LECTURE 25,26

UEI407

Laplace transform of a Derivative $\left[\frac{df(t)}{dt}\right]$

Since. $\int u\,dv = uv - \int v\,du$

Let $u = f(t)$ and $dv = e^{-st}\,dt$

$\therefore du = f'(t)$ and $v = \frac{-1}{s}e^{-st}$

$$\therefore F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$= \left[-\frac{f(t)}{s}\,e^{-st}\right]_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right) f'(t)\,dt$$

$$= \frac{f(0)}{s} + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\,dt$$

$$\therefore s\,F(s) = f(0) + \int_0^\infty f'(t)\,e^{-st}\,dt$$

$$\int_0^\infty f'(t)\,e^{-st}\,dt = sF(s) - f(0)$$

$$\therefore\ LT\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

In general it can be written as

$$LT\left[\frac{df^{n}(t)}{dt^{n}}\right] = s^{n} F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- s^{n-1} f''(0) - \ldots\ldots\ldots\ldots - f^{n-1}(0) \quad (1)$$

To apply Laplace Transformation in circuit theory, the time t = 0 is considered during closing the switch. For initial condition the time t = 0– is considered. Therefore, Eq. (1) becomes

$$LT\left[\frac{df^{n}(t)}{dt^{n}}\right] = s^{n} F(s) - s^{n-1} f(0-) - s^{n-2} f'(0-) - s^{n-1} f''(0-) - \ldots\ldots - f^{n-1}(0-) \quad (2)$$

Laplace Transform of an integral $\int f(t)\,dt$

$$LT\left[\int f(t)\,dt\right] = \int_{0}^{\infty}\left[\int f(t)dt\right] e^{-st}\,dt$$

$$= \int_0^\infty u \; dv \qquad \textit{where} \;\; u = \int f(t)\, dt \quad \text{and } dv = e^{-st}\, dt$$

$$= uv - v\int du$$

$$= \left[\left(-\frac{1}{s}\, e^{-st}\right)\int f(t)\, dt\right]_0^\infty - \int_0^\infty \left(-\frac{1}{s}\, e^{-st}\right) f(t)\, dt$$

$$= \frac{1}{s}\int f(t)\; dt \,\Big|_0 + \frac{1}{s}\int_0^\infty f(t)\; e^{-st}\; dt$$

$$\therefore \; LT = \left[f(dt)\right] = \frac{1}{s}\int f(t)\; dt\,\Big|_0 + \frac{1}{s} F(s)$$

For application in circuit, the form of equation (3) will be (3)

$$\therefore \qquad LT \; \left[f(dt)\right] = \frac{1}{s}\int f(t)\, dt\,\Big|_{0-} + \frac{1}{s} F(s) \qquad (4)$$

Laplace Transform of Some Common Time Function

4.7.1 Unit step function

It is the most common driving function in electrical engineering and is denoted by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$\therefore$ The Laplace transformation of unit step function can be written as

$$F(s) = LT\,[u(t)\,] = \int_0^{\infty} u(t)\,e^{-st}\,dt = \int_0^{\infty} e^{-st}\,dt$$

$$= \left[-\frac{1}{s}e^{-st}\right]_0^{\infty} \qquad (5)$$

$$=$$

$$= -\left.\frac{e^{-st}}{s}\right|_{t\to\infty} + \frac{1}{s}e^{-0} = -\frac{1}{s}\left.\left(e^{-\sigma t}\ e^{-j\omega\,t}\right)\right|_{t\to\infty} + \frac{1}{s}$$

$$= -\frac{1}{s}\left.\left[e^{-\sigma t}\left(\cos\omega\,t - j\sin\omega\,t\right)\right]\right|_{t\to\infty} + \frac{1}{s} \qquad (6)$$

The both terms within the bracket are oscillating functions and hence can not be zero for $t\to\infty$. Therefore, for convergence $e^{-\sigma t}\to 0$ at $t=\infty$ when $\sigma > 0$.

Figure 1 shows the region of convergence.

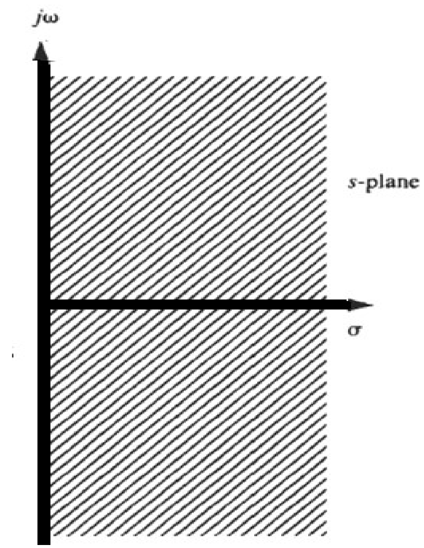

Fig. 1 : ROC for u(t)

$$\therefore\ F(s) = \frac{1}{s} \qquad (7)$$

Similarly, the Laplace transform of any constant C is $\frac{C}{s}$ (8)

Laplace transform of impulse function

The impulse function is obtained by differentiating unit step function u(t).

Thus $\delta\ (t) = \frac{du(t)}{dt}$

$$\therefore\ LT[\delta\ (t)\ ] = LT\left[\frac{du(t)}{dt}\right] = sF(s) - f\,(0)$$

If the system is initially relaxed, f (0) = 0

$$LT[\ \delta\ (t)\ ] = sF\ (s) = s\ LT\ [u\ (t)\ ] = s \times \frac{1}{s} = 1. \tag{9}$$

Ramp function

The ramp function is given by

$$f\ (t) = t$$

If F (s) be the Laplace transform of f (t) , it can be written as

$$F(s) = LT\ [\ f(t)\ ] = \int_0^\infty te^{-st}\ dt$$

Integrating by parts, equation (4.18) can be written as

$$F(s) = -\left.\frac{te^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st}\ dt = \frac{1}{s^2}$$

$$\therefore F\ (s) = \frac{1}{s^2} \tag{10}$$

The region of convergence is $\sigma > 0$.

Parabolic function

The Parabolic function in time domain is given by

$$f(t) = t^2$$

$$\therefore\ F(s) = LT = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty t^2 e^{-st}dt$$

$$= \left[\frac{t^2 e^{-st}}{-s}\right]_0^\infty + \frac{2}{s}\int_0^\infty t\,e^{-st}\,dt$$

$$\therefore\ F(s) = \frac{2}{s^2} \qquad \left[\because \int_0^\infty te^{-st}dt = \frac{1}{s^2}\right] \tag{11}$$

The region of convergence is $\sigma > 0$.

$f(t) = e^{at}u(t)$ with $a > 0$

This function is shown in Fig. 2. Mathematically this function is expressed by

$$f(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ e^{at} & \text{for} \quad t \geq 0 \end{cases} \tag{12}$$

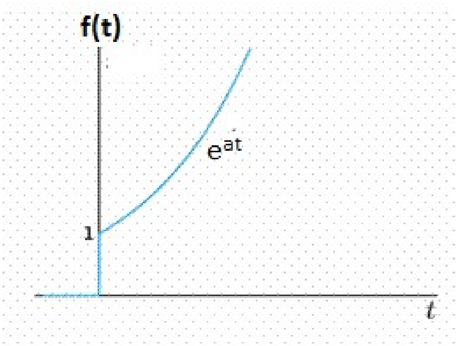

Fig. 2 $f(t) = e^{at}u(t)$ with $a > 0$

$$LT\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = -\left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty$$

At $t \to \infty$, $e^{-(s-a)t} = e^{-(\sigma + j\omega - a)t}\Big|_{t=\infty} = e^{-(\sigma - a)t}e^{-j\omega t}\Big|_{t=\infty}$

Therefore, $e^{-(\sigma - a)t}e^{-j\omega t}\Big|_{t=\infty}$ will be zero if and only if $\sigma > a$.

Hence $$\mathrm{LT}\left[f(t)\right] = \frac{0 - e^{-0}}{-(s-a)} = \frac{1}{s-a} \tag{13}$$
where Re(s) > a.

Figure 3 shows the ROC of $e^{at}u(t)$.

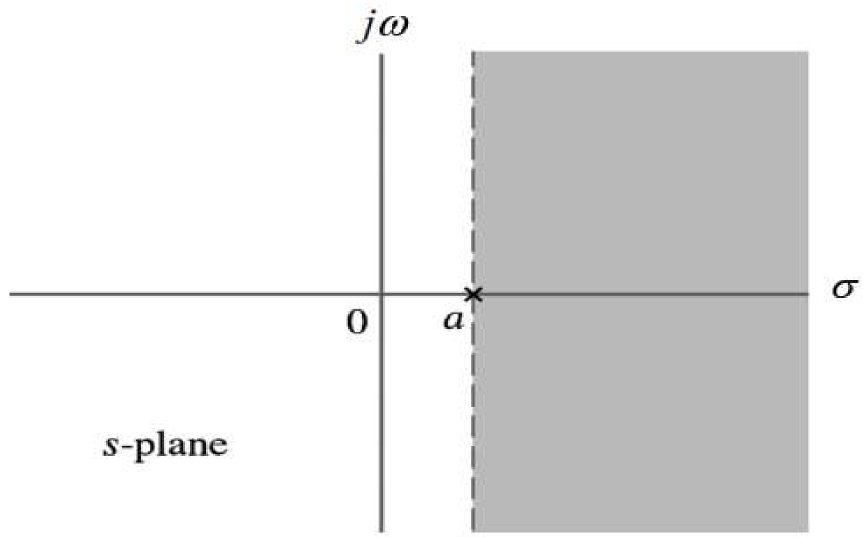

Fig. 3 ROC of $e^{at}u(t)$

(t) = $e^{-at}u(t)$

The function $f(t) = e^{-at}u(t)$ is decreasing in nature. At t = 0, f(t) = 1. Figure 4 represents the function $f(t) = e^{-at}u(t)$.

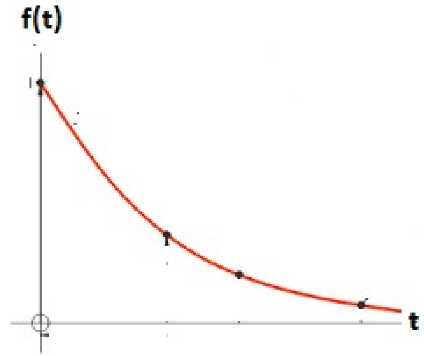

Figure 4

Mathematically, we can represent the function as follows:

$$f(t)=\begin{cases}0 & \text{for} \quad t<0 \\ e^{-at} & \text{for} \quad t>0\end{cases} \tag{14}$$

$$LT\left[f(t)\right]=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty e^{-at}e^{-st}dt=\int_0^\infty e^{-(s+a)t}dt=-\left[\frac{e^{-(s+a)t}}{s+a}\right]_0^\infty$$

At $t\to\infty$, $e^{-(s+a)t} = e^{-(\sigma+j\omega+a)t}\Big|_{t=\infty} = e^{-(\sigma+a)t}e^{-j\omega t}\Big|_{t=\infty}$

Therefore, $e^{-(\sigma+a)t}e^{-j\omega t}\Big|_{t=\infty}$ will be zero if and only if $\sigma > -a$.

Hence $$LT\left[f(t)\right] = \frac{0-e^{-0}}{-(s+a)} = \frac{1}{s+a} \tag{15}$$

where $Re(s) > -a$.

Figure 5 shows the ROC of $e^{-at}u(t)$.

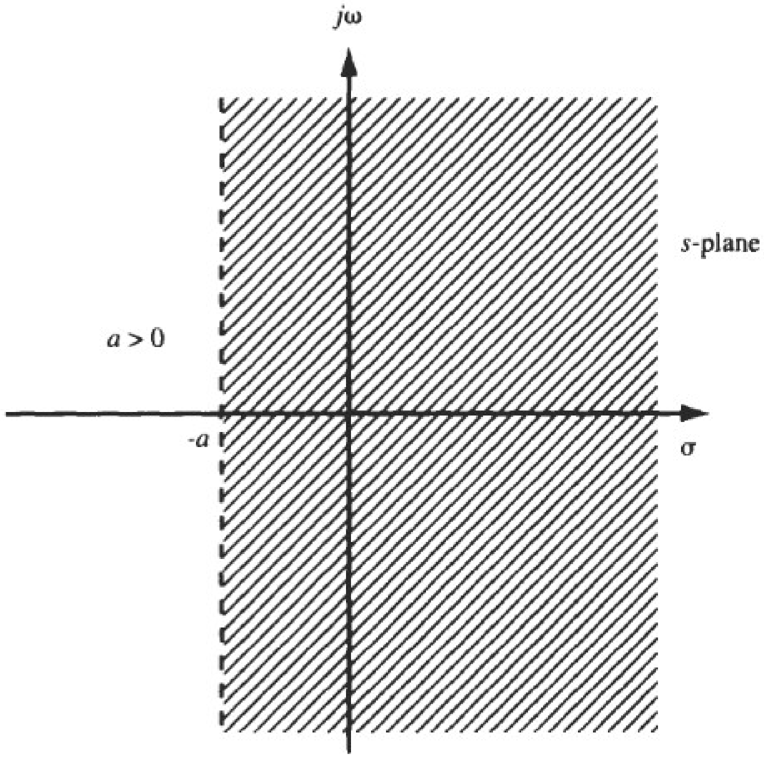

Fig. 5 ROC of $e^{-at}u(t)$

Sinusoidal function

This function is very common for circuits containing ac source. The sinusoidal function in time domain is given by

$$f(t)= \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\therefore\ F(s) = LT[\, f(t)\,] = LT[\sin \omega t\,]$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{j\omega t}\, e^{-st}\, dt - \int_0^\infty e^{-j\omega t}\, e^{-st}\, dt\right]$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}\, dt - \int_0^\infty e^{-(s+j\omega)t}\, dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\omega}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$$

$$\therefore\ LT[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

Cosine function

In time domain it is represented by

f (t) = cosωt

$\therefore$ F (s) = LT [f (t)] = LT [cosωt]

$$= \int_0^\infty \cos\omega t\, e^{-st}\, dt$$

$$= \int_0^\infty \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) e^{-st}\, dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega)t}\, dt + \int_0^\infty e^{-(s+j\omega)t}\, dt \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega} \right]$$

$$= \frac{1}{2}.\left(\frac{2s}{s^2+\omega^2}\right) = \frac{s}{s^2+\omega^2}$$

$$\therefore \text{ LT } [\cos\omega t\ ] = \frac{s}{s^2+\omega^2}$$

Hyperbolic Sine and Cosine functions

The hyperbolic sine function is represented by

$$f(t) = \sinh bt = \frac{1}{2}\left(e^{bt} - e^{-bt}\right)$$

$$\therefore \text{ LT}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \frac{1}{2}\left(\int_0^\infty e^{bt} - e^{-bt}\right)e^{-st}\,dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-b)t}dt - \int_0^\infty e^{-(s+b)t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-b} - \frac{1}{s+b}\right] = \frac{1}{2}\left[\frac{2b}{s^2 - b^2}\right]$$

$$\therefore\ F(s) = \frac{b}{s^2 - b^2}$$

Similarly, $LT\left[\cosh bt\right] = LT\left[\frac{1}{2}\left(e^{bt} + e^{-bt}\right)\right] = \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right)$

$$\therefore\ LT[\cosh bt] = \frac{s}{s^2 - b^2}$$

Damped Sine and Cosine Functions

(a) The damped sine function is represented

$f(t) = e^{-at} \sin bt$

$$= e^{-at}\left[\frac{e^{jbt} - e^{-jbt}}{2j}\right]$$

$$= \frac{1}{2j}\left[e^{-(a-jb)} - e^{-(a+jb)t}\right]$$

$$\therefore \text{ LT } [e^{-at}\sin bt] = \frac{1}{2j}\left[\frac{1}{s+(a-jb)} - \frac{1}{s+(a+jb)}\right] = \frac{1}{2j}\left[\frac{2jb}{(s+a)^2+b^2}\right]$$

$$= \frac{b}{(s+a)^2+b^2}$$

$$\therefore \text{ LT}[e^{-at}\cos bt] = \frac{s+a}{(s+a)^2+b^2}$$

Damped Hyperbolic Sine and Cosine Function

(a) The damped hyperbolic since function is represented as

$$f(t) = e^{-at}\sinh bt$$

$$= e^{-at}\ \frac{1}{2}\left[e^{bt} - e^{-bt}\right]$$

$$= \frac{1}{2}\left[e^{-(a-b)t} - e^{-(a+b)t}\right]$$

$$\therefore \text{LT}\,[e^{-at}\,\text{sinhbt}] = \int_0^\infty e^{-at}\,\text{sinbt}\,e^{-st}\,dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-[(s+a)-b]t}\,dt - \int_0^\infty e^{-[(s+a)+b]t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+a-b} - \frac{1}{s+a+b}\right]$$

$$\therefore \text{LT}\,[e^{-at}\,\text{sinhbt}] = \frac{b}{(s+a)^2 - b^2}$$

Similarly,

$$\text{LT}\,[e^{-at}\,\text{coshbt}] = \frac{s+a}{(s+a)^2 - b^2}$$

Laplace transform of t^n

Let $f(t) = t^n$

$$\therefore F(s) = LT[f(t)] = \int_0^\infty t^n e^{-st} dt$$

Let $x = st$,

$\therefore dt = \frac{1}{s} dx$ and $t = \frac{x}{s}$

$$\therefore F(s) = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x} \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$$

Since $\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$

$$\therefore LT[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$