CHAPTER 5 (8 LECTURES)
POLYNOMIAL INTERPOLATION AND APPROXIMATIONS

1. Introduction

Polynomials are used as the basic means of approximation in nearly all areas of numerical analysis. They are used in the solution of equations and in the approximation of functions, of integrals and derivatives, of solutions of integral and differential equations, etc. Polynomials have simple structure, which makes it easy to construct effective approximations and then make use of them. For this reason, the representation and evaluation of polynomials is a basic topic in numerical analysis. We discuss this topic in the present chapter in the context of polynomial interpolation, the simplest and certainly the most widely used technique for obtaining polynomial approximations.

Definition 1.1 (Polynomial). *A polynomial $P_n(x)$ of degree $\leq n$ is, by definition, a function of the form*

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{1.1}$$

with certain coefficients $a_0, a_1, \cdots, a_n$. This polynomial has (exact) degree n in case its leading coefficient a_n is nonzero.

The power form (1.1) is the standard way to specify a polynomial in mathematical discussions. It is a very convenient form for differentiating or integrating a polynomial. But, in various specific contexts, other forms are more convenient. For example, the following shifted power form may be helpful.

$$P(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n. \tag{1.2}$$

It is good practice to employ the shifted power form with the center c chosen somewhere in the interval $[a, b]$ when interested in a polynomial on that interval.

Definition 1.2 (Newton's form). *A further generalization of the shifted power form is the following Newton form*

$$P(x) = a_0 + a_1(x-c_1) + a_2(x-c_1)(x-c_2) + \cdots + a_n(x-c_1)(x-c_2)\cdots(x-c_n).$$

This form plays a major role in the construction of an interpolating polynomial. It reduces to the shifted power form if the centers $c_1, \cdots, c_n$, all equal c, and to the power form if the centers $c_1, \cdots, c_n$, all equal zero.

2. Lagrange Interpolation

In this chapter, we consider the interpolation problems. Suppose we do not know the function f, but a few information (data) about f. Now we try to compute a function g that approximates f.

2.1. **Polynomial Interpolation.** The polynomial interpolation problem, also called Lagrange interpolation, can be described as follows: Given $(n+1)$ data points (x_i, y_i), $i = 0, 1, \cdots, n$ find a polynomial P of lowest possible degree such

$$y_i = P(x_i), \quad i = 0, 1, \cdots, n.$$

Such a polynomial is said to interpolate the data. Here y_i may be the value of some unknown function f at x_i, i.e. $y_i = f(x_i)$.
One reason for considering the class of polynomials in approximation of functions is that they uniformly approximate continuous function. The following fundamental theorem also tells us that we can approximation a continuous function with a polynomial.

Theorem 2.1 (Weierstrass Approximation Theorem). *Suppose that f is defined and continuous on $[a, b]$. For any $\varepsilon > 0$, there exists a polynomial $P(x)$ defined on $[a, b]$ with the property that*

$$|f(x) - P(x)| < \varepsilon, \qquad \forall x \in [a, b].$$

Another reason for considering the class of polynomials in approximation of functions is that the derivatives and indefinite integrals of a polynomial are easy to compute.

2.2. **Linear Interpolation.** We determine a polynomial

$$P(x) = ax + b \tag{2.1}$$

where a and b are arbitrary constants satisfying the interpolating conditions $f(x_0) = P(x_0)$ and $f(x_1) = P(x_1)$. We have

$$f(x_0) = P(x_0) = ax_0 + b$$
$$f(x_1) = P(x_1) = ax_1 + b.$$

Lagrange interpolation: Solving for a and b, we obtain

$$\begin{aligned} a &= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\ b &= \frac{f(x_0)x_1 - f(x_1)x_0}{x_1 - x_0} \end{aligned}$$

Substituting these values in equation (2.1), we obtain

$$\begin{aligned} P(x) &= \frac{f(x_0) - f(x_1)}{x_0 - x_1}\, x + \frac{f(x_0)x_1 - f(x_1)x_0}{x_1 - x_0} \\ \implies P(x) &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \\ \implies P(x) &= l_0(x)f(x_0) + l_1(x)f(x_1) \end{aligned}$$

where $l_0(x) = \dfrac{x - x_1}{x_0 - x_1}$ and $l_1(x) = \dfrac{x - x_0}{x_1 - x_0}$.
These functions $l_0(x)$ and $l_1(x)$ are called the Lagrange Fundamental Polynomials and they satisfy the following conditions.

$$l_0(x) + l_1(x) = 1.$$
$$l_0(x_0) = 1,\ l_0(x_1) = 0$$
$$l_1(x_0) = 0,\ l_1(x_1) = 1$$
$$\implies l_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Higher-order Lagrange interpolation: In this section we take a different approach and assume that the interpolation polynomial is given as a linear combination of $n + 1$ polynomials of degree n. This time, we set the coefficients as the interpolated values, $\{f(x_i)\}_{i=0}^{n}$, while the unknowns are the polynomials. We thus let

$$P_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x),$$

where $l_i(x)$ are $n+1$ polynomials of degree n. Note that in this particular case, the polynomials $l_i(x)$ are precisely of degree n (and not $\leq n$). However, $P_n(x)$, given by the above equation may have a lower degree. In either case, the degree of $P_n(x)$ is n at the most. We now require that $P_n(x)$ satisfies the interpolation conditions

$$P_n(x_j) = f(x_j),\ 0 \leq j \leq n.$$

By substituting x_j for x we have

$$P_n(x_j) = \sum_{i=0}^{n} f(x_i) l_i(x_j),\ 0 \leq j \leq n.$$

Therefore we may conclude that $l_i(x)$ must satisfy

$$l_i(x_j) = \delta_{ij},\ i, j = 0, 1, \cdots, n$$

where δ_{ij} is the Kronecker delta, defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Each polynomial $l_i(x)$ has $n+1$ unknown coefficients. The conditions given above through delta provide exactly $n+1$ equations that the polynomials $l_i(x)$ must satisfy and these equations can be solved in order to determine all $l_i(x)$'s. Fortunately there is a shortcut. An obvious way of constructing polynomials $l_i(x)$ of degree n that satisfy the condition is the following:

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

The uniqueness of the interpolating polynomial of degree $\leq n$ given $n+1$ distinct interpolation points implies that the polynomials $l_i(x)$ given by above relation are the only polynomials of degree n.
Note that the denominator does not vanish since we assume that all interpolation points are distinct. We can write the formula for $l_i(x)$ in a compact form using the product notation.

$$\begin{aligned} l_i(x) &= \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} \\ &= \frac{W(x)}{(x-x_i)W'(x_i)}, \ i = 0, 1, \cdots, n \end{aligned}$$

where

$$\begin{aligned} W(x) &= (x-x_0)\cdots(x-x_{i-1})(x-x_i)(x-x_{i+1})\cdots(x-x_n) \\ \therefore \ W'(x_i) &= (x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n). \end{aligned}$$

Thus the Lagrange interpolating polynomial can be written as

$$P_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}.$$

Example 1. *Use Lagrange interpolation to find the unique polynomial of degree* 3 *or less, that agrees with the following data: Also estimate* $y(1.5)$.

x_i	-1	0	1	2
y_i	3	-4	5	-6

Sol. The Lagrange fundamental polynomials are given by

$$\begin{aligned} l_0(x) &= \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} = -\frac{1}{6}(x^3-3x^2+2x). \\ l_1(x) &= \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} = \frac{1}{2}(x^3-2x^2-x+2). \\ l_2(x) &= \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)} = -\frac{1}{2}(x^3-x^2-2x). \\ l_3(x) &= \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} = \frac{1}{6}(x^3-x). \end{aligned}$$

The interpolating polynomial in the Lagrange form is therefore

$$\begin{aligned} P_3(x) &= y_0l_0(x) + y_1l_1(x) + y_2l_2(x) + y_3l_3(x) \\ &= 3l_0(x) - 4l_1(x) + 5l_2(x) - 6l_3(x) \\ &= -6x^3 + 8x^2 + 7x - 4. \\ \therefore y(1.5) &\approx P_3(1.5) = 4.25. \end{aligned}$$

Example 2. *Prove that for given a real-valued function* $f(x)$ *and* $n+1$ *distinct points* $x_0, x_1, \cdots, x_n$, *there exists a unique polynomial* $P_n(x)$ *of degree* $\leq n$ *which interpolates the unknown function* $f(x)$ *at given points* $x_0, x_1, \cdots, x_n$.

Sol. Let there are two such polynomials P_n and Q_n such that

$$P_n(x_i) = f(x_i)$$

$$Q_n(x_i) = f(x_i), \ 0 \le i \le n.$$

Define

$$S_n(x) = P_n(x) - Q_n(x)$$

Since for both P_n and Q_n, degree $\le n$, which implies the degree of S_n is also $\le n$.
Also

$$S_n(x_i) = P_n(x_i) - Q_n(x_i) = f(x_i) - f(x_i) = 0, \ 0 \le i \le n.$$

This implies S_n has at least $n+1$ zeros which is not possible as degree of S_n is at most n.
This implies

$$S_n(x) = 0, \quad \forall x$$

$$\implies P_n(x) = Q_n(x), \quad \forall x.$$

Therefore interpolating polynomial is unique.

Example 3. *Let $f(x) = \sqrt{x - x^2}$ and $P_2(x)$ be the Lagrange interpolating polynomial on $x_0 = 0$, x_1 and $x_2 = 1$. Find the largest value of x_1 in $(0, 1)$ for which $f(0.5) - P_2(0.5) = -0.25$.*

Sol. If $f(x) = \sqrt{x - x^2}$ then our nodes are $[x_0, x_1, x_2] = [0, x_1, 1]$ and $f(x_0) = 0$, $f(x_1) = \sqrt{x_1 - x_1^2}$ and $f(x_2) = 0$. Therefore

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - x_1)(x - 1)}{x_1},$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x - 1)}{x_1(x_1 - 1)},$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - 1)}{(1 - x_1)}.$$

$$\begin{aligned}
\therefore \ P_2(x) &= l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) \\
&= \frac{(x - x_1)(x - 1)}{x_1}.0 + \frac{x(x - 1)}{x_1(x_1 - 1)}.\sqrt{x_1 - x_1^2} + \frac{x(x - 1)}{(1 - x_1)}.0 \\
&= -\frac{x(x - 1)}{\sqrt{x_1(1 - x_1)}}.
\end{aligned}$$

If we now consider $f(x) - P_2(x)$, then

$$f(x) - P_2(x) = \sqrt{x - x^2} + \frac{x(x - 1)}{\sqrt{x_1(1 - x_1)}}.$$

Hence $f(0.5) - P_2(0.5) = -0.25$ implies

$$\sqrt{0.5 - 0.5^2} + \frac{0.5(0.5 - 1)}{\sqrt{x_1(1 - x_1)}} = -0.25$$

Solving for x_1 gives

$$\begin{aligned}
x_1^2 - x_1 &= -1/9 \\
\text{or } (x_1 - 1/2)^2 &= 5/36 \\
\text{which gives } x_1 &= \frac{1}{2} - \sqrt{\frac{5}{36}} \text{ or } x_1 = \frac{1}{2} + \sqrt{\frac{5}{36}}.
\end{aligned}$$

The largest of these is therefore

$$x_1 = \frac{1}{2} + \sqrt{\frac{5}{36}} \approx 0.8727.$$

2.3. **Error Analysis for Polynomial Interpolation.** We are given nodes $x_0, x_1, \cdots, x_n$, and the corresponding function values $f(x_0), f(x_1), \cdots, f(x_n)$, but the we don't know the expression for the function. Let $P_n(x)$ be the polynomial of order $\leq n$ that passes through the $n+1$ points $(x_0, f(x_0))$, $(x_1, f(x_1))$,$\cdots$, $(x_n, f(x_n))$.
Question: What is the error between $f(x)$ and $P_n(x)$ even we don't know $f(x)$ in advance?

Definition 2.2 (Truncation error)**.** *The polynomial $P_n(x)$ coincides with $f(x)$ at all nodal points and may deviates at other points in the interval. This deviation is called the truncation error and we write*

$$E_n(f; x) = f(x) - P_n(x).$$

Theorem 2.3. *Suppose that $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a, b]$ and $f \in C^{n+1}[a, b]$. Let $P_n(x)$ be the unique polynomial of degree $\leq n$ that passes through $n + 1$ distinct points then prove that*

$$\forall x \in [a, b], \ \exists \xi = \xi(x) \in (a, b)$$

such that

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n+1)!} \ f^{(n+1)}(\xi).$$

Proof. Let $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a, b]$ and $f \in C^{n+1}[a, b]$.
Let $P_n(x)$ be the unique polynomial of degree $\leq n$ that passes through $n + 1$ discrete points.
Since $f(x_i) = P_n(x_i), \forall i = 1, 2, \cdots, n$; which implies $f(x) - P_n(x) = 0$.
Now for any t in the domain, define a function $g(t)$,$t \in [a, b]$,

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)} \tag{2.2}$$

Now $g(t) \in C^{n+1}[a, b]$ as $f \in C^{n+1}[a, b]$ and $P(x) \in C^{n+1}[a, b]$.
Now $g(t) = 0$ at $t = x, x_0, x_1, \cdots, x_n$. Therefore $g(t)$ satisfy the conditions of generalized Rolle's Theorem which states that between $n + 2$ zeros of a function, there is at least one zero of $(n + 1)$th derivative of the function. Hence there exists a point ξ such that

$$g^{(n+1)}(\xi) = 0$$

where $\xi \in (a, b)$ and may depend on x.
Now differentiate function $g(t)$, $(n + 1)$ times to obtain

$$\begin{aligned} g^{(n+1)}(t) &= f^{(n+1)}(t) - P_n^{(n+1)}(t) - [f(x) - P_n(x)]\frac{(n+1)!}{(x - x_0)(x - x_1) \cdots (x - x_n)} \\ &= f^{(n+1)}(t) - [f(x) - P_n(x)]\frac{(n+1)!}{(x - x_0)(x - x_1) \cdots (x - x_n)} \end{aligned}$$

Here $P_n^{(n+1)}(t) = 0$ as $P_n(x)$ is a n-th degree polynomial.
Now $g^{(n+1)}(\xi) = 0$ and then solving for $f(x) - P_n(x)$, we obtain

$$f(x) - P(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

■

Corollary 2.4. *To find the maximum error, we need to find the maxima of right side which contain two factors: one is products of factors of the form $x - x_i$ and second is $f^{(n+1)}(\xi)$. In practice we try to find two separate bounds for both the terms.*

The next example illustrates how the error formula can be used to prepare a table of data that will ensure a specified interpolation error within a specified bound.

Example 4. *Suppose a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent $x-$values, the step size, is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?*

Sol. Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated, x be in $[0,1]$, and suppose i satisfies $x_i \le x \le x_{i+1}$.
The error in linear interpolation is

$$|f(x) - P_1(x)| = \left|\frac{1}{2}f^2(\xi)(x-x_i)(x-x_{i+1})\right| = \frac{|f^2(\xi)|}{2}|(x-x_i)(x-x_{i+1})|.$$

The step size is h, so $x_i = ih, x_{i+1} = (i+1)h$, and

$$|f(x) - P_1(x)| \le \frac{1}{2}|f^2(\xi)|\;|(x-x_i)(x-x_{i+1})|.$$

Hence

$$\begin{aligned}|f(x) - P_1(x)| &\le \frac{1}{2}\max_{\xi\in[0,1]} e^\xi \max_{x_i\le x\le x_{i+1}} |(x-x_i)(x-x_{i+1})| \\ &\le \frac{e}{2}\max_{x_i\le x\le x_{i+1}} |(x-x_i)(x-x_{i+1})|.\end{aligned}$$

We write $g(x) = (x-x_i)(x-x_{i+1})$, for $x_i \le x \le x_{i+1}$.
For simplification, we write

$$\begin{aligned}x - x_i &= th \\ x - x_{i+1} &= x - (x_i + h) = (t-1)h.\end{aligned}$$

Thus

$$\begin{aligned}g(t) &= h^2 t(t-1) \\ g'(t) &= h^2(2t-1).\end{aligned}$$

The only critical point for g is at $t = \frac{1}{2}$, which gives $g\left(\frac{1}{2}\right) = \frac{h^2}{4}$.
Since $g(x_i) = 0$ and $g(x_{i+1}) = 0$, the maximum value of $|g'(x)|$ in $[x_i, x_{i+1}]$ must occur at the critical point which implies that

$$|f(x) - P_1(x)| \le \frac{e}{2}\max_{x_i\le x\le x_{i+1}} |g(x)| \le \frac{e}{2}\cdot\frac{h^2}{4} = \frac{eh^2}{8}.$$

Consequently, to ensure that the the error in linear interpolation is bounded by 10^{-6}, it is sufficient for h to be chosen so that

$$\begin{aligned}\frac{eh^2}{8} &\le 10^{-6} \\ h &\le 0.00172.\end{aligned}$$

Thus $n = \dfrac{b-a}{h} > \dfrac{1-0}{0.00172} \approx 583.$
Hence we need to take $h = \dfrac{1-0}{583} = 0.001.$

Example 5. *Determine the spacing h in a table of equally spaced values of the function $f(x) = \sqrt{x}$ between 1 and 2, so that interpolation with a quadratic polynomial will yield an accuracy of 5×10^{-4}.*

Sol. Let $x_0, x_1, \ldots$ be the equally spaced numbers in $[1,2]$,.
Let x_{i-1}, x_i, x_{i+1} are three eqispaced points with spacing h. The truncation error of the quadratic interpolation is given by

$$f(x) - P_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_{i-1})(x-x_i)(x-x_{i+1}),$$

where $|f^{(3)}(x)| = \dfrac{3}{8x^{5/2}}$.
It is given that that the maximum error is

$$\max_{1\le x\le 2}|f(x) - P_2(x)| = \max_{1\le\xi\le 2}\left|\frac{f^{(3)}(\xi)}{3!}\right| \max_{x_{i-1}\le x\le x_{i+1}} |(x-x_{i-1})(x-x_i)(x-x_{i+1})| \le 5\times 10^{-4}.$$

Now

$$\max_{1\le x\le 2} |f^{(3)}(x)| = \frac{3}{8}.$$

To simplify the calculation, let

$$\begin{aligned}
x - x_i &= th \\
x - x_{i-1} &= x - (x_i - h) = (t+1)h \\
\text{and } x - x_{i+1} &= x - (x_i + h) = (t-1)h.
\end{aligned}$$

$$\therefore |(x - x_{i-1},)(x - x_i)(x - x_{i+1})| = h^3|t(t+1)(t-1)| = g(t) \text{ (say)}.$$

Now $g(t)$ attains its extreme values if

$$\frac{dg}{dt} = 0$$

which gives $t = \pm\frac{1}{\sqrt{3}}$. At the end points of the interval $[x_{i-1}, x_{i+1}]$, the function g becomes zero. For both the values of $t = \pm\frac{1}{\sqrt{3}}$, we obtain $\max\limits_{x_{i-1}\le x\le x_{i+1}} |g(t)| = h^3 \frac{2}{3\sqrt{3}}$.

Thus the maximum error satisfy

$$\begin{aligned}
\frac{3}{8}\frac{1}{6}\frac{2h^3}{3\sqrt{3}} &< 5\times 10^{-4} \\
h &< 0.275 \\
n &= \frac{2-1}{0.275} \approx 4.
\end{aligned}$$

3. Newton's divided difference interpolation

Suppose that $P_n(x)$ is the n-th order Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \cdots, x_n$. Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations. The divided differences of f with respect to $x_0, x_1, \cdots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \tag{3.1}$$

for appropriate constants $a_0, a_1, \cdots, a_n$.
Now we determine the first of these constants a_0. For this we substitute $x = x_0$ in $P_n(x)$ and we obtain

$$a_0 = P_n(x_0) = f(x_0).$$

Similarly, when $P_n(x)$ is evaluated at x_1, the only nonzero terms in the evaluation of $P_n(x_1)$ are the constant and linear terms,

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1),$$

so

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

The ratio $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$, is called first divided difference of $f(x)$ and in general

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}.$$

The remaining divided differences are defined recursively.
The second divided difference of three points, x_i, x_{i+1}, x_{i+2}, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

Now if we substitute $x = x_2$ and the values of a_0 and a_1 in Eqs. (3.1), we obtain

$$\begin{aligned}
P(x_2) = f(x_2) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
\implies a_2 &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}.
\end{aligned}$$

Now we define the second divided difference as

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} \\ &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}. \\ &= a_2. \end{aligned}$$

The process ends with the single n-th divided difference, defined below

$$\begin{aligned} a_n &= f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \\ &= \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}. \end{aligned}$$

We can write the Newton's divided difference formula in the following fashion (and we will prove in next Theorem).

$$\begin{aligned} P_n(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \\ &\quad \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}) \\ &= f(x_0) + \sum_{i=1}^{n} f[x_0, x_1, \cdots, x_i] \prod_{j=0}^{i-1} (x - x_j). \end{aligned}$$

The generation of the divided differences is outlined in following Table.

$f(x)$	First divided differences	Second divided differences	Third divided differences
$f[x_0]$			
	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
	$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
	$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
$f[x_3]$		$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
	$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
$f[x_4]$		$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
	$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$		
$f[x_5]$			

Example 6. *We have the following four data points:*

x	-1	0	1	2
y	3	-4	5	-6

Find a polynomial in Newton's form to interpolate the data and evaluate $f(1.5)$ (the same exercise was done by Lagrange interpolation).

x	$y=f(x)$	first d.d.	second d.d.	third d.d.
$x_0=-1$	$f(x_0)=3$			
$x_1=0$	$f(x_1)=-4$	$f[x_0,x_1]=-7$		
$x_2=1$	$f(x_2)=5$	$f[x_1,x_2]=9$	$f[x_0,x_1,x_2]=8$	
$x_3=2$	$f(x_3)=-6$	$f[x_2,x_3]=-11$	$f[x_1,x_2,x_3]=-10$	$f[x_0,x_1,x_2,x_3]=-6$

Sol. To write the Newton's form, we draw divided difference (d.d.) table as following.

$$\begin{aligned}
P_3(x) &= f(x_0)+(x+1)f[-1,0]+(x+1)(x-0)f[-1,0,1]+(x+1)(x-0)(x-1)f[-1,0,1,2] \\
&= 3-7(x+1)+8x(x+1)-6x(x+1)(x-1) \\
&= -4+7x+8x^2-6x^3. \\
\therefore f(1.5) &\approx P_3(1.5)=4.25.
\end{aligned}$$

Note that x_i can be re-ordered but must be distinct. When the order of some x_i are changed, one obtain the same polynomial but in different form.

Theorem 3.1. *Let $f\in C^n[a,b]$ and $x_0,\cdots,x_n$ are distinct numbers in $[a,b]$. Let $P_n(x)$ be the interpolating polynomial in Newton form. Then prove that there exists a point $\xi\in(a,b)$ such that*

$$f[x_0,x_1,x_2,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!}.$$

Proof. Let $f\in C^n[a,b]$ and $x_0,\cdots,x_n$ are distinct numbers in $[a,b]$.
Let

$$P_n(x)=f(x_0)+\sum_{i=1}^{n} f[x_0,x_1,\cdots,x_i](x-x_0)(x-x_1)\cdots(x-x_{i-1})$$

be the interpolating polynomial of f in Newton divided difference form. Define

$$g(x)=f(x)-P_n(x).$$

Since $P_n(x_i)=f(x_i)$ for $i=0,1,\cdots,n$, the function g has $n+1$ distinct zeros in $[a,b]$. By the generalized Rolle's Theorem there exists a point $\xi\in(a,b)$ such that

$$g^{(n)}(\xi)=f^{(n)}(\xi)-P_n^{(n)}(\xi)=0.$$

Here

$$P_n^{(n)}(x)=n!\ f[x_0,x_1,\cdots,x_n].$$

Therefore

$$f[x_0,x_1,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!}.$$

■

Example 7. *Let $f(x)=x^n$ for some integer $n\geq 0$. Let $x_0,x_1,\cdots,x_m$ be $m+1$ distinct numbers. What is $f[x_0,x_1,\cdots,x_m]$ for $m=n$? For $m>n$?*

Sol. Since we can write

$$f[x_0,x_1,\cdots,x_m]=\frac{f^{(m)}(\xi)}{m!},$$

$$\therefore f[x_0,x_1,\cdots,x_n]=\frac{n!}{n!}=1.$$

If $m>n$, then $f^{(m)}(x)=0$ as $f(x)$ is a monomial of degree n, thus $f[x_0,x_1,\cdots,x_m]=0$.

3.1. **Newton's interpolation for equally spaced points.** Newton's divided-difference formula can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing. Let $n+1$ points $x_0, x_1, \cdots, x_n$ are arranged consecutively with equal spacing h.
Let

$$h = \frac{x_n - x_0}{n} = x_{i+1} - x_i, \ i = 0, 1, \cdots, n.$$

Then each $x_i = x_0 + ih, \ i = 0, 1, \cdots, n$.
For any $x \in [a, b]$, we can write $x = x_0 + sh, \ s \in \mathbb{R}$.
Then $x - x_i = (s - i)h$.
Now Newton's interpolating polynomial is given by

$$\begin{aligned} P_n(x) &= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \ (x - x_0) \cdots (x - x_{k-1}) \\ &= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \ (s-0)h \ (s-1)h \ \cdots (s-k+1)h \\ &= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \ s(s-1) \cdots (s-k+1) \ h^k \\ &= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \ k! \binom{s}{k} h^k \end{aligned}$$

where the binomial formula

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

Now we introduce the forward difference operator

$$\triangle f(x_i) = f(x_{i+1}) - f(x_i).$$

$$\triangle^k f(x_i) = \triangle^{k-1} \triangle f(x_i) = \triangle^{k-1}[f(x_{i+1}) - f(x_i)], \ i = 0, 1, \cdots, n-1.$$

Using the $\triangle$ notation, we can write

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \triangle f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{h}\triangle f(x_1) - \frac{1}{h}\triangle f(x_0)}{2h} = \frac{1}{2!h^2} \triangle^2 f(x_0).$$

In general

$$f[x_0, x_1, \cdots, x_k] = \frac{1}{k! h^k} \triangle^k f(x_0).$$

Therefore

$$P_n(x) = P_n(x_0 + sh) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \triangle^k f(x_0).$$

This is the Newton's forward divided difference interpolation.
If the interpolation nodes are arranged recursively as $x_n, x_{n-1}, \cdots, x_0$, a formula for the interpolating polynomial is similar to previous result. In this case, Newton's divided difference formula can be written as

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}] \ (x - x_n) \cdots (x - x_{n-k+1}).$$

If nodes are equally spaced with spacing

$$h = \frac{x_n - x_0}{n}, \ x_i = x_n - (n-i)h, \ i = n, n-1, \cdots, 0.$$

Let $x = x_n + sh$.
Therefore

$$\begin{aligned}
P_n(x) &= f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}]\ (x - x_n) \cdots (x - x_{n-k+1}) \\
&= f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}]\ (s)h\ (s+1)h\ \cdots (s+k-1)h \\
&= f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1} \cdots, x_{n-k}]\ (-1)^k \binom{-s}{k}\ h^k k!
\end{aligned}$$

where the binomial formula is extended to include all real values s,

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}.$$

Like-wise the forward difference operator, we introduce the backward-difference operator by symbol ∇ (nabla) and

$$\begin{aligned}
\nabla f(x_i) &= f(x_i) - f(x_{i-1}). \\
\nabla^k f(x_i) &= \nabla^{k-1} \nabla f(x_i) = \nabla^{k-1}[f(x_i) - f(x_{i-1})].
\end{aligned}$$

Then

$$\begin{aligned}
f[x_n, x_{n-1}] &= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = \frac{1}{h} \nabla f(x_n). \\
f[x_n, x_{n-1}, x_{n-2}] &= \frac{f[x_n, x_{n-1}] - f[x_{n-1}, x_{n-2}]}{x_n - x_{n-2}} = \frac{\frac{1}{h}\nabla f(x_n) - \frac{1}{h}\nabla f(x_{n-1})}{2h} = \frac{1}{2! h^2} \nabla^2 f(x_n).
\end{aligned}$$

In general

$$f[x_n, x_{n-1}, x_{n-2} \cdots, x_{n-k}] = \frac{1}{k! h^k} \nabla^k f(x_n).$$

Therefore by using the backward-difference operator, the divided-difference formula can be written as

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} \binom{-s}{k} (-1)^k\ \nabla^k f(x_n).$$

This is the Newton's backward difference interpolation formula.

Example 8. *Using the following data and Newton's forward difference interpolation, find the interpolating polynomial.*

x	*0*	*2*	*4*	*6*
$f(x)$	*-3*	*5*	*21*	*45*

Sol. We will use Newton's forward interpolation. The forward difference table is given below.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3	8	8	0
2	5	16	8	
4	21	24		
6	45			

Here $x_0 = 0$, $h = 2$, $x = x_0 + sh$ gives $s = \dfrac{x-0}{2} = \dfrac{x}{2}$.
The Newton's forward difference polynomial is given by

$$\begin{aligned} P_2(x) &= P_2(x_0 + sh) \\ &= f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) \\ &= -3 + (x/2)(8) + \frac{(x/2)(x/2-1)}{2}(8) + 0 \\ &= -3 + 2x + x^2. \end{aligned}$$

Example 9. *Using the following data and Newton's backward difference interpolation, approximate $f(175)$.*

x	140	150	160	170	180
$f(x)$	3.685	4.854	6.302	8.076	10.225

Sol. We will use Newton's backward interpolation. The backward difference table is given below.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
140	3.685	1.169			
150	4.854	1.169			
160	6.302	1.448	0.279		
170	8.076	1.774	0.326	0.047	
180	10.225	2.149	0.375	0.049	0.002

Here $x_4 = 180$, $h = 10$, $x = 175 = x_4 + sh$ gives $s = \dfrac{175-180}{10} = -0.5$.
Thus we have

$$\begin{aligned} f(175) \approx P_2(175) &= f(x_4) - (-s)\nabla f(x_4) + \frac{(-s)(-s-1)}{2!}\nabla^2 f(x_4) - \frac{(-s)(-s-1)(-s-2)}{3!}\nabla^3 f(x_4) \\ &\quad + \frac{(-s)(-s-1)(-s-2)(-s-3)}{3!}\nabla^4 f(x_4) \\ &= 9.1005. \end{aligned}$$

Example 10. *Show that the cubic polynomials*

$$P(x) = 3 - 2(x+1) + 0(x+1)(x) + (x+1)(x)(x-1)$$

and

$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + (x+2)(x+1)(x)$$

both interpolate the given data. Why does this not violate the uniqueness property of interpolating

x	-2	-1	0	1	2
$f(x)$	-1	3	1	-1	3

polynomials?

Sol. In the formulation of $P(x)$, second point -1 is taken as initial point x_0 while in the formulation of $Q(x)$ first point is taken as initial point.
Also (alternatively without drawing the table) $P(-2) = Q(-2) = -1,\ P(-1) = Q(-1) = 3,\ P(0) = Q(0) = 1,\ P(1) = Q(1) = -1,\ P(2) = Q(2) = 3.$
Therefore both the cubic polynomials interpolate the given data. Further the interpolating polynomials are unique but format of a polynomial is not unique. If $P(x)$ and $Q(x)$ are expanded, they are identical.
The forward difference table is:

x	$f(x)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$
-2	-1				
-1	3	4			
0	1	-2	-3		
1	-1	-2	0	1	
2	3	4	3	1	0

4. Curve Fitting : Principles of Least Squares

Least-squares, also called "regression analysis", is one of the most commonly used methods in numerical computation. Essentially it is a technique for solving a set of equations where there are more equations than unknowns, i.e. an overdetermined set of equations. Least squares is a computational procedure for fitting an equation to a set of experimental data points. The criterion of the "best" fit is that the sum of the squares of the differences between the observed data points, (x_i, y_i), and the value calculated by the fitting equation, is minimum. The goal is to find the parameter values for the model which best fits the data. The least squares method finds its optimum when the sum E, of squared residuals

$$E = \sum_{i=1}^{n} {e_i}^2$$

is a minimum. A residual is defined as the difference between the actual value of the dependent variable and the value predicted by the model. Thus

$$e_i = y_i - f(x_i).$$

Least square fit of a straight line: Suppose that we are given a data set $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ of observations from an experiment. We are interested in fitting a straight line of the form $f(x) = a+bx$, to the given data. Now residuals is given by

$$e_i = y_i - (a + bx_i).$$

Note that e_i is a function of parameters a and b. We need to find a and b such that

$$E = \sum_{i=1}^{n} e_i^2$$

is minimum. The necessary condition for the minimum is given by

$$\frac{\partial E}{\partial a} = 0, \ \frac{\partial E}{\partial b} = 0.$$

The conditions yield

$$\frac{\partial E}{\partial a} = \sum_{i=1}^{n} [y_i - (a + bx_i)](-2) = 0$$

$$\implies \sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \tag{4.1}$$

$$\frac{\partial E}{\partial b} = \sum_{i=1}^{n} [y_i - (a + bx_i)](-2x_i) = 0$$

$$\implies \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2. \tag{4.2}$$

These equations (4.1-5.2) are called normal equations, which are to be solved to get desired values for a and b.

Example 11. *Obtain the least square straight line fit to the following data*

x	0.2	0.4	0.6	0.8	1
$f(x)$	0.447	0.632	0.775	0.894	1

Sol. The normal equations for fitting a straight line $y = a + bx$ are

$$\sum_{i=1}^{5} f(x_i) = 5a + b\sum_{i=1}^{5} x_i$$

$$\sum_{i=1}^{5} x_i f(x_i) = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} x_i^2$$

From the data, we have $\sum_{i=1}^{5} x_i = 3$, $\sum_{i=1}^{5} x_i^2 = 2.2$, $\sum_{i=1}^{5} f(x_i) = 3.748$, and $\sum_{i=1}^{5} x_i f(x_i) = 2.5224$.
Therefore

$$5a + 3b = 3.748, \quad 3a + 2.2b = 2.5224.$$

The solution of this system is $a = 0.3392$ and $b = 0.684$. The required approximation is $y = 0.3392 + 0.684x$.

Least square error= $\sum_{i=1}^{5}[f(x_i) - (0.3392 + 0.684x_i)^2] = 0.00245$.

Example 12. *Find the least square approximation of second degree for the discrete data*

x	-2	-1	0	1	2
$f(x)$	15	1	1	3	19

Sol. We fit a second degree polynomial $y = a + bx + cx^2$.
By principle of least squares, we minimize the function

$$E = \sum_{i=1}^{5}[y_i - (a + bx_i + cx_i^2)]^2.$$

The necessary condition for the minimum is given by

$$\frac{\partial E}{\partial a} = 0, \ \frac{\partial E}{\partial b} = 0, \ \frac{\partial E}{\partial c} = 0.$$

The normal equations for fitting a second degree polynomial are

$$\sum_{i=1}^{5} f(x_i) = 5a + b\sum_{i=1}^{5} x_i + c\sum_{i=1}^{5} x_i^2$$

$$\sum_{i=1}^{5} x_i f(x_i) = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} x_i^2 + c\sum_{i=1}^{5} x_i^3$$

$$\sum_{i=1}^{5} x_i^2 f(x_i) = a\sum_{i=1}^{5} x_i^2 + b\sum_{i=1}^{5} x_i^3 + c\sum_{i=1}^{5} x_i^4.$$

We have $\sum_{i=1}^{5} x_i = 0$, $\sum_{i=1}^{5} x_i^2 = 10$, $\sum_{i=1}^{4} x_i^3 = 0$, $\sum_{i=1}^{5} x_i^4 = 34$, $\sum_{i=1}^{5} f(x_i) = 39$, $\sum_{i=1}^{5} x_i f(x_i) = 10$, $\sum_{i=1}^{5} x_i^2 f(x_i) = 140$.
From given data

$$5a + 10c = 39$$
$$10b = 10$$
$$10a + 34c = 140.$$

The solution of this system is $a = \dfrac{-37}{35}$, $b = 1$, and $c = \dfrac{31}{7}$.

The required approximation is $y = \dfrac{1}{35}(-37 + 35x + 155x^2)$.

Example 13. *Use the method of least square to fit the curve $f(x) = c_0 x + c_1/\sqrt{x}$. Also find the least square error.*

x	0.2	0.3	0.5	1	2
$f(x)$	16	14	11	6	3

Sol. By principle of least squares, we minimize the error

$$E(c_0, c_1) = \sum_{i=1}^{5}[f(x_i) - c_0 x_i - \frac{c_1}{\sqrt{x_i}}]^2$$

We obtain the normal equations

$$c_0 \sum_{i=1}^{5} x_i^2 + c_1 \sum_{i=1}^{5} \sqrt{x_i} = \sum_{i=1}^{5} x_i f(x_i)$$

$$c_0 \sum_{i=1}^{5} \sqrt{x_i} + c_1 \sum_{i=1}^{5} \frac{1}{x_i} = \sum_{i=1}^{5} \frac{f(x_i)}{\sqrt{x_i}}.$$

We have

$$\sum_{i=1}^{5} \sqrt{x_i} = 4.1163, \sum_{i=1}^{5} \frac{1}{x_i} = 11.8333, \sum_{i=1}^{5} x_i^2 = 5.38$$

$$\sum_{i=1}^{5} x_i f(x_i) = 24.9, \sum_{i=1}^{5} \frac{f(x_i)}{\sqrt{x_i}} = 85.0151.$$

The normal equations are given by

$$5.3c_0 + 4.1163c_1 = 24.9$$

$$4.1163c_0 + 11.8333c_1 = 85.0151.$$

Whose solution is $c_0 = -1.1836, \; c_1 = 7.5961.$
Therefore, the least square fit is given as

$$f(x) = \frac{7.5961}{\sqrt{x}} - 1.1836x.$$

The least square error is given by

$$E = \sum_{i=1}^{5}[f(x_i) - \frac{7.5961}{\sqrt{x_i}} + 1.1836x_i]^2 = 1.6887$$

Example 14. *Obtain the least square fit of the form $y = ab^x$ to the following data*

x	1	2	3	4	5	6	7	8
$f(x)$	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Sol. The curve $y = ab^x$ takes the form $Y = A + Bx$ after taking log on base 10, where $Y = \log y$, $A = \log a$ and $B = \log b$.
Hence the normal equations are given by

$$\sum_{i=1}^{8} Y_i = 8A + B \sum_{i=1}^{8} x_i$$

$$\sum_{i=1}^{8} x_i Y_i = A \sum_{i=1}^{x} x_i + B \sum_{i=1}^{8} x_i^2$$

From the data, we form the following table. Substituting the values, we obtain

$$8A + 36B = 3.7393, \quad 36A + 204B = 22.7385$$

$$\implies A = 0.1656, \quad B = 0.1407$$

$$\implies a = 1.4642, \quad b = 1.3826.$$

The required curve is $y = (1.4642)(1.3826)^x$.

x	y	$Y = \log y$	xY	x^2
1	1.0	0.0	0.0	1
2	1.2	0.0792	0.1584	4
3	1.8	0.2553	0.7659	9
4	2.5	0.3979	1.5916	16
5	3.6	0.5563	2.7815	25
6	4.7	0.6721	4.0326	36
7	6.6	0.8195	5.7365	49
8	9.1	0.9590	7.6720	64
Σ 36	30.5	3.7393	22.7385	204

Exercises

(1) Find the unique polynomial $P(x)$ of degree 2 or less such that

$$P(1) = 1, \; P(3) = 27, \; P(4) = 64$$

using Lagrange interpolation. Evaluate $P(1.05)$.

(2) For the given functions $f(x)$, let $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.6$. Construct Lagrange interpolation polynomials of degree at most one and at most two to approximate $f(1.4)$, and find the absolute error.
   (a) $f(x) = \sin \pi x$
   (b) $f(x) = \sqrt[3]{x - 1}$
   (c) $f(x) = \log_{10}(3x - 1)$
   (d) $f(x) = e^{2x} - x$.

(3) Let $P_3(x)$ be the Lagrange interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$ and $(2, 2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.

(4) Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.
   (a) $f(x) = e^{2x} \cos 3x, \; x_0 = 0, \; x_1 = 0.3, \; x_2 = 0.6, \; n = 2.$
   (b) $f(x) = \sin(\ln x), \; x_0 = 2.0, \; x_1 = 2.4, \; x_2 = 2.6, \; n = 2.$

(5) Use the Lagrange interpolating polynomial of degree two or less and four-digit chopping arithmetic to approximate $\cos 0.750$ using the following values. Find an error bound for the approximation.

$$\cos 0.698 = 0.7661, \; \cos 0.733 = 0.7432, \; \cos 0.768 = 0.7193, \; \cos 0.803 = 0.6946.$$

The actual value of $\cos 0.750$ is 0.7317 (to four decimal places). Explain the discrepancy between the actual error and the error bound.

(6) If linear interpolation is used to interpolate the error function

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dt,$$

show that the error of linear interpolation using data (x_0, f_0) and (x_1, f_1) cannot exceed $\dfrac{(x_1 - x_0)^2}{2\sqrt{2\pi e}}$.

(7) Using Newton's divided difference interpolation, construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

$$f(0.43) \text{ if } f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.4816.$$

(8) Show that the polynomial interpolating (in Newton's form) the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

(9) Let $f(x) = e^x$, show that $f[x_0, x_1, \ldots, x_m] > 0$ for all values of m and all distinct equally spaced nodes $\{x_0 < x_1 < \cdots < x_m\}$.

(10) Show that the interpolating polynomial for $f(x) = x^{n+1}$ at $n+1$ nodal points $x_0, x_1, \cdots, x_n$ is given by

$$x^{n+1} - (x - x_0)(x - x_1)\cdots(x - x_n).$$

(11) The following data are given for a polynomial $P(x)$ of unknown degree

x	0	1	2	3
$f(x)$	4	9	15	18

Determine the coefficient of x^3 in $P(x)$ if all fourth-order forward differences are 1.

(12) Verify that the polynomials $P(x) = 5x^3 - 27x^2 + 45x - 21$, $Q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$ interpolate the data

x	1	2	3	4
y	2	1	6	47

and explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

(13) Let $i_0, i_1, \cdots, i_n$ be a rearrangement of the integers $0, 1, \cdots, n$. Show that $f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}] = f[x_0, x_1, \cdots, x_n]$.

(14) Let $f(x) = 1/(1+x)$ and let $x_0 = 0,\ x_1 = 1,\ x_2 = 2$. Calculate the divided differences $f[x_0, x_1]$ and $f[x_0, x_1, x_2]$. Using these divided differences, give the quadratic polynomial $P_2(x)$ that interpolates $f(x)$ at the given node points $\{x_0, x_1, x_2\}$. Graph the error $f(x) - P_2(x)$ on the interval $[0, 2]$.

(15) Construct the interpolating polynomial that fits the following data using Newton's forward and backward

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	-1.5	-1.27	-0.98	-0.63	-0.22	0.25

difference interpolation. Hence find the values of $f(x)$ at $x = 0.15$ and 0.45.

(16) For a function f, the forward-divided differences are given by

$x_0 = 0.0$	$f[x_0]$		
$x_1 = 0.4$	$f[x_1]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2] = \dfrac{50}{7}$
$x_1 = 0.4$	$f[x_2] = 6$	$f[x_1, x_2] = 10$	

Determine the missing entries in the table.

(17) A fourth-degree polynomial $P(x)$ satisfies $\Delta^4 P(0) = 24$, $\Delta^3 P(0) = 6$, and $\Delta^2 P(0) = 0$, where $\Delta P(x) = P(x+1) - P(x)$. Compute $\Delta^2 P(10)$.

(18) Show that

$$f[x_0, x_1, x_2, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}.$$

(19) Use the method of least squares to fit the linear and quadratic polynomial to the following data.

x	-2	-1	0	1	2
$f(x)$	15	1	1	3	19

(20) By the method of least square fit a curve of the form $y = ax^b$ to the following data.

x	2	3	4	5
y	27.8	62.1	110	161

(21) Use the method of least squares to fit a curve $y = c_0/x + c_1\sqrt{x}$ to the following data.

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

(22) Experiment with a periodic process provided the following data :

t°	0	50	100	150	200
y	0.754	1.762	2.041	1.412	0.303

Estimate the parameter a and b in the model $y = a + b\sin t$, using the least square approximation.

Appendix A. Algorithms

Algorithm (Lagrange Interpolation):

- Read the degree n of the polynomial $P_n(x)$.
- Read the values of $x(i)$ and $y(i) = f(x_i)$, $i = 1, \ldots, n$.
- Read the point of interpolation p.
- Calculate the Lagrange's fundamental polynomials $l_i(x)$ using the following loop:
  `for i=1 to n`
  $l(i) = 1.0$
  `for j=1 to n`
  `if` $j \neq i$
  $l(i) = \dfrac{p - x(j)}{x(i) - x(j)} * l(i)$
  `end j`
  `end i`
- Calculate the approximate value of the function at $x = p$ using the following loop:
  `sum=0.0`
  `for i=1 to n`
  `sum = sum +` $l(i) * y(i)$
  `end i`
- Print `sum`.

Algorithm (Newton's Divided-Difference Interpolation):
Given n distinct interpolation points $x_0,\ x_1, \cdots, x_n$, and the values of a function $f(x)$ at these points, the following algorithm computes the matrix of divided differences:
$D = zeros(n, n)$;
`for` $i = 1 : n$
`D(i,1) = y(i);`
`end` i
`for` $j = 2 : n$,
`for` $k = j : n$,
$D(k, j) = (D(k, j-1) - D(k-1, j-1))/(x(k) - x(k-j+1))$;
`end` i
`end` j.
Now compute the value at interpolating point p using nesting:
$fp = D(n, n)$;
`for` $i = n-1 : -1 : 1$
$fp = fp * (p - x(i)) + D(i, i)$;
`end` i
Print `Matrix` D `and` fp.

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