LECTURE-15,16

UEI407

Nyquist Rate

The minimum sampling rate required for perfect reconstruction of sampled signal at the receiver is known as Nyquist rate .

Nyquist rate = $2F_{max}$ where F_{max} is the highest frequency component.

Sampling Theorem

If sampling frequency is greater than or equal to Nyquist rate of the message signal then perfect reconstruction of sampled signal at the receiver is possible. As per this theorem, we can write that

Sampling rate $\geq$ Nyquist rate, $2F_{max}$.

A relationship between t and n of continuous–time and discrete–time signals is obtained during periodic sampling.

Aliasing

The phenomenon of aliasing occurs if the sampling frequency is less than Nyquist rate. If sampling rate is less than or greater than Nyquist rate , it is called under sampling or over sampling. Aliasing phenomenon occurs only for under sampling.

Consider a continuous signal in the time and frequency domain shown in Figure 1.

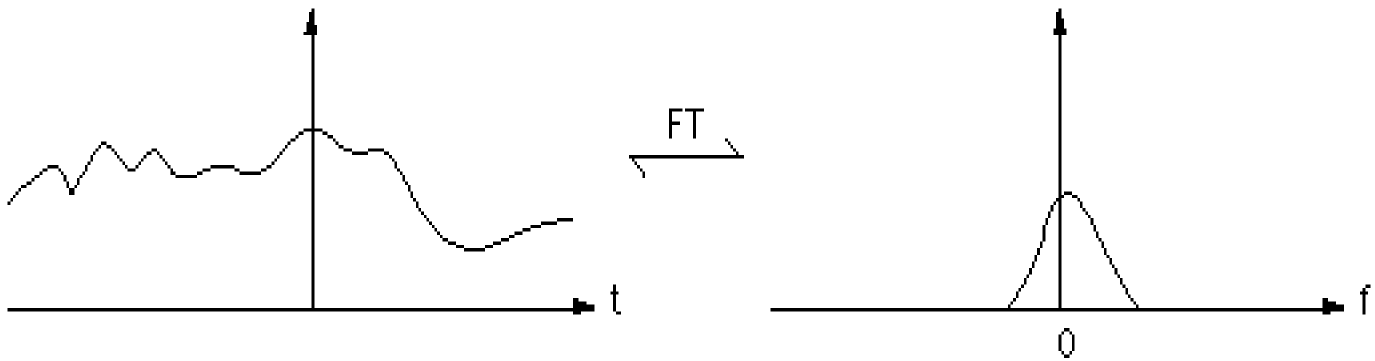


Figure 1

If the sampling frequency is too low, the frequency spectrum overlaps, and become corrupted shown in Figure 2.

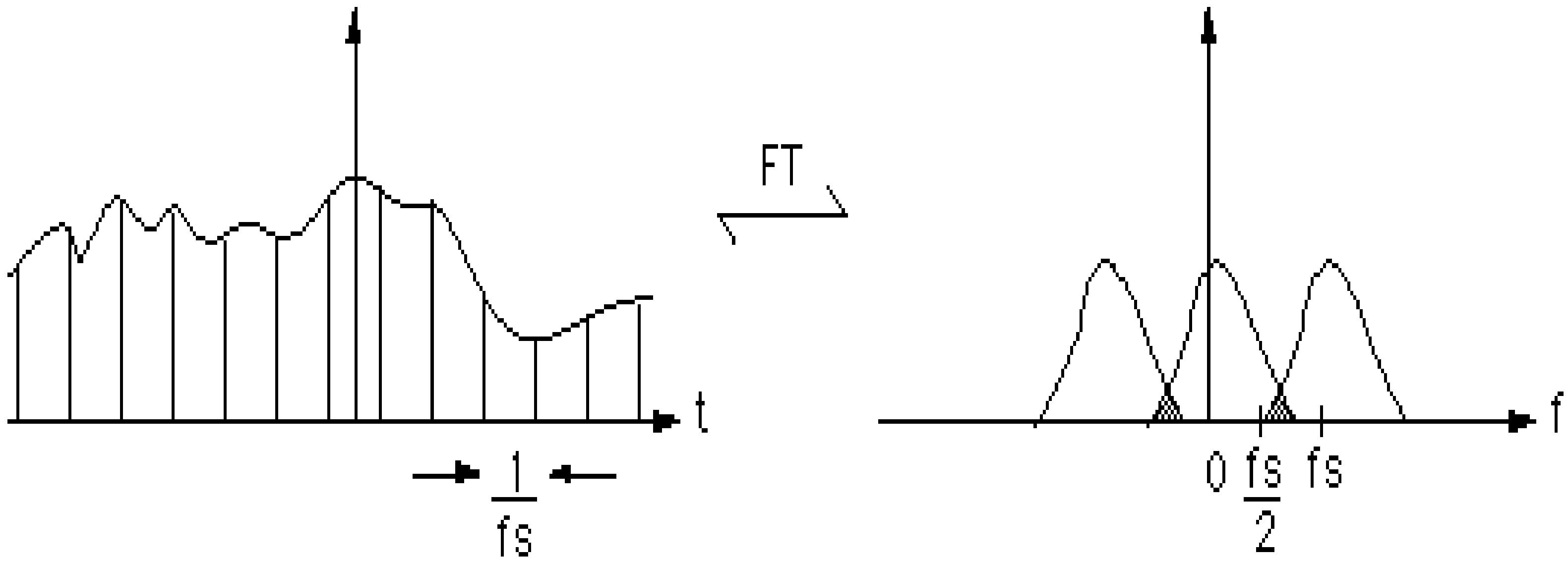


Figure 2

Discrete Time Signal as Weighted Impulses

A discrete time signal $x(n) = \{1,2,\underset{\uparrow}{3},1,-2\}$ is considered here to be expressed in the form of weighted impulses. Here $x(-2) = 1$, $x(-1) = 2$, $x(0) = 3$, $x(1) = 1$, $x(2) = -2$

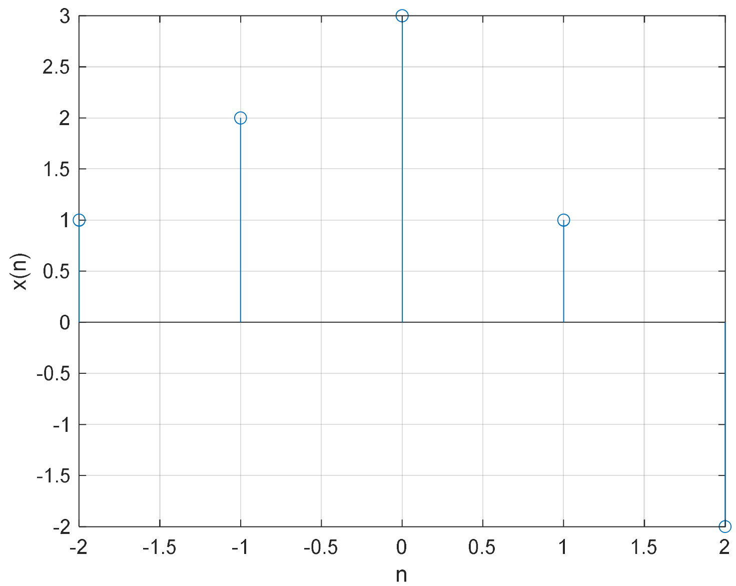


Figure 3: Plot of x(n)

Mathematically, the unit sample sequence is expressed by

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

In general,

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

where ‘k’ can be positive or negative.

Therefore, $\delta(n+2) = \delta(n+1) = \delta(n) = \delta(n-1) = \delta(n-2) = 1$.

x(n) and δ(n+2)

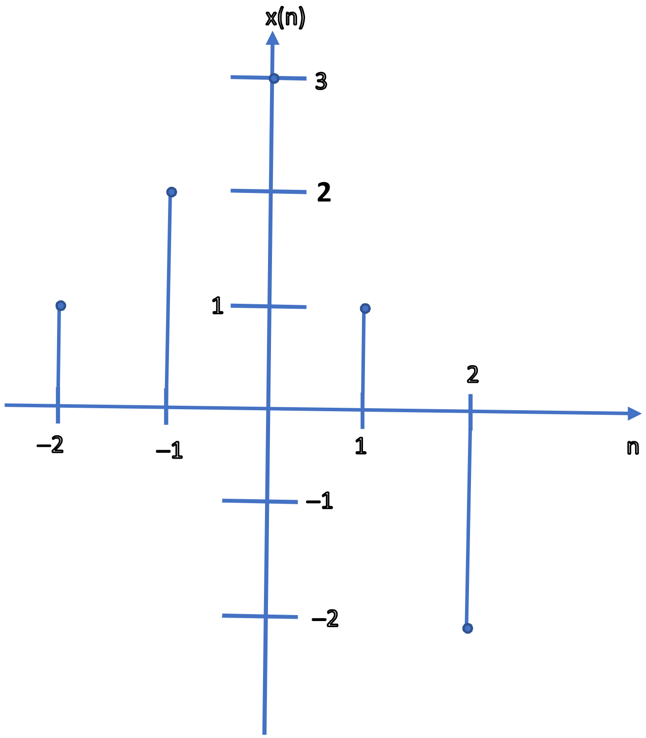


Figure 4

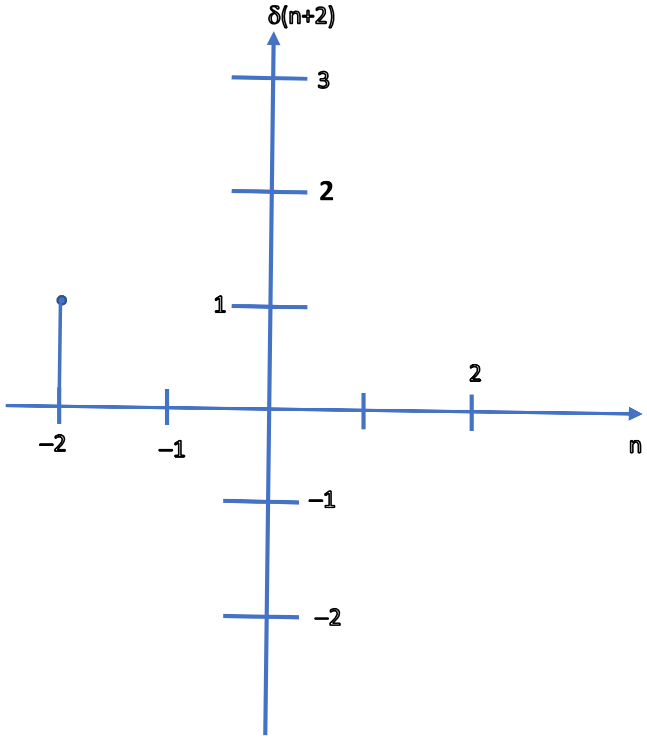


Figure 5

$x(n)\,\delta(n+2)=1\times1=x(-2)\,\delta(n+2)=1$

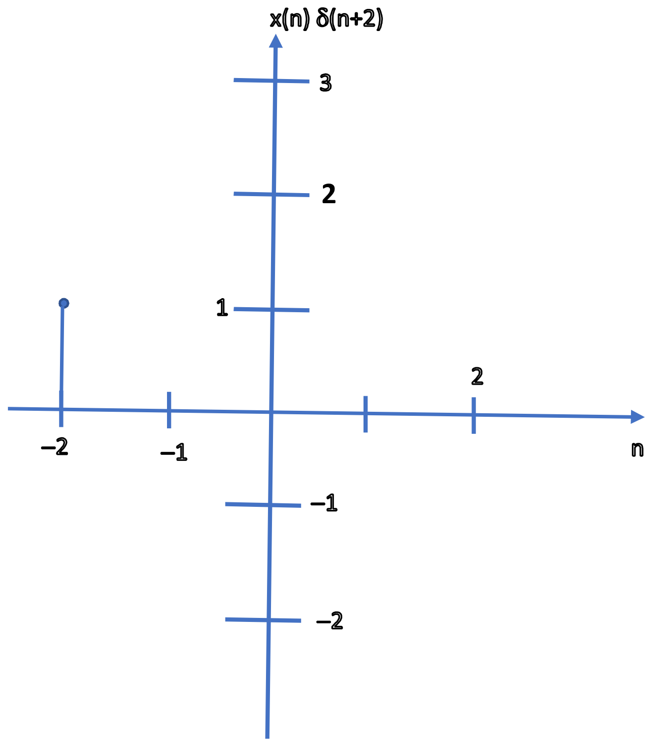


Figure 6

x(n) and δ(n+1)

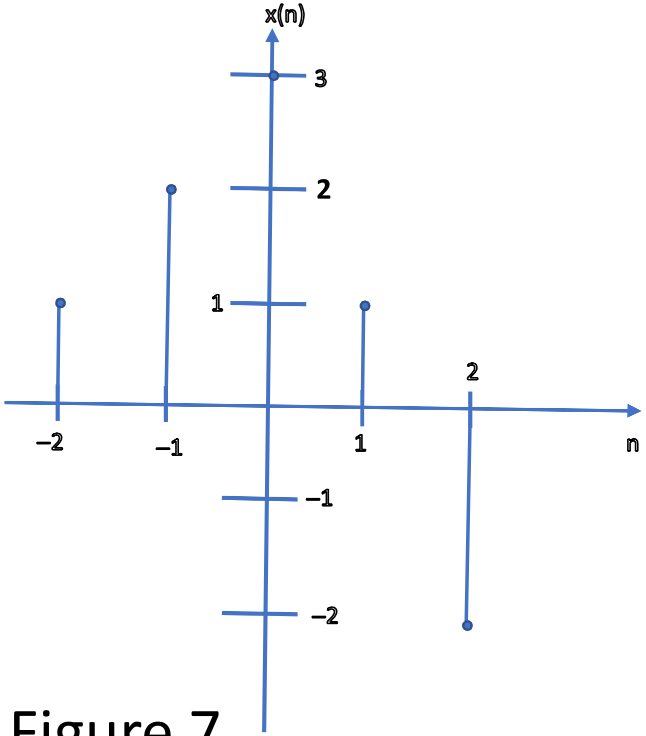


Figure 7

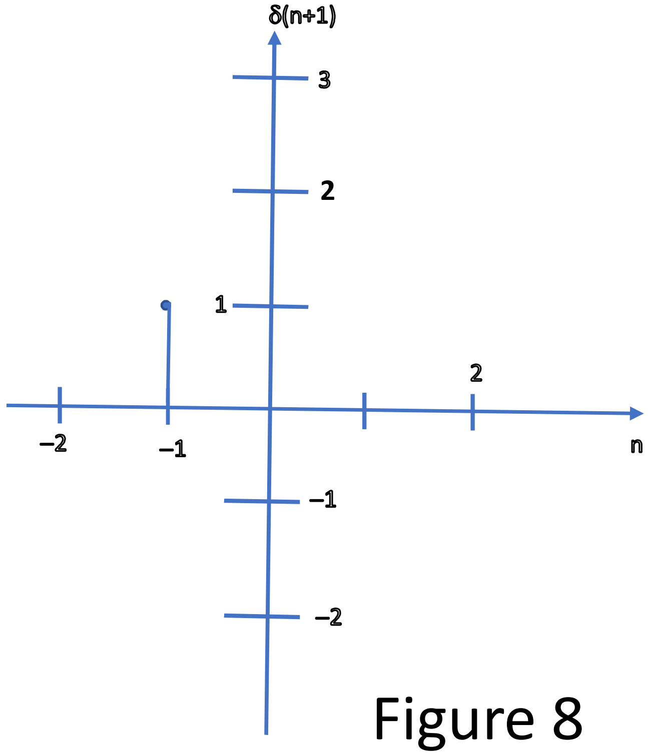


Figure 8

$x(n)\ \delta(n+1)=2\times1=x(-1)\ \delta(n+1)=2$

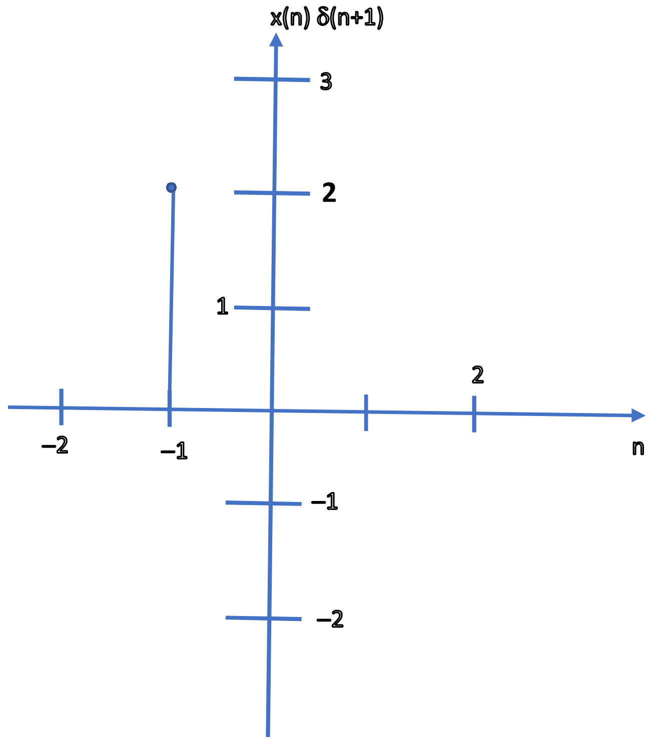


Figure 9

Therefore, $x(-2)\,\delta(n+2) = 1 = x(-2)$

$x(-1)\,\delta(n+1) = 2 = x(-1)$

Similarly,

$x(0)\,\delta(n) = 3 = x(0)$

$x(1)\,\delta(n-1) = 1 = x(1)$

$x(2)\,\delta(n-2) = -2 = x(2)$

$x(-2) = 1,\ x(-1) = 2,\ x(0) = 3,\ x(1) = 1,\ x(2) = -2$

Therefore, x(n) can be expressed as

$$x(n) = x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2)$$

$$= \sum_{k=-2}^{2} x(k)\delta(n-k)$$

In general,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

The above expression shows that x(n) is equal to the summation of unit samples where the amplitudes of unit samples are basically sample values of x(n).

Linear Convolution

Linear convolution is a very powerful technique, which is used to analyze the LTI systems. It has been shown that $x(n)$ can be expressed as sum of weighted impulses.

The response of $y(n)$ of the discrete system for the input $x(n)$ is expressed by

$$y(n) = T[x(n)]$$

i.e.,

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

The above expression has been obtained after substituting x(n) as a sum of weighted impulses. The above expression can be expressed as

$$y(n) = T[\cdots\cdots + x(-3)\delta\,(n+3) + x(-2)\delta\,(n+2) + x(-1)\delta\,(n+1) + x(0)\delta(n)$$
$$+ x(1)\delta\,(n-1)\; + x(2)\,\delta\,(n-2) + x(3)\delta\,(n-3) + \cdots\cdots]$$

The above expression shows the linearity property, which states that the output due to linear convolution of inputs, is same as the sum of outputs due to individual inputs. In this expression sample values ……, $x(-3)$, $x(-2)$,$x(-1)$,$x(0)$,$x(1)$,$x(2)$,$x(3)$,…… etc. are constants. The scaling property states that if $y(n) = T[a\delta(n)]$ then $y(n) = aT[\delta(n)]$. On the basis of scaling property, the above expression can be expressed as

$$y(n) = \cdots\cdots + x(-3)T\left[\delta\,(n+3)\right] + x(-2)T\left[\delta\,(n+2)\right] + x(-1)T\left[\delta\,(n+1)\right]$$
$$+\, x(0)T\left[\delta(n)\right] + x(1)\,T\left[\delta\,(n-1)\right] \;+ x(2)\,T\left[\delta\,(n-2)\right] + x(3)\,T\left[\delta\,(n-3)\right] + \cdots$$

The above expression can be expressed as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T\left[\delta(n-k)\right] \qquad (1)$$

The response of the system due to unit sample sequence $\delta(n)$ is expressed by $T[\delta(n)] = h(n)$ where **h(n) is the unit sample response or impulse response of the system**.

For shift invariant discrete time systems, Eq. (1) can be expressed as

$$T[\delta(n-k)] = h(n-k) \qquad (2)$$

where ‘k’ represents the shift in samples. Therefore, if the excitation of the shift invariant system is delayed by ‘k’ samples, its response will also be delayed by the same number of samples. Therefore, the final expression becomes .

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad (3)$$

Therefore, the response of linear time invariant (LTI) system due to an input x(n) can be completely characterized by the unit sample response h(n). The above expression is basically linear convolution of x(n) and h(n) giving y(n). Therefore,

Linear Convolution:

$$y(n) = x(n) * h(n) \tag{4}$$

i.e.,
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5}$$

where $y(n)$ is the response, $x(n)$ is the input and $h(n)$ depends on the characteristics of the system.

To complete the linear convolution the following steps are followed:

- Folding: Sequence $h(k)$ is folded at $k = 0$ so that $h(-k)$ is obtained.
- Shifting: $h(-k)$ is shifted depending upon the value of 'n' in $y(n)$.
- Multiplication: $x(k)$ and $h(n-k)$ are then multiplied on sample to sample basis.
- Summation: The sequences $x(k)$ and $h(n-k)$ are multiplied over all values of 'k' so that $y(n)$ is obtained.

Obtain y(n) by convolving the following two sequences x(n) and h(n) where

$x(n) = \{1,1,1,1\}$ and $h(n) = \{2,2\}$

Since upward arrow (↑) has not been shown in x(n) and h(n), the first sample in the sequence is 0 th sample and the sample values are shown below:

$x(0) = 1,\ x(1) = 1,\ x(2) = 1$ and $x(3) = 1$

$h(0) = 3$ and $h(1) = 3$

Figure E1 and Figure E2 show x(n) and h(n) respectively.

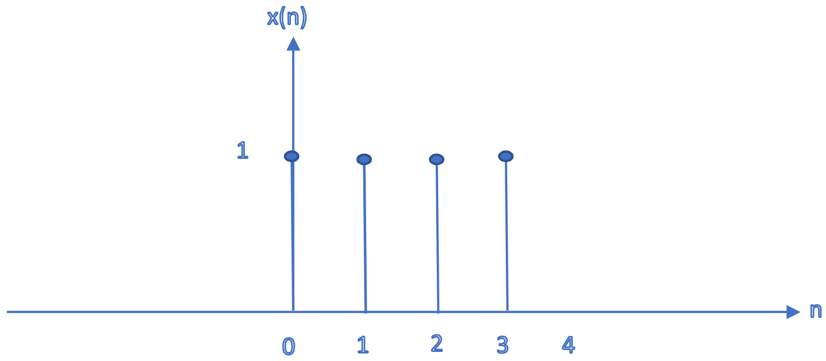


Figure E1

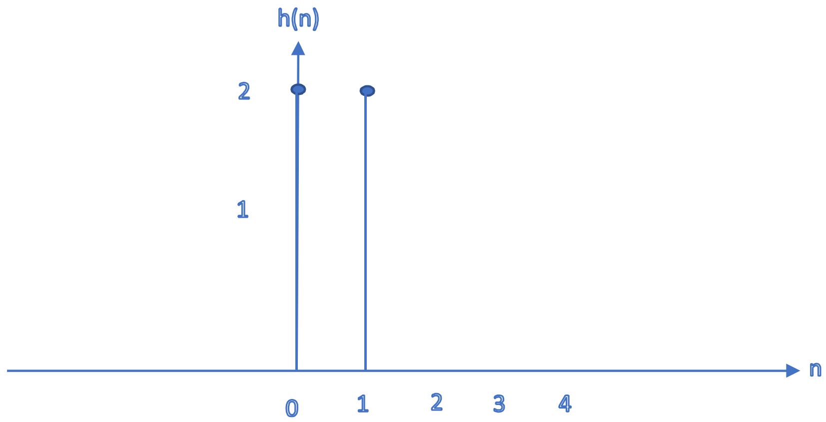


Figure E2

The convolution of x(n) and h(n) is expressed by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

For n = 0, $y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$

Figure E3 shows x(k).To get h(–k), h(k) is folded around k = 0. Figure E4 shows h(–k). The product x(k)h(–k) is obtained by multiplying sequences x(k) and h(–k) shown in Figure E5.

Figure E3

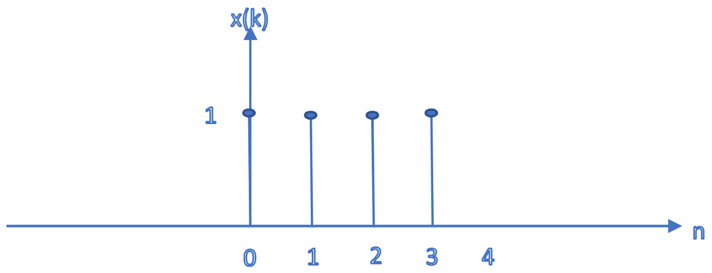


Figure E4

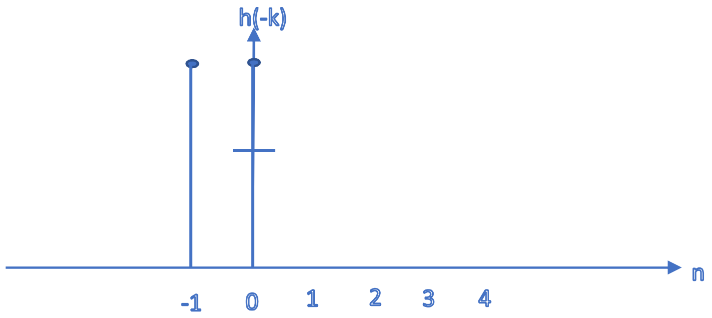
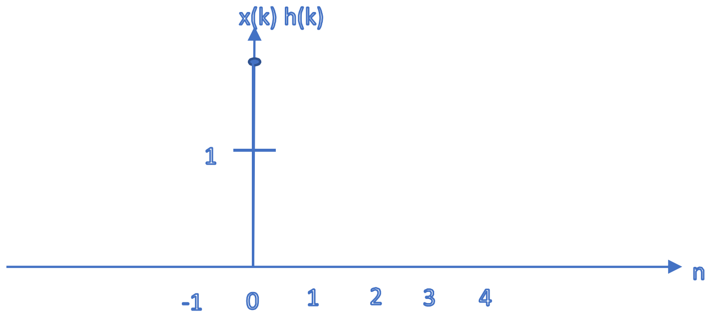


Figure E5



From Figure E5, $y(0) = \sum_{\text{over all samples}} x(k)h(-k) = 0 + 2 + 0 + 0 + 0 = 2$

For n=1, $y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \sum_{k=-\infty}^{\infty} x(k)h(-(k\text{-}1))$

Figure E6 shows x(k). Figure E7 shows h(1–k), which is obtained by shifting h(1–k) to the right by one sample. Figure E8 shows the product of sequences x(k)h(1–k).

Figure E6

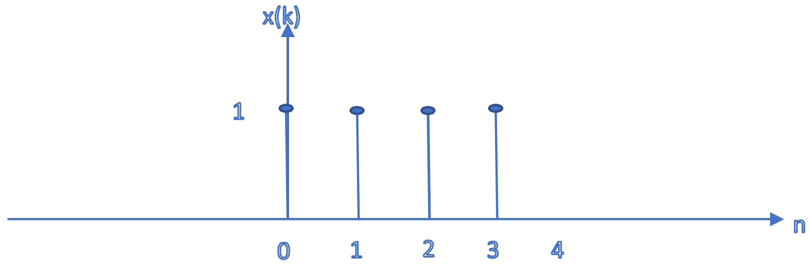


Figure E7

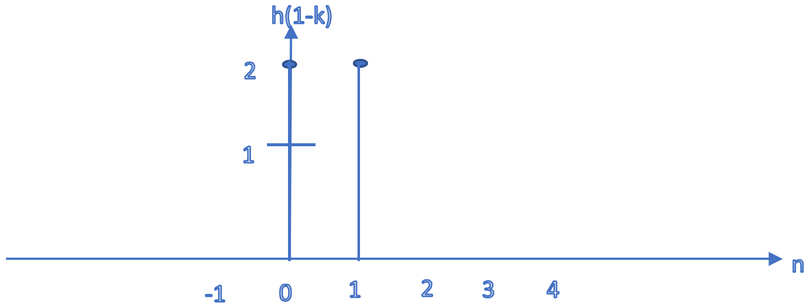
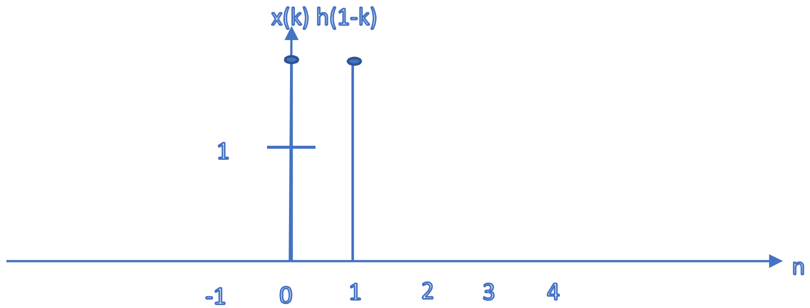


Figure E8



$$\therefore\ y(1) = \sum_{\text{over all samples}} x(k)h(1-k) = 2+2+0+0 = 4$$

For n =2, $$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$

Figure E9 shows x(k). Figure E10 shows h(2–k), which is obtained by shifting h(–k) to the right by two samples. Figure E11 shows the product of sequences x(k)h(2–k).

Figure E9

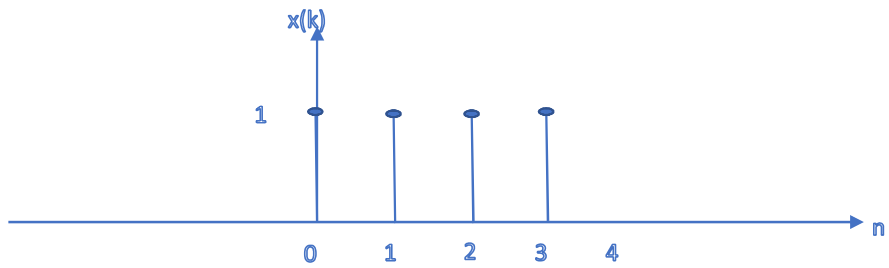


Figure E10

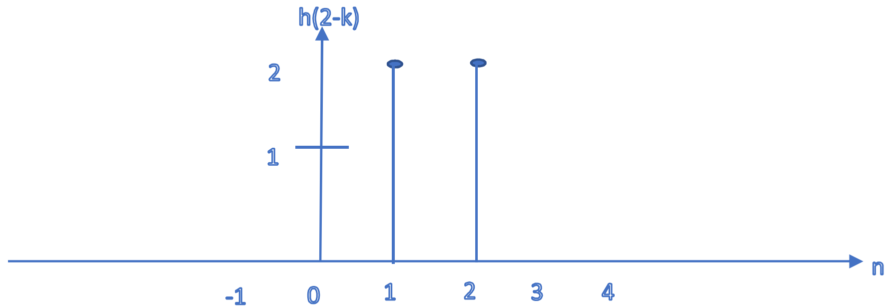
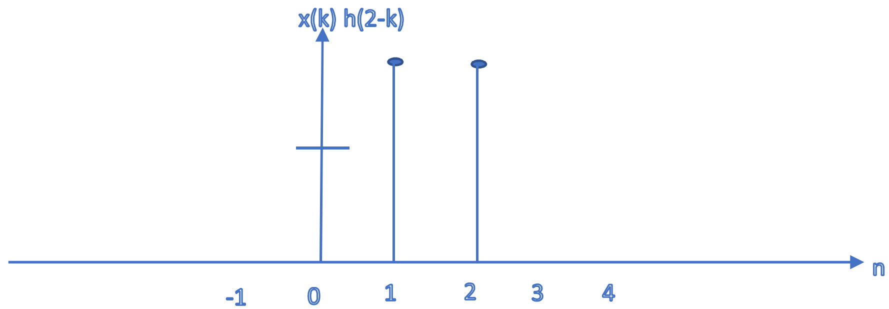


Figure E11



Therefore, $$y(2) = \sum_{\text{over all samples}} x(k)h(2-k) = 0+2+2+0 = 4$$

For n = 3, $$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

Figure E12 shows x(k). Figure E13 shows h(3–k), which is obtained by shifting h(–k) to the right by three samples. Figure E14 shows the product of sequences x(k)h(3–k).

Figure E12

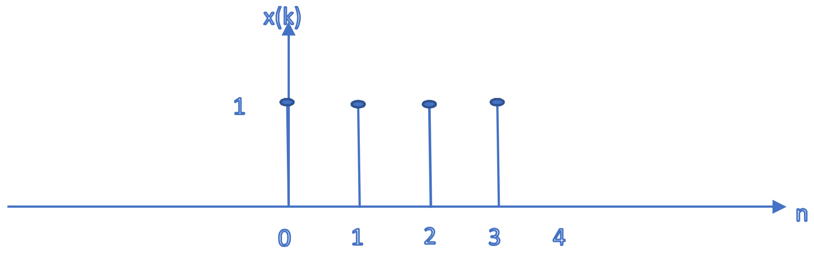


Figure E13

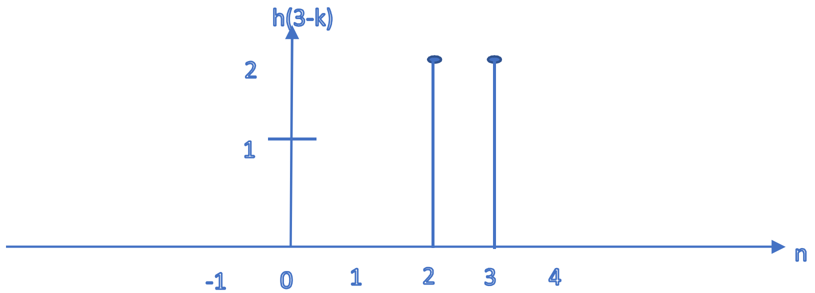
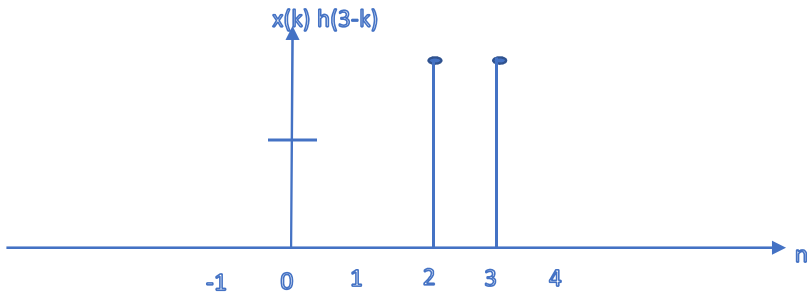


Figure E14



Therefore, $y(3) = \sum_{\text{over all samples}} x(k)h(3-k) = 0 + 0 + 2 + 2 = 4$

For n=4, $y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$

Figure E15 shows x(k). Figure E16 shows h(4–k), which is obtained by shifting h(–k) to the right by four samples. Figure E17 shows the product of sequences x(k)h(4–k).

Figure E15

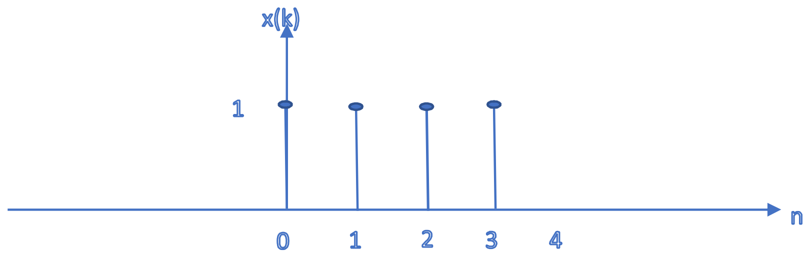


Figure E16

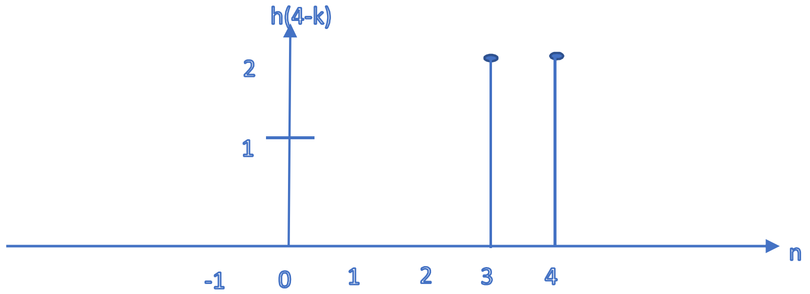
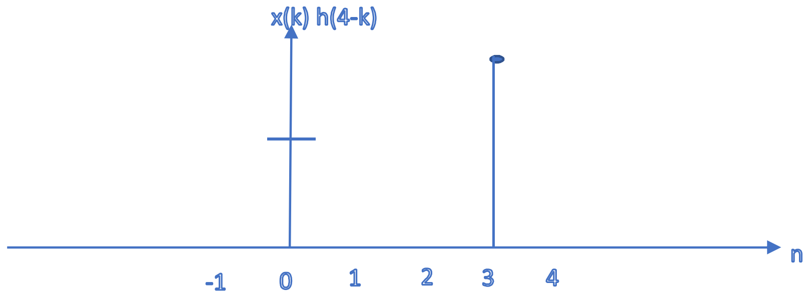


Figure E17



Therefore, $y(4) = \sum_{\text{over all samples}} x(k)h(4-k) = 0+0+0+2+0 = 2$

For n =5, $y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k)$

Figure E18 shows x(k).Figure E19 shows h(5–k), which is obtained by shifting h(–k) to the right by five samples. Figure E20 shows the product of sequences x(k)h(5–k).

Figure E18

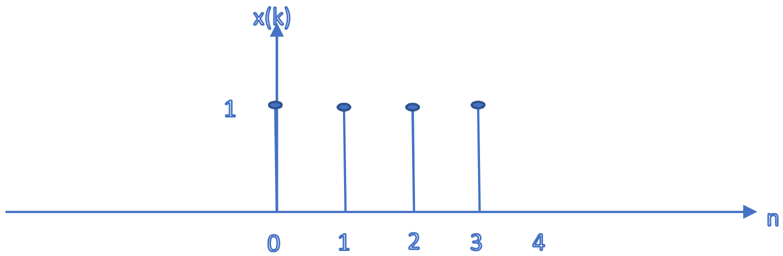


Figure E19

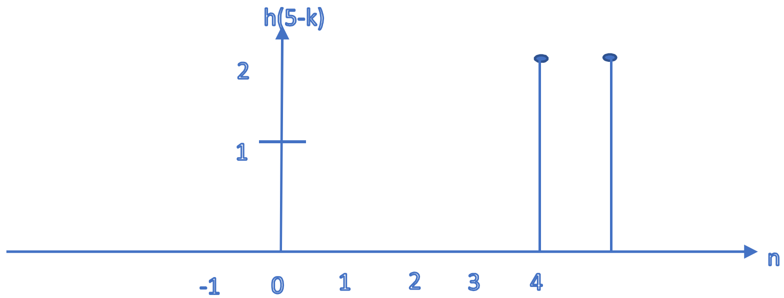
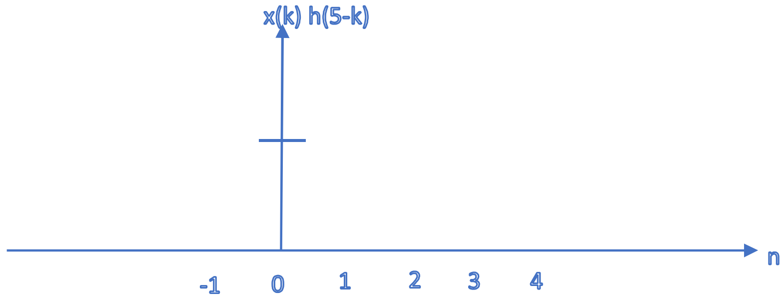


Figure E20



Therefore, $y(5) = \sum_{\text{over all samples}} x(k)h(5-k) = 0+0+0+0+0 = 0$

Hence for $n \geq 5$, y(n) becomes 0.

For $n = -1, \quad y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$

Figure E21 shows x(k). Similarly, for n = –1, h(–1–k) i.e., h[–(k+1)] i.e., h(–k) is advanced by one sample i.e., h(–k) is shifted to the left by one sample shown in Fig. E22. Figure E23 shows the product sequences of x(k)h(–1–k).

Figure E21

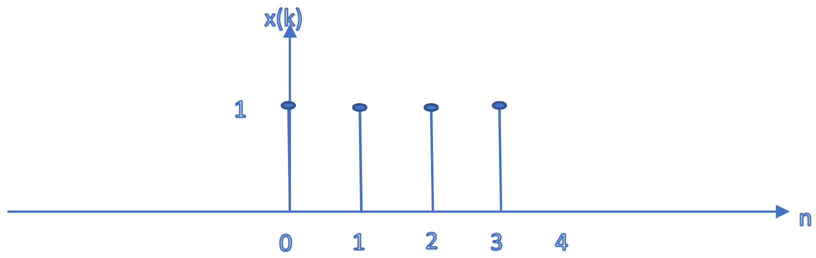


Figure E22

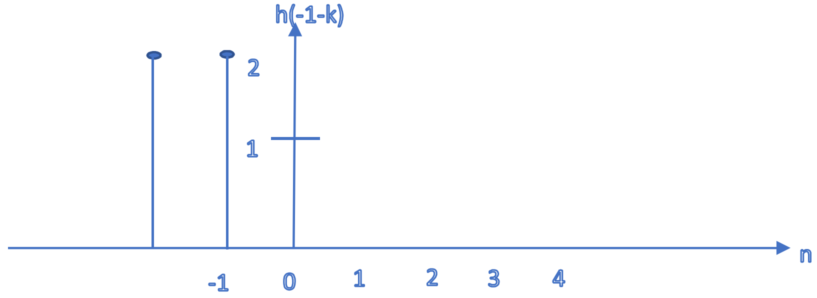
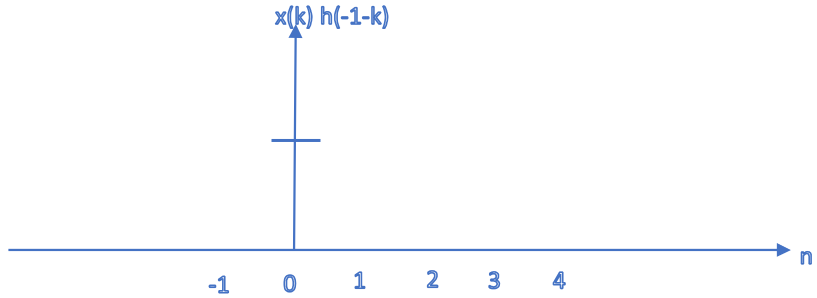


Figure E23



Therefore, $$y(-1) = \sum_{\text{over all samples}} x(k)h(-1-k) = 0+0+0+0+0 = 0$$

Hence for $n \le -1$, y(n) becomes zero.

$$\therefore y(n) = \{\cdots\cdots,0,0\underset{\uparrow}{2},4,4,4,2,0,0,\cdots\cdots\}$$

$$\therefore y(n) = \{\underset{\uparrow}{2},4,4,4,2\}$$

Figure E24 shows the plot of y(n) = x(n)*h(n).

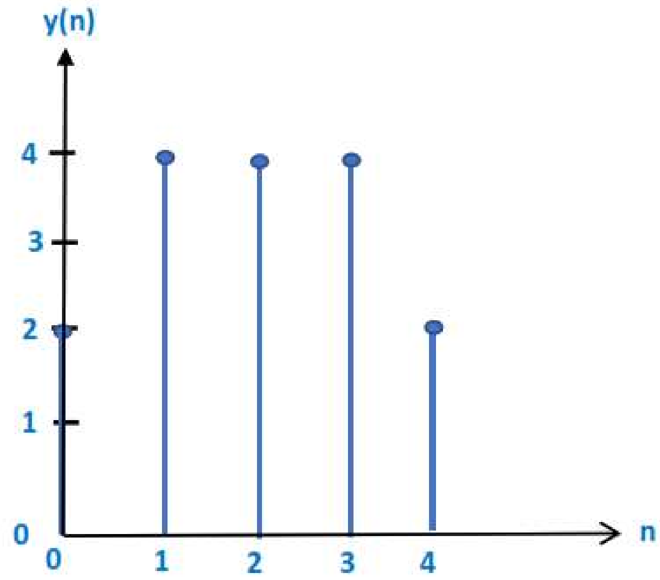


Figure E24 : y(n) = x(n)*h(n)

Obtain y(n) by convolving x(n) = {1,1,1,1} and h(n) = {2,2} by using basic convolution equation.

Given that x(n) = {1,1,1,1} and h(n) = {2,2}.

$\therefore x(0) = 1,\ x(1) = 1,\ x(2) = 1,\ x(3) = 1$ and $h(0) = 2,\ h(1) = 2.$

The linear convolution of x(n) and h(n) is expressed as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

In the present case x(k) = 0 for k < 0 and x(k) = 0 for k > 3. Therefore,

lower limit of k is 0 and upper limit of k is 3.

$$\therefore\ y(n) = \sum_{k=0}^{3} x(k)h(n-k)$$

Similarly, h(k) = 0 for k < 0 and k > 1.

Therefore, lower limit of k is 0 and upper limit of k is 1.

The lower index of ‘n’ is 0 + 0 = 0 and highest index of ‘n’ is 3 + 1 = 4.

For n = 0:

$$y(0) = \sum_{k=0}^{3} x(k)h(-k) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) + x(3)h(-3)$$

$$= \ 1\times 2 + 0 + 0 + 0 \ = \ 2$$

For n = 1:

$$y(1) = \sum_{k=0}^{3} x(k)h(1-k) = x(0)h(1) + x(1)h(0) + x(2)h(-1) + x(3)h(-2)$$
$$= 1\times 2 + 1\times 2 + 1\times 0 + 1\times 0 = 4$$

For n = 2:

$$y(2) = \sum_{k=0}^{3} x(k)h(2-k) = x(0)h(2) + x(1)h(1) + x(2)h(0) + x(3)h(-1)$$
$$= 1\times 0 + 1\times 2 + 1\times 2 + 1\times 0 = 4$$

For n = 3:

$$y(3) = \sum_{k=0}^{3} x(k)h(3-k) = x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0)$$
$$= 1\times 0 + 1\times 0 + 1\times 2 + 1\times 2 = 4$$

For n = 4:

$$y(4) = \sum_{k=0}^{3} x(k)h(4-k) = x(0)h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1)$$

$$= 1\times 0 + 1\times 0 + 1\times 0 + 1\times 2 = 2$$

$$\therefore y(n) = x(n) * h(n) = \{2,4,4,4,2\}$$