Lecture 25: Numerical Analysis (UMA011)

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$AX = b$

$\downarrow$

Sufficient condition

Strictly diagonal dominant (weaker condition)

$\Downarrow$

for any choice of $X^{(0)}$, $\{X^k\}_{k=0}^{\infty} \to X$

Stronger condition

iff

$A \to$ not

$\not\Downarrow$

$\{X^k\}_k \not\to X$

System of linear equations: Matrix representation of iterative methods

Gauss-Seidel method:

The Gauss-Seidel method is given by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right)$$

$$a_{ii} x_i^{(k)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}$$

$$a_{ii} x_i^{(k)} + \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} = b_i - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}$$

$$i = 1, 2, 3 \; --- \; n$$

$$a_{11} x_1^{(k)} = b_1 - \sum_{j=2}^{n} a_{1j} x_j^{(k-1)}$$

$$a_{22} x_2^{(k)} + a_{21} x_1^{(k)} = b_2 - \sum_{j=3}^{n} a_{2j} x_j^{(k-1)}$$

$$a_{33} x_3^{(k)} + \sum_{j=1}^{2} a_{3j} x_j^{(k)} = b_3 - \sum_{j=4}^{n} a_{4j} x_j^{(k-1)}$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$a_{nn} x_n^{(k)} + \sum_{j=1}^{n-1} a_{nj} x_j^{(k)} = b_n$$

Here,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad X^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}$$

we define

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{22} & \cdots & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & & & \\ 0 & \vdots & \cdots & 0 & a_{n-1,n} \\ & & & & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn-1} & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$D x^{(k)} + L x^{(k)} = b - U x^{(k-1)}$$

$$(D+L) x^{(k)} = -U x^{(k-1)} + b$$

$$I x^{(k)} = -(D+L)^{-1} U x^{(k-1)} + (D+L)^{-1} b$$

$$\boxed{x^{(k)} = T_g x^{(k-1)} + C_g}$$

$$T_g = -(D+L)^{-1} U$$

$$C_g = (D+L)^{-1} b$$

System of linear equations: Matrix representation of iterative methods

SOR method:

The SOR method is given by

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right)$$

$$a_{ii}x_i^{(k)} = (1-\omega)a_{ii}x_i^{(k-1)} + \omega\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right)$$

$$a_{ii}x_i^{(k)} + \omega\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} = (1-\omega)a_{ii}x_i^{(k-1)} + \omega b_i - \omega\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}$$

$$DX^{(k)} + \omega L X^{(k)} = (1-\omega)DX^{(k-1)} + \omega b - \omega U X^{(k-1)}$$

$$(D+\omega L)\,x^{(k)} = \big((1-\omega)D - \omega U\big)\,x^{(k-1)} + \omega b$$

$$x^{(k)} = \underbrace{(D+\omega L)^{-1}\big((1-\omega)D - \omega U\big)}\,x^{(k-1)} + (D+\omega L)^{-1}\omega b.$$

$$\boxed{x^{(k)} = T_\omega\, x^{(k-1)} + c_\omega},$$

where

$$T_\omega = (D+\omega L)^{-1}\big((1-\omega)D - \omega U\big)$$

$$c_\omega = (D+\omega L)^{-1}\omega b.$$

for any iterative method to solve the linear system of equations, we get matrix Rep.

$$\begin{array}{ccc} X^{(k)} & = & T\, X^{(k-1)} + C \\ \downarrow & & \quad\downarrow \\ X & & \quad X \end{array}$$

$$X = T X + C$$

System of linear equations: Matrix representation of iterative methods

Result:(Stronger condition for the convergence of iterative methods):

For any $X^{(0)} \in \mathbb{R}^n$, the sequence $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} = TX^{(k-1)} + C$, for each $k \geq 1$ converges to unique solution $X = TX + C$ iff $\underbrace{\rho(T)}_{} < 1$.

spectral radius of T =

max{| eigenvalues of T |}

System of linear equations:

Example:

Check whether you can apply Gauss-Seidel iterative techniques to solve the following linear system of equations.

$$
\begin{aligned}
2x_1 - x_2 + x_3 &= -1\\
2x_1 + 2x_2 + 2x_3 &= 4\\
-x_1 - x_2 + 2x_3 &= -5.
\end{aligned}
$$

Solution. Let the given system of linear eq^n's be

$$Ax = b$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix associated with Gauss-Seidel method

$$T_g = -(D+L)^{-1} U$$

$$T_g = -\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

$$\Rightarrow T_g = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix} \checkmark$$

To eigen values of T_g

$$|T_g - \lambda I| = \begin{vmatrix} 0-\lambda & 1/2 & -1/2 \\ 0 & -1/2-\lambda & -1/2 \\ 0 & 0 & -1/2-\lambda \end{vmatrix} = 0$$

$$-\lambda\left(\left(-\frac{1}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right)-0\right)=0$$

$$\lambda\left(\frac{1}{4}+\lambda^2+\lambda\right)=0$$

$$\lambda=0,\quad \lambda=-\frac{1}{2},-\frac{1}{2}$$

$$\Rightarrow \quad \rho(T_g)=\max\left\{|0|,\left|-\frac{1}{2}\right|,\left|-\frac{1}{2}\right|\right\}=\frac{1}{2}<1$$

$\Rightarrow$ Gauss-Seidel method is applicable on the given system of linear equations.

System of linear equations:

Exercise:

Check whether you can apply Gauss-Seidel iterative techniques to solve the following linear systems.

1.

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &= -1\\
3x_1 + 2x_2 + 2x_3 &= 1\\
x_1 + 2x_2 + 2x_3 &= 1
\end{aligned}
$$

2.

$$
\begin{aligned}
x_1 + 2x_2 - 2x_3 &= 7\\
x_1 + x_2 + x_3 &= 2\\
2x_1 + 2x_2 + x_3 &= 5
\end{aligned}
$$