Lecture 17: Numerical Analysis (UMA011)

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System of linear equations

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots a_{nn}x_n = b_n.$$

This is a linear system of n equation in n unknowns $x_1, x_2 \ldots\ldots x_n$. This system can simply be written in the matrix equation form

$$Ax=b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

System of linear equations

Direct Methods

1 Gauss Elimination

2 LU Decomposition(Factorization)

System of linear equations

Gauss Elimination

To solve larger system of linear equation, we consider a following $n \times n$ system

$$\begin{array}{rcll}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n & = & b_1 & (E_1) \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n & = & b_2 & (E_2)
\end{array}$$

$$\begin{array}{rcll}
a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n & = & b_i & (E_i)
\end{array}$$

$$\begin{array}{rcll}
a_{n1}x_1 + a_{n2}x_2 + a_{n3} + \cdots + a_{nn}x_n & = & b_n & (E_n).
\end{array}$$

Here E_i denote the i-th row of the coefficients matrix A, $i = 1, 2, \cdots, n$.

Augmented Matrix is given by $[A:b] = \begin{matrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & -a_{1n} & : b_1 \\ a_{21} & a_{22} & & & -a_{2n} & : b_2 \\ - & - & - & & - & \\ a_{n1} & a_{n2} & & \cdots & a_{nn} & : b_n \end{bmatrix}$

If $a_{11} = 0$, then find p s.t. $a_{p1} \neq 0$, $E_1 \Leftrightarrow E_p$

Let $a_{11} \neq 0$ and eliminate x_1 from $E_2, E_3, \cdots, E_n$.

Define multipliers $m_{j1} = \dfrac{a_{j1}}{a_{11}}$, for each $j = 2, 3, \cdots, n$.

We write each entry in E_j as $E_j - m_{j1} E_1$ and b_j as $b_j - m_{j1} b_1; j = 2, 3, \cdots, n$.

$\Rightarrow [A:b] = \begin{matrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_n \end{matrix} \begin{bmatrix} a_{11} & a_{12} & - & - & - & a_{1n} & : b_1 \\ 0 & a_{22}^* & & \cdots & - & a_{2n}^* & : b_2^* \\ 0 & a_{32}^* & & & & & \\ \vdots & \vdots & & & & & \\ 0 & a_{n2}^* & - & - & - & a_{nn}^* & : b_n^* \end{bmatrix}$

If $a_{22}^* = 0$ then find $a_{p2}^* \neq 0$ and $E_2 \leftrightarrow E_p$

If $a_{22}^* \neq 0$, then

$$E_3 \to E_3 - \frac{a_{32}^*}{a_{22}^*} E_2$$

$$E_4 \to E_4 - \frac{a_{42}^*}{a_{22}^*} E_2$$

$$-\ -\ -\ -\ -\ -$$

$$E_n \to E_n - \frac{a_{n2}^*}{a_{22}^*} E_2$$

$$E_j \to E_j - \frac{a_{j2}^*}{a_{22}^*} E_2$$

$$j = 3, 4 \cdots n$$

$$[A:b] = \begin{matrix} F_1 \\ E_2 \\ E_3 \\ \vdots \\ E_n \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{4n} & : & b_1 \\ 0 & a_{22} & a_{23}^* & - & - & - & a_{2n}^* & : & b_2^* \\ 0 & 0 & a_{33}^{**} & - & - & & a_{3n}^{**} & : & b_3^{**} \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ 0 & 0 & a_{n3}^{**} & - & - & - & - & a_{nn}^{**} & : & b_n^{**} \end{bmatrix}$$

We repeat this procedure and follow a sequential procedure for $i = 2, 3, \cdots, n$ and perform the operations

$$(E_j - (a_{ji}/a_{ii})E_i) \rightarrow (E_j) \quad \text{for each } j = i+1, i+2, \ldots, n,$$

provided $a_{ii} \neq 0$. This eliminates (changes the coefficient to zero) x_i in each row below the ith for all values of $i = 1, 2, \ldots, n-1$. The resulting matrix has the form:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & \overset{=b_1}{a_{1,n+1}} \\ 0 & a_{22} & \cdots & a_{2n} & : & a_{2,n+1}\,^{=b_2} \\ \vdots & & \ddots & \vdots & : & \vdots \\ 0 & \cdots\cdots & 0 & a_{nn} & : & \overset{=b_n}{a_{n,n+1}} \end{bmatrix}, \rightarrow \text{upper triangular Matrix}$$

Using backward Substitution,

Solving the n-th equation for x_n gives

$$x_n = \frac{b_n}{a_{nn}}.$$

Solving the $(n-1)$st equation for x_{n-1} and using the known value for x_n yields (back substitution)

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}.$$

Continuing this process, we obtain

$$x_i = \frac{b_i - a_{i,i+1}x_{i+1} - a_{i,i+2}x_{i+2} - \cdots - a_{in}x_n}{a_{ii}} = \frac{b_i - \sum\limits_{j=i+1}^{n} a_{ij}x_j}{a_{ii}},$$

for each $i = n-1, n-2, \cdots, 2, 1$.

System of linear equations

Example

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear system. (The exact solution to each system is $x_1 = -1$, $x_2 = 1$, $x_3 = 3$.)

$$-x_1 + 4x_2 + x_3 = 8$$

$$\frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 = 1$$

$$2x_1 + x_2 + 4x_3 = 11.$$

$-0.7 \qquad 1.1 \qquad 2.9$

Solution:

Solution

$[A : b] = \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix} \begin{bmatrix} -1 & 4 & 1 & : & 8 \\ 1\cdot7 & 0\cdot67 & 0\cdot67 & : & 1 \\ 2 & 1 & 4 & : & 11 \end{bmatrix}$

$E_2 \rightarrow E_2 - \dfrac{1\cdot7}{-1} E_1 \quad \sim \quad E_2 \rightarrow E_2 + 1\cdot7 \; E_1$

$E_3 \rightarrow E_3 - \dfrac{2}{-1} E_1 \quad \sim \quad E_3 \rightarrow E_3 + 2 E_1$

$\sim \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix} \begin{bmatrix} -1 & 4 & 1 & : & 8 \\ 0 & 7\cdot5 & 2\cdot4 & : & 15 \\ 0 & \boxed{9} & 6 & : & 27 \end{bmatrix}$

$E_3 \rightarrow E_3 - \dfrac{9}{7\cdot5} E_2 \quad \sim \quad E_3 \rightarrow E_3 - 1\cdot2 \; E_2$

$$\sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \begin{pmatrix} -1 & 4 & 1 & : & 8 \\ 0 & 7.5 & 2.4 & : & 15 \\ 0 & 0 & 3.1 & : & +9 \end{pmatrix}$$

by backward Sub.

$$3.1 x_3 = 9$$

$$x_3 = \frac{9}{3.1} = 2.9$$

$$7.5 x_2 + 2.4 x_3 = 15$$

$$7.5 x_2 + 2.4 \times 2.9 = 15$$

$$x_2 = \frac{8.04}{7.5} = \frac{8.0}{7.5} = 1.1$$

$$-x_1 + 4 x_2 + x_3 = 8$$

$$-x_1 + 4.4 + 2.9 = 8$$

$$-x_1 = 0.7 \Rightarrow x = \begin{bmatrix} .0.7 \\ 1.1 \\ 2.9 \end{bmatrix}$$

$$x_1 = -0.7$$

Ans.

System of linear equations:

Exercise:

1 Using four-digit arithmetic operations, solve the following system of equations by Gaussian elimination with and without partial pivoting

$$0.729x_1 + 0.81x_2 + 0.9x_3 = 0.6867$$
$$x_1 + x_2 + x_3 = 0.8338$$
$$1.331x_1 + 1.21x_2 + 1.1x_3 = 1.000.$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!