Course: UMA 035 (Optimization Techniques)

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Alternative optimal solutions

If the optimal table, the value of $Z_j - C_j$ is 0 corresponding to a non-basic variable then alternative solutions of the problem may exist. To find the alternative solutions, enter such a non-basic variable.

Case (i) **If there exist a leaving variable then alternative solution exists.**

Case (ii) **If there does not exist a leaving variable then alternative solution does not exist.**

Example:

Construct a Simplex table for the following LPP by considering S_3, x_1 and x_2 as first, second and third basic variables respectively. Check that the obtained solution is optimal or not. If not then find the optimal solution. Also, find alternative solutions, if exist.

Max $(4x_1+10x_2)$

Subject to

$2x_1+x_2+S_1=10$

$2x_1+5x_2+S_2=20$

$2x_1+3x_2+S_3=18$

$x_1, x_2, S_1, S_2, S_3>=0.$

Solution:

$$\begin{array}{c} \begin{array}{ccc} S_3 & x_1 & x_2 \end{array} \\ B = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$$

$$B^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{5}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Since, S_3 is first basic variable so its column in table will be

1

0

0

and the corresponding value of $Z_j - C_j$ will be 0.

Since, x_1 is second basic variable so its column in table will be

0

1

0

and the corresponding value of $Z_j - C_j$ will be 0.

Since, x_2 is third basic variable so its column in table will be

0

0

1

and the corresponding value of $Z_j - C_j$ will be 0.

		4	10	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		0	0			0		
0	S_3	0	0			1		
4	x_1	1	0			0		
10	x_2	0	1			0		

Column of S_1

B^{-1}* Coefficients of S_1 in constraints

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{5}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{8} \\ -\frac{1}{4} \end{bmatrix}$$

Column of S_2

B^{-1}* Coefficients of S_2 in constraints

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{5}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{8} \\ \frac{1}{4} \end{bmatrix}$$

Column of Solution

B^{-1}* RHS

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{5}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{15}{4} \\ \frac{5}{2} \end{bmatrix}$$

		4	10	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Minimum Ratio

$Z_j - C_j$ =		0	0			0		
0	S_3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	
4	x_1	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$	
10	x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{5}{2}$	

		4	10	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		0	0	0	2	0		
0	S_3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	
4	x_1	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$	
10	x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{5}{2}$	

Optimal solution:

$S_3=3$

$x_1=\frac{15}{4}$

$x_2=\frac{5}{2}$

Remaining are 0 i.e., $S_1= S_2=0$

Optimal Value:

$4x_1+10x_2=4*\frac{15}{4}+10*\frac{5}{2}=40$

Basic variable **Non-Basic variable** **Basic Variable**

		4	10	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		0	0	0	2	0		
0	S_3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	
4	x_1	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$	
10	x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{5}{2}$	

It is obvious that S_1 is a non-basic variable and the value of $Z_j - C_j$ is 0 corresponding to it. So, alternative solution may exist.

		4	10	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		0	0	0	2	0		
0	S_3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	3/-
4	x_1	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$	$\frac{15}{4}/\frac{5}{8}$
10	x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{5}{2}$	$\frac{5}{2}$/-
$Z_j - C_j$ =		0	0	0	2	0		
0	S_3	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1	6	$6/\frac{4}{5}=\frac{30}{4}$
0	S_1	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	0	6	$6/\frac{8}{5}=\frac{30}{8}$
10	x_2	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	4	$4/\frac{2}{5}=\frac{20}{2}$
$Z_j - C_j$ =		0	0	0	2	0		
0	S_3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	
4	x_1	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$	
10	x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{5}{2}$	

Table 3 is same as Table 1. No need to proceed further as Tables will be repeated.

	First optimal solution	**Second optimal solution**	**Remaining optimal solutions**
x_1	$\frac{15}{4}$	0	$a_1(\frac{15}{4})+a_2(0)$
x_2	$\frac{5}{2}$	4	$a_1(\frac{5}{2})+a_2(4)$
S_1	0	6	$a_1(0)+a_2(6)$
S_2	0	0	$a_1(0)+a_2(0)$
S_3	3	6	$a_1(3)+a_2(6)$

where,

$a_1>=0$,

$a_2>=0$,

$a_1+ a_2=1$.

Simplex method/Big-M method/Two Phase method	Dual Simplex method
RHS should be positive or 0 in the starting table. (Solution is feasible)	RHS should be negative or 0 in the starting table. (Solution is not feasible)
$Z_j - C_j$ should be negative or 0 in the starting table (Solution is not optimal)	$Z_j - C_j$ should be positive or 0 in the starting table (Solution is optimal)
First find entering variable	First find leaving variable
That variable enters corresponding to which minimum of negative values of first row ($Z_j - C_j$) exist	That variable leaves corresponding to which minimum negative values of last column (solution column) values exist
That variable enters corresponding to which the ratio of the elements of last column (solution) and positive elements of the entering column is minimum.	That variable leaves corresponding to which the ratio of the elements of first row ($Z_j - C_j$) and negative elements of the leaving row is maximum.
Optimal solution when all elements of first row ($Z_j - C_j$) are greater than or equal to 0.	Optimal solution when all elements of last column (Solution column) are greater than or equal to 0.

Example

Solve the following LPP by dual simplex method.

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2>=3$

$4x_1+3x_2>=6$

$x_1+2x_2>=3,$

$x_1>=0,\ x_2>=0.$

Solution

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2-S_1=3$

$4x_1+3x_2-S_2=6$

$x_1+2x_2-S_3=3$

$x_1>=0,\ x_2>=0,\ S_1>=0,\ S_2>=0,\ S_3>=0\ .$

Maximize $(-2x_1-x_2)$

Subject to

$3x_1+x_2-S_1=3$

$4x_1+3x_2-S_2=6$

$x_1+2x_2-S_3=3$

$x_1>=0,\ x_2>=0,\ S_1>=0,\ S_2>=0,\ S_3>=0\ .$

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =								
		3	1	-1	0	0		
		4	3	0	-1	0		
		1	2	0	0	-1		

Maximize $(-2x_1 - x_2)$

Subject to

$-3x_1 - x_2 + S_1 = -3$

$-4x_1 - 3x_2 + S_2 = -6$

$-x_1 - 2x_2 + S_3 = -3$

$x_1 >= 0$, $x_2 >= 0$, $S_1 >= 0$, $S_2 >= 0$, $S_3 >= 0$.

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =								
		-3	-1	1	0	0		
		-4	-3	0	1	0		
		-1	-2	0	0	1		

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =								
	S_1	-3	-1	1	0	0		
	S_2	-4	-3	0	1	0		
	S_3	-1	-2	0	0	1		

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =								
	S_1	-3	-1	1	0	0	-3	
	S_2	-4	-3	0	1	0	-6	
	S_3	-1	-2	0	0	1	-3	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =								
0	S_1	-3	-1	1	0	0	-3	
0	S_2	-4	-3	0	1	0	-6	
0	S_3	-1	-2	0	0	1	-3	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =				0	0	0		
0	S_1	-3	-1	1	0	0	-3	
0	S_2	-4	-3	0	1	0	-6	
0	S_3	-1	-2	0	0	1	-3	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		2	1	0	0	0		
0	S_1	-3	-1	1	0	0	-3	
0	S_2	-4	-3	0	1	0	-6	
0	S_3	-1	-2	0	0	1	-3	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		2	1	0	0	0		
0	S_1	-3	-1	1	0	0	-3	
0	S_2	-4	-3	0	1	0	-6	
0	S_3	-1	-2	0	0	1	-3	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		2	1	0	0	0		Maximum
0	S_1	-3	-1	1	0	0	-3	$\left\{\frac{2}{-4}, \frac{1}{-3}\right\}$
0	S_2	-4	-3	0	1	0	-6	$= -\frac{1}{3}$
0	S_3	-1	-2	0	0	1	-3	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		2	1	0	0	0		Maximum
0	S_1	-3	-1	1	0	0	-3	$\left\{\frac{2}{-4}, \frac{1}{-3}\right\}$
0	S_2	-4	-3	0	1	0	-6	$= -\frac{1}{3}$
0	S_3	-1	-2	0	0	1	-3	

$R_1 \rightarrow R_1 - (1)(R_3/-3) \Rightarrow R_1 \rightarrow R_1 + \left(\frac{1}{3}\right)(R_3)$

$R_2 \rightarrow R_2 - (-1)(R_3/-3) \Rightarrow R_2 \rightarrow R_1 - \left(\frac{1}{3}\right)(R_3)$

$R_3 \rightarrow R_3 / (-3) \Rightarrow R_3 \rightarrow \left(\frac{1}{-3}\right)(R_3)$

$R_4 \rightarrow R_4 - (-2)(R_3/-3) \Rightarrow R_4 \rightarrow R_4 - \left(\frac{2}{3}\right)(R_3)$

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		$\frac{2}{3}$	0	0	$\frac{1}{3}$	0		
0	S_1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1	
-1	x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2	
0	S_3	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	1	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		$\frac{2}{3}$	0	0	$\frac{1}{3}$	0		
0	S_1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1	
-1	x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2	
0	S_3	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	1	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		$\frac{2}{3}$	0	0	$\frac{1}{3}$	0		Maximum $\left\{\frac{\frac{2}{3}}{-\frac{5}{3}}, \frac{\frac{1}{3}}{-\frac{1}{3}}\right\}$ $= \frac{\frac{2}{3}}{-\frac{5}{3}}$
0	S_1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1	
-1	x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2	
0	S_3	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	1	

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		$\frac{2}{3}$	0	0	$\frac{1}{3}$	0		
0	S_1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1	
-1	x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2	
0	S_3	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	1	

$R_1 \to R_1 - \left(\frac{2}{3}\right)\left(R_2/\left(-\frac{5}{3}\right)\right) \Rightarrow R_1 \to R_1 + \left(\frac{2}{5}\right)(R_2)$

$R_2 \to R_2/\left(-\frac{5}{3}\right) \Rightarrow R_2 \to -\left(\frac{3}{5}\right)(R_2)$

$R_3 \to R_3 - \left(\frac{4}{3}\right)\left(R_2/\left(-\frac{5}{3}\right)\right) \Rightarrow R_3 \to R_3 + \left(\frac{4}{5}\right)(R_2)$

$R_4 \to R_4 - \left(\frac{5}{3}\right)\left(R_2/\left(-\frac{5}{3}\right)\right) \Rightarrow R_4 \to R_4 + (R_2)$

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		0	0	$\frac{2}{5}$	$\frac{1}{5}$	0		
-2	x_1	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	
-1	x_2	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	$\frac{6}{5}$	
0	S_3	0	0	1	-1	1	0	

Optimal solution:

$x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, $S_3 = 0$

Remaining are 0 i.e., $S_1 = S_2 = 0$

Optimal value is $2x_1 + x_2 = 2\left(\frac{3}{5}\right) + \frac{6}{5} = \frac{12}{5}$

Pattern for examination

		-2	-1	0	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	S_3	Solution	Maximum Ratio
$Z_j - C_j$ =		2	1	0	0	0		Maximum $\left\{\frac{2}{-4}, \frac{1}{-3}\right\}$ $= -\frac{1}{3}$
0	S_1	-3	-1	1	0	0	-3	
0	S_2	-4	-3	0	1	0	-6	
0	S_3	-1	-2	0	0	1	-3	
$Z_j - C_j$ =		$\frac{2}{3}$	0	0	$\frac{1}{3}$	0		Maximum $\left\{\frac{\frac{2}{3}}{-\frac{5}{3}}, \frac{\frac{1}{3}}{-\frac{1}{3}}\right\}$ $= \frac{\frac{2}{3}}{-\frac{5}{3}}$
0	S_1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1	
-1	x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2	
0	S_3	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	1	
$Z_j - C_j$ =		0	0	$\frac{2}{5}$	$\frac{1}{5}$	0		
-2	x_1	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	
-1	x_2	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	$\frac{6}{5}$	
0	S_3	0	0	1	-1	1	0	

Optimal solution:

$x_1=\frac{3}{5}$, $x_2=\frac{6}{5}$, $S_3=0$

Remaining are 0 i.e., $S_1= S_2=0$

Optimal value is $2x_1+x_2=2(\frac{3}{5})+\frac{6}{5}=\frac{12}{5}$

Example

Solve the following LPP by dual simplex method.

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2=3$

$4x_1+3x_2>=6$

$x_1+2x_2>=3,$

$x_1>=0,\ x_2>=0.$

Solution

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2>=3$

$3x_1+x_2<=3$

$4x_1+3x_2>=6$

$x_1+2x_2>=3,$

$x_1>=0,\ x_2>=0.$

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2-S_1=3$

$3x_1+x_2+S_2=3$

$4x_1+3x_2-S_3=6$

$x_1+2x_2-S_4=3$

$x_1>=0,\ x_2>=0,\ S_1>=0,\ S_2>=0,\ S_3>=0,\ S_4>=0$.

Maximize $(-2x_1-x_2)$

Subject to

$3x_1+x_2-S_1=3$

$3x_1+x_2+S_2=3$

$4x_1+3x_2-S_3=6$

$x_1+2x_2-S_4=3$

$x_1>=0,\ x_2>=0,\ S_1>=0,\ S_2>=0,\ S_3>=0,\ S_4>=0$.

Maximize $(-2x_1-x_2)$

Subject to

$-3x_1-x_2+S_1=-3$

$3x_1+x_2+S_2=3$

$-4x_1-3x_2+S_3=-6$

$-x_1-2x_2+S_4=-3$

$x_1>=0,\ x_2>=0,\ S_1>=0,\ S_2>=0,\ S_3>=0,\ S_4>=0$.

DO YOURSELF