

Lecture 23: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

weaker condition

$$AX = b$$

↓

Sufficient condition.

A is strictly diagonal dominant

⇔

$$\{x^{(k)}\}_{k=0}^{\infty} \rightarrow x$$

↓
exact value

A is not S.D.D

⇔

Gauss-Seidel method:

Example:

Solve the following linear system using Gauss-Seidel with $x^{(0)} = 0$ and iterating until $\|x^{(k)} - x^{(k-1)}\|_{\infty} \leq 10^{-2}$:

$$x^{(0)} = (0, 0, 0)^t$$

$$9x_1 + x_2 + x_3 = 10 \quad - \textcircled{1}$$

$$2x_1 + 10x_2 + 3x_3 = 19 \quad - \textcircled{2}$$

$$3x_1 + 4x_2 + 11x_3 = 0. \quad - \textcircled{3}$$

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix}$$

$$|9| > |1| + |1|$$

$$|10| > |2| + |3|$$

$$|11| > |3| + |4|$$

$\Rightarrow A$ is S.D.D.

from eqn ①

$$x_1^{(k+1)} = \frac{1}{9} (10 - (x_2^{(k)} + x_3^{(k)}))$$

from eqn ②

$$x_2^{(k+1)} = \frac{1}{10} (19 - 2x_1^{(k+1)} - 3x_3^{(k)})$$

from eqn ③

$$x_3^{(k+1)} = \frac{1}{11} (-3x_1^{(k+1)} - 4x_2^{(k+1)})$$

$$x_1^{(1)} = \frac{1}{9}(10-0) = \frac{10}{9} = 1.1111$$

$$x_2^{(1)} = \frac{1}{10}(19 - 2 * 1.1111 - 3(0)) = \frac{1}{10}(19 - 2.2222) = 1.6778$$

$$x_3^{(1)} = \frac{1}{11}(-3 * 1.1111 - 4 * 1.6778) = -0.9131$$

$$x^{(1)} = (1.1111, 1.6778, -0.9131)^t$$

for $x^{(2)}$

$$x_1^{(2)} = \frac{1}{9}(10 - x_2^{(1)} - x_3^{(1)}) = \frac{1}{9}(10 - 1.6778 + 0.9131) = 1.0261$$

$$x_2^{(2)} = \frac{1}{10}(19 - 2 * 1.0261 - 3 * (-0.9131)) = 1.9687$$

$$x_3^{(2)} = \frac{1}{11}(-3 * 1.0261 - 4 * 1.9687) = -0.9588$$

$$x^{(2)} = (1.0261, 1.9687, -0.9588)$$

$$\|x^{(2)} - x^{(1)}\|_{\infty} = \max \{ |1.0261 - 1.1111|, |1.9687 - 1.6778|, |-0.9588 + 0.9131| \} \neq 10^{-2}$$

$$\underline{x^{(3)}} \quad x_1^{(3)} = \frac{1}{9}(10 - x_2^{(2)} - x_3^{(2)}) = \frac{1}{9}(10 - 1.9687 + 0.9588) = 0.9989$$

$$x_2^{(3)} = \frac{1}{10}(19 - 2 * 0.9989 - 3 * (-0.9588)) = 1.9878$$

$$x_3^{(3)} = \frac{1}{11}(-3 * 0.9989 - 4 * 1.9878) = -1.0953$$

Table for Gauss-Seidel method:-

K	0	1	2	3	
$x_1^{(k)}$	0	1.1111	1.0261	0.9989	
$x_2^{(k)}$	0	1.6778	1.9687	1.9878	
$x_3^{(k)}$	0	-0.9131	-0.9588	-0.9953	

SOR method:

(Successive over Relaxation Method)

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right)$$

$$\begin{aligned} x_i^{(k+1)} &= \tilde{x}_i^{(k)} + \omega (x_i^{(k+1)} - \tilde{x}_i^{(k)}) \\ &= (1-\omega) \tilde{x}_i^{(k)} + \omega x_i^{(k+1)} \end{aligned}$$

$$\left[x_i^{(k+1)} = (1-\omega) \tilde{x}_i^{(k)} + \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right) \right] \quad \text{SOR method}$$

If w

$w < 1$	Under Relaxation Method
$w = 1$	Gauss-seidel method.
$w > 1$	<u>SOR</u>

SOR method:

Convergence conditions for SOR method: Positive definite matrix

A matrix A is **positive definite** if it is symmetric and if each of its leading principal submatrices has a positive determinant.

For example: $A = \begin{bmatrix} [2] & -1 & 0 \\ -1 & [2] & -1 \\ 0 & -1 & [2] \end{bmatrix}$ is positive definite.

$$A^t = A$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A \rightarrow \text{Symmetric Matrix}$$

$$|2| > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0 \quad \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(4 - 1) + 1(-2) = 6 - 2 = 4 > 0$$

SOR method:

Convergence conditions for SOR method

If A is a **positive definite matrix** and $0 < \omega < 2$, then the SOR method converges for any choice of initial approximate vector $x^{(0)}$.

$$Ax = b$$

$$\omega < 1$$
$$\omega > 1$$

SOR method:

Example:

Solve the following linear system using SOR method with $\omega = 1.25$, $x^{(0)} = (1, 1, 1)^t$ and iterating until $\|x^{(k)} - x^{(k-1)}\|_{\infty} < 10^{-1}$:

$$4x_1 + 3x_2 = 24 \quad - \textcircled{1}$$

$$3x_1 + 4x_2 - x_3 = 30 \quad - \textcircled{2}$$

$$-x_2 + 4x_3 = -24. \quad - \textcircled{3}$$

which has exact solution $(3, 4, -5)^t$.

$$x_1^{(k+1)} = \frac{\omega}{4} (24 - 3x_1^{(k)}) + (1-\omega) x_1^{(k)}$$

$$x_2^{(k+1)} = \frac{\omega}{4} (30 - 3x_1^{(k+1)} + x_3^{(k)}) + (1-\omega) x_2^{(k)}$$

$$x_3^{(k+1)} = \frac{\omega}{4} (-24 + x_2^{(k+1)}) + (1-\omega) x_3^{(k)}$$

$$x_1^{(1)} = \frac{1.25}{4} (24 - 3 \times 1) + (1 - 1.25) \times 1 = 6.3125 \checkmark$$

$$x_2^{(1)} = \frac{1.25}{4} (30 - 3 \times 6.3125 + 1) + (1 - 1.25) \times 1 = 3.5195$$

$$x_3^{(1)} = \frac{1.25}{4} (-24 + 3.5195) + (1 - 1.25) \times 1 = -6.6502$$

$$x_1^{(2)} = \frac{1.25}{4} (24 - 3 * 3.5195) - 0.25 * 6.3125 = 2.6223$$

$$x_2^{(2)} = \frac{1.25}{4} (30 - 3 * 2.6223 + (-6.6502)) - 0.25 * 3.5195 = 3.9585$$

$$x_3^{(2)} = \frac{1.25}{4} (-24 + 3.9585) - 0.25 * (-6.6502) = -4.6004$$

iterates until $\|x^{(k)} - x^{(k-1)}\|_{\infty} \leq 10^{-1}$

System of linear equations:

Exercise:

- 1 Perform first four iterations of Gauss-Seidel and SOR method with $\omega = 1.1$ to the following linear system of equations with $x^{(0)} = 0$:

$$4x_1 + x_2 - x_3 = 3$$

$$2x_1 + 7x_2 + x_3 = 19$$

$$x_1 - 3x_2 + 12x_3 = 31.$$

Compare the iterations for both methods.