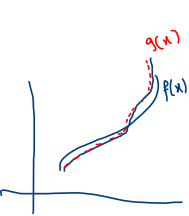# Lecture 28: Numerical Analysis (UMA011)

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Approximate Root

Approximate Solution of linear system of equations

Problem ✓

{

Apply Numerical Methods

⇓

Approximate solution

Problem (function)
(complicated)
‖ approximated by
simpler form }
(Approximated)

$f(x)$ at $x = 1.3$
⇓ $f(1.3)$?
find app. form $p(x)$ then $p(1.3)$

Polynomial approximation:

Approximation of function:

With whom
to app ?

$f(x)$

→ Weirestrass App <u>th m</u> :- Any continuous function $f(x)$

can be approximated by a polynomial $f(x)$

on $[a, b]$

i.e. for any $\epsilon > 0$, $|f(x) - P(x)| < \epsilon$

let
exact — $f(x)$
App. — $P(x)$ ✓

$$\frac{d}{dx} f(x) = Q(x) \approx \frac{df}{dx}$$

$$\frac{d}{dx} f(x)$$

$$f(p) \approx P(p)$$

\#   How to approximation ?

→ Taylor's polynomial.

for f(x), the taylor's polynomial about any pt x_0 is

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i \checkmark$$

nth degree Taylor's polynomial.

Polynomial approximation:

Taylor's polynomial:

For example: Let $f(x) = e^x$ ✓

$$P_0(x) = \sum_{i=0}^{0} \frac{f^{(i)}(0)}{i!}(x-0)^i = \frac{f^{(0)}(0)}{0!}(x-0)^0 = 1$$

$$P_1(x) = \sum_{i=0}^{1} \frac{f^{(i)}(0)}{i!}(x-0)^i = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}\cdot x^1$$

$$= 1 + \frac{1\cdot x}{1} = 1+x \checkmark$$
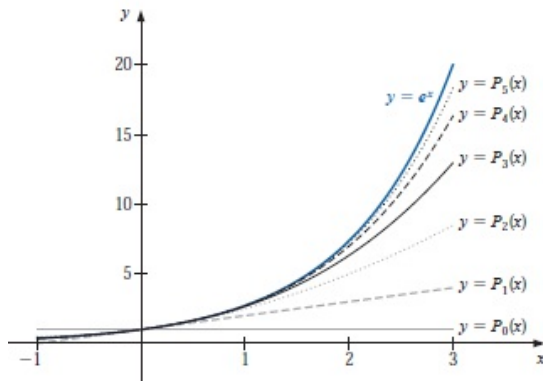
$$P_2(x) = \sum_{i=0}^{2} \frac{f^{(i)}(0)}{i!}(x-0)^i = 1 + x + \frac{x^2}{2!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{6}, \quad - \ - \ \underline{\quad\quad}$$

Polynomial approximation:

Approximation of function:

From the following graph of the polynomial approximation we can see that for the higher-degree polynomials, the error becomes less and it approaches to the function $f(x) = e^x$.

$$\lim_{n \to \infty} P_n(x) = f(x)$$

Polynomial approximation:

Limitations:

i By Taylor's polynomials approximation, we can get better approximation <u>only for higher order differentiable functions</u>.

ii This approximation <u>does not</u> give <u>better for all functions</u>.

Polynomial approximation:

Limitations:

e.g. $f(x) = \dfrac{1}{x}$

$f'(x) = \dfrac{-1}{x^2}$

$f''(x) = \dfrac{(-1)^2 \cdot 2}{x^3}$

$f'''(x) = \dfrac{(-1)^3 \, 3!}{x^4}$

$f^{iv}(x) = \dfrac{(-1)^4 \, 4!}{x^5}$

— - - - -

$f^{(i)}(x) = \dfrac{(-1)^i \, i!}{x^{i+1}}$

Taylor's polynomial about $x = 1$ is

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(1)}{i!} (x-1)^i = \sum_{i=0}^{n} \frac{(-1)^i \, i!}{i!} (x-1)^i$$

$$= \sum_{i=0}^{n} (-1)^i (x-1)^i$$

Now, to app. $f(3) = \dfrac{1}{3} = 0.333 \rightarrow$ exact value.

$$P_n(x) = \sum_{i=0}^{n} (-1)^i (x-1)^i$$

for $n = 2$

$$P_2(x) = \sum_{i=0}^{2} (-1)^i (x-1)^i$$

$$= 1 + (-1)^1 (x-1)^1 + (-1)^2 (x-1)^2$$

$$= 1 - x + 1 + (x-1)^2$$

App. value. $P_2(3) = 1 - 3 + 1 + 4 = 3$

Table.

| n	0	1	2	3	4	5
$P_n(3)$	1	-1	3	-5	11	-21

→ not best app.

Polynomial interpolation:

Interpolation:

Given $(n + 1)$ points data points (x_i, y_i), $i = 0, 1, 2, \cdots, n$, at which function $f(x)$ passes, then approximate in form of a polynomial function $P(x)$ of degree atmost n such that $f(x_i) = P(x_i), i = 0, 1, 2, \cdots, n$.

Using this polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.
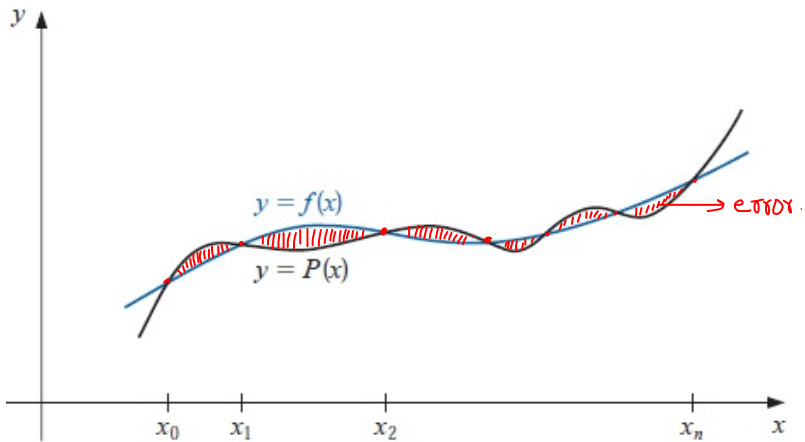
Here $y_i = f(x_i)$

Given pts are (x_0, y_0), (x_1, y_1)
$- - - - - \cdot (x_n, y_n)$
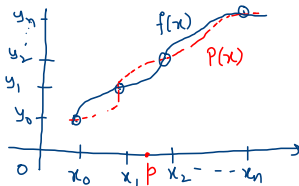
$P(p) \approx f(p)$

↳ interpolate at p bet^n x_0 & x_n

$y = f(x)$

$y = P(x)$

error

Polynomial interpolation:

> ### Result:
>
> **(Existence and Uniqueness)** Given a real-valued function $f(x)$ and $n+1$ distinct points $x_0, x_1, \cdots, x_n$, there exists a unique polynomial $P_n(x)$ of degree $\leq n$ which interpolates the unknown $f(x)$ at points $x_0, x_1, \cdots, x_n$.

The approximated polynomial using given pts at which function passes is unique

Polynomial interpolation:

Methods to interpolate:

i Lagrange Interpolation
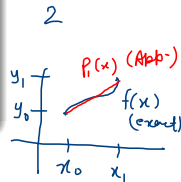
ii Newton's interpolation

Polynomial interpolation:

Lagrange Interpolating polynomials:

Linear Interpolation: Approximate a function that passes through two points $(x_0, y_0), (x_1, y_1)$ i.e. $y_0 = f(x_0), y_1 = f(x_1)$ by polynomial of atmost one degree i.e. linear polynomial $P_1(x)$. Interpolating polynomial should satisfy
$f(x_0) = P_1(x_0), f(x_1) = P_1(x_1)$.

2

$P_1(x)$ (App⁻)

$f(x)$ (exact)

Interpolating polynomial should satisfy
$$f(x_0) = \checkmark P_1(x_0) \quad , \quad f(x_1) = P_1(x_1) \checkmark$$

Define the function $L_0(x) = \dfrac{x - x_1}{x_0 - x_1}$, $L_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

The linear interpolating polynomial is given by $P_1(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$

$$P_1(x) = \quad L_0(x)\, f(x_0) \; + \quad L_1(x)\, f(x_1)$$

$$L_i(x) = \frac{x - x_j}{x_i - x_j} \qquad i, j = 0, 1$$
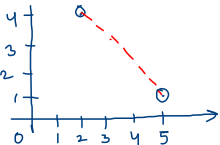$$j \neq i \; .$$

$$P_1(x) \approx f(x)$$

Lagrange Interpolating polynomials:

Example:

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Solution:

Linear interpolating polynomial which passes through $\underset{(x_0, y_0)}{(2,4)}$, $\underset{(x_1, y_1)}{(5,1)}$ is

$$P_1(x) = l_0(x) \, f(x_0) + l_1(x) \, f(x_1), \text{ where } l_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$= \frac{(x-5)}{-3} \, 4 + \frac{x-2}{3} \, 1 \qquad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$y_i = f(x_i)$$

$$= -\frac{4}{3}(x-5) + \frac{1}{3}(x-2) = \frac{1}{3}(-3x + 18)$$

$$= 6 - x$$

Lagrange Interpolating polynomials:

> ### Example:
>
> **1** Find the unique polynomial $P(x)$ of degree 1 such that
>
> $$P(1) = 1, \ P(3) = 27,$$
>
> using Lagrange interpolation. Evaluate $P(1.05)$.
>
> **2** For the function $f(x) = \sin(\pi x)$, let $x_0 = 1.25$, and $x_1 = 1.6$ Construct linear Lagrange interpolation polynomials of degree at most one to approximate $f(1.4)$, and find the absolute error.