Numerical Analysis

Solution of Some Exercises : Chapter 1[1]
Floating Point Arithmetic and Errors

1. Compute the absolute error and relative error in approximations of x by x^*.

(a) $x = \pi,\ x^* = 22/7$.
(b) $x = \sqrt{2},\ x^* = 1.414$.
(c) $x = 8!,\ x^* = 39900$.

Sol. For calculations, you can use either calculator or Octave or Matlab.

(a) Absolute error: $|x - x^*| = |\pi - 22/7| = 0.001264489$.
Relative error: $\dfrac{|x - x^*|}{|x|} = \dfrac{|\pi - 22/7|}{\pi} = 0.000402499$.

(b) Absolute error: $|x - x^*| = |\sqrt{2} - 1.414| = 0.000213562$.
Relative error: $\dfrac{|x - x^*|}{|x|} = \dfrac{|\sqrt{2} - 1.414|}{\sqrt{2}} = 0.000151011$.

(c) Absolute error: $|x - x^*| = |8! - 39900| = 420$.
Relative error: $\dfrac{|x - x^*|}{|x|} = \dfrac{|8! - 39900|}{8!} = 0.010416667$.

2. Find the largest interval in which x^* must lie to approximate x with relative error at most 10^{-4} for each value of x.

(a) π.
(b) e.
(c) $\sqrt{3}$.
(d) $\sqrt[3]{7}$.

Sol.

(a) The relative error is $E_r = \dfrac{|x - x^*|}{|x|}$, where $x = \pi$. Now

$$\begin{aligned}
\frac{|\pi - x^*|}{|\pi|} &\leq 10^{-4} \\
\therefore\ |\pi - x^*| &\leq \pi \cdot 10^{-4} \\
\implies -\pi \cdot 10^{-4} &\leq \pi - x^* \leq \pi \cdot 10^{-4} \\
\implies -\pi - \pi \cdot 10^{-4} &\leq -x^* \leq -\pi + \pi \cdot 10^{-4} \\
\implies \pi - \pi \cdot 10^{-4} &\leq x^* \leq \pi + \pi \cdot 10^{-4}.
\end{aligned}$$

The interval can be written as $[3.141278494, 3.141906813]$.

(b) Similarly

$$e - e \cdot 10^{-4} \leq x^* \leq e + e \cdot 10^{-4}.$$

The interval can be written as $[2.71801, 2.718553657]$.

(c)

$$\sqrt{3} - \sqrt{3} \cdot 10^{-4} \leq x^* \leq \sqrt{3} + \sqrt{3} \cdot 10^{-4}.$$

The interval can be written as $[1.731877602, 1.732224013]$.

(d)

$$\sqrt[3]{7} - \sqrt[3]{7} \cdot 10^{-4} \leq x^* \leq \sqrt[3]{7} + \sqrt[3]{7} \cdot 10^{-4}.$$

The interval can be written as $[1.91273989, 1.913122476]$.

[1] Lecture Notes of Dr. Paramjeet Singh

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3. A rectangular parallelepiped has sides of length 3 cm, 4 cm, and 5 cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

Sol. The lengths are measured to measured to the nearest centimeter means that if one side is $a = 3$, then $a \in [2.5, 3.5)$. Similarly other two sides $b = 4$, $b \in [3.5, 4.5)$ and $c = 5$, $c \in [4.5, 5.5)$.
Using smallest value for each length, the lower bound for the volume is $V_{\min} = 2.5 \times 3.5 \times 4.5 = 39.375$.
Using largest values, the upper bound for volume is $V_{\max} = 3.5 \times 4.5 \times 5.5 = 86.625$.
Similarly using minimum values, the lower bound for the surface area is $S_{\min} = 2(2.5 \times 3.5 + 3.5 \times 4.5 + 2.5 \times 4.5) = 71.5$.
The upper bound is for the surface area is $S_{\max} = 2(3.5 \times 4.5 + 4.5 \times 5.5 + 5.5 \times 3.5) = 119.5$.

4. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

(a) $\sqrt{3} + (\sqrt{5} + \sqrt{7})$.

(b) $(121 - 0.327) - 119$.

(c) $-10\pi + 6e - \dfrac{3}{62}$.

(d) $\dfrac{\pi - 22/7}{1/17}$.

Sol.

(a) True value (by taking six digits): $x = \sqrt{3} + (\sqrt{5} + \sqrt{7}) = 6.61387$.
Approximate value (by three-digit rounding) $x^* = 1.73 + (2.24 + 2.65) = 1.73 + 4.89 = 6.62$.
Absolute error: $|x - x^*| = 0.00613$.
Relative error: $\dfrac{|x - x^*|}{|x|} = 0.0009268$.

(b) True value (by taking six digits): $x = (121 - 0.327) - 119 = 120.673119 = 1.673$.
Approximate value (by three-digit rounding) $x^* = (121 - 0.327) - 119 = 121 - 119 = 2$.
Absolute error: $|x - x^*| = 0.327$.
Relative error: $\dfrac{|x - x^*|}{|x|} = 0.1955$.

(c) True value (by taking six digits): $x = -10\pi + 6e - \dfrac{3}{62} = -31.4159 + 16.3097 - 0.048387 = -15.1546$.
Approximate value (by three-digit rounding) $x^* = -31.4 + 16.3 - 0.048 = -15.1$.
Absolute error: $|x - x^*| = 0.0546$.
Relative error: $\dfrac{|x - x^*|}{|x|} = 0.0036029$.

(d) True value (by taking six digits): $x = \dfrac{\pi - 22/7}{1/17} = \dfrac{3.14159 - 3.14286}{0.058824} = -0.021590$.
Approximate value (by three-digit rounding) $x^* = \dfrac{3.14 - 3.14}{0.059} = 0$.
Absolute error: $|x - x^*| = 0.02159$.
Relative error: $\dfrac{|x - x^*|}{|x|} = 1$.

5. Use four-digit rounding arithmetic and the formula to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

$$\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0.$$

Sol. The quadratic formula states that the roots of $ax^2 + bx + c = 0$ are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Using the above formula, the roots of given eq. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$ are approximately (using long format)

$$x_1 = 92.24457962731231, \quad x_2 = 0.00542037268770.$$

We use four-digit rounding arithmetic to find approximations to the roots. We write the approximations of root as x_1^* and x_2^*. These approximations are given by

$$\begin{aligned} x_{1,2}^* &= \frac{\frac{123}{4} \pm \sqrt{(-\frac{123}{4})^2 - 4 \cdot \frac{1}{3} \cdot \frac{1}{6}}}{2 \cdot \frac{1}{3}} \\ &= \frac{30.75 \pm \sqrt{30.75^2 - 4 \cdot 0.3333 \cdot 0.1667}}{2 \times 0.3333} \\ &= \frac{30.75 \pm \sqrt{945.6 - 1.3333 \cdot 0.1667}}{0.6666} \\ &= \frac{30.75 \pm \sqrt{945.6 - 0.2222}}{0.6666} \\ &= \frac{30.75 \pm \sqrt{945.4}}{0.6666} \\ &= \frac{30.75 \pm 30.75}{0.6666} \end{aligned}$$

Therefore we find the first root by taking plus sign:

$$x_1^* = \frac{30.75 + 30.75}{0.6666} = \frac{61.50}{0.6666} = 92.26.$$

This root has the absolute error

$$|x_1 - x_1^*| = 0.015420373,$$

and relative error

$$|x_1 - x_1^*|/|x_1| = \frac{92.24457962731231 - 92.26}{92.24457962731231} = 0.000167168.$$

We find the second root by taking minus sign

$$x_2^* = \frac{30.75 - 30.75}{0.6666} = 0.$$

This root has the following absolute error

$$|x_2 - x_2^*| = 0.00542037268770 - 0 = 0.00542037268770,$$

and relative error

$$\frac{|x_2 - x_2^*|}{|x_2|} = 1.0.$$

We obtained a very large relative error, since the calculation of second root involved the subtraction of nearly equal numbers. In order to get a more accurate approximation to x_2, we need to use an alternate quadratic formula by rationalize the expression to calculate x_2^* and approximation is given by

$$x_2^* = \frac{-2c}{b - \sqrt{b^2 - 4ac}} = 0.005420,$$

which has the relative error (very small now)

$$\frac{|x_2 - x_2^*|}{|x_2|} = \frac{|0.00542037268770 - 0.005420|}{0.00542037268770} = 0.00006876.$$

6. Find the root of smallest magnitude of the equation $x^2 - 1000x + 25 = 0$ using quadratic formula. Work in floating-point arithmetic using a four-decimal place mantissa.

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Sol. Given that,

$$x^2 - 1000x + 25 = 0$$
$$\implies x = \frac{1000 \pm \sqrt{10^6 - 10^2}}{2}$$

Now in four digit mantissa

$$10^6 = 0.1000e7 \ \& \ 10^2 = 0.1000e3$$

Therefore

$$\sqrt{10^6 - 10^2} = 0.1000e4.$$

Hence roots are given by

$$x_1 = \left(\frac{0.1000e4 + 0.1000e4}{2}\right) = 0.1000e4$$

and

$$x_2 = \left(\frac{0.1000e4 - 0.1000e4}{2}\right) = 0.0000e4.$$

One of the roots becomes zero due to the limited precision allowed in computation. In this equation since b^2 is much larger than $4ac$. Hence b and $\sqrt{b^2 - 4ac}$ become two equal numbers. Calculation of x_2 involves the subtraction of nearly two equal numbers which will cause serious loss of significant figures.
To obtain a more accurate 4-digit rounding approximation for x_2, we change the formulation by rationalizing the numerator or we know that in quadratic equation $ax^2 + bx + c = 0$, the product of the roots is given by c/a, therefore the smaller root may be obtained by dividing (c/a) by the largest root.
Therefore first root is given by 0.1000e4 and second root is given as

$$\frac{25}{0.1000e4} = \frac{0.2500e2}{0.1000e4} = 0.2500e - 1.$$

7. The derivative of $f(x) = \dfrac{1}{(1-3x^2)}$ is given by $\dfrac{6x}{(1-3x^2)^2}$. Do you expect to have difficulties evaluating this derivative at $x = 0.577$? Try it using 3- and 4-digit arithmetic with chopping.
Sol. The true value of the derivative is given by

$$f'(0.577) = \frac{6 \cdot 0.577}{(1 - 3(0.577)^2)^2} = 2352910.7926.$$

Using 3-digit chopping:

$$\begin{aligned}
6x &= 6 \times 0.577 = 3.46 \\
3x^2 &= 3(0.577)^2 = 3 \times 0.332 = 0.996 \\
1 - 3x^2 &= 0.004 \\
(1-3x^2)^2 &= 0.000016 \\
\frac{6x}{(1-3x^2)^2} &= 216000.
\end{aligned}$$

Relative error $E_r = 90.81\%$.

Using 4-digit chopping:

$$\begin{aligned}
6x &= 6 \times 0.577 = 3.462 \\
3x^2 &= 3(0.577)^2 = 3 \times 0.3329 = 0.9987 \\
1 - 3x^2 &= 0.0013 \\
(1-3x^2)^2 &= 0.00000169 \\
\frac{6x}{(1-3x^2)^2} &= 2048000
\end{aligned}$$

Relative error $E_r = 12.95\%$.

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8. Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to find the x-intercept of the line:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}, \text{ and } x = x_0 - \frac{(x_1 - x_0) y_0}{y_1 - y_0}.$$

(a) Show that both formulas are algebraically correct.

(b) Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better and why?

Sol.

(a) We should be a bit careful here to avoid dividing by 0. It is potentially unsafe to write the equation of the line is

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0},$$

because potentially $x_0 = x_1$.
However, we are told that $y_0 \neq y_1$, so we can instead write the equation of the line as

$$\frac{x - x_0}{y - y_0} = \frac{x_1 - x_0}{y_1 - y_0}.$$

We can cross-multiply and write this instead as

$$x - x_0 = (y - y_0) \left(\frac{x_1 - x_0}{y_1 - y_0} \right).$$

The x-intercept is the point on the line at which $y = 0$, so we can substitute $y = 0$ into this equation and get

$$x = x_0 - \left(\frac{x_1 - x_0}{y_1 - y_0} \right) y_0,$$

which is the given formula.

We can get the second form by simplifying the right side.

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}.$$

(b) The first formula gives the answer -0.00658, while the second formula gives the answer -0.0100.
In this case, the second formula is better. The first one involved subtracting $x_0 y_1 - x_1 y_0$. Because $x_0 y_1 = 6.24$ and $x_1 y_0 = 6.25$, the result of the subtraction has only one significant digit.
We can check this by working to 10 significant digits. In that case, the first formula gives -0.0115789474 and the second gives -0.0115789470. Surely the answer is closer to -0.01 than to -0.00658.

9. Verify that the functions $f(x)$ and $g(x)$ are identical functions.

$$f(x) = 1 - \sin x, \quad g(x) = \frac{\cos^2 x}{1 + \sin x}.$$

(a) Which function should be used for computations when x is near $\pi/2$? Why?

(b) Which function should be used for computations when x is near $3\pi/2$? Why?

Sol.

$$f(x) = (1 - \sin x) \times \frac{1 + \sin x}{1 + \sin x} = \frac{\cos^2 x}{1 + \sin x}.$$

Thus both the functions are same.

(a)

$$\begin{aligned} f(x) &= 1 - \sin x \\ f'(x) &= 1 - \cos x \\ \kappa(1.001\pi/2) &= \frac{1.001\pi f'(1.001\pi/2)}{f(1.001\pi/2)} \\ &= 1.2765 \times 10^6. \end{aligned}$$

Thus function evaluation will become ill-conditioned near $\pi/2$. We prefer second formula which does not involve cancellation of two similar digits.

(b) Similarly when x is near $3\pi/2$, we use first formula as $1 + \sin x \approx 0$.

10. Consider the identity

$$\int_0^x \sin(xt)dt = \frac{1 - \cos(x^2)}{x}.$$

Explain the difficulty in using the right-hand fraction to evaluate this expression when x is close to zero. Give a way to avoid this problem and be as precise as possible.

Sol. There is loss of significance at $x \approx 2k\pi$, k integer, because then $\cos x \approx 1$.
We can avoid this problem at $x \approx 0$ by the alternative expression

$$\frac{1 - \cos x^2}{x} = \frac{2\sin^2 \frac{x^2}{2}}{x}.$$

For $x \approx 2k\pi$, $k \neq 0$, integer, this only moves the problem (in a less severe form) into the evaluation of $\sin x$, because this is achieved by mapping the argument into a fixed interval of length 2π by the identity

$$\sin x = \sin(x - 2k\pi).$$

Alternatively we can rewrite the expression as

$$\begin{aligned} \frac{1 - \cos x^2}{x} &= \frac{(1 - \cos x^2)(1 + \cos x^2)}{x(1 + \cos x^2)} \\ &= \frac{1 - \cos^2 x^2}{x(1 + \cos x^2)} \\ &= \frac{\sin^2 x^2}{x(1 + \cos x^2)}. \end{aligned}$$

Another Example: How to write $\sin x - \tan x$ as $x \to 0$.

$$\begin{aligned} \sin x - \tan x &= \sin x - \frac{\sin x}{\cos x} \\ &= \tan x(\cos x - 1) \\ &= \tan x \left(-2\sin^2 \frac{x}{2}\right). \end{aligned}$$

11. Assume 3-digit mantissa with rounding

(a) Evaluate $y = x^3 - 3x^2 + 4x + 0.21 \quad$ for $x = 2.73$.

(b) Evaluate $y = [(x - 3)x + 4]x + 0.21 \quad$ for $x = 2.73$.

Compare and discuss the errors obtained in part (a) and (b).

Sol. Exact value of the expression $y = 2.73^3 - 3 \cdot 2.73^2 + 4 \cdot 2.73 + 0.21 = 9.117717$.

(a)

$$\begin{aligned} y(2.73) &= 2.73^3 - 3 \cdot 2.73^2 + 4 \cdot 2.73 + 0.21 \\ &= 7.45 \cdot 2.73 - 3 \cdot 7.45 + 10.9 + 0.21 \\ &= 20.3 - 22.4 + 10.9 + 0.21 = 9.01. \end{aligned}$$

Relative error$= \dfrac{|9.117717 - 9.01|}{9.117717} \times 100 = 1.18\%.$

(b)

$$\begin{aligned} y(2.73) &= [(2.73 - 3)2.73 + 4]2.73 + 0.21 \\ &= [(-0.27)(2.73) + 4]2.73 + 0.21 \\ &= [-0.737 + 4]2.73 + 0.21 \\ &= 3.26 \times 2.73 + 0.21 = 8.90 + 0.21 = 9.11. \end{aligned}$$

Relative error$= \dfrac{|9.117717 - 9.11|}{9.117717} \times 100 = 0.0846\%.$
In part (b) nesting is done.

12. How many multiplications and additions are required to determine a sum of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_i \, b_j \; ?$$

Modify the sum to an equivalent form that reduces the number of computations.
Sol.

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{i} a_i \, b_j \;\; &= \;\; a_1 b_1 + \\ & \quad\;\; a_2 b_1 + a_2 b_2 + \\ & \quad\;\; a_3 b_1 + a_3 b_2 + a_3 b_3 + \\ & \quad\;\; \cdots\cdots \\ & \quad\;\; a_n b_1 + a_n b_2 + \cdots + a_n b_n. \end{aligned}$$

For each i there are i multiplications since j ranges from 1 to i. There are exactly n iterations so there are a total of $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$ multiplications.
There are $i - 1$ additions for each i so there are a total of $\sum_{i=1}^{n} (i-1) = \dfrac{n(n-1)}{2}$ additions at each i. Then we must add these sums together for $n - 1$ more additions.
Therefore, there are $\dfrac{n(n+1)}{2} + \dfrac{n(n-1)}{2} + n - 1 = n^2 + n - 1$ total computations.
We can modify the sum as follows:

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{i} a_i \, b_j &= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j \right) \\ &= a_1 b_1 + a_2(b_1 + b_2) + a_3(b_1 + b_2 + b_3) + \cdots + a_n(b_1 + b_2 + \cdots + b_n). \end{aligned}$$

Thus the number of multiplications required is reduced to n, while the number of additions hasn't changed.

13. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial, and let x_0 be given. Construct an algorithm to evaluate $P(x_0)$ using nested multiplication.
Sol. We write

$$P(x) = ((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_3)x + a_2)x + a_1 x + a_0.$$

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This form requires n multiplications. The following algorithm computes $P(x_0)$ using the nested arithmetic.
Input: $n, a_0, a_1, \cdots, a_n, x_0$
Output: $y = P(x_0)$
Step 1: Set $y = a_n$
Step 2: For $i = n-1, n-2, \cdots, 0$
set $y = x_0 y + a_i$.
Step 3: Output (y);
STOP.

14. Construct an algorithm that has as input an integer $n \geq 1$, numbers $x_0, x_1, \cdots, x_n$, and a number x and that produces as output the product $(x - x_0)(x - x_1) \cdots (x_0 - x_n)$.

Sol. Input : $n, x_0, x_1, \cdots, x_n, x$.
Output: $\prod\limits_{i=0}^{n} (x - x_i)$.
Step 1: Set $p = (x - x_0)$.
$i = 1$.
Step 2: While $i \leq n$ set
$p = p * (x - x_i)$;
$i = i + 1$.
Step 3: Output p.
STOP.

15. Consider the stability (by calculating the condition number) of $\sqrt{1+x} - 1$ when x is near 0. Rewrite the expression to rid it of subtractive cancellation.

Sol. Let $x = 0.0001$

$$\begin{aligned} x_0 &= 0.0001 \\ x_1 &= x_0 + 1 \\ x_2 &= \sqrt{x_1} \\ x_3 &= 1 \\ x_4 &= x_2 - x_3 \end{aligned}$$

Now to check the effect of x_3 on x_2, we consider the function $f_3(x_3) = x_2 - x_3$

$$\kappa(x_3) = \left| \frac{x_3 f'(x_3)}{f(x_3)} \right| = \left| \frac{x_3}{x_2 - x_3} \right| \approx 20000.$$

We obtain a very larger condition number, which shows that the last step is not stable. Now we modify the above algorithm

$$\begin{aligned} f(x) &= \frac{\sqrt{1+x} - 1 \times \sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \frac{x}{\sqrt{x+1} + 1} \\ x_0 &= 0.0001 \\ x_1 &= x_0 + 1 \\ x_2 &= \sqrt{x_1} \\ x_3 &= x_2 + 1 \\ x_4 &= x_0 / x_3 \end{aligned}$$

Consider the function $f_3(x_3) = \dfrac{x_0}{x_3}$, $\kappa = \left| \dfrac{x_3 f'(x_3)}{f(x_3)} \right|$, which is a small number. Hence the above algorithm is stable.

16. Show that the computation of

$$f(x) = \frac{e^x - 1}{x}$$

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is unstable for small value of x. Rewrite the expression to make it stable.

Sol. Consider

$$f(x) = \frac{e^x - 1}{x}$$

so that there is potential loss of significance when x is small. One possible algorithm is

$$\begin{aligned} x_0 : &= x = 0.0001 \\ x_1 : &= e^{x_0} \\ x_2 : &= x_1 - 1 \\ x_3 : &= \frac{x_2}{x_0}. \end{aligned}$$

We calculate the condition numbers of each step. Condition number of second step is

$$\kappa(x_0) = \left| \frac{x_0 e^{x_0}}{e^{x_0}} \right| = 1.$$

Condition number of third step is

$$\kappa(x_1) = \left| \frac{x_1 \times 1}{x_1 - 1} \right| \to \infty \text{ as } x_0 \text{ is small.}$$

To find an alternative algorithm, we write the Taylor's expansion of e^x about $x = 0$. We have

$$f(x) = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \cdots$$

In this expression now there is no subtraction of nearly two equal numbers, neither division by a small number. Thus this form is stable.

17. Suppose that a function $f(x) = \ln(x+1) - \ln(x)$, is computed by the following algorithm for large values of x using six digit rounding arithmetic

$$\begin{aligned} x_0 : &= x = 12345 \\ x_1 : &= x_0 + 1 \\ x_2 : &= \ln x_1 \\ x_3 : &= \ln x_0 \\ f(x_3) := x_4 : &= x_2 - x_3. \end{aligned}$$

By considering the condition $\kappa(x_3)$ of the subproblem of evaluating the function, show that such a function evaluation is not stable. Also propose the modification of function evaluation so that algorithm will become stable.

Sol. As discussed in previous question, by taking $x = 0.0001$ and writing algorithm, we obtain a very large condition number which makes our algorithm unstable. Therefore we need to modify the pattern.

Loss of significance occurs as $x \approx y$, $\ln x - \ln y = \ln \dfrac{x}{y}$.

There is still a loss of significance, since $\ln 1 = 0$, but this is the best we can do if the input is x and y; to avoid loss of significance completely, we would have to ask the user to give, say, y and $x - y$ as data; then according to our problem

$$\ln(1+x) - \ln x = \ln \frac{1+x}{x} = \ln\left(1 + \frac{1}{x}\right) = \frac{1}{x} + \cdots$$