Course: UMA 035 (Optimization Techniques)

Instructor: Dr. Amit Kumar,

Associate Professor,

School of Mathematics,

TIET, Patiala

Email: amitkumar@thapar.edu

Mob: 9888500451

Decision variables

The variables used to formulate an LPP (if LPP is not given)

or

the variables used in the given LPP (if LPP is given)

are called decision variables.

Example:

Maximize $(3x_1 - 2x_2 + 5x_3 - 7x_4)$

Subject to

$x_1 - 4x_2 + 2x_3 - 3x_4 <= -5,$

$-2x_1 - x_2 + 5x_3 - x_4 >= 7,$

$2x_1 - 3x_2 - x_3 + 6x_4 = 10,$

$x_1 >= 2,\ x_2 >= -8,\ x_3 <= 6.$

Decision variables: x_1, x_2, x_3, x_4

Classification of decision variables

Restricted decision variables and unrestricted decision variables

Restricted decision variables

Restricted decision variables are those decision variables for which the minimum value is a finite real number

or

the maximum value is a finite real number

or

both the minimum value and the maximum value are finite real numbers

Unrestricted decision variables

Unrestricted decision variables are those decision variables for which the minimum value is – infinity and the maximum value is + infinity

Example:

Maximize $(3x_1 - 2x_2 + 5x_3 - 7x_4)$

Subject to

$x_1 - 4x_2 + 2x_3 - 3x_4 <= -5,$

$-2x_1 - x_2 + 5x_3 - x_4 >= 7,$

$2x_1 - 3x_2 - x_3 + 6x_4 = 10,$

$x_1 >= 2,\ x_2 >= -8,\ x_3 <= 6.$

In the considered LPP,

- **The minimum value of the first variable x_1 is 2 i.e., a finite real number. So, x_1 is a restricted decision variable.**
- **The minimum value of the second variable x_2 is – 8 i.e., a finite real number. So, x_2 is a restricted decision variable.**
- **The maximum value of the third variable x_3 is 6 i.e., a finite real number. So, x_3 is a restricted decision variable.**
- **The minimum value and the maximum values of the fourth variable x_4 are – infinity and + infinity respectively. So, x_4 is an unrestricted decision variable.**

Standard form of a LPP

- The minimum value of each variable should be zero.
- RHS (Constant term) of all the constraints should be either zero or positive.
- Sign of all the constraints should be = sign.

Example:

Maximize $(3x_1 - 2x_2 + 5x_3 - 7x_4)$

Subject to

$x_1 - 4x_2 + 2x_3 - 3x_4 <= -5,$

$-2x_1 - x_2 + 5x_3 - x_4 >= 7,$

$2x_1 - 3x_2 - x_3 + 6x_4 = 10,$

$x_1 >= 2,\ x_2 >= -8,\ x_3 <= 6.$

LPP is not in standard form

How to change in Standard form?

Step 1: All variables >=0

First variable

$x_1>=2$

Minimum value of x_1 is 2. But according to condition the minimum value should be 0.

Minimum value can be transformed into 0 as follows:

$x_1>=2$

may be written as

$x_1 – 2>=0$

Assume

$x_1 – 2=y_1$

i.e.,

$x_1=y_1+2$

We will replace x_1 with y_1+2 in the given LPP.

Second variable

$x_2>= –8$

The minimum value of x_2 is –8.

But according to condition, it should be 0.

The minimum value can be transformed into 0 as follows:

$x_2>= –8$

may be written as

$x_2+8>=0$

Assume

$x_2+8=y_2$

i.e., $x_2=y_2-8$

We will replace x_2 with y_2-8 in the given LPP.

Third variable

$x_3<= 6$

The minimum value of x_3 is –infinity.

But according to condition, it should be 0.

The minimum value can be transformed into 0 as follows:

$x_3<= 6$

may be written as

$0<= 6 - x_3$

Assume

$6-x_3=y_3$

i.e., $x_3=6 - y_3$

We will replace x_3 with $6 - y_3$ in the given LPP.

Fourth variable

The minimum value of x_4 is –infinity and the maximum value is +infinity

x_4 will be either zero or negative or positive.

Since, zero, every negative number and every positive number may be written as subtraction of two non-negative numbers.

So, x_4 may be written as

$x_4 = x_5 - x_6$

where,

$x_5 >= 0$ **and** $x_6 >= 0$

Finally,

- $x_1 = 2 + y_1$
- $x_2 = y_2 - 8$
- $x_3 = 6 - y_3$
- $x_4 = x_5 - x_6$

The LPP

Maximize $(3x_1 - 2x_2 + 5x_3 - 7x_4)$

Subject to

$x_1 - 4x_2 + 2x_3 - 3x_4 <= -5,$

$-2x_1 - x_2 + 5x_3 - x_4 >= 7,$

$2x_1 - 3x_2 - x_3 + 6x_4 = 10,$

$x_1 >= 2,\ x_2 >= -8,\ x_3 <= 6.$

Will be transformed into

Maximize $(3(2+y_1) - 2(y_2 - 8) + 5(6 - y_3) - 7(x_5 - x_6)$

Subject to

$(2+y_1) - 4(y_2 - 8) + 2(6 - y_3) - 3(x_5 - x_6) <= -5,$

$-2(2+y_1) - (y_2 - 8) + 5(6 - y_3) - (x_5 - x_6) >= 7,$

$2(2+y_1) - 3(y_2 - 8) - (6 - y_3) + 6(x_5 - x_6) = 10,$

$(2+y_1) >= 2,\ (y_2 - 8) >= -8,\ (6 - y_3) <= 6,\ x_5 >= 0,\ x_6 >= 0$

Maximize $(6+3y_1 - 2y_2+16 + 30 - 5y_3 - 7x_5 + 7x_6)$

Subject to

$2+y_1 - 4y_2+32 + 12 -2 y_3 - 3x_5 + 3x_6 <= -5,$

$-4 - 2y_1 - y_2+8 + 30 - 5y_3 - x_5 +x_6 >= 7,$

$4+2y_1 - 3y_2+24 - 6 + y_3 +6x_5 - 6x_6 = 10,$

$y_1>=2-2, y_2>= -8+8, 6 -6<= y_3, x_5>=0, x_6>=0$

Maximize $(3y_1 - 2y_2 -5y_3 - 7x_5 + 7x_6+52)$

Subject to

$y_1 - 4y_2 -2 y_3 - 3x_5 + 3x_6 <= -5-46,$

$-2y_1 - y_2 - 5y_3 - x_5 + x_6 >= 7-34,$

$2y_1 - 3y_2+ y_3 +6x_5 - 6x_6 = 10-22,$

$y_1>=0, y_2>= 0, 0<= y_3, x_5>=0, x_6>=0$

Maximize $(3y_1 - 2y_2 - 5y_3 - 7x_5 + 7x_6+52)$

Subject to

$y_1 - 4y_2 -2 y_3 - 3x_5 + 3x_6 <= -51,$

$-2y_1 - y_2 - 5y_3 - x_5 + x_6 >= -27,$

$2y_1 - 3y_2+ y_3 +6x_5 - 6x_6 = -12,$

$y_1>=0, y_2>= 0, 0<= y_3, x_5>=0, x_6>=0$

Step 2

RHS should be positive

Maximize $(3y_1 – 2y_2 – 5y_3 – 7x_5 + 7x_6+52)$

Subject to

$y_1 – 4y_2 –2 y_3 – 3x_5 + 3x_6 <= –51,$

$–2y_1 – y_2 – 5y_3 – x_5 + x_6 >= –27,$

$2y_1 – 3y_2+ y_3 +6x_5 – 6x_6 = –12,$

$y_1>=0, y_2>= 0, 0<= y_3, x_5>=0, x_6>=0$

Maximize $(3y_1 – 2y_2- 5y_3 – 7x_5 + 7x_6+52)$

Subject to

$–y_1 + 4y_2 +2 y_3 + 3x_5– 3x_6 >= 51,$

$2y_1 + y_2 + 5y_3 + x_5 – x_6 <= 27,$

$–2y_1 + 3y_2- y_3 +6x_5 + 6x_6 = 12,$

$y_1>=0, y_2>= 0, 0<= y_3, x_5>=0, x_6>=0$

Step 3

Sign of each constraints should be =

Maximize $(3y_1 – 2y_2 - 5y_3 – 7x_5 + 7x_6 + 52)$

Subject to

$–y_1 + 4y_2 + 2y_3 + 3x_5 – 3x_6 >= 51$,

$2y_1 + y_2 + 5y_3 + x_5 – x_6 <= 27$,

$–2y_1 + 3y_2 - y_3 + 6x_5 + 6x_6 = 12$,

$y_1>=0, y_2>= 0, 0<= y_3, x_5>=0, x_6>=0$

Maximize $(3y_1 – 2y_2 – 5y_3 – 7x_5 + 7x_6 + 52)$

Subject to

$–y_1 + 4y_2 + 2y_3 + 3x_5 + 3x_6 – S_1 = 51$,

$2y_1 + y_2 + 5y_3 + x_5 + x_6 + S_2 = 27$,

$–2y_1 + 3y_2 – y_3 + 6x_5 + 6x_6 = 12$,

$y_1>=0, y_2>= 0, 0<= y_3, x_5>=0, x_6>=0, S_1>=0, S_2>=0$

Remark: **The non-negative variable which is added to make <= sign into = sign is called Slack variable (in the considered LPP S_2 is a Slack variable)**

Remark: **The non-negative variable which is subtracted to make >= sign into = sign is called Surplus variable (in the considered LPP S_1 is a Surplus variable)**

Conclusion:

Given LPP

Maximize $(3x_1 - 2x_2 + 5x_3 - 7x_4)$

Subject to

$x_1 - 4x_2 + 2x_3 - 3x_4 <= -5,$

$-2x_1 - x_2 + 5x_3 - x_4 >= 7,$

$2x_1 - 3x_2 - x_3 + 6x_4 = 10,$

$x_1 >= 2,\ x_2 >= -8,\ x_3 <= 6.$

Standard form

Maximize $(3y_1 - 2y_2 - 5y_3 - 7x_5 + 7x_6 + 52)$

Subject to

$-y_1 + 4y_2 + 2y_3 + 3x_5 + 3x_6 - S_1 = 51,$

$2y_1 + y_2 + 5y_3 + x_5 + x_6 + S_2 = 27,$

$-2y_1 + 3y_2 - y_3 + 6x_5 + 6x_6 = 12,$

$y_1 >= 0,\ y_2 >= 0,\ 0 <= y_3,\ x_5 >= 0,\ x_6 >= 0,\ S_1 >= 0,\ S_2 >= 0$

where,

- $x_1 = 2 + y_1$
- $x_2 = y_2 - 8$
- $x_3 = 6 - y_3$
- $x_4 = x_5 - x_6$

Feasible solution of a LPP

A feasible solution of a LPP is a set of non-negative variables which satisfies all the constraints of the considered LPP.

To find feasible solution of a LPP, it is necessary that the minimum value of each decision variable of the considered LPP should be non-negative (either zero or positive).

RHS of constraints may be positive or negative or zero

Sign of constraints may be >= or = or <=

Example:

Maximize $(3x_1 - 2x_2 + 5x_3 - 7x_4)$

Subject to

$x_1 - 4x_2 + 2x_3 - 3x_4 <= -5,$

$-2x_1 - x_2 + 5x_3 - x_4 >= 7,$

$2x_1 - 3x_2 - x_3 + 6x_4 = 10,$

$x_1 >= 2,\ x_2 >= -8,\ x_3 <= 6.$

In the considered LPP,

- **The minimum value of the first variable x_1 is 2 i.e., a positive number. So, there is no need to change it.**

- **The minimum value of the second variable x_2 is – 8 i.e., a negative number. So, there is need to change it.**

- **The minimum value of the third variable x_3 is – infinity i.e., a negative number. So, there is need to change it.**

- **The minimum value of the fourth variable x_4 is – infinity i.e., a negative number. So, there is need to change it.**

How to change?

First variable

No need to cahnge

Second variable

$x_2 >= -8$

may be written as

$x_2+8>=0$

Assume $x_2+8=y_2$

i.e.,

$x_2=y_2-8$

Third variable

$x_3<= 6$ **may be written as**

$0<= 6 - x_3$

Assume $6-x_3=y_3$,
i.e.,
$x_3=6 - y_3$

Fourth variable

x_4 will be either zero or negative or positive.

Since, Zero, every negative number and every positive number may be written as subtraction of two non-negative numbers.

So, x_4 may be written as

$x_4 = x_5 - x_6$

where,

$x_5 >= 0$ **and** $x_6 >= 0$

Finally,

- **No change in x_1**
- $x_2 = y_2 - 8$
- $x_3 = 6 - y_3$
- $x_4 = x_5 - x_6$

The LPP

Maximize $(3x_1 – 2x_2 + 5x_3 – 7x_4)$

Subject to

$x_1 – 4x_2 + 2x_3 - 3x_4 <= –5,$

$–2x_1 – x_2 + 5x_3 – x_4 >= 7,$

$2x_1 – 3x_2 – x_3 +6x_4 = 10,$

$x_1>=2, x_2>= –8, x_3<= 6.$

Will be transformed into

Maximize $(3x_1 – 2(y_2–8) + 5(6 – y_3) – 7(x_5 – x_6)$

Subject to

$x_1 –4(y_2–8) + 2(6 – y_3) –3(x_5 – x_6) <= -5,$

$–2x_1 – (y_2–8) + 5(6 – y_3) – (x_5 – x_6) >= 7,$

$2x_1 – 3(y_2–8) – (6 –y_3) +6(x_5 – x_6) = 10,$

$x_1>=2, (y_2-8)>= –8, (6 – y_3)<= 6, x_5>=0, x_6>=0$

Maximize $(3x_1 – 2y_2+16 + 30 - 5y_3 – 7x_5 + 7x_6)$

Subject to

$x_1 – 4y_2+32 + 12 –2 y_3 – 3x_5 + 3x_6 <= –5,$

$–2x_1 – y_2+8 + 30 – 5y_3 – x_5 + x_6 >= 7,$

$2x_1 –3y_2+24 – 6 + y_3 +6x_5 – 6x_6 = 10,$

$x_1>=2, y_2>= –8+8, 6 –6<= y_3, x_5>=0, x_6>=0$

Maximize $(3x_1 – 2y_2 – 5y_3 – 7x_5 + 7x_6+46)$

Subject to

$x_1 – 4y_2 –2 y_3 – 3x_5 – 3x_6 <= –5–44$,

$–2x_1 – y_2 – 5y_3 –x_5 – x_6 >= 7–38$,

$2x_1 – 3y_2 + y_3 +6x_5 – 6x_6 = 10–18$,

$x_1>=2$, $y_2>= –8+8$, $6 –6<= y_3$, $x_5>=0$, $x_6>=0$

Maximize $(3x_1 – 2y_2 - 5y_3 – 7x_5 + 7x_6+46)$

Subject to

$x_1 – 4y_2 –2 y_3 – 3x_5 – 3x_6 <= –49$,

$–2x_1 – y_2 – 5y_3 – x_5 – x_6 >= –31$,

$2x_1 - 3y_2 + y_3 +6x_5 – 6x_6 = –8$,

$x_1>=2$, $y_2>= 0$, $0<= y_3$, $x_5>=0$, $x_6>=0$

In the problem variables are x_1, y_2, y_3, x_5 and x_6.

Since, minimum value of each variable is either 0 or a positive real number. So, we may now discuss about feasible solution.

Some examples

Maximize $(x_1+2x_2-x_3)$

Subject to

$x_1+x_2+x_3<=5$

$x_1>=2$, $x_2>=0$, $x_3>=0$

- **$x_1=2$, $x_2=3$, $x_3=0$ is a feasible solution.**
- **$x_1=1$, $x_2=4$, $x_3=0$ is not a feasible solution as $x_1>=2$ is not satisfying.**

- **$x_1=5$, $x_2=0$, $x_3=0$ is a feasible solution.**
- **$x_1=1$, $x_2=0$, $x_3=4$ is not a feasible solution as $x_1>=2$ is not satisfying.**

- **$x_1=5$, $x_2=-1$, $x_3=0$ is not a feasible solution as $x_2>=0$ is not satisfying.**

Maximize $(x_1+2x_2-x_3)$

Subject to

$x_1+x_2+x_3<=5$

$x_1>=-2$, $x_2>=0$, $x_3>=0$

Since minimum value of x_1 is negative. So, to discuss the feasible solution of the given LPP, there is a need to transform x_1 into new variable.

$x_1>=-2$

may be written as

$x_1+2>=0$

Assuming $x_1+2=y_1$ i.e., $x_1=y_1-2$, the LPP is

Maximize $(y_1-2+2x_2-x_3)$

Subject to

$y_1-2+x_2+x_3<=5$

$y_1-2>=-2$, $x_2>=0$, $x_3>=0$

Maximize $(y1+2x_2-x_3-2)$

Subject to

$y_1+x_2+x_3<=7$

$y_1>=0$, $x_2>=0$, $x_3>=0$

Now we may discuss feasible solutions

- **$y_1=7$, $x_2=0$ and $x_3=0$ is a feasible solution.**
- **$y_1=0$, $x_2=7$ and $x_3=0$ is a feasible solution.**
- **$y_1=0$, $x_2=0$ and $x_3=7$ is a feasible solution.**
- **$y_1=9$, $x_2= -2$ and $x_3=0$ is not a feasible solution.**

Conclusion:

Standard Form	Feasible solution
Minimum value of each variable is 0	Minimum value of each variable is 0 or positive
RHS should be 0 or positive	-
Sign of all constraints should be =	-