Group Theory

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Examples

Q:- Consider the set $\mathbb{Q}$ of rational numbers and let $*$ be the operation defined on $\mathbb{Q}$ by:

$$a * b = a + b - ab$$

a) find i) $3 * 4$ ii) $2 * (-5)$ iii) $7 * (1/2)$

b) Is $(\mathbb{Q}, *)$ a semigroup? Is it commutative?

c) Find the identity element for $*$.

d) Do any of the elements in $\mathbb{Q}$ have an inverse? What is it?

Ans- a) i) $3*4 = 3+4-(3)(4) = 7-12 = -5$

ii) $2*(-5) = 2+(-5)-(2)(-5) = 7$

iii) $7*(1/2) = 7+\frac{1}{2} - (7)(\frac{1}{2}) = 4$

b) Semigroup $\rightarrow$ Axioms closure & associativity

$\Rightarrow$ $a * b = a + b - ab$, $\forall a, b \in Q$

i) Closure: ✓

ii) Associativity: $(a*b)*c = a*(b*c)$ → To Prove

LHS $(a*b)*c = (a+b-ab)*c$

$= a+b-ab+c-(a+b-ab)c$

$= a+b-ab+c-ac-bc+abc$

RHS $a*(b*c) = a*(b+c-bc)$

$= a+b+c-bc-a(b+c-bc)$

$= a+b+c-bc-ab-ac+abc$

LHS = RHS $\therefore (Q, *)$ is a semigroup.

Commutative? $a * b = b * a$ → To prove

LHS $\quad a * b = a + b - ab$ ✓

RHS $\quad b * a = b + a - ba$ ✓

LHS = RHS, ∴ $(Q, *)$ is commutative

c) $a * e = a$

$\cancel{a} + e - ae = \cancel{a}$

$e - ae = 0$

$e(1-a) = 0$

$e = 0$

∴ 0 is the identity element.

d) $a * \underline{a^{-1}} = e$

$a * a^{-1} = 0$

$a + a^{-1} - aa^{-1} = 0$

$a = aa^{-1} - a^{-1}$

$a = a^{-1}(a-1)$

$a^{-1} = \frac{a}{a-1}$

∴ if $a \neq 1$ then inverse of a exists & it is $a/(a-1)$.

Q: Consider the group $G = \{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.

a) Find the multiplication table of G.

b) Find 2^{-1}, 3^{-1}, 6^{-1}.

c) Find the orders and subgroups generated by 2 & 3.

d) Is G cyclic?

Ans: a)

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	$\boxed{1}$	3	5
3	3	6	2	5	$\boxed{1}$	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	$\boxed{1}$

$a * e = a$

$e = 1$

b) $2 * 4 = 1$

$(2)^{-1} = 4$

$(3)^{-1} = 5$

$(6)^{-1} = 6$

c) Subgroup generated by 2.

$2^1 = \underline{2}$, $2^2 = \underline{4}$, $2^3 = \underline{1}$, $2^4 = 2$, $2^5 = 4$,

$2^6 = 1$

$gp(2) = \{1, 2, 4\}$, $|2| = 3$ or $o(gp(2)) = 3$

Subgroup generated by 3.

$3^1 = 3$, $3^2 = 2$, $3^3 = 6$, $3^4 = 4$, $3^5 = 5$, $3^6 = 1$

$gp(3) = \{1, 2, 3, 4, 5, 6\}$, $|3| = 6$

$\quad\quad\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad\quad\quad G$

d) Yes, G is cyclic

3 is the generator


Thank you!!!
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