Group Theory

Examples

1. Consider the set Q of rational Numbers and let $*$ be the operation on Q defined by:

$$a*b = a+b-ab$$

a) Find i) $3*4$ ii) $2*(-5)$ iii) $7*(1/2)$

b) Is $(Q,*)$ a semigroup? Is is Commutative?

c) Find the Identity element for $*$.

d) Do any of the elements in Q have an inverse? What is it?

a) i) $3 * 4 = 3 + 4 - 3 \times 4 = 7 - 12 = -5$

$a * b = a + b - ab$

ii) $2 * (-5) = 2 - 5 + 10 = 7$

iii) $7 * (1/2) = 7 + \frac{1}{2} - \frac{7}{2}$

$$\frac{14 + 1 - 7}{2} = \frac{8}{2} = 4$$

b) for Semigroup

(1) Closure

$a * b = a + b - ab$

$\forall a, b \in Q$

It is closed.

Yes it is a Semigroup.

(1) Associative

$(a*b)*c = a*(b*c)$

L.H.S

$(a+b-ab)*c$

$= a+b-ab+c-ac-bc+abc$

R.H.S

$a*(b+c-bc)$

$= a+b+c-bc-ab-ac+abc$

Yes

Commutative

$$a * b = a + b - ab = b + a - ab$$
$$= b * a$$

Hence it is Commutative.

c) Identity element

$a * e = a \quad \forall a \in Q$

$$\cancel{a} + e - ae = \cancel{a}$$

$$e(1-a) = 0$$

$$e = 0$$

0 is the identity element.

d) Inverse element

$$a * (a^{-1}) = e$$

$$a * (a^{-1}) = 0$$

$$a + a^{-1} - a \cdot a^{-1} = 0$$

$$a = a^{-1}(a-1)$$

$$a^{-1} = \frac{a}{a-1}$$

Thus if $a \neq 1$, then inverse of a exists and it is

$$\boxed{a^{-1} = \frac{a}{a-1}}$$

② Consider the group $G = \{1, 2, 3, 4, 5, 6\}$ under Multiplication Modulo 7.

a) Find the Multiplication Table of G.

b) Find $2^{-1}, 3^{-1}, 6^{-1}$.

c) Find the orders and subgroups generated by 2 and 3.

d) Is G cyclic?

Sol^n:-

a)

$*_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$a * e = a$

$e = 1$

$a \cdot a^{-1} = e$

$2 \cdot 4 = 1$

b) $2^{-1} = 4$

$3^{-1} = 5$

$6^{-1} = 6$

c) $2^1 = 2$ $3^1 = 3$ $1^1 =$ $4^1 =$ 5^1 6^1

$2^2 = 4$ $3^2 = 2$ $1^2 =$ 4^2

$2^3 = 1$ $3^3 = 6$

$2^4 = 2$ $3^4 = 4$

$2^5 = 4$ $3^5 = 5$

$2^6 = 1$ $3^6 = 1$ $1^6 =$ $4^6 =$ 5^6 6^6

$gp(1) = \{1\}$

$gp(2) = \{1, 2, 4\}$ $gp(3) = \{1, 2, 3, 4, 5, 6\}$

$|2| = 3.$ $|3| = 6$

d) Here, G is cyclic.

Generator is 3.


Thank you!!!
...