Lecture 33: Numerical Analysis (UMA011)

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Newton Divided Difference Interpolation:

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Newton's divided difference formula can be expressed as

$$\begin{aligned} P_n(x) &= f[x_0] + (x - x_0)f[x_0, x_1] + (\overset{sh}{x - x_0})(\overset{(s-1)h}{x - x_1})f[x_0, x_1, x_2] \\ &\quad + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, x_1, \cdots, x_n]. \end{aligned}$$

It can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing.

let the Step size be h

then nodes will be x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$

$+ \text{-----}\ x_n = x_0 + nh$

$$x_i = x_0 + ih$$

Newton Divided Difference Interpolation:

Newton Divided Difference Interpolation:

We take $h = x_{i+1} - x_i$, for $i = 0, 1, \cdots, n-1$ and let

$x_s = x = x_0 + sh$, then polynomial becomes

x_s ↓ interpolating point (Given)

$$P_n(x) = f[x_0] + s\,h\, f[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2] + \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0, x_1, \cdots, x_n]$$

$$= f[x_0] + sh\, f[x_0, x_1] + \binom{s}{2} 2!\, h^2 f[x_0, x_1, x_2] + \binom{s}{3} 3!\, h^3 f[x_0, x_1, x_2, x_3] + \cdots \binom{s}{n} h^n f[x_0, x_1, \cdots x_n]$$

$$P_n(x) = \underbrace{f[x_0]}_{= f(x_0)} + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k f[x_0, x_1, \cdots, x_k]$$

Here

$$\binom{s}{2} = \frac{s!}{(s-2)!\,2!} = \frac{s(s-1)\cancel{(s-2)!}}{\cancel{(s-2)!}\,2!} = \frac{s(s-1)}{2!}$$

&

$$\binom{s}{3} = \frac{s(s-1)(s-2)}{3!}$$

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Forward Differences

Notations:

x_0
x_1
x_2

$$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x_1)-f(x_0)}{h} = \frac{\triangle f(x_0)}{h},$$ delta

$$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{1}{2h}\left(\frac{\triangle f(x_1)-\triangle f(x_0)}{h}\right) = \frac{\triangle^2 f(x_0)}{2h^2} = \frac{\Delta(\Delta f(x_0))}{2h^2}$$

and, in general $f[x_0, x_1, \cdots, x_k] = \dfrac{\triangle^k f(x_0)}{k!\, h^k}$.

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k \frac{\Delta^k f(x_0)}{k!\, h^k}$$

Newton Forward Difference Formula

Newton Forward Difference Formula:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \triangle^k f(x_0), \text{ where } x = x_0 + sh.$$

unknown

known values

Newton Forward Difference Interpolation:

Newton Forward Difference Table:

Forward Difference table

$\Delta f(x_0) = f(x_1) - f(x_0)$

i	x_i	f_i	Δf	$\Delta^2 f$	$\Delta^3 f$	Δ^4
0	x_0	f_0				
			Δf_0			
1	x_1	f_1		$\Delta^2 f_0$		
			Δf_1		$\Delta^3 f_0$	
2	x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
			Δf_2		$\Delta^3 f_1$	
3	x_3	f_3		$\Delta^2 f_2$		
			Δf_3			
4	x_4	f_4				

Newton Backward Difference Formula

Reordered the nodes

$x_0 \to x_n$

$x_1 \to x_{n-1}$

$x_2 \to x_{n-2}$

$\vdots \qquad \vdots$

$x_{n-1} \to x_1$

$x_n \to x_0$

$x_{n-1} = x_n - h$

$x_{n-2} = x_n - 2h$

Newton Backward Difference Formula:

If the interpolating nodes are reordered from last to first as $x_n, x_{n-1}, \cdots, x_0$, we can write the interpolating polynomial as $P_n(x)$

$$= f[x_n] + (x - x_n)f[x_n, x_{n-1}] + (x - x_n)(x - x_{n-1})f[x_n, x_{n-1}, x_{n-2}] + \cdots + (x - x_n)(x - x_{n-1})\cdots(x - x_1)f[x_n, x_{n-1}, \cdots, x_0].$$

$sh \qquad (s+1)h$

If nodes are equally spaced with $x = x_n + sh$, then

$$P_n(x) = f[x_n] + s\,h\,f[x_n, x_{n-1}] + s(s+1)\,h^2\,f[x_n, x_{n-1}, x_{n-2}] + \cdots + s(s+1)\cdots(s+n-1)\,h^n\,f[x_n, x_{n-1}, \cdots, x_0].$$

$(x - x_n)$

$= x_n + sh - x_n$

$= sh$

$(x - x_{n-1})$

$= (\cancel{x_n} + sh) - (\cancel{x_n} - h)$

Backward Differences

Notations:

$$f[x_n, x_{n-1}] = \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}} = \frac{f(x_n)-f(x_{n-1})}{h} = \frac{\nabla f(x_n)}{h},$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{f[x_n,x_{n-1}]-f[x_{n-1},x_{n-2}]}{x_n-x_{n-2}} = \frac{1}{2h}\left(\frac{\nabla f(x_n)-\nabla f(x_{n-1})}{h}\right)$$

$$= \frac{\nabla f(x_n) - \nabla f(x_{n-1})}{2\,h^2} = \frac{\nabla^2 f(x_n)}{2h^2}$$

and, in general $f[x_n, x_{n-1}, \cdots, x_0] = \dfrac{\nabla^k f(x_n)}{k!\,h^k}$.

$\Delta f(x_0)$
$= f(x_1) - f(x_0)$
also
$\nabla f(x_1)$
$= f(x_1) - f(x_0)$
$\Rightarrow \Delta f(x_0) = \nabla f(x_1)$

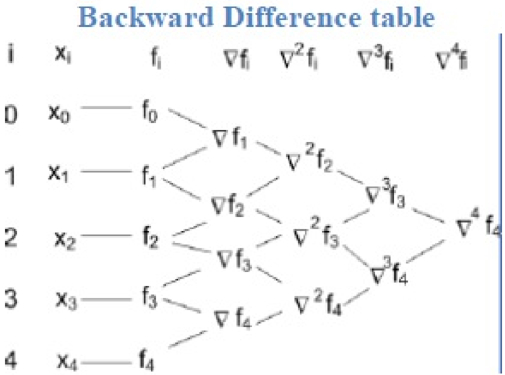
Newton Backward Difference Formula

Newton Backward Difference Formula:

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n), \text{ where } x = x_n + sh.$$

Newton Backward Difference Interpolation:

Newton Backward Difference Table:



Newton Forward-Backward Difference

Example:

Given the following data, estimate $f(1.83)$, $f(3.5)$ using Newton forward and backward difference interpolating polynomial:

x	1	3	5	7	9
$f(x)$	0	1.10	1.61	1.95	2.20

Solution:

Here, $h=2$

i	x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$
0	1	0				
			1.10			
1	3	1.10		-0.59		
			0.51		0.42	
2	5	1.61		-0.17		-0.34
			0.34		0.08	
3	7	1.95		-0.09		
			0.25			
4	9	2.20				

$$\nabla f(x_4) = f(x_4) - f(x_3)$$
$$= \Delta f(x_3)$$

Using Newton forward interpolation formula.

by taking $h = 2$, $x_s = 1.83 = x_0 + sh$

$1.83 = 1 + s(2)$

$0.83 = 2s$

$s = 0.415$

$$\left\{ \binom{s}{2} = \frac{s(s-1)}{2!} \right.$$

$$P_4(x_s) = f(x_0) + \sum_{k=1}^{4} \binom{s}{k} \Delta^k f(x_0)$$

$$= f(x_0) + \binom{s}{1} \Delta f(x_0) + \binom{s}{2} \Delta^2 f(x_0) + \binom{s}{3} \Delta^3 f(x_0)$$

$$+ \binom{s}{4} \Delta^4 f(x_0)$$

$$= 0 + (0.415)(1.10) + \frac{(0.415)(0.415-1)}{2!}(-0.59)$$

$$+ \frac{(0.415)(0.415-1)(0.415-2)}{3!}(0.42) + \frac{(0.415)(0.415-1)(0.415-3)(0.415-4)}{4!} * (-0.34)$$

$$= 0.5676$$

Using Newton's backward difference interpolation

by taking $h=2$, $x_s = x_n + sh = x_4 + sh$

$$3.5 = 9 + s(2)$$

$$s = \frac{3.5-9}{2} = -2.75$$

$$P_4(x_s) = f(x_4) + \sum_{k=1}^{4} \nabla^k f(x_n) \binom{-s}{k} =$$

$$= f(x_4) + \nabla^1 f(x_4) \binom{-s}{1} + \nabla^2 f(x_4) \binom{-s}{2} + \nabla^3 f(x_4) \binom{-s}{3}$$

$(-s)(-s-1)$

$$+ \nabla^4 f(x_4) \binom{-s}{4}$$

$$= 2.20 + (0.25)(2.75) + (-0.09) \frac{(2.75)(2.75-1)}{2!}$$

$$+ \frac{(2.75)(2.75-1)(2.75-2)}{3!} * 0.08 + \frac{(2.75)(2.75-1)(2.75-2)(2.75-3)}{4!} * (-0.34)$$

$$= - - -$$

Example:

For a function f, the forward-divided-differences are given by

$x_0 = 0.0$	$f(x_0)$	$\triangle f(x_0)$	$\triangle^2 f(x_0) = \frac{50}{7}$
$x_1 = 0.4$	$f(x_1)$	$\triangle f(x_1) = 10$	
$x_2 = 0.4$	$f(x_2) = 6$		

Determine the missing entries in the table.

Solution: Here $f(x_2) = 6$

$$\Delta f(x_1) = 10$$

$$\Rightarrow f(x_2) - f(x_1) = 10$$

$$6 - f(x_1) = 10$$

$$\Rightarrow \boxed{f(x_1) = -4}$$

Also $\Delta^2 f(x_0) = \frac{50}{7}$

$$\Rightarrow \Delta f(x_1) - \Delta f(x_0) = \frac{50}{7}$$

$$10 - \Delta f(x_0) = \frac{50}{7}$$

$$\Rightarrow \Delta f(x_0) = 10 - \frac{50}{7} \Rightarrow \boxed{\Delta f(x_0) = \frac{20}{7}}$$

and $\quad \Delta f(x_0) = f(x_1) - f(x_0)$

$$\frac{20}{7} = -4 - f(x_0)$$

$$\frac{20}{7} + 4 = -f(x_0)$$

$$\Rightarrow \boxed{f(x_0) = -\frac{48}{7}}$$

Newton Forward-Backward Difference

Exercise:

1. Construct the interpolating polynomial that fits the following data using Newton's forward and backward difference interpolation. Hence find the values of $f(x)$ at $x = 0.15$ and 0.45.

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	-1.5	-1.27	-0.98	-0.63	-0.22	0.25

2. The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2	3
$f(x)$	4	9	15	18

Determine the coefficient of x^3 in $P(x)$ if all fourth-order forward differences are 1.

Exercise:

3. Suppose that $f(x) = \cos x$ to be approximated on $[0, 1]$ by an interpolating polynomial on $n + 1$ equally spaced points. What step size h ensure that linear interpolation gives an absolute error of at most 10^{-6} for all $x \in [0, 1]$.