Lecture 22,23

UEI407

Time integration

This property states that Fourier Transform of the integral of a function f(t) from –∞ to t is $\frac{1}{j\omega}$ times the Fourier Transform of f(t) if and only if f(t) has no dc values i.e., F(0) = 0. Mathematically, we can write

if f(t) ↔ F(ω), according to this property it can be represented as,

$$\int_{-\infty}^{\infty} f(t)dt \leftrightarrow \frac{1}{j\omega}F(\omega) \text{ provide } F(0) = 0$$

and $$\int_{-\infty}^{t} f(t)\mathrm{dt} \leftrightarrow \frac{1}{\mathrm{j}\omega}F(\omega) + \pi F(0)\delta(\omega) \text{ provided } F(0) \neq 0.$$

Proof:

Let us consider $g(t) = \int f(\lambda)\, \mathrm{d}\lambda = \int_{-\infty}^{t} f(\lambda)\mathrm{d}\lambda$ such that $\frac{\mathrm{dg}(t)}{\mathrm{dt}} = f(t)$.

We can write,

$$F[f(t)] = F\left[\frac{\mathrm{dg}(t)}{\mathrm{dt}}\right] = F(\omega)$$

i.e., $\mathrm{j}\omega\ \mathrm{G}(\omega) = F(\omega)$

i.e., $\mathrm{G}(\omega) = \frac{1}{\mathrm{j}\omega} F(\omega)$

i.e., $\mathrm{F}\left[\int_{-\infty}^{t} f(\lambda)\mathrm{d}\lambda\right] = \frac{1}{\mathrm{j}\omega} F(\omega)$

G(ω) must exist to have a transform of g(t) and hence there should not be no dc term. Its presence will not allow the satisfaction of integrability condition.

Therefore, $\int_{-\infty}^{t} f(t)\mathrm{dt}$ must tend to zero when $t\to\infty$. i.e., we can write

$$\int_{-\infty}^{\infty} f(t)\mathrm{dt} = 0 \text{ or, } \underset{t\to\infty}{\mathrm{Lt}}\ \mathrm{g}(t) = 0.$$

The signal will not be energy signal if this condition is not satisfied and hence, its transform will have impulse at $\omega = 0$.

The function $u(t-\lambda)$ equals unity upto $\lambda = t$ and it equals zero beyond $\lambda = t$. Therefore, we can write

$$g(t) = \int_{-\infty}^{t} f(\lambda)\mathrm{d}\lambda = \int_{-\infty}^{t} f(\lambda)u(t-\lambda)\mathrm{d}\lambda = \int_{-\infty}^{\infty} f(\lambda)u(t-\lambda)\mathrm{d}\lambda = f(t) * u(t)$$

Using time convolution property, we an write

$$G(\omega) = F(\omega) \cdot F[u(t)] = F(\omega)\left[\pi\delta(\omega) + \frac{1}{\mathrm{j}\omega}\right]$$

But $\quad F(\omega)\delta(\omega) = \pi \mathrm{F}(0)\delta(\omega) + \frac{F(\omega)}{\mathrm{j}\omega}$

Hence F(0) = 0, therefore the above expression becomes

$$G(\omega) = \frac{F(\omega)}{\mathrm{j}\omega}$$

Fourier Transform of f(–t)

This property states that if $f(t) \leftrightarrow F(\omega)$ then $f(-t) \leftrightarrow F(-\omega)$.

Proof:

We know that $F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt$

Therefore, $F[f(-t)] = \int_{-\infty}^{\infty} f(-t) e^{-j\omega t} \, dt$

Let us put x = –t. Therefore, dx = –dt.

Now t = –∞, x = ∞ and when t = ∞, x = –∞.

$$F[f(-t)] = \int_{-\infty}^{\infty} f(x) e^{j\omega x} \, (-dx) = \int_{\infty}^{-\infty} f(x) e^{j\omega x} \, dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-j\omega(-x)} \, (-dx) = F(-\omega)$$

Symmetry Properties of Fourier Transform

Magnitude and Phase Spectra

We can prove for a function f(t) which is a real function of variable ‘t’ that

(i) Magnitude spectrum $|F(\omega)|$ is an even function of ω and

(ii) Phase spectrum Arg $\{F(\omega)\}$ is an odd function of ω.

Proof:

We know that $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$

$$\therefore F^*(\omega) = \left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right]^* = \int_{-\infty}^{\infty} f^*(t)e^{j\omega t}\,dt$$

Since f(t) is real, we can write that $f^*(t) = f(t)$.

$$\therefore F^*(\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt$$

Therefore, the integral on RHS is same as F(ω) except that ω has been replaced by (–ω). Therefore,

$$F^*(\omega) = F(-\omega)$$

If $F(\omega) = |F(\omega)|e^{j\theta(\omega)}$, then

$$F(-\omega) = |F(\omega)|e^{-j\theta(\omega)}$$

This means that $|F(-\omega)| = |F(\omega)|$. Therefore, the magnitude spectrum is **even function of** ω. Also, Arg{F(–ω)} = –Arg{F(ω)} i.e., phase spectrum is **odd function of** ω.

Real and Imaginary parts of F(ω) – Symmetry

Since $F^*(\omega) = F(-\omega)$

Let us write $F(\omega) = \mathrm{Re}[F(\omega)] + j\mathrm{Im}[F(\omega)]$. Therefore, $F(-\omega)$ is complex conjugate of $F(\omega)$.

Therefore, $F(-\omega) = \mathrm{Re}[F(\omega)] - j\,\mathrm{Im}[F(\omega)]$

Again, $F(\omega) = \mathrm{Re}[F(-\omega)] + j\mathrm{Im}[F(-\omega)]$

From above two equations

$\mathrm{Re}[F(-\omega)] = \mathrm{Re}[F(\omega)]$

and $\mathrm{Im}[F(-\omega)] = -\mathrm{Im}[F(\omega)]$

Therefore, real part of $F(\omega)$ is an even function of ω and imaginary part of $F(\omega)$ is an odd function of ω.

Fourier Transform of Even and Odd functions

For a function f(t) which is real for all values of t, we can show that if f(t) is even, $F(\omega)$ is real and if f(t) is odd, $F(\omega)$ is purely imaginary.

Proof:

We know that $F^*(\omega) = F(-\omega)$ for any real function f(t).

$$F(-\omega) = \mathrm{Re}[F(\omega)] - j\mathrm{Im}[F(\omega)]$$

If f(t) is an even function, $f(t) = f(-t)$.

Therefore, Fourier Transform of $f(-t)$ = Fourier transform of f(t).

Hence, $F(-\omega) = F(\omega)$

i.e., $\mathrm{Re}[F(\omega)] - j\,\mathrm{Im}[F(\omega)] = \mathrm{Re}[F(\omega)] + j\,\mathrm{Im}[F(\omega)]$

The equation will be valid if and only if

$\mathrm{Im}[F(\omega)] = 0.$

Therefore for even real function f(t), $F(\omega)$ has no imaginary part.

If f(t) is an odd function, $f(t) = -f(-t)$.

Therefore, Fourier Transform of $f(-t) = -$ Fourier transform of f(t).

Fourier Transform of $f(t) = F(\omega)$ and Fourier Transform of $f(-t) = F(-\omega) = F^*(\omega)$

Hence, $F(-\omega) = -F(\omega)$

i.e., $\mathrm{Re}[F(\omega)] - j\,\mathrm{Im}[F(\omega)] = -\mathrm{Re}[F(\omega)] - j\,\mathrm{Im}[F(\omega)]$

The above equation will be valid if and only if

$\mathrm{Re}[F(\omega)] = 0.$

Therefore for odd real function f(t), $F(\omega)$ has no real part.

Fourier Transform of Periodic Signals

The periodic functions like sine wave, triangular wave and saw tooth wave etc exist for $t = -\infty$ to $t = \infty$ are truly periodic. A perpetual periodic function has infinite area under the rectified condition and therefore the does not satisfy the absolute integrability condition. Hence direct use of Fourier Transform gives an inaccurate result.

To find the Fourier Transform any of the following two approaches must be taken into account:

- We use Fourier Transforms after making use of the exponential Fourier series of the function.
- The Fourier transform of the periodic function is found out for a limited period of time say 'τ' and let τ tends to infinity. Now we can obtain Fourier Transform of the perpetual periodic signal.

Energy Density and Power Spectral density

In signal processing and identification, the concept of energy and power of a signal are often used.

Energy in a Signal

The energy over t = ∞ and t = –∞ expanded in resistance (R) after applying a voltage e(t) can be expressed as

$$E = \int_{-\infty}^{\infty} \frac{e^2(t)}{R} \mathrm{dt}$$

This energy is called as normalized energy for R = 1Ω.

Therefore, the energy in a signal f(t) by analogy we can write

$$E = \int_{-\infty}^{\infty} f^2(t) \mathrm{dt} \quad \text{for a real function f(t).}$$

The signal is said to be energy signal if the energy is finite.

For energy signal, $\int_{-\infty}^{\infty} f^2(t) \mathrm{dt} < \infty$ and it always satisfies the absolute integrability criterion

$$\int_{-\infty}^{\infty} |f(t)| \mathrm{dt} < \infty$$

Power in a Signal

There are signals which have infinite energy when integrated over infinite time but the power in the signal is finite over a fairly long time. These signals are called power signals. For power signals

$$\int_{-\infty}^{\infty} f^2(t)\mathrm{dt} = \infty$$

But P $=\frac{1}{T}\int_{-\infty}^{\infty} f^2(t)\mathrm{dt}$ and P is finite. If f(t) is complex we take |f(t)| and we take a fairly long time interval to make a reliable estimate of the average power in the signal. Periodic signals, noise signals and modulated signals are the important power signals.

Energy Spectrum and Energy Spectral Density

We can assume that the energy is distributed in the various complex exponentials to make the signals and Figure 1 shows the imagined graph. The energy in the signal component from frequency ω_1 to $(\omega_1 + \Delta\omega)$ is given by

$$\therefore\ dE = s(\omega)\, d\omega$$

The total energy in the signal is given by

$$E = \int_{-\infty}^{\infty} s(\omega) d\omega$$

The energy in band of frequencies ω_1 to ω_2 is given by

$$E = \int_{\omega_1}^{\omega_2} s(\omega) d\omega$$

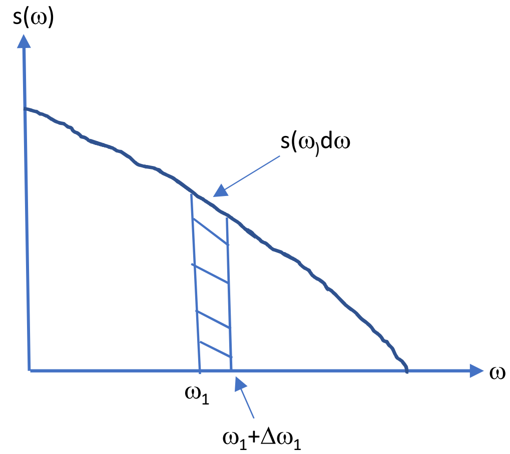


Figure 1

We can prove that for any real signal $s(\omega) = \frac{|F(\omega)|^2}{\pi}$. Therefore, the main part of the energy signal lies in those parts of frequency spectrum in which $|F(\omega)|^2$ is relatively large.

Proof:

We know that $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{j}\omega\,\mathrm{t}}\ \mathrm{d}\omega$ and energy in the signal is $E = \int_{-\infty}^{\infty} f^2(t)\mathrm{dt}$.

For real signal f(t), we can write $f^2(t) = f(t).f(t)$ and hence

$$E = \int_{-\infty}^{\infty} f(t)\cdot f(t)\ \mathrm{dt} = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{j}\omega\,\mathrm{t}}\ \mathrm{d}\omega\right]\mathrm{dt}$$

i.e., $$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} e^{\mathrm{j}\omega\,\mathrm{t}}\ \mathrm{dt}\right]\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F(-\omega)\ \mathrm{d}\omega$$

Again, for real signal $F(-\omega) = F^*(\omega)$ = conjugate of $F(\omega)$.

$$\therefore E = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F^*(\omega)\ \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

$F(\omega)$ is even function of frequency for real signals.

Hence, $E = 2\times\frac{1}{2\pi}\int_{0}^{\infty}|F(\omega)|^2\ \mathrm{d}\omega = \int_{0}^{\infty}\frac{|F(\omega)|^2}{\pi}\ \mathrm{d}\omega$

$$\therefore s(\omega) = \frac{|F(\omega)|^2}{\pi}$$

Therefore , we can draw either two sided s(ω) graph by considering $s(\omega) = \frac{|F(\omega)|^2}{2\pi}$ using the limits of integral from $-\infty$ to ∞ or one sided s(ω) graph by considering $s(\omega) = \frac{|F(\omega)|^2}{\pi}$ using the limits of integral from 0 to ∞. Figure 2 shows a typical one sided energy density graph which is for a gate function.

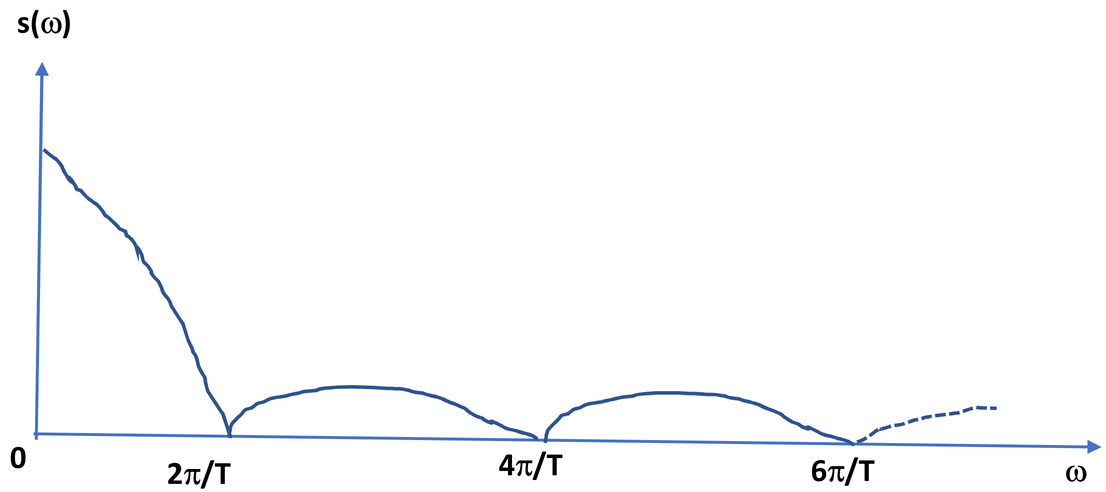


Figure 2

Parseval's Relation for Energy Signals

This relation states that the total energy in a signal may be obtained by calculating energy per unit time f(t) integrating it from –∞ to +∞ or by computing the energy per unit frequency $\frac{|F(\omega)|^2}{2\pi}$ integrating it from –∞ to +∞ frequency range. The result in both cases will remain same. The following is Parseval's relation:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{d\omega}$$

Substituting ω = 2πf or dω = 2π df, the relation $E = \int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{d\omega}$ can be expressed in slightly different way.

$$E = \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{df}$$

Therefore, for small change in frequency (Δf) the energy is given by $|F(\omega)|^2$ df.

1 Proof of Parseval's Relation for Complex f(t)

We can write for any signal

$$|f(t)|^2 = f(t)f^*(t)$$

Therefore, $\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{dt} = \int_{-\infty}^{\infty} f(t)f^*(t)\mathrm{dt} = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-\mathrm{j}\omega\ \mathrm{t}}\ \mathrm{d}\omega\right]\ \mathrm{dt}$

i.e., $\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\left[\int_{-\infty}^{\infty} f(t)e^{-\mathrm{j}\omega\ \mathrm{t}}\ \mathrm{dt}\right]\ \mathrm{d}\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)\ \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Power Spectral Density

The power spectral density having infinite energy but finite power has been shown in Figure 14. the power (normalized power) in the frequency range $\Delta\omega$ is given by

$$dP = G_1(\omega)\, d\omega$$

and total power is

$$P = \int_{-\infty}^{\infty} G_1(\omega) d\omega$$

where $G_1(\omega)$ is one sided power spectral density.

Parseval's Relation for Periodic Signals

If a periodic function f(θ) with period 2π is expressed in Fourier series as

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\ ,$$

it can be shown that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2 d\theta = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

This is called Parseval's theorem. This theorem is useful in computing the effective value of given periodic function f(θ).

Proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}(a_n^2 \cos^2 n\,\theta + b_n^2 \sin^2 n\,\theta)\right] d\theta$$

Since $\int_{-\pi}^{\pi} \cos n\theta\ d\theta = \int_{-\pi}^{\pi} \sin n\theta\ d\theta = \int_{-\pi}^{\pi} \cos n\theta\ \sin n\theta\ d\theta = 0$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{a_n^2}{2}(1+\cos 2n\theta) + \frac{b_n^2}{2}(1-\cos 2n\theta)\right)\right] d\theta$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Nyquist Theorem

This theorem states that a signal which has Fourier Transform having only frequencies upto certain maximum value f_m , we can obtain the analog signal f(t) from the sampled signal $f^*(t)$ by passing the sampled signal $f^*(t)$ through a low pass filter provided that the sampling frequency f_s is more than twice the maximum frequency f_m present in the signal i.e., $f_s > 2f_m$.

The required conditions for recovery of the analog signal f(t) from the sampled signal $f^*(t)$ are given below:

- The analog signal should be band limited to some maximum frequency f_m.
- The sampling rate (f_s) must be greater than $2f_m$.
- We will be able to get hold an ideal low pass filter.

Proof of Nyquist Theorem

The Fourier transform of impulse sampled signal is given by

$$f^*(t) = f(t) \cdot \delta_{\mathrm{PT}}(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - \mathrm{nT})$$

$$f^*(t) = \sum_{n=-\infty}^{\infty} f(\mathrm{nt}) \delta(t - \mathrm{nT})$$

Fourier transform of $f^*(\mathrm{nt})$ can be obtained using frequency convolution theorem

$$F[f^*(t)] = \frac{F[f(t)] * F\left[\delta_{\mathrm{Pt}}(t)\right]}{2\pi}$$ where $F[f(t)] = F(\omega)$

We can prove that

$$F\left[\delta_{\mathrm{PT}}(t)\right] = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - \mathrm{n}\omega_0)$$

Therefore, $F[f^*(t)] = \frac{F(\omega) * \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - \mathrm{n}\omega_0)}{2\pi} = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - \mathrm{n}\omega_0)$

where $\omega_0 = \frac{2\pi}{T}$ is sampling Frequency in rad/sec.

We conclude that the Fourier transform of sampled signal consists of several versions of $F(\omega)$which are displaced from each other by ω_0.

$$\therefore F[f^*(t)] = \frac{1}{T}[\cdots + F(\omega + 2\omega_0) + F(\omega + \omega_0) + F(\omega) + F(\omega + \omega_0) + F(\omega + 2\omega_0) + \cdots]$$

Figure 3 shows the Fourier transform of sampled signal. From Figure 3 we can recover $F(\omega)$ if and only if

$\omega_0 - \omega_m > \omega_m$

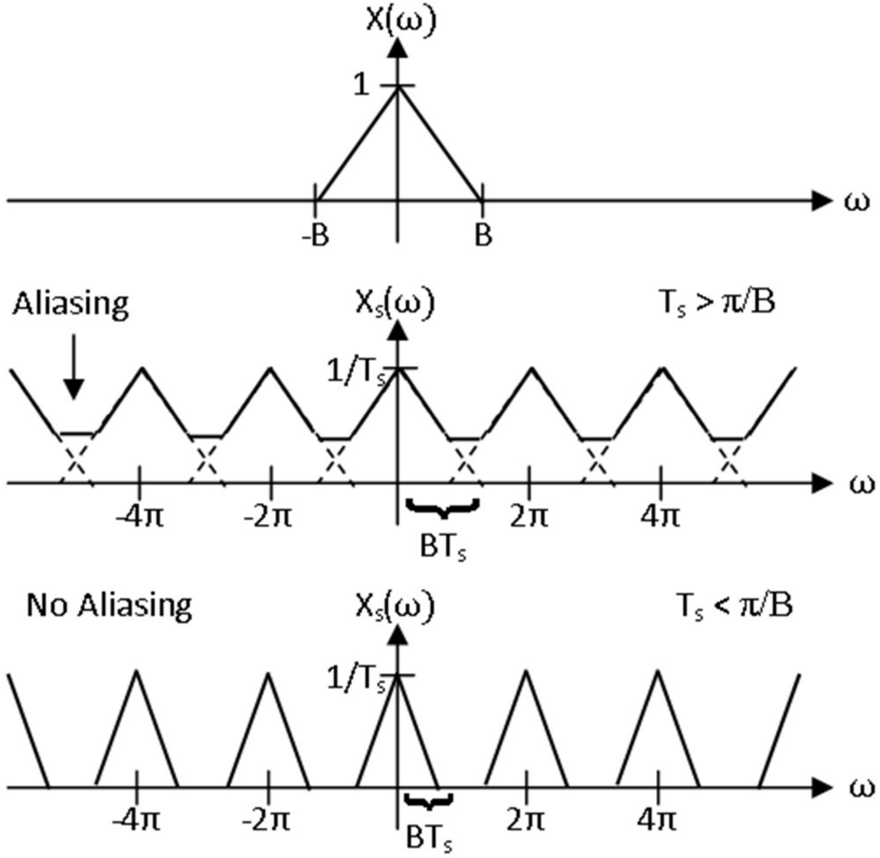
i.e., $\omega_0 > 2\omega_m$

i.e., $f_0 > 2f_m$

Therefore, for recovery of signal,

$f_s > 2f_m$

where f_s is the sampling frequency.



X(ω)
1
ω
-B
B
Aliasing
$X_s(\omega)$
$T_s > \pi/B$
$1/T_s$
ω
-4π
-2π
2π
4π
BT_s
No Aliasing
$X_s(\omega)$
$T_s < \pi/B$
$1/T_s$
ω
-4π
-2π
2π
4π
BT_s

Figure 3

Practical Implementation

Figure 4 shows that the signal before reaching the sampling switch it comes to an low pass filter (LPF) which makes the signal band limited. This filter is also known as anti–aliasing filter. After sampling of the signal another filter for recovering f(t) from $f^*(t)$ which is known as reconstruction filter.

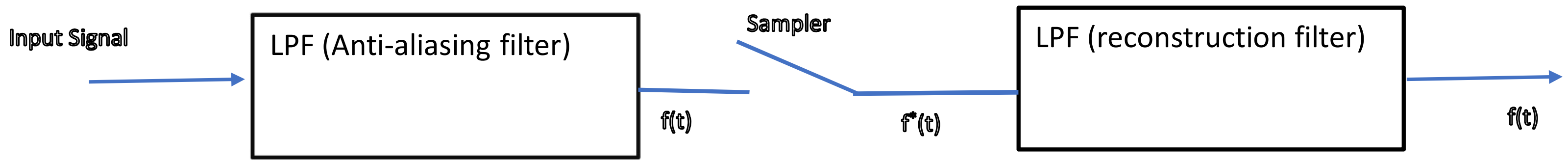


Figure 4

System Analysis Using Fourier Transform

Figure 5 shows a system having impulse response. The system function H(ω) of the system is the Fourier transform of h(t).

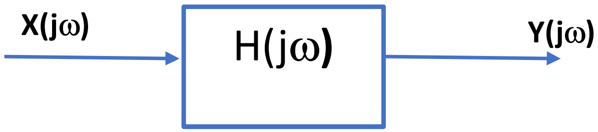


Figure 5

In the figure, the input is x(t) and its output y(t) can be obtained by convolving x(t) with h(t).

We can write $Y(\omega) = X(\omega)H(\omega)$

i.e., $y(t) = F^{-1}[X(\omega)H(\omega)]$

where X(ω) is the Fourier transform of x(t), Y(ω) is the Fourier transform of y(t) and H(ω) is the system function

Relation between differential equation and system function

Let us consider the following differential equation relating the input x(t) and y(t)

$$a_3 \frac{d^3y(t)}{dt^3} + a_2 \frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_3 \frac{d^3x(t)}{dt^3} + b_2 \frac{d^2x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Taking Fourier transform of both sides , we get

$$a_3(j\omega)^3 Y(\omega) + a_2(j\omega)^2 Y(\omega) + a_1(j\omega) Y(\omega) + a_0 Y(\omega)$$
$$= b_3(j\omega)^3 X(\omega) + b_2(j\omega)^2 X(\omega) + b_1(j\omega) X(\omega) + b_0 X(\omega)$$

i.e., $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b_3(j\omega)^3 + b_2(j\omega)^2 + b_1(j\omega) + b_0}{a_3(j\omega)^3 + a_2(j\omega)^2 + a_1(j\omega) + a_0}$