Q-1

A continuous signal is band limited to 10 KHz. The signal is quantized in 4 levels of a PCM system with the probabilities 0.5, 0.25, 0.125, 0.125. Determine the Entropy and rate of information.

Solution:

- Each quantized level is act as information symbols .

$$H(X) = -\sum_{j=1}^{M} p(x_j) \log_2 \left(p(x_j)\right)$$

$$H(X) = -[0.5 \log_2(0.5) + 0.25 \log_2(0.25) + 0.125 \log_2(0.125) + 0.5 \log_2(0.125)]$$

$$H(X) = 1.75 \; bits/symbols$$

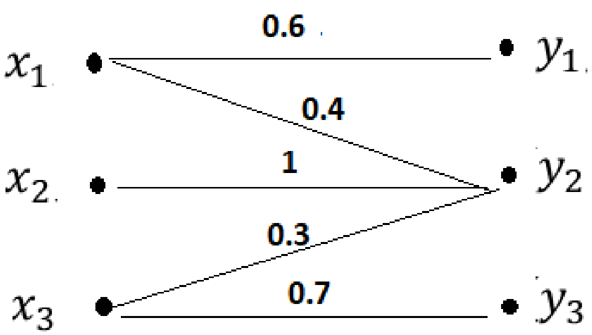
- Signal is band limited to 10 kHz then sampling rate = $2 \times 10 = 20 \; kHz$
- It means it 20,000 samples per sec
- Hence rate of information is R = $r\, H = 20{,}000 \; \frac{symbols}{sec} \times \; 1.75 \; \frac{bits}{symbols}$

Q-2

A discrete source transmits symbols or message $x_1, x_2, and\ x_3\ \ with\ the\ probabilities\ 0.25, 0.5\ and\ 0.25.$ The channel matrix is shown below

$$P(Y/X) = \begin{bmatrix} p(y_1/x_1) & p(y_2/x_1) & p(y_3/x_1) \\ p(y_1/x_2) & p(y_2/x_2) & p(y_3/x_2) \\ p(y_1/x_3) & p(y_2/x_3) & p(y_3/x_3) \end{bmatrix}$$

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$



$$P(X) = [0.25 \quad 0.5 \quad 0.25]$$

Determine all Entropies and mutual Information

Solution-Q2

$$P(Y/X) = \begin{array}{cccc} y_1 & y_2 & y_3 & \\ \left[\begin{matrix} p(y_1/x_1) & p(y_2/x_1) & p(y_3/x_1) \\ p(y_1/x_2) & p(y_2/x_2) & p(y_3/x_2) \\ p(y_1/x_3) & p(y_2/x_3) & p(y_3/x_3) \end{matrix}\right] & \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \end{array}$$

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

$$P(X) = [0.25 \quad 0.5 \quad 0.25]$$

Using Bayes Rule, the joint probability matrix P(X,Y) can be obtained by multiplying the rows of P(Y/X) by p(x1), p(x2) and p(x3)

$$p(x_1, y_1 = p(x_1)\ p(\frac{y_1}{x_1})$$

$$\boldsymbol{P(X,Y) = \begin{bmatrix} 0.6 \times 0.25 & 0.4 \times 0.25 & 0 \\ 0 & 1 \times 0.5 & 0 \\ 0 & 0.3 \times 0.25 & 0.7 \times 0.25 \end{bmatrix} = \begin{bmatrix} 0.15 & 0.1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.075 & 0.175 \end{bmatrix}}$$

Sum of all entries of P(X,Y) = 1

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

$$P(X) = [0.25 \quad 0.5 \quad 0.25]$$

Given Probabilities

$$\boldsymbol{P(X,Y)} = \begin{bmatrix} \mathbf{0.15} & \mathbf{0.1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.075} & \mathbf{0.175} \end{bmatrix}$$

Determine in First Step

Bayes Rule

$$p(x_1, y_1) = p(x_1)\ p(\frac{y_1}{x_1})$$

$p(y_1)$ $\quad$ $p(y_2)$ $\quad$ $p(y_3)$

The probabilities p(y1) p(y2) and p(y3) can be obtained by adding the columns of P(X,Y)

$$p(y_1) = p(x_1, y_1) + p(x_2, y_1) + p(x_3, y_1) = \mathbf{0.15}$$

$$p(y_2) = 0.05 + 0.5 + 0.075 = \mathbf{0.675}$$

$$p(y_3) = \mathbf{0.175}$$

$$\boldsymbol{P(Y)} = [\mathbf{0.15} \quad \mathbf{0.675} \quad \mathbf{0.175}]$$

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

$$P(X) = [0.25 \quad 0.5 \quad 0.25]$$

Given Probabilities

$$\boldsymbol{P(X,Y)} = \begin{bmatrix} \mathbf{0.15} & \mathbf{0.1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.075} & \mathbf{0.175} \end{bmatrix}$$

Determine in First Step

Bayes Rule

$$p(x_1, y_1) = p(y_1)\ p(\frac{x_1}{y_1})$$

$$\boldsymbol{P(Y)} = [\mathbf{0.15} \quad \mathbf{0.675} \quad \mathbf{0.175}]$$

The conditional probability matrix P(X/Y) can be obtained by dividing the columns of P(X,Y) by p(y1), p(y2) and p(y3)

$$\boldsymbol{P(X/Y)} = \begin{bmatrix} \mathbf{0.15/0.15} & \mathbf{0.1/.0.675} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.5/.0.675} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.075/.0.675} & \mathbf{0.175/0.175} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0.1481} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.7407} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.1112} & \mathbf{1} \end{bmatrix}$$

The sum of all columns is 1

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix} \qquad P(X) = [0.25 \quad 0.5 \quad 0.25]$$

$$\boldsymbol{P(Y) = [0.15 \quad 0.675 \quad 0.175]}$$

$$\boldsymbol{P(X,Y) = \begin{bmatrix} 0.15 & 0.1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.075 & 0.175 \end{bmatrix}}$$

$$\boldsymbol{P(X/Y) = \begin{bmatrix} 1 & 0.1481 & 0 \\ 0 & 0.7407 & 0 \\ 0 & 0.1112 & 1 \end{bmatrix}}$$

$$H(X) = -\sum_{j=1}^{M} p(x_j)\log_2\left(p(x_j)\right)$$

$$= \ -(0.25 * \log_2(0.25) \ + 0.5 * \log_2(0.5) \ + \ 0.25 * \log_2(0.25) \ )$$

H(X) = 1.5 bits /message

$$H(Y) = -\sum_{j=1}^{M} p(y_j)\log_2\left(p(y_j)\right)$$

H(Y) = 1.2333 bits /message

$$P(X,Y) = \begin{bmatrix} 0.15 & 0.1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.075 & 0.175 \end{bmatrix}$$

$$P(X) = [0.25 \quad 0.5 \quad 0.25]$$

$$P(Y) = [0.2 \quad 0.625 \quad 0.175]$$

$$P(X/Y) = \begin{bmatrix} 1 & 0.08 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0.12 & 1 \end{bmatrix}$$

Joint Entropy

$$H(X,Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j, y_k)\, log\left(\frac{1}{p(x_j, y_k)}\right)$$

H(X, Y) = 1.9631 bits /message

Conditional Entropy

$$H(X/Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j, y_k)\, log\left(\frac{1}{p\left(\frac{x_j}{y_k}\right)}\right)$$

H(X/ Y) = 0.7297 bits /message

$$H(Y/X) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j, y_k)\, log\left(\frac{1}{p\left(\frac{y_k}{x_j}\right)}\right)$$

H(Y/ X) = 0. 4631 bits /message

Mutual Information

$$I(X,Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j, y_k)\ log\left(\frac{p(x_j, y_k)}{p(x_j)\ p(y_k)}\right)$$

I(X,Y) = 0. 7703 bits /message

H(X, Y) = 1.9631 bits /message

H(X) = 1.5 bits /message

H(X/ Y) = 0.7297 bits /message

H(Y) = 1.2334 bits /message

H(Y/ X) = 0. 4631 bits /message

I(X,Y) = 0. 7703 bits /message

$$\mathrm{H}(X, Y) = H(X) + H\left(\frac{Y}{X}\right) = 1.5 + 0.4631 = 1.9631$$

$$\mathrm{H}(X, Y) = H(Y) + H\left(\frac{X}{Y}\right) = 1.2334 + 0.7297 = 1.9631$$

$$\mathrm{I}(X, Y) = H(X) - H\left(\frac{X}{Y}\right) = 1.5 - 0.7297 = 0.7703$$

$$\mathrm{I}(X, Y) = H(Y) - H\left(\frac{Y}{X}\right) = 1.2334 - 0.4631 = 0.7703$$

$$\mathrm{I}(X, Y) = H(X) + H(Y) - H(X, Y) = 1.5 + 1.2334 - 1.9631 = 0.7703$$

Solution - Q3

Consider a discrete memory less channel with independent input and output

$$P(X,Y) = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix} \longrightarrow \begin{matrix} x_1 = 0.5 \\ x_2 = 0.5 \end{matrix} \qquad P(X) = [0.5 \quad 0.5\,]$$

$$y_1 = 0.5 \qquad y_2 = 0.5 \qquad P(Y) = [0.5 \quad 0.5\,]$$

$$\boldsymbol{H(X) = log_2(M) = log_2(2) = 1\; bit\; per\; message}$$

H(X/ Y) = H(Y/X) = 0

$$\boldsymbol{H(Y) = log_2(M) = log_2(2) = 1\; bit\; per\; message}$$

$$\boldsymbol{H(X,Y) = 4 \times 0.25\;\; log_2\left(\frac{1}{0.25}\right) = 2\; bit/message}$$

$$\boldsymbol{I(X,Y) = 0.25\; log_2\left(\frac{0.25}{0.5 \times 0.5}\right) = 0}$$

$$I(X,Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j, y_k)\;\; log\left(\frac{p(x_j, y_k)}{p(x_j)\;\; p(y_k)}\right)$$

$$\mathbf{I}(\boldsymbol{X,Y}) = \boldsymbol{H(X) + H(Y) - H(X,Y) = 1 + 1 - 2 = 0}$$

$$H(X) = -\sum_{j=1}^{M} p(x_j)\log_2\left(p(x_j)\right)$$ Entropy of the transmitter

$$\boldsymbol{H(X,Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j,y_k)\, log\left(\frac{1}{p(x_j,y_k)}\right)}$$ **Joint Entropy**

$$\boldsymbol{H(X/Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j,y_k)\, log\left(\frac{1}{p\left(\frac{x_j}{y_k}\right)}\right)}$$ **Conditional Entropy**

$$\boldsymbol{H(Y/X) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j,y_k)\, log\left(\frac{1}{p\left(\frac{y_k}{x_j}\right)}\right)}$$ Conditional Entropy

$$I(X,Y) = \sum_{j=1}^{M}\sum_{k=1}^{N} p(x_j,y_k)\ log\left(\frac{p(x_j,y_k)}{p(x_j)\ p(y_k)}\right)$$ Mutual Information

$$\mathbf{H}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{H}(\boldsymbol{X}) + \boldsymbol{H}\left(\frac{\boldsymbol{Y}}{\boldsymbol{X}}\right)$$

$$\mathbf{H}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{H}(\boldsymbol{Y}) + \boldsymbol{H}\left(\frac{\boldsymbol{X}}{\boldsymbol{Y}}\right)$$

$$\mathbf{I}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{H}(\boldsymbol{X}) - \boldsymbol{H}\left(\frac{\boldsymbol{X}}{\boldsymbol{Y}}\right)$$

$$\mathbf{I}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{H}(\boldsymbol{Y}) - \boldsymbol{H}\left(\frac{\boldsymbol{Y}}{\boldsymbol{X}}\right)$$