Lecture 38: Numerical Analysis (UMA011)

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Recall,

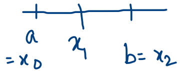
Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)\right], \quad h=b-a$$

$$\text{error} = -\frac{h^3}{12} f''(c), \quad c \in (a,b)$$

Simpson's $\frac{1}{3}$ rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0)+f(x_2)+4f(x_1)\right], \quad h=\frac{b-a}{2}$$



$$\text{error} \quad -\frac{h^5}{90} f^{iv}(c), \quad c \in (a,b)$$

Numerical Quadrature:Measuring Precision:

Degree of Precision:

The degree of precision of a quadrature formula is the largest positive integer n such that the formula is exact for x^k, for $k = 0, 1, 2, \cdots, n$.

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i) \checkmark$$

If the approximation quadrature formula is exact for $f(x) = x^0, x^1, x^2, --- x^n$ then degree of precision of this formula is n

Remark:- ① The degree of precision of Trapezoidal rule is 1

$\because$ error in trap. rule is $-\frac{h^3}{12} f''(c)$, $c \in (a,b)$

It gives exact value for x^k, for $k = 0, 1$

② The degree of precision of Simpson's $\frac{1}{3}$rd rule is 3

Numerical Quadrature:

Degree of Precision: Example

The quadrature formula $\int_0^2 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1 and c_2.

$-(*)$ $= \sum_{i=0}^{2} c_i f(x_i)$

Solution:- The given formula is exact for $f(x) = x^0, x^1, x^2$

So, Put $f(x) = x^0, x^1, x^2$ in $(*)$

for $f(x) = x^0 = 1$, we get

$$\int_0^2 1\, dx = c_0 + c_1 + c_2$$

$$\Rightarrow (x)_0^2 = c_0 + c_1 + c_2 \Rightarrow c_0 + c_1 + c_2 = 2 \quad -(1)$$

for $f(x) = x^1$

$$\int_0^2 x\,dx = C_0(0) + C_1(1) + C_2(2)$$

$$\left(\frac{x^2}{2}\right)_0^2 = C_1 + 2C_2$$

$$\left(\frac{4}{2} - 0\right) = 2 = C_1 + 2C_2 \quad - \quad (2)$$

for $f(x) = x^2$

$$\int_0^2 x^2\,dx = C_0(0) + C_1(1) + 4C_2$$

$$\left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} = C_1 + 4C_2 \quad - \quad (3)$$

Subtract ② from ③, we get

$$2C_2 = \frac{8}{3} - 2$$

$$2C_2 = \frac{2}{3}$$

$$\boxed{C_2 = \frac{1}{3}}$$

Put this value in eqⁿ ②, then

$$c_1 + \frac{2}{3} = 2$$

$$c_1 = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\boxed{c_1 = \frac{4}{3}}$$

Now, from eqn ①, we get

$$c_0 + c_1 + c_2 = 2$$

$$c_0 + \frac{1}{3} + \frac{4}{3} = 2$$

$$c_0 = 2 - \frac{5}{3}$$

$$\boxed{c_0 = \frac{1}{3}}$$

Thus,

$$\int_0^2 f(x)\,dx = \frac{1}{3} f(0) + \frac{4}{3} f(1) + \frac{1}{3} f(2)$$

$$= \frac{1}{3}\left(f(0) + f(2) + 4f(1)\right)$$

Degree of Precision: Example

Find the quadrature formula by method of undetermined coefficients $= \sum_{i=1}^{3} \alpha_i f(x_i)$

$$\int_0^1 \frac{f(x)}{\sqrt{x(1-x)}}dx = \alpha_1 f(0) + \alpha_2 f(1/2) + \alpha_3 f(1)$$ which is exact for polynomials of highest possible degree. Then use the formula to evaluate $\int_0^1 \frac{dx}{\sqrt{x - x^3}}$.

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$$\int_0^{\pi/2} \frac{2\sin\theta\cancel{\cos\theta}}{\cancel{\sqrt{\sin^2\theta\cos^2\theta}}}d\theta$$

$$= \int_0^{\pi/2} 2\,d\theta = (2\theta)_0^{\pi/2}$$

$$= \pi$$

Solution:- Let the given quadrature formula is exact for $f(x) = x^0$, then

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx = \alpha_1 + \alpha_2 + \alpha_3$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = \pi \quad - (1)$$

Put

$x = \sin^2\theta$

$$\int_0^{\pi/2} \frac{2\sin^3\theta \cancel{\cos\theta}}{\sqrt{\sin^2\theta\, \cancel{\cos^2\theta}}}\, d\theta$$

$$= \int_0^{\pi/2} 2\sin^2\theta\, d\theta$$

$$\int_0^{\pi/2} (1-\cos 2\theta)\, d\theta$$

$$\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2} = \frac{\pi}{2} - 0$$

The given formula is exact for $f(x) = x^1$

$$\Rightarrow \int_0^1 \frac{x\, dx}{\sqrt{x(1-x)}} = \alpha_1(0) + \alpha_2(1/2) + \alpha_3(1)$$

$$\frac{\pi}{2} = \frac{\alpha_2}{2} + \alpha_3 \quad - \quad (2)$$

The given formula is exact for $f(x) = x^2$

$$\int_0^1 \frac{x^2\, dx}{\sqrt{x(1-x)}} = \alpha_1(0) + \alpha_2(1/4) + \alpha_3(1)$$

$$= \frac{1}{4}\alpha_2 + \alpha_3$$

To solve $\int_0^1 \frac{x^2\,dx}{\sqrt{x(1-x)}}$, put $x = \sin^2\theta$

then we get $\int_0^1 \frac{x^2\,dx}{\sqrt{x(1-x)}} = \frac{3\pi}{8}$

$$\Rightarrow \quad \frac{\alpha_2}{4} + \alpha_3 = \frac{3\pi}{8} \quad - \quad 3$$

By solving 1, 2 & 3, we obtain

$$\alpha_1 = \frac{\pi}{4}, \quad \alpha_2 = \frac{\pi}{2}, \quad \alpha_3 = \frac{\pi}{4}$$

Thus, the quadrature formula becomes

$$\int_0^1 \frac{f(x)\,dx}{\sqrt{x(1-x)}} = \frac{\pi}{4} f(0) + \frac{\pi}{2} f(1/2) + \frac{\pi}{4} f(1)$$

for the polynomial of degree ≥ 3, and the values of $\alpha_1, \alpha_2, \& \alpha_3$ the formula is not exact.

Now, To evaluate $\int_0^1 \frac{dx}{\sqrt{x-x^3}} = \int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1+x)}}$

we use $\int_0^1 \frac{f(x)}{\sqrt{x(1-x)}} dx = \frac{\pi}{4} f(0) + \frac{\pi}{2} f(1/2) + \frac{\pi}{4} f(1)$

Here, we take $f(x) = \frac{1}{\sqrt{1+x}}$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} = \frac{\pi}{4}\left(\frac{1}{\sqrt{1+0}}\right) + \frac{\pi}{2}\left(\frac{1}{\sqrt{1+1/2}}\right) + \frac{\pi}{4}\left(\frac{1}{\sqrt{1+1}}\right)$$

$$= 2.62331$$

Numerical Quadrature:

To solve

$\int_0^{10} e^x dx$ by Trap.

$= \frac{h}{2}(e^0 + e^{10})$

$h = 10 - 0 = 10$

then error is

$-\frac{(10)^3}{12} e^c$

$\Rightarrow$ if h is large

then error is increased

Composite integration:

Now, we discuss a piecewise approach to numerical integration that uses the low-order Newton-Cotes formulas

Composite Trapezoidal Rule:

We divide the interval $[a, b]$, into n subintervals with step size $h = \frac{b-a}{n}$, and taking nodal points $a = x_0 < x_1 < \cdots < x_n = b$, where $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$.

To reduce the value of h we use piecewise approach i.e. composite Integration

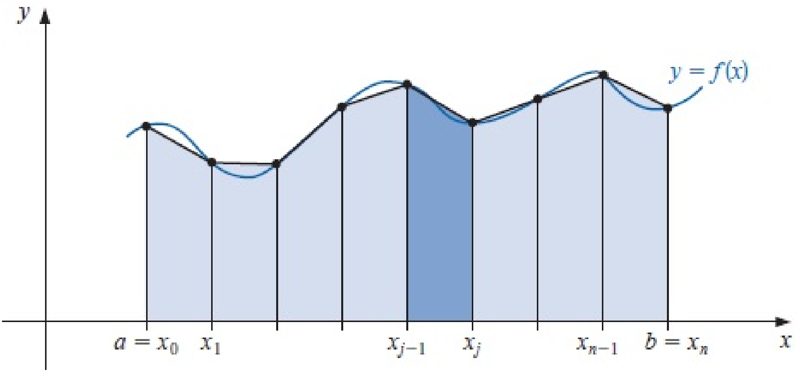
$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + --- \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$\approx \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + --- \frac{h}{2}[f(x_{n-1}) + f(x_n)]$$

$$\approx \frac{h}{2}[f(x_0) + f(x_n) + 2(f(x_1) + f(x_2) + --- f(x_{n-1}))], \quad h = \frac{b-a}{n}$$

Numerical Quadrature:

Composite Trapezoidal Rule:



Numerical Quadrature:

Degree of Precision: Exercise:

1. Find the constants c_0, c_1, and x_1 so that the quadrature formula $\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision.
2. Determine constants a, b, c, and d that will produce a quadrature formula $\int_{-1}^1 f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$ that has degree of precision 3.