Roll Number: ____________

Thapar Institute of Engineering and Technology, Patiala

School of Mathematics

Mid Semester Examination

B.E. (Second Year): Semester-II (2018 – 19) (COE/ECE/ENC)	Course Code: UMA031
Date: March 13, 2019	Course Name: Optimization Techniques
	Day/Time: Wednesday/8:00-10:00 AM
Time: 2 Hours, M. Marks: 30	Name of Faculty: Amit Kumar, Meenakshi Rana, Vikas Sharma, Sanjeev Kumar, Navdeep Kailey, Mamta Gulati, Isha Dhiman, Jolly Puri.

Note: (1) **Attempt all the questions.**
(2) **Calculator without graphing mode and alphanumeric memory is permitted.**

1. (a) A firm buys raw material for oil of two types I and II, to sell them as finished product after machining and refining. The purchasing cost of one unit of raw material is Rs. 3 and Rs. 4 for type I and II and selling costs are Rs. 8 and Rs. 10 respectively. The per hour capacity of machines used for machining and refining fro two products is given below:

Capacity/hour ↓	Type I	Type II
Machining	30	50
Refining	45	30

The running costs for machining and refining are Rs. 30 and Rs. 22.5 per hour respectively. Formulate the linear programming problem to find out the product mix to maximise the profit.

(b) For the above formulated LPP, find all the basic feasible solutions.

[4 + 2 marks]

2. (a) Find all the optimal solutions (also alternative optima) of the following LPP using simplex method:

$$Max \quad z = x_1 + 2x_2 + 3x_3, \quad s.t. \; \boxed{x_1 + 2x_2 + 3x_3 \leq 10}, \; x_1 + x_2 \leq 5, \; x_1, x_2, x_3 \geq 0.$$

(b) State and prove weak duality theorem

[4 + 4 marks]

3. (a) Solve the following linear programming graphically

$$Max \quad z = x_1 + 2x_2, \quad s.t. \; x_1 + x_2 \geq 2, \; -x_1 + x_2 \leq 3, \; x_1 \leq 3, \; x_1, x_2 \geq 0.$$

(b) Write the dual of the above LPP and use complementary slackness theorem to find an optimal solution of the dual.

[3 + 5 marks]

4. (a) Use dual simplex method (without using artificial variables) to solve the following LPP

$$Min \quad z = 2x_1 + 3x_2, \quad s.t. \; x_1 + x_2 = 2, \; 2x_1 + x_2 \geq 3, \; x_1, x_2 \geq 0.$$

(b) Show graphically that the following sets are convex:

i. $\{(x, y) \mid y \leq \sin(x), y \geq 1/2, 0 \leq x \leq \pi\}$.

ii. $\{(x, y) \mid x + y \leq 1 \text{ or } x \geq 2\}$.

(c) Show that the set $S = \{X \in \mathbb{R}^n \mid AX = b, X \geq 0, b \in \mathbb{R}^m, A \text{ is an } m \times n \text{ matrix}\}$ is convex set.

[4 + 2 + 2 marks]

———— End of question paper ————

Q-1 (a) x_1 = Units of I
x_2 = Units of II] (1/2)

Objective fn. :-

(1) $$\text{Max } z = \left[8 - \left(3 + \frac{30}{30} + \frac{22.5}{45}\right)\right]x_1 + \left[10 - \left(4 + \frac{30}{50} + \frac{22.5}{30}\right)\right]x_2$$

$$= 3.5x_1 + 4.65x_2$$

$7/2 \qquad 93/20$

Constraints :-

(1) $\leftarrow$ $\frac{x_1}{30} + \frac{x_2}{50} \leq 1 \Rightarrow 50x_1 + 30x_2 \leq 1500$

(1) $\leftarrow$ $\frac{x_1}{45} + \frac{x_2}{30} \leq 1 \Rightarrow 30x_1 + 45x_2 \leq 1350.$

i.e, $2x_1 + 3x_2 \leq 90$

Non-negativity :- $x_1, x_2 \geq 0$.] $\rightarrow$ (1/2)

(b) Constraints are :-

$5x_1 + 3x_2 \leq 150$. , $6x_1 + 9x_2 \leq 270$.

$5x_1 + 3x_2 + s_1 = 150$.

$6x_1 + 9x_2 + s_2 = 270$.

N.B.V	B.V	BFS (Yes / No)	
$x_1 = x_2 = 0.$	$s_1 = 150$ $s_2 = \cancel{270}$ 90.	Yes.	$\rightarrow$ 1/2
$x_1 = s_1 = 0.$	$x_2 = 50.$ $s_2 = \cancel{-180}$ -60	NO	
$x_2 = s_1 = 0$	$x_1 = 30$ $s_2 = \cancel{90}$ 30	Yes.	$\rightarrow$ 1/2
$s_1 = s_2 = 0.$	$x_1 = 20$ $x_2 = 50/3$	Yes.	$\rightarrow$ 1/2
$x_1 = s_2 = 0$	$x_2 = 30$ $s_1 = 60$	Yes.	$\rightarrow$ 1/2
$x_2 = s_2 = 0$	$x_1 = 45$ $s_1 = -75$	No.	

(a) Max $z = x_1 + 2x_2 + 3x_3$

S.T.

$x_1 + 2x_2 + 3x_3 + s_1 = 10$

$x_1 + x_2 + s_2 = 5$

$x_1, x_2, x_3, s_1, s_2 \geq 0$.

C_B	x_B	x_1	x_2	x_3 ↓	s_1	s_2	Soln
← 0	s_1	1	2	3 → Pivot.	1	0	10
0	s_2	1	1	0	0	1	5
	$z_j - c_j$	−1	−2	−3	0	0	$z = 0$

1

C_B	x_B	x_1 ↓	x_2	x_3	s_1	s_2	Soln
3	x_3	1/3	2/3	1	1/3	0	10/3
← 0	s_2	1 → Pivot,	1	0	0	1	5
	$z_j - c_j$	0	0	0	1	0	10.

1

All $z_j - c_j \geq 0$ ⇒ optimal soln.

Soln. is $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{10}{3}$, $z = 10$.

→ $z_1 - c_1 = 0$ & $z_2 - c_2 = 0$, will give alternate optimas.

C_B	x_B	x_1 ↓	x_2	x_3	s_1	s_2	Soln
3	x_3	$-1/3$	0	1	$1/3$	$-2/3$	0
← 2	x_2	1	1	0	0	1	5.
	$z_j - c_j$	0	0	0	1	0	$z = 10$.

$\frac{1}{2}$

Give 1 mark if only one soln is found

$\boxed{\text{Soln is } x_1 = 0, \; x_2 = 5, \; x_3 = 0, \; z = 10.}$

C_B	x_B	x_1	x_2	x_3	s_1	s_2	Soln
3	x_3	0	$1/3$	1	$1/3$	$-1/3$	$5/3$
1	x_1	1	1	0	0	1	5.
	$z_j - c_j$	0	0	0	1	0	$z = 10$.

$\boxed{\text{Soln: } x_1 = 5, \; x_2 = 0, \; x_3 = 5/3, \; z = 10}$.

→ Moreover the plane formed by ~~the~~ the convex linear combination of these three pts. is also optima.

$\frac{1}{2}$ (or In other words) Plane $x_1 + 2x_2 + 3x_3 = 10$ in I^{st} quadrant is ~~whole~~ optim

Weak duality theorem:-

For an primal-dual pair given by,

Primal:- Max $z = C^T x$
s.t. $Ax \leq b$, $x \geq 0$. $\Big] \rightarrow (1)$

2-Marks for Statement

dual:- Min $w = b^T y$
$A^T y \geq c$, $y \geq 0$. $\Big] \rightarrow (2)$

If this primal-dual pair is not written here and mentioned in proof, give full marks.

If x and y are feasible solns. of the primal-dual pair, then. $c^T x \leq b^T y$.

In other words, ~~Sb~~ In a primal-dual pair the objective fn. value of maximization problem is always less than or equal to the objective fn. value of the minimization problem, if both primal and dual are feasible.

Proof:- 1

(1/2) [x is feasible for (1) $\Rightarrow$ $Ax \leq b$, $x \geq 0 \rightarrow (3)$

(1/2) [y is feasible for (2) $\Rightarrow$ $A^T y \geq c$, $y \geq 0 \rightarrow (4)$.

(3) $\Rightarrow$ (1/2) [$Ax \leq b \Rightarrow x^T A^T \leq b^T$ and $y \geq 0$

$\Rightarrow x^T A^T y \leq b^T y \rightarrow (5)$.

(4) $\Rightarrow$ $A^T y \geq c \Rightarrow y^T A \geq c^T$ & $x \geq 0$

$\Rightarrow$ $y^T A x \geq c^T x. \rightarrow$ (6)

Now, $x^T A^T y$ & $y^T A x$ are scalars and

$(y^T A x)^T = x^T A^T y.$

$\Rightarrow$ $x^T A^T y = y^T A x. \rightarrow$ (7)

1/2

1

Combining (5), (6) & (7) we get

$$\boxed{c^T x \leq b^T y}$$

Hence proved.

1/2

Q-3 (a) Max $z = x_1 + 2x_2$

$x_1 + x_2 \geq 2$ $\equiv$ $x_1 + x_2$

$-x_1 + x_2 \leq 3.$

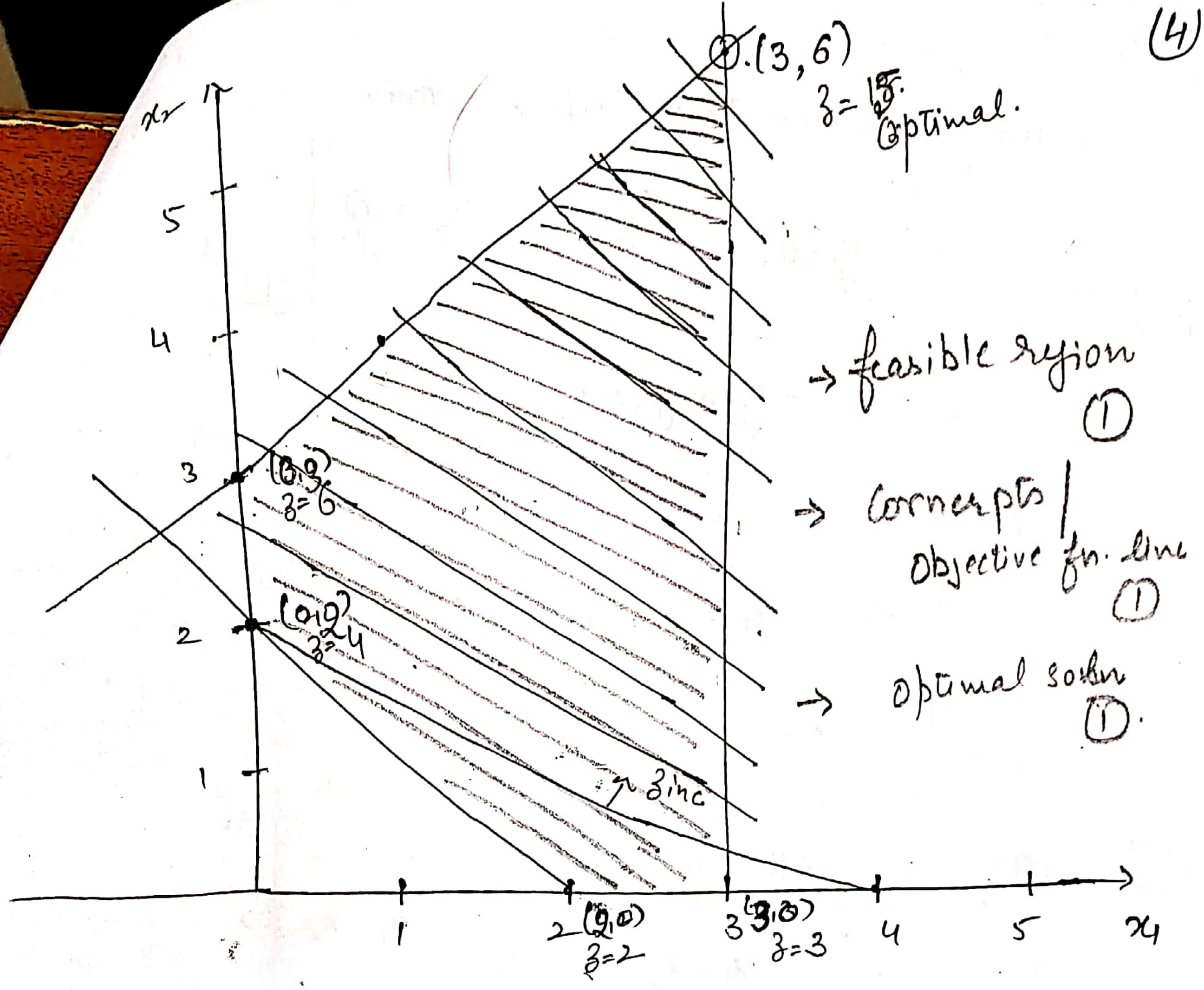
$x_1 \leq 3$

$x_1, x_2 \geq 0.$

$\rightarrow$ Objective fn. line for

$z = x_1 + 2x_2 = 4$ (say).

or find all the corner pts (since the region is bdd.)



Optimal soln. is

$x_1 = 3,\ x_2 = 6,\ z = 15.$

(b) Dual problem is

1 ← Min $w = 2y_1 + 3y_2 + 3y_3$.

1 ← $y_1 - y_2 + y_3 \geq 1$

$\equiv y_1 - y_2 + y_3 - s_1' = 1$

$y_1 + y_2 \geq 2$

$y_1 + y_2 - s_2' = 2$

1 ← $y_1 \leq 0,\ y_2 \geq 0,\ y_3 \geq 0$.

Primal

Max $z = x_1 + 2x_2$

$x_1 + x_2 - s_1 = 2$

$-x_1 + x_2 + s_2 = 3$

$x_1 + s_3 = 3$

$x_1, x_2, s_1, s_2, \geq$

Optimal soln. of primal is :-

$x_1 = 3,\ x_2 = 6,\ s_1 = 7,\ s_2 = 0,\ s_3 = 0$

$z = 15$.

$z_j - c_j \leq 0 \Rightarrow$ optimal. , But not feasible (3)

C_B	x_B	x_1	x_2	s_1	s_2	s_3	soln
0	s_1	0	$1/2$	1	0	$1/2$	$1/2$
0	s_2	0	$-1/2$	0	1	$-1/2$	$-1/2$
2	x_1	1	$1/2$	0	0	$-1/2$	$3/2$
	$z_j - c_j$	0	-2	0	0	-1	$z = 3$

(1)

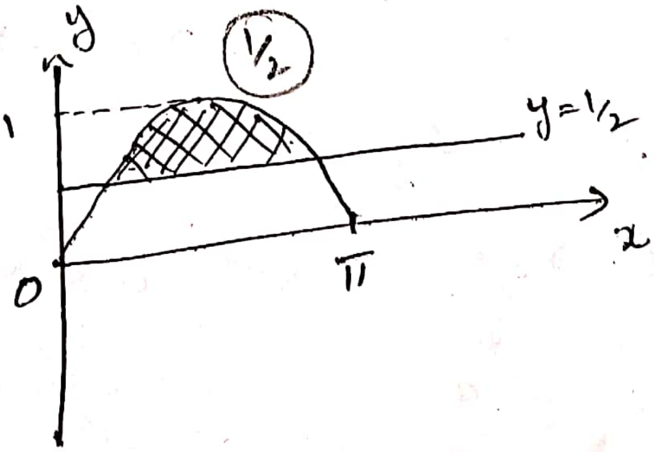
C_B	x_B	x_1	x_2	s_1	s_2	s_3	soln
0	s_1	0	0	1	1	0	0
0	s_3	0	1	0	-2	1	1
2	x_1	1	1	0	-1	0	2.
	$z_j - c_j$	0	-1	0	-2	0	$z = 4$

(1).

Soln. is feasible $\Rightarrow$ the Soln. is

$$\boxed{x_1 = 2,\quad x_2 = 0,\quad z = 4}.$$

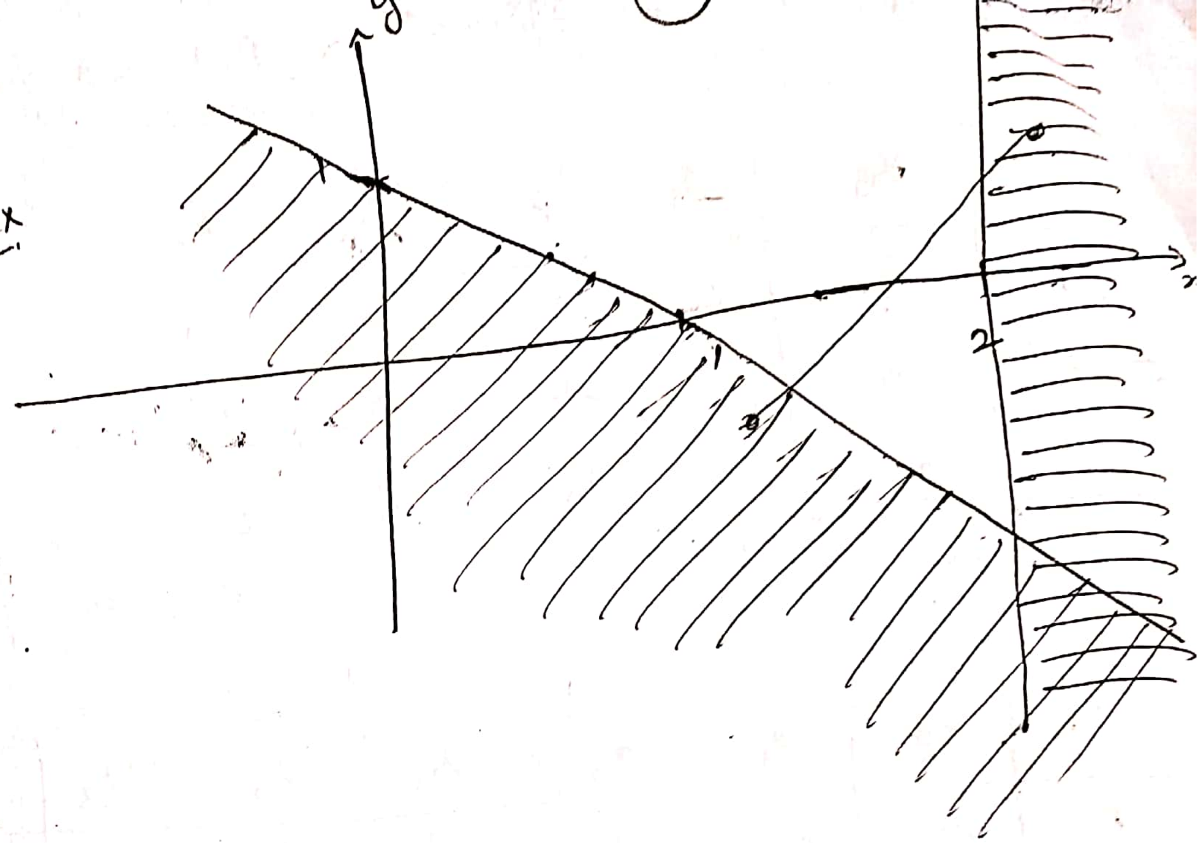
(b) (i)



(1/2)

Convex set. (1/2)

(ii)



Not convex

(1/2)

(c) $S = \{ x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0,\ b \in \mathbb{R}^m,\ A = m\times$ ma

Let $x, y \in S \Rightarrow Ax = b,\ x \geq 0$
$Ay = b,\ y \geq 0$ (1/2)

for $0 \leq \lambda \leq 1$, let $z = \lambda x + (1-\lambda) y$.

$Az = A(\lambda x + (1-\lambda)y) = \lambda Ax + (1-\lambda)Ay$
$= \lambda b + (1-\lambda) b = b.$ (1/2)

$\Rightarrow Az = b.$

Also $z = \underset{\geq 0}{\lambda x} + (1-\lambda)\underset{\geq 0}{y}$ and $0 \leq \lambda \leq 1$ (1/2)

$\Rightarrow z \geq 0 \Rightarrow z \in S$

Hence S is a convex set.