Numerical Analysis

Solution of Exercises : Chapter 5[1]
Interpolation and Approximations

1. Find the unique polynomial $P(x)$ of degree 2 or less such that

$$P(1) = 1,\ P(3) = 27,\ P(4) = 64$$

using Lagrange interpolation. Evaluate $P(1.05)$.

Sol. Given $x_0 = 1,\ x_1 = 3, x_2 = 4,\quad f(x_0) = P(1) = 1,\ f(x_1) = P(3) = 27,\ f(x_2) = P(4) = 64.$
Lagrange polynomial is

$$\begin{aligned} P_2(x) &= l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) \\ &= \frac{(x-3)(x-4)}{(1-3)(1-4)} \times 1 + \frac{(x-1)(x-4)}{(3-1)(3-4)} \times 27 + \frac{(x-1)(x-3)}{(4-1)(4-3)} \times 64 \\ &= 8x^2 - 19x + 12. \end{aligned}$$

Thus $P(1.05) \approx 8(1.05)^2 - 19(1.05) + 12 = 11.895.$

2. For the given functions $f(x)$, let $x_0 = 1,\ x_1 = 1.25$, and $x_2 = 1.6$. Construct Lagrange interpolation polynomials of degree at most one and at most two to approximate $f(1.4)$, and find the absolute error.

(a) $f(x) = \sin \pi x$.
(b) $f(x) = \sqrt[3]{x-1}$.
(c) $f(x) = \log_{10}(3x - 1)$.
(d) $f(x) = e^{2x} - x$.

Sol.

(a) To construct one degree polynomials, we need to data points. Since the interpolated value $x = 1.4$ lies between x_1 and x_2, so we use these two values and the one degree polynomial is

$$\begin{aligned} P_1(x) &= \frac{x-1.6}{1.25-1.6}\sin(1.25\pi) + \frac{x-1.25}{1.6-1.25}\sin(1.6\pi) \\ &= -0.6967x + 0.16425. \\ P_1(1.4) &\approx -0.6967(1.4) + 0.16425 = -0.8117. \end{aligned}$$

Absolute error is

$$E_a = |f(1.4) - P_1(1.4)| = |\sin(1.4\pi) - (-0.8117)| = 0.1394.$$

Further, we use all three data points to construct second degree polynomial and is given by

$$\begin{aligned} P_2(x) &= \frac{(x-1.25)(x-1.6)}{(1-1.25)(1-1.6)}\sin\pi + \frac{(x-1)(x-1.6)}{(1.25-1)(1.25-1.6)}\sin(1.25\pi) + \frac{(x-1)(x-1.25)}{(1.6-1)(1.6-1.25)}\sin(1.6\pi) \\ &= 3.5524x^2 - 10.8213x + 7.2689 \\ P_2(1.4) &= -0.91822. \end{aligned}$$

Absolute error

$$|P_2(1.4) - \sin(1.4\pi)| = |-0.91822 + 0.95106| = 0.03284.$$

(b) Same as part (a).
(c) Same as part (a).

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(d) The one degree polynomial is

$$\begin{aligned} P_1(x) &= \frac{x-1.6}{1.25-1.6}(e^{2\times 1.25}-1.25)+\frac{x-1.25}{1.6-1.25}(e^{2\times 1.6}-1.6) \\ P_1(1.4) &\approx 16.075. \end{aligned}$$

Absolute error is

$$E_a = |P_1(1.4) - f(1.4)| = |16.075 - 15.0446| = 1.0304.$$

Further, we use all three data points to construct second degree polynomial and is given by

$$\begin{aligned} P_2(x) &= \frac{(x-1.25)(x-1.6)}{(1-1.25)(1-1.6)}(e^{2\times 1}-1)+\frac{(x-1)(x-1.6)}{(1.25-1)(1.25-1.6)}(e^{2\times 1.25}-1.25) \\ &\quad \frac{(x-1)(x-1.25)}{(1.6-1)(1.6-1.25)}(e^{2\times 1.6}-1.6) \\ &= 26.8534x^2 - 42.2465x + 21.7821. \\ P_2(1.4) &= 15.2698. \end{aligned}$$

Absolute error

$$|P_2(1.4) - f(1.4)| = |15.2698 - 15.0446| = 0.2252.$$

3. Let $P_3(x)$ be the Lagrange Lagrange interpolating polynomial for the data $(0,0)$, $(0.5,y)$, $(1,3)$ and $(2,2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.
Sol. We have

$$\begin{aligned} P_3(x) &= \sum_{i=0}^{3} l_i(x) f(x_i) \\ &= \frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)}\times 0+\frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)}\times y \\ &\quad \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)}\times 3+\frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)}\times 2. \end{aligned}$$

Coefficient of x^3 in above expression is

$$\frac{y}{0.375} - \frac{3}{0.5} + \frac{2}{3} = 6,$$

which gives $y = 4.25$.

4. Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.

(a) $f(x) = e^{2x}\cos 3x,\ x_0 = 0,\ x_1 = 0.3,\ x_2 = 0.6,\ n = 2.$

(b) $f(x) = \sin(\ln x),\ x_0 = 2.0,\ x_1 = 2.4,\ x_2 = 2.6,\ n = 2.$

Sol.

(a) We have

$$f(x) = e^{2x}\cos 3x.$$

You can construct the interpolating polynomial as given in previous problems and the answer is

$$P_2(x) = -11.22x^2 - 2.924x + 1.$$

A bound for the error can be found as

$$\begin{aligned} |f(x) - P_2(x)| &= \left|\frac{f^3(\xi)}{6}(x-0)(x-0.3)(x-0.6)\right| \\ &= \left|\frac{-e^{2\xi}(9\sin(3\xi)+46\cos(3\xi))}{6}(x)(x-0.3)(x-0.6)\right| \\ &\le \left|\frac{-e^{2\times 0.6}(9\sin(3\times 0.6)+46)}{6}0.0103923\right| \\ &= 0.31493 \end{aligned}$$

(b) You can construct the interpolating polynomial as given in previous problems. The error is given by

$$\begin{aligned}
|f(x) - P_2(x)| &= \left|\frac{f^3(\xi)}{6}(x-2)(x-2.4)(x-2.6)\right| \\
&= \left|\frac{3\sin(\ln(\xi)) + \cos(\ln(\xi))}{6\xi^3}(x-2.0)(x-2.4)(x-2.6)\right| \\
&\leq \left|\frac{4}{6\ 2^3}0.016901\right| \\
&= 0.0014084.
\end{aligned}$$

5. Use the Lagrange interpolating polynomial of degree two or less and four-digit chopping arithmetic to approximate $\cos 0.750$ using the following values. Find an error bound for the approximation.

$$\cos 0.698 = 0.7661,\ \cos 0.733 = 0.7432,\ \cos 0.768 = 0.7193,\ \cos 0.803 = 0.6946.$$

The actual value of $\cos 0.750$ is 0.7317 (to four decimal places). Explain the discrepancy between the actual error and the error bound.

Sol. We have $x_0 = 0.698,\ x_1 = 0.733,\ x_2 = 0.768$, and $x_3 = 0.803$, respectively. The Lagrange interpolating polynomial takes the form

$$\begin{aligned}
P_3(x) &= \sum_{i=0}^{3} l_i(x) f(x_i) \\
&= \frac{(x-0.733)(x-0.768)(x-0.803)}{(0.698-0.733)(0.698-0.768)(0.698-0.803)} \times \cos(0.698) \\
&\quad + \frac{(x-0.698)(x-0.768)(x-0.803)}{(0.733-0.698)(0.733-0.768)(0.733-0.803)} \times \cos(0.733) \\
&\quad \frac{(x-0.698)(x-0.733)(x-0.803)}{(0.768-0.698)(0.768-0.733)(0.768-0.803)} \times \cos(0.768) \\
&\quad + \frac{(x-0.698)(x-0.733)(x-0.768)}{(0.803-0.0.698)(0.803-0.733)(0.803-0.768)} \times \cos(0.803).
\end{aligned}$$

Thus, by means of 4-digit chopping (say), the answer is

$$P_3(0.750) = 0.7313.$$

The actual value of $\cos 0.750$ is 0.7317 which differ from computed value by using 4-digit chopping and absolute error is 0.0004.
The corresponding error bound is

$$\begin{aligned}
|f(0.750) - P_3(0.750)| &= \left|\frac{f^4(\xi)}{4!}(0.750-0.698)(0.750-0.733)(0.750-0.768)(0.750-0.803)\right| \\
&\leq 4 \times 10^{-8}.
\end{aligned}$$

It is intuitive that the round-off error is responsible for the discrepancy.

6. If linear interpolation is used to interpolate the error function

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

show that the error of linear interpolation using data (x_0, f_0) and (x_1, f_1) cannot exceed $\dfrac{(x_1 - x_0)^2}{2\sqrt{2\pi e}}$.

Sol. The error in linear interpolation is

$$|f(x) - P_1(x)| = \left|\frac{1}{2} f^2(\xi)(x - x_0)(x - x_1)\right|.$$

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Thus
$$\max|f(x)-P_1(x)| \le \frac{1}{2}\max_{\xi\in[x_0,x_1]}|f''(\xi)| \max_{x_0\le x\le x_1}|(x-x_0)(x-x_1)|.$$
Now
$$\begin{aligned} f(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dt \\ f'(x) &= \frac{2}{\sqrt{\pi}}(e^{-x^2}-1) \\ f''(x) &= \frac{-4x}{\sqrt{\pi}}e^{-x^2}. \\ f'''(x) &= \frac{-4e^{-x^2}}{\sqrt{\pi}}[1-2x^2]. \end{aligned}$$
Now for the maximum value of $f''(x)$, we fit $f'''(x)=0$ which gives $x=\pm\frac{1}{\sqrt{2}}$.
Thus
$$\max|f^2(x)| = \frac{4}{\sqrt{2\pi}}e^{-1/2}.$$
Now we find the maxima of the product $|(x-x_0)(x-x_1)|$. For simplification, we write
$$\begin{aligned} g(x) &= (x-x_0)(x-x_1) \\ g'(x) &= 2x-x_0-x_1). \end{aligned}$$
$g'(x)=0$ gives $x=\frac{x_0+x_1}{2}$. Thus
$$\max|(x-x_0)(x-x_1)| = \frac{(x_1-x_0)^2}{4}.$$
So finally we have
$$\max|f(x)-P_1(x)| \le \frac{1}{2}\frac{4}{\sqrt{2\pi}}e^{-1/2}\frac{(x_1-x_0)^2}{4} = \frac{(x_1-x_0)^2}{2\sqrt{2\pi e}}.$$

7. Using Newton divided difference interpolation, construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
$$f(0.43) \text{ if } f(0)=1, f(0.25)=1.64872, f(0.5)=2.71828, f(0.75)=4.4816.$$

Sol.

x_i	$f(x_i)$	first d.d.	second d.d.	third d.d.
0	1	2.5949	3.3667	2.9111
0.25	1.64872	4.2782	5.5501	
0.5	2.71828	7.0533		
0.75	4.4816			

We would like to calculate $f(0.43)$. The value $x=0.43$ lies between 0.5 and 0.75. Therefore degree one polynomial is
$$P_1(x) = f(0.5)+(x-0.5)f[0.5,0.75] = 2.71828+(x-0.5)\times 7.0533$$
$$P_1(0.43) = 2.71828+(0.43-0.5)\times 7.0533 = 2.224549.$$
Similarly
$$P_2(0.43) = 1.64872+(0.43-0.25)\times 4.2782+(0.43-0.25)(0.43-0.5)\times 5.5501 = 2.34886474.$$
Also
$$\begin{aligned} P_3(x) &= 1+(x-0)\times 2.5949+(x)(x-0.25)\times 3.36672+(x)\,(x-0.25)\,(x-0.5)2.9111 \\ P_3(x) &= 1+(0.43-0)\times 2.5949+(0.43)(0.43-0.25)\times 3.36672+(0.43)\,(0.43-0.25)\,(0.43-0.5)2.9111 = 2.3606. \end{aligned}$$

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8. Show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

Sol.

x	$f(x)$	first d. d.	second d.d.	third d.d.	fourth d.d.	fifth d.d.
-1	1	3	2	-1	0	0
-2	4	7	-1	-1	0	
0	11	5	-4	-1		
1	16	-3	-7			
2	13	-17				
3	-4					

From the table it is clear that the interpolating polynomial has degree 3 as third differences are equal and therefore remaining all differences of higher-order will be zero.

9. Let $f(x) = e^x$, show that $f[x_0, x_1, \cdots, x_m] > 0$ for all values of m and all distinct equally spaced nodes $\{x_0 < x_1 < \cdots < x_m\}$.
Sol. As

$$f[x_0, x_1, \cdots, x_m] = \frac{f^m(\xi)}{m!} = \frac{e^\xi}{m!} > 0.$$

10. Denoting the interpolating polynomial $f(x)$ on the set of distinct points $x_0, x_1, \cdots, x_n$ by $\sum\limits_{k=0}^{n} l_k(x)f(x_k)$, find an expression for $\sum\limits_{k=0}^{n} l_k(0)x_k^{n+1}$.
Sol. For Lagrange interpolation, we have the following error formula

$$\begin{aligned} f(x) - P_n(x) &= \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!}\, f^{(n+1)}(\xi) \\ f(x) &= P_n(x) + \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!}\, f^{(n+1)}(\xi) \\ &= \sum_{k=0}^{n} l_k(x)f(x_k) + \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!}\, f^{(n+1)}(\xi). \end{aligned}$$

Let $f(x) = x^{n+1}$,

$$x^{n+1} = \sum_{k=0}^{n} l_k(x)x_k^{n+1} + \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!}\,(n+1)!$$

$$\implies x^{n+1} = \sum_{k=0}^{n} l_k(x)x_k^{n+1} + (x-x_0)\cdots(x-x_n).$$

Now put $x = 0$ to obtain

$$\sum_{k=0}^{n} l_k(0)x_k^{n+1} = (-1)^n x_0x_1\cdots x_n.$$

11. The following data are given for a polynomial $P(x)$ of unknown degree

x	0	1	2	3
$f(x)$	4	9	15	18

Determine the coefficient of x^3 in $P(x)$ if all fourth-order forward differences are 1.
Sol.

x	$f(x)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$
0	4	5	1	-4
1	9	6	-3	
2	15	3		
3	18			

As all fourth-order forward differences are 1, therefore polynomial will have degree four. Newton interpolation for equally spaced points is

$$P_n(x) = P_n(x_0 + sh) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \triangle^k f(x_0).$$

Here $x_0 = 0,\ h = 1,\ x = x_0 + sh = s$. We can write x for variable s.

$$P(x) \quad = \quad 4 + x \times 5 + \frac{x(x-1)}{2} \times 1 + \frac{x(x-1)(x-2)}{6} \times (-4) + \frac{x(x-1)(x-2)(x-3)}{24} \times 1.$$

From the above expression the coefficient of x^3 is $-11/12$.

12. Verify that the polynomials $P(x) = 5x^3 - 27x^2 + 45x - 21$, $Q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$ interpolate the data

x	1	2	3	4
y	2	1	6	47

and explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

Sol.

x	$f(x)$	$P(x)$	$Q(x)$
1	2	2	2
2	1	1	1
3	6	6	6
4	47	47	47

Both the polynomials are interpolating polynomials. Their degree differs and thus the uniqueness part is not violated. In other words, from given four points, you will get three degree polynomial which is $P(x)$ and $Q(x)$ is just another polynomial passing through all given data points.

13. Let $i_0, i_1, \cdots, i_n$ be a rearrangement of the integers $0, 1, \cdots, n$. Show that

$$f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}] = f[x_0, x_1, \cdots, x_n].$$

Sol. Let $P(x)$ and $Q(x)$ be two polynomials, such that $P(x)$ interpolates f at $\{x_0, x_1, \cdots, x_n\}$ and $Q(x)$ interpolates f at $\{x_{i_0}, x_{i_1}, \cdots, x_{i_n}\}$. Therefore

$$P(x) = f(x_0) + \sum_{k=0}^{n} f[x_0, x_1, \cdots, x_k](x - x_0) \cdots (x - x_{k-1})$$

$$= f[x_0, x_1, \cdots, x_n]x^n + \text{lower order terms}$$

$$Q(x) = f(x_{i_0}) + \sum_{k=0}^{n} f[x_{i_0}, \cdots, x_{i_k}](x - x_{i_0}) \cdots (x - x_{i_{k-1}})$$

$$= f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}]x^n + \text{lower order terms}$$

Since the polynomial interpolating the same nodes is unique, therefore

$$f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}] = f[x_0, x_1, \cdots, x_n].$$

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14. Let $f(x) = 1/(1+x)$ and let $x_0 = 0,\ x_1 = 1,\ x_2 = 2$. Calculate the divided differences $f[x_0, x_1]$ and $f[x_0, x_1, x_2]$. Using these divided differences, give the quadratic polynomial $P_2(x)$ that interpolates $f(x)$ at the given node points $\{x_0, x_1, x_2\}$. Graph the error $f(x) - P_2(x)$ on the interval $[0, 2]$.

Sol.

x	$f(x)$	first d.d.	second d.d.
0	1	$f[x_0, x_1] = -0.5$	$f[x_0, x_1, x_2] = 0.1667$
1	0.5	$f[x_1, x_2] - 0.1667$	
2	0.3333		

From the table we obtain

$$P_2(x) = 1 - 0.5x + 0.1667(x)(x-1)$$

which is a second degree polynomial. Now

$$f(x) - P_2(x) = \frac{f^3(\xi)}{3!} x(x-1)(x-2).$$

Calculate third derivative of f and make a (rough) plot.

15. Construct the interpolating polynomial that fits the following data using Newton forward and backward

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	-1.5	-1.27	-0.98	-0.63	-0.22	0.25

difference interpolation. Hence find the values of $f(x)$ at $x = 0.15$ and 0.45.

Sol. Refer to Example 9 from Notes.

x	$f(x)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$	$\Delta^5 f(x_i)$
0	-1.5	0.23	0.06	0	0	0
0.1	-1.27	0.29	0.06	0	0	
0.2	-0.98	0.35	0.06	0		
0.3	-0.63	0.41	0.06			
0.4	-0.22	0.47				
0.5	0.25					

Use forward difference interpolation to calculate the first value and use backward difference interpolation to calculate the second value which lies near the end of table.

16. For a function f, the forward-divided differences are given by

$x_0 = 0.0$	$f[x_0]$		
$x_1 = 0.4$	$f[x_1]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2] = \dfrac{50}{7}$
$x_1 = 0.4$	$f[x_2] = 6$	$f[x_1, x_2] = 10$	

Determine the missing entries in the table.

Sol.

$$\begin{aligned} f[x_1, x_2] &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ 10 &= \frac{6 - f(x_1)}{0.7 - 0.4} \\ f(x_1) &= 3. \end{aligned}$$

Also

$$
\begin{aligned}
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\
\frac{50}{7} &= \frac{10 - f[x_0, x_1]}{0.7 - 0.0} \\
f[x_0, x_1] &= 5.
\end{aligned}
$$

Further

$$
\begin{aligned}
f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
5 &= \frac{3 - f(x_0)}{0.4 - 0.0} \\
f(x_0) &= 1.
\end{aligned}
$$

17. A fourth-degree polynomial $P(x)$ satisfies $\Delta^4 P(0) = 24$, $\Delta^3 P(0) = 6$, and $\Delta^2 P(0) = 0$, where $\Delta P(x) = P(x+1) - P(x)$. Compute $\Delta^2 P(10)$.

Sol. If $P(x)$ is a fourth-degree polynomial, it can be easily proved that $\Delta P(x)$ is a third-degree polynomial. Furthermore, we can prove that $\Delta^2 P(x)$ is a second-degree polynomial. Let

$$Q(x) = \Delta^2 P(x) = ax^2 + bx + c,$$

we have

$$\Delta^3 P(x) = \Delta Q(x) = Q(x+1) - Q(x) = 2ax + a + b,$$
$$\Delta^4 P(x) = \Delta(\Delta Q(x)) = 2a.$$

Using the given values, we obtain

$$c = 0,$$
$$a + b = 6,$$
$$2a = 24.$$

Therefore $c = 0$, $b = -6$, $a = 12$, and $\Delta^2 P(10) = Q(10) = 1140$.

18. EXTRA : Using the following table for $\tan x$, approximate its value at 0.71 using Newton's interpolation.

x_i	0.70	72	0.74	0.76	0.78
$\tan x_i$	0.84229	0.87707	0.91309	0.95045	0.98926

Sol. As the point $x = 0.71$ lies in the beginning, we will use Newton's forward interpolation. The forward difference table is:

x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$
0.70	0.84229				
0.72	0.87707	0.03478			
0.74	0.91309	0.03602	0.00124		
0.76	0.95045	0.03736	0.00134	0.0001	
0.78	0.98926	0.03881	0.00145	0.00011	0.00001

Here $x_0 = 0.70$, $h = 0.02$, $x = 0.71 = x_0 + sh$ gives $s = 0.5$.
The Newton's forward difference polynomial is given by

$$
\begin{aligned}
P_4(x) &= P_4(x_0 + sh) \\
&= f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f(x_0).
\end{aligned}
$$

Substituting the values from table (first entries of each column starting from second), we obtain

$$P_3(0.71) = \tan(0.71) = 0.8596.$$

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19. Use the method of least squares to fit the linear and quadratic polynomial to the following data.

x	-2	-1	0	1	2
$f(x)$	15	1	1	3	19

Sol. The normal equations for fitting a straight line $y = a + bx$ are

$$\begin{aligned}\sum_{i=1}^{5} f(x_i) &= 5a + b\sum_{i=1}^{5} x_i \\ \sum_{i=1}^{5} x_i\, f(x_i) &= a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} x_i^2.\end{aligned}$$

After simplification we get as

$$a = 7.8, \quad b = 1,$$

The required approximation is $y = 7.8 + x$.

20. By the method of least square fit a curve of the form $y = ax^b$ to the following data.

x	2	3	4	5
y	27.8	62.1	110	161

Sol. The curve $y = ax^b$ takes the form $Y = A + Bx$ after taking log, where $Y = \log y, A = \log a$. The normal equations for fitting a straight line $Y = A + bx$ are

$$\begin{aligned}\sum_{i=1}^{4} Y_i &= 4A + b\sum_{i=1}^{4} X_i \\ \sum_{i=1}^{4} X_i\, Y_i &= a\sum_{i=1}^{4} X_i + b\sum_{i=1}^{4} X_i^2.\end{aligned}$$

After simplification we get as

$$A = 1.9988 \Rightarrow a = 0.6925, \quad b = 1.9302,$$

The required approximation is $y = 0.6925\,(x)^{1.9302}$.

21. Use the method of least squares to fit a curve $y = c_0/x + c_1\sqrt{x}$ to the following data.

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

Sol. We obtain the normal equations

$$\sum_{i=1}^{6} \frac{y_i}{x_i} = c_0 \sum_{i=1}^{6} \frac{1}{x_i^2} + c_1 \sum_{i=1}^{6} \frac{1}{\sqrt{x_i}}$$

$$\sum_{i=1}^{6} y_i\sqrt{x_i} = c_0 \sum_{i=1}^{6} \frac{1}{\sqrt{x_i}} + c_1 \sum_{i=1}^{6} x_i.$$

Using given data, we obtain

$$c_0 = 1.9733, \quad c_1 = 3.2818.$$

Hence the required least square approximation is

$$y = \frac{1.9733}{x} + 3.2878\sqrt{x}.$$

CONTINUED

22. Hooke's law states that when a force is applied to a spring constructed of uniform material, the length of the spring is a linear function of that force written as

$$F(l) = k(l - E).$$

The function F is the force required to stretch the spring l units, where the constant $E = 5.3$ is the length of the unstretched spring in inches and k is the spring constant.

(a) Suppose measurements are made of the length l, in inches, for applied weights $F(l)$, in pounds, as given in the following table.

$F(l)$	l
2	7.0
4	9.4
6	12.3

Find the least squares approximation for k.

(b) Additional measurements are made, giving more data:

$F(l)$	l
3	8.3
5	11.3
8	14.4
10	15.9

Compute the new least squares approximation for k. Which of (a) or (b) best fits the total experimental data?

Sol.

(a) A linear function $F(l) = k(l - E)$ takes the form $y = b\,x + a$, where $b = k$, and $a = -kE$
The normal equations for fitting a linear function $y = a + bx$ are

$$\begin{aligned} \sum_{i=1}^{3} y_i &= 3a + b\sum_{i=1}^{3} x_i \\ \sum_{i=1}^{3} x_i\, y_i &= a\sum_{i=1}^{3} x_i + b\sum_{i=1}^{3} x_i^2. \end{aligned}$$

After simplification we get as

$$b = 0.7525 \Rightarrow k = 0.7525, \quad a = -3.1988, \Rightarrow -kE = -3.1988$$

The required approximation is

$$F(l) = 0.7525(l - 5.3).$$

(b) As above.