

# Lecture 8

# Complex Exponential Signals

The complex signal is represented by

$$x(t) = e^{j\omega t}$$

Again by Euler's theorem

$$x(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

The real part of  $x(t)$  is  $\cos\omega t$  and its imaginary part is  $\sin\omega t$ . The complex signal  $x(t) = e^{j\omega t}$  is periodic with period  $T = 2\pi/\omega$  and it is periodic for any value of  $\omega$ .

# Sinusoidal Signal

The expression for a continuous time sinusoidal function is given by

$$x(t) = A \cos(\omega t + \phi)$$

where  $A$  is the amplitude having real values,  $\omega$  is the angular frequency in radians per second and  $\phi$  is the phase angle. The signal  $x(t) = A \cos(\omega t + \phi)$  is periodic with period  $T_p$  where  $T_p = 2\pi/\omega$

is known as fundamental period. The reciprocal of fundamental period ( $T_p$ ) is known as fundamental frequency ( $f$ ) having unit Hz.

Therefore,  $f=1/T$

Again,  $\omega = 2\pi f$

where  $\omega$  is known as the fundamental angular frequency.

Figure 1 shows the signal  $x(t)$ .

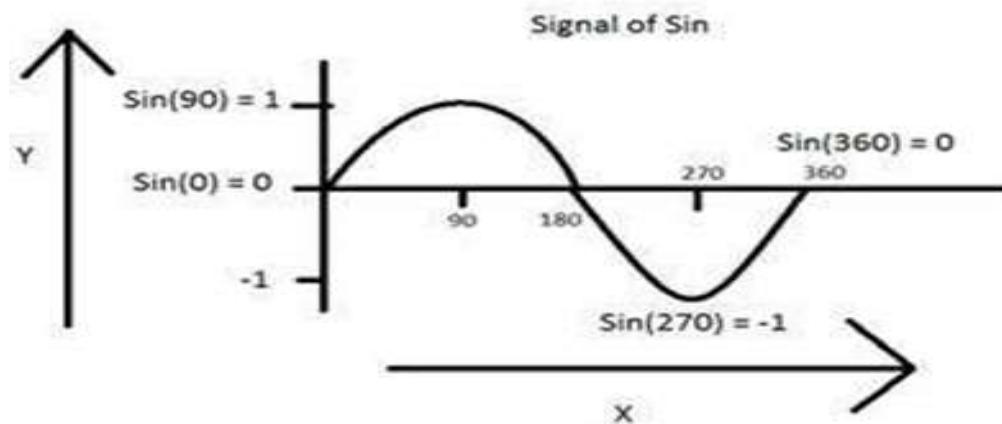


Figure 1

$$\text{Again, } x(t) = A \cos(\omega t + \phi) = \operatorname{Re} \{A e^{j(\omega t + \phi)}\}$$

where the notation **Re** denotes “real part of”. The notation **Im** denotes imaginary part of.

$$\text{Therefore, } x(t) = A \sin(\omega t + \phi) = \operatorname{Im} \{A e^{j(\omega t + \phi)}\}$$

# Classification of Continuous Time Systems

A system is classified as follows:

- Static/Dynamic
- Causal/Non-causal
- Time invariant/ Time variant
- Linear/Non-linear
- Invertible/Non-invertible
- Feedback System

# Static and Dynamic Systems

If the output of a continuous time system depends only on the present input and not on the past or future input, the system is known as static system. There can not be any energy storage elements like capacitor and inductor in a static system. Therefore, we can say that static system is a system without memory.

On the other hand, if the output of a system depends on the present as well as past values of inputs, the system is called a dynamic system. A dynamic system is a system with memory.

- On the other hand, if the output of a system depends on the present as well as past values of inputs, the system is called a dynamic system. A dynamic system is a system with memory.
- Since the voltage across a resistor  $v = iR$ , the resistor is a static system. On the other hand, for a capacitor  $v = \frac{1}{C} \int idt$  and for an inductor  $v = L \frac{di}{dt}$ . Therefore, inductor and capacitor are dynamic system.  $y(t) = 3x(t)$  and  $y(t) = 2x^3(t)$  represent static system whereas  $y(t) = 2x(t) + 3x(t-2)$  represents a dynamic system.

# Causal and Non-causal Systems

A system is said to be causal if its output depends on the present and past values of the input but it does not depend on the future values of input. Therefore, a causal system cannot produce an output before application of input. A causal system is non-anticipatory. On the other hand, a system whose output depends on the future input is known as non-causal system.

The output  $y(t) = x(t) + 2x(t-1) + x(t-2)$  and  $y(t) = x(t) + \frac{1}{2} \int_0^t x(\tau) d\tau$  represent causal system. On the other hand,  $y(t) = x(t+1)$ ,  $y(t) = x(t) + 2x(t+1)$  and  $y(t) = \frac{dx(t)}{dt}$  represent non-causal system.

$$y(t) = x(t) + 2x(t-1) + x(t-2)$$

For  $t = 0$ ,  $y(0) = x(0) + 2x(-1) + x(-2)$

For  $t = 1$ ,  $y(1) = x(1) + 2x(0) + x(-1)$

For  $t = 2$ ,  $y(2) = x(2) + 2x(1) + x(0)$

Therefore, for any value of  $t$ , system output depends on the present and past values of input. Hence the system is causal.

$$y(t) = x(t + 1)$$

For  $t = 0$ ,  $y(0) = x(1)$

For  $t = 1$ ,  $y(1) = x(2)$

For  $t = 2$ ,  $y(2) = x(3)$

Therefore, for any value of  $t$ , system output depends on the future values of input. Hence the system is non causal.

$$y(t) = x(t) + 2x(t + 1)$$

For  $t = 0$ ,  $y(0) = x(0) + 2x(1)$

For  $t = 1$ ,  $y(1) = x(1) + 2x(2)$

For  $t = 2$ ,  $y(2) = x(2) + 2x(3)$

Therefore, for any value of  $t$ , system output depends on the present and future values of input. Hence the system is non causal.

The output  $y(t) = \frac{dx(t)}{dt}$  is called non-causal because

$$\frac{dx(t)}{dt} = \lim_{\Delta t \rightarrow 0} \left[ \frac{x(t + \Delta t) - x(t)}{\Delta t} \right]$$

# Time invariant and Time variant Systems

Let us consider a continuous time system having output  $y(t)$  for an input  $x(t)$ . A input signal having a time shift is applied to the system, the system is said to be time invariant if and only if the output of the system has the time shift.

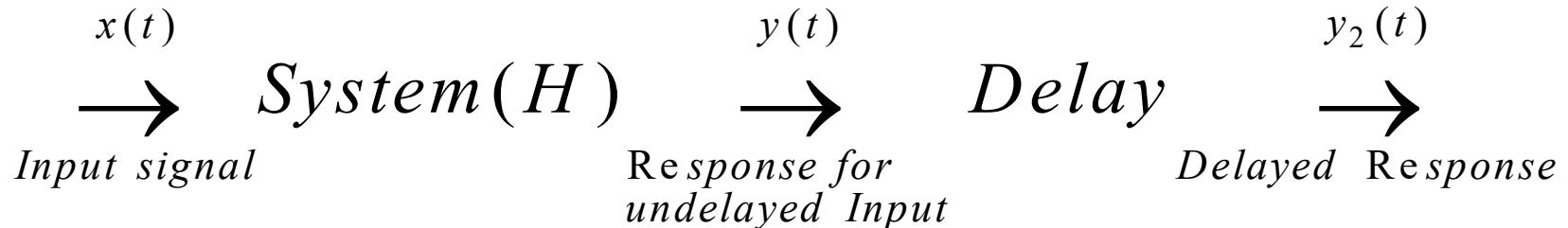
Mathematically, for a time invariant system , we can write

$$x(t) \rightarrow y(t)$$

$$x(t-t_0) \rightarrow y(t-t_0) \quad \text{for every input signal } x(t) \text{ and every time shift } t_0.$$

In time invariant system, if  $y(t) = H\{x(t)\}$  then  $y(t-t_0) = H\{x(t-t_0)\}$

A system  $H$  is time invariant if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed ) version of the response based on the unshifted (or undelayed) input.



If  $y_1(t) = y_2(t)$ , the system is time invariant. Otherwise, it will be time variant.

Procedure to test for time invariance:

1. Delay the input signal by  $t_0$  units of time (say) and determine the response of the system for this delayed input signal. Let this response be  $y_1(t)$ .
2. Delay the response due to the unshifted input by  $t_0$  units of time (say). Let this delayed response be  $y_2(t)$ .
3. Check whether  $y_1(t) = y_2(t)$ . If they are equal, the system is time invariant. Otherwise, it is time variant.

Check whether the following signals are time invariant or not.

- (a)  $y(t) = 2x(t)$
- (b)  $y(t) = 2tx(t)$
- (c)  $y(t) = x(t^2)$
- (d)  $y(t) = 2x(-t)$

Solution:

(a)  $y(t) = 2x(t)$

(i)  $x(t)$  is delayed by  $t_0$  units of time and it becomes  $x(t-t_0)$ . The output of the system for this delayed input becomes  $y_1(t) = 2x(t-t_0)$ .

(ii) The input is kept unshifted and for this unshifted input the output becomes  $y(t) = 2x(t)$ . Now  $y(t)$  is delayed by  $t_0$  units of time and the response of the system becomes  $y_2(t) = y(t-t_0) = 2x(t-t_0)$ .

(iii) Therefore,  $y_1(t) = y_2(t)$  and the system becomes time invariant.

$$(b) y(t) = 2tx(t)$$

- (i)  $x(t)$  is delayed by  $t_0$  units of time and it becomes  $x(t-t_0)$ . The output of the system for this delayed input becomes  $y_1(t) = 2t x(t-t_0)$ .
- (ii) The input is kept unshifted and for this unshifted input the output becomes  $y(t) = 2tx(t)$ . Now  $y(t)$  is delayed by  $t_0$  units of time and the response of the system becomes  $y_2(t) = y(t-t_0) = 2(t-t_0)x(t-t_0)$ .
- (iii) Therefore,  $y_1(t) \neq y_2(t)$  and the system becomes time variant.

$$(c) y(t) = x(t^2)$$

- (i)  $x(t)$  is delayed by  $t_0$  units of time and it becomes  $x(t^2 - t_0)$ . The output of the system for this delayed input becomes  
 $y_1(t) = x(t^2 - t_0)$ .
- (ii) The input is kept unshifted and for this unshifted input the output becomes  $y(t) = x(t^2)$ . Now  $y(t)$  is delayed by  $t_0$  units of time and the response of the system becomes  $y_2(t) = y(t-t_0) = x(t-t_0)^2$ .
- (iii) Therefore,  $y_1(t) \neq y_2(t)$  and the system becomes time variant.

$$(d) y(t) = 2x(-t)$$

- (i)  $x(t)$  is delayed by  $t_0$  units of time and it becomes  $x(-t-t_0)$ . The output of the system for this delayed input becomes  $y_1(t) = 2x(-t-t_0)$ .
- (ii) The input is kept unshifted and for this unshifted input the output becomes  $y(t) = 2x(-t)$ . Now  $y(t)$  is delayed by  $t_0$  units of time and the response of the system becomes  $y_2(t) = y(t-t_0) = 2x(-(t-t_0)) = 2x(-t+t_0)$ .
- (iii) Therefore,  $y_1(t) \neq y_2(t)$  and the system becomes time variant.

For example, the differential equations

$$\frac{dy(t)}{dt} + 5y(t) = x(t) \quad \text{and} \quad \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = x(t) + \frac{dx(t)}{dt}$$

represents time invariant system.

The differential equation  $\frac{dy(t)}{dt} + t^2y(t) = x(t)$  represents a time variant system.

For a time invariant system, any co-efficient of its representing differential equation should not vary with time.