

Group Theory

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Contents

- Symmetric Group-Definition
- Cyclic Notation
- Composition

Symmetric Group

- A one-to-one and onto mapping f of a set $X = \{1, 2, \dots, n\}$ onto itself is called a **permutation**.
- Such a permutation may be denoted as:

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}, \text{ where } j_i = f(i)$$

- The set of all such permutations is generally denoted by S_n .

Symmetric Group (Cont..)

- Here, S_n is called the symmetric group of degree n under operation composition of function.
- Since there are $n!$ such permutation operations, the order (number of elements) of the symmetric group S_n is $n!$.

Example

- Let us consider the symmetric group S_3 . Let $X = \{1, 2, 3\}$.
- Then, S_3 has $3! = 6$ elements:

$$\left[\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right]$$

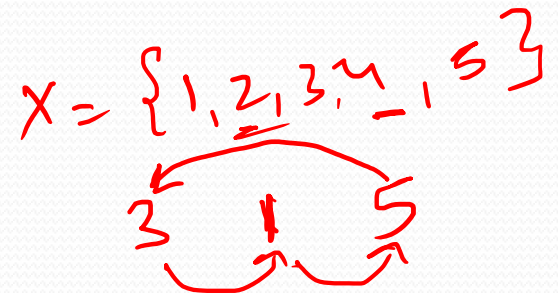
Cycle Notation

- Let $\{a_1, a_2, \dots, a_k\}$ be k distinct numbers between 1 and n . Then, the cycle (a_1, a_2, \dots, a_k) denotes the element of S_n that maps a_1 to a_2 , a_2 to a_3 , ..., a_{k-1} to a_k , a_k to a_1 , and leaves the remaining $n - k$ numbers fixed.
- The length of the cycle (a_1, a_2, \dots, a_k) is k .

Example

- let us consider the permutation $(315) \in S_5$.
- Then,

$$(315) = \begin{pmatrix} \boxed{1} & 2 & \boxed{3} & 4 & \boxed{5} \\ 5 & 2 & \boxed{1} & 4 & \boxed{3} \end{pmatrix}$$



Composition

$$\cancel{f \circ g}(u) = \cancel{f}(g(u))$$

- The group operation in a symmetric group is composition of function, denoted by the symbol \circ .
- The composition $f \circ g$ of permutations f and g maps any element $x \in X$ to $f(g(x))$.

Example

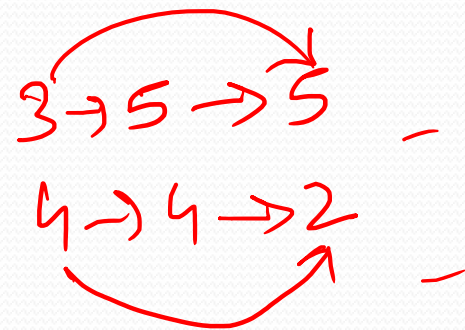
- Let us consider S_6 .

- Let $f = (1423) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 2 & 5 & 6 \end{pmatrix}$

- And $g = (16235) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 4 & 1 & 2 \end{pmatrix}$

Then, $f \circ g = (1423)(16235) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 2 & 4 & 3 \end{pmatrix}$

$$g \circ f = (16235)(1423) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 6 & 5 \end{pmatrix}$$



Composition (Cont..)

- A cycle of length $l = r \cdot p$, taken to the r^{th} power, will decompose into r cycles of length p .

Example

- Let us consider the permutation $(234561) \in S_6$. $l = 6$
- $r = 2, p = 3$, then

$$\begin{aligned}
 (234561)^2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} \\
 &= (135)(246)
 \end{aligned}$$

Handwritten notes: $1 \rightarrow 3 \rightarrow 5$, $2 \rightarrow 4 \rightarrow 6$, $(135)(246)$

