Lecture 24: Numerical Analysis (UMA011)

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$$X_{n\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\|X\|_\infty = \max_{1 \le i \le n} \{|x_i|\}$$

$$\|X\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$$\|X - y\|$$

System of linear equations: Matrix representation of iterative methods

$\|x\|$

Matrix norm:

A matrix norm on a set of all $n \times n$ matrices is a real-valued function, $\|.\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

$\|A\|_{n\times n}$

$\|A\|$

$f : A \to \mathbb{R}.$

$A - O$

- (i) $\|A\| \geq 0$;
- (ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;
- (iii) $\|\alpha A\| = |\alpha| \|A\|$;
- (iv) $\|A + B\| \leq \|A\| + \|B\|$;
- (v) $\|AB\| \leq \|\cancel{A}B\|$. $\|A\|+\|B\|$

$\|A\|_\infty$

$\|A\|_2$

The distance between $n \times n$ matrices A and B with respect to this matrix norm is $\|A - B\|$.

Matrix norm

Matrix norm in l_∞-space

If $A = (a_{ij})$ is $n \times n$ matrix, then the l_∞−norm is given by

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \end{bmatrix}$$

$$\max_{1 \le i \le n} \{ |a_{i1}| + |a_{i2}| + \cdots |a_{in}| \}$$

$$\max \{ (|a_{11}| + |a_{12}| + \cdots |a_{1n}|), (|a_{21}| + |a_{22}| \cdots |a_{2n}|) \cdots (|a_{n1}| + |a_{n2}| \cdots |a_{nn}|)$$

Matrix norm

Example:

Determine $\|A\|_\infty$ norm for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$.

Solution:

$$\|A\|_\infty = \max_{1 \le i \le 3} \sum_{j=1}^{3} |a_{ij}|$$

$$= \max_{1 \le i \le 3} \{ |a_{i1}| + |a_{i2}| + |a_{i3}| \}$$

$$= \max \{ |1| + |2| + |-1|, \ |0| + |3| + |-1|, \ |5| + |-1| + |1| \}$$

$$= \max \{4, 4, 7\} = 7$$

Matrix norm

To find $||A||_2$

$||A||_2 = ?$

$AX = b$

Eigenvalues and Eigenvectors:

If A is a square matrix, the characteristic polynomial of A is given by $|A - \lambda I| = p(\lambda)$(say). The zeros of $p(\lambda)$ are the eigenvalues for the matrix A.

If λ is an eigenvalue of A and $X \neq 0$ satisfies $(A - \lambda I)X = 0$, the X is an eigenvector corresponding to eigenvalue λ.

$\lambda \in \mathbb{R}$

$|A - \lambda I| = p(\lambda)$

$p(\lambda) = 0$

$\lambda_1, \lambda_2, \lambda_3$

$\downarrow \quad \downarrow \quad \downarrow$

$X_1 \quad X_2 \quad X_3$

$(A - \lambda_1 I) = B$

$(A - \lambda_1 I) X = 0$

$AX = \lambda_1 I X$

$= \lambda_1 X$

$BX_1 = 0$

$X_1 \neq 0$

$AX = 0$

$|A| \neq 0$

$X = 0$

$X \neq 0$

$|A| = 0$

Matrix norm

Example:

Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}.$$

Solution: Let $\lambda \in \mathbb{R}$

$$|A - \lambda I| = \left| \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|$$

$$= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)\left((1-\lambda)(4-\lambda)+2\right) = p(\lambda)$$

$$(2-\lambda)\left(\lambda^2 - 5\lambda + 4 + 2\right)$$

$$p(\lambda) = (2-\lambda)(\lambda^2-5\lambda+6)$$

To find zeros of $p(\lambda)$

Take $(2-\lambda)(\lambda^2-5\lambda+6)=0$

$$\lambda = 2, 3, 2$$

$$\lambda = 2, 2, 3 \rightarrow \text{eigenvalues of } A.$$

Call it $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 2$.

let X_1 be the eigenvector corresponding to $\lambda_1 = 3$

s.t. $(A-3I)X_1 = 0$

let $X_1 = (x_1, x_2, x_3)^t$

$$(A-3I)X_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad 0$$

$-x_1 = 0$ $\qquad$ $x_1 - 2x_2 + 2x_3 = 0$, $\qquad$ $x_1 - x_2 + x_3 = 0$

$x_1 = 0$ $\qquad$ $-2x_2 + 2x_3 = 0$ $\qquad$ $-x_2 + x_3 = 0$

$\Rightarrow x_2 = x_3$ $\qquad$ $\Rightarrow x_2 = x_3$

$$X_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

Let X_2 be eigenvector corresponding to $\lambda_2 = 2$

s.t $(A - \lambda_2 I)X_2 = 0$ $\qquad$ let $X_2 = (x_1, x_2, x_3)^t$

$$(A-2I)X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 - x_2 + 2x_3 = 0$$

$$x_2 = x_1 + 2x_3$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Matrix norm

Spectral radius:

The spectral radius $\rho(A)$ of a matrix A is defined by
$\rho(A) = \max|\lambda|$, where λ is an eigenvalue of A.

$\rho(A)$

$A_{n\times n}$

Matrix norm

Matrix norm in l_2-space

If $A = (a_{ij})$ is $n \times n$ matrix, then the l_2-norm is given by

$$\|A\|_2 = \sqrt{\rho(A^t A)}\,,$$

where A^t is the transpose of A.

$A^t A = B$

AA^t X

Matrix norm

Example:

Determine $\|A\|_2-$norm for the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

Solution:

To find $\|A\|_2$,

Take $A^tA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$

To find e.values of A^tA. $|A^tA - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left(30+\lambda^2-11\lambda-16\right) - 2\left(10-2\lambda+4\right) - 1\left(8+6-\lambda\right) = 0$$

$$(3-\lambda)\left(\lambda^2-11\lambda+14\right) - 2\left(14-2\lambda\right) - \left(14-\lambda\right) = 0$$

$$3\lambda^2 - 33\lambda + 42 - \lambda^3 + 11\lambda^2 - 14\lambda - 28 + 4\lambda - 14 + \lambda = 0$$

$$-\lambda^3 + 14\lambda^2 - 42\lambda = 0$$

$$-\lambda\left(\lambda^2 - 14\lambda + 42\right) = 0$$

$$\lambda = 0, \quad \lambda = \frac{14 \pm \sqrt{196-168}}{2}$$

$\lambda = 0$, $\lambda = 7 + \sqrt{7}$, $7 - \sqrt{7}$

$\max\{0, 7 \pm \sqrt{7}\} = 7 + \sqrt{7}$

$\Rightarrow \quad \|A\|_2 = \sqrt{7 + \sqrt{7}}$ Ans.

Matrix norm

Convergent Matrices:

The matrix $A_{n\times n}$ is convergent if $\lim_{k\to\infty}(A^k)_{ij} = 0$ for each $1 \le i, j \le n$.

$A^k \to 0$

$A^k = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^k$

A

$A^2 \quad A^3 \quad A^4 \quad A^5 \quad \cdots$

$\lim_{k\to\infty} A^k = 0$

Matrix norm

Example:

Show that $A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ is convergent matrix.

Solution:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ \frac{2}{8} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ \frac{1}{4} & 1/4 \end{bmatrix} = \begin{bmatrix} 1/2^2 & 0 \\ \frac{2}{2^3} & \frac{1}{2^2} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{1}{8}+\frac{1}{16} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} 1/2^3 & 0 \\ \frac{3}{2^4} & \frac{1}{2^3} \end{bmatrix}$$

$$A^k = \begin{bmatrix} \frac{1}{2^k} & 0 \\ \frac{k}{2^{k+1}} & \frac{1}{2^k} \end{bmatrix}$$

$$\lim_{k\to\infty} A^k = \lim_{k\to\infty} \begin{bmatrix} \frac{1}{2^k} & 0 \\ \frac{k}{2^{k+1}} & \frac{1}{2^k} \end{bmatrix} \to 0$$

System of linear equations:

Exercise:

1. Compute the eigenvalues and associated eigenvectors for the following matrix: $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. Also find $\|A\|_\infty$ and $\|A\|_2$.
2. Let $A_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ and $A_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 16 & \frac{1}{2} \end{bmatrix}$. Show that A_1 is not convergent, but A_2 is convergent.