

Lecture 7: Numerical Analysis (UMA011)

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Root-finding problem

Bisection method: Example

Show that $f(x) = x^3 + 2x^2 - 3x - 1 = 0$ has a root in $[1, 2]$ and use the bisection method to determine an approximation to the root i.e. is accurate to atleast within 10^{-2} . *tolerance*

Solution: $f(x)$ is a cubic poly, so it is continuous

at $[1, 2]$ and $f(1) = 1 + 2 - 3 - 1 < 0$

$$f(2) = 2^3 + 2(2)^2 - 3(2) - 1 = 8 + 8 - 6 - 1 > 0$$

$$\text{so } f(1) * f(2) < 0$$

from IVT, we have $f(x) = 0$ has a root in $[1, 2]$

-ve +ve
[1, 2]

Bisection method $p_1 = \frac{1+2}{2} = 1.5$

The root lie in betⁿ either [1, 1.5] or [1.5, 2]

check the sign of $f(1.5) = +ve$.

By IVT, the root lie in [1, 1.5].

So $p_2 = \frac{1+1.5}{2} = 1.25$ \downarrow
[1, 1.25] and [1.25, 1.5]

$|p_1 - p_2| < 10^{-2}$

The root lie in betⁿ either [1, 1.25] or [1.25, 1.5]

check the sign of $f(1.25) = +ve$

By IVT, the root lie in [1, 1.25]

$$p_3 = \frac{1+1.25}{2} = 1.125$$

$$|p_n - p_{n-1}| < 10^{-2}$$

0.32 \oplus \ominus
0.001

10^{-2}

4

$$|p_7 - p| < \frac{2^{-1}}{2^7}$$

$$\frac{1}{2^7} < 10^{-2}$$

n	a	b	p _n	p _n - p _{n-1}	f(p _n)
1	1	2	1.5		+
2	1	1.5	1.25	1.5 - 1.25 $\leq 10^{-2}$	+
3	1	1.25	1.125	1.25 - 1.125 $\leq 10^{-2}$	-
4	1.125	1.25	1.1875	1.125 - 1.1875 $\leq 10^{-2}$	-
5	1.1875	1.25	1.21875	1.1875 - 1.21875 $\leq 10^{-2}$	+
6	1.1875	1.21875	1.203125	1.21875 - 1.203125 $\leq 10^{-2}$	-
7	1.203125	1.21875	1.1953125	1.203125 - 1.1953125 $\leq 10^{-2}$	

$$p_7 = 1.1953125$$

Approximate
not

Bisection method

Maximum error bound

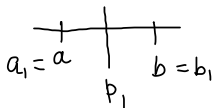
Suppose that $f \in C[a, b]$ and $f(a) * f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with $|p_n - p| \leq \frac{b-a}{2^n}$, when $n \geq 1$.

$$\sqrt{|p_n - p|} \leq ()$$

Proof:

length of $[a, b]$ is $[a_1, b_1]$

$$= |b_1 - a_1| = |b - a|$$



After applying bisection method and IVT, we get the root lie in $[a_2, b_2]$ (say)

$$\text{length of } [a_2, b_2] = |b_2 - a_2| = \frac{1}{2} |b_1 - a_1|$$

Again, applying bisection method on $[a_2, b_2]$ and then applying IVT, we get $[a_3, b_3]$ in which root of $f(x)=0$ lie.

$$\begin{aligned}\text{length of the interval } [a_3, b_3] &= |b_3 - a_3| = \frac{1}{2} |b_2 - a_2| \\ &= \frac{1}{2^2} |b_1 - a_1|\end{aligned}$$

Continue this process upto n - iterations, we get the root lie in $[a_n, b_n]$

$$\begin{aligned}\text{The length of interval } [a_n, b_n] &= |b_n - a_n| \\ &= \frac{1}{2^{n-1}} |b_1 - a_1|\end{aligned}$$

$$|b_n - a_n| = \frac{1}{2^{n-1}} |b - a| \quad (*)$$

Now,

$$|p_n - p| = \left| \frac{a_n + b_n}{2} - p \right|$$

$$\left\{ \begin{array}{l} \text{As } p \in [a_n, b_n] \text{ then } a_n \leq p \leq b_n \\ \phantom{\text{As } p \in [a_n, b_n] \text{ then }} -a_n \geq -p \\ \phantom{\text{As } p \in [a_n, b_n] \text{ then }} \Rightarrow -p \leq -a_n \end{array} \right.$$

$$\Rightarrow |p_n - p| \leq \left| \frac{a_n + b_n}{2} - a_n \right| = \left| \frac{a_n + b_n - 2a_n}{2} \right|$$

$$|p_n - p| \leq \frac{|b_n - a_n|}{2}$$

from $(*)$, we get

$$|p_n - p| \leq \frac{1}{2} \left(\frac{1}{2^{n-1}} |b_1 - a_1| \right)$$

$$|p_n - p| \leq \frac{1}{2^n} |b - a|.$$

Bisection method

Example

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Solution:

from prev. result, we have

$$\frac{|b_1 - a_1|}{2^n} \leq 10^{-3}$$

$$\frac{1}{2^n} \leq 10^{-3}$$

$$2^n \geq 10^3$$

$$n \log 2 \geq 3 \log 10$$

$$n \geq \frac{3 \log_{10}}{\log_2}$$

$$n \geq 9.965$$

$$n = 10 \checkmark$$

Exercise:

- 1** Use intermediate value theorem to get the first positive root of $x - 2^{-x} = 0$ and hence apply bisection method to find the root accurate to within 10^{-1} .
- 2** Using the bisection method, determine the point of intersection of the curves given by $y = 3x$ and $y = e^x$ in the interval $[0, 1]$ with an accuracy 0.1.
- 3** Find a bound for the number of iterations needed to achieve an approximation of $(25)^{1/3}$ by the bisection method with an accuracy 10^{-2} . Hence find the approximation with given accuracy.