Lecture 9: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Background :-

$f(x) = 0 \qquad [a, b]$

$g(x) = x$

$g(x)$

$p_0 \in [a, b]$

$[3, 4]$

$p_1 = g(p_0) \neq p_0$

$\langle p_n \rangle \to p$

$p_n = g(p_{n-1})$

$n \geq 1$

$p_2 = g(p_1) \neq p_1$

$\frac{1}{n} \to 0$

$\vdots$

$p \to$ exact fixed pt.

$\lim_{n \to \infty} \frac{1}{n} = 0$

Fixed point iteration

$[a, b]$

$f(x) = 0$

a b

$g(x) \in [a, b]$

Convergence conditions satisfied by $g(x)$:

(i) (existence) If $g \in C[a, b]$ and $g(x) \in [a, b], \forall x \in [a, b]$, then $g(x)$ has at least one fixed point in $[a, b]$.

(ii) (uniqueness) If, in addition, $g'(x)$ exists in (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a, b)$, then there is exactly one fixed point in $[a, b]$.

$|g'(x)| < 1$

3.2 3.8

$[3, 4]$

(iii) (convergence) If conditions of (i) and (ii) are satisfied, then for any number $p_0 \in [a, b]$, the sequence defined by $p_n = g(p_{n-1})$, $n \geq 1$ converges to the unique fixed point p in $[a, b]$.

$\langle p_n \rangle \to p$ $\qquad \lim_{n \to \infty} |p_n - p| \to 0$

Fixed point iteration

Proof of (i):

If $g(a) = a$ or $g(b) = b$, the result is true.

If $g(a) \neq a$ and $g(b) \neq b$, then $g(a) > a$ and $g(b) < b$

Now, define $h(x) = g(x) - x$

$$h(a) = g(a) - a > 0$$

$$h(b) = g(b) - b < 0$$

and also $h(x)$ is continuous function on $[a, b]$

then from IVT, $\exists\ c \in (a, b)$ s.t. $h(c) = 0$

$\Rightarrow \quad g(c) - c = 0$

$g(c) = c$

$\Rightarrow$ c is a fixed pt. for g in (a,b)

Fixed point iteration

Proof of (ii):

Let p and q be two fixed point for $g(x)$ on $[a,b]$ ie $g(p)=p$, $g(q)=q$

Mean value Theorem

$$\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)|$$

$c \in (x,y)$

Now,

$$|p-q| = |g(p)-g(q)| = |g'(c)|\,|p-q|, \quad c \in (p,q) \subseteq [a,b]$$

$$< 1\ |p-q|$$

$|p-q| < |p-q| \rightarrow$ It is not true

It is a contradiction. So, supposition is wrong.

There is only one fixed pt. for $g(x)$ in $[a, b]$.

Fixed point iteration

Proof of (iii):

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(c_n)|\,|p_{n-1} - p| \qquad c_n \in (p_{n-1}, p)$$

$$\le k\,|p_{n-1} - p|$$

$$|p_n - p| \le k\,|p_{n-1} - p|, \qquad n \ge 1$$

$$\le k \cdot k\,|p_{n-2} - p| = k^2\,|p_{n-2} - p|$$

$$\le k^2\, k\,|p_{n-3} - p| = k^3\,|p_{n-3} - p|$$

$$\cdots$$

$$|p_n - p| \leq k^n |p_{n-n} - p|$$

$$|p_n - p| \leq k^n |p_0 - p|$$

$$\lim_{n \to \infty} |p_n - p| \leq \lim_{n \to \infty} k^n |p_0 - p|$$

$$\lim_{n \to \infty} |p_n - p| \leq |p_0 - p| \lim_{n \to \infty} k^n = 0 \qquad \because k < 1$$

$$\Rightarrow \lim_{n \to \infty} |p_n - p| = 0$$

$$\lim_{n \to \infty} p_n = p$$

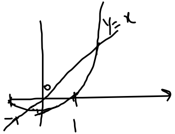
Fixed point iteration

Example:

Show that $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on the interval $[-1, 1]$.

Solution: i) $g(x) = \dfrac{x^2-1}{3}$ is continuous on $[-1,1]$.

T.P.
$g(x) \in [-1,1]$
$\forall x \in [-1,1]$



(ii) $g'(x) = \dfrac{2x}{3} = 0$

$x = 0$

$g''(x) = \dfrac{2}{3} > 0$

at $x = 0$ min value of g
$g(0) = \dfrac{-1}{3} \in [-1,1]$

at $x = \pm 1$ max value of g
$g(-1) = g(1) = 0 \in [-1,1]$

$\Rightarrow g(x) \in [-1,1] \ \forall x \in [-1,1]$

$$g'(x) = \frac{2x}{3} > 0 \quad \text{on } [0,1]$$

$$< 0 \quad \text{on } [-1,0]$$

$\Rightarrow$ $g(x)$ is increasing on $[0,1]$

$g(x)$ is decreasing on $[-1,0]$

Max value of g at $[0,1]$ ie $g(1) = 0$

Min " " " " ie $g(0) = -1/3$

Again Max " " " at $[-1,0]$ ie $g(-1) = 0$

Min - - - - - - $[-1,0]$ ie $g(0) = -1/3$

$\in [-1,1]$

Fixed point iteration

Solution(continued):

(iii.) $\quad |g'(x)| = \left|\frac{2x}{3}\right| < 1 \quad \forall\, x \in [-1, 1]$

$\Rightarrow$ $g(x)$ has a unique fixed pt. in $[-1, 1]$.

Fixed point iteration

Exercise:

1. Show that $g(x) = 2^{-x}$ has a unique fixed point on the interval $\left[\frac{1}{3}, 1\right]$.