Solids and Structures
UES017

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Axial Deformations in Bars

(Uniform, Tapered and Stepped Bars)

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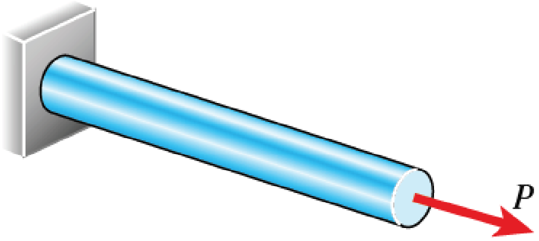
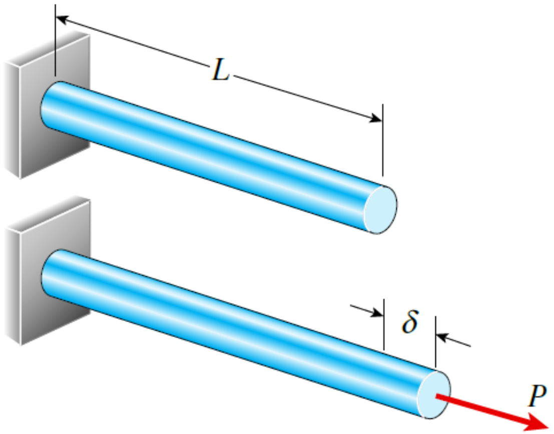
Axial Deformation of a Bar: Prismatic Bar

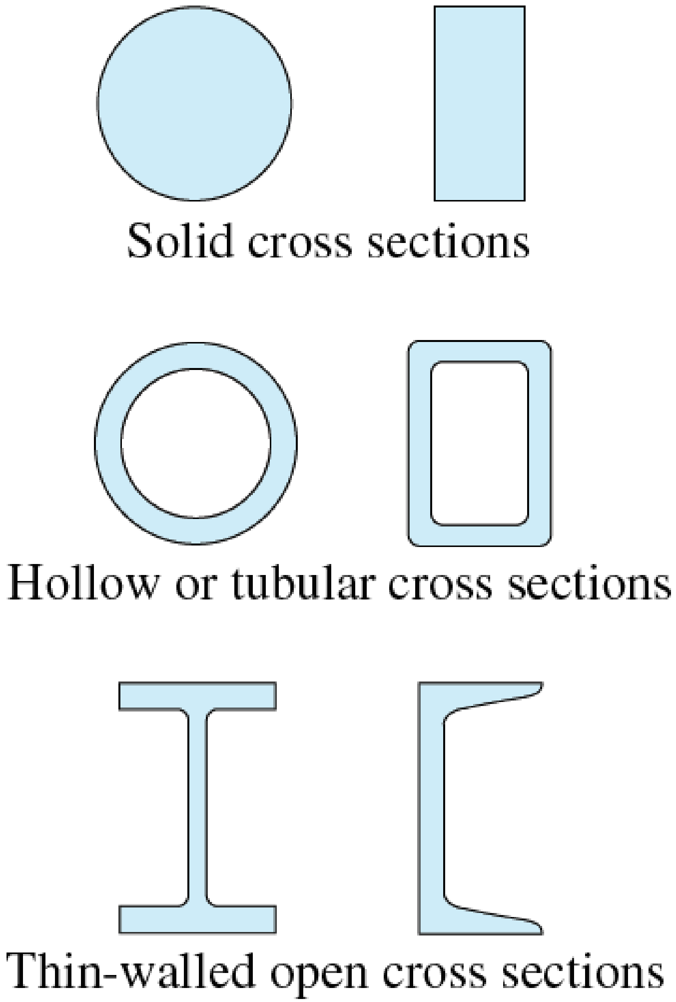
- Axially loaded bars elongate under tensile loads and shorten under compressive loads.
- To analyze this behavior, consider the prismatic bar.
- A **prismatic bar** is a structural member having a straight longitudinal axis and constant cross section throughout its length.
- Bars may have variety of cross-sections such as circular, tubular, rectangular, triangular, channel, etc

Prismatic bar of circular cross section





Typical cross sections of structural members



Axial Deformation of a Bar: Prismatic Bar

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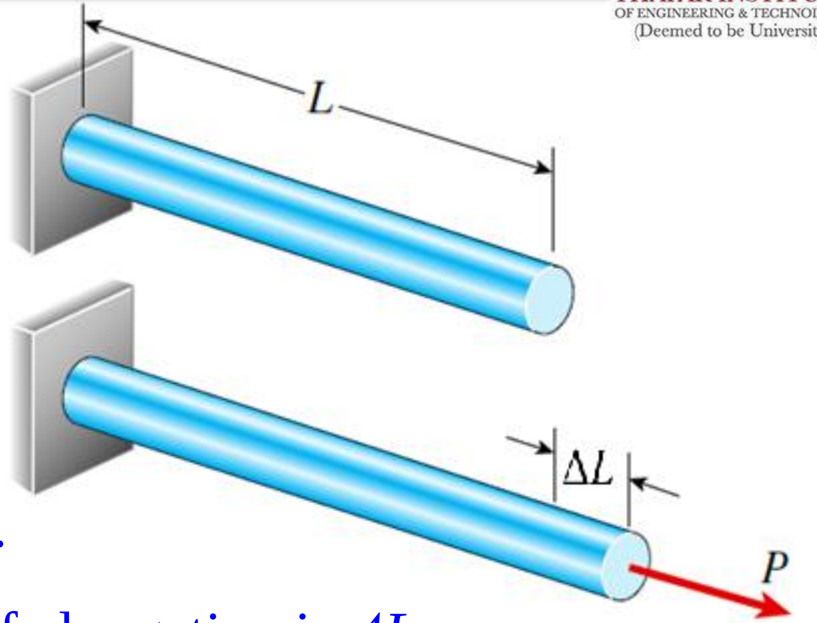
- Assumptions:
  - ✓ The material is homogeneous (i.e., constant E), elastic and isotropic,
  - ✓ The axial member is prismatic (uniform cross-sectional area A),
  - ✓ has a constant internal force (i.e., loaded only by forces at its ends).



- The bar has original length as *L*, area as *A* and Elastic modulus as *E*.
- After application of axial load P, the bar elongates and the amount of elongation is *ΔL*

Axial Stress in the bar $\sigma = \dfrac{P}{A}$ …(1)

Axial Strain in the bar $\varepsilon = \dfrac{\Delta L}{L}$ …(2)

Hooke's law $\sigma = E\varepsilon$ …(3)

From (1) to (3)

$$\Delta L = \varepsilon L = \frac{\sigma}{E} L = \frac{PL}{EA}$$

Thus, Axial Deformation of a Prismatic Bar

$$\boxed{\Delta L = \frac{PL}{EA}}$$

Axial Deformation of a Bar: Principle of Superposition

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*"The **principle of superposition** states that if a body is acted upon by a number of forces on various segments of a body, then the net effect on the body is the algebraic sum of effects caused by each of the loads acting independently on the respective segment of the body"*

Principle of Superposition on axial deformation of the bar is implemented when

- The member is subjected to various axial loads at intermediate points
- The axial member consists of various cross-sectional areas or materials, (the axial member must be divided into segments that satisfy the assumptions of prismatic bars).

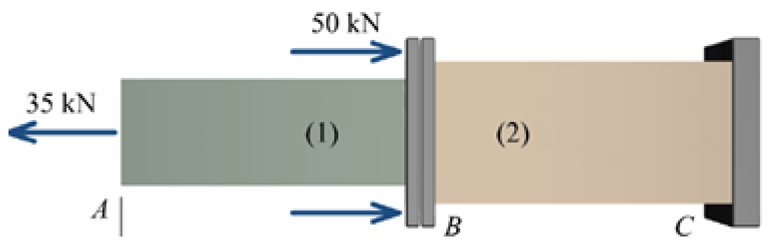
The overall deformation of the axial member can be determined by algebraically adding the segment deformations:

$$\boxed{\Delta L = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}}$$ for n segments

$$\boxed{\Delta L = \frac{P_1 L_1}{E_1 A_1} + \frac{P_2 L_2}{E_2 A_2} + \frac{P_3 L_3}{E_3 A_3}}$$ for 3 segments

Deformation of a Compound Bar: Illustrations

Illustration: An axial member consisting of two bars is supported at C and loaded as shown in Fig. Bar (1) has a cross-sectional area of 600 mm², length of 1000 mm and an elastic modulus of 30 GPa. Bar (2) has a cross-sectional area of 900 mm², length of 1200 mm and an elastic modulus of 15 GPa. Determine the deflection of point A.

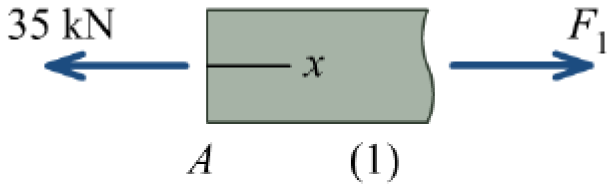


Given Data: $L_1 = 1000mm, \quad L_2 = 1200mm, \quad A_1 = 600mm^2, \quad A_2 = 900mm^2, \quad E_1 = 30GPa, \quad E_2 = 15GPa$

To find: $\Delta L_A = ?$

Solution:

- ✓ To find the internal axial force in member (1)
- ✓ Draw a FBD that cuts through member (1)



$$\boxed{F_1 = 35kN} \quad \text{(tensile)}$$

Deformation of a Compound Bar: Illustrations

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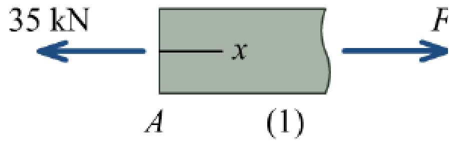
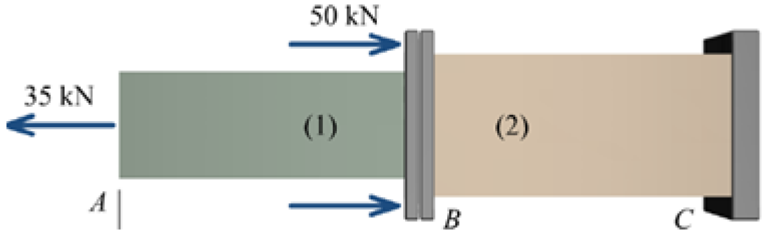
- ✓ To find the internal axial force in member (2)
- ✓ Draw a FBD that cuts through member (2)

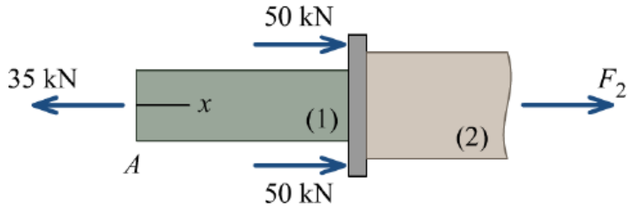




$\boxed{F_1 = 35kN}$ (tensile)

$$\Sigma F_x = -35 \text{ kN} + 50 \text{ kN} + 50 \text{ kN} + F_2 = 0$$

$\boxed{F_2 = -65kN}$ *(compressive in nature)*



Deflection at Point A

$$\Delta L_A = \left[\frac{F_1 L_1}{E_1 A_1} + \frac{F_2 L_2}{E_2 A_2} \right]$$

$$L_1 = 1000mm, \quad L_2 = 1200mm, \quad A_1 = 600mm^2, \quad A_2 = 900mm^2, \quad E_1 = 30GPa, \quad E_2 = 15GPa$$

$$F_1 = 35kN, \quad F_2 = -65kN.$$

$$\Delta L_A = \left[\frac{35 \times 10^3 \times 1000}{30 \times 10^3 \times 600} - \frac{65 \times 10^3 \times 1200}{15 \times 10^3 \times 900} \right] \Rightarrow \boxed{\Delta L_A = 3.83mm}$$

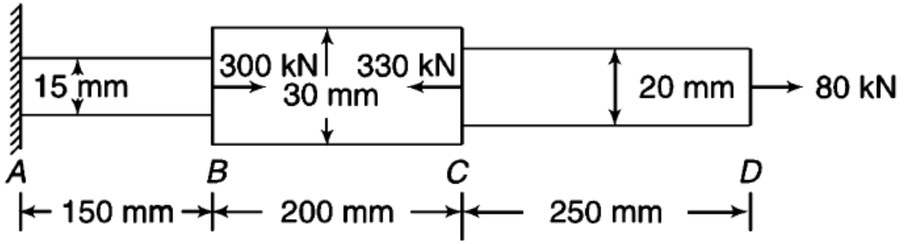
Thus, the deflection at point A is 3.83 mm (elongation)

Deformation of a Compound Bar: Illustrations

Illustration: A steel circular bar has three segments and loaded as shown in Figure. Consider E = 205 Gpa. Determine:

(a) Total elongation of the bar

(b) The length of the middle segment to have zero elongation of the bar



Given A steel circular bar having three segments with different forces

$E = 205$ GPa.

To find

— Total elongation

— Length of middle segment for zero elongation of bar

Solution:

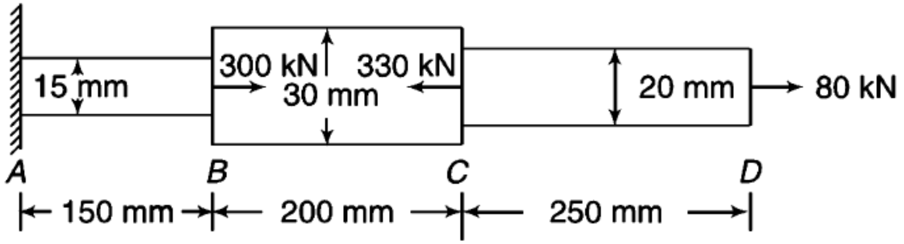
✓ Calculate the forces in individual members

✓ In order to evaluate forces, draw free body diagram for each section

✓ In Section CD, the force is 80 kN tensile

Deformation of a Compound Bar: Illustrations

- Draw free body diagram for each section
  - ✓ In Section CD, the force is 80 kN tensile (F_1 = 80 kN)
  - ✓ In Section BC, the force is 250 kN Compressive (F_2 = -250 kN)
  - ✓ In section AB, the force is 50 kN tensile (F_3 = 50 kN)



(a) Total Elongation

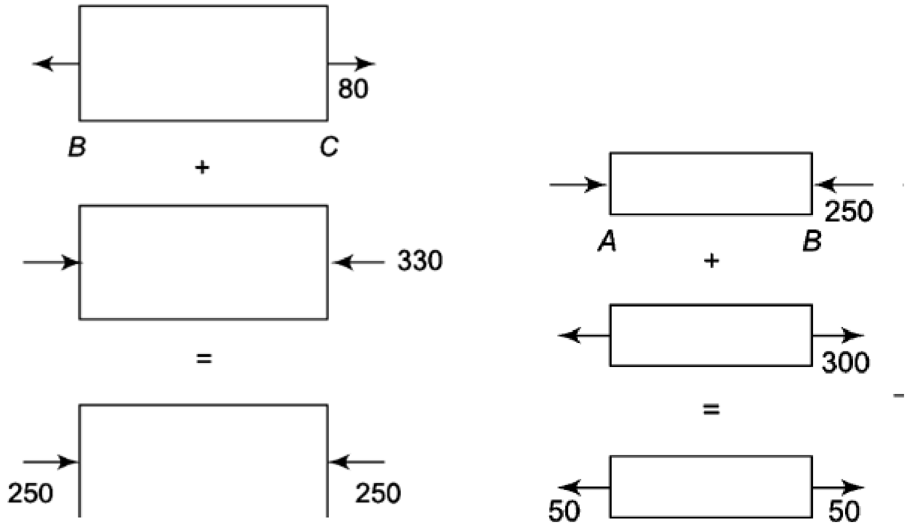
$$\Delta L = \frac{1}{E}\left[\frac{F_1L_1}{A_1} + \frac{F_2L_2}{A_2} + \frac{F_3L_3}{A_3}\right]$$

$$A_1 = \frac{\pi}{4}(15)^2,\quad A_2 = \frac{\pi}{4}(30)^2,\quad A_3 = \frac{\pi}{4}(20)^2.$$

$$E = 205\text{ GPa}$$

$$L_1 = 250mm,\quad L_2 = 200mm,\quad L_3 = 150mm$$

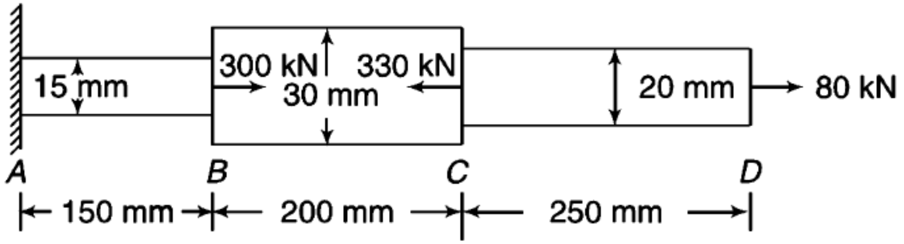
$$\boxed{\Delta L = \frac{1\times10^3}{(\pi/4)205000}\left[\frac{80\times250}{(20)^2} - \frac{250\times200}{(30)^2} + \frac{50\times150}{(15)^2}\right] = 0.173mm}$$



Thus, the deflection at point D is 0.173 mm (elongation)

Deformation of a Compound Bar: Illustrations

(a) Length of middle segment if total deflection is zero



$$\Delta L = \frac{1}{E}\left[\frac{F_1 L_1}{A_1} + \frac{F_2 L_2}{A_2} + \frac{F_3 L_3}{A_3}\right]$$

$$F_1 = 80kN, \quad F_2 = -250kN, \quad F_3 = 50kN.$$

$$A_1 = \frac{\pi}{4}(20)^2, \quad A_2 = \frac{\pi}{4}(30)^2, \quad A_3 = \frac{\pi}{4}(15)^2.$$

$$L_1 = 250mm, \quad L_2 = ?, \quad L_3 = 150mm$$

$$E = 205 \text{ GPa}$$

$$\Delta L = \frac{1 \times 10^3}{(\pi/4)205000}\left[\frac{80 \times 250}{(20)^2} - \frac{250 \times L_2}{(30)^2} + \frac{50 \times 150}{(15)^2}\right] = 0$$

$$\left[50 - \frac{250 \times L_2}{(30)^2} + 33.33\right] = 0 \Rightarrow \boxed{L_2 = 300mm}$$

Thus, the length of middle segment should be 300 mm in order to ensure that total deformation of the bar is zero under applied load

Deformation of a Compound Bar: Illustrations

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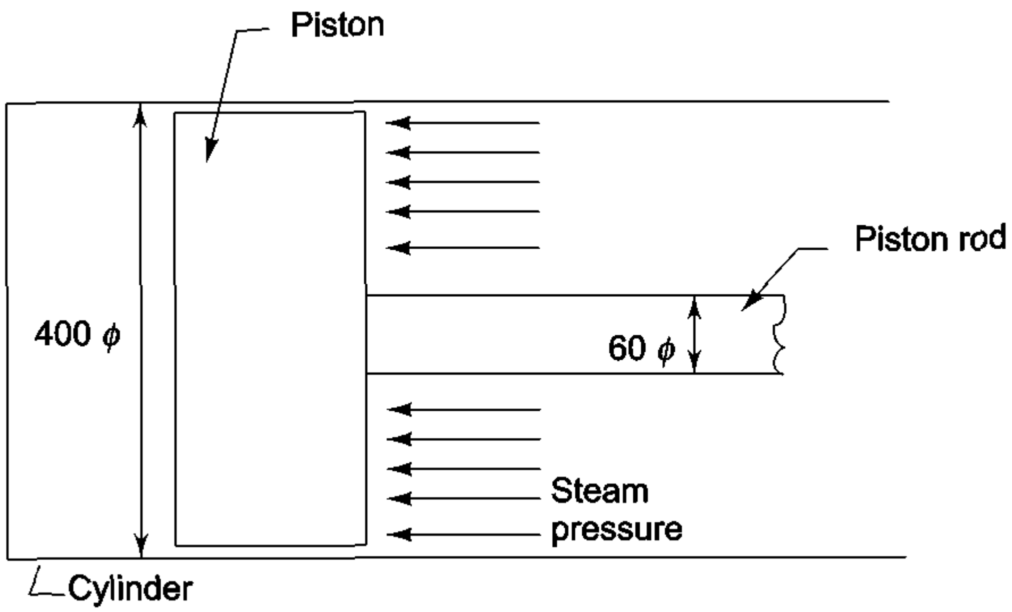
Illustration: The piston of a steam engine is 60 mm diameter and operates in a cylinder of diameter 400 mm. The piston rod is 1 m long. What is the maximum pressure that can be allowed in the cylinder, if the stress in the rod is limited to 100 N/mm^2? What will be the change in the length of the piston at this pressure? E = 200 GPa.

Solution: p = maximum permissible pressure
F = Force on the piston rod
A = Area on which pressure is acting

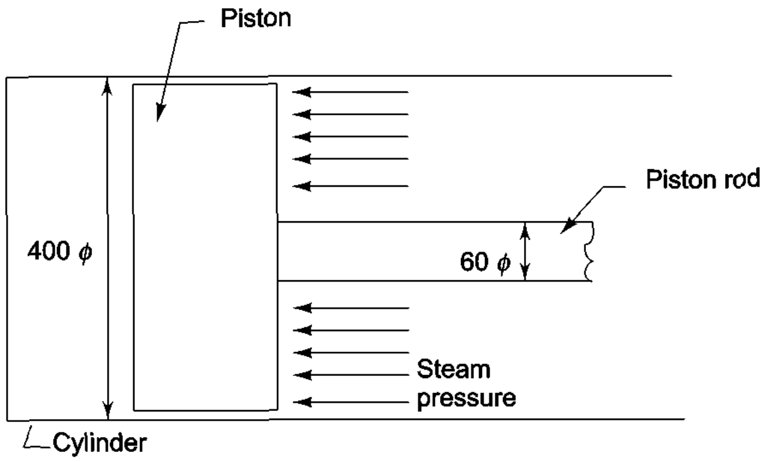
$$\because A = \frac{\pi}{4}(400)^2 - \frac{\pi}{4}(60)^2$$

$$\therefore F = p\left[\frac{\pi}{4}(400)^2 - \frac{\pi}{4}(60)^2\right]$$



Deformation of a Compound Bar: Illustrations

$$\therefore F = p\left[\frac{\pi}{4}(400)^2 - \frac{\pi}{4}(60)^2\right] \qquad A_{rod} = \frac{\pi}{4}(60)^2 = 900\pi$$



Stress in the rod

$$\sigma = \frac{F}{A_{rod}} = \frac{p\left[\frac{\pi}{4}(400)^2 - \frac{\pi}{4}(60)^2\right]}{\frac{\pi}{4}(60)^2} = \frac{p\left[(400)^2 - (60)^2\right]}{(60)^2} = \frac{p\left[(400)^2 - (60)^2\right]}{(60)^2} = 43.44p$$

But, Stress in the rod is limited to 100 MPa $\quad \therefore 100 = 43.44p \Rightarrow p = 2.30 N/mm^2$

$$\therefore F = 39100\pi p = 39100\pi \times 2.30 = 282523.43N$$

Elongation in the rod $\quad \boxed{\Delta L = \frac{FL}{EA}} \Rightarrow \boxed{\Delta L = \frac{282523.43 \times 1000}{200000 \times 900\pi} = 0.4996mm}$

Thus, change in the length of the piston rod is 0.4996 mm

Deformation of a Tapered Bar: Conical Section

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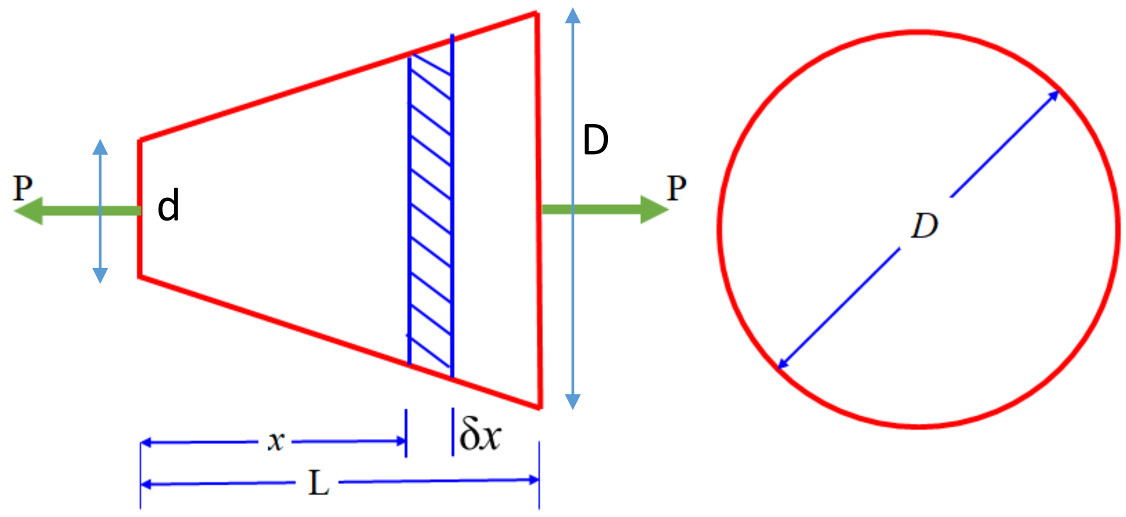
- Consider a bar of Conical Section under axial load
  - ✓ D: Diameter at larger end
  - ✓ d: diameter at smaller end
  - ✓ L: Length of the bar
  - ✓ E: Young's modulus of the bar material.



Consider a small length δx at a distance x from the small end

$$\text{The diameter at a distance } x \text{ from the small end} = d + \frac{D-d}{L} \cdot x$$

$$\text{The extension of the small length} = \frac{P \cdot \delta x}{\frac{\pi}{4}\left(d + \frac{D-d}{L}x\right)^2 \cdot E}$$

Deformation of a Tapered Bar: Conical Section

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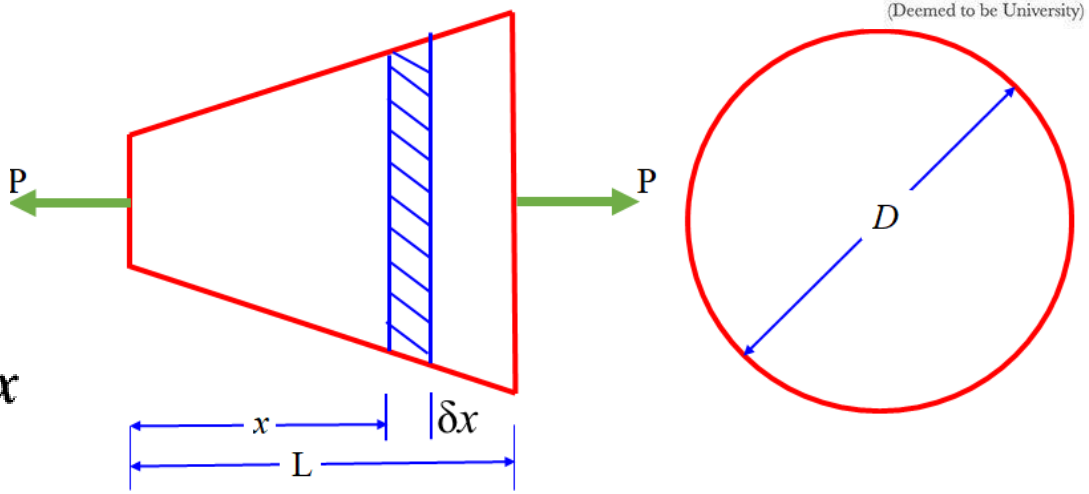
$$\text{Extension of the whole rod} = \int_0^L \frac{4P}{\pi(d+(D-d)x/L)^2 \cdot E} \cdot dx$$

$$= \frac{4P}{\pi E}\int_0^L \left(d + \frac{D-d}{L}\cdot x\right)^{-2} \cdot dx$$



$$= -\frac{4P}{\pi E}\cdot\frac{L}{(D-d)}\left(\frac{1}{(d+(D-d)x/L)}\right)_0^L = \frac{4PL}{\pi E(D-d)}\left(\frac{1}{d}-\frac{1}{D}\right) = \frac{4PL}{\pi E(D-d)}\left(\frac{D-d}{dD}\right) = \frac{4PL}{\pi EdD}$$

Thus, Deformation of a Tapered Bar of conical section is $\boxed{\Delta L = \dfrac{4PL}{\pi EdD}}$

Self Assessment

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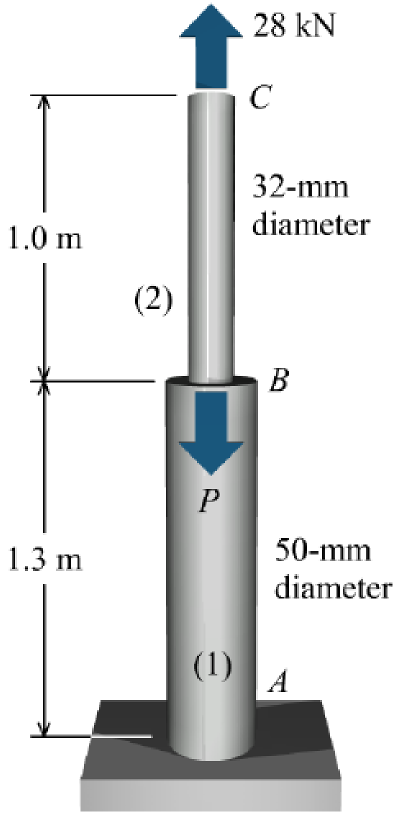
Exercise 1: A tapering conical bar of 1m length has diameters of 20 mm and 50 mm at the two ends. Determine elongation of the bar due to an axial tensile load of 250 kN. Take E = 205 GPa.

Exercise 2: A member ABC made of Aluminum (*E* = 70 GPa) supports a load of 28 kN, as shown in Fig. Determine: (a) the value of load *P* such that the deflection of joint *C* is zero. (b) the corresponding deflection of joint *B*.



Exercise 3: A steel rod, 20 mm diameter and 800 mm long, is rigidly attached to an aluminum rod, 40 mm diameter and 1 m long. The combination is subjected to a tensile load of 40 kN. Determine the stress in the materials and the total elongation of the bar. *E* for steel = 200 GPa, *E* for aluminium = 70 GPa.

Answers (1): 1.55 mm, (2) 80.6 kN, 0.497 mm, (3) 31.8 MPa, 0.964 mm