

Lecture 10: Numerical Analysis (UMA011)

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Fixed point iteration

Convergence through graphics:

$$g(x)$$

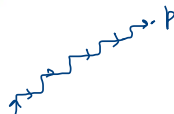
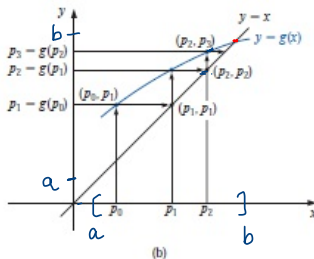
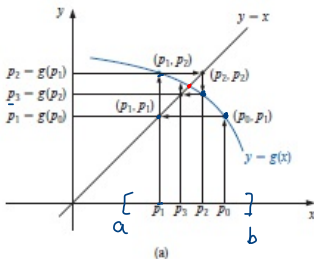
$$p_n = g(p_{n-1})$$

$$p_0$$

$$p_1 = g(p_0)$$

$$p_2 = g(p_1)$$

$$p_3 = g(p_2)$$

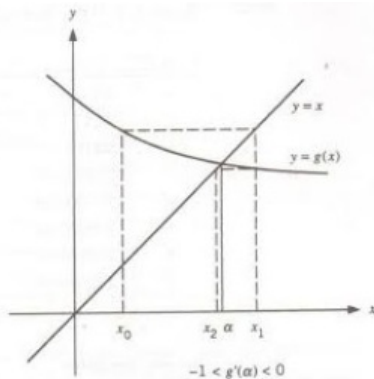
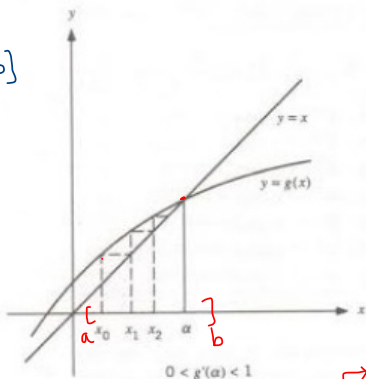


Fixed point iteration

$|g'(x)| < 1$ is required:

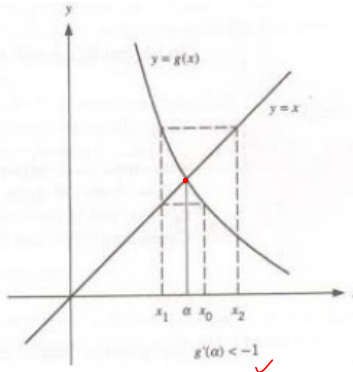
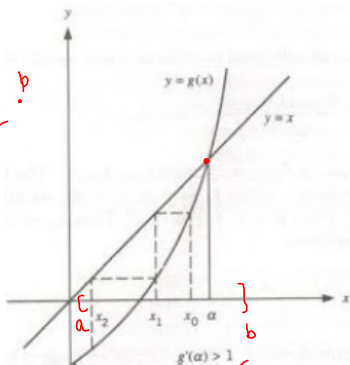
$$-1 < g'(x) < 1$$

$$x \in [a, b]$$



Fixed point iteration

$|g'(x)| < 1$ is required:



Fixed point iteration

Converse is not true:

If the conditions for the convergence of a fixed point given in previous result are satisfied then there is a guarantee for the existence and uniqueness of a fixed point on a given interval but if we have one fixed point in a given interval then condition may or may not be satisfied.

$$p_n = g(p_{n-1})$$

↓
p

$$g(x) \quad [a, b]$$

Fixed point iteration

Counter example:

Show that the conditions for the convergence of a fixed point do not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exist.

conditions of

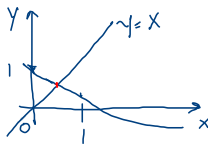
$g(x)$

✗

⇒ it has a
fixed pt.

$g(x)$

Solution:



⇒ $g(x)$ has a ^{unique} fixed
pt. on $[0, 1]$

i) $g(x) = 3^{-x}$ is cont. function on $[0, 1]$

(ii) $g'(x) = -3^{-x} \ln(3) < 0 \quad \forall x \in [0, 1]$

$\Rightarrow g(x)$ is decreasing function on $[0,1]$

\Rightarrow Max. value of $g(x) = g(0) = 3^0 = 1 \in [0,1]$

Min value ' ' = $g(1) = 3^{-1} = \frac{1}{3} \in [0,1]$

$\Rightarrow g(x) = 3^{-x} \in [0,1] \quad \forall x \in [0,1]$

(iii)

$$g'(x) = 3^{-x} (-1) \ln(3)$$

$$|g'(x)| = |-3^{-x} \ln(3)|$$

$$g'(0) = |-3^0 \ln(3)| = |-\ln(3)| = 1.09 > 1$$

The convergence conditions by $g(x)$ are not satisfied.

Fixed point iteration

Example of FPI:

Find the root of an equation $x^3 + 4x^2 - 10 = 0$ by using fixed point iteration method with the accuracy of 10^{-2} .

$$f(x) = 0$$
$$x = g(x)$$

Solution: $f(x) = x^3 + 4x^2 - 10 = 0$

Step 1

$$f(0) = -ve$$

"find interval"

$$f(1) = 1 + 4 - 10 = -ve$$

$$f(2) = 8 + 16 - 10 > 0$$

The root of $f(x) = 0$ lie in $[1, 2]$ ✓

Step 2 "find Appropriate $g(x)$ "

$$\textcircled{1} \quad x = x + x^3 + 4x^2 - 10$$

$$x = g_1(x) = x^3 + 4x^2 + x - 10 \quad \times$$

$g_1(x)$ is a cont. f'n on $[1, 2]$

$$g_1(1) = 1 + 4 + 1 - 10 = -4 \notin [1, 2] \quad \checkmark$$

$\Rightarrow g_1(x)$ does not satisfy the cond.

$\textcircled{2}$

$$x = x - (x^3 + 4x^2 - 10)$$

$$g_2(x) = -x^3 - 4x^2 + x + 10 \quad \times$$

$$g_2(1) = -1 - 4 + 1 + 10 = 6 \notin [1, 2]$$

$$(3) \quad x^3 + 4x^2 - 10 = 0$$

$$x = (10 - 4x^2)^{1/3} = g_3(x) \quad \times$$

$$g_3(1) = (10 - 4)^{1/3}$$

$$g_3(2) = (10 - 16)^{1/3} = (-6)^{1/3}$$

$$= -1.8 \notin [1, 2]$$

(4)

$$x^2 = \frac{10 - x^3}{4}$$

$$x = \frac{\sqrt{10 - x^3}}{2} = g_4(x) \quad \times$$

$$g_4 \in C[1, 2]$$

$$g_4(1) = \frac{\sqrt{10-1}}{2} = \frac{3}{2} \in [1, 2]$$

$$g_4(2) = \frac{\sqrt{10-8}}{2} = \frac{1}{\sqrt{2}} \notin [1, 2]$$

⑤

$$x^3 + 4x^2 - 10 = 0$$

$$x^2 + 4x = \frac{10}{x}$$

$$x = \sqrt{\frac{10}{x} - 4x} = g_5(x)$$

$$g_5 \in C[1, 2]$$

$$g_5(1) = \sqrt{10-4} = \sqrt{6} \notin [1, 2]$$

⑥

$$x + 4 = \frac{10}{x^2}$$

$$\Rightarrow x = \frac{10}{x^2} - 4 = g_6(x)$$

$$g_6 \in C[1, 2]$$

$$g_6(1) = 10 - 4 = 6 \notin [1, 2]$$

(7)

$$x^2(x+4) = 10$$

$$x^2 = \frac{10}{x+4}$$

$$x = \sqrt{\frac{10}{x+4}} = g_7(x)$$

$$g_7 \in C[1, 2] \checkmark$$

$$g_7(1) = \sqrt{\frac{10}{5}} = \sqrt{2} \in [1, 2]$$

$$g_7(2) = \sqrt{\frac{10}{6}} = \sqrt{\frac{5}{3}} \in [1, 2]$$

" will continue in next lecture "

Fixed point iteration

Exercise:

- 1 Show that the conditions for the convergence of a fixed point do not ensure a unique fixed point of $g(x) = \frac{x^2-1}{3}$ on the interval $[3, 4]$, even though a unique fixed point on this interval does exist.
- 2 Find the root of an equation $x^3 - 2x^2 - 5 = 0$ by using fixed point iteration method with the accuracy of 10^{-2} .