given by the equation

$$\mathbf{R} = \mathbf{a} + \mathbf{r} \quad (1)$$

where $r = x'l + y'm + z'n$

- Equation (1) may be rewritten as $R = a + x'l + y'm + z'n$

- Taking derivatives with respect to time to find the velocity

$$\dot{\mathbf{R}} = \dot{\mathbf{a}} + (\dot{x}'\mathbf{l} + \dot{y}'\mathbf{m} + \dot{z}'\mathbf{n}) + (x'\dot{\mathbf{l}} + y'\dot{\mathbf{m}} + z'\dot{\mathbf{n}}) \quad (2)$$

First term: velocity of the origin of the moving system, Second term: velocity of P relative to the moving system, Third term: due to rotary motion with velocity ω.

Vector Approach: Position and Velocity Analysis

- Also, $\dot{\mathbf{l}} = \omega \times \mathbf{l}, \quad \dot{\mathbf{m}} = \omega \times \mathbf{m}, \quad \dot{\mathbf{n}} = \omega \times \mathbf{n}$
- Therefore, equation (2) becomes, $\dot{\mathbf{R}} = \dot{\mathbf{a}} + (\dot{x}'\mathbf{l} + \dot{y}'\mathbf{m} + \dot{z}'\mathbf{n}) + \omega(x'\mathbf{l} + y'\mathbf{m} + z'\mathbf{n}$
- The above equation can be written in the form, $\mathbf{v}_p = \mathbf{v}_a + \mathbf{v}^R + \omega \times \mathbf{r} \longrightarrow 3$
- The second term known as relative velocity which an observer attached to the moving system would report for point *P*. This also implies that the velocity relative to the coincident point *Q* on the moving body since the observer may be stationed at the point *Q* on the moving body.
- Now the absolute velocity of coincident point *Q* on the moving system which coincides with the point *P* at any instant may be written as

$$\begin{aligned} \mathbf{v}_{qo} &= \mathbf{v}_{qa} + \mathbf{v}_{ao} \\ &= \mathbf{v}_{ao} + \mathbf{v}_{qa} \\ &= \mathbf{v}_a + \omega \times \mathbf{r} \end{aligned}$$

- and equation (3) changes to

$$\mathbf{v}_p = \mathbf{v}_{qo} + \mathbf{v}^R$$

Vector Approach: Position and Velocity Analysis

- Thus absolute velocity of the point *P* moving relative to a moving reference system is equal to the velocity of the point relative to the moving system plus the absolute velocity of a coincident point fixed to the moving reference system
- The above equation may be written as

$$\mathbf{v}_{po} = \mathbf{v}_{qo} + \mathbf{v}_{pq}$$
$$\mathbf{v}_{po} = \mathbf{v}_{pq} + \mathbf{v}_{qo}$$

$$\boxed{\text{Vel. of } P \text{ rel. to } O = \text{Vel. of } P \text{ rel. to } Q + \text{Vel. of } Q \text{ rel. to } O}$$

Vector Approach: Acceleration Analysis

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- Equation (3) is $\mathbf{v}_p = \mathbf{v}_b + \mathbf{v}^R + \omega \times \mathbf{r}$
- Differentiating it to obtain the acceleration of *P*,

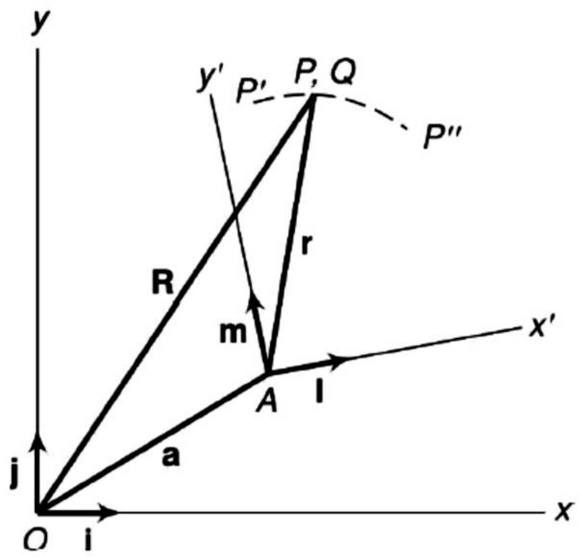
$$\dot{\mathbf{v}}_\mathbf{p} = \dot{\mathbf{v}}_\mathbf{b} + \dot{\mathbf{v}}^\mathbf{R} + \dot{\omega} \times \mathbf{r} + \omega \times \dot{\mathbf{r}}$$

where $\dot{\omega}$ is the angular acceleration of rotation of the moving system.

and $\dot{V}^R$ is obtained by differentiating $(\dot{x}' l + \dot{y}' m + \dot{z}' n)$

$$\begin{aligned}\dot{\mathrm{v}}^\mathrm{R} &= (\ddot{x}'\mathrm{l} + \ddot{y}'\mathrm{m} + \ddot{z}'\mathrm{n}) + (\dot{x}'\dot{\mathrm{l}} + \dot{y}'\dot{\mathrm{m}} + \dot{z}'\dot{\mathrm{n}}) \\ &= (\ddot{x}'\mathrm{l} + \ddot{y}'\mathrm{m} + \ddot{z}'\mathrm{n}) + \omega\,(\dot{x}'\mathrm{l} + \dot{y}'\mathrm{m} + \dot{z}'\mathrm{n}) \\ &= \mathrm{f}^\mathrm{R} + \omega \mathrm{X} \mathrm{v}^\mathrm{R}\end{aligned}$$

$$\begin{aligned}\omega \times \dot{r} &= \omega \times \frac{\mathrm{d}}{\mathrm{dt}}\,(x'\mathrm{l} + y'\mathrm{m} + z'\mathrm{n}) \\ &= \omega \times \mathrm{v}^\mathrm{R} + \omega \times (\omega \times \mathrm{r})\end{aligned}$$



Adopted from SS Ratan, "Theory of Machines"

Vector Approach: Acceleration Analysis

- But $\dot{\mathbf{v}}_\mathbf{p} = \mathbf{f}_\mathbf{p}$ and $\dot{\mathbf{v}}_\mathbf{b} = \mathbf{f}_\mathbf{b}$
- Therefore

$$\mathbf{f}_\mathbf{p} = \mathbf{f}_\mathbf{b} + (\mathbf{f}^\mathbf{R} + \omega \times \mathbf{v}^\mathbf{R}) + \dot{\omega} \times \mathbf{r} + [\omega \times \mathbf{v}^\mathbf{R} + \omega \times (\omega \times \mathbf{r})]$$

$$= \mathbf{f}_\mathbf{b} + \mathbf{f}^\mathbf{R} + 2\omega \times \mathbf{v}^\mathbf{R} + \dot{\omega} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \longrightarrow 4$$

- Now absolute acceleration of *Q*, the coincident point may be written as

$$\mathbf{f}_{qa} = \mathbf{f}_{qb} + \mathbf{f}_{ba}$$

$$= \mathbf{f}_{ba} + \mathrm{f}_{qb}$$

$$= \mathbf{f}_\mathbf{b} + \frac{\mathbf{d}}{\mathbf{dt}}(\omega \times \mathbf{r})$$

$$= \mathbf{f}_\mathbf{b} + \dot{\omega} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r})$$

- Equation (4) reduces to

$$\mathbf{f}_\mathbf{p} = \mathbf{f}_\mathbf{qa} + \mathbf{f}^\mathbf{R} + 2\omega \mathbf{X} \mathbf{v}^\mathbf{R}$$

$$= \mathbf{f}_\mathbf{qa} + \mathbf{f}^\mathbf{R} + \mathbf{f}^\mathbf{C}$$

Vector Approach: Acceleration Analysis

- Equation (4) reduces to

$$\mathbf{f_p} = \mathbf{f_{qa}} + \mathbf{f}^{\mathbf{R}} + 2\omega \mathbf{X} \mathbf{v}^{\mathbf{R}}$$

$$= \mathbf{f_{qa}} + \mathbf{f}^{\mathbf{R}} + \mathbf{f}^{\mathbf{C}}$$

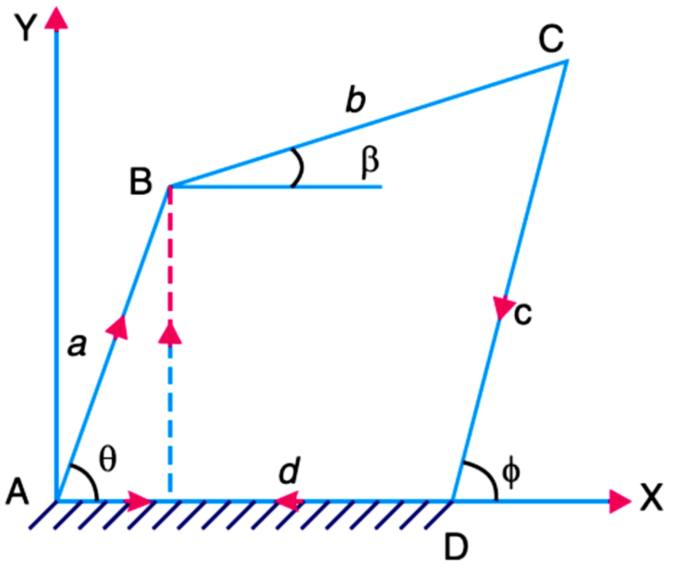
where f_{qa} is the absolute acceleration of Q, f^R is the acceleration of P relative to the moving system or relative to Q, and f^C is known as the Coriolis component of acceleration

- Above Equation may be written as

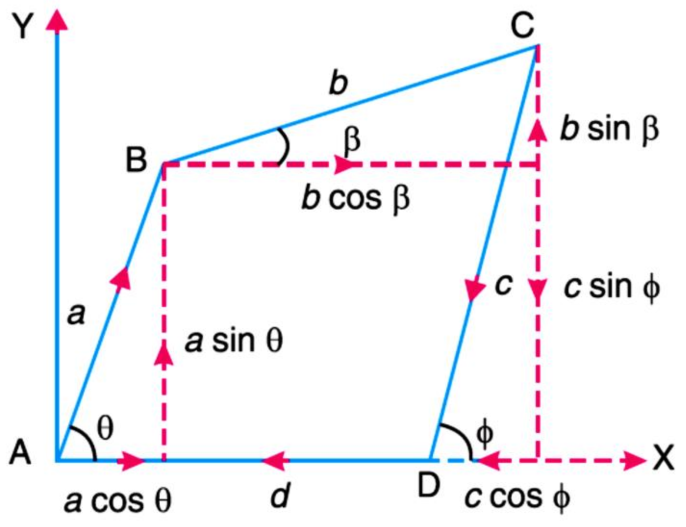
$$\mathbf{f}_{pa} = \mathbf{f}_{qa} + \mathbf{f}_{pq} + \mathbf{f}^{c}$$
$$\mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa} + \mathbf{f}^{c}$$

Acc. of P rel. to A = Acc. of P rel. to Q + Acc. of Q rel. to A + Coriolis Acc.

Analysis of Four-bar Chain Mechanism



Consider links as vectors



Adopted from R S Khurmi, " A Text Book of Theory of Machines"

Displacement/ Position Analysis:

- For equilibrium of the mechanism, the sum of the components along *X*-axis and along *Y*-axis must be equal to zero
- Taking sum of components along *X*-axis

$$a\cos\theta + b\cos\beta - c\cos\phi - d = 0 \quad \text{or} \quad b\cos\beta = c\cos\phi + d - a\cos\theta \qquad (1)$$

Analysis of Four-bar Chain Mechanism

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- Squaring both sides

$$b^2\cos^2\beta = (c\cos\phi + d - a\cos\theta)^2 = c^2\cos^2\phi + d^2 + 2c\,d\cos\phi + a^2\cos^2\theta - 2a\,c\cos\phi\cos\theta - 2a\,d\cos\theta \longrightarrow (2)$$

- Taking sum of components along *Y*-axis

$$a\sin\theta + b\sin\beta - c\sin\phi = 0 \quad \text{or} \quad b\sin\beta = c\sin\phi - a\sin\theta \longrightarrow (3)$$

- Squaring both sides

$$b^2\sin^2\beta = (c\sin\phi - a\sin\theta)^2 = c^2\sin^2\phi + a^2\sin^2\theta - 2a\,c\sin\phi\sin\theta \longrightarrow (4)$$

- Adding equation (2) and (4)

$$b^2(\cos^2\beta + \sin^2\beta) = c^2(\cos^2\phi + \sin^2\phi) + d^2 + 2\,c\,d\cos\phi + a^2(\cos^2\theta + \sin^2\theta) - 2\,a\,c(\cos\phi\cos\theta + \sin\phi\sin\theta) - 2\,a\,d\cos\theta$$

$$\Rightarrow b^2 = c^2 + d^2 + 2\,c\,d\cos\phi + a^2 - 2\,a\,c(\cos\phi\cos\theta + \sin\phi\sin\theta) - 2\,a\,d\cos\theta$$

$$\Rightarrow 2\,a\,c(\cos\phi\cos\theta + \sin\phi\sin\theta) = a^2 - b^2 + c^2 + d^2 + 2\,c\,d\cos\phi - 2a\,d\cos\theta$$

$$\Rightarrow \cos\phi\cos\theta + \sin\phi\sin\theta = \frac{a^2 - b^2 + c^2 + d^2}{2\,a\,c} + \frac{d}{a}\cos\phi - \frac{d}{c}\cos\theta \longrightarrow (5)$$

Analysis of Four-bar Chain Mechanism

- Let $\frac{d}{a} = k_1;\ \frac{d}{c} = k_2;\ \frac{a^2 - b^2 + c^2 + d^2}{2\,a\,c} = k_3$ → 6
- Equation (5) may be written as

$$\cos\phi\cos\theta + \sin\phi\sin\theta = k_1\cos\phi - k_2\cos\theta + k_3 \quad \text{or} \quad \cos(\theta - \phi) = k_1\cos\phi - k_2\cos\theta + k_3 \rightarrow 7$$

- Equation (7) is known as Freudenstein's equation
- Simplifying equation (7) we can write an expression for ϕ and β as

$$\phi = 2\tan^{-1}\left[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right] \rightarrow 8$$

where

$$A = k - a\,(d - c)\cos\theta - cd$$

$$B = -2ac\sin\theta$$

$$C = k - a\,(d + c)\cos\theta + cd$$

$$a^2 - b^2 + c^2 + d^2 = 2k$$

$$\beta = 2\tan^{-1}\left[\frac{-E \pm \sqrt{E^2 - 4DF}}{2\,D}\right] \rightarrow 9$$

where

$$D = k' - a(d + b)\cos\theta + bd$$

$$E = 2ab\sin\theta$$

$$F = k' - a(d - b)\cos\theta - bd$$

Analysis of Four-bar Chain Mechanism

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Velocity Analysis:

- Let ω_1 = Angular velocity of link $AB = d\theta/dt$, ω_2 = Angular velocity of link $BC = d\beta/dt$, ω_3 = Angular velocity of link $CD = d\phi/dt$
- Differentiating Equation (1) with respect to time

$$-a\sin\theta\times\frac{d\theta}{dt}-b\sin\beta\times\frac{d\beta}{dt}+c\sin\phi\times\frac{d\phi}{dt}=0 \Rightarrow -a\,\omega_1\sin\theta-b\,\omega_2\sin\beta+c\,\omega_3\sin\phi=0 \quad (10)$$

- Differentiating Equation (3) with respect to time

$$a\cos\theta\times\frac{d\theta}{dt}+b\cos\beta\times\frac{d\beta}{dt}-c\cos\phi\times\frac{d\phi}{dt}=0 \Rightarrow a\,\omega_1\cos\theta+b\,\omega_2\cos\beta-c\,\omega_3\cos\phi=0 \quad (11)$$

- Multiplying the equation (10) by cosβ and equation (11) by sinβ

$$-a\,\omega_1\sin\theta\cos\beta-b\,\omega_2\sin\beta\cos\beta+c\,\omega_3\sin\phi\cos\beta=0 \quad (12)$$

$$a\,\omega_1\cos\theta\sin\beta+b\,\omega_2\cos\beta\sin\beta-c\,\omega_3\cos\phi\sin\beta=0 \quad (13)$$

- Adding equation (12) and (13)

$$a\,\omega_1\sin(\beta-\theta)+c\,\omega_3\sin(\phi-\beta)=0 \Rightarrow \boxed{\omega_3=\frac{-a\,\omega_1\sin(\beta-\theta)}{c\sin(\phi-\beta)}} \quad (13)$$

Analysis of Four-bar Chain Mechanism

Velocity Analysis contd..:

- Multiplying the equation (10) by *cosϕ* and equation (11) by *sinϕ*

$$-a\,\omega_1 \sin\theta\cos\phi - b\,\omega_2 \sin\beta\cos\phi + c\,\omega_3 \sin\phi\cos\phi = 0 \quad \rightarrow \quad (14)$$

$$a\,\omega_1 \cos\theta\sin\phi + b\,\omega_2 \cos\beta\sin\phi - c\,\omega_3 \cos\phi\sin\phi = 0 \quad \rightarrow \quad (15)$$

- Adding equation (14) and (15)

$$a\,\omega_1 \sin(\phi-\theta) + b\,\omega_2 \sin(\phi-\beta) = 0$$

$$\boxed{\omega_2 = \frac{-a\,\omega_1 \sin(\phi-\theta)}{b\sin(\phi-\beta)}} \quad \rightarrow \quad (16)$$

- From equations (13) and (16), we can find ω_3 and ω_2, if a, b, c, θ, ϕ, β and ω_1 are known

Analysis of Four-bar Chain Mechanism

Acceleration Analysis:

- Let α_1 = Angular acceleration of link $AB = d\omega_1/dt$, α_2 = Angular acceleration of link $BC = d\omega_2/dt$, α_3 = Angular acceleration of link $CD = d\omega_3/dt$

- Differentiating Equation (10) with respect to time

$$-a\left[\omega_1 \cos\theta \times \frac{d\theta}{dt} + \sin\theta \times \frac{d\omega_1}{dt}\right] - b\left[\omega_2 \cos\beta \times \frac{d\beta}{dt} + \sin\beta \times \frac{d\omega_2}{dt}\right] + c\left[\omega_3 \cos\phi \times \frac{d\phi}{dt} + \sin\phi \times \frac{d\omega_3}{dt}\right] = 0$$

$$\Rightarrow -a\,\omega_1^2 \cos\theta - a\sin\theta\,\alpha_1 - b\,\omega_2^2 \cos\beta - b\sin\beta\,\alpha_2 + c\,\omega_3^2 \cos\phi + c\sin\phi\,\alpha_3 = 0 \quad \rightarrow (17)$$

- Differentiating Equation (11) with respect to time

$$a\left[\omega_1 \times -\sin\theta \times \frac{d\theta}{dt} + \cos\theta \times \frac{d\omega_1}{dt}\right] + b\left[\omega_2 \times -\sin\beta \times \frac{d\beta}{dt} + \cos\beta \times \frac{d\omega_2}{dt}\right] - c\left[\omega_3 \times -\sin\phi \times \frac{d\phi}{dt} + \cos\phi \times \frac{d\omega_3}{dt}\right] = 0$$

$$-a\omega_1^2 \sin\theta + a\cos\theta\,\alpha_1 - b\,\omega_2^2 \sin\beta + b\cos\beta\,\alpha_2 + c\,\omega_3^2 \sin\phi - c\cos\phi\,\alpha_3 = 0 \quad \rightarrow (18)$$

Analysis of Four-bar Chain Mechanism

Acceleration Analysis contd..

- Multiplying the equation (17) by *cosϕ* and equation (18) by *sinϕ*

$$-a\,\omega_1^2\cos\theta\cos\phi - a\,\alpha_1\sin\theta\cos\phi - b\,\omega_2^2\cos\beta\cos\phi - b\,\alpha_2\sin\beta\cos\phi + c\,\omega_3^2\cos^2\phi + c\,\alpha_3\sin\phi\cos\phi = 0 \quad (19)$$

$$-a\,\omega_1^2\sin\theta\sin\phi + a\,\alpha_1\cos\theta\sin\phi - b\,\omega_2^2\sin\beta\sin\phi + b\,\alpha_2\cos\beta\sin\phi + c\,\omega_3^2\sin^2\phi - c\,\alpha_3\cos\phi\sin\phi = 0 \quad (20)$$

- Adding equation (19) and equation (20)

$$-a\,\omega_1^2(\cos\phi\cos\theta + \sin\phi\sin\theta) + a\,\alpha_1(\sin\phi\cos\theta - \cos\phi\sin\theta) - b\,\omega_2^2(\cos\phi\cos\beta + \sin\phi\sin\beta) + b\,\alpha_2(\sin\phi\cos\beta - \cos\phi\sin\beta)$$
$$+c\,\omega_3^2(\cos^2\phi + \sin^2\phi) = 0$$

$$-a\,\omega_1^2\cos(\phi-\theta) + a\,\alpha_1\sin(\phi-\theta) - b\,\omega_2^2\cos(\phi-\beta) + b\,\alpha_2\sin(\phi-\beta) + c\,\omega_3^2 = 0$$

$$\boxed{\alpha_2 = \frac{-a\,\alpha_1\sin(\phi-\theta) + a\,\omega_1^2\cos(\phi-\theta) + b\,\omega_2^2\cos(\phi-\beta) - c\,\omega_3^2}{b\sin(\phi-\beta)}} \quad (21)$$

Analysis of Four-bar Chain Mechanism

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Acceleration Analysis contd..

- Multiplying the equation (17) by *cosβ* and equation (18) by *sinβ*

$$-a\,\omega_1^2\cos\theta\cos\beta - a\,\alpha_1\sin\theta\cos\beta - b\,\omega_2^2\cos^2\beta - b\,\alpha_2\sin\beta\cos\beta + c\,\omega_3^2\cos\phi\cos\beta + c\,\alpha_3\sin\phi\cos\beta = 0 \quad (22)$$

$$-a\,\omega_1^2\sin\theta\sin\beta + a\,\alpha_1\cos\theta\sin\beta - b\,\omega_2^2\sin^2\beta + b\,\alpha_2\cos\beta\sin\beta + c\,\omega_3^2\sin\phi\sin\beta - c\,\alpha_3\cos\phi\sin\beta = 0 \quad (23)$$

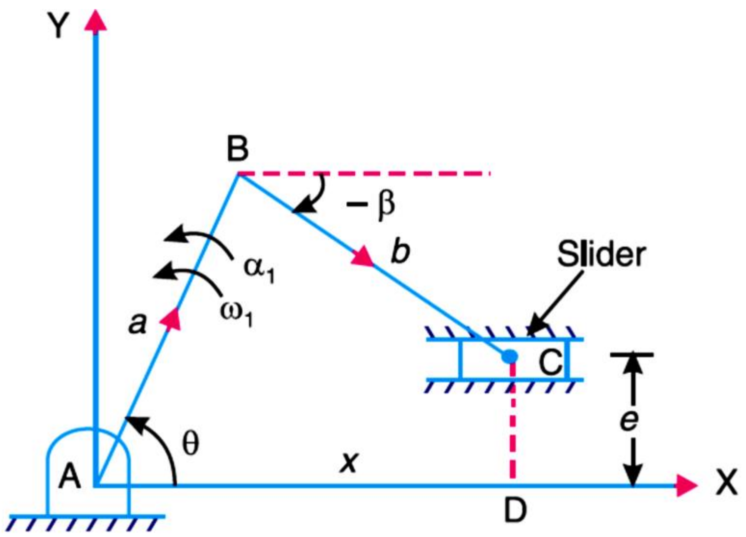
- Adding equation (22) and (23)

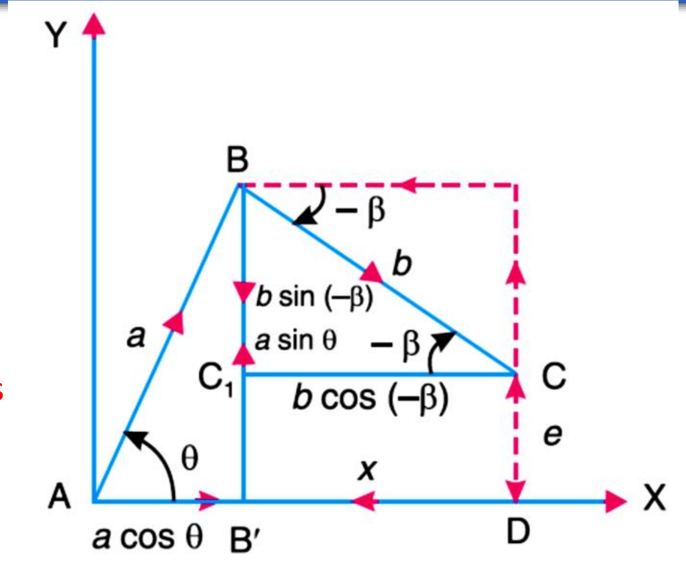
$$-a\,\omega_1^2(\cos\beta\cos\theta + \sin\beta\sin\theta) + a\,\alpha_1(\sin\beta\cos\theta - \cos\beta\sin\theta) - b\,\omega_2^2(\cos^2\beta + \sin^2\beta)$$
$$+ c\,\omega_3^2(\cos\phi\cos\beta + \sin\phi\sin\beta) + c\,\alpha_3(\sin\phi\cos\beta - \cos\phi\sin\beta) = 0$$

$$-a\,\omega_1^2\cos(\beta-\theta) + a\,\alpha_1\sin(\beta-\theta) - b\,\omega_2^2 + c\,\omega_3^2\cos(\phi-\beta) + c\,\alpha_3\sin(\phi-\beta) = 0$$

$$\boldsymbol{\alpha_3} = \frac{-a\,\alpha_1\sin(\beta-\theta) + a\,\omega_1^2\cos(\beta-\theta) + b\,\omega_2^2 - c\,\omega_3^2\cos(\phi-\beta)}{c\sin(\phi-\beta)} \quad (24)$$

Analysis of Slider-crank Mechanism





Consider links as vectors

Adopted from R S Khurmi, " A Text Book of Theory of Machines"

Displacement/ Position Analysis:

- For equilibrium of the mechanism, the sum of the components along *X*-axis and along *Y*-axis must be equal to zero
- Taking sum of components along *X*-axis

$$a\cos\theta + b\cos(-\beta) - x = 0 \quad \text{or} \quad b\cos\beta = x - a\cos\theta \qquad (1)$$

β in clockwise direction from *X*-axis is taken -ve

Analysis of Slider-crank Mechanism

- Squaring both sides

$$b^2\cos^2\beta = x^2 + a^2\cos^2\theta - 2xa\cos\theta \quad (2)$$

- Taking sum of components along *Y*-axis

$$b\sin(-\beta) + e + a\sin\theta = 0 \quad \text{or} \quad -b\sin\beta + e = a\sin\theta \Rightarrow b\sin\beta = e - a\sin\theta \quad (3)$$

- Squaring both sides

$$b^2\sin^2\beta = e^2 + a^2\sin^2\theta - 2ea\sin\theta \quad (4)$$

- Adding equation (2) and (4)

$$b^2(\cos^2\beta + \sin^2\beta) = x^2 + e^2 + a^2(\cos^2\theta + \sin^2\theta) - 2xa\cos\theta - 2ea\sin\theta$$

$$\Rightarrow b^2 = x^2 + e^2 + a^2 - 2xa\cos\theta - 2ea\sin\theta$$

$$\Rightarrow x^2 + (-2a\cos\theta)x + a^2 - b^2 + e^2 - 2ea\sin\theta = 0$$

$$\Rightarrow x^2 + k_1 x + k_2 = 0 \quad (5)$$

where $k_1 = -2a\cos\theta$, $k_2 = a^2 - b^2 + e^2 - 2ea\sin\theta$ (6)

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- The equation (5) is quadratic equation in *x*. Its two roots are

$$x = \frac{-k_1 \pm \sqrt{k_1^2 - 4k_2}}{2} \longrightarrow 7$$

- From this expression, the output displacement *x* may be determined if the values of *a*, *b*, *e* and θ are known. The position of the connecting rod *BC* (i.e. angle β) is given by both sides

$$\sin(-\beta) = \frac{a\sin\theta - e}{b} \Rightarrow \sin\beta = \frac{e - a\sin\theta}{b} \Rightarrow \beta = \sin^{-1}\left(\frac{e - a\sin\theta}{b}\right) \longrightarrow 8$$

- Special case: When the slider lies on the *X*-axis, i.e. the line of stroke of the slider passes through the axis of rotation of the crank, then eccentricity, *e* = 0. In such a case, equations (6) and (8) may be written as

$$k_1 = -2a\cos\theta \quad \text{and} \quad k_2 = a^2 - b^2$$

and

$$\boxed{\beta = \sin^{-1}\left(\frac{-a\sin\theta}{b}\right)}$$

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Velocity Analysis:

- Let ω_1 = Angular velocity of link AB = $d\theta/dt$, ω_2 = Angular velocity of link BC = $d\beta/dt$, V_S = Linear velocity of the slider = dx/dt
- Differentiating Equation (1) with respect to time

$$b \times -\sin\beta \times \frac{d\beta}{dt} = \frac{dx}{dt} - a \times -\sin\theta \times \frac{d\theta}{dt} \Rightarrow -a\,\omega_1 \sin\theta - b\,\omega_2 \sin\beta - \frac{dx}{dt} = 0 \longrightarrow 9$$

- Differentiating Equation (3) with respect to time

$$b\cos\beta \times \frac{d\beta}{dt} = -a\cos\theta \times \frac{d\theta}{dt} \Rightarrow a\,\omega_1 \cos\theta + b\,\omega_2 \cos\beta = 0 \longrightarrow 10$$

- Multiplying the equation (9) by cosβ and equation (10) by sinβ

$$-a\,\omega_1 \sin\theta\cos\beta - b\,\omega_2 \sin\beta\cos\beta - \frac{dx}{dt} \times \cos\beta = 0 \longrightarrow 11$$

$$a\,\omega_1 \cos\theta\sin\beta + b\,\omega_2 \cos\beta\sin\beta = 0 \longrightarrow 12$$

From equation (13), the linear velocity of the slider (V_S) may be determined.

- Adding equation (11) and (12)

$$a\,\omega_1 \sin(\beta-\theta) = \frac{dx}{dt} \times \cos\beta \Rightarrow \boxed{\frac{dx}{dt} = \frac{a\,\omega_1 \sin(\beta-\theta)}{\cos\beta}} \longrightarrow 13$$

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Velocity Analysis contd..:

- The angular velocity of the connecting rod *BC* (i.e. ω_2) may be determined from equation (10) and it is given by

$$\boxed{\omega_2 = \frac{-a\,\omega_1 \cos\theta}{b\cos\beta}}$$

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Acceleration Analysis:

- Let α_1 = Angular acceleration of crank AB = $d\omega_1/dt$, α_2 = Angular acceleration of connecting rod = $d\omega_2/dt$, α_S = Linear acceleration of slider = d^2x/dt^2

- Differentiating Equation (9) with respect to time

$$-a\left[\omega_1\cos\theta\times\frac{d\theta}{dt}+\sin\theta\times\frac{d\omega_1}{dt}\right]-b\left[\omega_2\cos\beta\times\frac{d\beta}{dt}+\sin\beta\times\frac{d\omega_2}{dt}\right]-\frac{d^2x}{dt^2}=0$$

$$\Rightarrow\quad -a\left[\alpha_1\sin\theta+\omega_1^2\cos\theta\right]-b\left[\alpha_2\sin\beta+\omega_2^2\cos\beta\right]-\frac{d^2x}{dt^2}=0 \longrightarrow 14$$

- Differentiating Equation (10) with respect to time

$$a\left[\omega_1\times-\sin\theta\times\frac{d\theta}{dt}+\cos\theta\times\frac{d\omega_1}{dt}\right]+b\left[\omega_2\times-\sin\beta\times\frac{d\beta}{dt}+\cos\beta\times\frac{d\omega_2}{dt}\right]=0$$

$$\Rightarrow\quad a\left[\alpha_1\cos\theta-\omega_1^2\sin\theta\right]+b\left[\alpha_2\cos\beta-\omega_2^2\sin\beta\right]=0 \longrightarrow 15$$

Analysis of Slider-crank Mechanism

Acceleration Analysis contd..

- Multiplying the equation (14) by *cosβ* and equation (15) by *sinβ*

$$-a\left[\alpha_1 \sin\theta\cos\beta+\omega_1^2\cos\theta\cos\beta\right]-b\left[\alpha_2\sin\beta\cos\beta+\omega_2^2\cos^2\beta\right]-\frac{d^2x}{dt^2}\times\cos\beta=0 \quad 16$$

$$a\left[\alpha_1\cos\theta\sin\beta-\omega_1^2\sin\theta\sin\beta\right]+b\left[\alpha_2\cos\beta\sin\beta-\omega_2^2\sin^2\beta\right]=0 \quad 17$$

- Adding equation (16) and equation (17)

$$a\left[\alpha_1(\sin\beta\cos\theta-\cos\beta\sin\theta)-\omega_1^2(\cos\beta\cos\theta+\sin\beta\sin\theta)\right]-b\omega_2^2(\cos^2\beta+\sin^2\beta)-\frac{d^2x}{dt^2}\times\cos\beta=0 \quad 18$$

$$a\alpha_1\sin(\beta-\theta)-a\omega_1^2\cos(\beta-\theta)-b\omega_2^2-\frac{d^2x}{dt^2}\times\cos\beta=0 \quad 19$$

$$\boxed{\frac{d^2x}{dt^2}=\frac{a\alpha_1\sin(\beta-\theta)-a\omega_1^2\cos(\beta-\theta)-b\omega_2^2}{\cos\beta}} \quad 20$$

- From this equation, the linear acceleration of the slider (α_S) may be determined

Analysis of Slider-crank Mechanism

Acceleration Analysis contd..

- The angular acceleration of the connecting rod *BC* (i.e. α_2) may be determined from equation (10) and it is given by

$$\boxed{\alpha_2 = \frac{a\,(\alpha_1 \cos\theta - \omega_1^2 \sin\theta) - b\,\omega_2^2 \sin\beta}{b\cos\beta}}$$

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Thank You

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