Deflections in Beams –Double Integration Method

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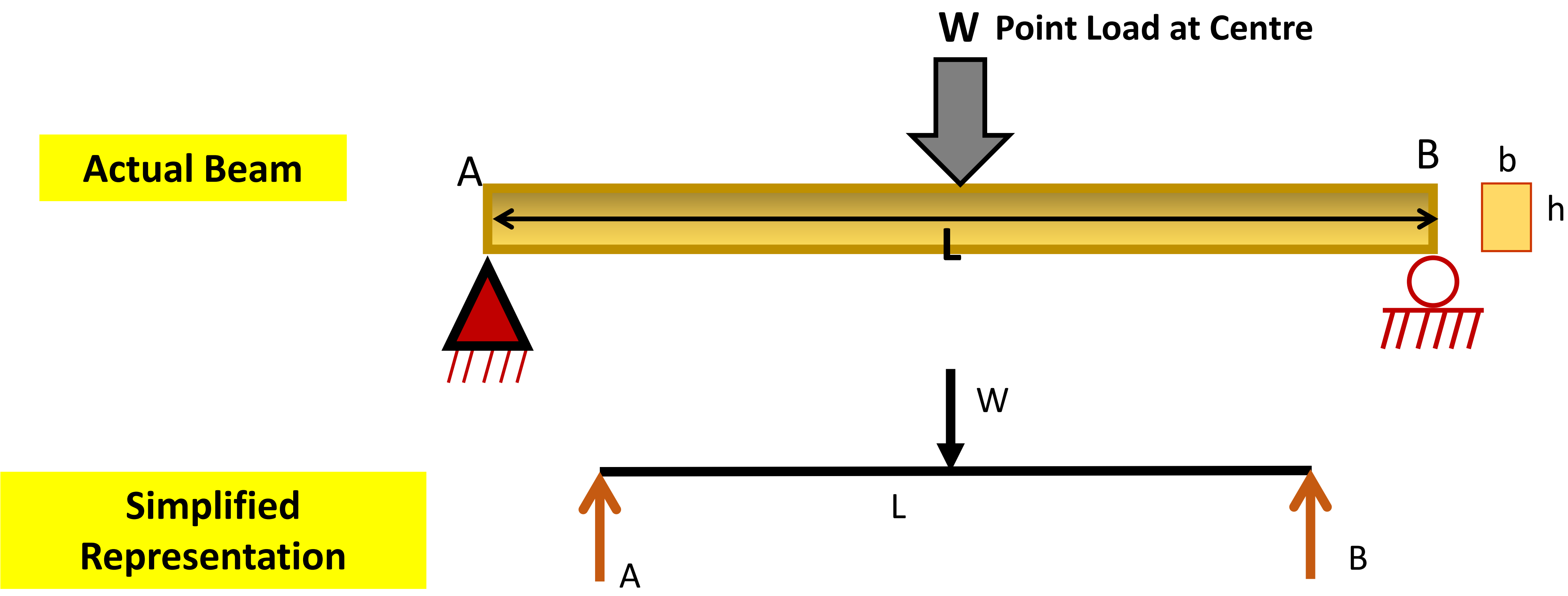
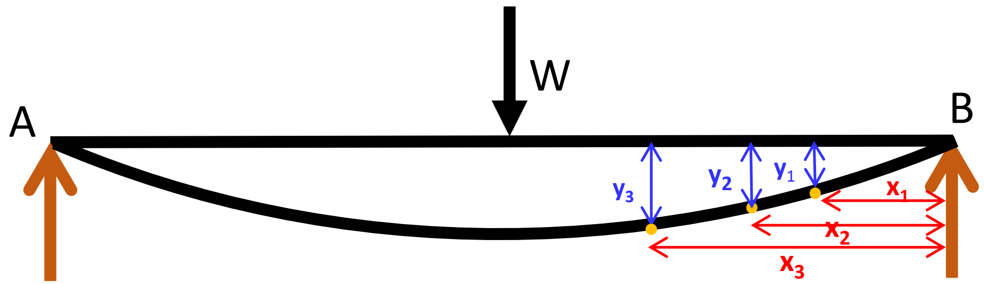
DEFLECTIONS OF BEAMS



- A loaded beam is initially straight but no longer remains straight.
- In due course of time, there is a deflection of its axis caused by loads acting on it.

Deflections in a Simply Supported Beam

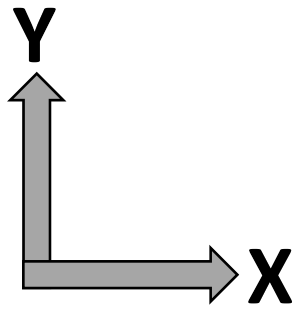
Deflected Beam



At location **(x_1)**, Vertical deflection **(y_1) = ?**

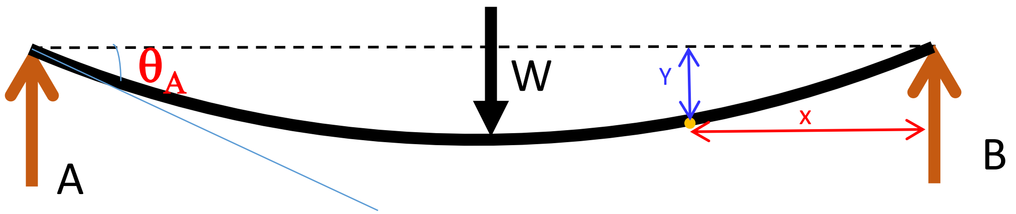
At location **(x_2)**, Vertical deflection **(y_2) = ?**

At location **(x_3)**, Vertical deflection **(y_3) = ?**



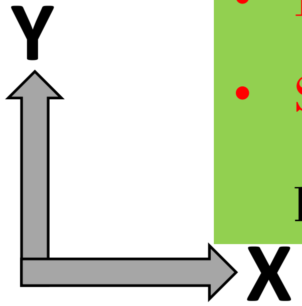
Deflection: Shifting of a point from its initial position in the transverse direction

Deflection in a Simply Supported Beam



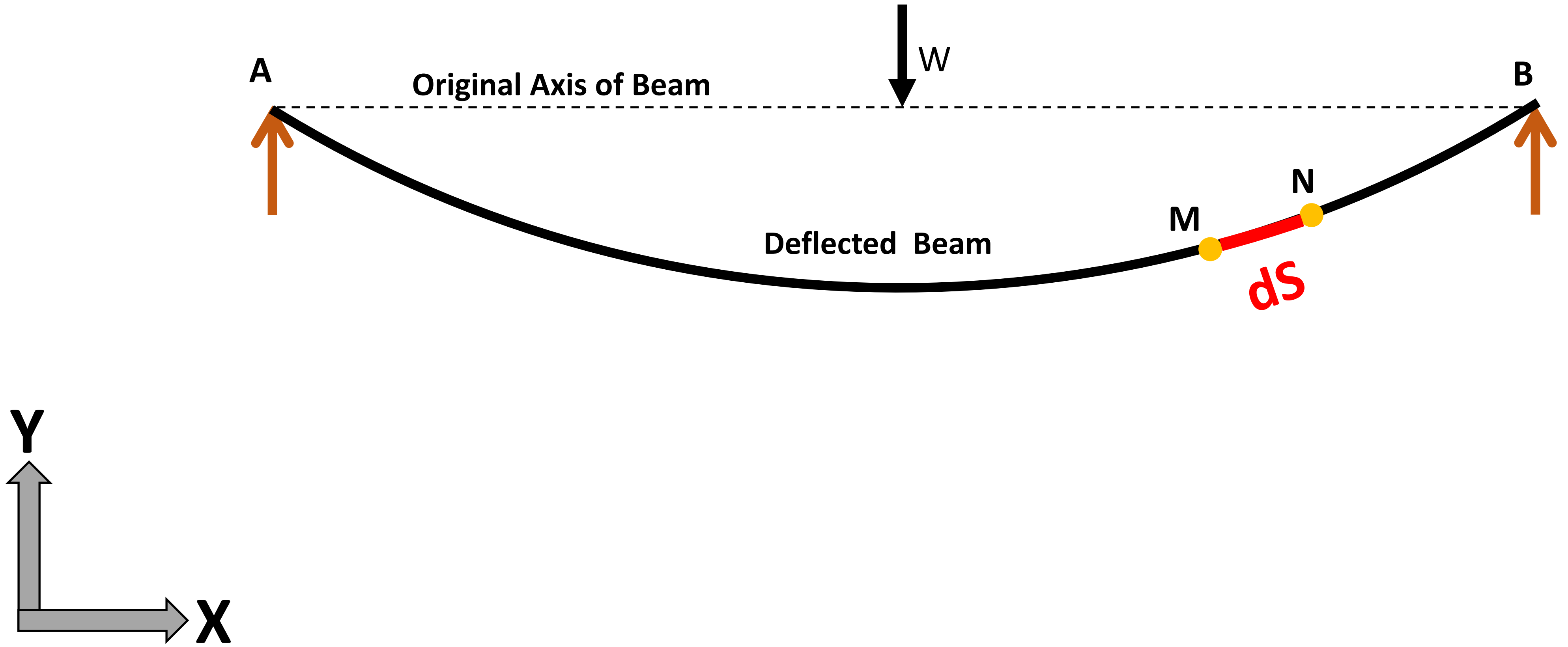
Find the vertical deflection **(Y)** in the beam at any given location **(x)**

$$Y = f(x)$$

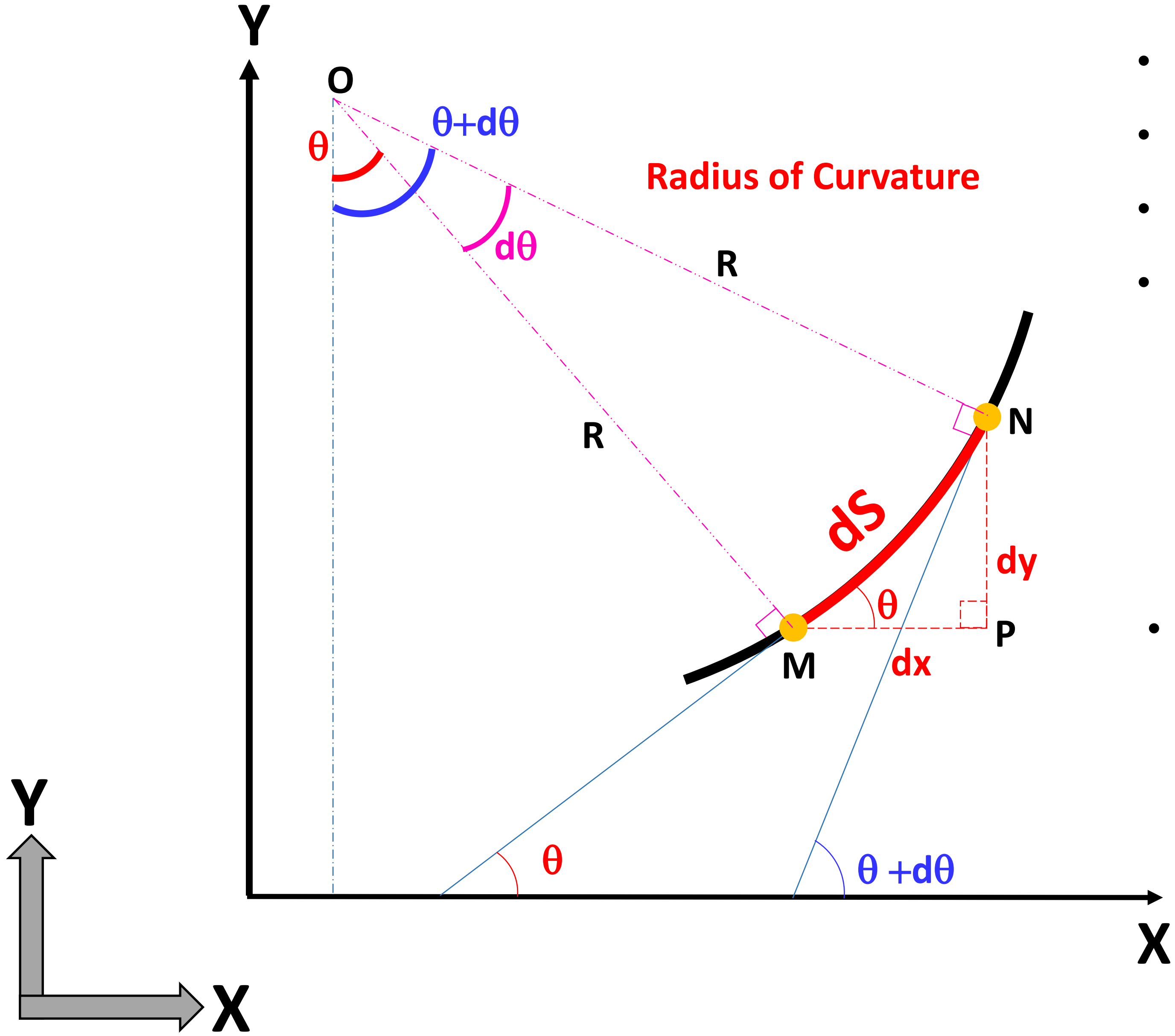


- Deflection is the shifting of a point from its initial position in the transverse direction
- Slope at a point is the angle made by the tangent drawn on the deflected beam at that point with the original axis of the beam

Relationship between Slope, Deflection & Radius of Curvature



Relationship between Slope, Deflection & Radius of Curvature



- Consider a very small portion (MN) of the beam.
- Draw tangents at point M & N.
- Let the slope at M and N be θ and θ+dθ respectively.
- Draw normal at Point M and N intersecting each other at O.

For Small values of angle $d\theta$

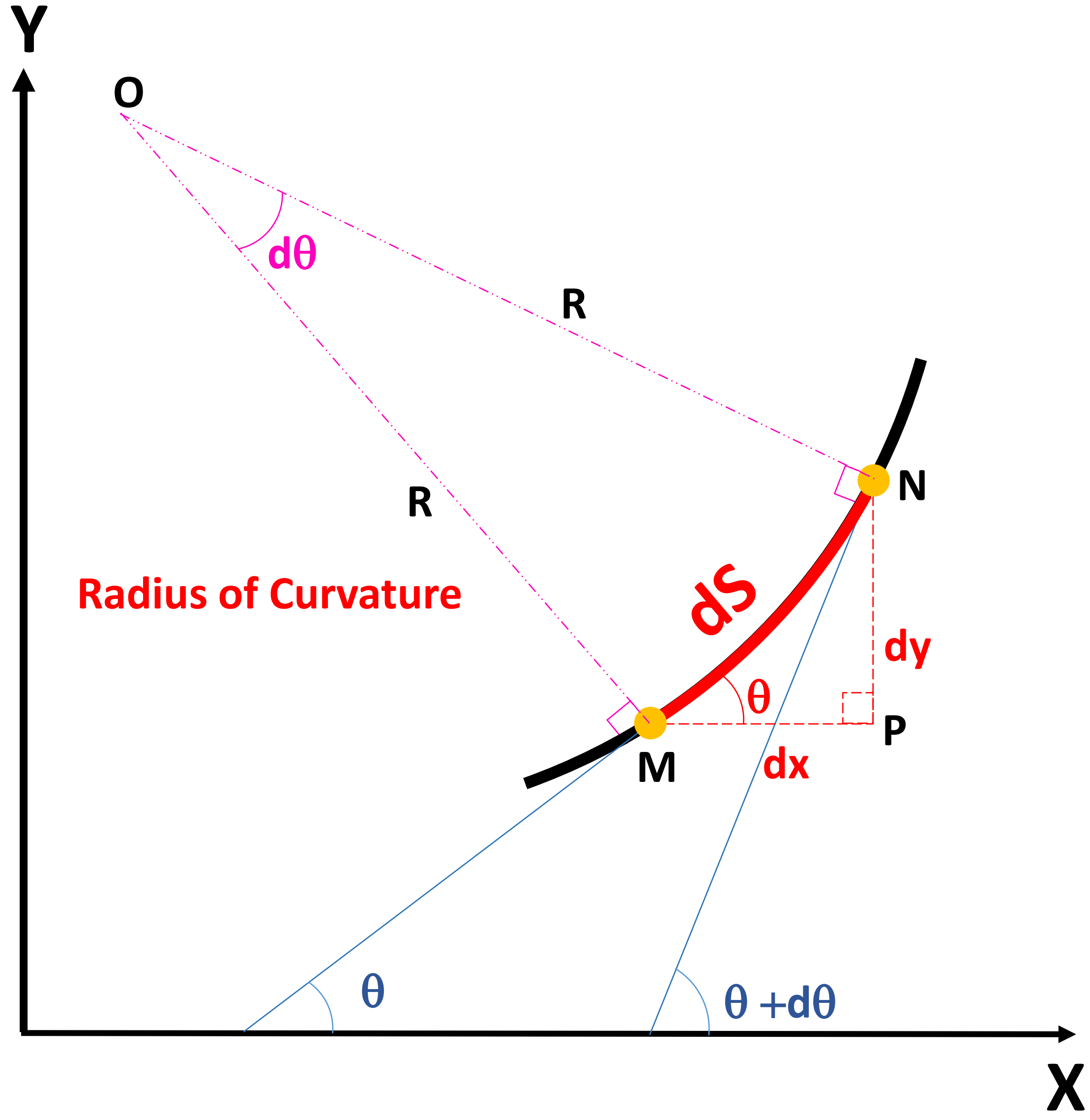
$Arc\ MN \approx Chord\ MN$= dS

- In △ MPN,

$$\sin\theta = \frac{dy}{dS}\ , \qquad \cos\theta = \frac{dx}{dS}, \qquad \tan\theta = \frac{dy}{dx}$$

Since, $Angle = \frac{Length\ of\ Arc}{Radius}$

Relationship between Slope, Deflection & Radius of Curvature



$$\therefore\ d\theta = \frac{MN}{Radius} = \frac{dS}{R}$$

$$\frac{1}{R} = \frac{d\theta}{dS}$$

Divide numerator & denominator with dx

$$\frac{1}{R} = \frac{d\theta/_{dx}}{dS/_{dx}} = \frac{-\,d\theta/_{dx}}{sec\theta}$$

From △ MNP, $tan\theta = \frac{dy}{dx}$

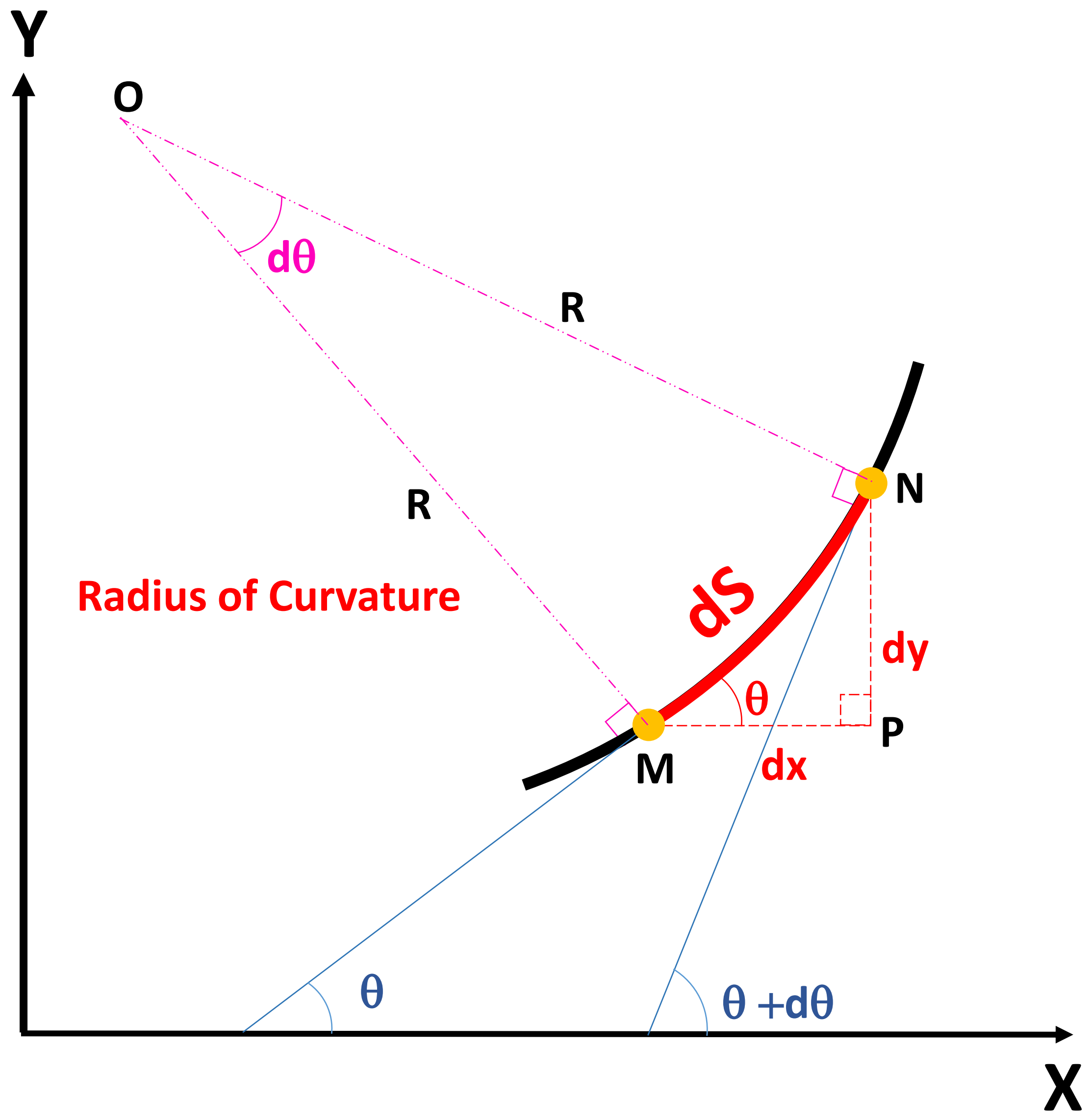
Differentiating w.r.t. x

$$sec^2\theta.\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\therefore\ \frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{sec^2\theta}$$

Substitute $\frac{d\theta}{dx}$

Relationship between Slope, Deflection & Radius of Curvature



$$\frac{1}{R} = \frac{-\,{d\theta}/{dx}}{sec\theta} = \frac{-\frac{d^2y}{dx^2}}{sec^3\theta}$$

$$\frac{1}{R} = \frac{-\frac{d^2y}{dx^2}}{(sec^2\theta)^{3/2}} = \frac{-\frac{d^2y}{dx^2}}{(1+tan^2\theta)^{3/2}}$$

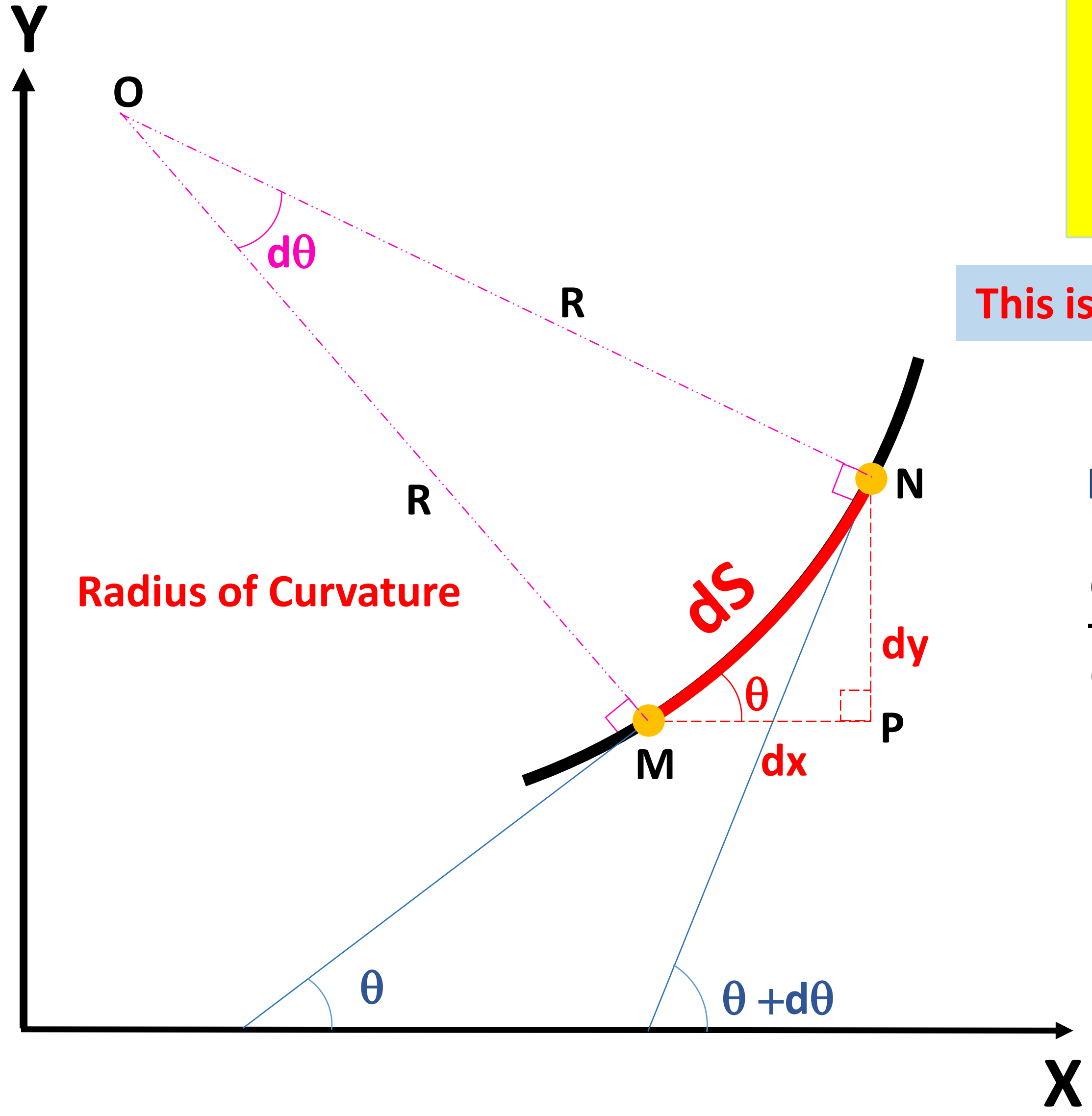
For very small value of θ, $tan^2\theta$ is extremely small and can be neglected

$$\Rightarrow \frac{1}{R} = -\frac{d^2y}{dx^2}$$

$$\frac{M}{I} = \frac{\sigma}{Y} = \frac{E}{R}$$ {Pure Bending Moment Equation}

$$EI\frac{d^2y}{dx^2} = -M$$

Slope and Deflection Equations



$$EI\frac{d^2y}{dx^2} = -M$$ .... Eqn. 1

This is Differential Equation of Flexure

Where EI- Flexural Rigidity

Integrate Eqn. 1

$$\frac{dy}{dx} = -\frac{1}{EI}\int M\,dx$$ .... Eqn. 2

Slope equation

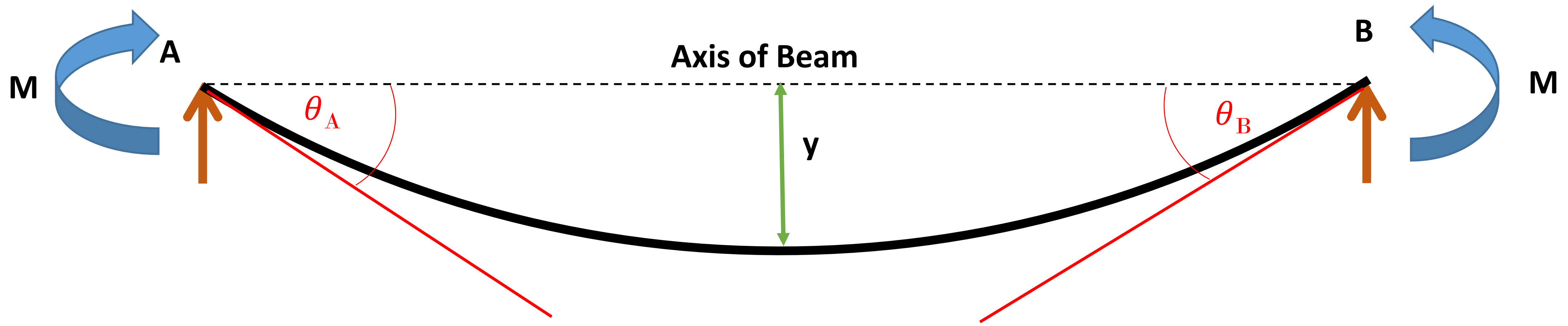
Integrate Eqn. 2

$$y = -\frac{1}{EI}\int\left(\int M\,dx\right)dx$$

Deflection equation

This method of finding Slopes and Deflections is called "Double Integration Method"

Sign Conventions to be Followed

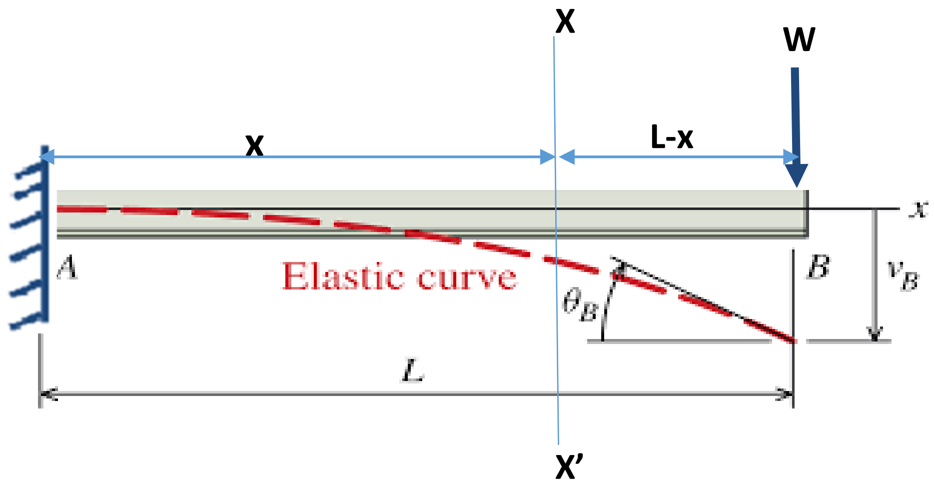


- Distances measured along the length of the beam are treated as positive when measured from left to right and negative when measured from right to left.
- Downward deflections are assumed to be positive.
- Slope is negative when the tangent makes an angle with the axis of the beam in the anti clockwise direction and positive when the angle is made in the clockwise direction.
- In the above figure M, θ_A and y are positive and θ_B is negative

Double Integration Method

CANTILEVER BEAMS
Case I: Carrying a point load at the end

Consider a cantilever beam 'AB' of length 'L' carrying a point **load** 'W' at the free end 'B'



❖ Aim is to determine the slope and deflection at the free end i.e y_B and θ_B **at the free end 'B'**

❖ Slope and deflection are zero at the fixed end

Step 1: Consider the origin at Fixed End 'A'
Step 2: Write the equation of Bending Moment (M) at any section at a section XX' at a distance 'x' from the fixed end

M = - W (L-x) ....(1)

Step 3: Apply the Differential Equation of Flexure

$$\boxed{EI\frac{d^2y}{dx^2} = -M}$$ ....(2)

Substituting the Value of 'M' from (1) in (2)

$$EI\frac{d^2y}{dx^2} = -[-W(L-x)]$$

$$EI\frac{d^2y}{dx^2} = W(L-x)$$

Step 4: Integrating the above equation to get θ i.e. dy/dx

$$EI\frac{dy}{dx} = WLx - \frac{Wx^2}{2} + C_1$$ ....(3)

Step 5: Integrating equation (3) again to determine Deflection i.e. 'y' at any point on the beam (Double Integration)

$$EIy = \frac{WLx^2}{2} - \frac{Wx^3}{6} + C_1 x + C_2 \qquad \text{..................(4)}$$

where C_1 and C_2 are constants of integration

Step 6: Determine the constants of integration C_1 and C_2 by applying boundary conditions

Boundary Conditions: At Fixed End i.e. at x= 0, dy/dx =0 and y =0 in (3) and (4)

$$\Rightarrow \boxed{C_1 = C_2 = 0} \qquad \text{..................(5)}$$

Substituting $C_1 = C_2 = 0$ in (3) and (4) to get final equations for Slope and Deflection

$$\frac{dy}{dx} = \frac{1}{EI}\left[WLx - \frac{W(x)^2}{2}\right] \quad \text{..................(6)}$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{2} - \frac{Wx^3}{6}\right] \quad \text{..................(7)}$$

To get slope and deflection at Free End 'B' , substitute x= L in (6) and (7) equations

$$\theta_B = \left(\frac{dy}{dx}\right)_{at\ x=L\ at\ B} = \frac{1}{EI}\left[\frac{WL^2}{2}\right] = \frac{WL^2}{2EI}$$

$$\left(y_{at\ x=L\ at\ B}\right) = \frac{1}{EI}\left[\frac{WL^3}{3}\right] = \frac{WL^3}{3EI}$$

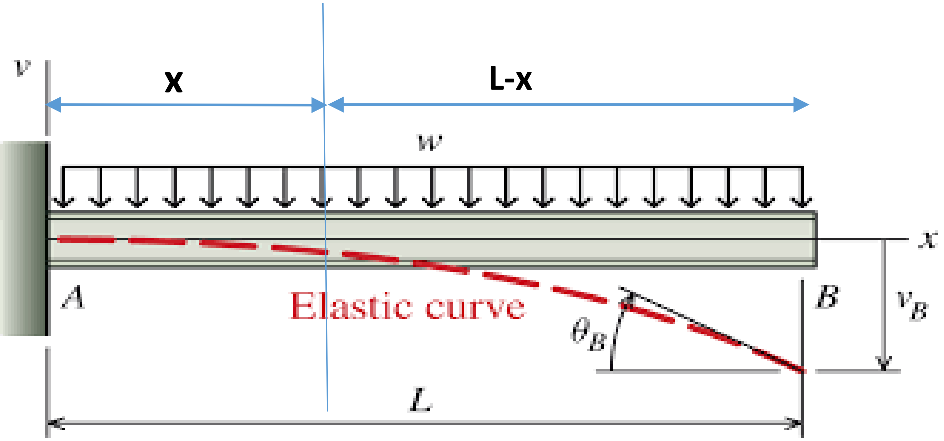
CANTILEVER BEAMS

Case II: Carrying a uniformly distributed load ('w' kN/m) throughout the length of the beam

Step 1: Consider the origin at Fixed End 'A'

Step 2: Write the equation of Bending Moment (M) at any section at a section XX' at a distance 'x' from the fixed end

$$M = -\ w\ (L-x)\frac{(L-x)}{2} = -w\frac{(L-x)^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$



Step 3: Apply the Differential Equation of Flexure

$$\boxed{EI\frac{d^2y}{dx^2} = -M} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting the Value of 'M' from (1) in (2)

$$EI\frac{d^2y}{dx^2} == w\frac{(\mathrm{L}-\mathrm{x})^2}{2}$$

Step 4: Integrating the above equation to get θ i.e. dy/dx

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{w(L-x)^3}{6} + C_1\right] \qquad \text{..................(3)}$$

This Equation (3) is used to get Slope 'θ' at any desired point on the beam.

Step 5: Integrating equation (3) again to determine Deflection i.e. 'y' at any point on the beam (Double Integration)

$$y = \frac{1}{EI}\left[-\frac{w(L-x)^4}{24} + C_1x + C_2\right] \quad \text{..................(4)}$$

Step 6: Determine the constants of integration C_1 and C_2 by applying boundary conditions

Boundary Conditions: At Fixed End i.e. at x= 0, dy/dx =0 and y =0 in (3) and (4), we get

$$C_1 = \frac{wL^3}{6} \quad \text{..................(5)}$$

$$C_2 = \frac{-wL^4}{24} \quad \text{..................(6)}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{w(L-x)^3}{6} + \frac{wL^3}{6}\right] \quad \text{..................(7)}$$

$$y = \frac{1}{EI}\left[-\frac{w(L-x)^4}{24} + \frac{wL^3}{6}x - \frac{-wL^4}{24}\right] \quad \text{..................(8)}$$

To get slope and deflection at Free End 'B' , substitute x= L in (7) and (8) equations

$$\theta_{\mathrm{B}} = \left(\frac{dy}{dx}\right)_{at\ x=L\ at\ B} = \frac{1}{EI}\left[\frac{wL^3}{6}\right] = \frac{wL^3}{6EI}$$

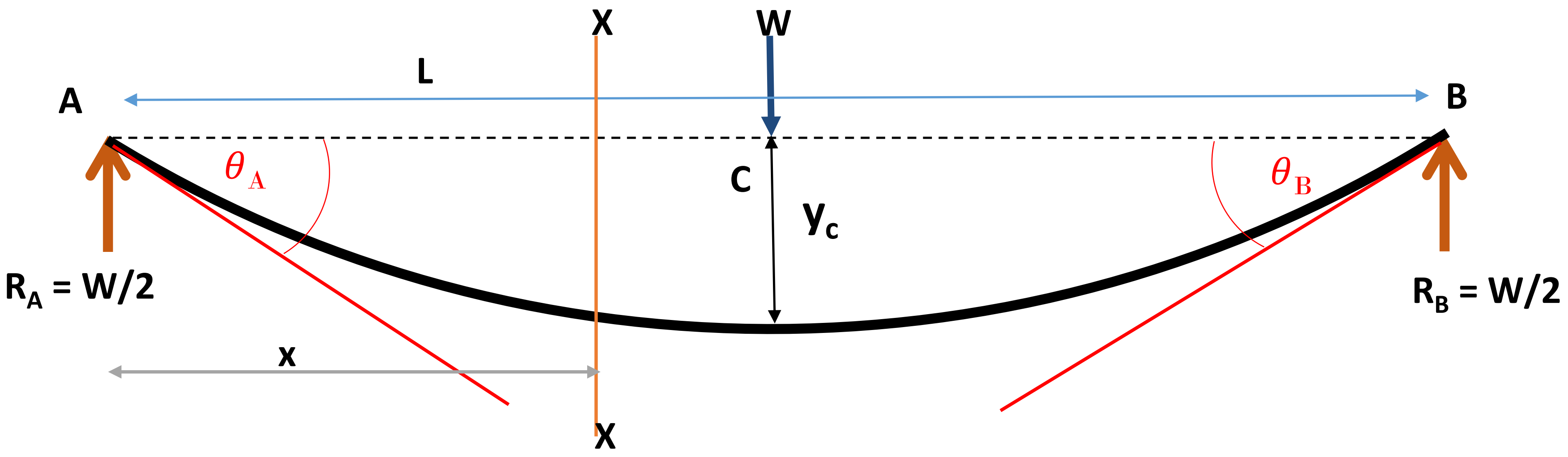
$$\left(y_{at\ x=L\ at\ B}\right) = \frac{1}{EI}\left[\frac{wL^4}{8}\ \right] = \frac{wL^4}{8EI}$$

Summary of Steps in Double Integration Method

1. Using Statics -- Find the Support Reactions
2. Establish coordinates for writing BM equation
3. Write the Equation of BM at a section 'XX'
4. Substitute the equation of BM in Differential Equation of Flexure
5. Integrate twice to get Equations of Slope and Deflection at a Point
6. Apply the Boundary conditions to get Constants of Integration
7. Solve for calculating Slope and Deflection at any desired point
8. Generally it is desired to obtain Maximum Slope and Maximum Deflections due to applied loading on various kinds of beams

SIMPLY SUPPORTED BEAMS

Case I: Carrying a point load 'W' at the Centre



Step 1: Using Statics -- Find the Support Reactions

$\sum V = 0 \Longrightarrow R_A + R_B = W$

$\sum M_A = 0 \Longrightarrow -W\ L/2 + R_B\ L = 0 \Longrightarrow \boxed{R_B = R_A = W/2}$

Step 2: Establish coordinates for writing BM equation

Consider Origin at 'A'

Step 3: Write the Equation of BM at a section 'XX'

$$M = R_A\, x = \frac{W}{2}\, x \qquad \text{.................(1)}$$

Step 4: Apply the Differential Equation of Flexure

$$\boxed{EI\,\frac{d^2y}{dx^2} = -M} \qquad \text{.................(2)}$$

Substituting the Value of 'M' from (1) in (2)

$$EI\frac{d^2y}{dx^2} = -\frac{W\,\mathrm{x}}{2}$$

Step 5: Integrating the above equation to get θ i.e. dy/dx

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{W(x)^2}{4} + C_1\right] \quad \text{.................(3)}$$

This Equation (3) is used to get Slope 'θ' at any desired point on the beam.

Step 5: Integrating equation (3) again to determine Deflection i.e. 'y' at any point on the beam (Double Integration)

$$y = \frac{1}{EI}\left[-\frac{W(x)^3}{12} + C_1 x + C_2\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Step 6: Determine the constants of integration C_1 and C_2 by applying boundary conditions

Boundary Conditions: At x= L/2, dy/dx =0 and at x = 0, y= 0 in (3) and (4), we get

$$C_1 = \frac{WL^2}{16} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$C_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{W(x)^4}{4} + \frac{WL^2}{16}\right] \qquad \text{..................(7)}$$

$$y = \frac{1}{EI}\left[-\frac{W(x)^3}{12} + \frac{WL^2}{16}x\right] \qquad \text{..................(8)}$$

To get slope and deflection at any point , substitute value of x in (7) and (8) equations

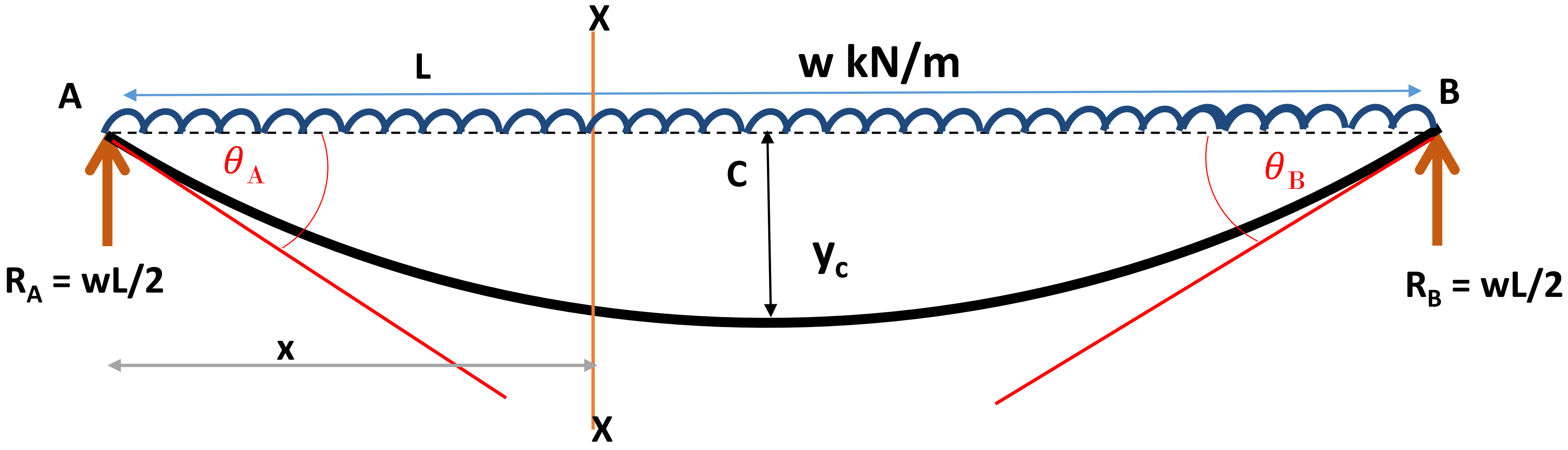
For slope at 'A' , put x=0 in (7) equation

$$\theta_A = \left(\frac{dy}{dx}\right)_{at\ x=0\ at\ A} = \frac{1}{EI}\left[\frac{WL^2}{16}\right] = \frac{WL^2}{16EI} = -\ \theta_B$$

To get deflection at centre i.e. at 'C' , substitute x= L/2 in (8) equation

$$\left(y_{at\ x=L/2\ at\ C}\right) = \frac{1}{EI}\left[-\frac{WL^3}{96} + \frac{WL^3}{32}\right] = \frac{WL^3}{48EI}$$

Case II: Carrying a uniformly distributed load (‘w’ kN/m) throughout the length of the beam



Step 1: Using Statics -- Find the Support Reactions

$\sum V = 0 \longrightarrow R_A + R_B = (wx)/2$

$\sum M_A = 0 \longrightarrow -W\, L^2/2 + R_B\, L = 0 \longrightarrow \boxed{R_B = R_A = wL2}$

Step 2: Establish coordinates for writing BM equation

Consider Origin at 'A'

Step 3: Write the Equation of BM at a section 'XX'

$$Mx = R_A x - \frac{W}{2} x^2 = \frac{W}{2} L - \frac{W}{2} x^2 \quad \text{.................(1)}$$

Step 4: Apply the Differential Equation of Flexure

$$\boxed{EI \frac{d^2 y}{dx^2} = -M} \quad \text{.................(2)}$$

Substituting the Value of 'M' from (1) in (2)

$$EI\frac{d^2y}{dx^2} = -\frac{wL}{2}\mathrm{x} + \frac{w}{2}\ \mathrm{x}^2$$

Step 5: Integrating the above equation to get θ i.e. dy/dx

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{w\,L(x)^2}{4} + \frac{wx^3}{6} + C_1\right] \quad \text{..................(3)}$$

This Equation (3) is used to get Slope 'θ' at any desired point on the beam.

Step 5: Integrating equation (3) again to determine Deflection i.e. ‘y’ at any point on the beam (Double Integration)

$$y = \frac{1}{EI}\left[-\frac{w\,L(x)^3}{12} + \frac{w\,(x)^4}{24} + C_1 x + C_2\right]$$ ………………..(4)

Step 6: Determine the constants of integration C_1 and C_2 by applying boundary conditions

Boundary Conditions: At x= L/2, dy/dx =0 and at x = 0, y= 0 in (3) and (4), we get

$$C_1 = \frac{wL^3}{24}$$ ………………..(5)

$$C_2 = 0$$ ………………..(6)

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{wL(x)^2}{4} + \frac{w(x)^3}{6} + \frac{wL^3}{24}\right]$$ ....(7)

$$y = \frac{1}{EI}\left[-\frac{wL(x)^3}{12} + \frac{w(x)^4}{24} + \frac{wL^3}{24}x\right]$$ ....(8)

To get slope and deflection at any point , substitute value of x in (7) and (8) equations

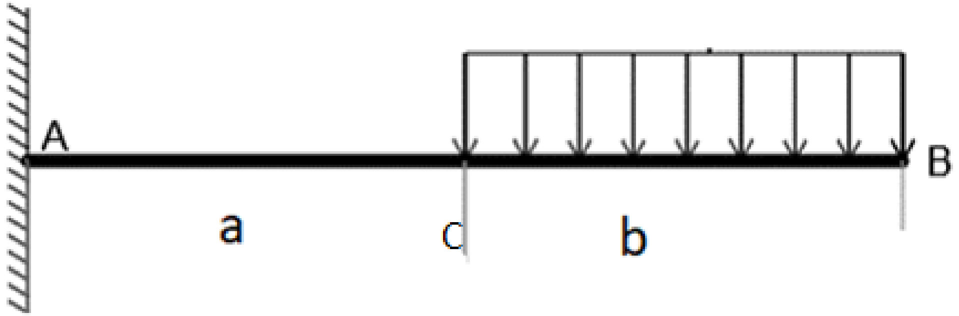
For slope at 'A' , put x=0 in (7) equation

$$\theta_A = \left(\frac{dy}{dx}\right)_{at\ x=0\ at\ A} = \frac{1}{EI}\left[\frac{wL^3}{24}\right] = \frac{wL^3}{24EI} = -\ \theta_B$$

To get deflection at centre i.e. at ‘C’ , substitute x= L/2 in (8) equation

$$\left(y_{at\ x=L/2\ at\ C}\right) = \frac{1}{EI}\left[-\frac{wL^4}{96} + \frac{wL^4}{384} + \frac{wL^4}{48}\right] = \frac{5wL^4}{384\ EI}$$

Concept of Discontinuity



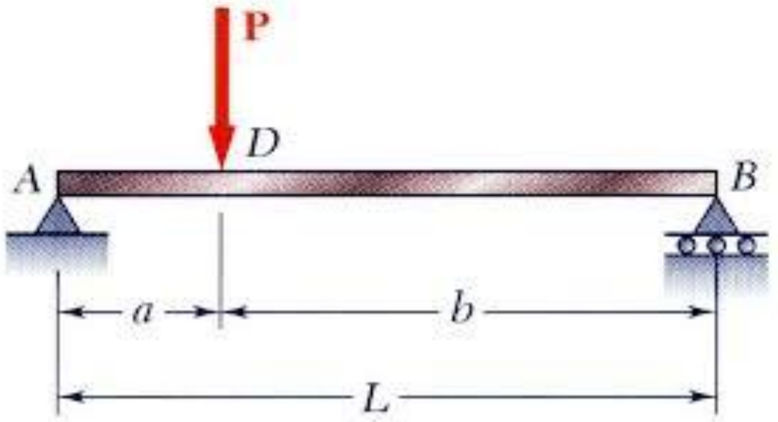
- Consider a beam AB as shown above with BM equation in Span AC different from Span CB
- There will be two different equations for two different spans for calculating deflections

$$EI\frac{d^2y}{d^2x} = -M$$

$$M_{AC} = R_A x - M_A$$

$$M_{CB} = R_A x - M_A - \frac{w(x-a)^2}{2}$$

Concept of Discontinuity



Step 1: Find reactions

$$R_B L - Pa = 0$$

$$R_B = \frac{Pa}{L} \qquad R_A = \frac{Pb}{L}$$

Step 2 : Assign coordinates for moment function

Origin at A

Region AD (0, a)

Region DB (a, L)

Step 3: Write BM equations in two spans and apply to Differential Equation of Flexure

Span AD

$$M_x = R_A x$$

$$EI\frac{d^2y}{d^2x} = -M$$

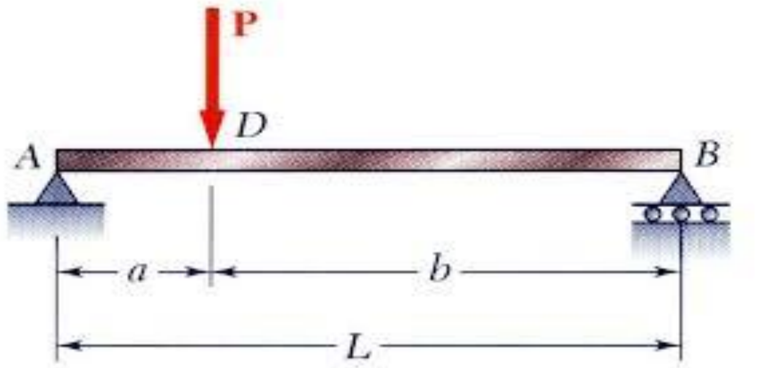
$$EI\frac{d^2y}{d^2x} = -R_A x = -\frac{Pbx}{L}$$

Span DB

$$M_x = R_A x - P(x-a)$$

$$EI\frac{d^2y}{d^2x} = -M$$

$$EI\frac{d^2y}{d^2x} = -R_A x + P(x-a)$$



Step 4: Integrate

$$EI\frac{dy}{dx} = -\frac{Pbx^2}{2L} + C_1$$

$$EIy = -\frac{Pbx^3}{6L} + C_1x + C_2$$

$$EI\frac{dy}{dx} = -\frac{Pbx^2}{L2} + \frac{P(x-a)^2}{2} + C_3$$

$$EIy = -\frac{Pbx^3}{L6} + \frac{P(x-a)^3}{6} + C_3x + C_4$$

Step 4 Determine the Constants of Integration by applying BC's

At x=0; y=0

$$EIy = -\frac{Pbx^3}{6L} + C_1 x + C_2 \quad \Longrightarrow \quad C_2 = 0$$

Slope and deflection at point D can be found using both sets of equations

Using Slope and Deflection Equations in both spans

$$\left| -\frac{Pbx^2}{2L} + C_1 \right|_{at\ x=a} = \left| -\frac{Pbx^2}{L2} + \frac{P(x-a)^2}{2} + C_3 \right|_{at\ x=a} \quad \Longrightarrow \quad C_1 = C_3$$

$$\left| -\frac{Pbx^3}{6L} + C_1 x + C_2 \right|_{at\ x=a} = \left| -\frac{Pbx^3}{L6} + \frac{P(x-a)^3}{6} + C_3 x + C_4 \right|_{at\ x=a} \quad \Longrightarrow \quad C_2 = C_4$$

$$C_2 = C_4$$

$$C_2 = 0 \Longrightarrow C_4 = 0$$

At x=L ; y= 0

Substitute in Deflection Equation

$$-\frac{Pbx^3}{L6} + \frac{P(x-a)^3}{6} + C_3 x + C_4 = 0$$

$$-\frac{PbL^3}{L6} + \frac{P(b)^3}{6} + C_3 L = 0 \Longrightarrow C_3 = \frac{Pb(L^2 - b^2)}{6L}$$

$$\Longrightarrow C_1 = \frac{Pb(L^2 - b^2)}{6L}$$

FINAL SLOPE AND DEFLECTION EQUATIONS

Span AD

$$EI\frac{dy}{dx} = -\frac{Pbx^2}{2L} + \frac{Pb(L^2 - b^2)}{6L}$$

$$EIy = -\frac{Pbx^3}{6L} + \frac{Pb(L^2 - b^2)x}{6L}$$

Span DB

$$EI\frac{dy}{dx} = -\frac{Pbx^2}{L2} + \frac{P(x-a)^2}{2} + \frac{Pb(L^2 - b^2)}{6L}$$

$$EIy = -\frac{Pbx^3}{L6} + \frac{P(x-a)^3}{6} + \frac{Pb(L^2 - b^2)x}{6L}$$

Drawbacks/Shortcomings of Double Integration Method

Double integration method is a lengthy technique to solve problems with discontinuity

Separate Differential equation for each Moment Equation and integrate

⇩

More BC's to solve

⇩

More Number of constants

⇩

Very Lengthy for more multiple spans and various kinds of loadings

⇩

SOLUTION

Generalized Moment Equation which represents moments on all Spans and Points irrespective of discontinuity

MACAULAY'S METHOD

MACAULAY'S METHOD

Generalized Moment Equation which represents moments on all Spans and Points irrespective of discontinuity

- Single Bending Moment Equation satisfying all BC's at a time is difficult in case of DI Method
- Macaulay's Method is an improvement over DI method
- Single Equation of moments is written for all loads acting on the beam and the constants of integration apply to all sections of the beam

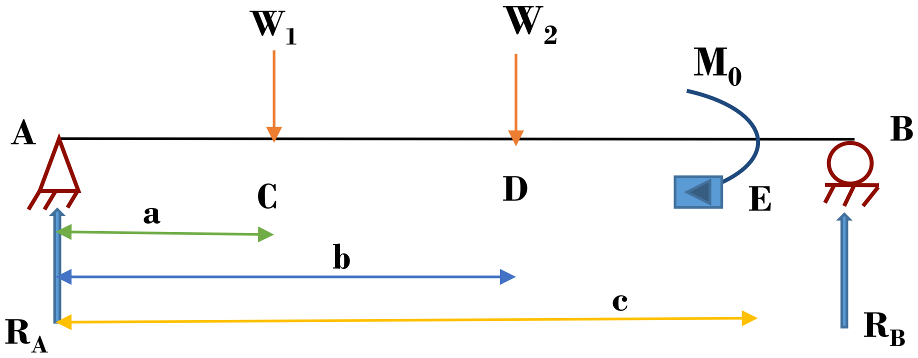
MACAULAYS METHOD

- Work is started always from the extreme left end of beam
- Constants of Integration are always written after the first term
- This method uses a Step Function of the form

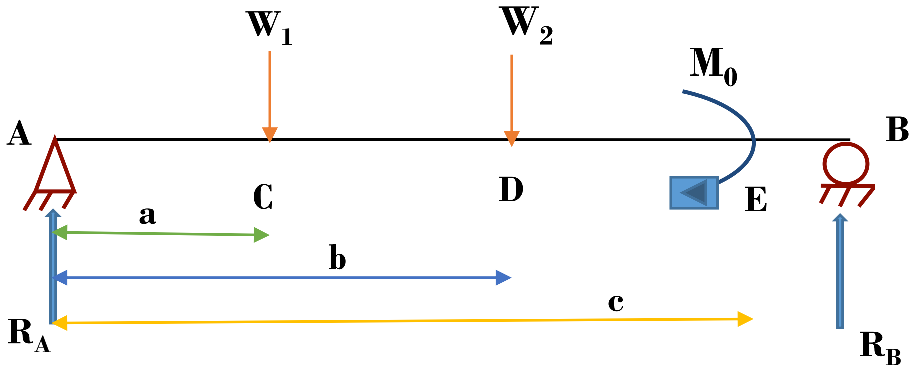
$$f_n(x) = (x - a)^n$$

$$\text{such that } fn(x) = 0 \; if \; x < a$$

$$\text{and } fn(x) = (x - a)^n \; if \; x > a$$



SPAN 'AB' = L



SPAN 'AB' = L

$$f_n(x) = (x - a)^n$$

where n = 0 for any applied moment on the beam

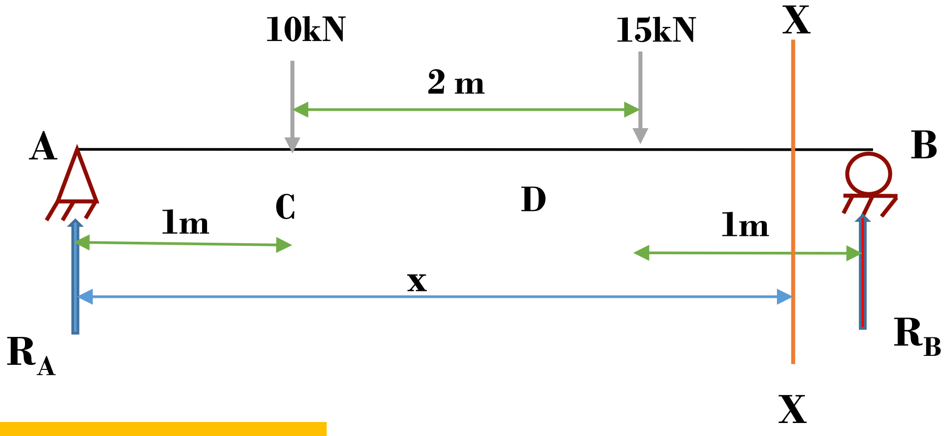
= 1 for point load

= 2 for UDL

- The negative terms inside the bracket are omitted
- The UDL should be extended up to the extreme right end of beam , if it is not so.
- Negative UDL should be applied for the extended part to make a balance
- Hence, the equation of moment (Origin at A always) can be written as

$$M = R_A x - W_1(x-a) - W_2(x-b) + M_0(x-c)^0$$

Example 1



Step 1: Find reactions

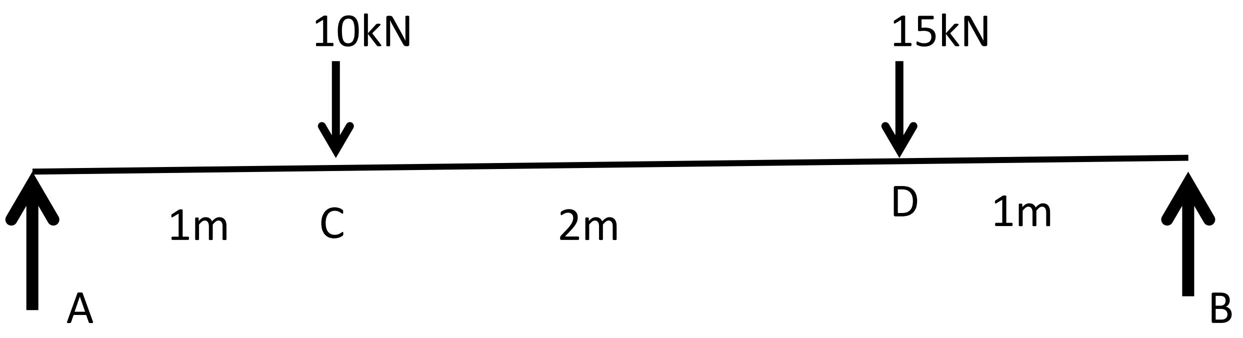
$$R_b \times 4 - 15 \times 3 - 10 \times 1 = 0$$

$$\boxed{R_B = 13.75} \qquad \boxed{R_A = 11.25}$$

Step 2: Write BM equation and apply to Differential Equation of Flexure

$$M = R_A x - 10 < x - 1 > -15 < x - 3 >$$

$$EI\frac{d^2y}{dx^2} = -R_A x + 10 < x - 1 > +15 < x - 3 >$$



Step 3: Integrate to get equation for Slope and Deflection

$$EI\frac{d^2y}{dx^2} = -R_A x + 10 < x - 1 > +15 < x - 3 >$$

$$EI\frac{dy}{dx} = \frac{-R_A x^2}{2} + \frac{10 < x - 1 >^2}{2} + \frac{15 < x - 3 >^2}{2} + C_1$$

$$EIy = \frac{-R_A x^3}{6} + \frac{10 < x - 1 >^3}{6} + \frac{15 < x - 3 >^3}{6} + C_1 x + C_2$$

Step 4: Applying BC's to get Constants of Integration

$at\ x = 0; y = 0$ $\qquad$ $EIy = 0 = C_2$

$$at\ x = 4; y = 0$$

$$EIy = \frac{-R_A x^3}{6} + \frac{10 < x-1 >^3}{6} + \frac{15 < x-3 >^3}{6} + C_1 x + C_2$$

$$0 = \frac{-R_A 4^3}{6} + \frac{10 < 4-1 >^3}{6} + \frac{15 < 4-3 >^3}{6} + C_1 4$$

$$0 = \frac{-11.25 \times 4^3}{6} + \frac{10(4-1)^3}{6} + \frac{15(4-3)^3}{6} + C_1 4$$

$$C_1 = 18.25$$

Step 5: Final Equations for Slope and Deflection

$$\boxed{EI\frac{dy}{dx} = \frac{-R_A x^2}{2} + \frac{10 < x-1 >^2}{2} + \frac{15 < x-3 >^2}{2} + 18.25}$$

$$\boxed{EIy = \frac{-R_A x^3}{6} + \frac{10 < x-1 >^3}{6} + \frac{15 < x-3 >^3}{6} + 18.25x}$$

$$\boxed{R_A = 11.25}$$

Slope and Deflection at ‘C’

$$EI\frac{dy}{dx} = \frac{-R_A x^2}{2} + \frac{10 < x - 1 >^2}{2} + \frac{15 < x - 3 >^2}{2} + 18.25$$

At ‘C’, x= 1, slope $\frac{dy}{dx}$ is calculated as

$$EI\frac{dy}{dx} = -11.25\frac{x^2}{2} + 10\cancel{\frac{(1-1)^2}{2}} + 15\cancel{\frac{(1-3)^2}{2}} + 18.25$$

- The negative terms inside the bracket are omitted

$$EI\frac{dy}{dx} = -11.25\frac{1^2}{2} + 18.25 = 12.625$$

At C, slope is calculated as $\frac{dy}{dx} = \frac{12.625}{EI}$

$$EIy = \frac{-R_A x^3}{6} + \frac{10 < x - 1 >^3}{6} + \frac{15 < x - 3 >^3}{6} + 18.25x$$

At 'C', x= 1, deflection 'y' is calculated as

$$EI\, y = -11.25\frac{\mathrm{x}^3}{6} + 10\cancel{\frac{(1-1)^3}{6}} + 15\cancel{\frac{(1-3)^3}{6}} + \ 18.25 \text{ x } 1$$

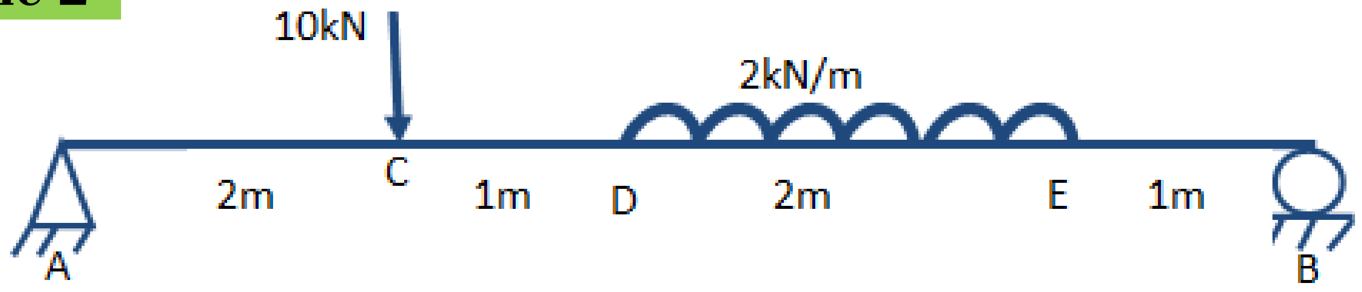
- The negative terms inside the bracket are omitted

$$EI\, y = -11.25\frac{1^3}{2} + \ 18.25 \ = 12.625$$

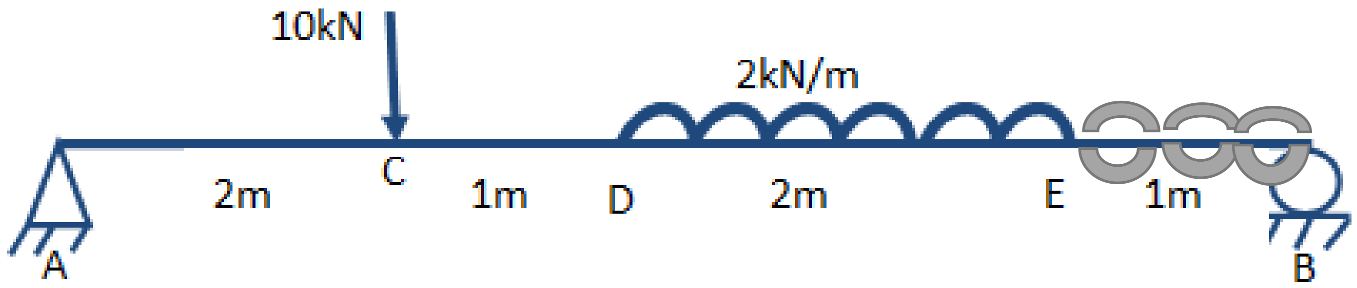
At C, deflection is calculated as $y = \dfrac{12.625}{EI}$

- Similarly, slopes and deflections at various points can be calculated by substituting values of 'x' like under the loads, maximum deflection etc.

Example 2

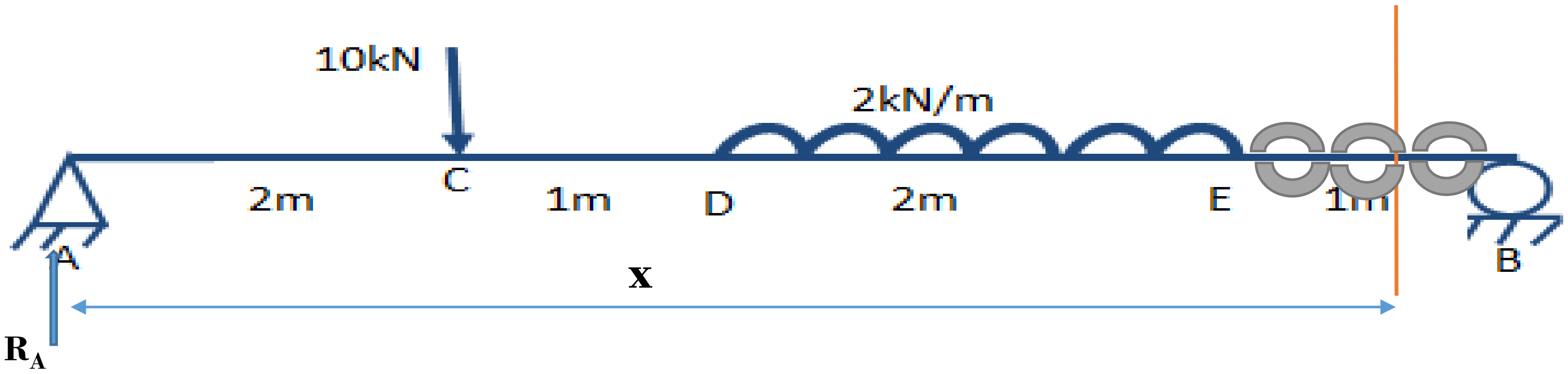


Modified to UDL till end and a negative UDL of same magnitude is applied at the bottom



Cross section of beam is 50mm x100 mm and E=200 GPa

- Develop the elastic curve equation
- Find deflection at centre of beam
- Determine the location of maximum deflection in beam



Step 1: Find reactions

$$\sum M_B = 0 \longrightarrow R_A \times 6 - 10 \times 4 - 2 \times 2 \times 2 = 0$$

$$\longrightarrow R_A = 8$$

Step 2: Write BM equation and apply to Differential Equation of Flexure

$$M = R_A x - 10 < x-1 > - \frac{2 < x-3 >^2}{2} + \frac{2 < x-5 >^2}{2}$$

$$EI\frac{d^2y}{dx^2} = -M$$

$$EI\frac{d^2y}{dx^2} = -R_A x + 10 < x-1 > + \frac{2 < x-3 >^2}{2} - \frac{2 < x-5 >^2}{2}$$

Step 3: Integrate to get equation for Slope and Deflection

$$EI\frac{d^2y}{dx^2} = -R_A x + 10 < x - 1 > + \frac{2 < x - 3 >^2}{2} - \frac{2 < x - 5 >^2}{2}$$

$$EI\frac{dy}{dx} = -\frac{R_A x^2}{2} + \frac{10 < x - 2 >^2}{2} + \frac{2 < x - 3 >^3}{6} - \frac{2 < x - 5 >^3}{6} + C_1$$

$$EIy = -\frac{R_A x^3}{6} + \frac{10 < x - 2 >^3}{6} + \frac{2 < x - 3 >^4}{24} + \frac{2 < x - 5 >^4}{24} C_1 x + C_2$$

Step 4: Applying BC's to get Constants of Integration

1st BC is at x=0m, y =0 $\longrightarrow$ $C_2 = 0$

Similarly 2nd BC is at x=6m, y =0

Similarly 2nd BC is at x=6m, y =0

$$EIy = -\frac{R_A x^3}{6} + \frac{10 < x-2 >^3}{6} + \frac{2 < x-3 >^4}{24} + \frac{2 < x-5 >^4}{24} C_1 x + C_2$$

$$0 = -\frac{R_A 6^3}{6} + \frac{10 < 6-2 >^3}{6} + \frac{2 < 6-3 >^4}{24} + \frac{2 < 6-5 >^4}{24} + C_1 6$$

$$0 = -\frac{R_A 6^3}{6} + \frac{10 < 6-2 >^3}{6} + \frac{2 < 6-3 >^4}{24} + \frac{2 < 6-5 >^4}{24} + C_1 6$$

$$0 = -\frac{8 \times 6^3}{6} + \frac{10(4)^3}{6} + \frac{2(3)^4}{24} + \frac{2(1)^4}{24} + C_1 6$$

$$\longrightarrow \quad C_1 = 29.11$$

Final Equation of Elastic Curve is given by

$$EIy = -\frac{R_A x^3}{6} + \frac{10 < x-2 >^3}{6} + \frac{2 < x-3 >^4}{24} + \frac{2 < x-5 >^4}{24} + 29.11x$$

According to given information,

$$A = 50 \times 100mm^2$$

$$I = \frac{50 \times 100^3}{12} mm^4 \qquad I = \frac{50 \times 100^3}{12 \times 10^{12}} m^4$$

$$EI = \frac{200 \times 10^9}{1000} \times \frac{50 \times 100^3}{12 \times 10^{12}} kNm^2 = \frac{10}{12} kNm^2$$

Deflection at the center of beam is calculated by substituting x= 3 is

$$\mathrm{EIy} = -\frac{\mathrm{R_A x^3}}{6} + \frac{10 < \mathrm{x} - 2 >^3}{6} + \frac{2 < \mathrm{x} - 3 >^4}{24} + \frac{2 < \mathrm{x} - 5 >^4}{24} + 29.11\mathrm{x}$$

$$\mathrm{y} = \frac{53}{\mathrm{EI}}$$

$$\mathrm{y} = 0.063\mathrm{m}$$

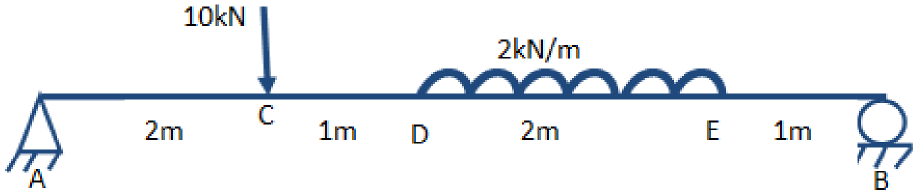
Location of Maximum Deflection i.e. where is y_{max} ????

Maximum Deflection occurs where dy/dx is ZERO

To find maximum deflection

Assume it in region CD

(x lies between 2m and 3m)



$$\frac{dy}{dx} = 0$$

$$EI\frac{dy}{dx} = -\frac{8x^2}{2} + \frac{10 < x-2 >^2}{2} + \frac{2 < x-3 >^3}{6} - \frac{2 < x-5 >^3}{6} + 29.11$$

$$0 = -\frac{8x^2}{2} + \frac{10 < x-2 >^2}{2} + 29.11$$

$$x = 2.88, 17.13$$

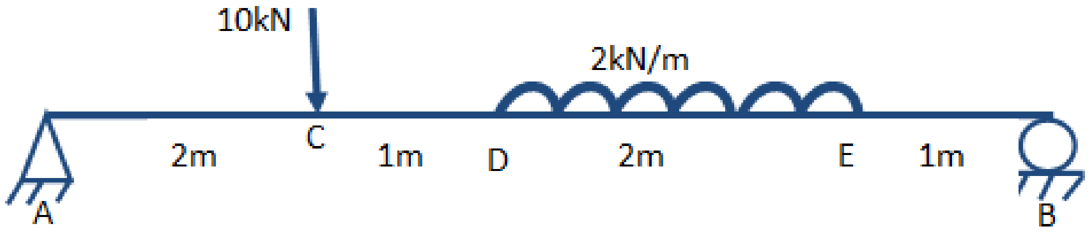
x= 17.13 m is not possible

Hence, y_{max} occurs at x= 2.88m

CHECK

If assumed in region DE

(x lies between 3m and 5m)



$$EI\frac{dy}{dx} = -\frac{8x^2}{2} + \frac{10 < x-2 >^2}{2} + \frac{2 < x-3 >^3}{6} - \frac{2 < x-5 >^3}{6} + 29.11$$

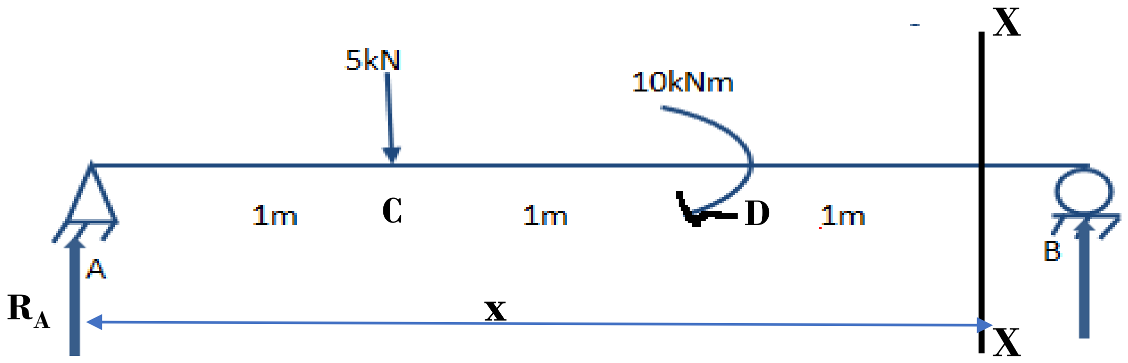
$$0 = -\frac{8x^2}{2} + \frac{10 < x-2 >^2}{2} + \frac{2 < x-3 >^3}{6} + 29.11$$

x= 1.5m, 8.9m, -4.47m

x= All values are not possible since x lies between 3m and 5m

Hence, y_{max} occurs at x= 2.88m

Example 3



Step 1: Find reactions

$$\sum M_B = 0$$

$$\Rightarrow R_A \times 3 - 5 \times 2 + 10 = 0$$

$$\Rightarrow R_A = 0$$

Step 2: Write BM equation and apply to Differential equation of flexure

Writing M at a section 'XX' distance 'x' from A

$$M = R_A x - 5 < x - 1 > +10 < x - 2 >^0$$

$$EI \frac{d^2 y}{dx^2} = -R_A x + 5 < x - 1 > -10 < x - 2 >^0$$

$$EI\frac{d^2y}{dx^2} = -R_A x + 5 < x-1 > -10 < x-2 >^0$$

Step 3: Integrate to get equation for Slope and Deflection

$$EI\frac{dy}{dx} = \frac{-R_A x^2}{2} + 5\frac{< x-1 >^2}{2} - 10 < x-2 >^1 + C_1$$

$$EIy = \frac{-R_A x^3}{6} + 5\frac{< x-1 >^3}{6} - \frac{10 < x-2 >^2}{2} + C_1 x + C_2$$

Step 4: Applying BC's to get Constants of Integration

1st BC is at x=0, y =0 → $C_2 = 0$

Similarly 2nd BC is at x=3m, y =0

Similarly 2nd BC is at x=3m, y =0

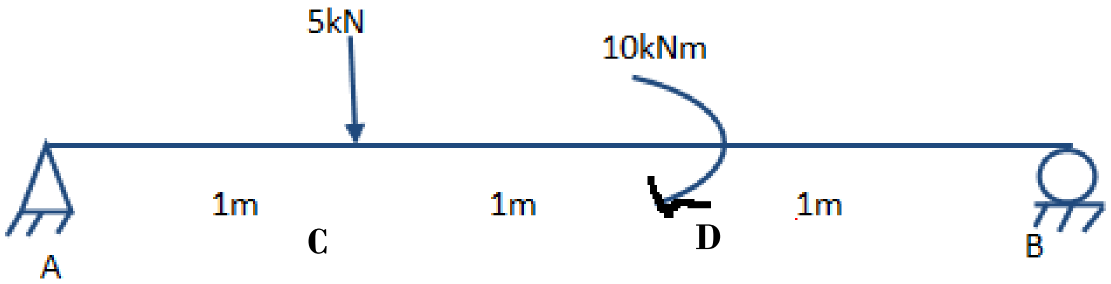
$$EIy = +5\frac{<3-1>^3}{6} - \frac{10<3-2>^2}{2} + C_1 3$$

$$C_1 = -0.55$$

Final Equation of Slope and Elastic Curve (Deflections) is given by

$$\boxed{EI\frac{dy}{dx} = +5\frac{<x-1>^2}{2} - 10<x-2>^1 - 0.55}$$

$$\boxed{EIy = +5\frac{<x-1>^3}{6} - \frac{10<x-2>^2}{2} - 0.55x}$$



To compute maximum deflection, assume maximum deflection in CD (where x lies between 1m and 2m)

$$EI\frac{dy}{dx} = 0 = +5\frac{<x-1>^2}{2} - 0.55$$

On solving, we get, x = 1.4m and 0.53 m

Hence, y_{max} occurs at x= 1.4m

If we assume maximum deflection in span DE

$$EI\frac{dy}{dx} = 0 = +5\frac{<x-1>^2}{2} - 10<x-2>^1 - 0.55$$

We get 'x' as Imaginary values.

Thank you