Course: UMA 035 (Optimization Techniques)

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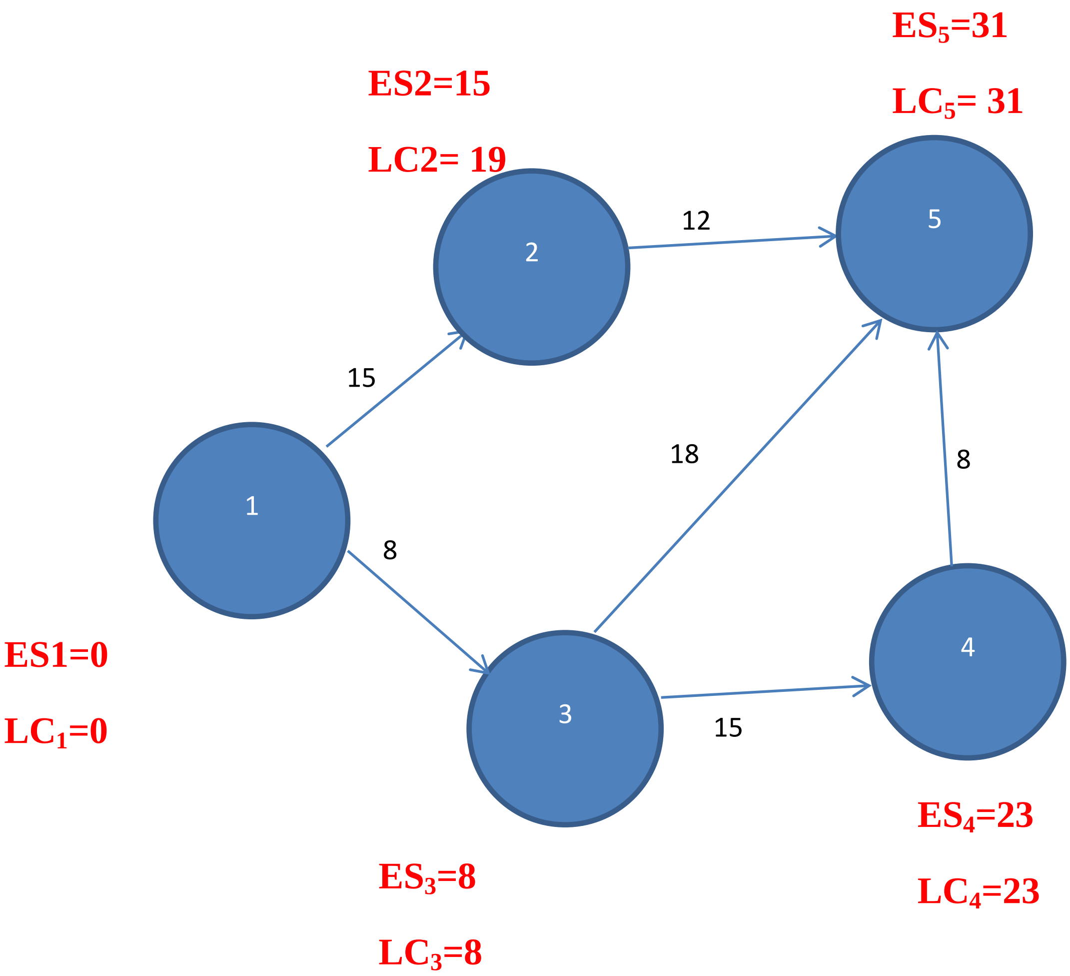
Critical activity

The $(i,j)^{th}$ activity between node i and node j is said to be a critical activity if

- $ES_i = LC_i$
- $ES_j = LC_j$
- $ES_j - ES_i = LC_j - LC_i = t_{ij}$



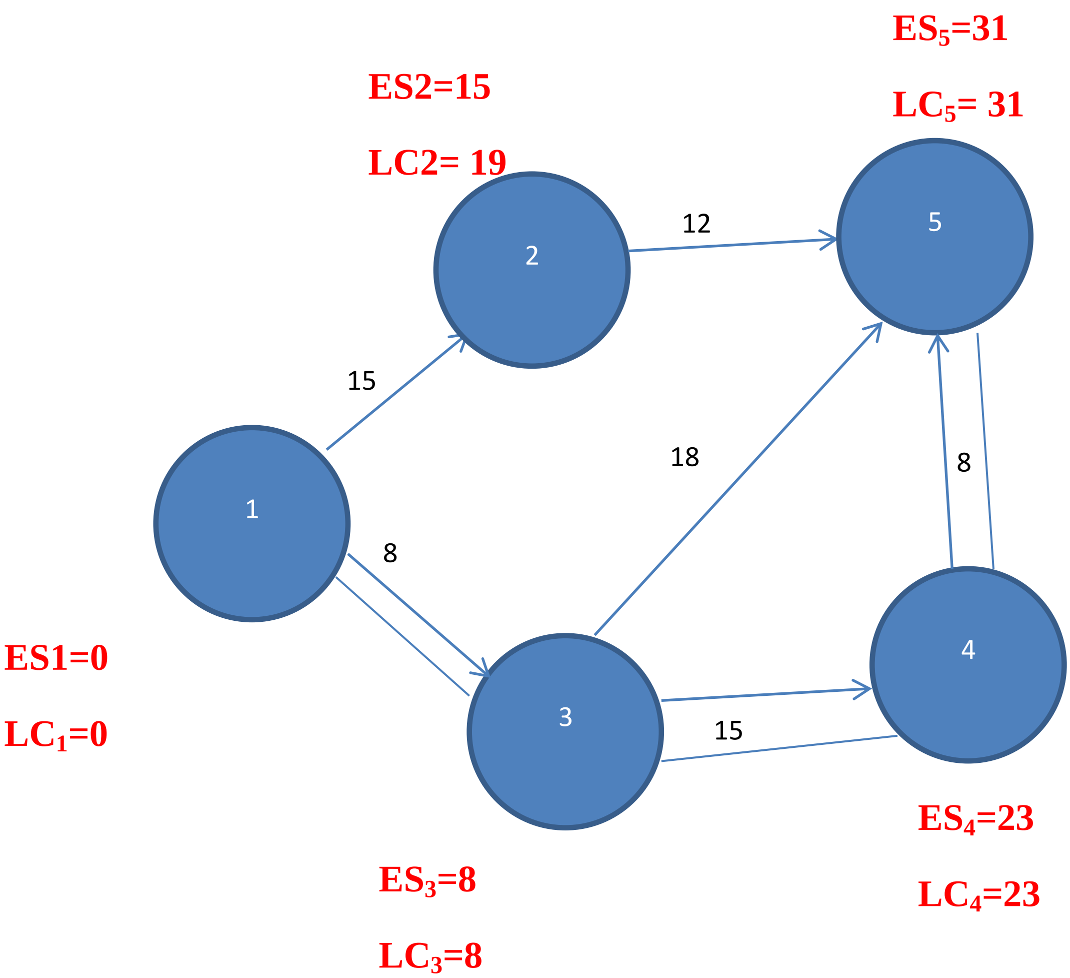
Critical activities

- **(1,3) as $ES_1=LC_1$, $ES_2=LC_2$, $ES_2- ES_1= LC_2- LC_1=t_{12}$**
- **(3,4) as $ES_3=LC_3$, $ES_4=LC_4$, $ES_4- ES_3= LC_4- LC_3= t_{34}$**
- **(4,5) as $ES_4=LC_4$, $ES_5=LC_5$, $ES_5- ES_4= LC_5- LC_4= t_{45}$**

Critical Path

A path between the origin and the destination, constructed by critical activities, is called a critical path.

Critical path is $1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$



Crash duration and crash cost of an activity

The minimum time required to complete an activity is known as the crash duration of the considered activity. Also, the corresponding cost is known as the crash cost of the considered activity.

Normal duration and normal cost of an activity

The time specified by a person to complete an activity (greater than or equal to crash duration) is known as the normal duration of the considered activity. Also, the corresponding cost is known as the normal cost of the considered activity.

Crash limit of an activity

The maximum time that can be reduced from an activity.

Crash limit = Normal duration − Crash duration

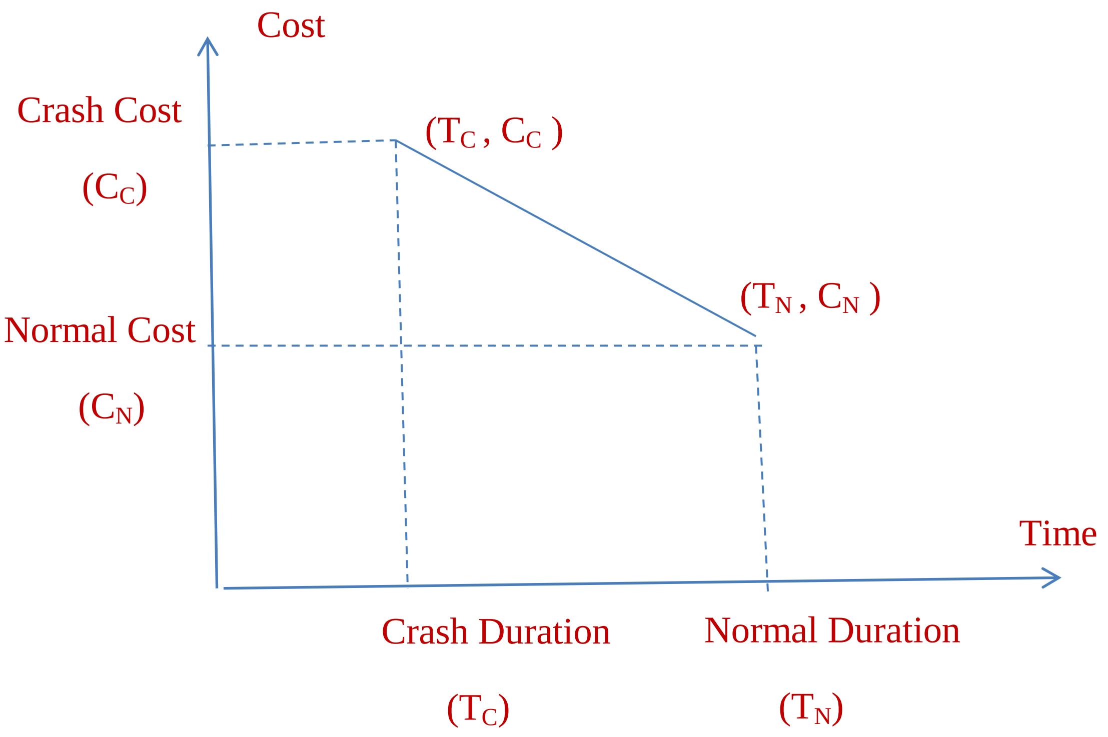
Relation between normal duration and crash duration

Normal duration >= Crash duration

Relation between normal cost and crash cost

Normal cost <= Crash cost

Slope of an activity



Magnitude of slope of the line joining the points (T_C,C_C) and (T_N,C_N)

$$= \left|\frac{C_N - C_c}{T_N - T_c}\right|$$

$$= \frac{\text{Change in Cost}}{\text{Change in Time}}$$

= Change in cost per unit time

= The cost required to reduce one unit time from the activity

Free float of a non-critical activity

Free float = $ES_j - ES_i - t_{ij}$

FF limit

FF limit=minimum {non-zero free floats}

Compression limit

Compression limit = minimum {FF limit, Crash limit}

Example

Activity	T_N	C_N	T_C	C_C
(1, 2)	15	600	12	1200
(1, 3)	8	700	5	1600
(2, 5)	12	750	6	1500
(3, 4)	15	650	12	1400
(3, 5)	18	700	3	1450
(4, 5)	8	500	5	950

(i) Find the cost for completing the project in 29 days.

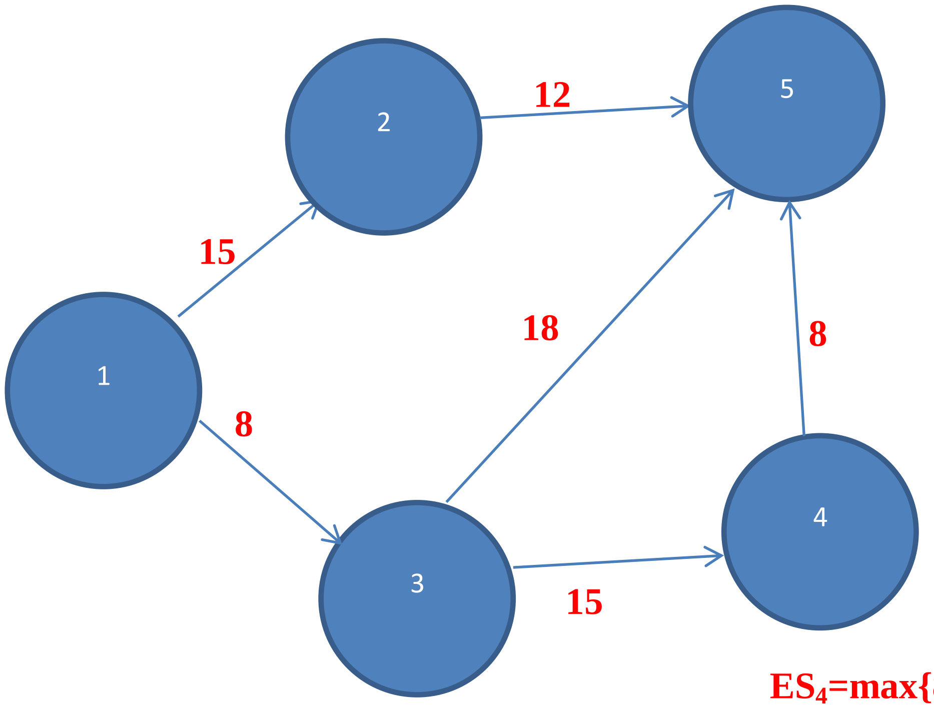
(ii) Find the cost for completing the project in 28 days.

(iii) Find the crash cost for completing the project (cost for completing the project in minimum time).

Solution:

ES_5=max{15+12, 8+18, 23+8}=31

ES2=max {0+15}=15



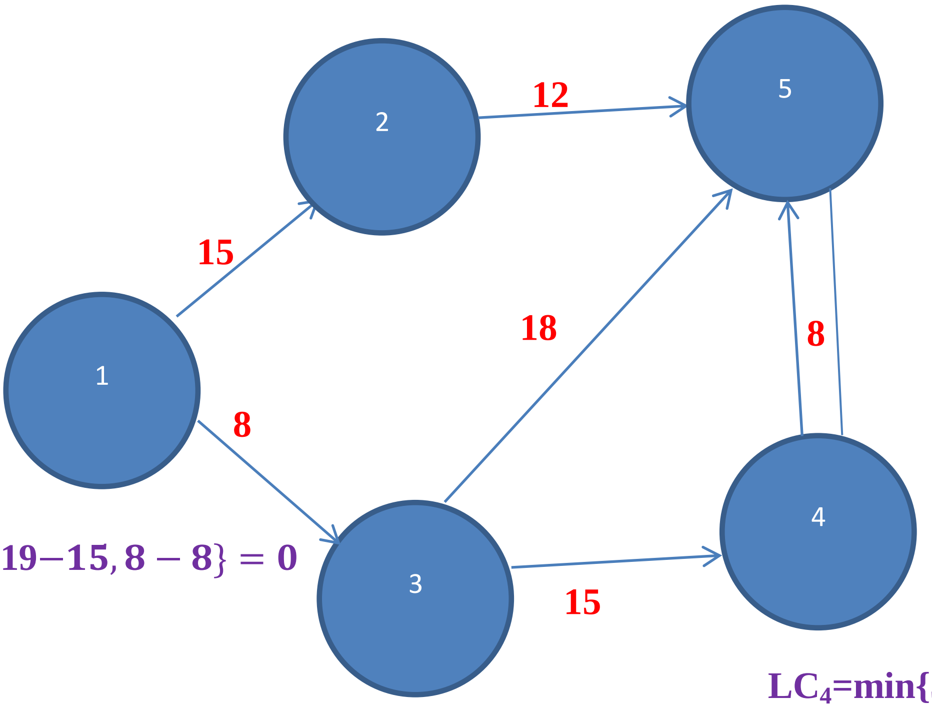
ES1=0

ES_4=max{8+15}=23

ES_3=max {0+8}=8

LC2= $\min\{31-12\}=19$

$LC_5= 31$



$LC_1= \min\{19-15, 8-8\} = 0$

$LC_4=\min\{31-8\}=23$

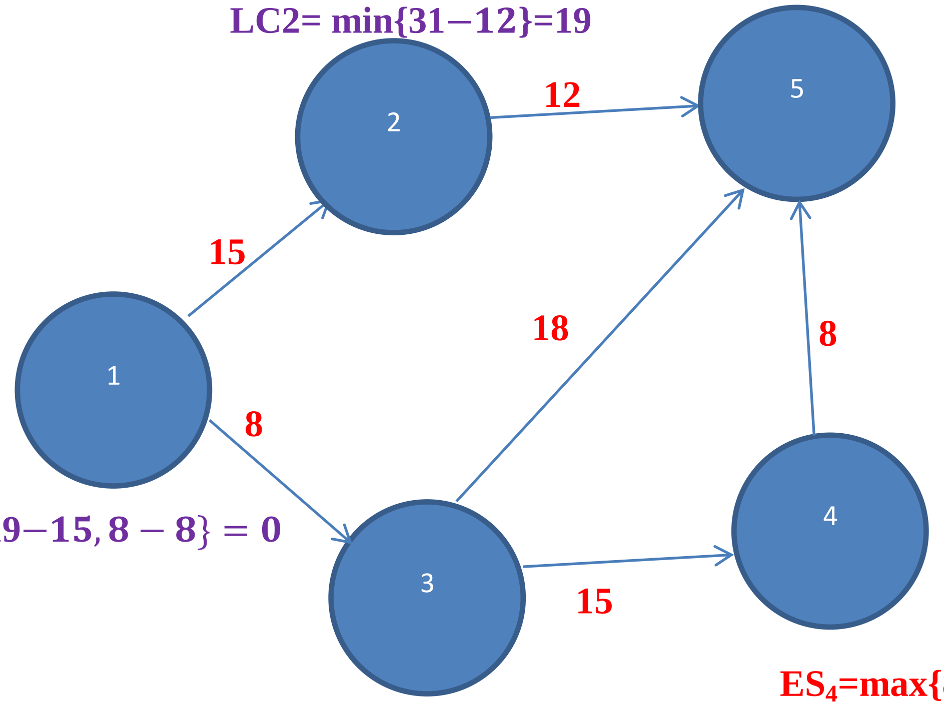
$LC_3= \min\{31-18, 23-15\}=8$

$ES_5=\max\{15+12, 8+18, 23+8\}=31$

ES2=$\max \{0+15\}=15$

$LC_5= 31$

LC2= $\min\{31-12\}=19$



ES1=0

$LC_1= \min\{19-15, 8-8\} = 0$

$ES_4=\max\{8+15\}=23$

$LC_4=\min\{31-8\}=23$

$ES_3=\max \{0+8\}=8$

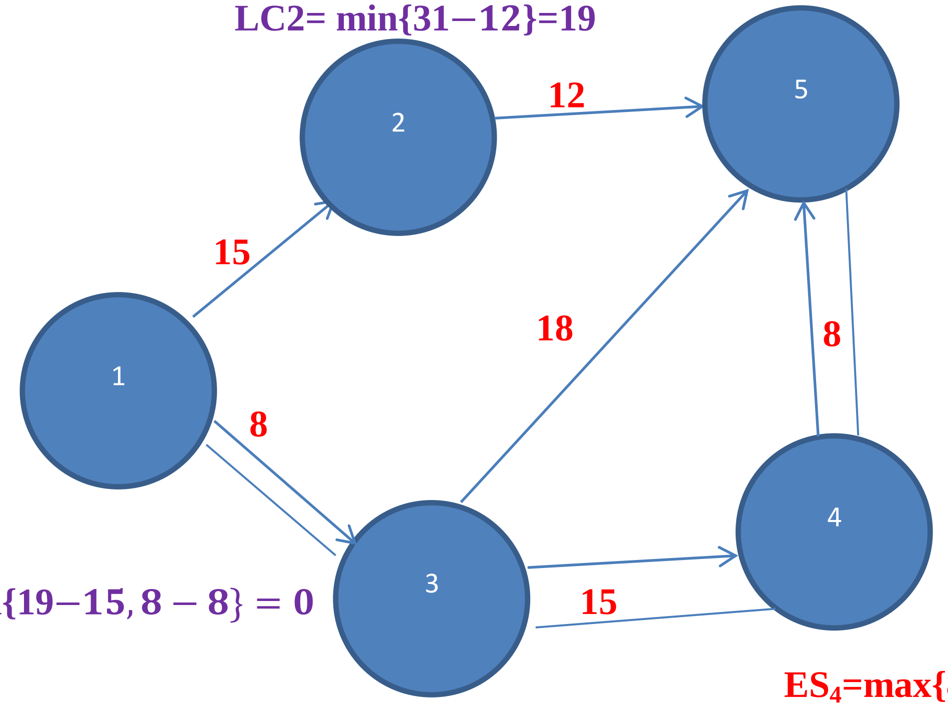
$LC_3= \min\{31-18, 23-15\}=8$

ES_5=max{15+12, 8+18, 23+8}=31

LC_5= 31

ES2=max {0+15}=15

LC2= min{31−12}=19



ES1=0

LC_1= $\min\{19-15, 8-8\} = 0$

ES_4=max{8+15}=23

LC_4=min{31−8}=23

ES_3=max {0+8}=8

LC_3= $\min\{31-18, 23-15\}$=8

Current duration for completing the project is 31 days

Cost for completing the project in 31 days is sum of all the given normal costs=600+700+750+650+700+500=3900

To complete the project in 29 days there is a need to reduce 2 days.

To reduce the time for a project, there is a need to reduce the time of an

activity from each critical path.

Since, in this case, there is only one critical path $1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$. So, to reduce the project completion time, there is a need to reduce the time of an activity on this critical path.

There are three activities (1, 3), (3, 4) and (4, 5) on this critical path. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,4)=$\left|\frac{1400-650}{15-12}\right| = 250$

Slope of (4,5)=$\left|\frac{950-500}{8-5}\right| = 150$

Since, slope of (4, 5) is minimum so we will reduce time of (4,5).

The maximum time that can be reduced from an activity at a time is equal to its compression limit.

Compression limit=minimum {Crash limit, FF limit}

Crash limit for (4,5)=Current duration of (4,5) −Crash duration of (4,5)

= 8−5=3

FF limit =minimum {non-zero free floats of non-critical activities}

FF of (1,2)=15−0−15=0

FF of (2,5)=31−15−12=4

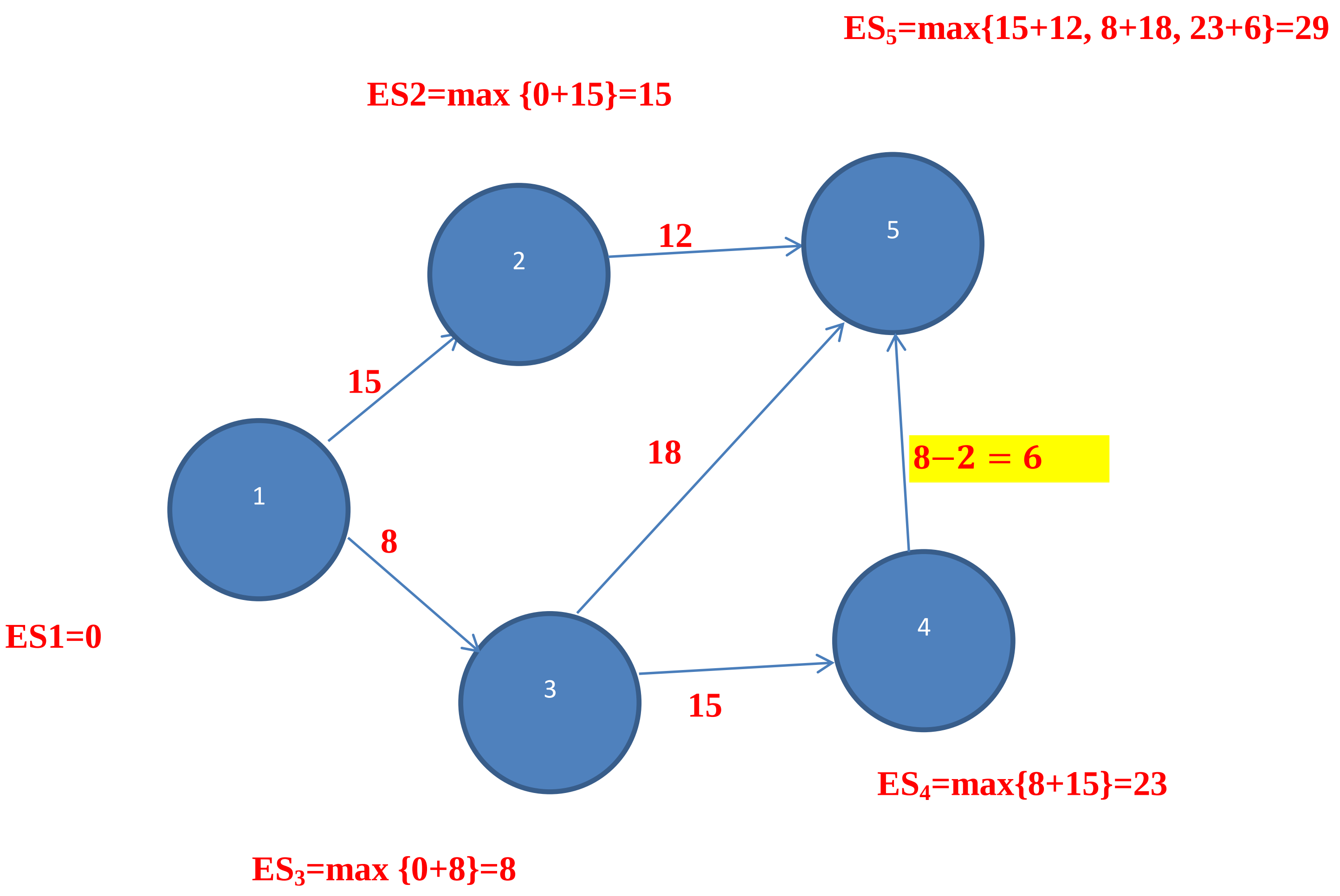
FF of (3,5)=31−8−18=5

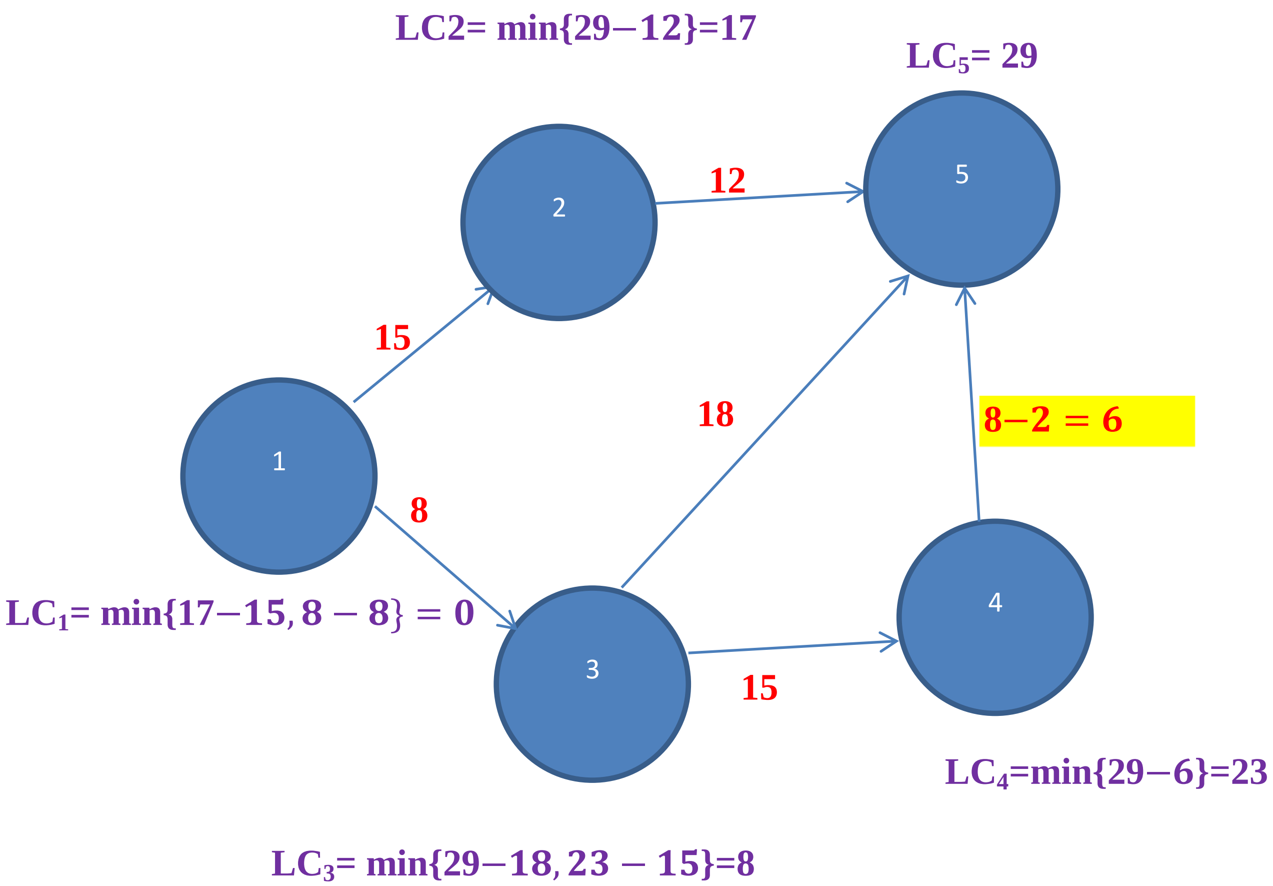
FF limit =minimum {4,5}=4

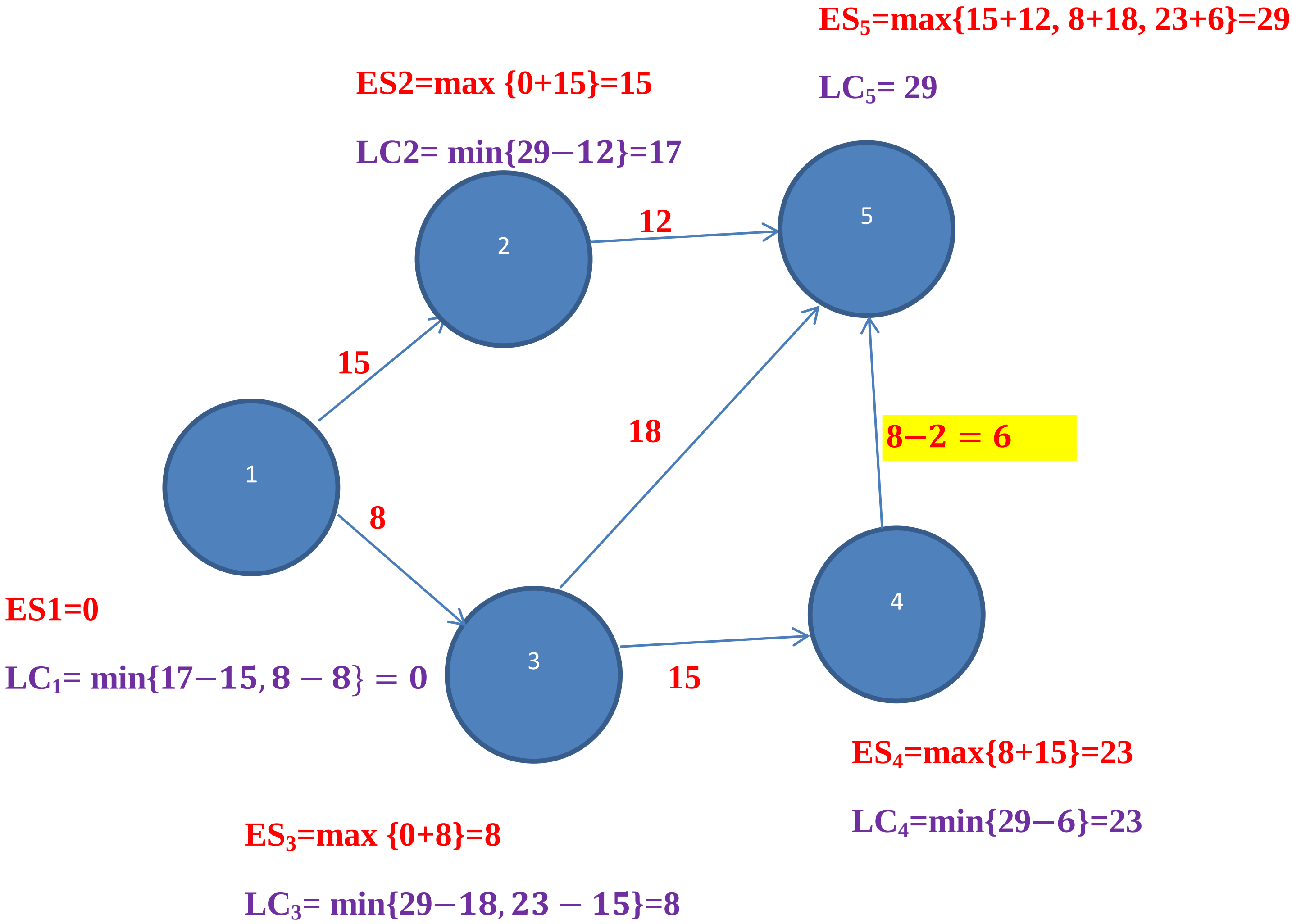
Compression limit=minimum {Crash limit, FF limit}={3,4}=3

Since, the compression limit is 3. So maximum 3 days can be reduced from the activity (4,5) at a time.

While, there is need to reduce only 2 days. So reduce 2 days from the activity (4,5).




LC2= min{29−12}=17
LC5= 29
12
2
5
15
18
8−2 = 6
1
8
LC1= min{17−15, 8 − 8} = 0
3
4
15
LC4=min{29−6}=23
LC3= min{29−18, 23 − 15}=8

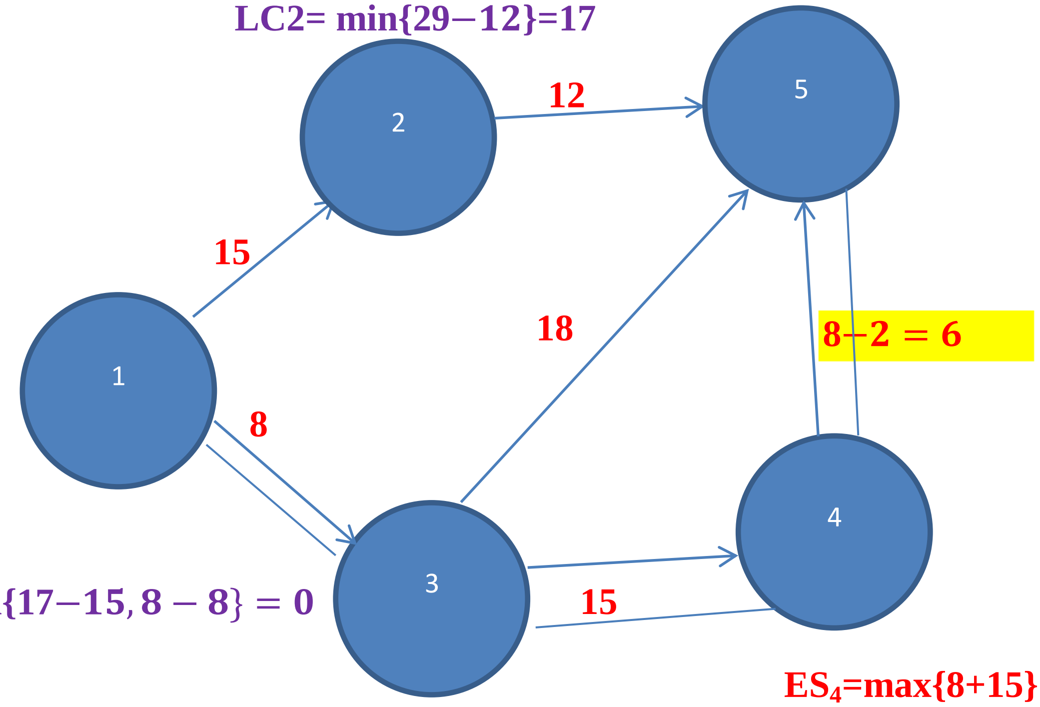

ES5=max{15+12, 8+18, 23+6}=29
LC5= 29
ES2=max {0+15}=15
LC2= min{29−12}=17
12
5
2
15
18
8−2 = 6
1
8
4
ES1=0
LC1= min{17−15, 8 − 8} = 0
3
15
ES4=max{8+15}=23
LC4=min{29−6}=23
ES3=max {0+8}=8
LC3= min{29−18, 23 − 15}=8

ES_5=max{15+12, 8+18, 23+6}=29

LC_5= 29

ES2=max {0+15}=15

LC2= min{29−12}=17



ES1=0

LC_1= min{17−15, 8 − 8} = 0

ES_4=max{8+15}=23

LC_4=min{29−6}=23

ES_3=max {0+8}=8

LC_3= min{29−18, 23 − 15}=8

Current duration for completing the project is 29 days

Cost for completing the project in 29 days is

=3900+extra cost for reducing 2 days from (4,5)

=3900+2*Slope of (4,5)

=3900+2*150

=4200

To complete the project in 28 days there is a need to reduce 1 day.

To reduce the time for a project, there is a need to reduce the time of an activity from each critical path.

Since, in this case, there is only one critical path 1⟶3 ⟶4⟶ 5 . So, to reduce the project completion time, there is a need to reduce the time of an activity on this critical path.

There are three activities (1, 3), (3, 4) and (4, 5) on this critical path. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,4)=$\left|\frac{1400-650}{15-12}\right| = 250$

Slope of (4,5)=$\left|\frac{950-500}{8-5}\right| = 150$

Since, slope of (4, 5) is minimum so we will reduce time of (4,5).

The maximum time that can be reduced from an activity at a time is equal to its compression limit.

Compression limit=minimum {Crash limit, FF limit}

Crash limit for (4,5)=Current duration of (4,5) $-$Crash duration of (4,5)

= 6$-$5=1

FF limit =minimum {non-zero free floats of non-critical activities}

FF of (1,2)=15$-$0$-$15=0

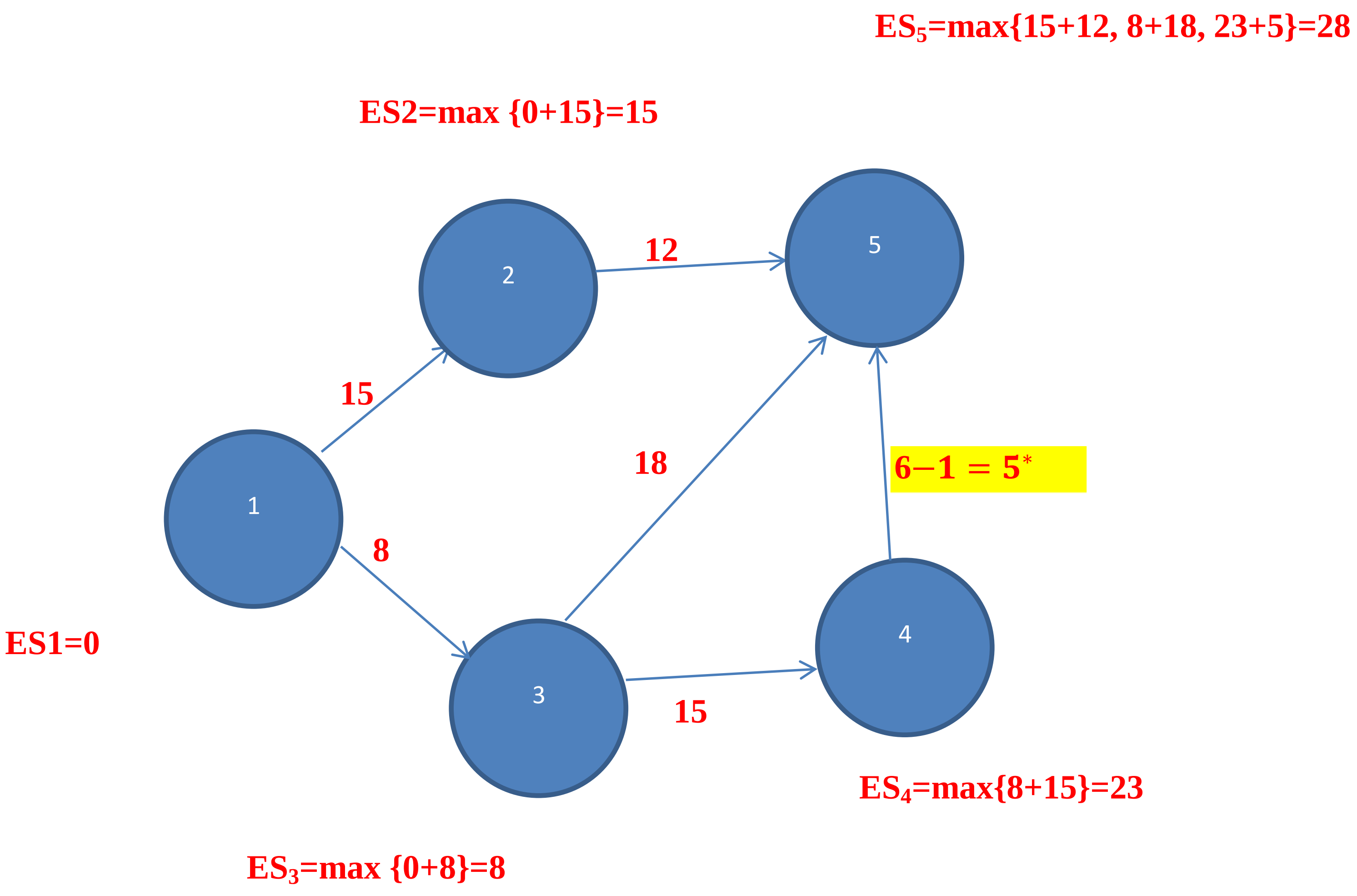
FF of (2,5)=29$-$15$-$12=2

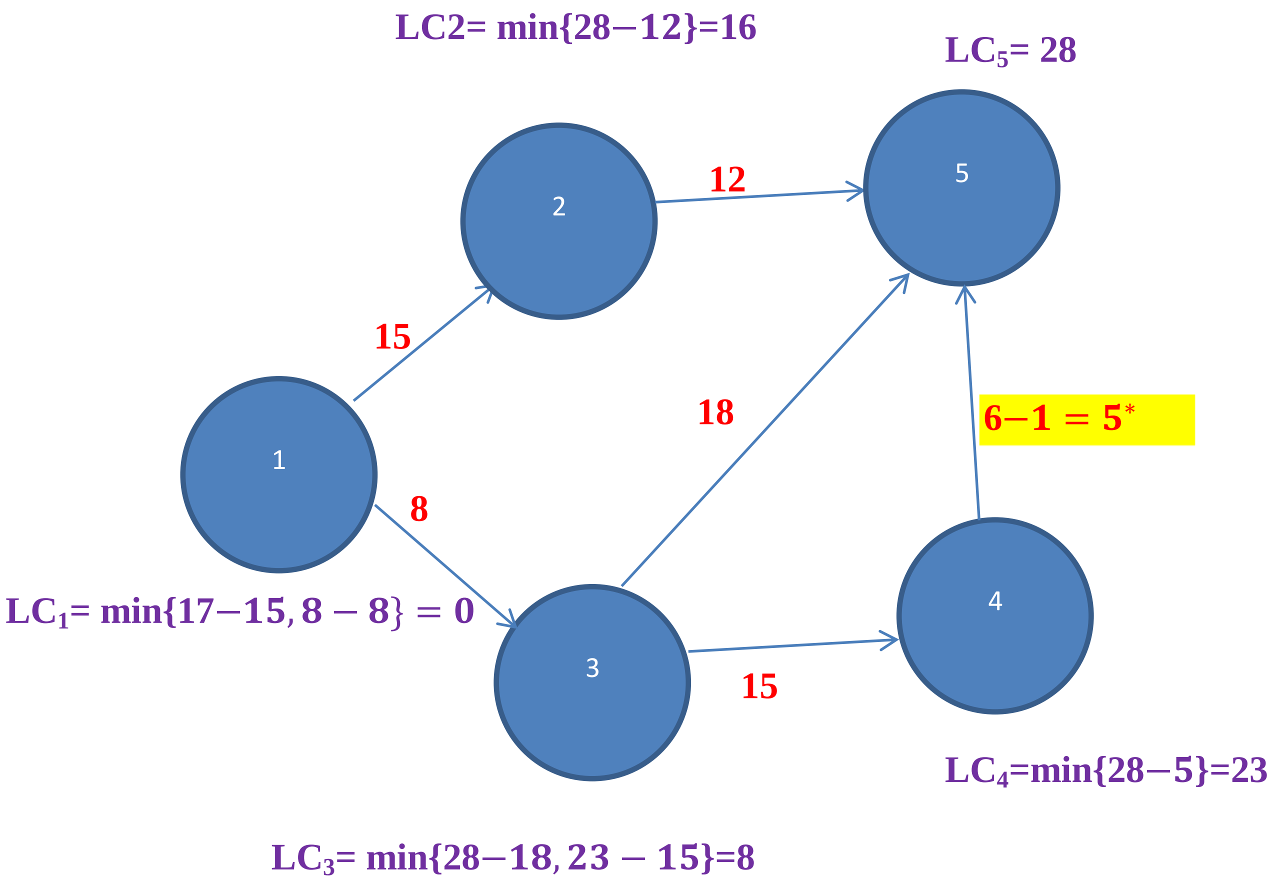
FF of (3,5)=29$-$8$-$18=3

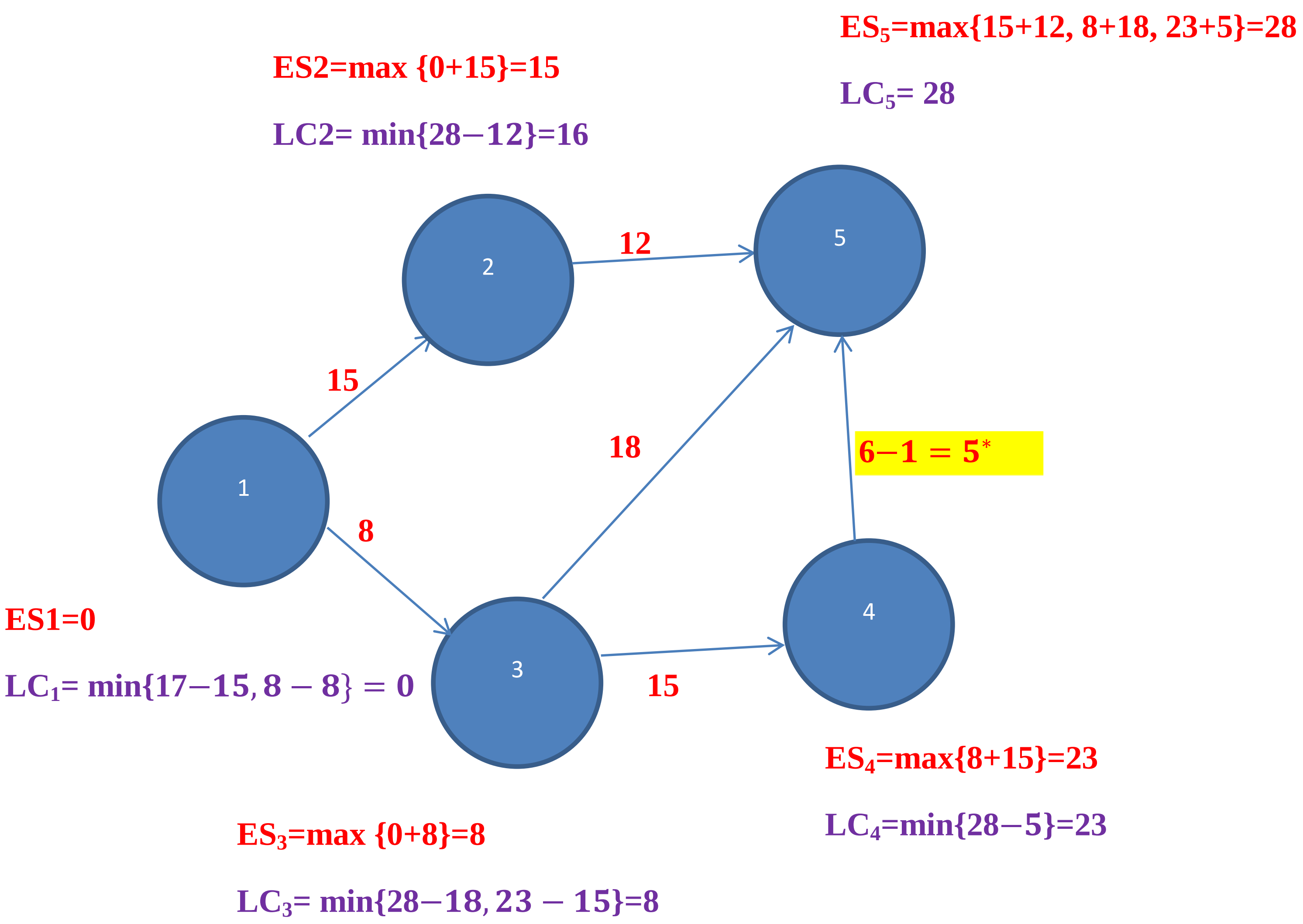
FF limit =minimum {2,3}=2

Compression limit=minimum {Crash limit, FF limit}={1,2}=1

Since, the compression limit is 1. So maximum 1 day can be reduced from the activity (4,5) at a time.




LC2= min{28−12}=16
LC5= 28
2
5
12
15
1
18
6−1 = 5*
8
LC1= min{17−15, 8 − 8} = 0
3
4
15
LC4=min{28−5}=23
LC3= min{28−18, 23 − 15}=8



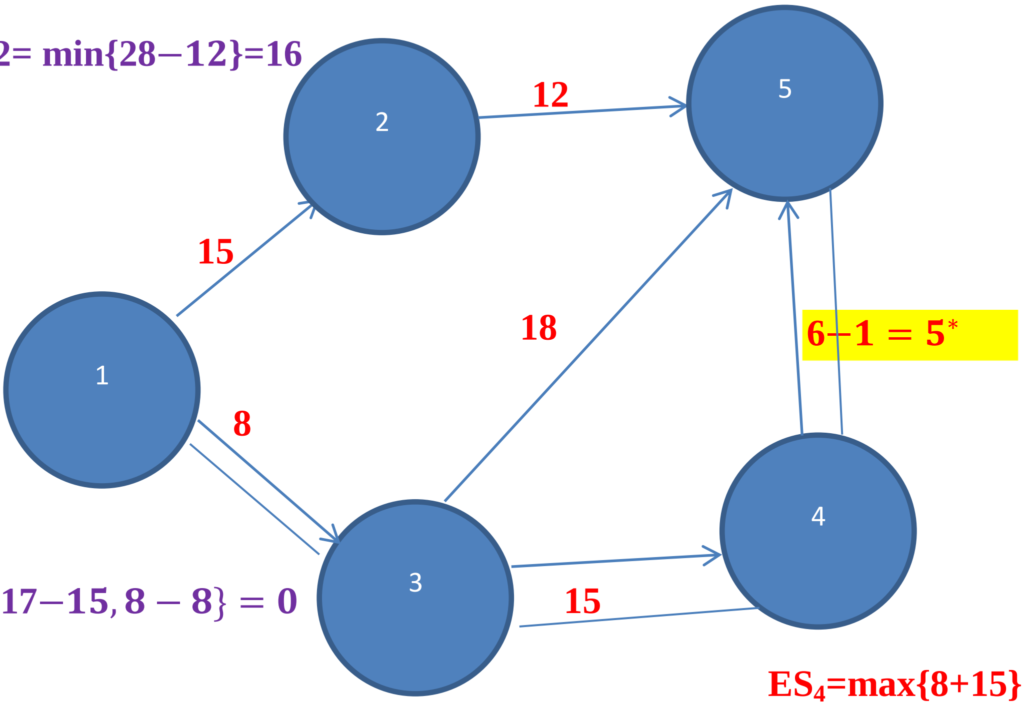
ES2=max {0+15}=15
LC2= min{28−12}=16
ES_5=max{15+12, 8+18, 23+5}=28
LC_5= 28
2
5
12
15
18
1
$6-1 = 5^*$
8
4
ES1=0
LC_1= min{17−15, 8 − 8} = 0
3
15
ES_4=max{8+15}=23
LC_4=min{28−5}=23
ES_3=max {0+8}=8
LC_3= min{28−18, 23 − 15}=8

ES_5=max{15+12, 8+18, 23+5}=28

LC_5= 28

ES2=max {0+15}=15

LC2= min{28−12}=16



ES1=0

LC_1= min{17−15, 8 − 8} = 0

ES_4=max{8+15}=23

LC_4=min{28−5}=23

ES_3=max {0+8}=8

LC_3= min{28−18, 23 − 15}=8

Current duration for completing the project is 28 days

Cost for completing the project in 28 days is

=4200+extra cost for reducing 1 day from (4,5)

=4200+1*Slope of (4,5)

=4200+1*150

=4350

To complete the project in 23 days there is a need to reduce 5 days.

To reduce the time for a project, there is a need to reduce the time of an activity from each critical path.

Since, in this case, there is only one critical path 1⟶3 ⟶4⟶ 5 . So, to reduce the project completion time, there is a need to reduce the time of an activity on this critical path.

There are three activities (1, 3), (3, 4) and (4, 5) on this critical path. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,4)=$\left|\frac{1400-650}{15-12}\right| = 250$

Slope of (4,5)=$\left|\frac{950-500}{8-5}\right| = 150$

Although, slope of (4, 5) is minimum, But, its current duration is equal to crash duration. So, time of this activity cannot be reduced further.

Since, the next activity with minimum slope of this critical path is (3,4). So, we will reduce time of (3,4).

The maximum time that can be reduced from an activity at a time is equal to its compression limit.

Compression limit=minimum {Crash limit, FF limit}

Crash limit for (3,4)=Current duration of (3,4) −Crash duration of (3,4)

= 15−12=3

FF limit =minimum {non-zero free floats of non-critical activities}

FF of (1,2)=15−0−15=0

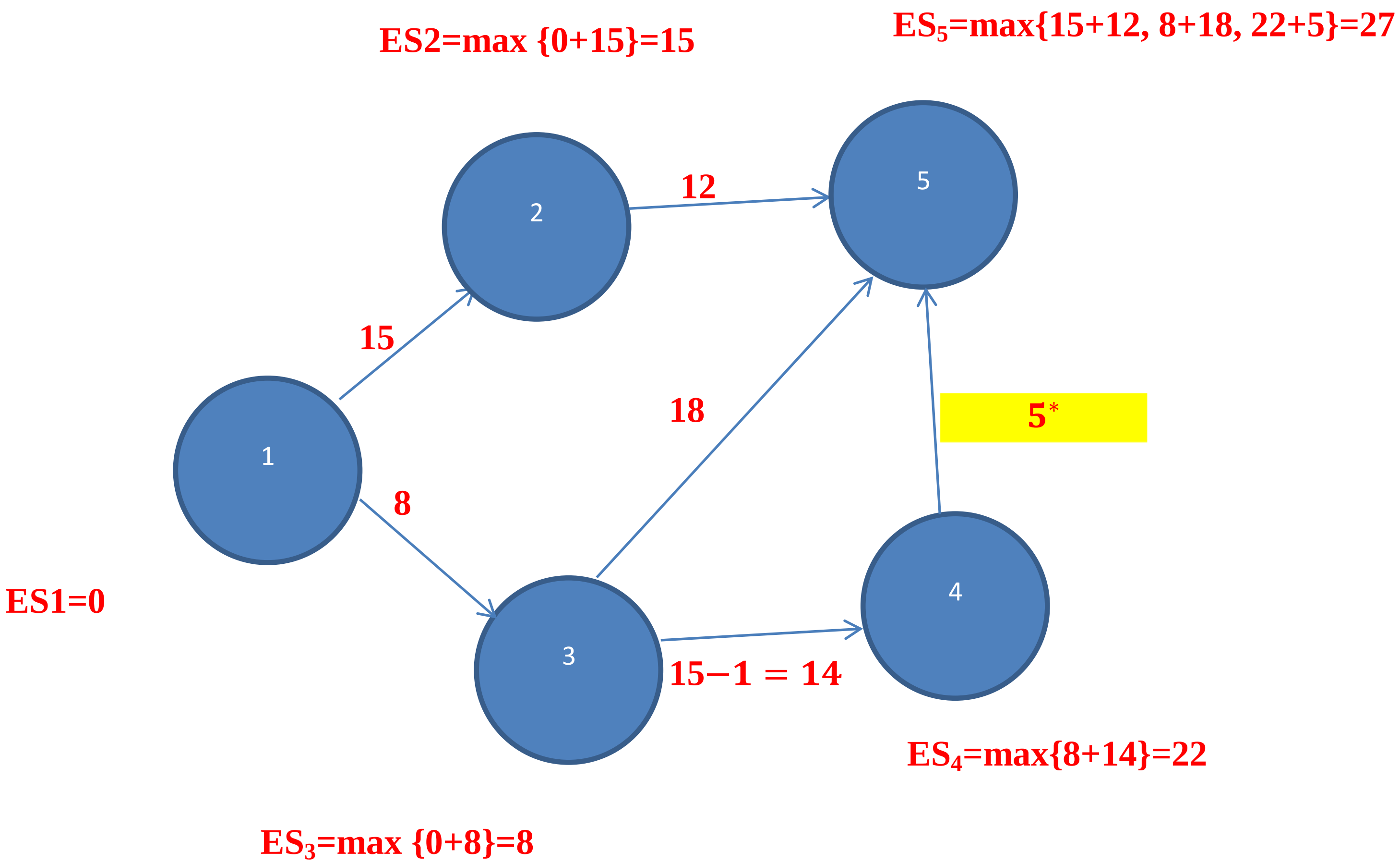
FF of (2,5)=28−15−12=1

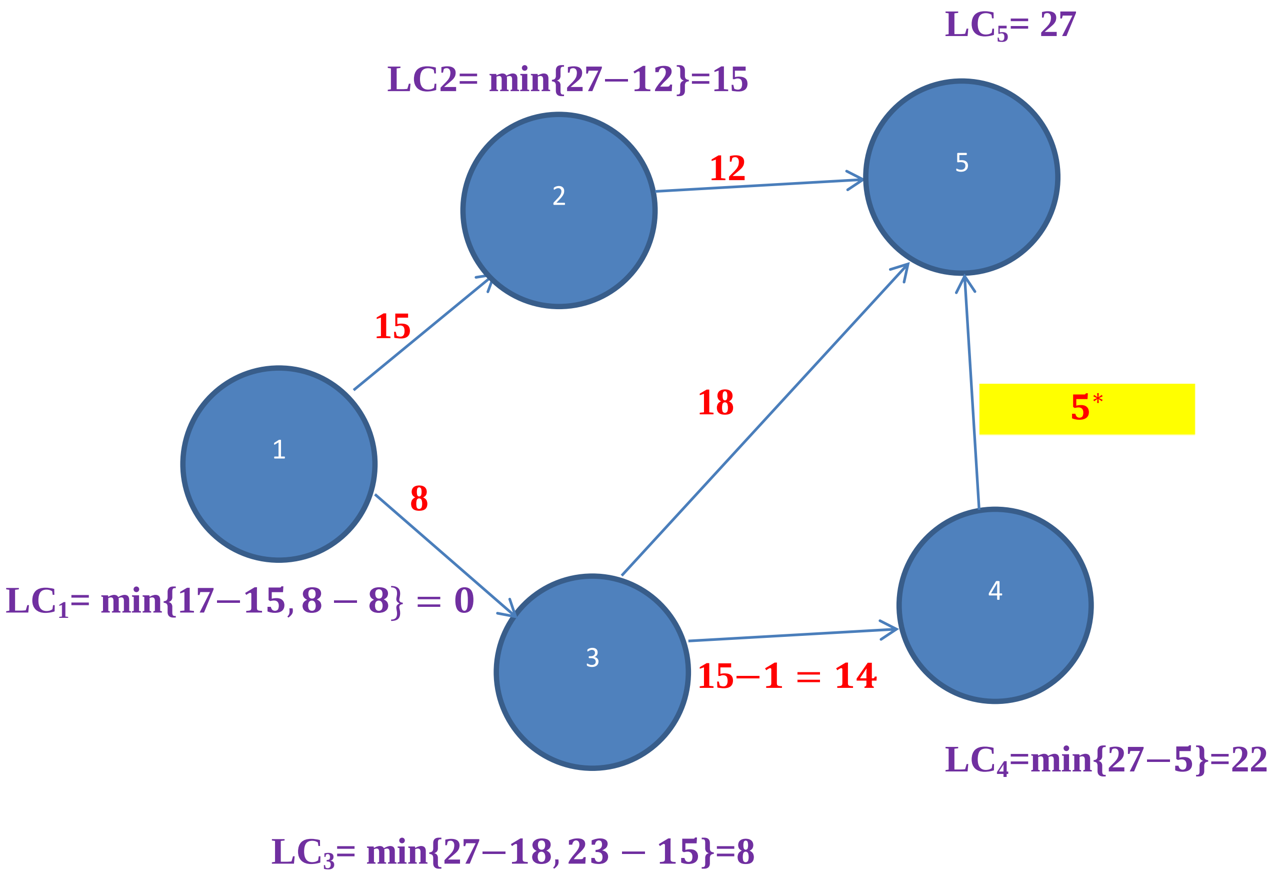
FF of (3,5)=28−8−18=2

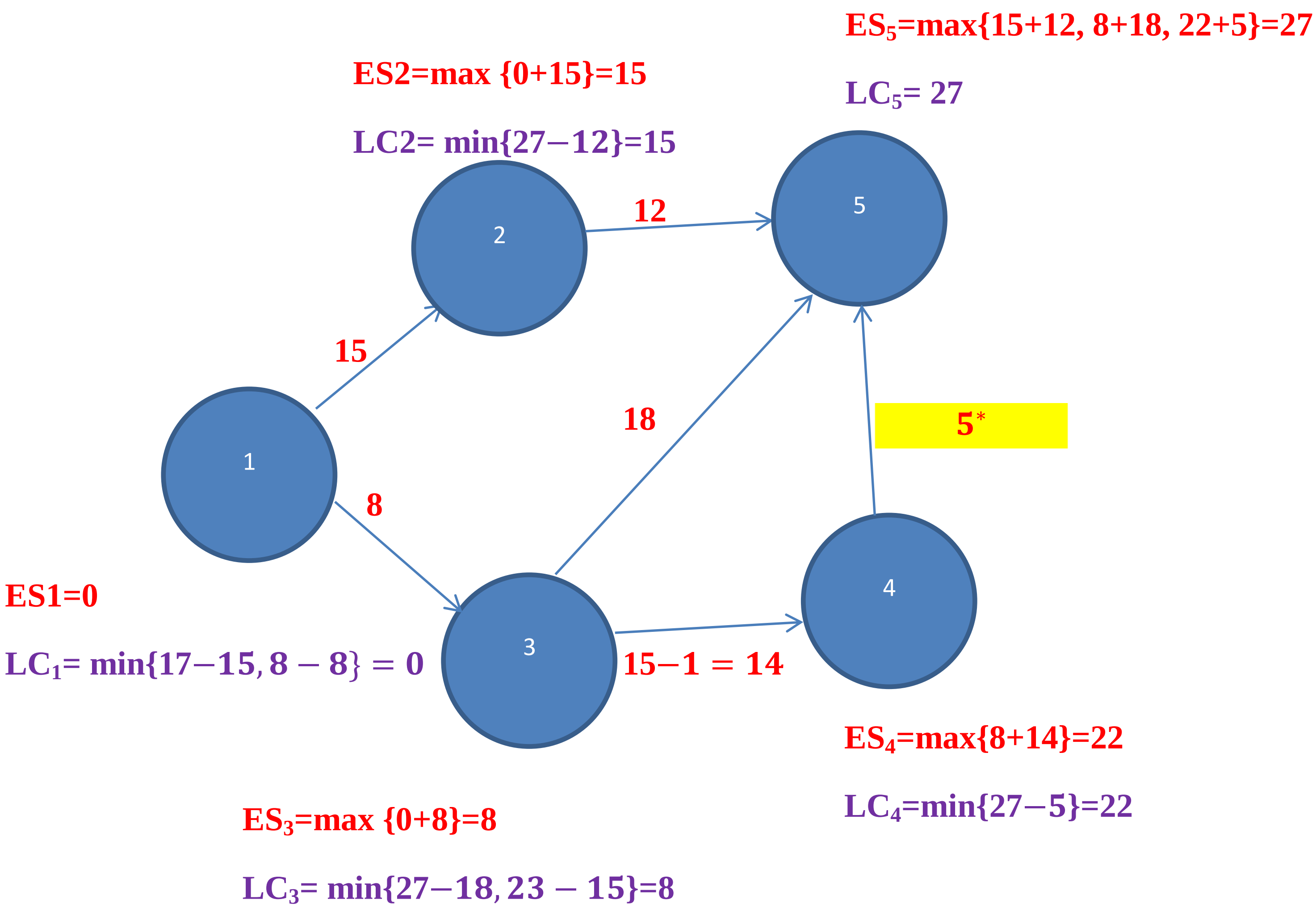
FF limit =minimum {1,2}=1

Compression limit=minimum {Crash limit, FF limit}={1,3}=1

Since, the compression limit is 1. So maximum 1 day can be reduced from the activity (3,4) at a time.


ES2=max {0+15}=15
ES5=max{15+12, 8+18, 22+5}=27
12
15
18
5*
8
ES1=0
15−1 = 14
ES4=max{8+14}=22
ES3=max {0+8}=8
1
2
3
4
5


LC5= 27
LC2= min{27−12}=15
12
15
18
5*
8
LC1= min{17−15, 8 − 8} = 0
15−1 = 14
LC4=min{27−5}=22
LC3= min{27−18, 23 − 15}=8
1
2
3
4
5



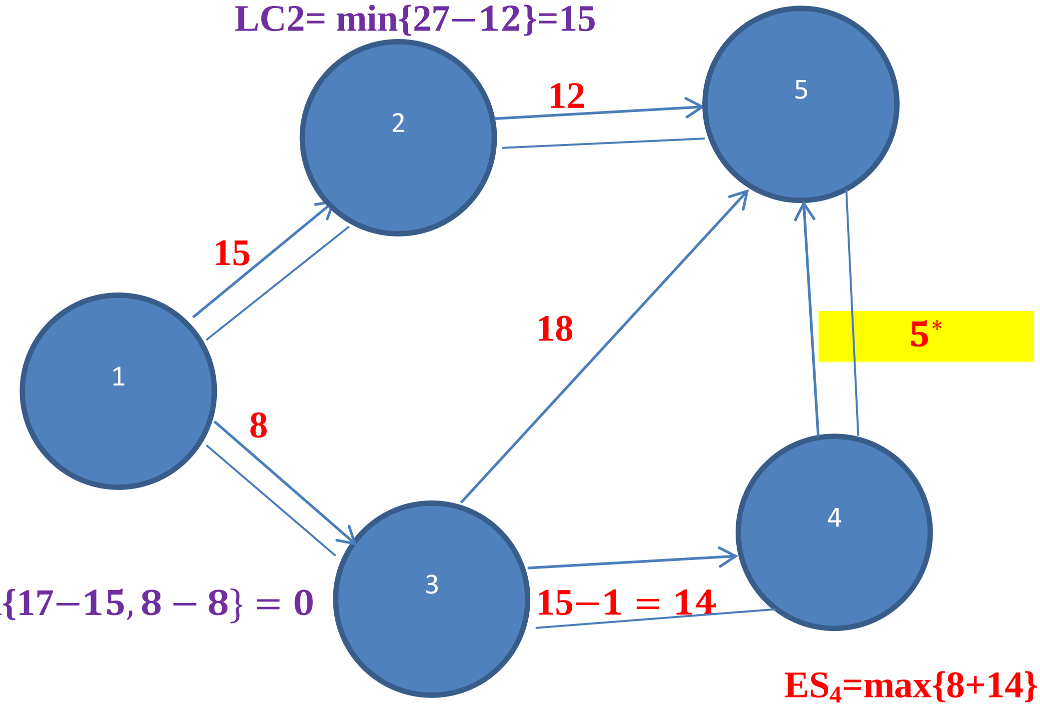
ES5=max{15+12, 8+18, 22+5}=27
LC5= 27
ES2=max {0+15}=15
LC2= min{27−12}=15
12
2
5
15
18
5*
1
8
4
ES1=0
LC1= min{17−15, 8 − 8} = 0
3
15−1 = 14
ES4=max{8+14}=22
LC4=min{27−5}=22
ES3=max {0+8}=8
LC3= min{27−18, 23 − 15}=8

ES_5=max{15+12, 8+18, 22+5}=27

LC_5= 27

ES2=max {0+15}=15

LC2= min{27−12}=15



ES1=0

LC_1= min{17−15, 8 − 8} = 0

ES_4=max{8+14}=22

LC_4=min{27−5}=22

ES_3=max {0+8}=8

LC_3= min{27−18, 23 − 15}=8

Current duration for completing the project is 27 days

Cost for completing the project in 27 days is

=4350+extra cost for reducing 1 day from (3,4)

=4350+1*Slope of (3,4)

=4350+1*250

=4600

To complete the project in 23 days there is a need to reduce 4 days.

To reduce the time for a project, there is a need to reduce the time of an activity from each critical path.

Since, in this case, there are two critical pats 1⟶3⟶ 4⟶ 5 and 1⟶2⟶ 5 . So, to reduce the project completion time, there is a need to reduce the time of an activity from both the critical paths.

There are three activities (1, 3), (3, 4) and (4, 5) on the critical path 1⟶3⟶ 4⟶ 5. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,4)=$\left|\frac{1400-650}{15-12}\right| = 250$

Slope of (4,5)=$\left|\frac{950-500}{8-5}\right| = 150$

Although, slope of (4, 5) is minimum, But, its current duration is equal to crash duration. So, time of this activity cannot be reduced further.

Since, the next activity with minimum slope of this critical path is (3,4). So, we will reduce time of (3,4).

There are two activities (1, 2) and (2, 5) on the critical path

1⟶2⟶ 5. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,2)= $\left|\frac{1200-600}{15-12}\right| = 200$

Slope of (2,5)= $\left|\frac{1500-750}{12-6}\right| = 125$

Since, slope of (2, 5) is minimum, So, the time of (2,5) will be reduced.

The maximum time that can be reduced from an activity at a time is equal to its compression limit.

Compression limit=minimum {Crash limit, FF limit}

Crash limit for (3,4)=Current duration of (3,4) −Crash duration of (3,4)

= 14−12=2

Crash limit for (2,5)=Current duration of (2,5) −Crash duration of (2,5)

= 12−6=6

FF limit =minimum {non-zero free floats of non-critical activities}

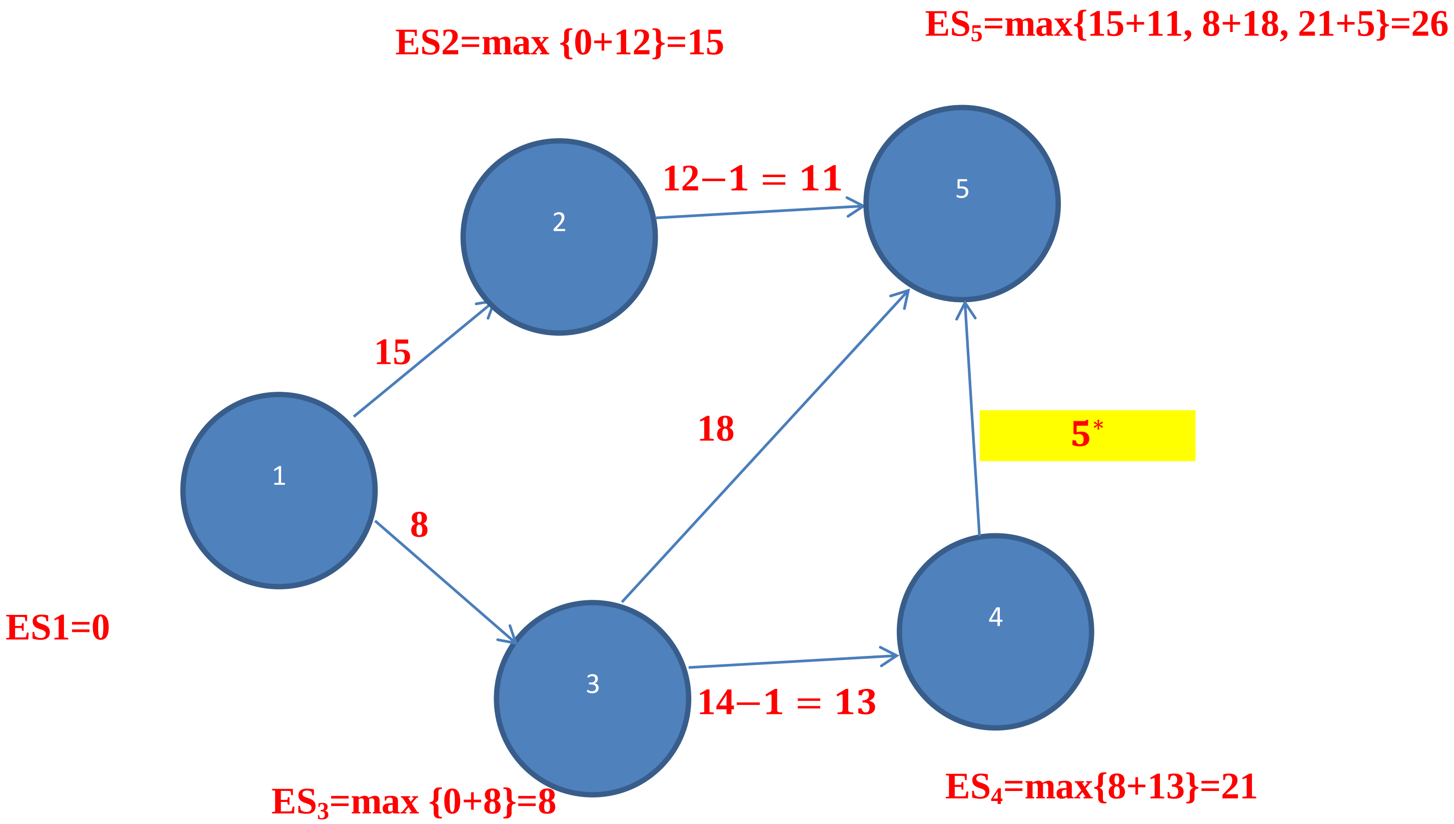
FF of (3,5)=27−8−18=1

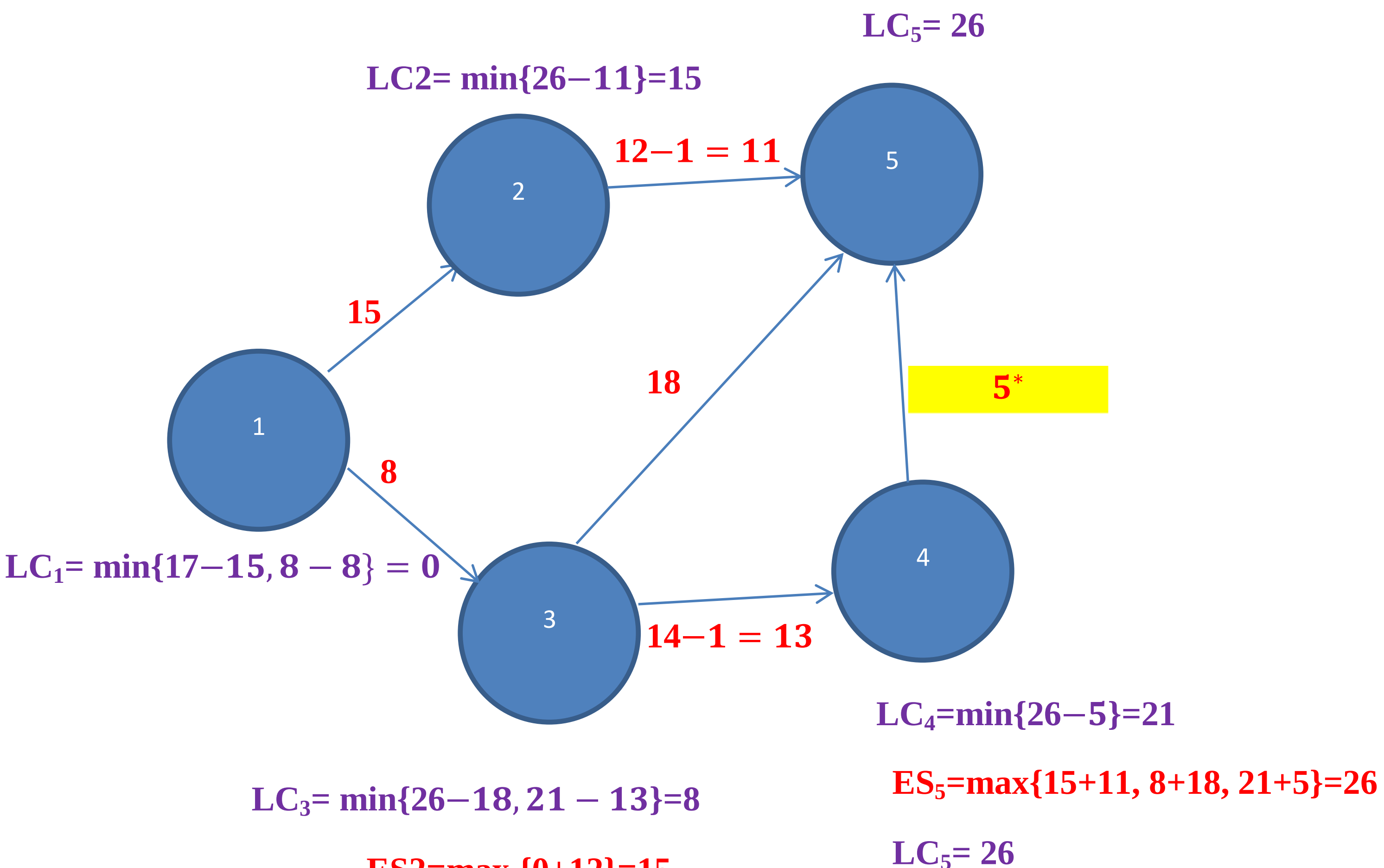
FF limit =minimum {1}=1

Compression limit=minimum {Crash limit, FF limit}={2,6,1}=1

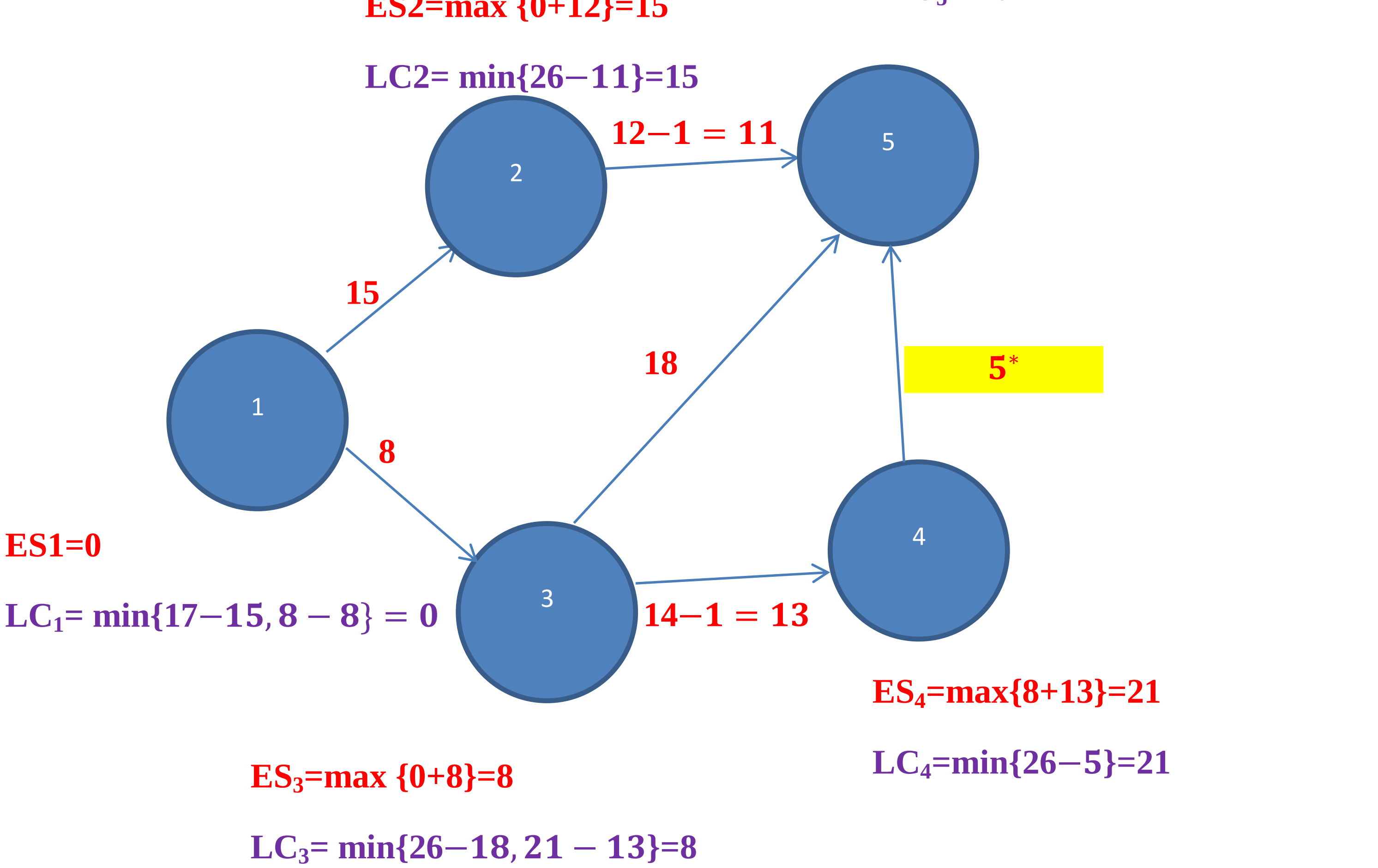
Since, the compression limit is 1. So maximum 1 day can be reduced from

the activity (3,4) and (2,5) at a time.





$LC_5 = 26$
LC2= min{26−11}=15
$12-1=11$
15
18
5^*
8
$LC_1 = \min\{17-15, 8-8\} = 0$
$14-1=13$
$LC_4 = \min\{26-5\} = 21$
$ES_5 = \max\{15+11, 8+18, 21+5\} = 26$
$LC_3 = \min\{26-18, 21-13\} = 8$
$LC_5 = 26$
1
2
3
4
5



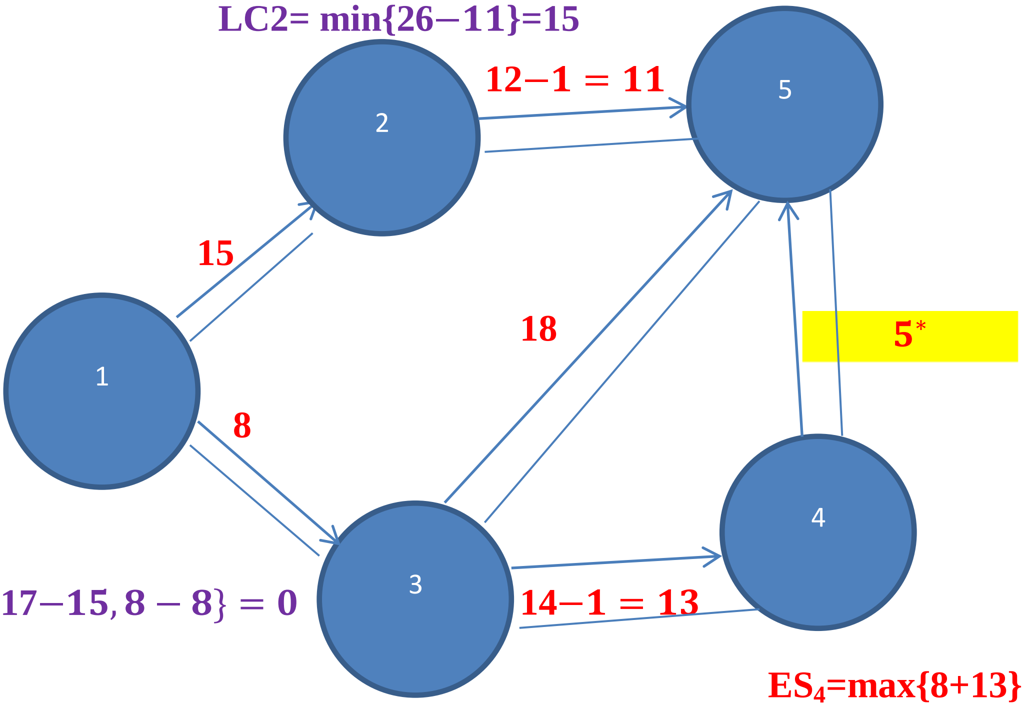
ES2=max {0+12}=15
LC2= min{26−11}=15
$12-1=11$
15
18
5^*
8
ES1=0
$LC_1 = \min\{17-15, 8-8\} = 0$
$14-1=13$
$ES_4 = \max\{8+13\} = 21$
$LC_4 = \min\{26-5\} = 21$
$ES_3 = \max\{0+8\} = 8$
$LC_3 = \min\{26-18, 21-13\} = 8$
1
2
3
4
5

ES_5=max{15+11, 8+18, 21+5}=26

LC_5= 26

ES2=max {0+12}=15

LC2= min{26−11}=15



ES1=0

LC_1= min{17−15, 8 − 8} = 0

ES_4=max{8+13}=21

LC_4=min{26−5}=21

ES_3=max {0+8}=8

LC_3= min{26−18, 21 − 13}=8

Current duration for completing the project is 26 days

Cost for completing the project in 26 days is

=4600+extra cost for reducing 1 day from (3,4)+ extra cost for reducing 1 day from (2,5)

=4350+1*Slope of (3,4)+ 1*Slope of (2,5)

=4350+1*250+1*125

=4975

To complete the project in 23 days there is a need to reduce 3 days.

To reduce the time for a project, there is a need to reduce the time of an activity from each critical path.

Since, in this case, there are three critical pats 1⟶3⟶ 4⟶ 5 , 1⟶2⟶ 5 and 1⟶3⟶ 5 . So, to reduce the project completion time, there is a need to reduce the time of an activity from both the critical paths.

There are three activities (1, 3), (3, 4) and (4, 5) on the critical path 1⟶3⟶ 4⟶ 5. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,4)=$\left|\frac{1400-650}{15-12}\right| = 250$

Slope of (4,5)=$\left|\frac{950-500}{8-5}\right| = 150$

Although, slope of (4, 5) is minimum, But, its current duration is equal to crash duration. So, time of this activity cannot be reduced further.

Since, the next activity with minimum slope of this critical path is (3,4). So, we will reduce time of (3,4).

There are two activities (1, 2) and (2, 5) on the critical path

$1 \longrightarrow 2 \longrightarrow 5$. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,2)=$\left|\frac{1200-600}{15-12}\right| = 200$

Slope of (2,5)=$\left|\frac{1500-750}{12-6}\right| = 125$

Since, slope of (2, 5) is minimum, So, the time of (2,5) will be reduced.

There are two activities (1, 3) and (3, 5) on the critical path

$1 \longrightarrow 3 \longrightarrow 5$. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,5)=$\left|\frac{1450-700}{18-3}\right| = 50$

Since, slope of (3, 5) is minimum, So, the time of (3,5) will be reduced.

Finally the selected activities are (3,4), (2,5) and (3,5)

It is pertinent to mention that as (1,3) is common for the critical paths $1 \longrightarrow 3 \longrightarrow 5$ and $1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$ so instead of reducing time of both (3,4) and (3,5), the time of (1,3) can be reduced if the following condition will be satisfied.

"Slope of (1,3)< Slope (3,4)+Slope(4,5)"

Since, it is satisfying so we will reduce time of (1,3).

Hence, final selected activities are (1,3) and (2,5).

The maximum time that can be reduced from an activity at a time is equal to its compression limit.

Compression limit=minimum {Crash limit, FF limit}

Crash limit for (1,3)=Current duration of (1,3) −Crash duration of (1,3)

= 8−5=3

Crash limit for (2,5)=Current duration of (2,5) −Crash duration of (2,5)

= 11−6=5

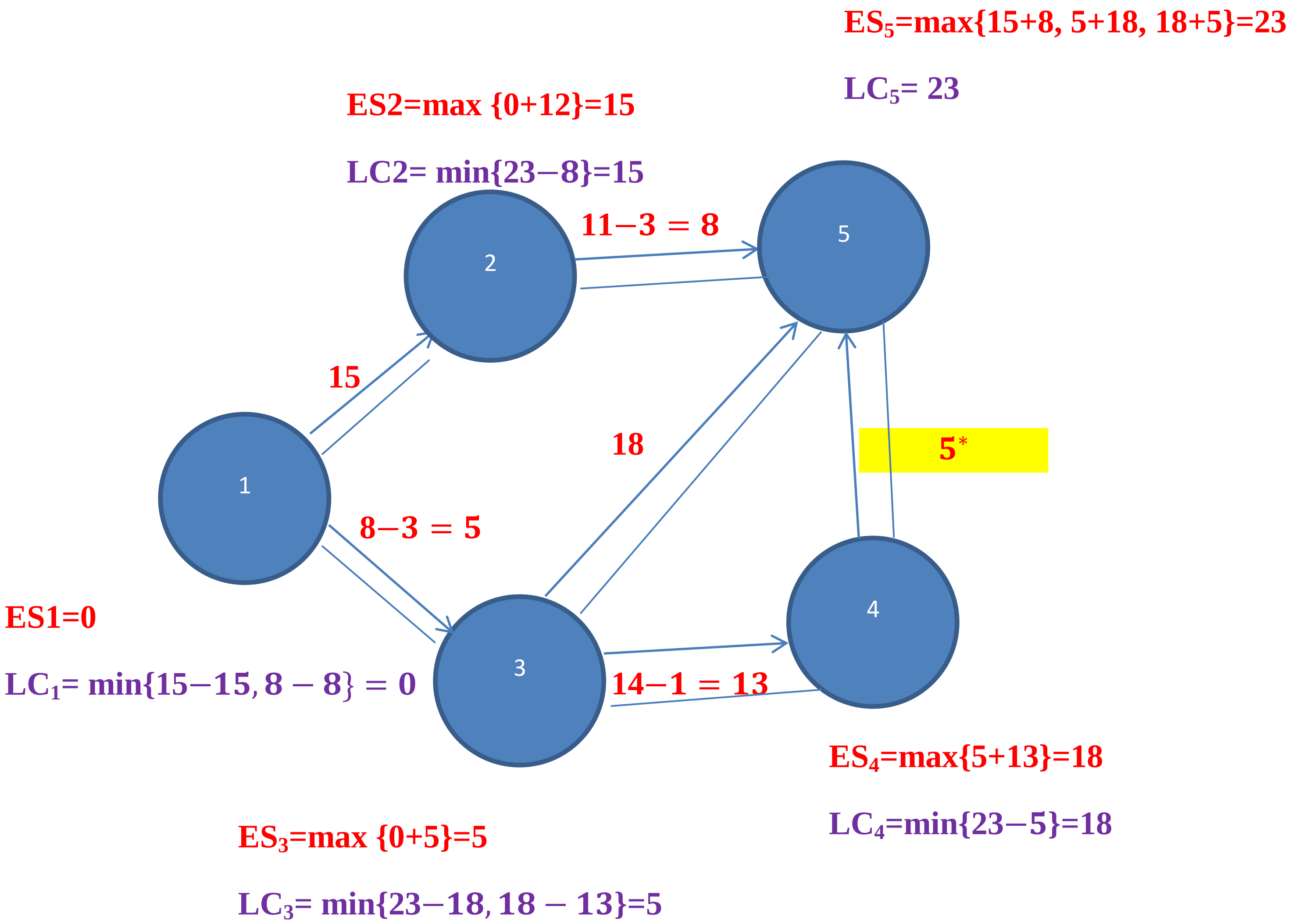
FF limit =minimum {non-zero free floats of non-critical activities}

Does not exist as no non-critical activity.

Compression limit=minimum {Crash limit}={3,5}=3

Since, the compression limit is 3. So maximum 3 days can be reduced from

the activity (1,3) and (2,5) at a time.



Current duration for completing the project is 23 days

Cost for completing the project in 23 days is

=4975+extra cost for reducing 3 days from (1,3)+ extra cost for reducing 3 day from (2,5)

=4975+3*Slope of (1,3)+ 3*Slope of (2,5)

=4975+3*300+3*125

=6250

To reduce the time for a project, there is a need to reduce the time of an activity from each critical path.

Since, in this case, there are three critical pats 1⟶3⟶ 4⟶ 5 , 1⟶2⟶ 5 and 1⟶3⟶ 5 . So, to reduce the project completion time, there is a need to reduce the time of an activity from both the critical paths.

There are three activities (1, 3), (3, 4) and (4, 5) on the critical path 1⟶3⟶ 4⟶ 5. Out of these activities that activity will be preferred for which slope is minimum.

Slope of (1,3)=$\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,4)=$\left|\frac{1400-650}{15-12}\right| = 250$

Slope of (4,5)=$\left|\frac{950-500}{8-5}\right| = 150$

Although, slope of (4, 5) is minimum, But, its current duration is equal to crash duration. So, time of this activity cannot be reduced further.

Since, the next activity with minimum slope of this critical path is (3,4). So, we will reduce time of (3,4).

There are two activities (1, 2) and (2, 5) on the critical path

$1 \longrightarrow 2 \longrightarrow 5$. **Out of these activities that activity will be preferred for which slope is minimum.**

Slope of (1,2)= $\left|\frac{1200-600}{15-12}\right| = 200$

Slope of (2,5)= $\left|\frac{1500-750}{12-6}\right| = 125$

Since, slope of (2, 5) is minimum, So, the time of (2,5) will be reduced.

There are two activities (1, 3) and (3, 5) on the critical path

$1 \longrightarrow 3 \longrightarrow 5$. **Out of these activities that activity will be preferred for which slope is minimum.**

Slope of (1,3)= $\left|\frac{1600-700}{8-5}\right| = 300$

Slope of (3,5)= $\left|\frac{1450-700}{18-3}\right| = 50$

Since, slope of (3, 5) is minimum, So, the time of (3,5) will be reduced.

Finally the selected activities are (3,4), (2,5) and (3,5)

It is pertinent to mention that as (1,3) is common for the critical paths $1 \longrightarrow 3 \longrightarrow 5$ and $1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$ so instead of reducing time of both (3,4) and (3,5), the time of (1,3) can be reduced if the following condition will be satisfied.

"Slope of (1,3)< Slope (3,4)+Slope(4,5)"

Since, it is satisfying so we will reduce time of (1,3). But as the current

Duration of (1,3) is equal to its crash duarion. So, we cannot reduce its time.

Hence, need to reduce time of all the three activities (3,4), (3,5) and (2,5).

The maximum time that can be reduced from an activity at a time is equal to its compression limit.

Compression limit=minimum {Crash limit, FF limit}

Crash limit for (3,4)=Current duration of (3,4) −Crash duration of (3,4)

= 13−12=1

Crash limit for (3,5)=Current duration of (3,5) −Crash duration of (3,5)

= 18−3=5

Crash limit for (2,5)=Current duration of (2,5) −Crash duration of (2,5)

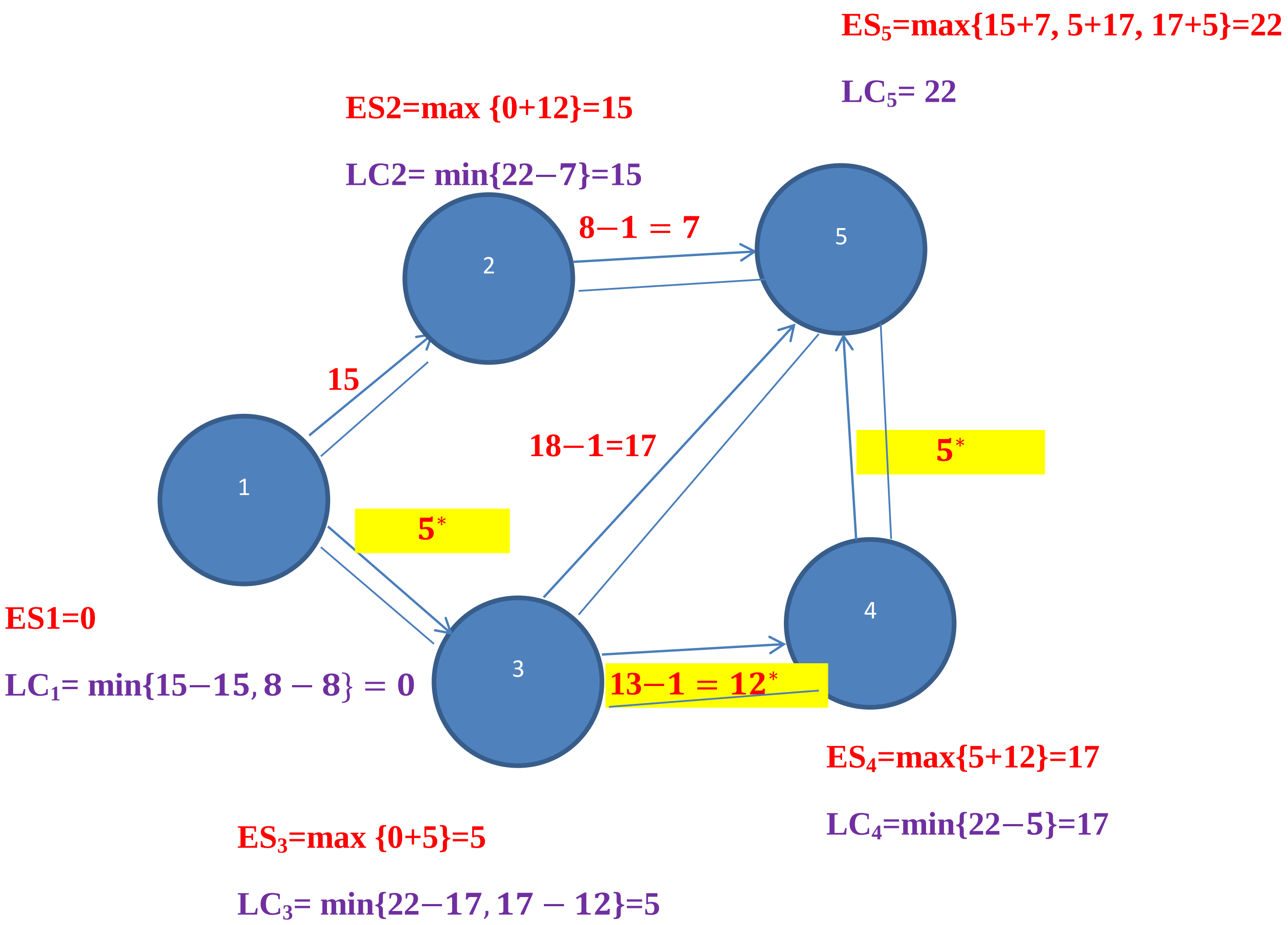
= 8−6=2

FF limit =minimum {non-zero free floats of non-critical activities}

Does not exist as no non-critical activity.

Compression limit=minimum {Crash limit}={1, 5, 2}=1

Since, the compression limit is 1. So maximum 1 day can be reduced from the activity (3,4), (3,5) and (2,5) at a time.



Current duration for completing the project is 22 days

Cost for completing the project in 22 days is

=6250+extra cost for reducing 1 day from (3,4)+ extra cost for reducing 1 day from (3,5)+ extra cost for reducing 1 day from (2,5)

=6250+1*Slope of (3,4)+ 1*Slope of (3,5)+ 1*Slope of (2,5)

=6250+1*250+1*125+1*50

=6675

To reduce the time for a project, there is a need to reduce the time of an activity from each critical path.

But as all the activities of the critical path 1 ⟶ 3 ⟶ 4 ⟶ 5 has reached at its crash duration. So, time of any activity on this critical path cannot be reduced.

Hence, project completion time cannot be reduced further.

Crash duration for completing the project is 22 days

Crash cost for completing the project is Rs. 6675.