*For reference purpose only

SET THEORY

- Ms. Rajanpreet Kaur Chahal
Assistant Professor, CSE Dept
Thapar Institute of Engineering & Technology
rajanpreet@thapar.edu

Topics covered

- Set Operations
- Bit representation of sets
- Multi-set
- Fuzzy Set
- Inclusion-Exclusion Principle
- Partition and Covering of a set

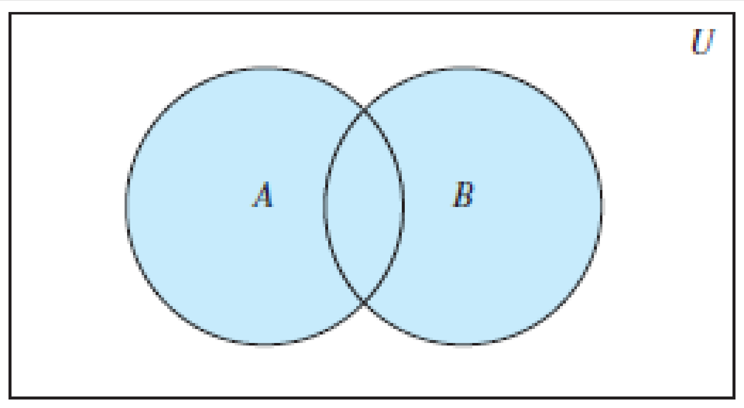
Set Operations

- Union
- Intersection
- Difference
- Symmetric Difference
- Cartesian Product
- Complement

Union

- Let A and B be two sets. The union of sets A and B, represented as $A \cup B$, is a set containing the elements that are either in A or in B, or in both. In other words, $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Venn diagram for $A \cup B$:

Shaded area represents $A \cup B$ →



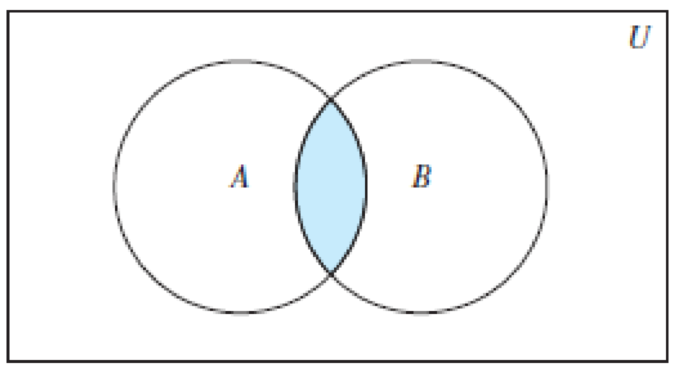
- Let $A = \{a,c,d\}$ and $B = \{c,d,e\}$ be two sets. Find their union.

  Ans: $A \cup B = \{a,c,d,e\}$

Intersection

- Let A and B be two sets. The intersection of sets A and B, represented as $A \cap B$, is a set containing the elements that are in both A and B. In other words, $A \cap B = \{x \mid x \in A \land x \in B\}$.
- Venn diagram for $A \cap B$:

Shaded area represents $A \cap B$ $\longrightarrow$



- Let $A = \{a,c,d\}$ and $B = \{c,d,e\}$ be two sets. Find their intersection.

  Ans: $A \cap B = \{c,d\}$

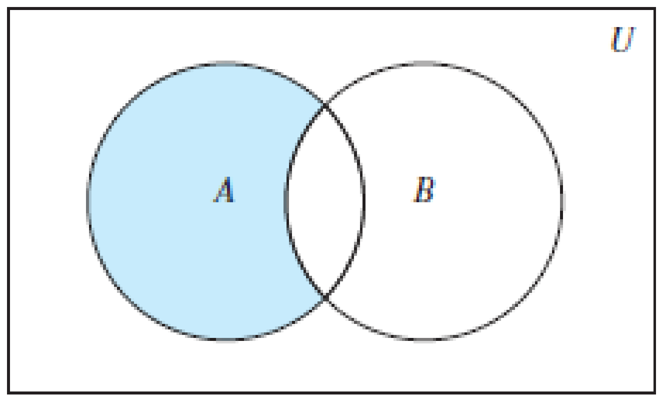
Disjoint Sets

- Two sets are known as disjoint if their intersection results in an empty set.
- For ex, let A = {1,3,5} and B = {2,4,6}. Here, A and B are disjoint because $A \cap B = \emptyset$.

Difference

- Let A and B be two sets. The difference of sets A and B, represented as $A - B$, is a set containing those elements that are in A but not in B. The difference of A and B is also known as the complement of B with respect to A.
- In other words, $A - B = \{x \mid x \in A \land x \notin B\}$.
- Venn diagram for $A – B$:

Shaded area represents $A - B$ ⟶



Difference

- Let $A = \{a,c,d\}$ and $B = \{c,d,e\}$ be two sets. Find $A – B$ and $B - A$.

Ans: $A – B = \{a\}$

$B – A = \{e\}$

***Note that A – B ≠ B – A. Hence, set difference is not commutative.**

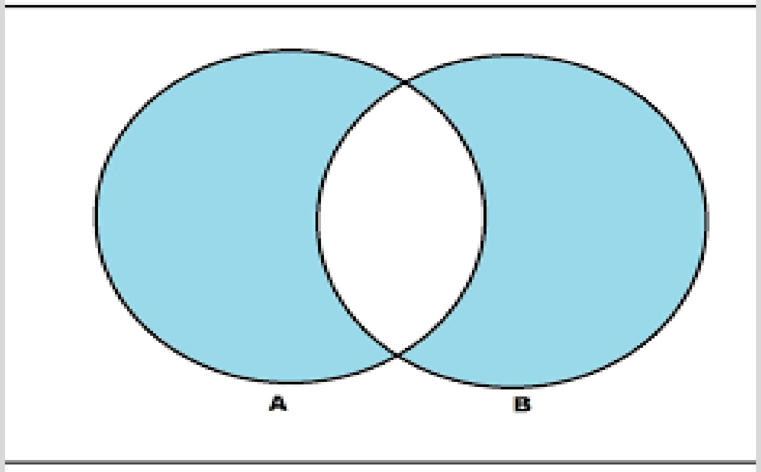
Symmetric Difference

- Let A and B be two sets. The symmetric difference of sets A and B, represented as $A \oplus B$, is a set containing those elements that are in either A or B, but not in both.
- In other words, $A \oplus B = (A - B) \cup (B - A)$, or

$$A \oplus B = (A \cup B) - (A \cap B)$$

- Venn diagram of $A \oplus B$:

Shaded area represents $A \oplus B$ →



Symmetric Difference

- Let $A = \{a,c,d\}$ and $B = \{c,d,e\}$ be two sets. Find $A \oplus B$.

  Ans: $A - B = \{a\}$, $B - A = \{e\}$. So, $A \oplus B = (A - B) \cup (B - A) = \{a,e\}$

  OR

  $A \cup B = \{a,c,d,e\}$, $A \cap B = \{c,d\}$. So, $A \oplus B = (A \cup B) - (A \cap B) = \{a,e\}$

Cartesian Product

- Let A and B be two sets. The cartesian product of sets A and B, represented as $A \times B$, is the set of all ordered pairs (a,b) where a $\in A$ and b $\in B$. In other words, $A \times B = \{(a,b) \mid a \in A \ \wedge \ b \in B\}$.
- For ex, consider sets $A = \{1,2,3\}$ and $B = \{a,b\}$. Find $A \times B$ and $B \times A$.

Ans: $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B \times A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

***Note: A × B ≠ B × A unless A = ∅ or B = ∅ (which makes A × B = ∅) or A = B.**

Cartesian Product

- Find $A \times B \times C$ where $A = \{a,b\}$, $B = \{c,d\}$ and $C = \{e,f,g\}$.

  Ans: $A \times B \times C = \{(a,c,e), (a,c,f), (a,c,g), (a,d,e), (a,d,f), (a,d,g), (b,c,e), (b,c,f), (b,c,g), (b,d,e), (b,d,f), (b,d,g)\}$

- Let $A = \{a,b\}$. Find A^2 and A^3.

  Ans: $A^2 = A \times A =$ **{(a,a), (a,b), (b,a), (b,b)}**

  $A^3 = A^2 \times A =$ **{(a,a,a), (a,a,b), (a,b,a), (a,b,b), (b,a,a), (b,a,b), (b,b,a), (b,b,b)}**

- A subset R of the cartesian product $A \times B$ is called a **relation** from set A to set B.

Cardinality of Cartesian Product

- The cardinality of the output set (i.e. set obtained after cartesian product) is equal to the product of cardinality of input sets (i.e. individual sets whose cartesian product is being taken).
- In other words, we can say that $|A \times B| = |A| \,.\, |B|$,

  where $|A|$ = cardinality of set *A*.
- For ex, find the number of elements in $A \times B \times C$ where $A = \{a,b\}$, $B = \{c,d\}$ and $C = \{e,f,g\}$.

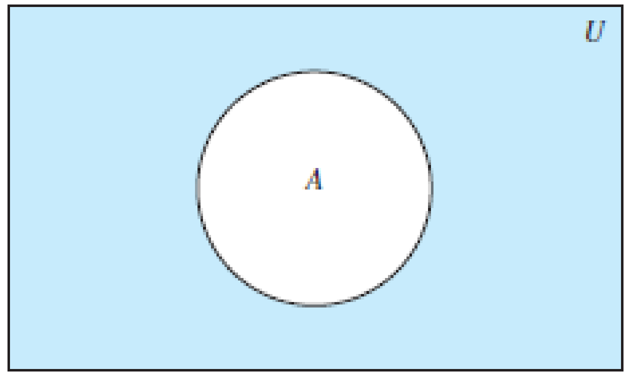
  Ans: $|A| = 2$, $|B| = 2$, $|C| = 3$

  Therefore, $|A \times B \times C| = 2.2.3 = 12$

Complement

- Let U be the universal set. The complement of a set is defined with respect to U.
- Let A be a set. The complement of A contains those elements which are present in U but not in A. Complement of A is denoted as $\bar{A}$.
- An element x belongs to $\bar{A}$ if and only if $x \notin A$. In other words, $\bar{A} = \{x \in U \mid x \notin A\}$.
- Venn diagram for $\bar{A}$:

Shaded area represents $\bar{A}$ $\longrightarrow$



Complement

- Let A be the set of positive integers greater than 5. Find $\bar{A}$.

  Ans: Here, U will be the set of all positive integers. So,

  $$\bar{A} = \{1,2,3,4,5\}$$

***Note: We can also define $\boldsymbol{A - B}$ in terms of complement and intersection, i.e.,**

$$\boldsymbol{A - B = A \cap \bar{B}}$$

Important points

- Complement of universal set U is $\emptyset$.
- Complement of $\emptyset$ is U.
- $A \cup \bar{A} = U$.
- $A \cap \bar{A} = \emptyset$. Hence, A and $\bar{A}$ are disjoint sets.

Bit representation of sets

- Also known as computer representation of sets.
- It means representing sets in a computer.
- The technique is to store elements by using an arbitrary ordering of elements in the universal set.
- Assumption: Universal set U is finite and of reasonable size (such that size of U is not larger than the memory size of the computer!)

Bit representation of sets

- Technique:
  - Specify an arbitrary ordering of elements in *U* (if not already given).
  - Let the ordering be $a_1, a_2, \ldots\ldots, a_n$.
  - Represent a subset *A* of *U* with a bit string of length *n*, where i^{th} bit of this bit string is 1 if $a_i \in$ A and 0 if $a_i \notin$ A.

Bit representation of sets

- Let $U = \{1,2,3,4,5,6,7,8,9,10\}$. Write the computer representation/bit strings of following subsets of U:
  - All odd integers

    Ans: The bit string to represent the subset of all odd integers in U, i.e. {1,3,5,7,9} will be of length 10 where bits in positions 1,3,5,7,9 will be 1 and all other bits will be 0, i.e., 1010101010
  - All even integers

    Ans: The subset is {2,4,6,8,10} and its bit representation is 0101010101.

Bit representation of sets

- All integers greater than 5

  Ans: The subset is {6,7,8,9,10} and its bit representation is 0000011111.

- All integers less than 5

  Ans: The subset is {1,2,3,4} and its bit representation is 1111000000.

Multi-set

- A multi-set is an unordered collection of elements/members where an element may occur more than once in the set.
- It is represented as $\{m_1.a_1, m_2.a_2, \ldots \ldots, m_n.a_n\}$ where m_i refers to the multiplicity of element a_i, i.e. the number of times element a_i occurs in the multi-set. Here, $i = 1,2,\ldots\ldots,n$.
- For ex, multi-set $A = \{1,2,3,1,1,2,3,4,3,5,3,3\}$ can be written as

  $A = \{3.1, 2.2, 4.3, 1.4, 1.5\}$

Operations on Multi-set

- Union
- Intersection
- Difference
- Sum

Union of Multi-sets

- Let P and Q be two multi-sets. The union of P and Q is the multi-set where multiplicity of each element is the maximum of its multiplicities in P and Q.
- For ex, let $A = \{3.p, 2.q, 1.r\}$ and $B = \{2.p, 3.q, 4.s\}$. Find $A \cup B$.

  Ans: $A \cup B = \{3.p, 3.q, 1.r, 4.s\}$

Intersection of Multi-sets

- Let P and Q be two multi-sets. The intersection of P and Q is the multi-set where multiplicity of each element is the minimum of its multiplicities in P and Q.
- For ex, let $A = \{3.p, 2.q, 1.r\}$ and $B = \{2.p, 3.q, 4.s\}$. Find $A \cap B$.

  Ans: $A \cap B = \{2.p, 2.q\}$

Difference of Multi-sets

- Let P and Q be two multi-sets. The difference of P and Q is the multi-set where multiplicity of an element is its multiplicity in P minus its multiplicity in Q, unless the difference comes out to be negative, in which case multiplicity is taken as 0.
- For ex, let $A = \{3.p, 2.q, 1.r\}$ and $B = \{2.p, 3.q, 4.s\}$. Find $A - B$ and $B - A$.

  Ans: $A - B = \{1.p, 1.r\}$

  $B - A = \{1.q, 4.s\}$

Sum of Multi-sets

- Let P and Q be two multi-sets. The sum of P and Q is the multi-set where multiplicity of an element is the sum of its multiplicities in P and Q.
- For ex, let $A = \{3.p, 2.q, 1.r\}$ and $B = \{2.p, 3.q, 4.s\}$. Find $A + B$.

  Ans: $A + B = \{5.p, 5.q, 1.r, 4.s\}$

Fuzzy Sets

- Fuzzy sets are used in Artificial Intelligence.
- Every element in the universal set *U* has a degree of membership in the fuzzy set *S*.
- The degree of membership is a real number between 0 and 1 (both inclusive). Elements with 0 degree of membership are not listed.
- A fuzzy set is represented as $\{\mu_1 a_1, \mu_2 a_2, \ldots \ldots, \mu_n a_n\}$ where μ_i refers to the degree of membership of element a_i in the given fuzzy set. Here, *i=1,2,.....,n.*
- For ex, let *H = {0.4 X, 0.6 Y, 0.7 Z, 0.5 W}* be the set of honest people. This means that *X* has *0.4* degree of membership in fuzzy set *H*, *Y* has *0.6* and so on.

Operations on Fuzzy Sets

- Complement
- Union
- Intersection

Complement of Fuzzy Set

- The complement of a fuzzy set S, written as $\bar{S}$, is the set in which the degree of membership of an element is 1 minus its degree of membership in the set S.
- For ex, consider $H = \{0.4\ X, 0.6\ Y, 0.7\ Z, 0.5\ W\}$ as the set of honest people. Find $\overline{H}$, i.e. the fuzzy set of people who are not honest.

  Ans: $\overline{H} = \{0.6\ X, 0.4\ Y, 0.3\ Z, 0.5\ W\}$

Union of Fuzzy Sets

- The union of two fuzzy sets *R* and *S*, written as $R \cup S$, is the fuzzy set in which the degree of membership of an element is the maximum of its degrees of membership in sets *R* and *S*.
- For ex, let $H = \{0.4\ X, 0.6\ Y, 0.7\ Z, 0.5\ W\}$ and $T = \{0.7\ X, 0.8\ Y, 0.3\ Z, 0.4\ W\}$ be two fuzzy sets. Find $H \cup T$.

  Ans: $H \cup T = \{0.7\ X, 0.8\ Y, 0.7\ Z, 0.5\ W\}$

Intersection of Fuzzy Sets

- The intersection of two fuzzy sets R and S, written as $R \cap S$, is the fuzzy set in which the degree of membership of an element is the minimum of its degrees of membership in sets R and S.
- For ex, let $H = \{0.4\ X, 0.6\ Y, 0.7\ Z, 0.5\ W\}$ and $T = \{0.7\ X, 0.8\ Y, 0.3\ Z, 0.4\ W\}$ be two fuzzy sets. Find $H \cap T$.

  Ans: $H \cap T = \{0.4\ X, 0.6\ Y, 0.3\ Z, 0.4\ W\}$

Cardinality & Relative Cardinality of Fuzzy Set

- Let $H = \{0.4\ X, 0.6\ Y, 0.7\ Z, 0.5\ W\}$ be the fuzzy set.
- Cardinality of $H = 0.4+0.6+0.7+0.5 = 2.2$
- Relatively cardinality of $H = 2.2/4 = 0.55$

Inclusion-Exclusion Principle

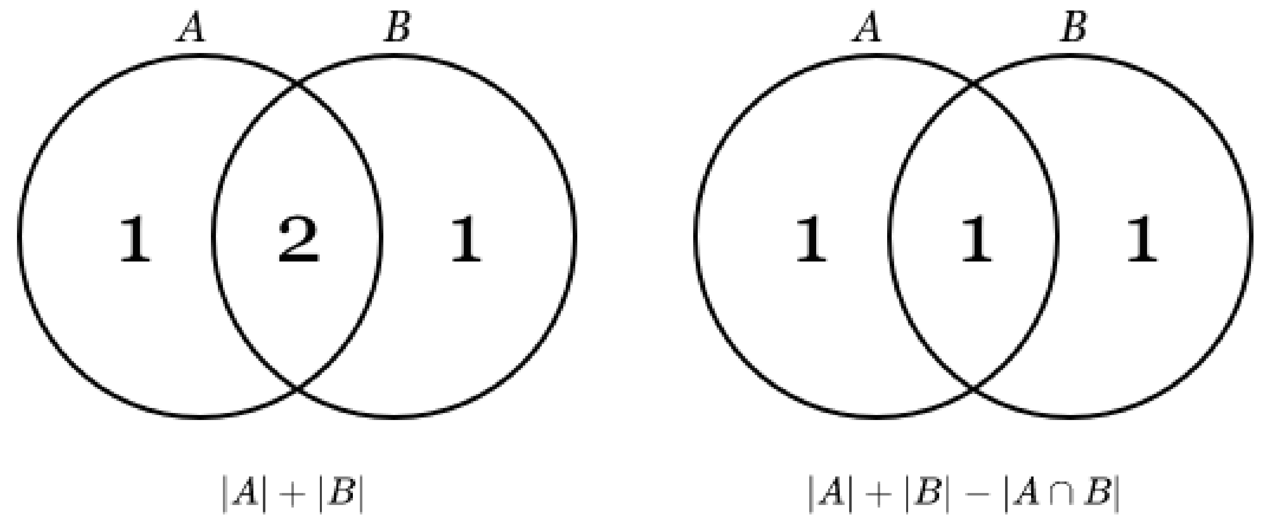
- Inclusion-exclusion principle states that the number of elements in the union of two sets is equal to the sum of the numbers of elements in the sets minus the number of elements in their intersection, i.e.,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Similarly, for three sets, inclusion-exclusion principle states that:

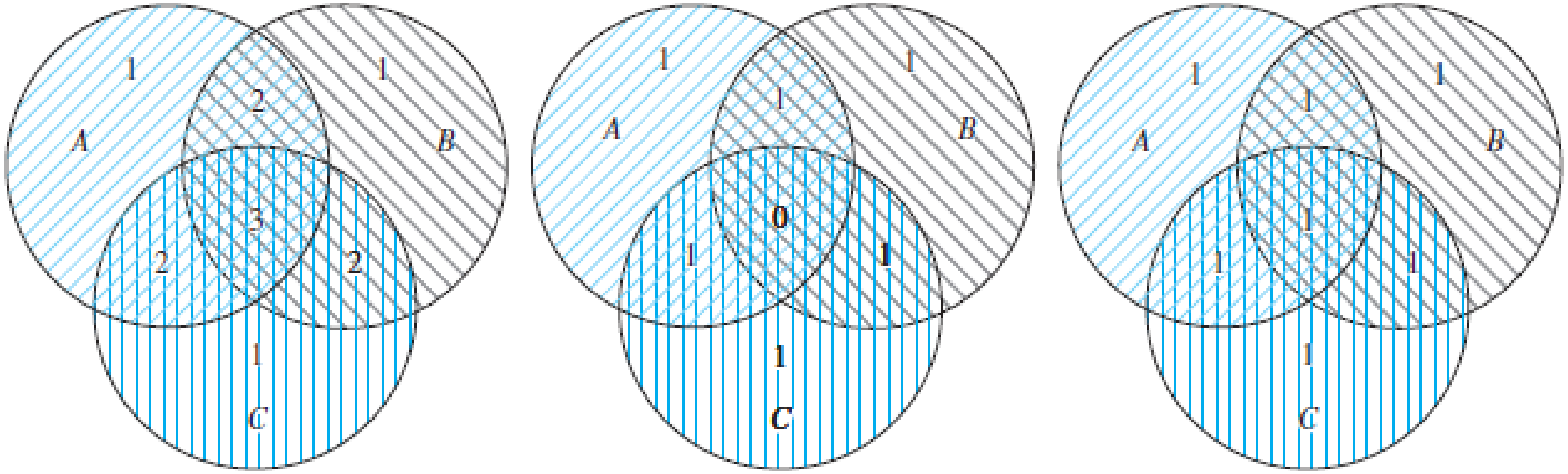
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion Principle for 2 sets



The number in each subset represents how many times that subset is counted.
Note that by adding the elements in sets A and B, their intersection is counted twice.

Inclusion-Exclusion Principle for 3 sets



(a) Count of elements by
$|A|+|B|+|C|$

(b) Count of elements by
$|A|+|B|+|C|-|A\cap B|-$
$|A\cap C|-|B\cap C|$

(c) Count of elements by
$|A|+|B|+|C|-|A\cap B|-$
$|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Inclusion-Exclusion Principle

- In general, inclusion-exclusion principle gives a formula to find the number of elements in union of *n* sets, where *n* is a positive integer.
- Number of terms in the formula for inclusion-exclusion principle of *n* sets is $2^n - 1$.
- For ex, find the number of terms in the inclusion-exclusion formula for 4 sets.

  Ans: Here, n =4. Therefore, number of terms in the formula will be $2^4 - 1 = 15$.

Inclusion-Exclusion Principle

- Write the formula for inclusion-exclusion principle of four sets.

Ans: $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

Inclusion-Exclusion Principle

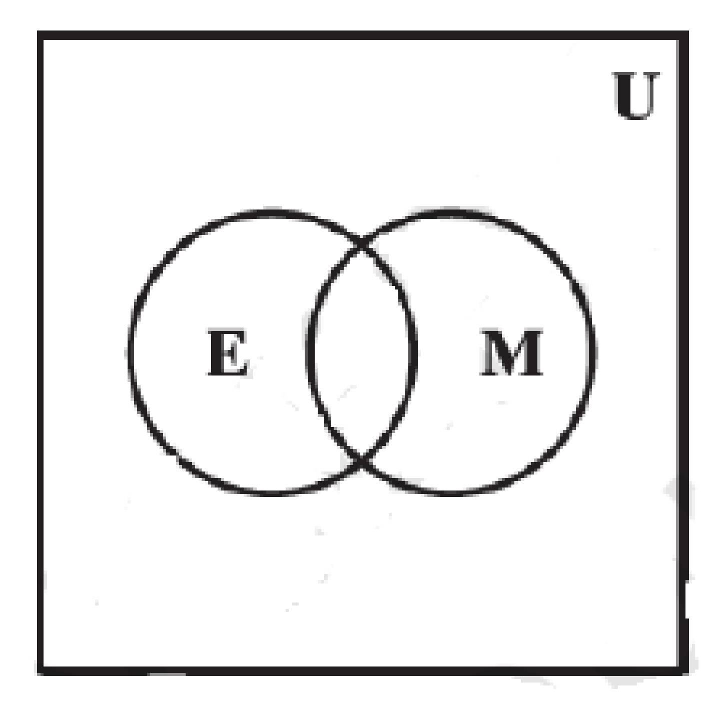
- Suppose that there are 1500 students at your school. Of these, 325 are taking a course in English, 455 are taking a course in Mathematics, and 129 are taking courses in both English and Mathematics. How many are not taking a course either in English or in Mathematics?

Ans: $|E| = 325$, $|M| = 455$, $|E \cap M| = 129$

No. of students taking a course in either English or Mathematics is:

$|E \cup M| = |E| + |M| - |E \cap M| = 325 + 455 - 129 = 651$

No. of students **not** taking a course in either English or Mathematics = 1500 – 651 = 849



Inclusion-Exclusion Principle

- A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

  Ans: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then,

  $|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

  $|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$

  Therefore, $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

  $|S \cap F \cap R| = 7$.

  Hence, there are seven students who have taken courses in Spanish, French, and Russian.

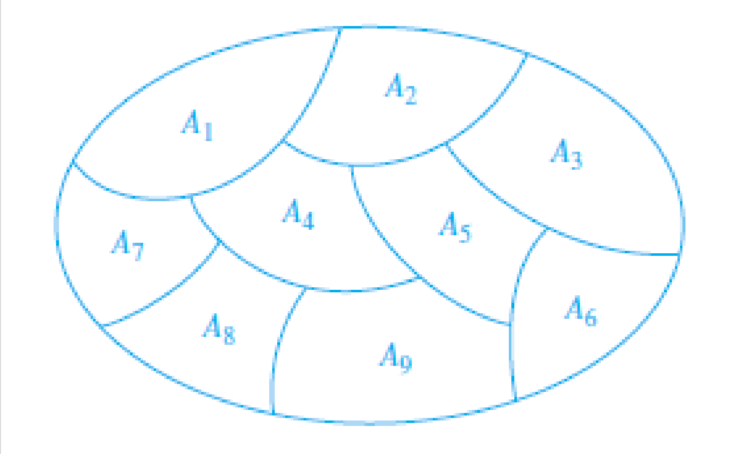
Partition of a set

- Partition of a set S is the collection of disjoint non-empty subsets of S whose union results in set S.
- In other words, we can say that the collection of subsets A_i, $i \in I$ (where I is the index set) is the partition of set S, if and only if:

  $A_i \neq \emptyset$ for $i \in I$,

  $A_i \cap A_j = \emptyset$ when $i \neq j$, and

  $\bigcup_{i \in I} A_i = S.$



- Index set is the set whose members label/index the elements of a set.

Partition of a set

- For ex, suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S.

Covering of a set

- Let S be a given set and $A = \{A_1, A_2, \ldots \ldots \ldots, A_m\}$ where each A_i is a non-empty subset of S and $\bigcup_{i=1}^{m} A_i = S$, then set A is called covering of set S and the subsets $A_1, A_2, \ldots \ldots \ldots, A_m$ are said to cover set S.

Partition and Covering of a set

- Let $S = \{a,b,c\}$ be a set. Consider the following collections of subsets of S. Determine which of the following are partitions of S.

  i) $A = \{\{a,b\}, \{b,c\}\}$

  ii) $B = \{\{a\}, \{a,c\}\}$

  iii) $C = \{\{a\}, \{b,c\}\}$

  iv) $D = \{\{a,b,c\}\}$

  v) $E = \{\{a\},\{b\}, \{c\}\}$

  vi) $F = \{\{a\}, \{a,b\}, \{a,c\}\}$

Partitions: C,D,E

Covering: A,C,D,E,F

Exercise…

- Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset {107 . personal computers, 44 . routers, 6 . servers} and B could be the multiset {14 . personal computers, 6 . routers, 2 . mainframes}.

a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment? **A ∪ B = {107 . personal computers, 44 . routers, 6 . servers, 2 . mainframes }**

b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment? **A ∩ B = {14 . personal computers, 6 . routers }**

c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment? **B – A = { 2 . mainframes }**

d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment? **A + B = {121 . personal computers, 50 . routers, 6 . servers, 2. mainframes }**

Exercise…

- Let F = {*0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita*} be the set of famous people and R = {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita} be the set of rich people.

  a) i) Find the set of people who are not famous.

  $\overline{F}$= {***0.4 Alice, 0.1 Brian, 0.6 Fred, 0.9 Oscar, 0.5 Rita***}

  ii) Find the set of people who are not rich.

  $\overline{R}$= {***0.6 Alice, 0.2 Brian, 0.8 Fred, 0.1 Oscar, 0.3 Rita***}

  b) Find the fuzzy set $F \cup R$ of rich or famous people.

  F ∪ R = {*0.6 Alice, 0.9 Brian, 0.4 Fred, 0.9 Oscar, 0.7 Rita*}

  c) Find the fuzzy set $F \cap R$ of rich and famous people.

  F ∩ R = {*0.4 Alice, 0.8 Brian, 0.2 Fred, 0.1 Oscar, 0.5 Rita*}

Exercise…

- How many positive integers not exceeding 1000 are divisible by 7 or 11?

Ans: $|A \cup B| = |A| + |B| - |A \cap B| = \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7.11} \right\rfloor$

$= 142 + 90 – 12 = 220$

Therefore, 220 positive integers not exceeding 1000 are divisible by either 7 or 11.

Exercise…

- How many elements are in $A1 \cup A2$ if there are 12 elements in $A1$, 18 elements in $A2$, and

  a) $A1 \cap A2 = \emptyset$? **12 + 18 = 30**

  b) $|A1 \cap A2| = 1$? **12 + 18 – 1 = 29**

  c) $|A1 \cap A2| = 6$? **12 + 18 – 6 = 24**

  d) $A1 \subseteq A2$? **All elements of A1 are contained in A2. Therefore, ans is 18.**

- Find the number of elements in $A1 \cup A2 \cup A3$ if there are 100 elements in $A1$, 1000 in $A2$, and 10,000 in $A3$ if

  a) $A1 \subseteq A2$ and $A2 \subseteq A3$. **All elements of A1 and A2 are contained in A3. Therefore, ans is 10,000.**

  b) the sets are pairwise disjoint. **100 + 1000 + 10000 = 11,100**

  c) there are two elements common to each pair of sets and one element in all three sets.

  100 + 1000 + 10000 – 2 – 2 – 2 + 1 = 11095

References

- Rosen H. K., Discrete Mathematics and its Applications, McGraw Hill (2011) 7th ed.

Questions ?