Course: UMA 035 (Optimization Techniques)

Instructor: Dr. Amit Kumar,

Associate Professor,

School of Mathematics,

TIET, Patiala

Email: amitkumar@thapar.edu

Mob: 9888500451

Integer Linear Programming Problem

Remark:

$$\frac{7}{4} = 1 + \frac{3}{4}$$

$$-\frac{7}{4} =?$$

$$-\frac{7}{4} = -1 - \frac{3}{4} = -1 - \frac{3}{4} - 1 + 1 = -2 + \frac{1}{4}$$

$$-\frac{10}{3} =?$$

$$-\frac{10}{3} = -3 - \frac{1}{3} = -3 - \frac{1}{3} + 1 - 1 = -4 + \frac{2}{3}$$

Gomory's cutting plane method

Example:

Max $(21x_1+11x_2)$

Subject to

$7x_1+4x_2<=13$

$x_1, x_2>=0$ and integers.

Max $(21x_1+11x_2)$

Subject to

$7x_1+4x_2+S_1=13$

$x_1, x_2>=0$ and integers, $S_1>=0$ and integer.

		21	11	0		
C_B	Basic Variables	x_1	x_2	S_1	Solution	Minimum Ratio
$Z_j - C_j$ =		-21	-11	0		
0	S_1	7	4	1	13	13/7
$Z_j - C_j$ =		0	1	3		
21	x_1	1	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	

Optimal solution is

$x_1=\frac{13}{7}$ and remaining are 0 i.e., $S_1=x_2=0$.

x_1 is not an integer.

So add the following constraint corresponding to non-basic variables of the row x_1.

−fractional part of first non-basic variable in x_1 row * first non-basic variable in x_1 row−fractional part of second non-basic variable in x_1 row * second non-basic variable in x_1 row …..+one new slack variable= − fractional part of value of x_1

$-\frac{4}{7}x_2-\frac{1}{7}S_1+S_2=-\frac{6}{7}$

		21	11	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	Solution	Minimum Ratio
Z_j-C_j =		0	1	3	0		
21	x_1	1	$\frac{4}{7}$	$\frac{1}{7}$	0	$\frac{13}{7}$	
0	S_2	0	$-\frac{4}{7}$	$-\frac{1}{7}$	1	$-\frac{6}{7}$	

RHS corresponding to S_2 is negative so need to use Dual Simplex method.

		21	11	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	Solution	Maximum Ratio
$Z_j - C_j$ =		0	1	3	0		
21	x_1	1	$\frac{4}{7}$	$\frac{1}{7}$	0	$\frac{13}{7}$	Maximum $\{\frac{\frac{4}{7}}{-\frac{4}{7}}, \frac{3}{-\frac{1}{7}}\}=\frac{\frac{4}{7}}{-\frac{4}{7}}$
0	S_2	0	$-\frac{4}{7}$	$-\frac{1}{7}$	1	$-\frac{6}{7}$	
$Z_j - C_j$ =		0	0	$\frac{11}{4}$	$\frac{7}{4}$		
21	x_1	1	0	0	1	1	
11	x_2	0	1	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{3}{2}$	

Solution is optimal but $x_2=\frac{3}{2}$ is not an integer.

Need to add the constraint

$-\frac{1}{4}S_1 -$ **fraction part of** $(-\frac{7}{4})S_2+S_3=-\frac{1}{2}$

$-\frac{1}{4}S_1 -$ **fraction part of** $(-1-\frac{3}{4})S_2+S_3=-\frac{1}{2}$

$-\frac{1}{4}S_1 -$ **fraction part of** $(-1-1+1-\frac{3}{4})S_2+S_3=-\frac{1}{2}$

$-\frac{1}{4}S_1 -$ **fraction part of** $(-2+\frac{1}{4})S_2+S_3=-\frac{1}{2}$

$-\frac{1}{4}S_1-\frac{1}{4}S_2+S_3=-\frac{1}{2}$

		21	11	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		0	0	$\frac{11}{4}$	$\frac{7}{4}$	0		
21	x_1	1	0	0	1	0	1	
11	x_2	0	1	$\frac{1}{4}$	$-\frac{7}{4}$	0	$\frac{3}{2}$	
0	S_3	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	1	$-\frac{1}{2}$	

		21	11	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		0	0	$\frac{11}{4}$	$\frac{7}{4}$	0		Maximum $\{\frac{\frac{11}{4}}{-\frac{1}{4}}, \frac{\frac{7}{4}}{-\frac{1}{4}}\} = \frac{\frac{7}{4}}{-\frac{1}{4}}$
21	x_1	1	0	0	1	0	1	
11	x_2	0	1	$\frac{1}{4}$	$-\frac{7}{4}$	0	$\frac{3}{2}$	
0	S_3	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	1	$-\frac{1}{2}$	
$Z_j - C_j$ =		0	0	1	0	7		
21	x_1	1	0	-1	0	4	-1	
11	x_2	0	1	2	0	-7	5	
0	S_2	0	0	1	1	-4	2	
$Z_j - C_j$ =		1	0	0	0	11		
0	S_1	-1	0	1	0	-4	1	
11	x_2	2	1	0	0	1	3	
0	S_2	1	0	0	1	0	1	

Optimal solution is

x_2=3

S_1=1

S_2=1

Optimal value is $21x_1+11x_2=21*0+11*3=33$

Example:

Max (x_1+3x_2)

Subject to

$2x_1-3x_2<=4$

$-x_1+2x_2<=7$

$3x_1+x_2<=9$

x_1, x_2, x_2>=0 and integers.

Max (x_1+3x_2)

Subject to

$2x_1-3x_2+S_1=4$

$-x_1+2x_2+S_2=7$

$3x_1+x_2+S_3=9$

x_1, x_2>=0 and integers, S_1>=0 and integer, S_2>=0 and integer, S_3>=0 and integer.

		1	3	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		-1	-3	0	0	0		
0	S_1	2	-3	1	0	0	4	4/-
0	S_2	-1	2	0	1	0	7	7/2
0	S_3	3	1	0	0	1	9	9/1
$Z_j - C_j$ =		$-\frac{5}{2}$	0	0	$\frac{3}{2}$	0		
0	S_1	$\frac{1}{2}$	0	1	$\frac{3}{2}$	0	$\frac{29}{2}$	$\frac{29}{2}/\frac{1}{2}$
3	x_2	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{7}{2}$	$\frac{7}{2}/-$
0	S_3	$\frac{7}{2}$	0	0	$-\frac{1}{2}$	1	$\frac{11}{2}$	$\frac{11}{2}/\frac{7}{2}$
$Z_j - C_j$ =		0	0	0	$\frac{8}{7}$	$\frac{5}{7}$	1	
0	S_1	0	0	1	$\frac{11}{7}$	$-\frac{1}{7}$	$\frac{96}{7}$	
3	x_2	0	1	0	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{30}{7}$	
1	x_1	1	0	0	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{11}{7}$	

Optimal solution is

$S_1=\frac{96}{7} = 13 + \frac{5}{7}$

$x_2=\frac{30}{7} = 4 + \frac{2}{7}$

$x_1=\frac{11}{7} = 1 + \frac{1}{7}$

$\max\{\frac{5}{7}, \frac{2}{7}, \frac{1}{7}\}=\frac{5}{7}$

Add the following constraint corresponding to S_1

$-$(**fractional part of** $\frac{11}{7}$)$*S_2 -$ (**fractional part of** $-\frac{1}{7}$)$*S_3+S_4=-$**fractional part of** $\frac{96}{7}$

$-(\frac{4}{7})*S_2 - (\frac{6}{7})*S_3+S_4=-\frac{5}{7}$

		1	3	0	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	S_4	Solution	Minimum Ratio
$Z_j - C_j$ =		0	0	0	$\frac{8}{7}$	$\frac{5}{7}$	0	1	
0	S_1	0	0	1	$\frac{11}{7}$	$-\frac{1}{7}$	0	$\frac{96}{7}$	
3	x_2	0	1	0	$\frac{3}{7}$	$\frac{1}{7}$	0	$\frac{30}{7}$	
1	x_1	1	0	0	$-\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{11}{7}$	
0	S_4	0	0	0	$-\frac{4}{7}$	$-\frac{6}{7}$	1	$-\frac{5}{7}$	

		1	3	0	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	S_4	Solution	Maximum Ratio
$Z_j - C_j$ =		0	0	0	$\frac{8}{7}$	$\frac{5}{7}$	0		
0	S_1	0	0	1	$\frac{11}{7}$	$-\frac{1}{7}$	0	$\frac{96}{7}$	Maximum $\{\frac{\frac{8}{7}}{-\frac{4}{7}}, \frac{\frac{5}{7}}{-\frac{6}{7}}\}=$ $\frac{\frac{5}{7}}{-\frac{6}{7}}$
3	x_2	0	1	0	$\frac{3}{7}$	$\frac{1}{7}$	0	$\frac{30}{7}$	
1	x_1	1	0	0	$-\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{11}{7}$	
0	S_4	0	0	0	$-\frac{4}{7}$	$-\frac{6}{7}$	1	$-\frac{5}{7}$	
$Z_j - C_j$ =		0	0	0	$\frac{2}{3}$	0	$\frac{5}{6}$		
0	S_1	0	0	1	$\frac{5}{3}$	0	$-\frac{1}{6}$	$\frac{83}{6}$	
3	x_2	0	1	0	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{25}{6}$	
1	x_1	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	
0	S_3	0	0	0	$\frac{2}{3}$	1	$-\frac{7}{6}$	$\frac{5}{6}$	

Optimal solution is

$S_1 = \frac{83}{7} = \mathbf{11} + \frac{6}{7}$

$x_2 = \frac{25}{6} = \mathbf{4} + \frac{1}{6}$

$x_1 = \frac{4}{3} = \mathbf{1} + \frac{1}{3}$

$S_3 = \frac{5}{6} = \mathbf{0} + \frac{5}{6}$

$\max\{\frac{6}{7}, \frac{1}{6}, \frac{1}{3}, \frac{5}{6}\} = \frac{6}{7}$

Add the following constraint corresponding to S_1

$-$(fractional part of $\frac{5}{3}$)$*S_2$ $-$ (fractional part of $-\frac{1}{6}$)$*S_4+S_5=-$fractional part of $\frac{83}{6}$

$-(\frac{2}{3})*S_2 - (\frac{5}{6})*S_4+S_5=-\frac{5}{6}$

		1	3	0	0	0	0	0		
C_B	**Basic Variables**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**	**S_4**	**S_5**	**Solution**	**Maximum Ratio**
$Z_j - C_j$ =		0	0	0	$\frac{2}{3}$	0	$\frac{5}{6}$	0		
0	S_1	0	0	1	$\frac{5}{3}$	0	$-\frac{1}{6}$	0	$\frac{83}{6}$	Maximum $\{\frac{\frac{5}{3}}{-\frac{2}{3}}, \frac{\frac{5}{6}}{-\frac{5}{6}}\}=\frac{\frac{5}{6}}{-\frac{5}{6}}$
3	x_2	0	1	0	$\frac{1}{3}$	0	$\frac{1}{6}$	0	$\frac{25}{6}$	
1	x_1	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{4}{3}$	
0	S_3	0	0	0	$\frac{2}{3}$	1	$-\frac{7}{6}$	0	$\frac{5}{6}$	
0	S_5	0	0	0	$-\frac{2}{3}$	0	$-\frac{5}{6}$	1	$-\frac{5}{6}$	
$Z_j - C_j$ =		0	0	0	0	0	0	0		
0	S_1	0	0	1	$\frac{9}{5}$	0	0	$-\frac{1}{5}$	14	
3	x_2	0	1	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$	4	
1	x_1	1	0	0	$-\frac{3}{5}$	0	0	$\frac{2}{5}$	1	
0	S_3	0	0	0	$\frac{8}{5}$	1	0	$-\frac{7}{5}$	2	
0	S_4	0	0	0	$\frac{4}{5}$	0	1	$-\frac{6}{5}$	1	

Optimal solution is

$S_1=14$

$x_2=4$

$x_1=1$

$S_3=2$

$S_4=1$

Remaining are 0

$S_2=S_5=0$

Optimal value is $x_1+3x_2=1+3*4=13$

Transportation Problem

Tabular representation of a transportation problem

	Destinations					Availability
		D_1	D_2	$\cdots$	D_n	
Sources	S_1	C_{11}	C_{12}		C_{1n}	a_1
	S_2	C_{21}	C_{22}		C_{2n}	a_2
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	S_m	C_{m1}	C_{m2}		C_{mn}	a_m
	Demand	b_1	b_2		b_n	

Sources:

S_1 , S_2, …, S_m

Availability of the product at the i^{th} source

a_i

Destinations:

D_1, D_2,…, D_n

Demand of the product at the j^{th} destination

b_j

Cost for supplying unit quantity of the product from the i^{th} source to the j^{th} destination

C_{ij}

Objective

To find the quantity of the product to be supplied from the i^{th} source to the j^{th} destination in such a manner that total transportation cost is minimum.

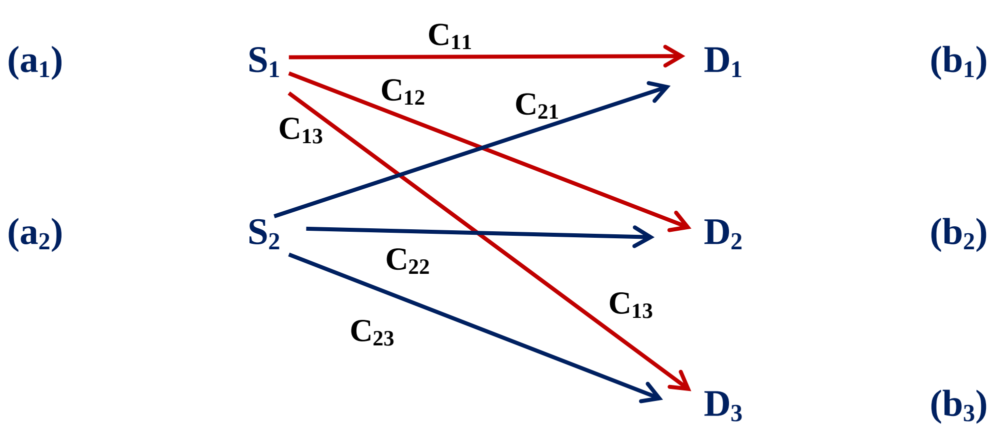
LPP formulation of a balanced transportation problem

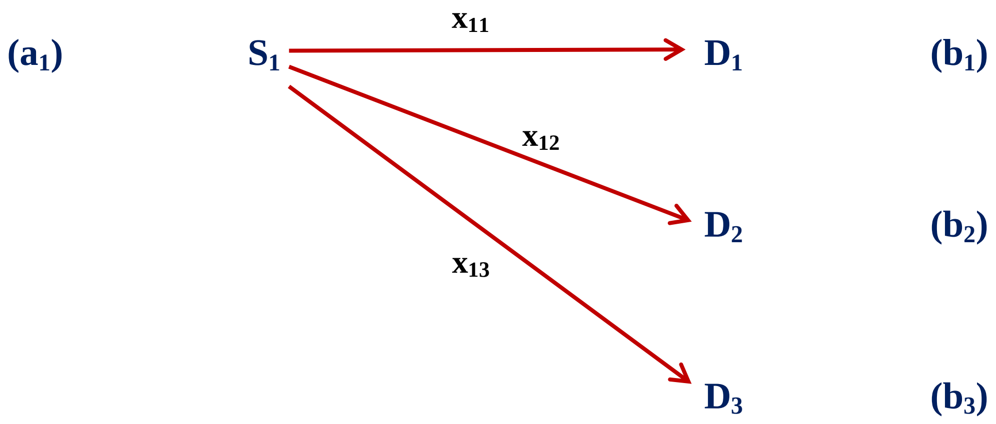
Let x_{ij} quantity of the product should be supplied from the i^{th} source to the j^{th} destination.

Balanced: $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Unbalanced: $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$

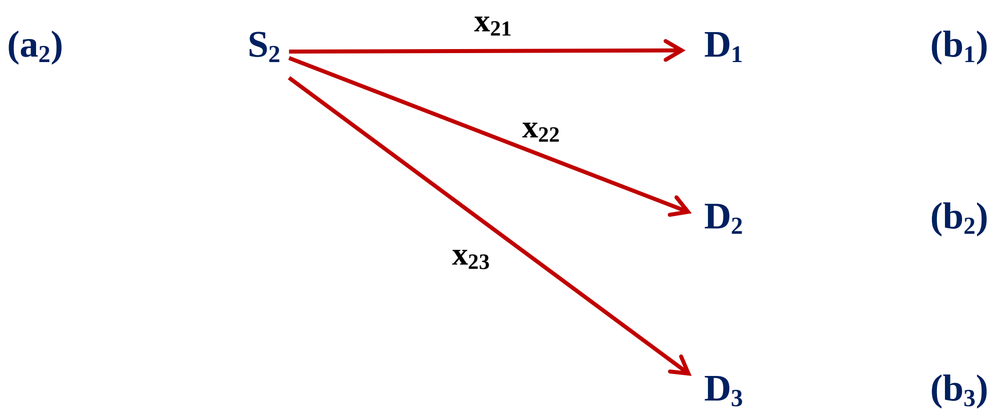
(Availability) **Sources** **Destinations** **Demand**





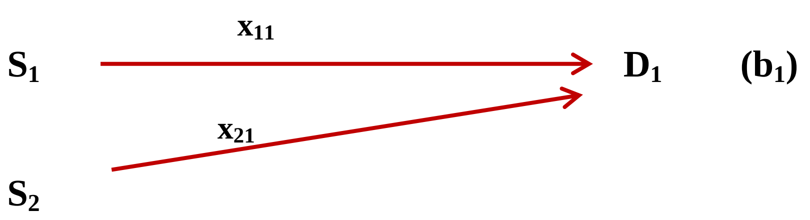
The quantity a_1 has been split into three parts x_{11} , x_{12} and x_{13}. Therefore,

$x_{11}+x_{12}+x_{13}=a_1$



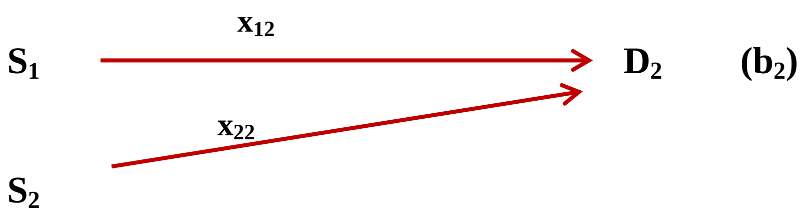
The quantity a_2 has been split into three parts x_{21} , x_{22} and x_{23}. Therefore,

$$x_{21} + x_{22} + x_{23} = a_2$$



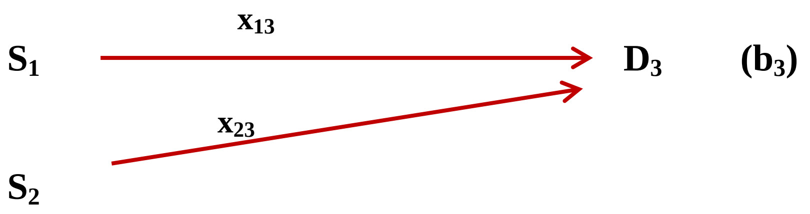
The total quantity of the product received at destination D_1 is $x_{11}+x_{21}$. Therefore,

$$x_{11}+x_{21}=b_1$$



The total quantity of the product received at destination D_1 is $x_{12}+x_{22}$. Therefore,

$$x_{12}+x_{22}=b_2$$



The total quantity of the product received at destination D_1 is $x_{12}+x_{22}$. Therefore,

$x_{12}+x_{23}=b_3$

Cost for supplying one unit quantity of the product from S_1 to D_1 = C_{11}

Cost for supplying x_{11} unit quantity of the product from S_1 to D_1 = $C_{11}x_{11}$

Cost for supplying one unit quantity of the product from S_1 to D_2 = C_{12}

Cost for supplying x_{12} unit quantity of the product from S_1 to D_2 = $C_{12}x_{12}$

Cost for supplying one unit quantity of the product from S_1 to D_3 = C_{13}

Cost for supplying x_{13} unit quantity of the product from S_1 to D_3 = $C_{13}x_{13}$

Cost for supplying one unit quantity of the product from S_2 to D_1 = C_{21}

Cost for supplying x_{21} unit quantity of the product from S_2 to D_1 = $C_{21}x_{21}$

Cost for supplying one unit quantity of the product from S_2 to D_2 = C_{22}

Cost for supplying x_{22} unit quantity of the product from S_2 to D_2 = $C_{22}x_{22}$

Cost for supplying one unit quantity of the product from S_2 to D_3 = C_{23}

Cost for supplying x_{23} unit quantity of the product from S_2 to D_3 = $C_{23}x_{23}$

Total transportation cost$=C_{11}x_{11}+C_{12}x_{12}+C_{13}x_{13}+C_{21}x_{21}+C_{22}x_{22}+C_{23}x_{23}$

LPP for a balanced transportation problem

Minimize $(C_{11}x_{11}+C_{12}x_{12}+C_{13}x_{13}+C_{21}x_{21}+C_{22}x_{22}+C_{23}x_{23})$

Subject to

$x_{11}+x_{12}+x_{13}=a_1$

$x_{21}+x_{22}+x_{23}=a_2$

$x_{11}+x_{21}=b_1$

$x_{12}+x_{22}=b_2$

$x_{13}+x_{23}=b_3$

$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}>=0$

OR

Minimize $\left(\sum_{i=1}^{2}\sum_{j=1}^{3} C_{ij}x_{ij}\right)$

Subject to

$\sum_{j=1}^{3} x_{ij} = a_i$, $i=1, 2$

$\sum_{i=1}^{2} x_{ij} = b_j$, $j=1, 2, 3$

$x_{ij}>=0$, $i=1,2$; $j=1,2,3$.

Replacing 2 sources with m sources and 3 destinations with n destinations, the above LPP may be written as

Minimize $\left(\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij}\right)$

Subject to

$\sum_{j=1}^{n} x_{ij} = a_i$, i =1, 2,…,m

$\sum_{i=1}^{m} x_{ij} = b_j$, j =1, 2, 3,…,n

$x_{ij}>=0$, i=1,2,…,m; j=1,2,3,…,n.

Number of Basic variables in a balanced transportation problem

Minimize $(C_{11}x_{11}+ C_{12}x_{12}+ C_{13}x_{13}+ C_{21}x_{21}+ C_{22}x_{22}+ C_{23}x_{23})$

Subject to

$x_{11}+ x_{12}+ x_{13} = a_1$ (1)

$x_{21}+ x_{22}+ x_{23}=a_2$ (2)

$x_{11}+ x_{21}=b_1$ (3)

$x_{12}+ x_{22}=b_2$ (4)

$x_{13}+ x_{23}=b_3$ (5)

$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}>=0$

[(1)+ (2)]−[(3)+(4)]

$(x_{11}+ x_{12}+ x_{13}+ x_{21}+ x_{22}+ x_{23})-(x_{11}+ x_{21}+ x_{12}+ x_{22}) = (a_1+a_2) -(b_1+ b_2)$

$x_{13}+x_{23}=b_3$ (since, $a_1+a_2= b_1+ b_2+b_3$)

This indicates that only four constraints are independent.

Some results for a transportation problem having 2 sources and 3 destinations

Number of variables = 2*3 = 6

Number of Constraints = 2+3 =5

Number of linearly independent constraints = 2+3−1=4

Number of basic variables = 2+3−1=4

Number of non-basic variables = total variables − basic variables = 6 – 4=2

Maximum number of basic feasible solutions =

$$C^{\text{no.of variables}}_{\text{no of linearly independent constraints}}$$

Some results for a transportation problem having m sources and n destinations

Number of variables = m*n

Number of Constraints = m+n

Number of linearly independent constraints = m+n−1

Number of basic variables = m+n−1

Number of non-basic variables = total variables − basic variables = mn – (m+n–1) = $(m-1)(n-1)$

Maximum number of basic feasible solutions =

$$C^{mn}_{m+n-1}$$

Dual for a transportation problem

Maximize $\left(\sum_{i=1}^{m} a_i\, u_i + \sum_{j=1}^{n} b_j\, v_j\right)$

Subject to

$u_i + v_j <= C_{ij}$

u_i **and** v_j **are unrestricted variables.**

u_i **are dual variables corresponding to the availability constraints**

v_j **are dual variables corresponding to the demand constraints**