Lecture 22: Numerical Analysis (UMA011)

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Iterative methods to solve System of linear equations

Jacobi Method

Consider the system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad - \quad (1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad - \quad (2)\\
\vdots \qquad\quad \vdots \qquad\quad\;\; \vdots \quad &= \;\vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad - \quad (n)
\end{aligned}
$$

$X = (x_1, x_2 \, - - - \, x_n)^t$

Given initial guess is $X^{(0)} = (x_1^{(0)}, x_2^{(0)} \, - - - \, x_n^{(0)})$

To find I^{st} iteration $\rightarrow$ $X^{(1)} = (x_1^{(1)}, x_2^{(1)} \, - - - \, x_n^{(1)})$

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a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad - \; 2\\
\vdots \qquad \vdots \qquad \vdots \;\; &= \;\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad - \; n
\end{aligned}$$

$X = (x_1, x_2 - - - x_n)^t$

Given initial guess is $X^{(0)} = (x_1^{(0)}, x_2^{(0)} - - - x_n^{(0)})$

To find I^{st} iteration $\rightarrow$ $X^{(1)} = (x_1^{(1)}, x_2^{(1)} - - - x_n^{(1)})$;

from eqⁿ ①

$$a_{11}x_1 = b_1 - (a_{12}x_2 + a_{13}x_3 - - - a_{1n}x_n)$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - (a_{12}x_2^{(k)} + a_{13}x_3^{(k)} - - - a_{1n}x_n^{(k)})\right)$$

$$= \frac{1}{a_{11}}\left(b_1 - \sum_{j=2}^{n} a_{1j}x_j^{(k)}\right)$$

2nd iteration

$$x^{(2)} = (x_1^{(2)}, x_2^{(2)} - - - x_n^{(2)})$$

(k+1)th iteration

$$x^{(k+1)} = (x_1^{(k+1)} - - - x_n^{(k+1)})$$

from eqⁿ ②

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - (a_{21}x_1^{(k)} + a_{23}x_3^{(k)} - - - a_{2n}x_n^{(k)})\right)$$

$$= \frac{1}{a_{22}}\left(b_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j^{(k)}\right)$$

$$- - - - - - - - - - -$$

from eqⁿ ⓝ

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left(b_n - (a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} - - - a_{n,n-1}x_{n-1}^{(k)})\right)$$

$$= \frac{1}{a_{nn}}\left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j^{(k)}\right)$$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\, x_j^{(k)}\right)} \longrightarrow \text{Jacobi Method.}$$

for the Given tolerance

Stopping criterian $\quad \| x^{(2)} - x^{(1)} \|_\infty = \max\{ |x_1^{(2)} - x_1^{(1)}|, |x_2^{(2)} - x_2^{(1)}| - - - |x_n^{(2)} - x_n^{(1)}| \}$

$\leq$ tolerance

in general $\quad \| x^{(n+1)} - x^{(n)} \|_\infty \leq$ tolerance

Iterative methods to solve System of linear equations

Gauss-Seidel Method

Consider the system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Given initial guess $X^0 = (x_1^{(0)}, x_2^{(0)} - - - x_n^{(0)})$

from eqn ①

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)} \cdots x_n^{(1)}\right)$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - \left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right)\right)$$

$$= \frac{1}{a_{11}}\left(b_1 - \sum_{j=2}^{n} a_{1j}\, x_j^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - \left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right)\right)$$

$$= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - \sum_{j=3}^{n} a_{2j}\, x_j^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - \left(a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)} - \cdots a_{3n}x_n^{(k)}\right)\right)$$

$$= \frac{1}{a_{33}}\left(b_3 - \sum_{j=1}^{2} a_{3j}\, x_j^{(k+1)} - \sum_{j=4}^{n} a_{3j}\, x_j^{(k)}\right)$$

— — — — — — — — — —

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left(b_n - \left(a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{n,n-1}x_{n-1}^{(k-1)}\right)\right)$$

$$= \frac{1}{a_{nn}}\left(b_n - \sum_{j=1}^{n-1} a_{nj}\, x_j^{(k+1)}\right)$$

In general,

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)}\right)}$$

Gauss-seidel Method

Iterative methods to solve System of linear equations

strictly diagonally dominant matrix:

A square matrix A is called diagonally dominant if

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}| \; \forall \; i.$$

A is called strictly diagonally dominant if

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}| \; \forall \; i.$$

$$\begin{bmatrix} a_{11} & a_{12} & - & a_{1n} \\ a_{21} & a_{22} & - & a_{2n} \\ - & - & - & \\ a_{n1} & - & - & a_{nn} \end{bmatrix}$$

$$|a_{11}| \geq |a_{12}| + |a_{13}| - + |a_{1n}|$$

Iterative methods to solve System of linear equations

Example:

Check whether the following matrices are strictly diagonal dominant or not: $A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -6 & 3 \\ -2 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

Sol To check A

$|4| > |-2| + |1|$ ✓

$|-6| > |1| + |3|$ ✓

$|5| > |-2| + |2|$ ✓

A is S.D.D.

To check B

$|-2| \not> |2| + |1|$

$\Rightarrow$ B is not S.D.D

Iterative methods to solve System of linear equations

Result:

If A is strictly diagonally dominant, then for any choice of $x^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{x^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $Ax = b$.

$Ax = b$

if Strictly diagonal dominant

then

$\{x^{(k)}\} \rightarrow x$

for any initial guess $x^{(0)}$

exact value

$x^{(k)} \rightarrow x$

Sequence of vectors generated by either Jacobi or Gauss-Seidel Method

Iterative methods to solve System of linear equations

Example:

Use Gauss-Seidel iterative technique to find approximate solutions to

$$
\begin{aligned}
10x_1 - x_2 + 2x_3 &= 6 \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\
2x_1 - x_2 + 10x_3 - x_4 &= -11 \\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}
$$

starting with $x^{(0)} = (0, 0, 0, 0)^t$ and iterating until $\|x^{(k)} - x^{(k-1)}\|_\infty < 10^{-3}$.

Solution

let the system of linear eqn is $AX=b$

where $A = \begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix}$

check whether A is strictly diagonal dominant or not.

$|10| > |-1| + |2| + |0|$ ✓

$|11| > |-1| + |-1| + |3|$ ✓

$|10| > |2| + |-1| + |-1|$ ✓

$|8| > |0| + |3| + |-1|$ ✓

$\Rightarrow$ A is strictly diagonal matrix.

Now, applying Gauss-seidel method

$$x_1^{(k+1)} = \frac{1}{10}\left(6 + x_2^{(k)} - 2x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{11}\left(25 + x_1^{(k+1)} + x_3^{(k)} - 3x_4^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{10}\left(-11 - 2x_1^{(k+1)} + x_2^{(k+1)} + x_4^{(k)}\right)$$

$$x_4^{(k+1)} = \frac{1}{8}\left(15 - 3x_2^{(k+1)} + x_3^{(k+1)}\right)$$

Given that $x^{(0)} = (0,0,0,0)^t = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_n^{(0)}\right)^t$

$$x_1^{(1)} = \frac{6}{10} = 0.6$$

$$x_2^{(1)} = \frac{1}{11}\left(25 + 0.6\right) = 2.3273$$

$$x_3^{(1)} = \frac{1}{10}\left(-11 - 2(0.6) + 2.3273\right) = -1.1$$

$$x_4^{(1)} = \frac{1}{8}\left(15 - 3(2.3273) + (-1.1)\right) = 1.875$$

Table using Gauss-Seidel Method:-

iterations k	0	1	2	3	4	5	
1st component $x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001	$\to 1$
2nd comp. $x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000	$\to 2$
3rd comp. $x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000	$\to -1$
4th comp. $x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000	$\to 1$

exact solⁿ is $(1,2,-1,1)^t$

Table using Jacobi Method:-

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

10 iterations are needed in Jacobi to get 10^{-3} accuracy. while getting same accuracy in just 5 iterations with G.S.M.

System of linear equations:

Exercise:

1. The linear system

$$\begin{aligned} x_1 - x_3 &= 0.2 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425 \\ x_1 - \frac{1}{2}x_2 + x_3 &= 2 \end{aligned}$$

has the solution $(0.9, -0.8, 0.7)^T$.

   a. Is the coefficient matrix strictly diagonally dominant?
   b. Perform four iterations of the Gauss-Seidel iterative method to approximate the solution. Take $x^{(0)} = 0$.