Lecture 18,19

FOURIER TRANSFORM

Introduction

- To study the behavior of linear circuits there is another method known as Fourier transform. It is an operation that converts a function from time domain to frequency domain. From a given suitable function of time there is one and only one Fourier transform.
- It is an integral transform. It is known that the line or discrete spectrum will be a continuous spectrum if T tends to infinity or $\omega = {}^{1}/_{T}$ tends to zero resulting in a single or non-recurring pulse.
- By utilizing Fourier transform, it is possible to analyze such kind of functions if the following condition is obeyed by the function:

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty \qquad (1)$$

If the time period (T) of a periodic signal f(t) becomes larger, the following effects in the discrete spectrum will be obtained.

- The fundamental frequency (ω) will be smaller.
- The spectrum will be denser.
- The individual frequency components will have smaller amplitude.
- The shape of frequency spectrum will remain unchanged.
- The discrete frequency spectrum will tend to be continuous frequency spectrum.

Therefore, Fourier series (exponential) will tend to Fourier transform under the above said circumstances.

The above said statements mathematically can be put as

(i) $T \to \infty$

(ii) ω is very small

(iii) $\omega \to d\omega$, a small change in ω

(iv) $\frac{1}{T} \to \frac{\omega}{2\pi} \to \frac{d\omega}{2\pi}$

(v) $n\omega \to \omega$

To derive the Fourier transform let us begin with exponential Fourier series as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega t} \tag{1}$$

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\omega t}\mathrm{dt} \tag{2}$$

where ω = $\frac{2\pi}{T}$= fundamental frequency (3)

Putting T→ ∞ , the Eq. (2) can be written as follows:

$$TF_n = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{dt} \tag{4}$$

where nω → ω

From Eq. (4), we can write

$$F_n = \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{dt} \tag{5}$$

Using Eq. (5), Eq. (1) can be written as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{dt}\right] e^{j\omega t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{dt}\right] e^{j\omega t} d\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{dt}\right] e^{j\omega t} d\omega$$

or $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$ (6)

where $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{dt}$ (7)

F(ω) is called the Fourier transform of the function f(t) represented by Eq. (7) whereas Eq. (6) gives the inverse Fourier transform. Therefore, Fourier transform converts a function from time to frequency domain whereas by inverse Fourier transform we can convert a function from frequency domain to time domain. The above two integrals are known as Fourier integrals. The integral represented by equation (7) is called analysis equation whereas the integral represented by equation (6) is known as synthesis equation.

Condition for the Existence of Fourier Integral:

The sufficient condition for the existence of Fourier transform of a function f(t) is that the area under the function f(t) be finite. Hence, for the existence of Fourier transform is

$$I = \int_{-\infty}^{\infty} |f(t)| \mathrm{dt} < \infty \qquad (8)$$

The above absolute integrability criteria must be satisfied to give the guarantee for the existence of Fourier transforms.

The direct substitution of f(t) in the Fourier integral may give absurd result for the functions for which the above condition is not satisfied.

Fourier Transform of Some Functions

Fourier Transform of Gate Function

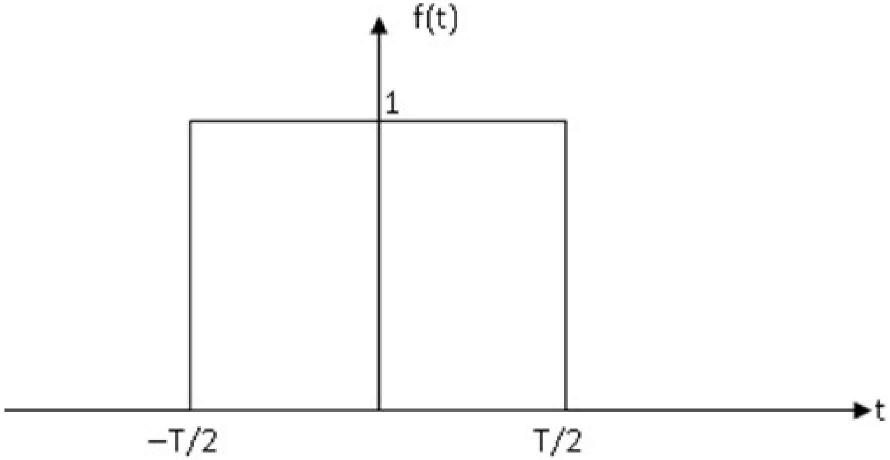


Figure 1: Gate function

Mathematically, it can be written as

$$f(t) = \left.\begin{array}{c} 0 \text{ for } \mathrm{t} < -\frac{T}{2} \\ 1 \text{ for } -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 \text{ for } \mathrm{t} > \frac{T}{2} \end{array}\right\}$$

Therefore, Fourier transform of f(t) is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{dt} \qquad = \int_{-\infty}^{\frac{-T}{2}} f(t)e^{-j\omega t}\mathrm{dt} + \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\omega t}\mathrm{dt} + \int_{\infty}^{\frac{T}{2}} f(t)e^{-j\omega t}\mathrm{dt}$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\omega t}\mathrm{dt} = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j\omega} = \frac{2j\,sin\left(\frac{\omega T}{2}\right)}{j\omega} = \frac{2}{\omega}sin\left(\frac{\omega T}{2}\right) \qquad (10)$$

$$\therefore F(\omega) = T\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} = T\frac{\sin x}{x} = \mathrm{T}\,\mathrm{Sa}(x) \qquad (11)$$

where $x = \frac{\omega T}{2}$ and $\mathrm{Sa}(x) = \frac{\sin x}{x}$, called sampling function.

Now $x \rightarrow 0$, $\mathrm{Sa}(x) \rightarrow 1$

and $\mathrm{Sa}(x) = 0$ when $x = \pm m\pi$ where ‘m’ is an integer.

Again x = $\frac{\omega T}{2}$ in the above case. Therefore, the function can also be calculated by finding ω at the zero crossings as follows:

$$\therefore \frac{\omega T}{2} = \pm m\pi$$

$$\therefore \omega = \pm \frac{2\pi}{T} m \tag{12}$$

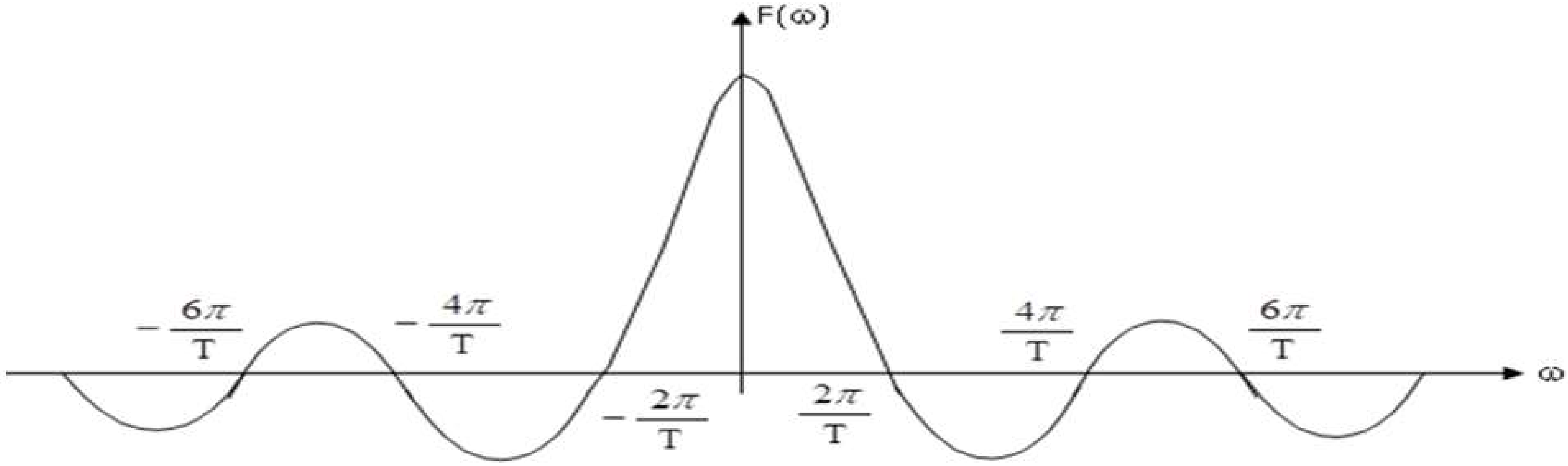
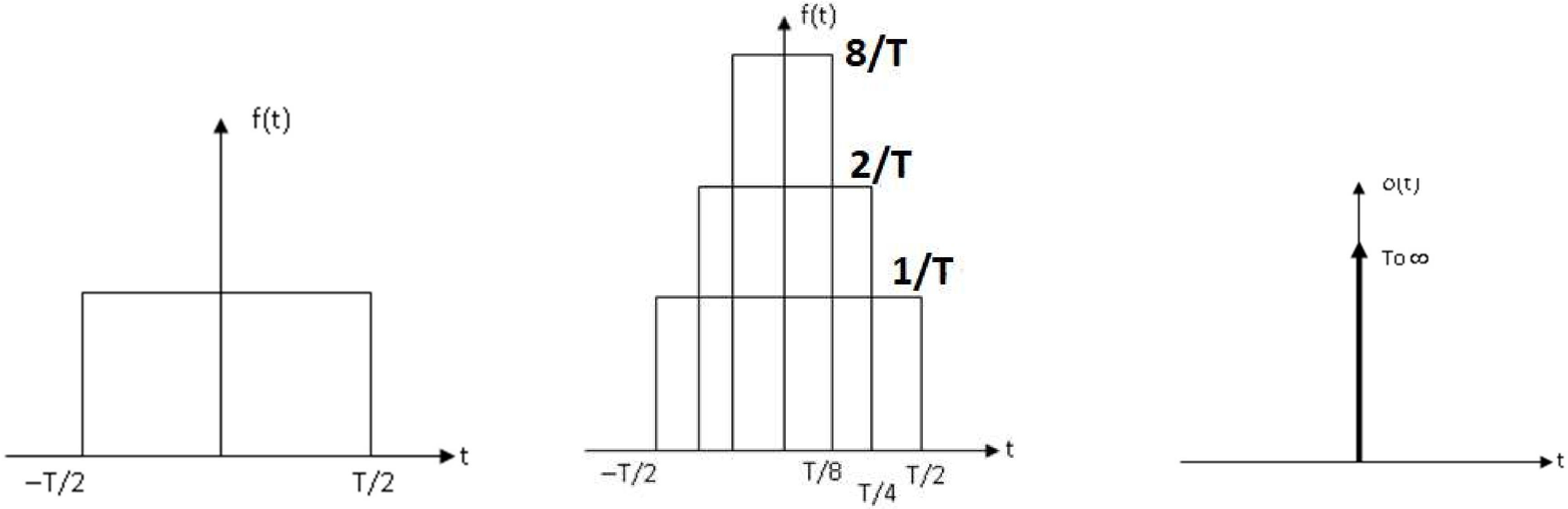


Figure 2 : Plot of F(ω) vs ω for Gate Function

$F(\omega)$ is generally a function of ω having magnitude $|F(\omega)|$ and a phase $\phi(\omega) = \mathrm{Arg}\{F(\omega)\}$. Therefore, the plot of $|F(\omega)|$ and $\mathrm{Arg}\{F(\omega)\}$ vs ω give the magnitude spectrum and phase spectrum.

Impulse Function

The impulse function can be obtained from gate function shown in Figure 3(a) by reducing the time while maintaining the area under the function to unity i.e., AT = 1, the height as shown in Figure 3(b). If T$\rightarrow$ 0, the function becomes impulse function as shown in Figure 3(c).



(a) Gate function

(b) T is reduced by T/2

(c) Impulse function

Figure 3

Therefore, it can be written,

$$\delta(t) = \underset{T \to 0}{\mathrm{Lt}} [Af(t)]$$

$$\therefore \delta(t) = \underset{T \to 0}{\mathrm{Lt}} \left[\frac{1}{T} f(t)\right]$$

The Fourier transform of gate function f(t) is given by

$$F(\omega) = T sa\left(\frac{\omega T}{2}\right)$$

$$F[\delta(t)] = \underset{T \to 0}{\mathrm{Lt}} F\left[\frac{1}{T} f(t)\right] = \underset{T \to 0}{\mathrm{Lt}} F\left[\frac{1}{T} f(t)\right] = \underset{T \to 0}{\mathrm{Lt}} \frac{1}{T} F(\omega) = \underset{T \to 0}{\mathrm{Lt}} \frac{1}{T} T\, \mathrm{Sa}\left(\frac{\omega T}{2}\right)$$

$$= \underset{T \to 0}{\mathrm{Lt}} \mathrm{Sa}\left(\frac{\omega T}{2}\right) = \underset{\frac{\omega T}{2} \to 0}{\mathrm{Lt}} \mathrm{Sa}\left(\frac{\omega T}{2}\right)$$

$$\therefore F[\delta(t)] = 1 \qquad (13)$$

Using the sampling property or shifting property, the Fourier transform of δ(t) can also be obtained. The sampling property of δ(t) is that if δ(t) is multiplied with any function f(t) and integrated over any period including t = 0 , the result becomes f(0) which is f(t) at t =0. This means

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{dt} = \int_{0-}^{0+} f(t)\delta(t)\mathrm{dt} = \int_{-t_1}^{t_2} f(t)\delta(t)\mathrm{dt} = \mathrm{f}(0) \qquad (14)$$

Similarly,

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\mathrm{dt} = \int_{-t_1}^{t_2} f(t)\delta(t-t_0)\mathrm{dt} = f(t_0) \qquad (15)$$

provided $t_1 < t_0 < t_2$

$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\,\mathrm{dt}$ [Here f(t) = $e^{-j\omega t}$]

i.e., $F(\omega) = e^{-j\omega t}\Big|_{t=0} = 1$ (16)

Shifted Impulse function

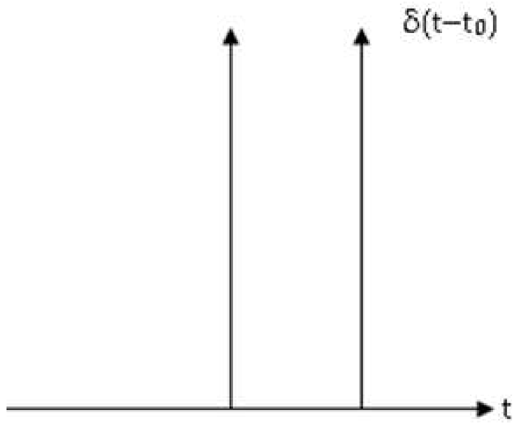


Figure . 4: Shifted unit impulse function

The Fourier transform of shifted unit impulse $\delta(t - t_0)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\ e^{-j\omega t}\, dt$$

Let $f(t) = e^{-j\omega t}$

Now at $t = t_0$, $\delta(t - t_0) = 1$ and $f(t_0) = e^{-j\omega t_0}$

$$\therefore F(\omega) = e^{-j\omega t_0}$$

$$\therefore F[\delta(t - t_0)] = e^{-j\omega t_0}$$

Fourier Transform of one sided exponential

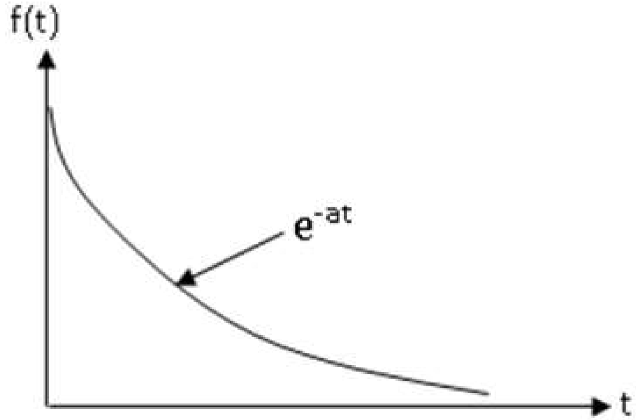


Figure 5 One sided exponential function

Mathematically, the function is given by

$$f(t) = \left.\begin{matrix} e^{-\mathrm{at}} \text{ for } \mathrm{t} \geq 0 \\ 0 \text{ for } \mathrm{t} < 0 \end{matrix}\right\}$$

The Fourier transform of the function is given by

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt}$$

$$= \int_{-\infty}^{0} f(t)\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt} + \int_{0}^{\infty} f(t)\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt}$$

$$= \int_0^\infty e^{-\mathrm{at}}\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt} = \int_0^\infty \mathrm{e}^{-(a+\mathrm{j}\omega)\,\mathrm{t}}\ \mathrm{dt}$$

$$= \left[\frac{\mathrm{e}^{-(a+\mathrm{j}\omega)\,\mathrm{t}}}{-(a+\mathrm{j}\omega)}\right]_0^\infty = \frac{1}{a+\mathrm{j}\omega}$$

$$\therefore F(\omega) = F[f(t)] = \frac{1}{a+\mathrm{j}\omega}$$

$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

And, $Arg[F(\omega)] = -\,tan^{-1}\left(\frac{\omega}{a}\right)$

The magnitude and phase spectrum are shown in Figure 6(a) and Figure 6(b) respectively

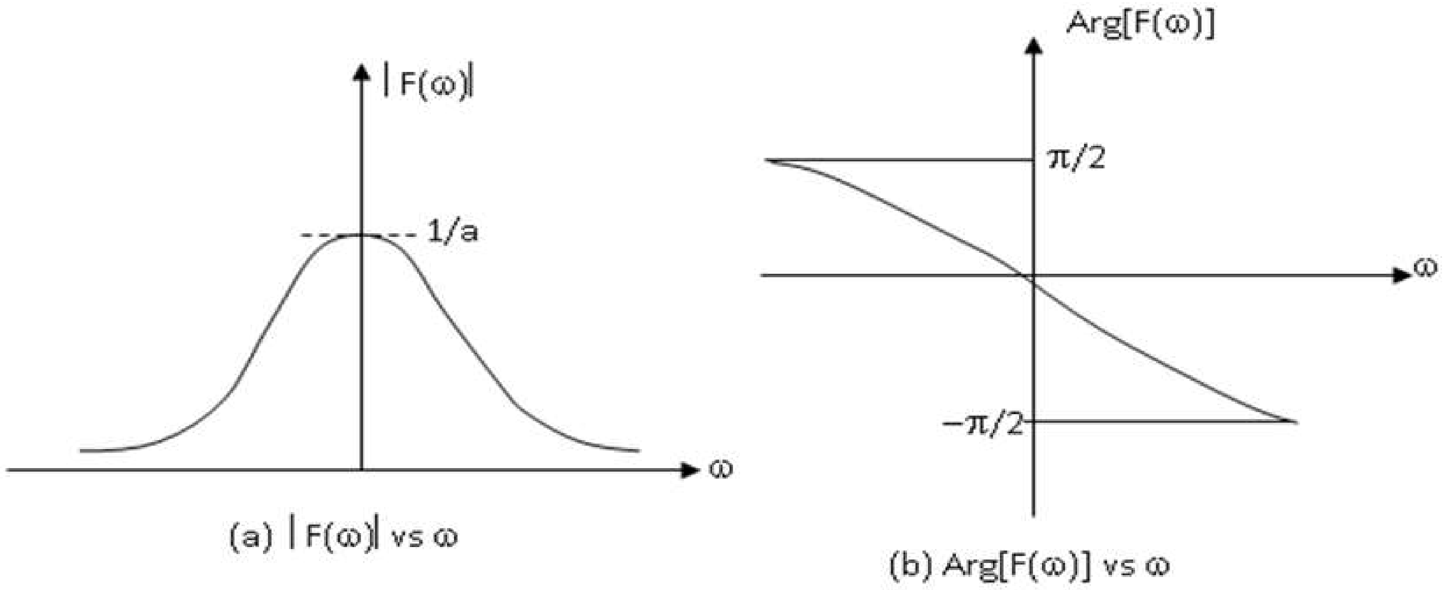


Figure 6 Magnitude and phase spectrum of one sided exponential function

Fourier Transform of two sided Exponential

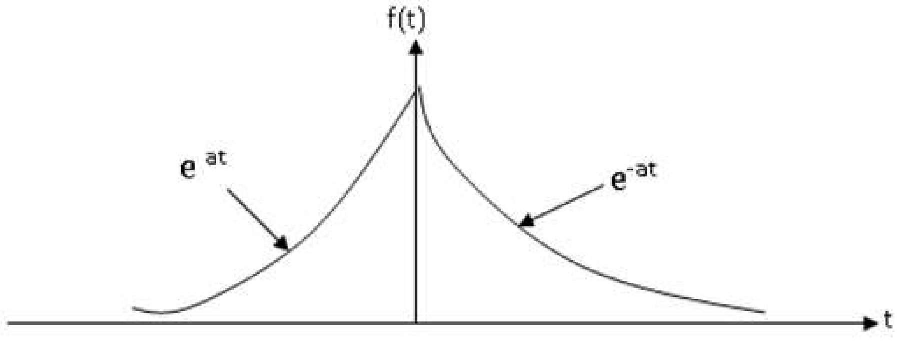


Figure 7 : One sided exponential function

Mathematically this function f(t) is represented as

$$f(t) = \left.\begin{matrix} e^{-\mathrm{at}} \text{ for } \mathrm{t} \geq 0 \\ e^{\mathrm{at}} \text{ for } \mathrm{t} \leq 0 \end{matrix}\right\}$$

The Fourier transform of the function is given by

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt}$$

$$= \int_{-\infty}^{0} f(t)\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt} + \int_{0}^{\infty} f(t)\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt}$$

$$= \int_{-\infty}^{0} e^{\mathrm{at}}\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt} + \int_{0}^{\infty} e^{-\mathrm{at}}\ \mathrm{e}^{-\mathrm{j}\omega\,\mathrm{t}}\,\mathrm{dt}$$

$$= \int_{-\infty}^{0} \mathrm{e}^{(a-\mathrm{j}\omega)\,\mathrm{t}}\,\mathrm{dt} + \int_{0}^{\infty} \mathrm{e}^{-(a+\mathrm{j}\omega)\,\mathrm{t}}\,\mathrm{dt}$$

$$= \left[\frac{\mathrm{e}^{(a-\mathrm{j}\omega)\,\mathrm{t}}}{(a-\mathrm{j}\omega\,)}\right]_{-\infty}^{0} + \left[\frac{\mathrm{e}^{-(a+\mathrm{j}\omega)\,\mathrm{t}}}{-(a+\mathrm{j}\omega)}\right]_{0}^{\infty} = \frac{1}{a-\mathrm{j}\omega} + \frac{1}{a+\mathrm{j}\omega}$$

$$= \frac{2\mathrm{a}}{a^2+\omega^2}$$

$$\therefore F(\omega) = F[f(t)] = \frac{2\mathrm{a}}{a^2+\omega^2}$$

The magnitude spectrum are shown in Figure 8.

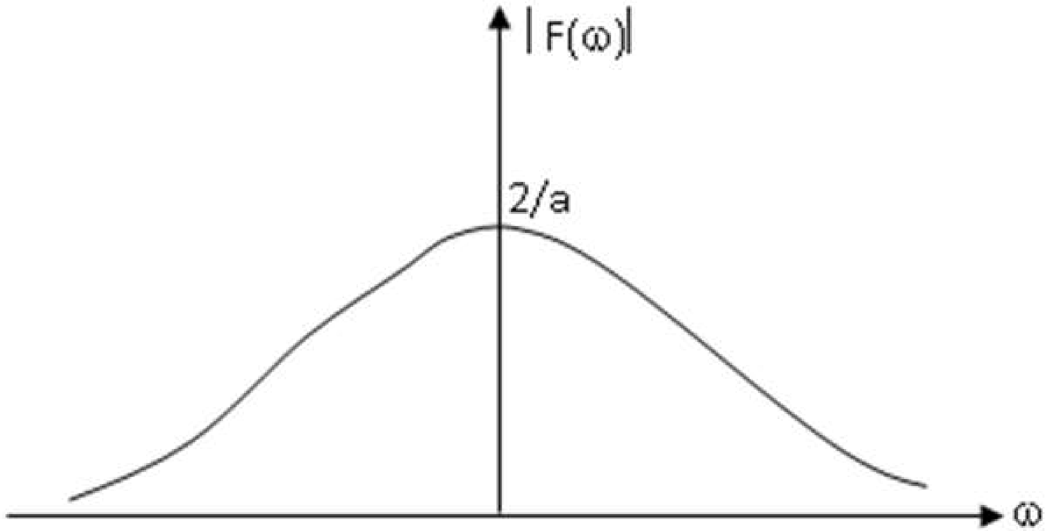


Figure 8 Magnitude spectrum of two sided exponential function

Fourier Transform of sgn(t) $e^{-a(t)}$

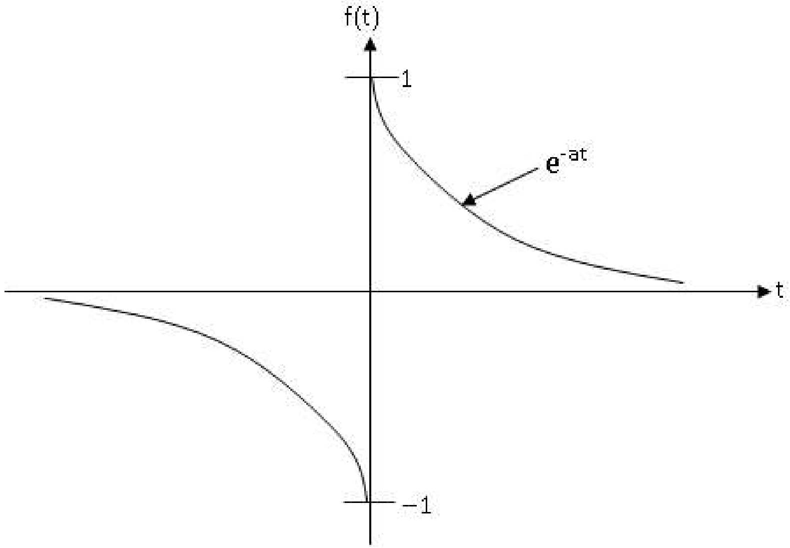


Figure .9 : One sided exponential function

represented as

$$f(t) = \begin{matrix} e^{-at} \text{ for } t \geq 0 \\ -e^{at} \text{ for } t \leq 0 \end{matrix}\Bigg\}$$

The Fourier transform of the function is given by

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\ e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} f(t)\ e^{-j\omega t}\, dt + \int_{0}^{\infty} f(t)\ e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} -e^{at}\ e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\ e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{0} e^{(a-j\omega)\,t}\ dt + \int_{0}^{\infty} e^{-(a+j\omega)\,t}\ dt$$

$$= -\left[\frac{e^{(a-j\omega)\,t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)\,t}}{-(a+j\omega)}\right]_{0}^{\infty} = -\frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= -\frac{2j\omega}{a^2+\omega^2}$$

$$\therefore F(\omega) = F[f(t)] = -\frac{2j\omega}{a^2+\omega^2}$$

The magnitude spectrum is shown in Figure 10.

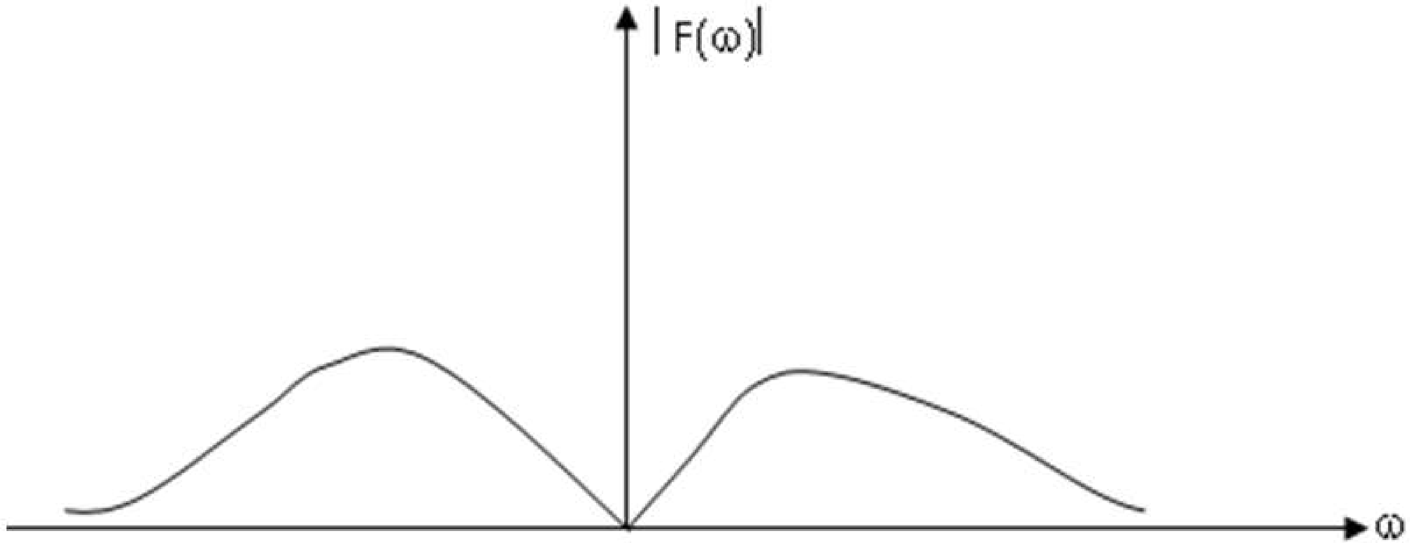


Figure 10 Magnitude of $sgn(t)\, e^{-a(t)}$

Fourier Transform of Signum Function

The signum function is denoted by sgn(t). It has magnitude unity for all t but its sign changes according to the sign of t. It is positive when t is positive. When t is negative, it is negative. This function is shown in Figure 11.

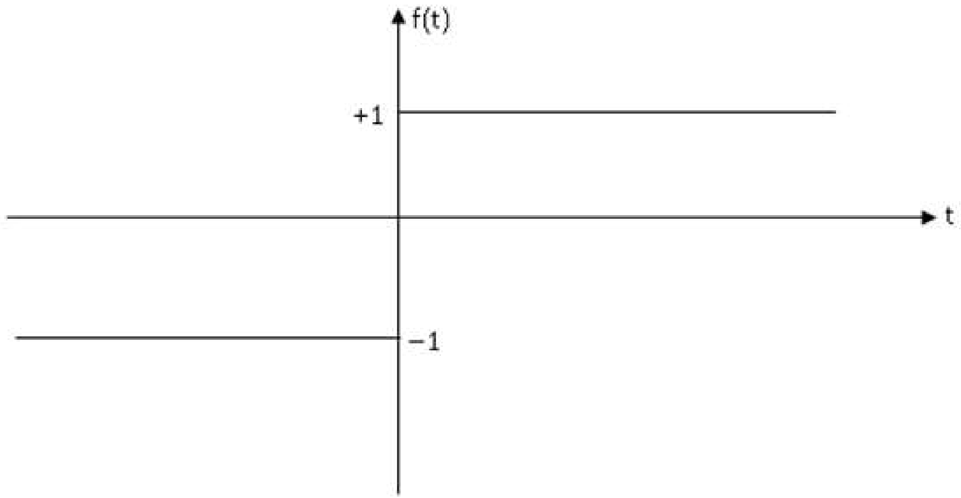


Figure 11 Graphical representation of sgn(t)

Since the area under the curve f(t)= sgn(t) is infinite, the function does not satisfy the condition $\int_{-\infty}^{\infty} |f(t)| dt < \infty$.

To get Fourier transform of sgn(t), let us take

$$F(\mathrm{sgn}(t)) = \underset{a \to 0}{Lt} \; F[\mathrm{sgn}(t) e^{-a(t)}]$$

$$= \underset{a \to 0}{Lt} \left(-\frac{2j\omega}{a^2 + \omega^2} \right) = \frac{2}{j\omega}$$

The magnitude spectrum is shown in Figure 12.

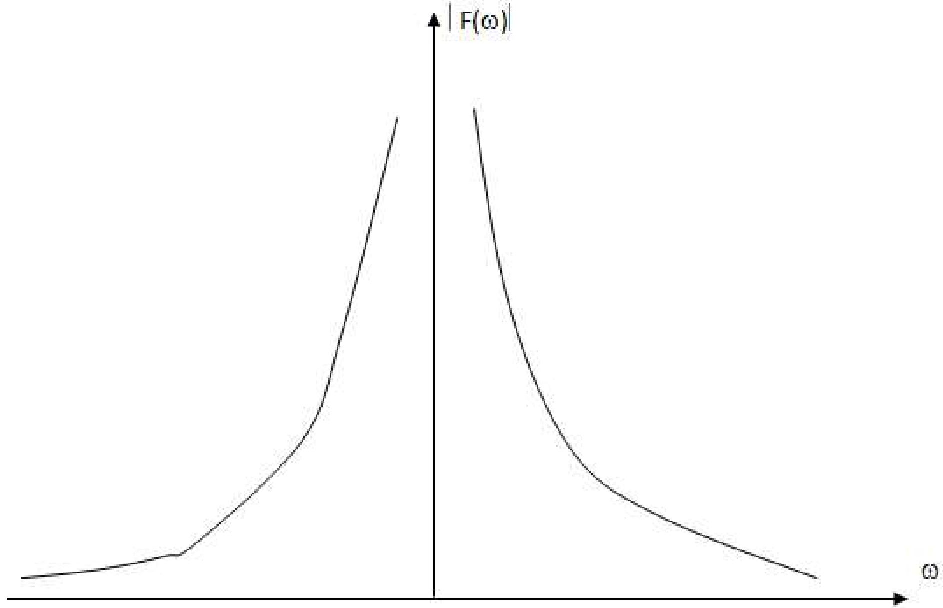


Figure 12 : Magnitude spectrum of sgn(t)

Fourier Transformation Theorem

Linearity

If $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$, it can be written that,

$F[a_1f_1(t)+ a_2f_2(t)] \leftrightarrow a_1F_1(\omega)+a_2F_2(\omega)$.

Proof:

$$F[f_1(t)] = \int_{-\infty}^{\infty} f_1(t)e^{-j\,\omega t}\,dt = F_1(\omega)$$

and

$$F[f_2(t)] = \int_{-\infty}^{\infty} f_2(t)e^{-j\,\omega t}\,dt = F_2(\omega)$$

$$\therefore F[a_1f_1(t) + a_2f_2(t)] = \int_{-\infty}^{\infty} [a_1f_1(t) + a_2f_2(t)]e^{-j\,\omega t}\,dt = a_1F_1(\omega) + a_2F_2(\omega)$$

$$F[a_1 f_1(t) + a_2 f_2(t)] = \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} a_1 f_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} a_2 f_2(t) e^{-j\omega t} dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2(t) e^{-j\omega t} dt$$

$$= a_1 F_1(\omega) + a_2 F_2(\omega)$$

Time scaling

If a function f(t) is expanded in time domain, its Fourier transform compressed in frequency domain along 'ω' axis and the magnitudes of |F(ω)| increase. Conversely, if a function is compressed in time domain, its Fourier transform expands in frequency domain along 'ω' axis and the magnitudes of |F(ω)| decrease

Proof:

Fourier transform of a function f(t) is given by

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\,\omega t}\,dt = F(\omega)$$

$$\therefore F[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-j\,\omega t}\,dt$$

Let x = at , $\therefore t = \frac{x}{a}$ and $dt = \frac{dx}{a}$

If t = ± ∞, x = ± ∞.

$$\therefore F[f(at)] = \int_{-\infty}^{\infty} f(x)e^{-j\,\omega\frac{x}{a}}\frac{1}{a}\,dx = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

If a>0, the above relation holds good.

Even if a<0, this relation still holds good.

In general, it can be written as

$$\therefore F[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

Time differentiation

If f(t) $\leftrightarrow$ F(ω), according to this property it can be said that,

$$\frac{df(t)}{dt} \leftrightarrow (j\omega)F(\omega)$$

$$\frac{d^2f(t)}{dt^2} \leftrightarrow (j\omega)^2F(\omega)$$

$$\frac{d^nf(t)}{dt^n} \leftrightarrow (j\omega)^nF(\omega)$$

Therefore, each derivative of f(t) in time domain is equivalent to multiplication of F.T. of f(t) by jω in frequency domain.

Proof:

$$F\left[\frac{df(t)}{dt}\right] = \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{j\omega t}\, dt$$

Let u = $e^{-j\omega t}$ and v=$\frac{df(t)}{dt}$

Since $\int uvdt = u\int vdt - \int\left[\frac{dv}{dt}\int vdt\right]dt$

$$\therefore F\left[\frac{df(t)}{dt}\right] = e^{-j\omega t}\, f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-j\omega)\, e^{-j\omega t}\, f(t)dt$$

$$= j\omega\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = (j\omega)F(\omega)$$

Time shifting property

If a function f(t) is shifted to the right by $t = t_0$, according to this property its Fourier transform is multiplied by $e^{-j\,\omega t_0}$. On the other hand, if the function is shifted to the left by an amount $t = t_0$, its Fourier transform is multiplied by $e^{j\,\omega t_0}$

Therefore, if $f(t) \leftrightarrow F(\omega)$, according to this property it can be written

$$f(t - t_0) \leftrightarrow e^{-j\,\omega t_0} F(\omega)$$

and $f(t + t_0) \leftrightarrow e^{j\,\omega t_0} F(\omega)$

Again, $\left|e^{-j\omega t_0}\right| = |\cos\omega t_0 - j\sin\omega t_0| = \sqrt{\cos^2\omega t_0 + \sin^2\omega t_0} = 1$. Therefore, delaying a function by an amount t_0 , the magnitude $|F(\omega)|$ remains unaltered. Only the phase spectrum becomes changed because

$$\mathrm{Arg}\left[e^{-j\omega\, t_0}\, F(\omega)\right] = -\omega t_0 + \mathrm{Arg}[F(\omega)]$$

$$F(f(t-t_0)) = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(x)e^{-j\omega(x+t_0)}dx \ [where\ x = t - t_0]$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx = e^{-j\omega t_0}F(\omega)$$

Similarly,

$$F(f(t+t_0)) = \int_{-\infty}^{\infty} f(t+t_0)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(x)e^{-j\omega(x-t_0)}dx \ [where\ x = t + t_0]$$

$$= e^{j\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx = e^{j\omega t_0}F(\omega)$$