# LECTURE 24

UEI407

# LAPLACE TRANSFORM

Laplace transformation is a very powerful method to solve differential equations which is involved in any branch of engineering.

In signals and systems or circuits and signals the two following steps are involved:

- Setting up the mathematical equations to describe the system and
- Solving these equations.

Differential equations can be solved either by classical method or by Laplace transform.

To solve the ordinary differential equations by classical methods, the following steps must be followed:

- Determination of the complementary function
- Determination of the arbitrary constants.

The classical method is very complicated to obtain the solution of differential equations when excitation functions are involved due to derivative terms. The Laplace transformation has found its superiority to solve these differential equations, as compared to the classical method.

# LAPLACE TRANSFORM(Contd..)

The chief advantage of this transform is that this method automatically takes care of initial conditions. It is not required to determine the general solution at first and then the particular solution. The direct solution of non-homogeneous differential equations is possible by Laplace transformation.

## Laplace Transform

It has been found that Fourier transform can be applied to a large variety of functions. It is also widely used as a mathematical tool in engineering. Due to the restriction

$$\int_{-\infty}^{\infty} f\ (t)\ dt\ < \infty$$

it is impossible to apply this transform for many functions, e.g. , ramp, parabolic etc. because the integral is not converging. So these kind of functions are not Fourier transformable. It is possible to handle such kind of functions by Laplace transform. The Laplace transform of a function f(t) in time domain t is given by

# LAPLACE TRANSFORM(Contd..)

$$F(s) = LT[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-st} dt \qquad (1)$$

$$s = \sigma + j\omega \qquad (2)$$

The notation LT indicates "Laplace transform of."

We can write equation (4.1) as

$$F(s) = LT[f(t)] = \int_{-\infty}^{0} f(t)\, e^{-st} dt + \int_{0}^{\infty} f(t)\, e^{-st} dt \qquad (3)$$

The Laplace transform can be applied to two sided as well as one sided functions. When applied to both sided functions, it is termed as bilateral transform (BLT). Most of the engineering functions are one sided functions i.e., these functions are zero before a particular instant of time (t = 0) and we can write Laplace transform of these functions as

# LAPLACE TRANSFORM(Contd..)

$$F\,(s) = LT[f\,(t)] = \int_{0}^{\infty} f(t)\; e^{-st} dt \qquad (4)$$

because f(t) = 0 for t < 0.

A function f(t) is said to be Laplace transformable when

$$\int_{-\infty}^{\infty} f(t)\; e^{-st}\, dt < \infty \qquad (5)$$

is satisfied for some limit σ where σ is the real part of s

## 2. Region of Convergence (ROC)

In Laplace transform we at first multiply the function f(t) by e−σt and then by e−jωt. The modified function [ f(t) e−σt] is created specially so that the integral converges which needs proper chosen value of σ and the region of σ for which the integral converges is called region of convergence (ROC) of the function f(t).

# 3 Inverse Laplace Transformation

Laplace transform permits to go from time domain to frequency domain whereas inverse Laplace transform allows to go from frequency domain to time domain. The inverse Laplace transform is represented by

$$LT^{-1}\left[F(t)\right] = f(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds \qquad (6)$$

Where $LT^{-1}$ is read as "inverse Laplace transform of".

# 4 Basic Properties of Laplace Transforms

**Property 1:**

The Laplace transform of the sum or difference of time functions is equal to the sum or difference of the Laplace transform of the individual time functions.

Proof:

$$LT\left[f_1(t)+f_2(t)+\ldots\ldots\ldots+f_n(t)\right]$$

$$=\int_0^{\infty}\left[f_1(t)+f_2(t)+\ldots\ldots\ldots+f_n(t)\right]e^{-st}dt$$

$$=\int_0^{\infty}f_1(t)e^{-st}dt+\int_0^{\infty}f_2(t)e^{-st}dt+\cdots\cdots+\int_0^{\infty}f_n(t)e^{-st}dt$$

$$=F_1(s)+F_2(s)+\cdots\cdots\cdots+F_n(s) \qquad (7)$$

**Property 2:**

The Laplace transform of product of a constant and a time function is equal to the product of the constant and the Laplace transformation of the time function.

Proof:

$$LT\,[af(t)] = \int_0^{\infty} af(t)e^{-st}dt = a\int_0^{\infty} f(t)e^{-st}dt = a\,F(s) \qquad (8)$$

**Property 3:**

If $F_1$ (s) and $F_2$ (s) be the Laplace transform of $f_1$ (t) and $f_2$ (t), then

$$LT\,[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s) \qquad (9)$$

where a1 and a2 are arbitrary constants.

Proof:

$$LT\,[a_1 f_1(t) + a_2 f_2(t)] = \int_0^\infty [a_1 f_1(t) + a_2 f_2(t)]\, e^{-st}\, dt$$

$$= a_1 \int_0^\infty f_1(t) e^{-st}\, dt + a_2 \int_0^\infty f_2(t)\, e^{-st}\, dt = a_1 F_1(s) + a_2 F_2(s)$$

**Property 4:** Scaling

$$LT\,[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

where 'a' is an arbitrary constant

Proof:

$$LT\left[f\left(at\right)\right]=\int_0^\infty f\left(at\right)e^{-st}\,dt=\frac{1}{a}\int_0^\infty f\left(x\right)e^{-\frac{sx}{a}}\,dx=\frac{1}{a}F\left(\frac{s}{a}\right)$$

[ Putting x = at ] (10)

**Property 5:** Time Shifting

$$LT\ [f(t-t_0)\ ]=e^{-t_0 s}\ F(s)$$

Proof:

$$LT\ [f(t-t_0)\ ]=\int_0^\infty f\left(t-t_0\right)e^{-st}\,dt$$

$$=\int_0^\infty f\left(t-t_0\right)e^{-st}dx=e^{-st_0}\int_0^\infty f\,(t)\,e^{-st}\,dt$$

$$=e^{-st_0}F\left(s\right) \qquad (11)$$

**Property 6:**

Frequency Shifting :

$$LT= \left[e^{-at}f(t)\right] = \int_0^\infty e^{at}\, f(t)\, e^{-st}\, dt = \int_0^\infty f(t) e^{-(s+a)t}\, dt$$

$= F(s + a)$ (4.12)

**Property 7:**

Time displacement theorem :

It states that if f (t) be Laplace transformable it can be written as

$$LT\,[f(t-T)] = e^{-sT} F(s)$$

where LT $[f(t)] = F(s)$ and the function $f(t-T)$ is the function $f(t)$ displaced by T .Therefore, it is evident that the displacement in the time domain is multiplied by $e^{-sT}$ in the s–domain.

Proof:

LT [f(t–T)]

$$= \int_0^\infty f(t-T)\, e^{-st}\, dt = \int_0^\infty f(x) e^{-st}\, dx \qquad \text{[ Putting } x = t - T\text{]}$$

$$= e^{-sT} F(s) \qquad (13)$$

**Property 8:**

Convolution:

It states that the convolution of two real functions is equal to multiplication of their respective functions.

If LT[$f_1(t)$] = $F_1$ (s) and LT [$f_2$ (t)] = $F_2$ (s), by convolution, we can write

$$LT\,[f_1(t) * f_2(t)] = F_1(s)\, F_2(s) \qquad (14)$$

The word "Convolve" means to revolve continuously. The two functions $f_1$ (t) and $f_2$ (t) are multiplied in such a way that one is continually moving with time (say T) relative to another.

$$\text{i.e.} \quad f_1(t)*f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau \qquad (15)$$

The mathematical expression given in equation (4.15) is known as convolution theorem.

Let $\quad F(s) = LT[\, f_1(t) * f_2(t)\,]$

$$\therefore \quad F(s) = \left[\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_0^\infty \left[\int_0^t f_1(t-\tau) f_2(\tau) u(t-\tau) e^{-st}\, d\tau\right] dt \qquad (16)$$

We know that u (t–τ ) = 0 for all t < τ. Therefore, inner integral is multiplied by u(t – T ) and the upper limit of the inner integral becomes t = ∞. We get from equation

$$\therefore \quad F(s) = \int_0^\infty f_2(\tau)\left[\int_0^\infty f_1(t-\tau)\, u(t-\tau) e^{-st}\, dt\right] d\tau$$

$$= \int_0^\infty f_2(\tau)\, e^{-s\tau}\, F_1(s)\, d\tau$$

$$= F_1(s) \int_0^\infty f_2(\tau)\, e^{-s\tau}\, d\tau = F_1(s)\, F_2(s)$$