Lecture 21: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

$$\underset{n\times n}{A}X = b$$

$$X = ?$$

$x_0 = 5$

$x_1 = 5.1$

$x_2 = 5.12$

$x_3 = 5.14$

$|x_2 - x_3|$

by taking difference between two value

$X \in \mathbb{R}$

$X \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cdots \mathbb{R}$

$X \in \mathbb{R}^2$

$(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$

Initial guess $X^{(0)} = (0, 0, 0 - 0)$

1st iteration $X^{(1)} = (1, 2, 1.1, - -)$

$X^{0} - X^{(1)} = ?$

Iterative methods to solve System of linear equations:

Distance between n−dimensional vectors

To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between n−dimensional column vectors.

$$X^{(n)} = (x_1^{(n)}, x_2^{(n)} \dots x_n^{(n)})^t$$

x, y

$|x-y|$

$$X^{(n+1)} = (x_1^{(n+1)}, x_2^{(n+1)} \dots x_n^{(n+1)})^t$$

$X = (x_1, x_2)$

$Y = (y_1, y_2)$

$$\|X-Y\|_2 = \sqrt{(x_2-y_2)^2 + (x_1-y_1)^2}$$

$$\|X-Y\|_\infty = \max |\,|x_1-y_1|, |x_2-y_2|\,|$$

$X = (x_1, x_2, x_3)$

$Y = (y_1, y_2, y_3)$

Norms

Vector norms

Let $\mathbb{R}^n$ denote the set of all $n-$dimensional column vectors with real-number components.
To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

$X \in \mathbb{R}^n$

$\mathbb{R}$

x, y

$|x-y|$

$|x|, x \in \mathbb{R}$

$\|X\|, X \in \mathbb{R}^n$

$|x-y| \geq 0$

$|x-y| = |y-x|$

$|x-y| \leq |x-z| + |z-y|$

Def^n of vector Norm.

$f : \mathbb{R}^n \to \mathbb{R}$

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$

A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, ie $X = (x_1, x_2 - - - x_n)^t$

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

$\|X\| \to$ distance bet^n vector X and 0

Norms

Vector norms

We will need only two specific norms on $\mathbb{R}^n$,

(i) $\|x\|_2 \geq 0$

(ii) $\|x\|_2 = 0$ iff $x = 0$

(iii) $\|\alpha x\| = \left(\sum_{i=1}^{n} (\alpha x_i)^2\right)^{1/2}$

$= |\alpha| \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

$|\alpha| \; \|x\|_2$

(iv) by using Cauchy-Schwarz inequality.

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

l_2-space

$x \in l_2$-space

$\|x\|_2$

l_∞-space

$x \in l_\infty$ space

$\|x\|_\infty$

$\|x - 0\|_2 = \sqrt{(x_1 - 0)^2 + (x_2 - 0)^2 + \cdots (x_n - 0)^2}$

$= \sqrt{x_1^2 + x_2^2 - - x_n^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

$\parallel^{ly}$ $\quad \|X-0\|_\infty = \max\{|x_1-0|, |x_2-0|, \ldots |x_n-0|\}$

$$= \max_{1\le i\le n} |x_i|$$

Graphical Representation

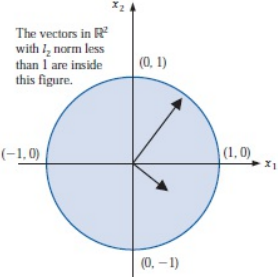
let $X \in \mathbb{R}^2$

$X = (x_1, x_2)^t$

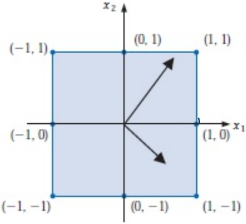
if $\|X\|_2 = 1$

then $\sqrt{x_1^2 + x_2^2} = 1$

let $\|X\|_2 = 1$



The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are inside this figure.



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.

let $\|X\|_\infty = 1$

then $\max\{|x_1 - 0|, |x_2 - 0|\} = 1$

$\Rightarrow \quad |x_1| \le 1$

$|x_2| \le 1$

Norms

Example:

Determine the l_2 norm and the l_∞ norm of the vector $x = (-1, 1, -2)^t$.

l_2- Norm, $\quad ||X||_2 = \left(\sum_{i=1}^{3} x_i^2 \right)^{1/2} = \sqrt{(-1)^2 + 1^2 + (-2)^2}$

$= \sqrt{6} = 2.45$

l_∞- Norm, $\quad ||X||_\infty = \max_{1 \le i \le 3} |x_i| = \max\{|-1|, |1|, |-2|\} = 2$

Norms

Distance between Vectors in $\mathbb{R}^n$:

If $x = (x_1, x_2, \cdots, x_n)^t$ and $y = (y_1, y_2, \cdots, y_n)^t$ are vectors in $\mathbb{R}^n$, then l_2 and l_∞ distances between x and y are defined by

$$\|x - y\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|x - y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|.$$

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$

$x^0 = (x_1^{(0)}, x_2^{(0)} - - - x_n^{(0)})$

$x^{(1)} = (x_1^{(1)}, x_2^{(1)} - - - x_n^{(1)})$

$\|x^{(1)} - x^0\|_\infty = \max\{|x_1^{(1)} - x_1^{(0)}|, |x_2^{(1)} - x_2^{(0)}|, - - |x_n^{(1)} - x_n^{(0)}|\}$

$\le$ tol (given)

Norms

Convergence of a sequence in $\mathbb{R}^n$:

The sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \cdots, n$.

$X^{(0)}$

$X^{(1)}$

$X^{(2)}$

$X^{(3)}$

$X^{(4)}$

$\vdots$

$X^{(n)}$

$\lim_{n\to\infty} X^{n} = X$?

$$X^{(n)} = (x_1^{(n)}, x_2^{(n)} \;---\; x_n^{(n)})$$

$$\downarrow n\to\infty \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$X = (x_1 \quad x_2 \quad x_n)$$

In case of real no's

$\lim_{n\to\infty} x_n = x$.

Convergence of a sequence in $\mathbb{R}^n$

Example:

Show that
$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t = \left(1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin(k)\right)^t$
converges to $x = (1, 2, 0, 0)^t$ with respect to l_∞ norm.

$$x^{(k)} = \left(1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin k\right) \to (1, 2, 0, 0)$$

$|a_n| \le |b_n| \to 0$ $\therefore$ component wise convergence

$\Downarrow$

0

$$1 \to 1$$

$$\lim_{k\to\infty} 2 + \frac{1}{k} = 2$$

$$\lim_{k\to\infty} \frac{3}{k^2} = 0$$

$$\lim_{k\to\infty} e^{-k}\sin k = \lim_{k\to\infty} \frac{\sin k}{e^k}$$

$$\le \lim_{k\to\infty} \frac{1}{e^k} = 0$$

System of linear equations:

Exercise:

1. Find l_∞ and l_2 norms of the vectors.
   a) $x = (3, -4, 0, \frac{3}{2})^t$.
   b) $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.
2. Find the limit of the sequence
   $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^t = \left(\frac{4}{k+1}, \frac{2}{k^2}, k^2 e^{-k}\right)^t$ with respect to l_∞ norm.