DIMENSIONAL ANALYSIS

- Mathematical technique which makes use of the study of dimensions for the solution of several engineering problems.

❖ Main applications of dimensional analysis (**DA**) are:

(i) To establish a relationship among the variables in a given phenomenon.

(ii) To conduct tests on models for getting the desired information i.e. models study.

- Each physical phenomenon can be expressed by an equation consisting of variables.

➢ These variables may be dimensional or non-dimensional.

- By using the methods of **DA**, a systematic relationship among the variables can be established.

- To know the performance of actual structures in advance, models are made and the tests are conducted on the models.

❖ If performance of model is satisfactory, only then actual structure is made.

PRIMARY QUANTITIES AND SECONDARY QUANTITIES

- To describe a given phenomenon, various physical quantities influencing the phenomenon can be described by a set of quantities, which are independent of each other.

❖ These independent quantities are known as <u>primary quantities</u>, also called fundamental quantities or fundamental dimensions (**FD**).

❖ Four fundamental dimensions are mass, length, time and temperature denoted by **M, L, T** and **θ** respectively

❖ Temperature is used as **FD** only in compressible fluids.

➢ All other quantities are known as secondary quantities, also called derived quantities, can be expressed in terms of primary quantities.

Examples: Area, volume, velocity - secondary quantities.

Definition of dimensions:

- Expression of a physical quantity in terms of fundamental dimensions.

❖ Dimensions of any physical quantity can be derived from the relationship of quantity with other variables or units of quantity.

Classification of physical quantities and their dimensions

- Various physical quantities involved in the problems of fluid flow may be classified as geometric, kinematic and dynamic quantities.

Geometric Quantities

- Most commonly used geometric quantities and their dimensions are Length **[L],** diameter **[L],** area **[L^2],** volume **[L^3],** curvature **(**1/R**)** **[L^{-1}],** slope **[$M^0L^0T^0$],** angle **[$M^0L^0T^0$]** etc.

Kinematic Quantities

- Time [**T**], velocity [**LT^{-1}**], acceleration [**LT^{-2}**], angular velocity [**T^{-1}**], angular acceleration [**T^{-2}**], frequency [**T^{-1}**], discharge [**L^3T^{-1}**], kinematic viscosity **[L^2 T^{-1}**]etc.

Dynamic Quantities

- Mass [**M**], force [**MLT^{-2}**], weight [**MLT^{-2}**], mass density (mass/volume) [**ML^{-3}**], specific weight (weight/volume) [**$ML^{-2}T^{-2}$**], specific gravity[**$M^0L^0T^0$**]**,** , intensity of pressure and shear stress [**$ML^{-1}T^{-2}$**], dynamic viscosity {$\tau = \mu(du/dy)$} [**$ML^{-1}T^{-1}$**], surface tension (F/L) [**MT^{-2}**], modulus of elasticity,momentum (mass $\times$ velocity) , work, energy and torque, Power (work/time), mass rate of flow (**ρQ**), weight rate of flow (**γQ**), Moment of a force (force $\times$ distance), moment of momentum (momentum $\times$ distance) etc.

DIMENSIONAL HOMOGENEITY

- An equation is said to be dimensionally homogeneous if dimensions of terms on left-hand side = dimensions of terms on the right hand.

❖ In other words, powers of fundamental dimensions are equal for a dimensionally homogeneous equation.

Example: Consider the eq. for time period of oscillation of a simple pendulum *i.e.* $t = 2\pi\sqrt{\frac{L}{g}}$

- Dimension of the term on the left-hand side = [**T**]
- Dimensions of the terms on the right-hand side = $\left[\frac{L}{LT^{-2}}\right]^{1/2}$

  ❖ **2π** is a dimensionless constant.

$\therefore$ Equation is dimensionally homogeneous.

➢ Some other dimensionally homogeneous equations are:

- Flow over a rectangular notch, $Q = \frac{2}{3} C_d L \sqrt{2g}\, H^{3/2}$
- Flow over a triangular notch, $Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\theta}{2} H^{5/2}$
- Head loss due to friction in pipes, $h_f = \frac{f_1 L V^2}{D(2g)}$

❖ C_d and f_1 are dimensionless.

Characteristics of dimensionally homogeneous Equations

(i) A dimensionally homogeneous eq. is independent of the units chosen for its measurement *i.e.* can be used in any one of the system of units (**CGS, MKS or SI**).

(ii) It is always possible to reduce a dimensionally homogeneous eq. to a non-dimensional form.

➢ Following dimensionally homogeneous eqs. can be converted into non-dimensional form:

$$\left[\frac{t}{\sqrt{L/g}}\right] = 2\pi, \qquad \left[\frac{Q}{\sqrt{g}H^{5/2}}\right] = \frac{8}{15} C_d \sqrt{2} \tan\frac{\theta}{2} = \text{constant}$$

Dimensionally non-homogeneous equations

❖ Ideally an equation should be dimensionally homogenous

❖ There are several equations which are non-homogeneous but are still used within their limited ranges.

Examples: Followings two equations are commonly used for finding mean velocity of flow, **V** in open channels.

$V = C\sqrt{R_h S}$ (Chezy's equation)

$V = \frac{1}{N} R_h^{2/3} S^{1/2}$ (Manning's equation)

➢ **C** = Chezy's constant, $\mathbf{R_h}$ = Hydraulic radius or Hydraulic mean depth (Area of flow/wetted perimeter) and **S** = Bed slope, **N** = Manning's constant.

❖ Both equations are dimensionally non-homogeneous.

- Consider Chezy's equation, $V = C\sqrt{R_h S}$
- Dimension of term on the left-hand side = $[LT^{-1}]$
- Dimensions of the terms on the right-hand side = $[L^{1/2}]$

❖ **C** is a constant and is dimensionless.

$\therefore$ Chezy's eq. is dimensionally non-homogeneous.

Observations:

(i) Dimensionally non-homogeneous equations can be used only in that system of unit in which the value of constant involved in the equation is given.

(ii) Dimensionally non-homogeneous eqs. can be converted into another system of unit by modifying the value of constant.

Example:

- Eq. $V = 10.8 R_h^{2/3} S^{1/2}$ is applicable in **FPS** system of units. The various terms have their usual meaning.Modify this equation for **SI** system of units.

Solution:

- The given eq. is dimensionally non-homogenous.
- Can be written as:

$$10.8 = \left(\frac{V}{R_h^{2/3} S^{1/2}} \right)$$

- Writing dimensions of terms on the **RHS**, to get

$$\left(\frac{V}{R_h^{2/3} S^{1/2}} \right) = \left[\frac{LT^{-1}}{L^{2/3}} \right] = \left[L^{1/3} T^{-1} \right]$$

- Constant **10.8** has units of ($ft^{1/3}/s$)

- Since 1 m = 3.28 ft, the value of constant **10.8** in **SI** units will be $10.8(1/3.28)^{1/3}$ = **7.27**

$\therefore$ Given equation in **SI** system of units is: $V = 7.27 R_h^{2/3} S^{1/2}$

METHODS OF DIMENSIONAL ANALYSIS

- If numbers of variables involved in a phenomenon are known, then the relationship among the variables can be determined by the following two methods:

- Rayleigh's method and Buckingham's method.

Rayleigh's method

- In this method, functional relationship among the variables is expressed in the form of an exponential equation:

- If **X** is some function of variables say $(X_1, X_2, X_3 ------, X_n)$ i.e.

$$X = f(X_1, X_2, X_3, ------, X_n) \qquad (1)$$

- **X** is called a dependent variable and $(X_1, X_2, X_3, ------, X_n)$ are called independent variables.

- According to Rayleigh's method, Eq. (1) can be expressed as:

$$X = K(X_1^a \times X_2^b \times X_3^c \times ------ \times X_n^n)$$

- exponents (**a, b, c, - - -,n**) are constants and are evaluated using the principle of dimensional homogeneity.

- **K** is a dimensionless constant - determined either from the physical characteristics of the problem or from experiments.

- To explain this method, consider the following problems.

Problem 1: Find an expression for drag force/resistance F_D on a smooth sphere of diameter **D** moving with constant velocity U_o in a stationary fluid of viscosity μ.

Solution: $F_D = f(D, U_o, \mu)$ **(1)**

- According to Rayleigh's method, Eq. (1) may be expressed as:

$$F_D = K(D^a \times U_o{}^b \times \mu^c)$$

- Exponents (**a, b, c, - - -,n**) are constants and are to be evaluated using the principle of dimensional homogeneity

- Writing dimensions of each variable, to write

$$F_D = [MLT^{-2}], \quad D = [L], \quad U_o = [LT^{-1}], \quad \mu = [ML^{-1}T^{-1}]$$

$$\therefore [MLT^{-2}] = [L]^a\ [LT^{-1}]^b\ [ML^{-1}T^{-1}]^c$$

- **K** is a dimensionless constant.

- Equating powers of **M**, **L** and **T**, to get

- Powers of **M**, **1 = c**

- Powers of **L**, **1** = (**a + b - c)**

- Powers of **T**, - **2** = - **b – c**

- On solving**, a = 1, b = 1 and c =1**

$$\therefore F_D = K(D^1 U_o{}^1 \mu^1)$$

- In this problem, **K = 3π**

$$\therefore F_D = 3\pi\mu U_o D$$

- **Stoke's Law**

Problem 2: Find an expression for drag force F_D on a smooth sphere of diameter **D** moving with a constant velocity U_o in a fluid of density ρ and viscosity μ.

Solution: The additional variable is ρ

∴ $F_D = f(D, U_o, \rho, \mu)$, can be written as

$$F_D = K\,(D^a \times U_o{}^b \times \rho_c \times \mu^d)$$

$$\therefore \; [MLT^{-2}] = [L]^a\,[LT^{-1}]^b\,[ML^{-3}]^c\,[ML^{-1}T^{-1}]^d$$

- ❖ Equating powers of **M**, **L** and **T**, to get
- Powers of **M**, **1 = c + d**
- Powers of **L**, **1** = (**a + b - 3c - d)**
- Powers of **T**, **-2** = - **b - d**
- ❖ There are four unknowns **a**, **b**, **c** and **d** but equations are three. Hence, it is not possible to find the values of **a**, **b**, **c** and **d.**
- ❖ Any three variables can be expressed in terms of fourth variable, which is the most significant in the problem.
- ❖ In this problem, viscosity is the most significant variable affecting the given phenomenon.

∴Expressing**a,b** and **c** in terms of **d (**power to viscosity), to get

- ❖ **c = (1 – d), b = (2 – d), a = (2 – d)**

$$\therefore F_D = K\,(D^{2-d} \times U^{2-d} \times \rho^{1-d} \times \mu^d)$$

$$= K\left[D^2 U^2 \rho \left(\frac{\mu}{\rho U D}\right)^d\right]$$

- can be further expressed as:

$$F_D = \rho U_o{}^2 D^2 \phi\left(\frac{\mu}{\rho U_o D}\right)$$

- Required expression.

Buckingham's method

- Rayleigh's method becomes increasingly tedious when more than four independent variables are involved in the phenomenon.
- ❖ Rayleigh's method should be used when the numbers of independent variables are either three or four.
- When four or more than four independent variables are involved in the phenomenon, then it is more convenient to use Buckingham's method, also known as Buckingham's **π** -theorem method.
- According to this method, "if there are **n** numbers of variables (independent and dependent) and if these variables are described by **m** fundamental dimensions, then the variables can be expressed into (**n-m**) independent dimensionless terms, each term is called a **π-term**".
- Each **π-term** contains (**m+1**) variables, **m** (number of fundamental dimensions) are also known as repeating variables.
- ❖ Each **π-**term is solved using the principle of dimensional homogeneity.
- ❖ Important step in this method is the selection of repeating variables as discussed below:

Method of selecting repeating variables

- Following points should be kept in mind while selecting repeating variables:

(i) Generally dependent variable is not selected as a repeating variable.

(ii) No two repeating variables should have the same dimensions.

(iii) Repeating variables should be chosen in such a manner that;

(a) First variable should represent the <u>geometric property</u> like length, diameter, height etc.

(b) Second variable should represent a <u>flow property</u> like velocity, acceleration, discharge etc.

(c) Third variable should represent a characteristic <u>fluidproperty</u> like mass density or viscosity.

- ❖ Viscosity is important only in case of viscous or laminar flow problems otherwise density is chosen as the repeating variable.

<u>To explain this method, let us consider the previous problem of finding drag force on a sphere:</u>

$\therefore$ $\quad \mathbf{F_D = f(D, U_o, \rho, \mu)} \qquad$ **(1)**

Or $\quad \mathbf{f_1(F_D, D, U_o, \rho, \mu)} = 0 \qquad$ **(2)**

- Writing dimensions of each variable as

$$F_D = [MLT^{-2}], D = [L], U_o = [LT^{-1}], \rho = [ML^{-3}], \mu = [ML^{-1}T^{-1}]$$

$\therefore$ No. of fundamental dimensions involved, **m = 3**

- Total number of variables = 5

$\therefore$ Number of π -terms = (5 - 3) = 2

- $\therefore$ Equation (2) can be written as: $f_1(\pi_1, \pi_2) = 0$ **(3)**

❖ Each π-term contains (m+1) variables, **m** is the number of fundamental dimensions and also called repeating variables.

❖ Taking **D**, $\mathbf{U_o}$ and ρ as repeating variables.

- Each π -term can be written as:

$$\pi_1 = D^{a_1} \times U_o^{b_1} \times \rho^{c_1} \times F_D \qquad (4)$$

$$\pi_2 = D^{a_2} \times U_o^{b_2} \times \rho^{c_2} \times \mu \qquad (5)$$

- exponents $\mathbf{a_i}$, $\mathbf{b_i}$ and $\mathbf{c_i}$ are constants, determined using the principle of dimensional homogeneity *i.e.* each π- term is solved using the principle of dimensional homogeneity.

<u>First π - term</u>

- Writing dimensions of each variable on both sides, to get

$$[M^0L^0T^0] = [L]^{a_1} [LT^{-1}]^{b_1} [ML^{-3}]^{c_1} [MLT^{-2}]$$

- Equating powers of **M**, **L** and **T** on both sides, to get

➢ Powers of **M**, $0 = C_1 + 1$

➢ Powers of **L**, $0 = a_1 + b_1 - 3c_1 + 1$

➢ Powers of **T,** $0 = - b_1 - 2$

❖ On solving, $\mathbf{a_1 = -2, b_1 = -2}$ **and** $\mathbf{c_1 = -1}$

- Substituting $\mathbf{a_1}$**,** $\mathbf{b_1}$ **and** $\mathbf{c_1}$in Eq. (4), to write

$$\pi_1 = D^2 \times U_o^{-2} \times \rho^{-1} \times F_D \qquad \Rightarrow \pi_1 = \frac{F_D}{\rho U_o^2 D^2}$$

- Similarly, solving **second π - term**, to get

  - $\mathbf{a_2 = -1, b_2 = -1, c_2 = -1}$

$$\pi_2 = D^{-1} \times U_o^{-1} \times \rho^{-1} \times \mu \qquad \Rightarrow \pi_2 = \frac{\mu}{\rho U_o D}$$

- Substituting the values of π_1 **and** π_2 in Eq. (3), to get

$$f_1\left(\frac{F_D}{\rho U_o^2 D^2}, \frac{\mu}{\rho U_o D}\right) = 0$$

- can be expressed as:

$$\frac{F_D}{\rho U_o^2 D^2} = \phi\left(\frac{\mu}{\rho U_o D}\right) \qquad \Rightarrow F_D = \rho U_o^2 D^2 \phi\left(\frac{\mu}{\rho U_o D}\right)$$

- Required expression (same).

❖ To reduce the calculations and to obtain an expression in the desired form, the following characteristics of π-terms are used:

CHARACTERISTICS OF π-TERMS

(i) If a variable is dimensionless it can be straightway assigned as one of the π-term.

(ii) If any two variables have the same dimensions, then their ratio will be one of the π-terms.

Example: If in a problem **L** and **B** are involved, then one of the π-terms will be **L/B** or **B/L**.

(iii) Reciprocal of a π-term, its square and its square root is non-dimensional *i.e.* its character will not change (it would remain dimensionless). Thus, any π-term may be replaced by $1/\pi, \pi^2, \sqrt{\pi}$

(iii) Product and division of any two π-terms is non-dimensional. Thus, any two π-terms say π_1 and π_2 may be replaced by $\pi_1 \times \pi_2, \pi_1/\pi_2$ **or** π_2/π_1

(iv) Multiplication, division, addition or subtraction of any π-term by any numerical constant does not change the character of π-term. Thus, for example any π-term may be replaced by $3\pi, \pi/3, (\pi+3)$ **or** $(\pi-3)$.

Illustration: To obtain the expression for drag force on a sphere in the standard from:

The equation is: $F_D = \rho U_o^2 D^2 \phi\left(\frac{\mu}{\rho U_o D}\right)$

$$\therefore F_D = \rho U_o{}^2 D^2 \phi\left(\frac{1}{Re}\right); \qquad Re = \frac{\rho U_o D}{\mu}$$

- **Re** is a function of drag coefficient C_D

$$\therefore F_D = C_D \rho U_o{}^2 D^2 \qquad \Rightarrow F_D = C_D\left(\frac{\pi}{4}\right) D^2 \frac{\rho U_o{}^2}{2} \qquad \Rightarrow F_D = C_D A \frac{\rho U_o{}^2}{2}$$

- Required expression.

MODELS STUDY

- A model is a small-scale replica of the actual structure or machine.
- Small-scale replica of the actual structure is known as model while the actual structure is called prototype.
- Models are generally smaller than the prototypes, but in some cases the models may be larger than the prototype.
- For example, performance of a carburetor and a wrist watch may require to be tested on a large scale model.

❖ Before any new project is undertaken for the first time, it is often desirable and advantageous to conduct model studies.

E**xamples**:

- Testing of models of building, rivers, spillways, objects moving through air (designing streamline shapes of objects) etc.
- A model study involves:

➢ Model design and fabrication

➢ Tests on model and analysis of test results

❖ For model design, similarities are to be established between the model and the prototype and for analysis of test results, model laws are applied.

SIMILITUDE (TYPES OF SIMILARITIES)

- Results obtained from model tests are applicable to prototype if there exist similitude between the model and the prototype.
- For this purpose, three types of similarities are to be established between the model and prototype viz.Geometric, Kinematic and Dynamic similarity

Geometric Similarity:

- Geometric similarity between the model and prototype implies similarity in shapes *i.e.*ratio of all the corresponding dimensions in model and prototype should be equal.

❖ Such a ratio is known as scale ratio.

- On geometric similarity basis, different scale ratios are linear scale ratio, area scale ratio and volume scale ratio.
- Linear scale ratio, $L_r = \frac{L_m}{L_p} = \frac{B_m}{B_p} = \frac{H_m}{H_p}$

➢ Subscripts **m** and **p** corresponds to model and prototype, respectively.

$\therefore$ Area scale ratio, $A_r = L_r^2$

Volume scale ratio, $\forall_r = L_r^3$

Kinematic Similarity

- is the similarity of motion i.e. both the systems should undergo same rates of change of motion.
- On kinematic similarity basis, we can have velocity scale ratio, acceleration scale ratio and discharge scale ratio etc.
- Thus, velocity scale ratio $\mathbf{V_r}$ can be expressed as:

$$V_r = \frac{(V_1)_m}{(V_1)_p} = \frac{(V_2)_m}{(V_2)_p}$$

- $(\mathbf{V_1})_\mathbf{m}$ is the velocity at point **1** in the model , $(\mathbf{V_1})_\mathbf{p}$ is the velocity at the corresponding point in the prototype, $(\mathbf{V_2})_\mathbf{m}$ and $(\mathbf{V_2})_\mathbf{p}$ are the corresponding values at point **2**.

- The corresponding points in the two systems are called homologous points.

- Velocity scale ratio can also be expressed as:

$$V_r = \frac{V_m}{V_p} = \frac{(L_m / T_m)}{(L_p / T_p)} = \frac{L_r}{T_r}$$, $\mathbf{T_r}$ is the time scale ratio $= \frac{T_m}{T_p}$

- Similarly, acceleration scale ratio, $\mathbf{a_r}$ and discharge scale ratio $\mathbf{Q_r}$ can be written as:

$$a_r = \frac{a_m}{a_p} = \frac{L_r}{T_r^2}, \qquad Q_r = \frac{Q_m}{Q_p} = \frac{L_r^3}{T_r}; \qquad Q_r = \frac{A_m V_m}{A_p V_p} = L_r^2 V_r$$

Dynamic Similarity

- is the similarity of forces i.e. ratio of all the forces acting at homologous points in the two systems should be equal.
- Force scale ratio $\mathbf{F_r}$ on the basis of dynamic similarity is given by

$$F_r = \frac{(F_i)_m}{(F_i)_p} = \frac{(F_v)_m}{(F_v)_p} = \frac{(F_g)_m}{(F_g)_p}$$

- $\mathbf{F_i}$, $\mathbf{F_v}$ and $\mathbf{F_g}$ are the inertia, viscous and gravity forces, respectively.

- When the two systems are geometrically, kinematically and dynamically similar, they are said to be completely similar or complete similitude exists between the two systems.

- <u>If the two systems are only dynamically similar, they are also said to be completely similar</u>.

DIMENSIONLESS NUMBERS

- Forces acting on a moving mass of fluid are: Inertia force (**F_i**), viscous force (**F_v**), Gravity force (**F_g**), pressure force (**F_p**), surface tension force (**F_s**) and elastic force (**F_e**).

  - ❖ Dimensionless numbersare obtained by dividing the inertia force with any one of the remaining five forces listed above. Accordingly, there are five dimensionless numbers *i.e.* Reynolds no. (**Re**), Froude no. (**Fr**), Euler no. (**Eu**), Weber no. (**We**), Mach no. (**Ma**)

  - ❖ Before determining expression for dimensionless numbers, let us first find expressions for these six forces:

  - Inertia force = Mass × Acceleration of the flowing fluid

  - can be expressed as: $F_i = \rho\forall \times a = \rho L^3\left(\frac{L}{T^2}\right) = \rho L^2\left(\frac{L}{T}\right)^2 = \rho L^2V^2$

  - ➢ **L** is the characteristic dimension, **V** is the velocity, **μ** is the dynamic viscosity

  - Viscous force, $F_v = \mu\frac{du}{dy} \times \textbf{Surface area} = \mu\frac{V}{L}L^2 = \mu VL$

  - Gravity force, **F_g** = Mass × Acc. due to gravity = $= \rho\forall \times g = \rho L^3 g$

  - Pressure force**,** **F_p** = Intensity of pressure × Area = **pL^2**

  - Surface tension force, **$F_s = \sigma L$**, **σ** is the surface tension

  - Elastic force, **F_e** = Bulk modulus of elasticity × Area = **KL^2**

Reynolds number (Re)

- ratio of inertia force of a flowing fluid to the viscous force of fluid *i.e.* **F_i/F_v**

$$\therefore R_e = \frac{\rho V^2L^2}{\mu VL} = \frac{\rho VL}{\mu} = \frac{VL}{\nu}$$

Froude number (Fr)

- ratio of square root of inertia force to the gravity force (**F_g**) of fluid

$$\therefore Fr = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho V^2L^2}{\rho L^3 g}} = \frac{V}{\sqrt{gL}}$$

Euler's number (Eu)

- ratio of square root of inertia force to pressure force (**F_p**)

$$\therefore \mathbf{Eu} = \sqrt{\frac{F_i}{F_p}} = \sqrt{\frac{\rho V^2 L^2}{pL^2}} = \frac{V}{\sqrt{p/\rho}}$$

Weber's number (We)

- ratio of square root of inertia force to surface tension force (**F_s**)

$$\therefore \mathbf{We} = \sqrt{\frac{F_i}{F_s}} = \sqrt{\frac{\rho V^2 L^2}{\sigma L}} = \frac{V}{\sqrt{(\sigma/\rho L)}}$$

Mach number (Ma)

- ratio of square root of inertia force to the elastic force **F_e**

$$\therefore \mathbf{Ma} = \sqrt{\frac{F_i}{F_e}} = \sqrt{\frac{\rho V^2 L^2}{KL^2}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}; \qquad C = \sqrt{\frac{K}{\rho}}$$

❖ **V** is the fluid velocity**; C** is the velocity of sound through that fluid medium whose**K** and ρ are considered.

- The ratio of fluid velocity to the velocity of sound in the fluid medium is also known as Mach number.

➢ When **V > C** *i.e.* **Ma > 1** flow is designated as supersonic.

➢ When **V = C** *i.e.* **Ma = 1**, flow is designated as sonic and

➢ When **V** >>C *i.e.* **Ma >> 1**, flow is hypersonic.

❖ If **Ma < 1**, the flow can be considered as incompressible **(V < C)** and If **Ma ≥ 1**, flow is compressible **(V ≥ C)**.

MODEL LAWS

- For most of the hydraulic problems of model studies, it is quite rare that all the forces are simultaneously predominant/ significant in the given problem.

- Generally, only one force in addition to inertia force is relatively more significant than the rest of forces, which may not either exist or may be of negligible magnitude.

❖ Various model laws have been developed depending upon the significant forces influencing the different flow situations.

❖ There are five model laws viz. Reynolds model law, Froude, Euler, Weber, Mach- model law

Reynolds Model Law: This law is applicable to flows which are influenced by viscous effects or used when there is dominant action of viscous forces. **Examples:** Boundary layer flows (Flow- through pipes, meters, over surfaces; completely submerged flows) etc.

Froude Model Law: applicable to flows that are influenced by gravity. **Examples:** Free surface flows (flow in channels, rivers, flow over spillways, notches, weirs etc.)

Euler Model Law: applicable to flows in which pressure drops are significant **Examples:** Flow through pipes, cavitation phenomenon etc.

Weber Model Law: applicable to those flow situations in which surface tension influences the flow **Examples:** Flow with interface (capillary rise), flow over notches and weirs with small heads etc.

Mach Model Law: applicable to flows where compressibility of fluid is significant. **Examples:** Aerodynamic testing, water hammer problems etc.

- ❖ In some problems, only one model law may be applicable where as in some other cases, more than one model law may be applicable.

Reynolds Model Law

- ❖ According to Reynolds model law, similarity of flow in model and prototype can be established if Reynolds number of model is equal to Reynolds number of prototype i.e. $(R_e)_m = (R_e)_p$

$$\therefore \left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p \qquad \text{OR} \qquad \frac{\rho_r V_r L_r}{\mu_r} = 1 \quad \text{or} \quad \frac{V_r L_r}{\nu_r} = 1$$

➢ quantities with subscript **r** represent the corresponding scale ratios.

Scale ratios for some physical quantities based on **RML(in terms of ρ_r, ν_r and L_r)**

- Velocity scale ratio, $V_r = \frac{\nu_r}{L_r}$
- Discharge ratio, $Q_r = V_r L_r^2 = \nu_r L_r$
- Intensity of pressure ratio, $p_r = \left(\frac{\text{Force}}{\text{Area}}\right)_r = \left(\frac{\rho L^2 V^2}{L^2}\right)_r = \rho_r V_r^2 = \frac{\rho_r \nu_r^2}{L_r^2}$
- Work/energy ratio, $E_r = (\text{Force} \times \text{distance})_r = (\rho L^3 V^2)_r = (\rho_r \nu_r^2 L_r)$
- Power ratio, $P_r = (\text{Force} \times \text{velocity})_r = (\rho L^2 V^3)_r = \frac{\rho_r \nu_r^3}{L_r}$
- Or Power = Work/Time

Froude Model Law

- According to Froude model law, $(F_r)_m = (F_r)_p$

$$\therefore \left(\frac{V}{\sqrt{Lg}}\right)_m = \left(\frac{V}{\sqrt{Lg}}\right)_p \text{ or } \left(\frac{V_r}{\sqrt{L_r g_r}}\right) = 1 \quad \text{or} \quad V_r = \sqrt{L_r g_r}$$

- If $g_r = 1$ (**g** is same for both model and prototype), then $V_r = \sqrt{L_r}$

<u>Scale ratios for some physical quantities based on **FML**</u>

- Time scale ratio, $T_r = \dfrac{L_r}{V_r} = \dfrac{L_r}{\sqrt{L_r}} = \sqrt{L_r}$
- Acceleration scale ratio, $a_r = \dfrac{V_r}{T_r} = \dfrac{\sqrt{L_r}}{\sqrt{L_r}} = 1$
- Discharge scale ratio, $Q_r = \dfrac{L_r^3}{T_r} = \dfrac{L_r^3}{\sqrt{L_r}} = L_r^{2.5}$

  or $Q_r = A_r V_r = Lr^2 \sqrt{L_r} = Lr^{2.5}$
- Force scale ratio, $F_r = \rho_r L_r^2 V_r^2 = L_r^2 L_r = L_r^3$ (when $\rho_r = 1$ i.e. $\rho_m = \rho_p$)
- Intensity of pressure scale ratio, $p_r = \rho_r V_r^2 = L_r$ when $\rho_r = 1$.
- Work, energy, torque or moment scale ratio, $W_r = L_r^3 L_r = L_r^4$
- Power scale ratio, $P_r = \dfrac{L_r^4}{T_r} = \dfrac{L_r^4}{\sqrt{L_r}} = L_r^{3.5}$

Euler Model Law

- According to Euler model law, $(E_u)_m = (E_u)_p$

$$\therefore \left(\frac{V}{\sqrt{p/\rho}}\right)_m = \left(\frac{V}{\sqrt{p/\rho}}\right)_p \quad \text{or} \quad \frac{V_r}{\sqrt{p_r/\rho_r}} = 1 \quad \text{or} \quad V_r = \sqrt{p_r} \quad (\text{when } \rho_r = 1)$$

Weber Model Law

- According to Weber model law, $(W_e)_m = (W_e)_p$

$$\therefore \left(\frac{V}{\sqrt{(\sigma/\rho L)}}\right)_m = \left(\frac{V}{\sqrt{(\sigma/\rho L)}}\right)_p \quad \text{or} \quad \frac{V_r}{\sqrt{(\sigma_r/\rho_r L_r)}} = 1$$

Mach Model Law

- According to Mach model law, $(M_a)_m = (M_a)_p$

$$\therefore \left(\frac{V}{\sqrt{K/\rho}}\right)_m = \left(\frac{V}{\sqrt{K/\rho}}\right)_p \quad \text{or} \quad \frac{V_r}{\sqrt{K_r/\rho_r}} = 1$$

Problems:

Q1: A pipe of diameter 1.5 m is required to transport oil of relative density 0.90 and viscosity 0.03 Stoke at the rate of 3 m^3/s. If a 150 mm diameter pipe with water at 20^{o}C (viscosity = 0.01 Stoke) is used to model the above flow, find velocity and discharge in the model.

Solution:

Model (Water)	Prototype (Oil)
Diameter = 150 mm	Diameter = 1.5 m
Viscosity = 0.01 Stoke	Viscosity = 0.03 Stoke
Velocity =?	Discharge = 3 m^3/s
Discharge =?	

- Here, Reynolds model law is applicable.

$$\therefore \left(\frac{VL}{\nu}\right)_m = \left(\frac{VL}{\nu}\right)_p \qquad \text{Here, } L = D \qquad \therefore \frac{V_r D_r}{\nu_r} = 1$$

$$L_r = D_r = \frac{D_m}{D_p} = \frac{150}{1.5 \times 1000} = \frac{1}{10}, \qquad \nu_r = \frac{\nu_m}{\nu_p} = \frac{0.01}{0.03} = \frac{1}{3}, \qquad \therefore V_r = \frac{\nu_r}{D_r} = \frac{1/3}{1/10} = \frac{10}{3}$$

$$V_r = \frac{V_m}{V_p}, \qquad V_p = \frac{Q_p}{A_p} = \frac{3}{\pi/4 \times 1.5^2} = 1.7 \text{ m/s} \qquad \Rightarrow V_m = \frac{10}{3} \times 1.7 = 5.67 \text{ m/s}$$

$$Q_r = V_r A_r = V_r D_r^2 = \left(\frac{10}{3}\right)\left(\frac{1}{10}\right)^2 = \frac{1}{30}$$

$$Q_r = \frac{Q_m}{Q_p} \qquad \Rightarrow Q_m = \frac{1}{30} \times 3 = 0.1 \text{ m}^3\text{/s}$$

- **Alternatively**

$$\left(\frac{VL}{\nu}\right)_m = \left(\frac{VL}{\nu}\right)_p \qquad \Rightarrow V_m = V_p\left(\frac{L_p}{L_m}\right)\left(\frac{\nu_m}{\nu_p}\right) \qquad (1)$$

- Substituting the values in **Eq. (1)** and solve for V_m, to get

$$V_m = 5.67 \text{ m/s}, \qquad Q_m = A_m V_m = \frac{\pi}{4}(0.15)^2 \times 5.66 = 0.1 \text{m}^3\text{/s}$$

Q2: A 1:50 spillway model has a discharge of 1.25 m^3/s. What is the corresponding prototype discharge? If a flood takes 12 h to occur in the prototype, how long will it take to occur in the model?

Solution:

- Here, Froude model law is applicable.
- Given, Q_m = 1.25 m^3/s, T_p = 12 h, $L_r = \frac{1}{50}$
- Q_p = ?, T_m = ?
- Using, $Q_r = L_r^{2.5}$ $\Rightarrow \frac{Q_m}{Q_p} = L_r^{2.5}$ $\Rightarrow Q_p = \frac{Q_m}{L_r^{2.5}} = 22097\ m^3/s$
- Using, $T_r = \sqrt{L_r}$ $\Rightarrow \frac{T_m}{T_n} = \sqrt{L_r}$ $\therefore T_m = T_p\sqrt{L_r} = 1.7\,h$

CLASSIFICATION OF MODELS

- Two categories - undistorted and distorted models.

Undistorted Models: are those which are geometrically similar to their prototypes *i.e.* scale ratio for the corresponding linear dimensions of model and prototype is same.

Distorted Models: are not geometrically similar to their prototypes. For a distorted model, different scale ratios for linear dimensions are adopted.

Example: For river models, two different scale ratios are taken, one for horizontal and the other for vertical dimensions (depth).

❖ If same scale ratio is taken, then depth of water in the model will be very-very small which may not be measured accurately.

Scale ratios for distorted models

- Scale ratio for horizontal dimensions, $(L_r)_H = \frac{L_m}{L_p} = \frac{B_m}{B_p}$
- Scale ratio for vertical dimensions, $(L_r)_V = \frac{H_m}{H_p}$

❖ Velocity scale ratio $\mathbf{V_r}$, area ratio $\mathbf{A_r}$ and discharge ratio $\mathbf{Q_r}$ for distorted models:

$$V_r = \frac{V_m}{V_p} = \frac{\sqrt{2g_m H_m}}{\sqrt{2g_p H_p}} = \sqrt{(L_r)_V}$$

$$A_r = \frac{A_m}{A_p} = \frac{B_m \times H_m}{B_p \times H_p} = (L_r)_H \times (L_r)_V$$

$$Q_r = \frac{Q_m}{Q_p} = \frac{A_m \times V_m}{A_p \times V_p} = (L_r)_H \times (L_r)_V \times \sqrt{(L_r)_V} = (L_r)_H \times (L_r)_V^{1.5}$$

Scale effect in models

- If similitude does not exist between a model and its prototype, then there will be some discrepancy between the results obtained from the model tests and those which will be indicated by the prototype after its construction. This discrepancy is known as scale effect.

Reasons for scale effect

(i) In some problems, if several forces have predominance, then similitude will be ensured only if all the pertinent forces are simultaneously satisfied. However, it is quite difficult to satisfy all the model laws involved in the phenomenon.

For example, if in particular problem, both viscous and gravity forces are predominant, then both Reynolds and Froude model laws should be satisfied simultaneously.

- The necessary condition by combining both the laws is:

$$\frac{V_r L_r}{\nu_r} = 1; \qquad V_r = \sqrt{L_r} \qquad\qquad \therefore \nu_r = L_r^{3/2} \qquad (1)$$

- Equation (1) indicates that once model scale is selected, a liquid of appropriate viscosity as required Eq. (1) must be found for the model testing or if a particular liquid is to be used for the model testing, then the model scale $\mathbf{L_r}$ has to be found in accordance with Eq. (1).

- Further, for model testing if same liquid as for the prototype is used then $\mathbf{L_r = 1}$ i.e. model must be as large as the prototype.

❖ Difficult to find liquids of any desired viscosity and also, models with full scale are impractical.

(ii) Scale effect may also be developed in cases where forces which have practically no effect on the prototype significantly affect the behaviour of its model.

Example: For flow over weir models, if head is less than about 15 mm, surface tension plays a significant part than in the full size weir having large head than in the model.

Application of model laws to some specific model studies:

- Submerged objects
- Partially submerged objects
- Models of rivers

(1) Submerged objects

- Consider the case of an aeroplane model testing in a wind tunnel, to find the drag (resistance to its motion).
- ❖ In this case, only viscous and inertia forces are involved.
- Gravity forces may be neglected (fluid being air has very small density)
- Surface tension is not present as there is no interface between a liquid and another liquid.
- If it is assumed that it is a low speed aeroplane (velocity of air in the wind tunnel is small), then the effects of compressibility can also be neglected.

- $\therefore$ Drag encountered by the aeroplane model is affected by the <u>viscous force only.</u>

$\therefore$ Drag $\mathbf{F_D}$ will thus be a function velocity $\mathbf{U_o}$, density $\boldsymbol{\rho}$ and viscosity $\boldsymbol{\mu}$ and some characteristic length **L** (to specify the size of model) i.e. $\mathbf{F_D = f(U_o, \rho, \mu, L)}$

- Using the method of dimensional analysis, the following dimensionless functional relationship is obtained.

$$\frac{F_D}{\rho U_o^2 L^2} = \phi\left(\frac{\rho U_o L}{\mu}\right) = \phi(Re)$$

- Eq. is true for both model and prototype.

- For similitude, the ratio of forces at homologous points must be same for both.

$\therefore$ Reynolds no. must be same for both.

- Consequently, function $\boldsymbol{\phi}$**(Re)** is same for both model and prototype.

$\therefore$ Dimensionless no. $\frac{F_D}{\rho U_o^2 L^2}$ is same for both model and prototype i.e.

$$\therefore \left(\frac{F_D}{\rho U_o^2 L^2}\right)_m = \left(\frac{F_D}{\rho U_o^2 L^2}\right)_p$$

- ❖ This eq. is valid only if tests on the model are carried out under the conditions that only Reynolds no. is same for the prototype and model.

Observation:

- If compressibility effects are also important for the prototype, then Mach no. must also be same in addition to Reynolds numbers for similitude between the model and prototype. Thus,

$$\left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p \quad \text{and} \quad \left(\frac{V}{\sqrt{K/\rho}}\right)_m = \left(\frac{V}{\sqrt{K/\rho}}\right)_p$$

- To satisfy both the conditions simultaneously, we have

$$\frac{\rho_r V_r L_r}{\mu_r} = 1 \quad \text{and} \quad \frac{V_r}{\sqrt{K_r/\rho_r}} = 1 \qquad \Rightarrow \mu_r = L_r\sqrt{K_r \rho_r}$$

- ❖ As the range of available fluids being limited, it is not possible to satisfy the above condition.
- In such cases, one of the forces is considered as having secondary effect and may be neglected. This would however result in introducing the scale effect which needs to be accounted for before making any predictions about prototype behaviour.

(2) Partially Submerged objects

- Consider the case of a ship moving through water, to find the drag
- ❖ Drag in this case is due to three causes/reasons: Viscosity, surface waves and wake formation.
- Due to velocity gradient and viscosity of fluid, viscous forces are set up between the fluid layers close to the surface of ship, resulting in causing frictional resistance to the motion of ship, known as skin frictional drag.
- Ship being only partly immersed in water, its motion gives rise to waves on the surface, resulting in imparting resistance to its motion.
- As ship moves, a part of flow towards the rear of ship separates from its surface so as to form a 'wake'. Eddies so formed considerably modify the pressure distribution around the moving ship, resulting in developing resistance to its motion, known as pressure or form drag.
- ❖ $\therefore$ Total drag on a ship moving through water is the sum of friction drag, drag due to surface waves and pressure drag.
- Since ships are generally streamlined, pressure drag is relatively small and the formation of surface waves is associated with gravitational action.
- (For this case, compressibility of water is insignificant and surface tension forces may be neglected if the model is not too small).
- ❖ Thus, resistance to the motion of ship is affected by viscosity as well as gravity.

$\therefore$ Total drag $\mathbf{F_D}$ encountered by the ship will thus be a function of velocity $\mathbf{U_o}$ of the ship, viscosity of water μ, density of water ρ, characteristic length $\mathbf{L}$ to specify the size of ship and acceleration due to gravity $\mathbf{g}$ i.e. $\mathbf{F_D = f(U_o, \rho, \mu, L, g)}$

- Using the method of dimensional analysis, the following dimensionless functional relationship is obtained.

$$\frac{F_D}{\rho U_o^2 L^2} = \phi\left(\frac{\rho U_o L}{\mu}, \frac{U_o^2}{gL}\right) = \phi\left(Re, Fr^2\right)$$

- Thus, for complete similarity between the prototype and its model, **Re** and **Fr** numbers must be same i.e.

$$\left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p \quad \textbf{and} \quad \left(\frac{V}{\sqrt{gL}}\right)_m = \left(\frac{V}{\sqrt{gL}}\right)_p$$

- In practice, $g_m = g_p$

$\therefore$ For both the conditions to be satisfied simultaneously, we have

$$\frac{V_r L_r}{\nu_r} = 1; \qquad V_r = \sqrt{L_r} \qquad\qquad \therefore \nu_r = L_r^{3/2}$$

(3) Models of rivers

- River models are frequently used to predict the behaviour of rivers when some works are undertaken across the river.
- ❖ Froude's model law is the basic similitude criteria. Surface roughness should also be given due considerations.
- River models may have either vertical or slope enhancement/enlargement/exaggeration. In vertically enlarged models, depth scale is enhanced (distorted). and in slope enhanced model, model is geometrically similar (undistorted) but is made relatively steeper by titling it.
- ❖ Surface roughness for river models is calculated using Manning's Eq. i.e.

$$V_r = \frac{1}{N_r}(R_h)_r^{2/3} S_r^{1/2}$$

Vertically enlarged river models (distorted):

- For these models, as depth scale is enlarged, let $(L_r)_H$ and $(L_r)_V$ represent the horizontal and vertical length (depth) scale ratios, respectively.
- ❖ Hydraulic radius is dependent on both horizontal and vertical dimensions, so it cannot be uniquely represented by a single scale ratio.
- ❖ For rivers, it can be assumed that , $(R_h)_r \cong (L_r)_V$
- Also, slope scale ratio, $S_r = \frac{(L_r)_V}{(L_r)_H}$

$\therefore$ Velocity scale ratio, $$V_r = \frac{1}{N_r}\frac{(L_r)_V^{7/6}}{(L_r)_H^{1/2}}$$

- According to Froude law, $\textbf{As} \quad V_r = (L_r)_V^{1/2}$

$\therefore$ Scale ratio for rugosity coefficient, $\therefore N_r = \dfrac{(L_r)_V^{2/3}}{(L_r)_H^{1/2}}$

- By suitably fixing the scale ratios $(L_r)_H$ and $(L_r)_V$, the value of N_r can be obtained.

Observation: For undistorted models, since $(L_r)_H = (L_r)_V$ $\therefore N_r = L_r^{1/6}$

Slope enlarged river models:

Models are geometrically similar (undistorted) but slope of models is increased by titling them more.Such models are therefore also known as tilted models.

- For such models, $V_r = \dfrac{1}{N_r}(R_h)_r^{2/3} S_r^{1/2}$ $\Rightarrow \sqrt{L_r} = \dfrac{1}{N_r} L_r^{2/3} S_r^{1/2}$

$$\therefore N_r = L_r^{1/6} S_r^{1/2}$$

- By suitably selecting the slope and length scale ratios, the corresponding value of N_r can be obtained.

Problem: To estimate the frictional head loss in a pipeline of diameter 1 m through which oil of specific gravity 0.80 and viscosity 0.02 P is required to be transported at the rate of 2000 litres per sec, a model test was conducted. The test was conducted on a 100 mm diameter pipe using water at 20°C. What is the discharge required for the model pipe? If head loss in 30 m length of model pipe is measured as 440 mm of water, determine the head loss in the prototype. Given, viscosity of water at 20°C = 1 cP.

Solution: The data given in the problem can be written in the following manner:

Model (Water)	Prototype (Oil)
Diameter = 100 mm	Diameter = 1 m
Viscosity at 20°C = 0.01 P	Specific gravity = 0.80
Discharge =?	Viscosity = 0.02 P
Head loss = 440 mm	Discharge =2000 lps =2 m^3/s
Length = 30 m	Head loss =?

- Here, resistance to flow, **R** is governed by $\dfrac{R}{\rho V^2 L^2} = \phi(Re)$
- For similitude, function **ϕ(Re)** must be same for both model and prototype i.e. Reynolds no. must be same for both model and prototype.
- Also, dimensionless no. $\dfrac{R}{\rho V^2 L^2}$ will be same for both model and prototype.

$\therefore$ First applying Reynolds model law, to write

$$\frac{\rho_r V_r L_r}{\mu_r} = 1$$

- Here, $L_r = \frac{L_m}{L_p} = \frac{D_m}{D_p} = \frac{10}{100} = \frac{1}{10}$; $\rho_r = \frac{\rho_m}{\rho_p} = \frac{1000}{0.80 \times 1000} = \frac{5}{4}$, $\mu_r = \frac{\mu_m}{\mu_p} = \frac{0.01}{0.02} = \frac{1}{2}$

$$\therefore V_r = \frac{1}{2} \times \frac{4}{5} \times 10 = 4$$

$$Q_r = A_r V_r = Lr^2 \times V_r \quad = \left(\frac{1}{10}\right)^2 \times 4 = \frac{1}{25}$$

$$\therefore Q_r = \frac{Q_m}{Q_p} \qquad \Rightarrow Q_m = \frac{1}{25} \times 2000 = 80 \text{ lps}$$

- Also, $\left(\frac{R}{\rho V^2 L^2}\right)_m = \left(\frac{R}{\rho V^2 L^2}\right)_p$; $\quad \therefore \left(\frac{R_r}{\rho_r V_r^2 L_r^2}\right) = 1$

$$\Rightarrow R_r = \rho_r V_r^2 L_r^2 \qquad = \frac{5}{4} \times 4^2 \times \left(\frac{1}{10}\right)^2 = \frac{1}{5}$$

- Frictional resistance **R** is also given by; $R = \Delta p A = (h_L \rho g) A$

$$\therefore R_r = (h_L)_r \rho_r g_r L_r^2; \qquad \therefore \frac{1}{5} = (h_L)_r \rho_r g_r L_r^2$$

$$\Rightarrow (h_L)_r = \frac{1}{5} \times \frac{4}{5} \times (10)^2 = 16; \qquad (g_r = 1)$$

$$(h_L)_r = \frac{(h_L)_m}{(h_L)_p} \qquad \Rightarrow (h_L)_p = \frac{440}{16} = 27.5 \text{ mm}$$