Course: UMA 035 (Optimization Techniques)

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Non-linear programming problem

A mathematical programming is said to be non-linear programming problem if either the objective function and/or at least one of the constraints is non-linear.

Example:

Maximize $(x_1^2 + x_2^2)$

Subject to

$x_1 + x_2 <= 1,$

$x_1 >= 0,\ x_2 >= 0.$

Example:

Maximize $(x_1 + x_2)$

Subject to

$x_1^2 + x_2^2 <= 1,$

$x_1 >= 0,\ x_2 >= 0.$

Example:

Maximize $(x_1^2 + x_2^2)$

Subject to

$x_1^2 + x_2^2 <= 1,$

$x_1 >= 0,\ x_2 >= 0.$

Optimal solution of a non-linear programming problem may also exist inside the feasible region.

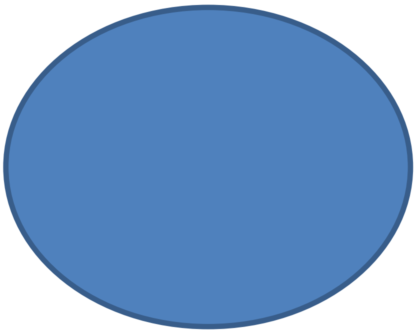
Example:

Minimize (x_1+ x_2)

Subject to

x_1^2+ x_2^2 <=1,

Solution

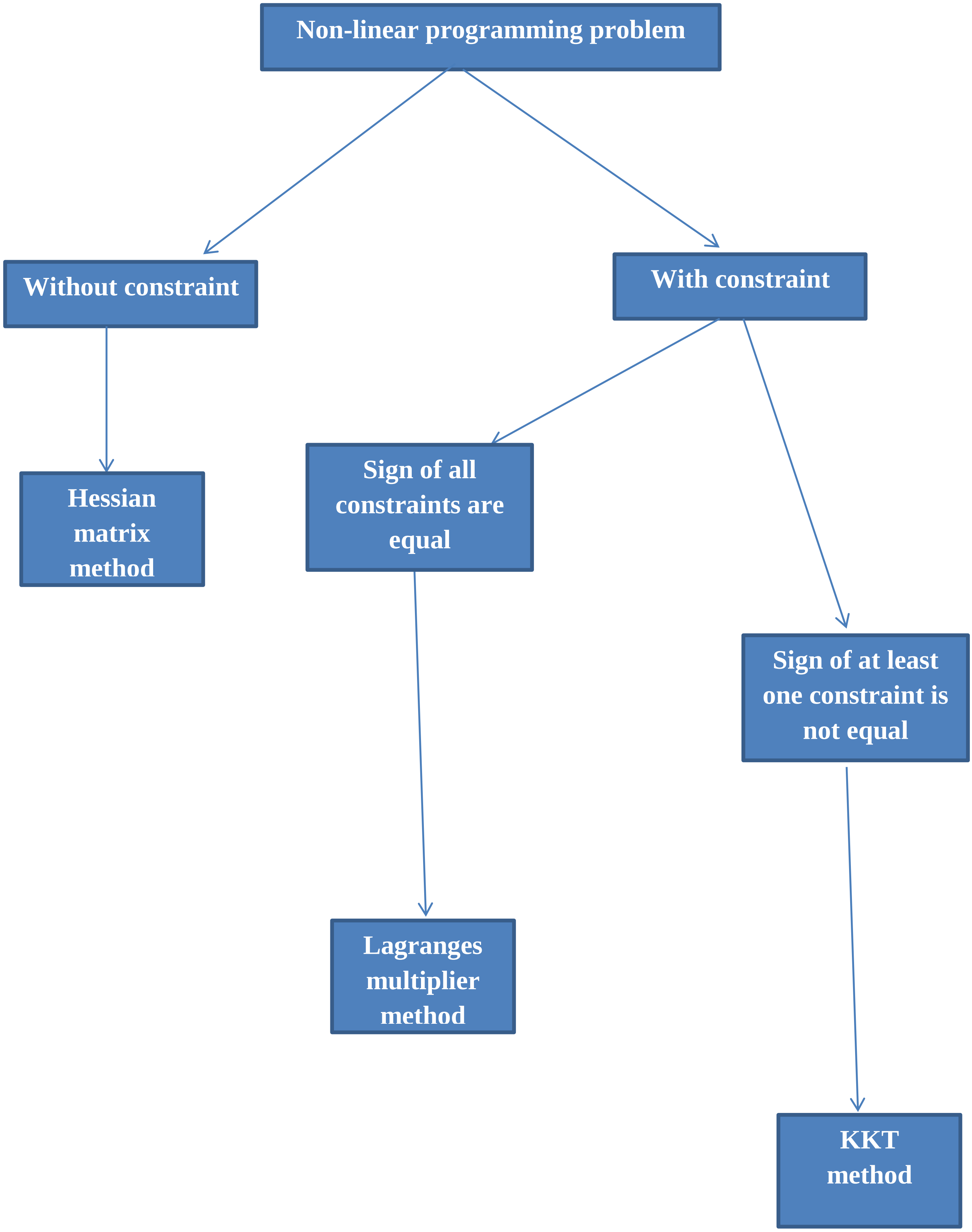


All the points lying on the boundary of the circle are extreme points. But at all these points the value of the objective function is 1.

While, the value of the objective function at center (0,0) is 0.

Hence, the value of the objective function is minimum inside the feasible region.

There may exist infinite points inside the feasible region. Therefore, it is difficult to find optimal solution of a non-linear programming problem.


Non-linear programming problem
Without constraint
With constraint
Hessian matrix method
Sign of all constraints are equal
Sign of at least one constraint is not equal
Lagranges multiplier method
KKT method

Non-linear without constraint

Stationary points

Put

$$\frac{\partial y}{\partial x_1} = 0,$$

$$\frac{\partial y}{\partial x_2} = 0,$$

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$$\frac{\partial y}{\partial x_n} = 0,$$

Solve the obtained equations to find values of x_1 , x_2, …, x_n.

(x_1 , x_2, …, x_n) is called a stationary point.

Nature of a stationary point

A stationary point will be of maxima if corresponding to the stationary point, the Hessian matrix is negative semi-definite or negative definite.

A stationary point will be of minima if corresponding to the stationary point, the Hessian matrix is positive semi-definite or positive definite.

A stationary point will be a saddle point if corresponding to the stationary point, the Hessian matrix is Indefinite.

Hessian matrix

$$H=\begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 y}{\partial x_n \partial x_1} & \frac{\partial^2 y}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_n^2} \end{bmatrix}$$

Positive definite

A matrix $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$**, where,** $a_{ij}=a_{ji}$ **is said to be**

positive definite if

$D_1=a_{11}>0$

$D_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0$

$D_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}>0$

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$D_n=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}>0$

Positive semi-definite

A matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, where, $a_{ij}= a_{ji}$ is said to be positive semi-definite if

$D_1= a_{11}>0$

$D_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geq 0$

$D_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \geq 0$

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$D_n= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} >=0$

Atleast one out of D_2 , D_3,…, D_n should be 0.

Negative definite

A matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, where, $a_{ij}= a_{ji}$ is said to be negative definite if

$D_1 = a_{11} < 0$

$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0$

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$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} < 0$ if n is odd

$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0$ if n is even

Negative semi-definite

A matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, where, $a_{ij} = a_{ji}$ is said to be negative semi-definite if

$D_1 = a_{11} < 0$

$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geq 0$

$$D_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \leq 0$$

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$$D_n=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} <=0 \text{ if n is odd}$$

$$D_n=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} >=0 \text{ if n is even}$$

Atleast one out of D_2 , D_3,…, D_n should be 0.

Example

Find the stationary points and classify for the following function

$f(x)=x_1^2+ x_2^2+3 x_3^2-2x_1x_2+x_1 x_3+16$

Solution

$$\frac{\partial y}{\partial x_1}=2x_1-2x_2+x_3$$

$$\frac{\partial y}{\partial x_2}=2x_2-2x_1$$

$$\frac{\partial y}{\partial x_3}=6x_3+x_1$$

Putting

$\frac{\partial y}{\partial x_1} = 0, \frac{\partial y}{\partial x_2} = 0, \frac{\partial y}{\partial x_3} = 0$, **we have**

$2x_1 - 2x_2 + x_3 = 0$

$2x_2 - 2x_1 = 0$

$6x_3 + x_1 = 0$

On solving

$x_1=0,\ x_2=0,\ x_3=0$

Hessian matrix=$\begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_1 \partial x_3} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \frac{\partial^2 y}{\partial x_2 \partial x_3} \\ \frac{\partial^2 y}{\partial x_3 \partial x_1} & \frac{\partial^2 y}{\partial x_3 \partial x_3} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix}$

$=\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 6 \end{bmatrix}$

$D_1=2>0$

$D_2=\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$

$D_3==\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 6 \end{bmatrix}$= **negative value**

Hessian matrix is indefinite.

$x_1=0,\ x_2=0,\ x_3=0$ is a saddle point.

Example

Find the stationary points and classify for the following function

$f(x)=2+2x_1+3x_2-x_1^2-x_2^2$

Solution

$\frac{\partial y}{\partial x_1}=2-2x_1$

$\frac{\partial y}{\partial x_2}=3-2x_2$

Putting

$\frac{\partial y}{\partial x_1}=0, \frac{\partial y}{\partial x_2}=0$, **we have**

$2-2x_1=0$

$3-2x_2=0$

On solving

$x_1=1,\ x_2=\frac{3}{2}$

Hessian matrix$=\begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix}$

$=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$D_1=-2<0$

$D_2=\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}=4>0$

Hessian matrix is negative definite.

$x_1=1$, $x_2=\frac{3}{2}$, is a point of maxima.

Maximum value $=2+2(1)+3(\frac{3}{2})-(1)^2-(\frac{3}{2})^2$

$=\frac{21}{4}$

Non-linear programming problems with equality constraints

Min/Max f(x)

Subject to

$g_i(x)=0$, $i=1,2,\ldots,m$

$x_j>=0$.

Lagrange's multiplier method

$$L(x, \lambda) = f(x) + \sum_{i=1}^{n} \lambda_i \, gi(x)$$

Put

$$\frac{\partial L}{\partial xi} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = 0$$

Find value of x and λ_i.

Construct a bordered Hessian matrix

$$H^B = \begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix}$$

where, 0 is null matrix of order m*m , m represents number of constraints.

$$\mathbf{P}=\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}$$

$$Q=\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

If D_i starts from sign of $(-1)^{m+1}$ and changes alternatively, where,

i >=2m+1. Then, point of maxima.

If sign of D_i is same as sign of $(-1)^m$ where, i >=2m+1. Then, point of minima.

Example

Use the Lagranges multiplier method to solve the following NLPP. Does the solution maximizes or minimizes the objective function?

Opt $f(x) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

Subject to

$x_1 + x_2 + x_3 = 20,$

$x_1 >= 0,\ x_2 >= 0,\ x_3 >= 0.$

Solution

The Lagrange function is

$L = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 + \lambda(x_1 + x_2 + x_3 - 20)$

$\frac{\partial L}{\partial x_1} = 4x_1 + 10 + \lambda$

$\frac{\partial L}{\partial x_2} = 2x_2 + 8 + \lambda$

$\frac{\partial L}{\partial x_3} = 6x_3 + 6 + \lambda$

$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 20$

Assuming

$\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x_2} = 0, \frac{\partial L}{\partial x_3} = 0, \frac{\partial L}{\partial \lambda} = 0$, **we have**

$4x_1 + 10 + \lambda = 0 \Rightarrow x_1 = -\left(\frac{10+\lambda}{4}\right)$

$2x_2 + 8 + \lambda = 0 \Rightarrow x_2 = -\left(\frac{8+\lambda}{2}\right)$

$6x_3 + 6 + \lambda = 0 \Rightarrow x_1 = -\left(\frac{6+\lambda}{6}\right)$

$$x_1 + x_2 + x_3 - 20 = 0 \Rightarrow -\left(\frac{10+\lambda}{4}\right) - \left(\frac{8+\lambda}{2}\right) - \left(\frac{6+\lambda}{6}\right) - 20 = 0 \Rightarrow \lambda = -30$$

$$x_1 = -\left(\frac{10+\lambda}{4}\right) = 5$$

$$x_2 = -\left(\frac{8+\lambda}{2}\right) = 11$$

$$x_3 = -\left(\frac{6+\lambda}{6}\right) = 4$$

$$0 = [0]$$

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$ $(g_1(x) = x_1 + x_2 + x_3 - 20)$

$$P^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Bordered Hessian Matrix

$$H^B = \begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$$

2m+1=2*1+1=3

$$D_3=\begin{vmatrix}0 & 1 & 1\\ 1 & 4 & 0\\ 1 & 0 & 2\end{vmatrix} = -6$$

$$D_4=\begin{vmatrix}0 & 1 & 1 & 1\\ 1 & 4 & 0 & 0\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 6\end{vmatrix} = -4$$

Both have same sign as $(-1)^m = -1$

Therefore, $x_1=5$, $x_2=11$ and $x_3=4$ is point of minimum.

Minimum value is

$2x_1^2+x_2^2+3x_3^2+10x_1+8x_2+6x_3-100=$

$2(5)^2+(11)^2+3(4)^2+10(5)+8(11)+6(4)-100=281$

Example

Use the Lagranges multiplier method to solve the following NLPP. Does the solution maximizes or minimizes the objective function?

Opt $f(x) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Subject to

$x_1 + x_2 + x_3 = 15,$

$2x_1 - x_2 + 2x_3 = 20,$

$x_1 >= 0,\ x_2 >= 0,\ x_3 >= 0.$

Solution

The Lagrange function is

$L = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 + \lambda_1(x_1 + x_2 + x_3 - 15) + \lambda_2(2x_1 - x_2 + 2x_3 - 20)$

$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 + \lambda_1 + 2\lambda_2$

$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 + \lambda_1 - \lambda_2$

$\frac{\partial L}{\partial x_3} = 2x_3 + \lambda_1 + 2\lambda_2$

$\frac{\partial L}{\partial \lambda_1} = x_1 + x_2 + x_3 - 15$

$\frac{\partial L}{\partial \lambda_2} = 2x_1 - x_2 + 2x_3 - 20$

Assuming

$\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x_2} = 0, \frac{\partial L}{\partial x_3} = 0, \frac{\partial L}{\partial \lambda_1} = 0, \frac{\partial L}{\partial \lambda_2} = 0$ **, we have**

$8x_1 - 4x_2 + \lambda_1 + 2\lambda_2 = 0$

$4x_2 - 4x_1 + \lambda_1 - \lambda_2 = 0$

$2x_3 + \lambda_1 + 2\lambda_2 = 0$

$x_1 + x_2 + x_3 - 15 = 0$

$2x_1 - x_2 + 2x_3 - 20 = 0$

On solving

$x_1 = \frac{11}{3}$

$x_2 = \frac{10}{3}$

$x_3 = 8$

$$\lambda_1 = -\frac{40}{9}$$

$$\lambda_2 = -\frac{52}{9}$$

m=2

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$(g_1(x) = x_1 + x_2 + x_3 - 15,$

$g_2(x) = 2x_1 - x_2 + 2x_3 - 20),$

$$P^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$Q=\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Bordered Hessian Matrix

$$H^B=\begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & 4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

2m+1=2*2+1=5

$$D_5=\begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & 4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix} = 72 > 0$$

Sign of $(-1)^m = 1$

There is only one determinant and sign of it is same as sign of $(-1)^m$.
Therefore, $x_1=\frac{11}{3}$, $x_2=\frac{10}{3}$ and $x_3=8$ is point of minimum.

Minimum value is

$4x_1^2+2x_2^2+x_3^2-4x_1x_2=$

$4(\frac{11}{3})^2+2(\frac{10}{3})^2+(8)^2-4(\frac{11}{3})(\frac{10}{3})=\frac{820}{9}$