Course: UMA 035 (Optimization Techniques)

Instructor: Dr. Amit Kumar,

Associate Professor,

School of Mathematics,

TIET, Patiala

Email: amitkumar@thapar.edu

Mob: 9888500451

State and prove Weak Duality Theorem

Statement

The value of the objective function of a minimization LPP corresponding to any of its feasible solution will always be greater than or equal to the value of the objective function of the dual problem corresponding to any feasible solution of the dual problem.

Proof

Let the LPP be

Minimize (CX)

Subject

AX>=b

X>=0.

Then, its dual will be

Maximize (b^TY)

Subject to

$A^TY \le C^T$

Y>=0

Let X_0 be a feasible solution of the considered LPP and Y_0 be a feasible solution of the dual problem.

Now we need to prove that $CX_0 >= b^T Y_0$

Since, X_0 and Y_0 are the feasible solutions of the considered LPP and its dual respectively. So, these will satisfy the constraints i.e.,

$AX_0 >= b$ (1)

$X_0 >= 0$ (2)

and

$A^T Y_0 <= C^T$ (3)

$Y_0 >= 0$ (4)

(Remark: We need b^T to prove the result)

Taking transpose of both sides of (1)

$(AX_0)^T >= (b)^T$

$(X_0)^T A^T >= b^T$ (5) (since $(AB)^T = B^T A^T$)

(Remark: We need $b^T Y_0$ to prove the result)

Multiplying both sides of (5) with Y_0

$(X_0)^T A^T Y_0 >= b^T Y_0$ (6)

(Remark: We have A^TY_0 in equation (3))

Multiplying both sides of (3) with $(X_0)^T$

$(X_0)^T A^TY_0<=(X_0)^T C^T$ (7)

From (6)

$b^T Y_0<=(X_0)^TA^T Y_0$

From (7)

$(X_0)^T A^TY_0<=(X_0)^T C^T$

Combining both

$b^T Y_0<=(X_0)^TA^T Y_0<=(X_0)^T C^T$

$b^T Y_0<=(X_0)^T C^T$

$b^T Y_0<=(CX_0)^T$ (8) (since $(AB)^T=B^TA^T$)

Since, the order of the matrix C is 1*n and the order of the matrix X_0 is n*1. So, order of CX_0 will be 1*1.

Transpose of a matrix of order 1*1 will be the matrix itself i.e., $(CX_0)^T= CX_0$

Hence, from (8),$b^T Y_0<=CX_0$

Optimal solution of a LPP using Complementary Slackness Theorem

Example

Minimize $(x_1 - 2x_2 + x_3)$

Subject to

$x_1 + 2x_2 - 2x_3 <= 4$

$x_1 - x_3 <= 3$

$2x_1 - x_2 + 2x_3 <= 2$

$x_1 >= 0,\ x_2 >= 0,\ x_3 >= 0.$

(i) **Write the dual of the given LPP.**

(ii) **Solve the given LPP.**

(iii) **Using the optimal solution of the given LPP and using Complementary Slackness Theorem, find the optimal solution of the dual problem.**

Solution:

(i)

Minimize $(x_1 - 2x_2 + x_3)$

Subject to

$-x_1 - 2x_2 + 2x_3 >= -4$

$-x_1 + x_3 >= -3$

$-2x_1 + x_2 - 2x_3 >= -2$

$x_1 >= 0$, $x_2 >= 0$, $x_3 >= 0$.

Dual problem

Maximize $(-4y_1 - 3y_2 - 2y_3)$

Subject to

$-y_1 - 2y_2 - y_3 <= 1$

$-2y_1 + y_3 <= -2$

$2y_1 + y_2 - 2y_3 <= 1$

$y_1 >= 0$,

$y_2 >= 0$,

$y_3 >= 0$.

(ii)

Maximize $(-x_1+2x_2-x_3)$

Subject to

$x_1+2x_2-2x_3+S_1=4$

$x_1-x_3+S_2=3$

$2x_1-x_2+2x_3+S_3=2$

$x_1>=0, x_2>=0, x_3>=0, S_1>=0, S_2>=0, S_3>=0.$

Solve yourself by Simplex method

Optimal solution

$x_2=6$

$S_2=7$

$x_3=4$

Remaining are 0 i.e., $x_1=S_1=S_3=0.$

(iii)

Constraints of the given problem with equal sign

$x_1+2x_2-2x_3+S_1=4 \longrightarrow$ used to find coefficient of y_1 in the objective function of dual

$x_1-x_3+S_2=3 \longrightarrow$ used to find coefficient of y_2 in the objective function of dual

$2x_1-x_2+2x_3+S_3=2 \longrightarrow$ used to find coefficient of y_3 in the objective function of dual

Constraints of the Dual problem with equal sign

$-y_1-2y_2-y_3+S_4=1 \longrightarrow$ used to find coefficient of x_1 in the objective function of given problem

$-2y_1+y_3+S_5=-2 \longrightarrow$ used to find coefficient of x_2 in the objective function of given problem

$2y_1+y_2-2y_3+S_6=1 \longrightarrow$ used to find coefficient of x_3 in the objective function of given problem

$S_1 \longrightarrow y_1$

$S_2 \longrightarrow y_2$

$S_3 \longrightarrow y_3$

$S_4 \longrightarrow x_1$

$S_5 \longrightarrow x_2$

$S_6 \longrightarrow x_3$

$S_1 \longrightarrow y_1$

S_2 (Basic variable) $\longrightarrow y_2=0$

$S_3 \longrightarrow y_3$

$S_4 \longrightarrow x_1$

$S_5=0 \longrightarrow x_2$ (Basic Variable)

$S_6=0 \longrightarrow x_3$ (Basic Variable)

Putting $y_2=0$, $S_5=0$ and $S_6=0$ in the dual problem,

$$-y_1-2\,y_2-\,y_3+S_4=1$$

$-2y_1 + y_3 + S_5 = -2$

$2y_1 + y_2 - 2y_3 + S_6 = 1$

We have

$-y_1 - y_3 + S_4 = 1$

$-2y_1 + y_3 = -2$

$2y_1 - 2y_3 = 1$

Solving these three equations,

$y_3 = 1$, $y_1 = \frac{3}{2}$, $S_4 = \frac{7}{2}$

Putting

$y_2 = 0$, $S_5 = 0$ **and** $S_6 = 0$ **and** $y_3 = 1$, $y_1 = \frac{3}{2}$, $S_4 = \frac{7}{2}$ **in the objective function of the dual problem**

$-4y_1 - 3y_2 - 2y_3 = -8.$

Sensitivity Analysis

Change in objective function coefficients

Max $(3x_1+2x_2+5x_3)$

Subject to

$x_1+2x_2+x_3+S_1=430$

$3x_1+2x_3+S_2=460$

$x_1+4x_2+S_3=420$

$x_1, x_2, S_1, S_2, S_3>=0.$

The optimal table for this LPP is as follows:

		3	2	5	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		4	0	0	1	2	0		
2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

(i) Find the new optimal solution if the objective function is $2x_1 + x_2 + 4x_3$ instead of $3x_1 + 2x_2 + 5x_3$.

(ii) Find the new optimal solution if the objective function is $3x_1 + 6x_2 + x_3$ instead of $3x_1 + 2x_2 + 5x_3$.

(iii) Within what range the coefficient of x_1 in the objective function can be changed without affecting the optimality.

(iv) Within what range the coefficient of x_2 in the objective function can be changed without affecting the optimality.

Solution

(i)

		2	1	4	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =									
1	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
4	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

		2	1	4	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		$\frac{15}{4}$	0	0	$\frac{1}{2}$	$\frac{1}{4}$	0		
1	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
4	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

All calculated values of $Z_j - C_j$ are >=0. So no change in optimal solution.

(ii)

		3	6	1	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =									
6	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
1	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

		3	6	1	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		-3	0	0	3	-1	0		
6	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
1	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

$Z_j - C_j < 0$ corresponding to x_1 and S_2. minimum from -3 and -1 is -3 . So x_1 is entering variable.

		3	6	1	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		-3	0	0	3	-1	0		
6	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
1	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

		3	6	1	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		-3	0	0	3	-1	0		
6	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	100/-
1	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	$230/\frac{3}{2}$
0	S_3	2	0	0	-2	1	1	20	20/2=10

S_3 is leaving variable.

		3	6	1	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		0	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$		
6	x_2	0	1	0	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{8}$	102.5	
1	x_3	0	0	1	$\frac{3}{2}$	$-\frac{1}{4}$	$-\frac{3}{4}$	215	
0	x_1	1	0	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	10	

New optimal solution is

x_2=102.5

x_3=215

x_1=10

(iii)

		C	2	5	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =									
2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

		C	2	5	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		$7-C$	0	0	1	2	0		
2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

Solution will be optimal if all values of $Z_j - C_j$ will be greater than or equal to 0.

Hence, $7-C>=0$

$C<=7$.

(iii)

		3	C	5	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =									
C	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

		3	C	5	0	0	0		
C_B	Basic Variables	x_1	x_2	x_3	S_1	S_2	S_3	Solution	Minimum Ratio
$Z_j - C_j$ =		$-\frac{C}{4} + \frac{9}{2}$	0	0	$\frac{C}{2}$	$-\frac{C}{4} + \frac{5}{2}$	0		
C	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
0	S_3	2	0	0	-2	1	1	20	

Solution will be optimal if all values of $Z_j - C_j$ will be greater than or equal to 0.

Hence,

$-\frac{C}{4} + \frac{9}{2}$**>=0 and** $\frac{C}{2}$**>=0 and** $-\frac{C}{4} + \frac{5}{2}$**>=0**

C<=18 and C>=0 and C<=10.

0<=C<=10.