Lecture 4: Numerical Analysis (UMA011)

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Error Analysis: Loss of Significance

Example:

Use four-digit rounding arithmetic and the formula for the roots of a quadratic equation, to find the most accurate approximations to the roots of the following quadratic equation. Compute the absolute and relative errors.

$$1.002x^2 + 11.01x + 0.01265 = 0.$$

x_1 x_2

x_1^* x_2^*

Solution: Exact roots of given quadratic eq^n

$x_1 = -0.00114907565991$

$x_2 = -10.9868748764359$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-11.01 \pm \sqrt{(11.01)^2 - 4(1.002)(0.01265)}}{2 \times 1.002}$$

$$= \frac{-11.01 \pm \sqrt{121.2 - (4.008)(0.01265)}}{2.004}$$

$$= \frac{-11.01 \pm \sqrt{121.2 - 0.05070}}{2.004} = \frac{-11.01 \pm \sqrt{121.1}}{2.004}$$

$$= \frac{-11.01 \pm 11.00}{2.004} \qquad \frac{-11.01 + 11.00}{2.004}$$

$$\frac{|x_1 - x_1^*|}{|x_1|}$$

$$x_1^* = \frac{-0.01}{2.004}, \qquad x_2^* = \frac{-22.01}{2.004}$$

$$x_1^* = -0.0050 \qquad x_2^* = -10.98$$

To find most accurate approximation to x_1

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}}$$

$$= \frac{\cancel{b^2}-(\cancel{b^2}-4ac)}{2a\left(-b-\sqrt{b^2-4ac}\right)} = \frac{4ac}{2a\left(-b-\sqrt{b^2-4ac}\right)}$$

$$x_1 = \frac{-2c}{b+\sqrt{b^2-4ac}}$$

$$x_1^{**} = \frac{-2\times 0.01265}{11.01+11.00} = -\frac{0.0253}{22.01} = -0.001149$$

Errors in Ist root

$$A.E. = |x_1 - x_1^{**}| =$$

$$R.E = \frac{|x_1 - x_1^{**}|}{|x_1|} =$$

Errors in 2nd root

$$A.E. = |x_2 - x_2^{*}| = 0.006874876$$

$$R.E = \frac{|\cancel{x}_2 - x_2^{*}|}{|x_2|} = 0.0006261$$

Error Analysis: Algorithms and Stability

Algorithms:

An algorithm is a procedure that describes a finite sequence of steps to be performed in a specified order.

one Criterion → Small changes in the initial data produce correspondingly small change in the final results

$y = f(x)$ input

output

$f(x+\Delta x) = y+\Delta y$

small small.

Stable or Unstable

An algorithm satisfies this property is called stable, otherwise it is unstable

Well-conditioned or ill-conditioned

A problem is well-conditioned if small changes in the input data can produce only small changes in the output otherwise it is ill-conditioned.

$$f(x+\Delta x) \downarrow y+\Delta y$$

$$f(x) = \cos x - \sin x$$

1) x
2) $\cos x$
3) $\sin x$
4) $\cos x - \sin x$

Error Analysis: Algorithms and Stability

Condition number:

The condition number is given by

$$\kappa = \frac{\text{relative change in the output}}{\text{relative change in the input}}$$

$<\sim\sim 1$

$>>> 1$

If condition no. of a problem is less than or near to one then that problem is well-conditioned.

and If it $>>> 1$ then problem is ill-conditioned

$f(x)$

$$K = \frac{R.E \text{ in } f(x)}{R.E \text{ in } x}$$

$$= \frac{\dfrac{f(x) - f(x^*)}{|f(x)|}}{\dfrac{|x - x^*|}{|x|}} = \frac{|f(x) - f(x^*)|}{|f(x)|} * \frac{|x|}{|x - x^*|}$$

$$= \frac{|f(x) - f(x^*)|}{|x - x^*|} * \frac{|x|}{|f(x)|}$$

$$\approx \frac{|f'(x)|\,|x|}{|f(x)|} = \left|\frac{x f'(x)}{f(x)}\right|$$

$$C.N = \left|\frac{x f'(x)}{f(x)}\right|$$

e.g. find condition no. of $f(x) = \frac{10}{1-x^2}$ ✓

$$C.N = \left| \frac{x f'(x)}{f(x)} \right|$$

$$f'(x) = \frac{10(-1)(-2x)}{(1-x^2)^2} = \frac{20x}{(1-x^2)^2}$$

$$= \frac{\left| x \dfrac{20x}{(1-x^2)^2} \right|}{\left| \dfrac{10}{1-x^2} \right|}$$

$$C.N. = \frac{20x^2}{10(1-x^2)} = \frac{2x^2}{(1-x^2)}$$

if $x \approx \pm 1$

$>>>>1$

Error Analysis

Exercise:

1. Use four-digit rounding arithmetic and the formula to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

$$\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0.$$

Ans. Most accurate roots are

$x_1^* = -92.26$, $x_2^* = 0$ (previous)

$x_2^{**} = 0.005420$

2. Compute and interpret the condition number for:
(i) $f(x) = \sin(x)$ for $x = 0.51\pi$ Ans. 0.05035
and (ii) $f(x) = \tan(x)$ for $x = 1.7$. Ans. -13.305