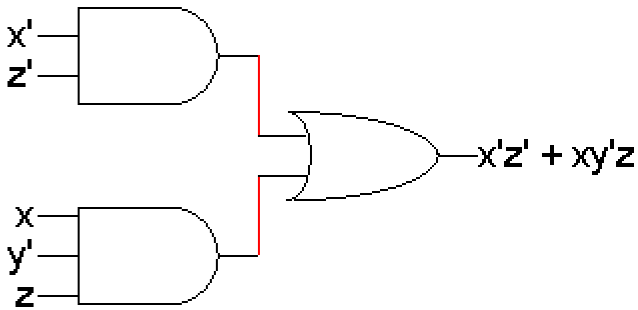
Karnaugh Maps for Simplification

Karnaugh Maps

- Boolean algebra helps us simplify expressions and circuits
- Karnaugh Map: A graphical technique for simplifying a Boolean expression into either form:
  - minimal sum of products (MSP)
  - minimal product of sums (MPS)
- Goal of the simplification.
  - There are a minimal number of product/sum terms
  - Each term has a minimal number of literals
- Circuit-wise, this leads to a *minimal* two-level implementation



Re-arranging the Truth Table

- A two-variable function has four possible minterms. We can re-arrange these minterms into a Karnaugh map

x	y	minterm
0	0	$x'y'$
0	1	$x'y$
1	0	xy'
1	1	xy

		y	
		0	1
X	0	$x'y'$	$x'y$
	1	xy'	xy

- Now we can easily see which minterms contain common literals
  - Minterms on the left and right sides contain y' and y respectively
  - Minterms in the top and bottom rows contain x' and x respectively

		y	
		0	1
X	0	$x'y'$	$x'y$
	1	xy'	xy

	y'	y
X'	$x'y'$	$x'y$
X	xy'	xy

Karnaugh Map Simplifications

- Imagine a two-variable sum of minterms:

$$x'y' + x'y$$

- Both of these minterms appear in the top row of a Karnaugh map, which means that they both contain the literal x'

		y
	$x'y'$	$x'y$
x	xy'	xy

- What happens if you simplify this expression using Boolean algebra?

$$\begin{aligned} x'y' + x'y &= x'(y' + y) && [\text{ Distributive }] \\ &= x' \bullet 1 && [\ y + y' = 1\ ] \\ &= x' && [\ x \bullet 1 = x\ ] \end{aligned}$$

More Two-Variable Examples

- Another example expression is $x'y + xy$
  - Both minterms appear in the right side, where y is uncomplemented
  - Thus, we can reduce $x'y + xy$ to just y

		y
	$x'y'$	$x'y$
X	xy'	xy

- How about $x'y' + x'y + xy$?
  - We have $x'y' + x'y$ in the top row, corresponding to x'
  - There's also $x'y + xy$ in the right side, corresponding to y
  - This whole expression can be reduced to $x' + y$

		y
	$x'y'$	$x'y$
X	xy'	xy

A Three-Variable Karnaugh Map

- For a three-variable expression with inputs x, y, z, the arrangement of minterms is more tricky:

		YZ 00	01	11	10
X	0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
	1	$xy'z'$	$xy'z$	xyz	xyz'

		YZ 00	01	11	10
X	0	m_0	m_1	m_3	m_2
	1	m_4	m_5	m_7	m_6

- Another way to label the K-map (use whichever you like):

			y	y
	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
X	$xy'z'$	$xy'z$	xyz	xyz'
		Z	Z	

			y	y
	m_0	m_1	m_3	m_2
X	m_4	m_5	m_7	m_6
		Z	Z	

Why the funny ordering?

- With this ordering, any group of 2, 4 or 8 adjacent squares on the map contains common literals that can be factored out

			y	y
	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
x	$xy'z'$	$xy'z$	xyz	xyz'
		z	z	

$$
\begin{aligned}
& x'y'z + x'yz \\
= \;& x'z(y' + y) \\
= \;& x'z \bullet 1 \\
= \;& x'z
\end{aligned}
$$

- "Adjacency" includes wrapping around the left and right sides:

			y	y
	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
x	$xy'z'$	$xy'z$	xyz	xyz'
		z	z	

$$
\begin{aligned}
& x'y'z' + xy'z' + x'yz' + xyz' \\
= \;& z'(x'y' + xy' + x'y + xy) \\
= \;& z'(y'(x' + x) + y(x' + x)) \\
= \;& z'(y'+y) \\
= \;& z'
\end{aligned}
$$

- We'll use this property of adjacent squares to do our simplifications.

K-maps From Truth Tables

- We can fill in the K-map directly from a truth table
  - The output in row *i* of the table goes into square m_i of the K-map
  - Remember that the rightmost columns of the K-map are "switched"

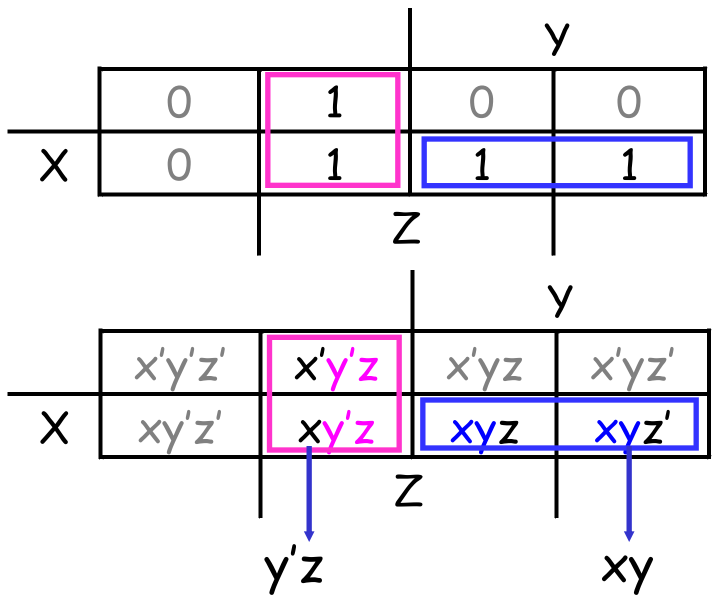
x	y	z	f(x,y,z)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

			Y	
	m_0	m_1	m_3	m_2
X	m_4	m_5	m_7	m_6
		Z		

			Y	
	0	1	0	0
X	0	1	1	1
		Z		

Reading the MSP from the K-map

- You can find the minimal SoP expression
  - Each rectangle corresponds to one product term
  - The product is determined by finding the common literals in that rectangle



$F(x,y,z) = y'z + xy$

Grouping the Minterms Together

- The most difficult step is grouping together all the 1s in the K-map
  - Make rectangles around groups of one, two, four or eight 1s
  - All of the 1s in the map should be included in at least one rectangle
  - Do *not* include any of the 0s
  - Each group corresponds to one product term

			Y	
	0	1	0	0
X	0	1	1	1
		Z		

For the Simplest Result

- Make as few rectangles as possible, to minimize the number of products in the final expression.
- Make each rectangle as large as possible, to minimize the number of literals in each term.
- Rectangles can be overlapped, if that makes them larger.

K-map Simplification of SoP Expressions

- Let's consider simplifying $f(x,y,z) = xy + y'z + xz$

- You should convert the expression into a sum of minterms form,
  - The easiest way to do this is to make a truth table for the function, and then read off the minterms
  - You can either write out the literals or use the minterm shorthand

- Here is the truth table and sum of minterms for our example:

x	y	z	f(x,y,z)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$\begin{aligned} f(x,y,z) &= x'y'z + xy'z + xyz' + xyz \\ &= m_1 + m_5 + m_6 + m_7 \end{aligned}$$

Unsimplifying Expressions

- You can also convert the expression to a sum of minterms with Boolean algebra
  - Apply the distributive law in reverse to add in missing variables.
  - Very few people actually do this, but it's occasionally useful.

$$
\begin{aligned}
xy + y'z + xz &= (xy \bullet 1) + (y'z \bullet 1) + (xz \bullet 1) \\
&= (xy \bullet (z' + z)) + (y'z \bullet (x' + x)) + (xz \bullet (y' + y)) \\
&= (xyz' + xyz) + (x'y'z + xy'z) + (xy'z + xyz) \\
&= xyz' + xyz + x'y'z + xy'z \\
&= m_1 + m_5 + m_6 + m_7
\end{aligned}
$$

- In both cases, we're actually "unsimplifying" our example expression
  - The resulting expression is larger than the original one!
  - But having all the individual minterms makes it easy to combine them together with the K-map

Making the Example K-map

- In our example, we can write $f(x,y,z)$ in two equivalent ways

$$f(x,y,z) = x'y'z + xy'z + xyz' + xyz$$

			y	y
	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
x	$xy'z'$	$xy'z$	xyz	xyz'
		z	z	

$$f(x,y,z) = m_1 + m_5 + m_6 + m_7$$

			y	y
	m_0	m_1	m_3	m_2
x	m_4	m_5	m_7	m_6
		z	z	

- In either case, the resulting K-map is shown below

			y	y
	0	1	0	0
x	0	1	1	1
		z	z	

Practice K-map 1

- Simplify the sum of minterms $m_1 + m_3 + m_5 + m_6$

			y	y
x				
		z	z	

			y	y
	m_0	m_1	m_3	m_2
x	m_4	m_5	m_7	m_6
		z	z	

Solutions for Practice K-map 1

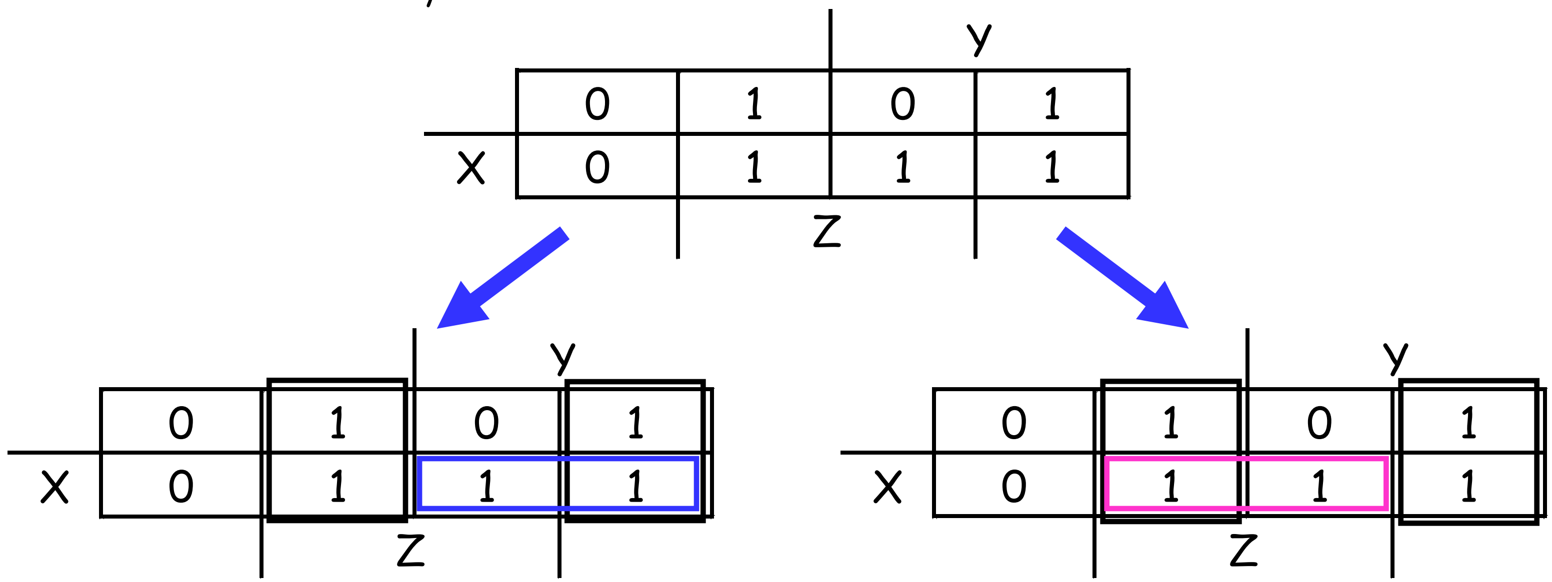
- Here is the filled in K-map, with all groups shown
  - The magenta and green groups overlap, which makes each of them as large as possible
  - Minterm m_6 is in a group all by its lonesome

			y	
	0	1	1	0
X	0	1	0	1
		Z		

- The final MSP here is $x'z + y'z + xyz'$

K-maps can be tricky!

- There may not necessarily be a *unique* MSP. The K-map below yields two valid and equivalent MSPs, because there are two possible ways to include minterm m_7

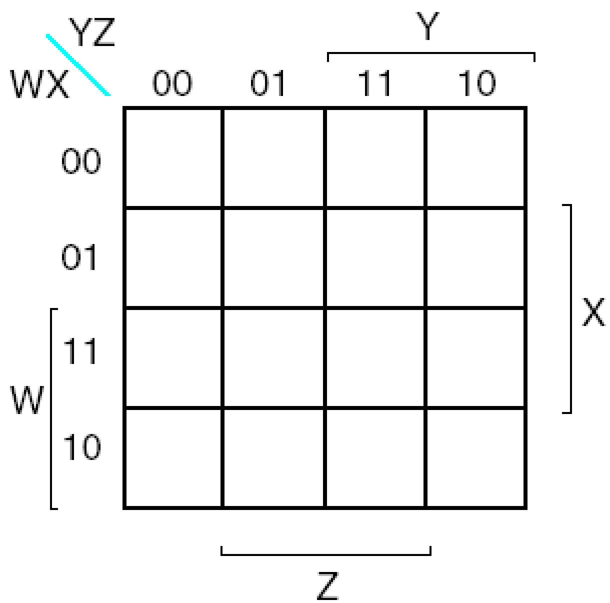


$y'z + yz' + xy$ $\qquad\qquad$ $y'z + yz' + xz$

- Remember that overlapping groups is possible, as shown above

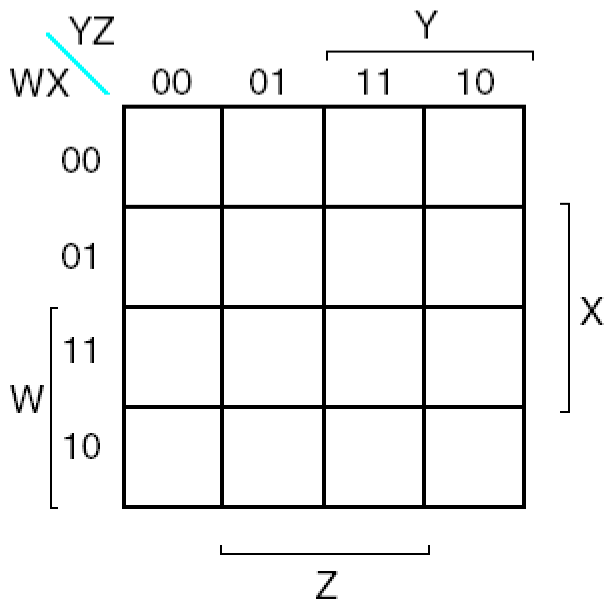
Four-variable K-maps – f(W,X,Y,Z)

- We can do four-variable expressions too!
  - The minterms in the third and fourth columns, *and* in the third and fourth rows, are switched around.
  - Again, this ensures that adjacent squares have common literals



- Grouping minterms is similar to the three-variable case, but:
  - You can have rectangular groups of 1, 2, 4, 8 or 16 minterms
  - You can wrap around *all four* sides

Four-variable K-maps



w'x'y'z'	w'x'y'z	w'x'yz	w'x'yz'
w'xy'z'	w'xy'z	w'xyz	w'xyz'
wxy'z'	wxy'z	wxyz	wxyz'
wx'y'z'	wx'y'z	wx'yz	wx'yz'

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

Example: Simplify $m_0+m_2+m_5+m_8+m_{10}+m_{13}$

- The expression is already a sum of minterms, so here's the K-map:

1	0	0	1
0	1	0	0
0	1	0	0
1	0	0	1

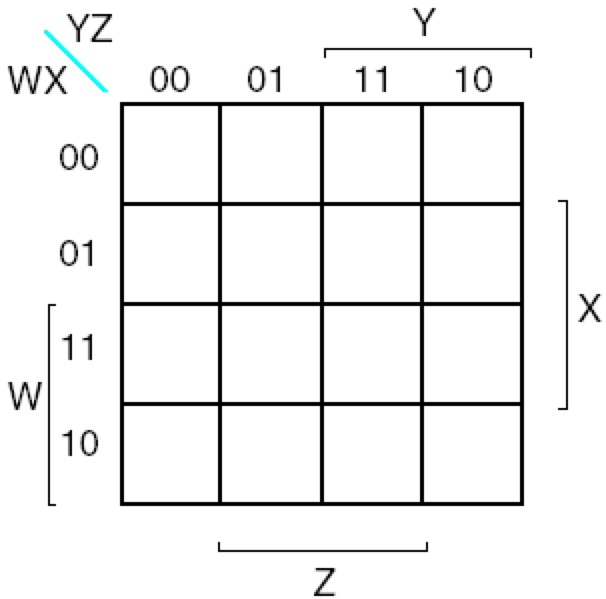
m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

- We can make the following groups, resulting in the MSP $x'z' + xy'z$

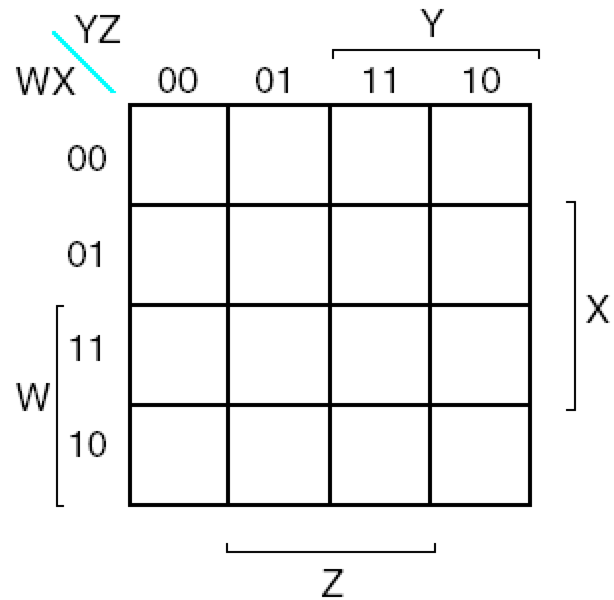
1	0	0	1
0	1	0	0
0	1	0	0
1	0	0	1

$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$
$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$
$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$
$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$

Five-variable K-maps – f(V,W,X,Y,Z)



V= 0



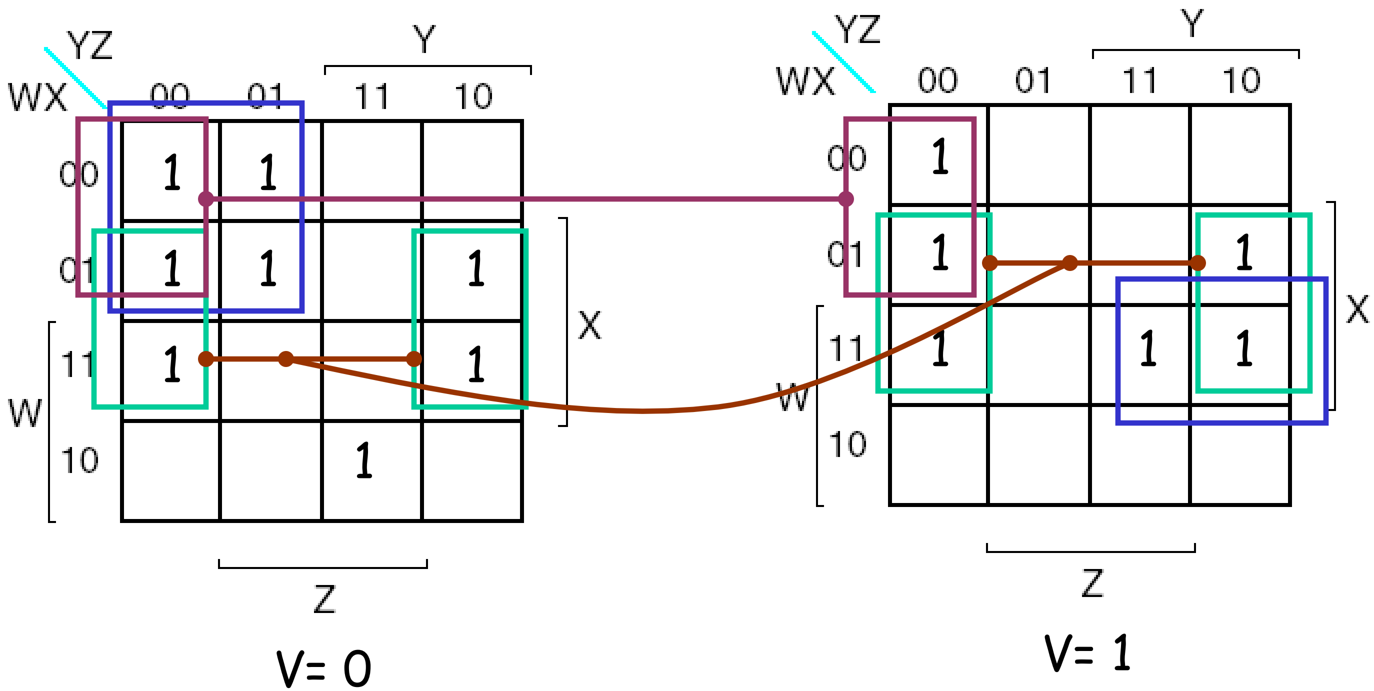
V= 1

	m_0	m_1	m_3	m_2
	m_4	m_5	m_7	m_6
W	m_{12}	m_{13}	m_{15}	m_{14}
W	m_8	m_9	m_{11}	m_{10}

(columns 3–4: y; rows 2–3: X; columns 2–3: Z)

	m_{16}	m_{17}	m_{19}	m_8
	m_{20}	m_{21}	m_{23}	m_{22}
W	m_{28}	m_{29}	m_{31}	m_{30}
W	m_{24}	m_{25}	m_{27}	m_{26}

Simplify f(V,W,X,Y,Z)=Σm(0,1,4,5,6,11,12,14,16,20,22,28,30,31)



$f = XZ'$ $\quad\quad$ $\Sigma m(4,6,12,14,20,22,28,30)$

$+ V'W'Y'$ $\quad\quad$ $\Sigma m(0,1,4,5)$

$+ W'Y'Z'$ $\quad\quad$ $\Sigma m(0,4,16,20)$

$+ VWXY$ $\quad\quad$ $\Sigma m(30,31)$

$+ V'WX'YZ$ $\quad\quad$ $m11$

PoS Optimization

- Maxterms are grouped to find minimal PoS expression

$x \backslash yz$	00	01	11	10
0	$x+y+z$	$x+y+z'$	$x+y'+z'$	$x+y'+z$
1	$x'+y+z$	$x'+y+z'$	$x'+y'+z'$	$x'+y'+z$

PoS Optimization

- $F(W,X,Y,Z) = \prod M(0,1,2,4,5)$

x \ yz	00	01	11	10
0	$x+y+z$	$x+y+z'$	$x+y'+z'$	$x+y'+z$
1	$x'+y+z$	$x'+y+z'$	$x'+y'+z'$	$x'+y'+z$

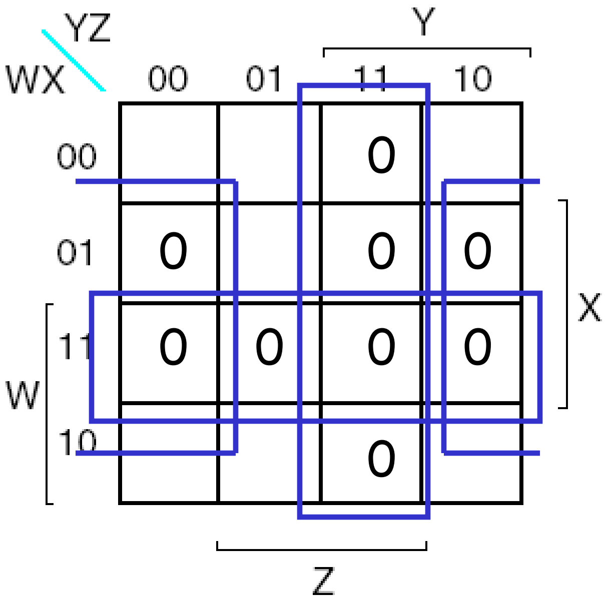
$F(W,X,Y,Z) = Y \cdot (X + Z)$

x \ yz	00	01	11	10
0	0	0	1	0
1	0	0	1	1

PoS Optimization from SoP

$F(W,X,Y,Z) = \Sigma m(0,1,2,5,8,9,10)$

$= \prod M(3,4,6,7,11,12,13,14,15)$



$F(W,X,Y,Z) = (W' + X')(Y' + Z')(X' + Z)$

Or,

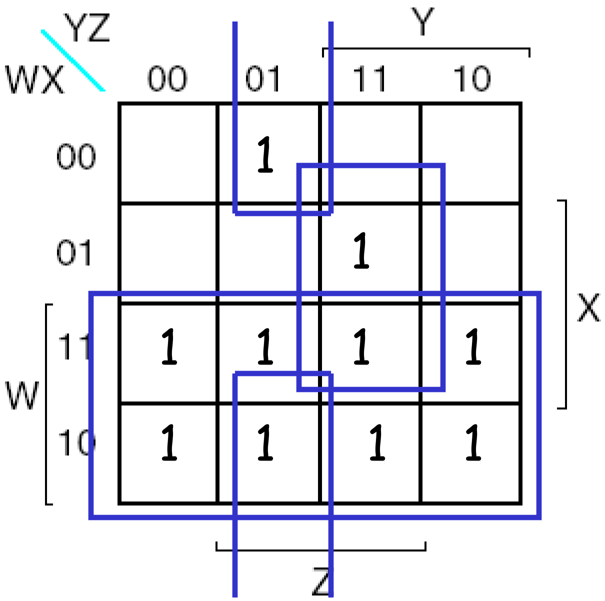
$F(W,X,Y,Z) = X'Y' + X'Z' + W'Y'Z$

Which one is the minimal one?

SoP Optimization from PoS

$F(W,X,Y,Z) = \prod M(0,2,3,4,5,6)$

$= \Sigma m(1,7,8,9,10,11,12,13,14,15)$



$F(W,X,Y,Z) = W + XYZ + X'Y'Z$

I don't care!

- You don't always need all 2^n input combinations in an n-variable function
  - If you can guarantee that certain input combinations never occur
  - If some outputs aren't used in the rest of the circuit

- We mark don't-care outputs in truth tables and K-maps with Xs.

x	y	z	f(x,y,z)
0	0	0	0
0	0	1	1
0	1	0	X
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	X
1	1	1	1

- Within a K-map, each X can be considered as either 0 or 1. You should pick the interpretation that allows for the most simplification.

Practice K-map

- Find a MSP for

$$f(w,x,y,z) = \Sigma m(0,2,4,5,8,14,15),\ d(w,x,y,z) = \Sigma m(7,10,13)$$

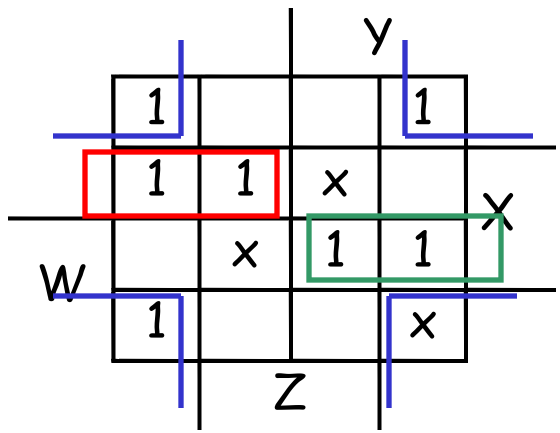
This notation means that input combinations wxyz = 0111, 1010 and 1101 (corresponding to minterms m_7, m_{10} and m_{13}) are unused.

			y		
	1	0	0	1	
	1	1	x	0	x
	0	x	1	1	
w	1	0	0	x	
		z			

Solutions for Practice K-map

- Find a MSP for:

$f(w,x,y,z) = \Sigma m(0,2,4,5,8,14,15)$, $d(w,x,y,z) = \Sigma m(7,10,13)$



$f(w,x,y,z) = x'z' + w'xy' + wxy$

K-map Summary

- K-maps are an alternative to algebra for simplifying expressions
  - The result is a MSP/MPS, which leads to a minimal two-level circuit
  - It's easy to handle don't-care conditions
  - K-maps are really only good for manual simplification of small expressions...
- Things to keep in mind:
  - Remember the correct order of minterms/maxterms on the K-map
  - When grouping, you can wrap around all sides of the K-map, and your groups can overlap
  - Make as few rectangles as possible, but make each of them as large as possible. This leads to fewer, but simpler, product terms
  - There may be more than one valid solution