dx

(t_1, y_1) $y - y_1 = \frac{y_2 - y_1}{t_2 - t_1}(t - t_1)$

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Numerical differentiation

Initial value Problem

- Problem of the form

$$y'(t) = f(t, y) \qquad a \le t \le b$$
$$y(a) = \alpha$$

is called the initial value problem.

e.g. $\frac{dy}{dt} = t^2 + y \qquad 0 < t \le 2$

$$y(0) = 0$$

- Euler's Method: Divide $[a, b]$ into n equal intervals, i.e. $t_i = t_0 + ih$, $i = 0, 1, 2, \dots, n$, $h = \frac{b-a}{n}$

$a = t_0, t_1, t_2, t_3, t_4, t_5 \dots t_n = b$

$$\frac{dy}{dt}(t_i) = \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} = \frac{y(t_{i+1}) - y(t_i)}{h}$$

$$\therefore \frac{dy}{dt} = f(t, y)$$

$$\frac{dy_i}{dt} = f(t_i, y_i)$$

$$\frac{y(t_{i+1}) - y(t_i)}{h} = f(t_i, y_i)$$

$$\boxed{y(t_{i+1}) = y(t_i) + h f(t_i, y_i)}$$

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Q Solve IVP using Euler's method with $h=1$

$$y' = \frac{x-y}{2}, \quad y(0)=1 \text{ on } [0,3]$$

Sol.

$$y_{i+1} = y_i + h\left(f(x_i, y_i)\right)$$

$$y(t_{i+1}) = y(t_i) + hf(x_i, y(t_i))$$

$0+1=1$

$t_1 = t_i + 1$

$t_i = 0$, $f(x,y) = \frac{x-y}{2}$

$$y(1) = y(0) + h f(0, y(0))$$

$$= 1 + \frac{0-1}{2} = 1 - 0.5 = 0.5$$

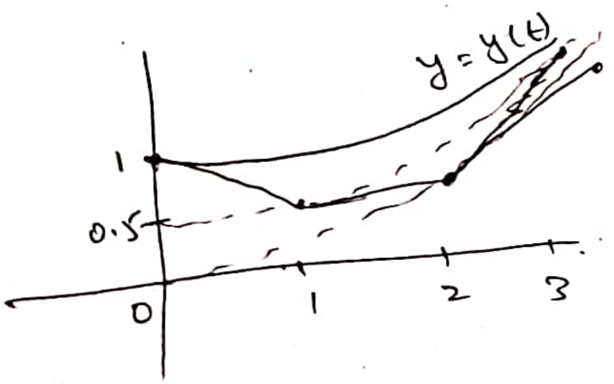
$$y(2) = y(1) + h f(1, y(1))$$

$$= y(1) + f(1, 0.5)$$

$$= 0.5 + \frac{1-0.5}{2} = 0.5 + \frac{0.5}{2} = 0.75$$

$$y(3) = y(2) + h f(2, y(2))$$

$$= 0.75 + \frac{2-0.75}{2} = 1.375$$



$x' = t(x+t) - 2$

$x(0) = 2$

Use the Euler method with stepsize $h = 0.2$ to compute $x(0.6)$?

$0 \quad 0.2 \quad 0.4 \quad 0.6$ $\qquad h = 0.2$

$f(t, x) = t(x+t) - 2$

$x(0.2) = x(0) + h f(0, x(0))$

$= 2 + 0.2 f(0, 2)$

$= 2 + 0.2 \left(0(0+2) - 2\right)$

$= 2 + 0.2(-2)$

$= 2 - 0.4 = 1.6$

$x(0.4) = x(0.2) + h f(0.2, x(0.2)) = x(0.2) + 0.2 f(0.2, 1.6))$

$= 1.6 + 0.2 \left(0.2(1.6 + 0.2) - 2 \right)$

$= 1.6 + 0.2 \left(0.2 (1.8) - 2 \right)$

$= 1.6 + 0.2 (0.36 - 2)$

$= 1.6 - 0.2 \times 1.64$

$= 1.6 - 0.328$

$= 1.272$

$x(0.6) = x(0.4) + h f(0.4, x(0.4))$

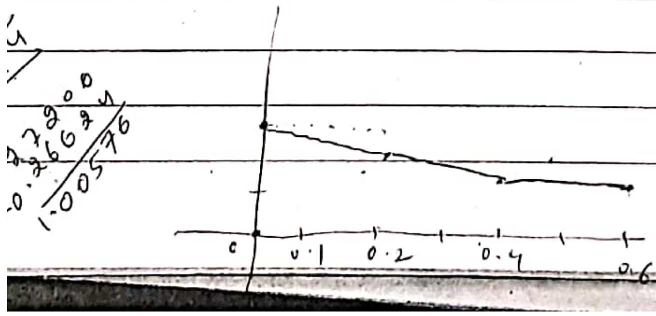
$= 1.272 + 0.2 f(0.4, 1.272)$

$= 1.272 + 0.2 \left(0.4(0.4 + 1.272) - 2 \right)$

$= 1.272 + 0.2 \left(0.4 \times 1.672 - 2 \right)$

$= 1.272 + 0.2(-1.3312)$

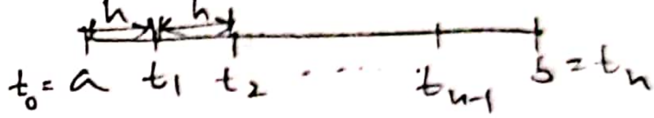
$= 1.00576$ Ans.



Modified Euler's Method

IVP $\quad y' = f(t,y), \; y(a) = \alpha, \quad a \le t \le b$

Divide interval $[a,b]$ into n equal sub-intervals using step length h



$$t_{i+1} = t_i + h$$

Prediction Step $\quad y(t_{i+1}) = y(t_i) + h\, f(t_i, y(t_i))$

Correction Step $\quad y(t_{i+1}) = y(t_i) + \frac{h}{2}\left[f(t_i, y(t_i)) + f(t_{i+1}, \underbrace{y(t_{i+1})}_{\text{from Prediction Step}})\right]$

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<u>Que</u> Use Modified Euler's method to find $y(0.2)$ and $y(0.4)$ with $h=0.2$ for IVP

$$y' = y + e^t \quad y(0) = 0.$$

<u>Solution</u> :-

0 0.2 0.4

$h = 0.2$

Prediction step

$$y(0.2) = y(0) + h\,f(0, y(0))$$

$$= 0 + 0.2\,f(0,0)$$

$$= 0 + 0.2\,(0 + e^0)$$

$$= 0.2$$

<u>Correction step</u>

$$y(0.2) = y(0) + \frac{0.2}{2}\left(f(0,0) + f(0.2, 0.2)\right)$$

$$= 0 + 0.1\left(1 + (0.2 + e^{0.2})\right)$$

$$= 0.1\,(1.2 + 1.2214)$$

$$= 0.2421$$

<u>Next</u>

Prediction step

$$y(0.4) = y(0.2) + h\,f(0.2, y(0.2))$$

$$= 0.2421 + 0.2\,(0.2421 + e^{0.2})$$

$$= 0.5348$$

<u>Correction step</u>

$$y(0.4) = y(0.2) + \frac{h}{2}\left[f(0.2, y(0.2)) + f(0.4, y(0.4)\right.$$

$$= 0.2421 + \frac{0.2}{2}\left[(0.2421 + e^{0.2}) + (e^{0.4} + 0.53\right.$$

$$= 0.2421 + 0.1\left[(1.4635) + 2.0266\right]$$

$$= 0.5911$$

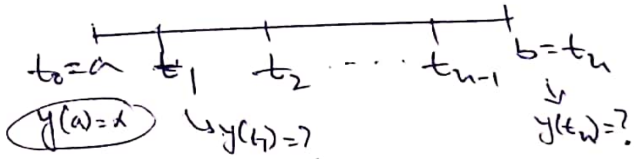
Runge-Kutta Method of order 4 (R.K Method of order 4)

IVP: $y' = f(t,y)$, $y(a) = \alpha$, $a \le t \le b$

$h \to$ step length

Divide $[a,b]$ into n equal parts with $h = \frac{b-a}{n}$

$$\boxed{t_{i+1} = t_i + h}$$



$$k_1 = h f(t_i, y(t_i))$$

$$k_2 = h f\left(t_i + \frac{h}{2}, y(t_i) + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(t_i + \frac{h}{2}, y(t_i) + \frac{k_2}{2}\right)$$

$$k_4 = h f(t_i + h, y(t_i) + k_3)$$

$$y(t_{i+1}) = y(t_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

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Q Using Runge-Kutta Method of order 4 to solve IVP in $[0.4, 0.8]$ for $y' = \sqrt{x+y}$, $y(0.4) = 0.41$ with step length 0.2

<u>Sol.</u> $f(x,y) = \sqrt{x+y}$, $x_0 = 0.4$, $y(x_0) = 0.41$, $h = 0.2$

<u>To find</u> $y(x_1) \equiv y(0.6)$,

$h = 0.2$ 0.2

$x_0 = 0.4$ $x_1 = 0.6$ $x_2 = 0.8$

$y(x_0) = 0.41$ $y(x_1) = ?$ $y(x_2) = ?$

$$x_1 = x_0 + h = 0.4 + 0.2 = 0.6$$

$$K_1 = h f(x_0, y(x_0)) = (0.2) f(0.4, 0.41) = (0.2)\sqrt{0.4 + 0.41} = 0.18$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y(x_0) + \frac{K_1}{2}\right) = (0.2) f\left(0.4 + \frac{0.2}{2}, 0.41 + \frac{0.18}{2}\right) = (0.2) f(0.5, 0.5)$$
$$= 0.2$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y(x_0) + \frac{K_2}{2}\right) = (0.2) f\left(0.4 + \frac{0.2}{2}, 0.41 + \frac{0.2}{2}\right) = (0.2) f(0.5, 0.51)$$
$$= 0.200996$$

$$K_4 = h f(x_0 + h, y(x_0) + K_3) = (0.2) f(0.4 + 0.2, 0.41 + 0.200996)$$
$$= (0.2) f(0.6, 0.610996) = 0.22009$$

Now $y(x_1) = y(x_0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$$= 0.41 + \frac{1}{6}(0.18 + 2(0.2) + 2(0.200996) + 0.22009) = 0.610347$$

~~[illegible]~~

$$\Rightarrow \boxed{y(0.6) = 0.610347}$$

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To find $y(x_2) \equiv y(0.8)$, $\boxed{x_2 = x_1 + h = 0.8}$

Then $k_1 = h f(0.6, y(0.6)) = (0.2) f(0.6, 0.610347)$

$= 0.220032$

$k_2 = (0.2) f\left(0.6 + \frac{0.2}{2}, 0.610347 + \frac{0.220032}{2}\right)$

$= (0.2) f(0.7, 0.720363) = 0.238358$

$k_3 = (0.2) f\left(0.7, 0.610347 + \frac{0.238358}{2}\right)$

$= (0.2) f(0.7, 0.729526) = 0.239128$

$k_4 = (0.2) f(0.6 + 0.2, 0.610347 + 0.239126)$

$= (0.2) f(0.8, 0.849473) = 0.256864$

$y(0.8) = y(0.6) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.610347 + \frac{1}{6}(0.220032 + 0.476716$

$= 0.610347 + \frac{1}{6}(1.431864)$

$= 0.848991$

Example 4. *Using Runge-Kutta fourth-order, solve* $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ *with* $y_0 = 1$ *at* $x = 0.2$ *and* 0.4.

Sol.

$$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}, x_0 = 0, y_0 = 1, h = 0.2$$
$$K_1 = hf(x_0, y_0) = 0.2f(0, 1) = 0.200$$
$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.2f(0.1, 1.1) = 0.19672$$
$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.2f(0.1, 1.09836) = 0.1967$$
$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.2f(0.2, 1.1967) = 0.1891$$
$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + 0.19599 = 1.196$$

Therefore $y(0.2) = 1.196$
Now $x_1 = x_0 + h = 0.2$

$$K_1 = hf(x_1, y_1) = 0.1891$$
$$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) = 0.2f(0.3, 1.2906) = 0.1795$$
$$K_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) = 0.2f(0.3, 1.2858) = 0.1793$$
$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2f(0.4, 1.3753) = 0.1688$$
$$\cdot\ y_2 = y(0.4) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.196 + 0.1792 = 1.3752.$$

Higher-order

Q Use R-K Method of order 4 to find approximate va of $y(1.1)$ and $y'(1.1)$ with step length $h=0.1$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 2x, \qquad y(1)=1, \quad y'(1)=1$$

Sol. Let $\frac{dy}{dx} = z = f_1(x,y,z)$ $\qquad y(1)=1$

$$\frac{dz}{dx} + yz + y = 2x \Rightarrow \frac{dz}{dx} = 2x - yz - y = f_2(x,y,z), \quad \underset{=y'(1)}{z(1)} = 1$$

$x_0=1, \; y_0=1, \; z_0=1$

$K_1 = h f_1(x_0, y_0, z_0)$
$= (0.1) f_1(1,1,1) = 0.1$

$K_2 = h f_1(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{L_1}{2})$
$= (0.1) f_1(1.05, 1.05, 1)$
$= (0.1)(1) = 0.1$

$K_3 = h f_1(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{L_2}{2})$
$= (0.1) f_1(1.05, 1.05, 1) = 0.1$

$K_4 = h f_1(x_0+h, y_0+K_3, z_0+L_3)$
$= (0.1) f_1(1.1, 1.1, 1)$
$= 0.1$

$L_1 = h f_2(x_0, y_0, z_0)$
$= (0.1) f_2(1,1,1) = (0.1)[2-1-1] = 0$

$L_2 = h f_2(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{L_1}{2})$
$= (0.1) f_2(1.05, 1.05, 1)$
$= (0.1)[2(1.05) - 1.05 - 1.05] = 0$

$L_3 = h f_2(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{L_2}{2})$
$= (0.1) f_2(1.05, 1.05, 1) = 0$

$L_4 = h f_2(x_0+h, y_0+K_3, z_0+L_3)$
$= (0.1) f_2(1.1, 1.1, 1) = 0$

$$y_1 = y(1.1) = y_0 + \frac{1}{6}(K_1+2K_2+2K_3+K_4) = 1 + \frac{1}{6}[0.1+0.2+0.2+0.1] = 1.1$$

$$z_1 = z(1.1) = z_0 + \frac{1}{6}(L_1+2L_2+2L_3+L_4) = 1 + \frac{1}{6}[0+0+0+0] = 1$$

$\therefore \; y(1.1) = 1.1$

$y'(1.1) = z(1.1) = 1$

Example 7. *Solve by using fourth-order Runge-Kutta method for $x = 0.2$.*

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2, \; y(0) = 1, \; y'(0) = 0.$$

Sol. Let

$$\frac{dy}{dx} = z = f(x, y, z)$$

Therefore

$$\frac{dz}{dx} = xz^2 - y^2 = g(x, y, z)$$

Now $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$

$$K_1 = hf(x_0, y_0, z_0) = 0.0$$
$$L_1 = hg(x_0, y_0, z_0) = -0.2$$
$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}) = -0.02$$
$$L_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}) = -0.1998$$
$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}) = -0.02$$
$$L_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}) = -0.1958$$
$$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.0392$$
$$L_4 = hg(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.1905$$

Hence

$$y_1 = y(0.2) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.9801$$
$$z_1 = y'(0.3) = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) = -0.1970$$