Z –TRANSFORM

The z-transform is used to take discrete time domain signals into a complex-variable frequency domain and it opens up new ways of solving problems and designing discrete domain applications. The inverse z-transform of rational z-images and of some other image functions can be computed. The used method is via partial fraction decomposition and symbolic computation.

6.1 z–Transform

X(z) denotes the z–transform of discrete time signal x(n) and it is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (6.1)$$

where z is the complex variable and the z–transform of x(n) is also represented by the operator ‘Z’ i.e.,

$$X(z) = Z[x(n)] \qquad (6.2)$$

In equation (6.2) X(Z) and x(n) are known as z–transform pair which is represented as

$$x(n) \overset{z}{\leftrightarrow} X(z) \qquad (6.3)$$

6.2 Region of Convergence

The region of convergence are the values of z for which the z–transform converges. Equation (6.1) shows that z–transform is an infinite power series which is not always convergent for all values of z. Therefore, the region of convergence should be mentioned along with the z–transformation.

Example 6.1 Find the z–transform of the following sequences

(i) $x_1(n) = \{1,2,3,4,5,6,7\}$, (ii) $x_2(n) = \{1,2,\ 3,\ 4,5,6,\underset{\uparrow}{7}\}$

(iii) $x_3(n) = \{1,2,\ \underset{\uparrow}{3},\ 4,5,6,7\}$

Solution:

(i) For the given sequence $x_1(n) = \{1,2,3,4,5,6,7\}$, we can write $x_1(0) = 1$, $x_1(1) = 2$, $x_1(2) = 3$, $x_1(3) = 4$, $x_1(4) = 5$, $x_1(5) = 6$ and $x_1(6) = 7$.

The z–transform of the sequence $x_1(n)$ is given by

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

In the present case n =0 to n = 6.

Therefore, $X_1(z) = \sum_{n=0}^{6} x_1(n) z^{-n} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 6z^{-5} + 7z^{-6}$

$$= 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + \frac{6}{z^5} + \frac{7}{z^6}$$

$X_1(z)$ has finite values for all values of z except at z = 0. For z = 0 , X(z) = ∞. Therefore, the region of convergence is entire ROC except z = 0.

(ii) For the given sequence $x_2(n) = \{1, 2, 3, 4, 5, 6, \underset{\uparrow}{7}\}$, we can write $x_2(-6) = 1$,

$x_2(-5) = 2$, $x_2(-4) = 3$, $x_2(-3) = 4$, $x_2(-2) = 5$, $x_2(-1) = 6$ and $x_2(0) = 7$.

The z–transform of the sequence $x_1(n)$ is given by

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

In the present case n =–6 to n =0.

Therefore,

$$X_2(z) = \sum_{n=-6}^{0} x_2(n) z^{-n} = z^6 + 2z^5 + 3z^4 + 4z^3 + 5z^2 + 6z^1 + 7$$

$X_2(z)$ has finite values for all values of z except at $z = \infty$. For $z = \infty$, $X_2(z) = \infty$.

Therefore, the region of convergence is entire ROC except **at** $z = \infty$.

(iii) For the given sequence $x_3(n) = \{1, 2, \underset{\uparrow}{3}, 4, 5, 6, 7\}$, we can write $x_3(-2) = 1$,

$x_3(-1) = 2$, $x_3(0) = 3$, $x_3(1) = 4$, $x_3(2) = 5$, $x_3(3) = 6$ and $x_3(4) = 7$.

The z–transform of the sequence $x_1(n)$ is given by

$$X_3(z) = \sum_{n=-\infty}^{\infty} x_3(n) z^{-n}$$

In the present case n =–2 to n =4.

Therefore,

$$X_3(z) = \sum_{n=-2}^{4} x_3(n) z^{-n} = z^2 + 2z^1 + 3 + 4z^{-1} + 5z^{-2} + 6z^{-3} + 7z^{-4}$$

$$= z^2 + 2z^1 + 3 + \frac{4}{z} + \frac{5}{z^2} + \frac{6}{z^3} + \frac{7}{z^4}$$

$X_3(z)$ has finite values for all values of z except at $z = 0$ and $z = \infty$. For $z = 0$, and $z = \infty$, $X_3(z) = \infty$. Therefore, the region of convergence is entire ROC except $z = 0$ and $z = \infty$.

6.3 Properties of z–transform

We have already introduced the z–transform. Let us now introduce the properties of z–transform.

6.3.1 Linearity

This property states that if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$ then

$$a_1 x_1(n) + a_2 x_2(n) \overset{z}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z) \qquad (6.4)$$

where a_1 and a_2 are constants.

Proof:

Let $x(n) = a_1 x_1(n) + a_2 x_2(n)$

The z–transform of x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n} = a_1 X_1(z) + a_2 X_2(z)$$

Since the superposition principle is applicable to z–transform, it is linear and the ROC of X(z) is the overlap or intersection of the individual regions of convergence of $X_1(z)$ and $X_2(z)$.

6.3.2 Time Shifting

Proof:

The z-transform of the sequence x(n-k) is given by

$$Z\left[x(n-k)\right]=\sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \qquad (6.5)$$

Let us put p = n-k. The limits of p becomes -∞ to +∞.

$$\therefore Z\left[x(n-k)\right]=\sum_{n=-\infty}^{\infty} x(p)z^{-(p+k)}$$

$$= z^{-k}\sum_{n=-\infty}^{\infty} x(p)z^{-p} = z^{-k}X(z) \qquad (6.6)$$

This property suggests that shifting the sequence by 'k' samples in time domain is equivalent to multiplying its z–transform by z^{-k}.

6.3.3 Scaling in z–domain

This property of z–transform states that if

$$x(n) \overset{z}{\leftrightarrow} X(z) \qquad \text{ROC : } r_1 < |z| < r_2$$

$$\text{then} \quad a^n x(n) \overset{z}{\leftrightarrow} X\left(\frac{z}{a}\right) \quad \text{ROC: } |a|r_1 < |z| < |a|r_2 \qquad (6.7)$$

where 'a' is a constant.

Proof:

The z–transform of $a^n x(n)$ is given by

$$Z[x(n-k)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\left(a^{-1}z\right)^{-n}$$

$$= X\left(a^{-1}z\right) = X\left(\frac{z}{a}\right)$$

Therefore, equation (6.7) suggests that the scaling in z–domain is equivalent to multiplying by a^n in time domain.

If ROC of X(z) is $r_1 < |z| < r_2$, then ROC of $X\left(\frac{z}{a}\right)$ will be

$$r_1 < \left|\frac{z}{a}\right| < r_2 \quad \text{i.e.,} \quad |a|r_1 < |z| < |a|r_2$$

6.3.4 Time Reversal

This property of z–transform states that if $x(n) \overset{z}{\leftrightarrow} X(z)$ ROC : $r_1 < |z| < r_2$,

then $x(-n) \overset{z}{\leftrightarrow} X\left(z^{-1}\right)$ ROC : $\frac{1}{r_1} < |z| < \frac{1}{r_2}$ (6.8)

Proof:

The z–transform of x(–n) is given by

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n} \tag{6.9}$$

Substituting $-n = p$ in right hand side of equation (6.9), we get

$$Z[x(-n)] = \sum_{p=\infty}^{-\infty} x(p) z^{p} = \sum_{p=-\infty}^{\infty} x(p) \left(z^{-1}\right)^{p} = X\left(z^{-1}\right)$$

Therefore, the folding in time domain is equivalent to replacing z by z^{-1} in z– domain. Since replacement of z by z^{-1} is called inversion, folding or reflection in time domain is equivalent to inversion in z–domain.

6.3.5 Differentiation in z−domain

This property of z−transform states that if $x(n) \overset{z}{\leftrightarrow} X(z)$, then

$$nx(n) \overset{z}{\leftrightarrow} -z \frac{d\{X(z)\}}{dz} \tag{6.10}$$

Proof:

The z–transform of x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Taking differentiation of both sides with respect to z, we get

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} \frac{d}{dz}\left[x(n) z^{-n}\right] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}\left[z^{-n}\right] = \sum_{n=-\infty}^{\infty} (-n) x(n) z^{-n+1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} [nx(n)] z^{-n} = -z^{-1} Z[nx(n)]$$

i.e., $Z\left[nx(n)\right] = -z\dfrac{d}{dz}X(z)$ (6.11)

From equation (6.11), we can conclude that ROC of Z[nx(n)] is same as that of X(z).

In general we can write $n^{k}x(n) \overset{z}{\leftrightarrow} \left[-z\dfrac{d\{X(z)\}}{dz}\right]^{k}$ (6.11a)

6.3.6 Convolution in Time Domain

This property of z–transform states that if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$ then

$$x_1(n) * x_2(n) \overset{z}{\leftrightarrow} X_1(z) \cdot X_2(z) \tag{6.12}$$

Proof:

Let x(n) = $x_1(n) * x_2(n)$ which represents the convolution of $x_1(n)$and $x_2(n)$.

Therefore, we can write

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \tag{6.13}$$

The z–transform of x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[x_1(n) * x_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right\} z^{-n}$$

Interchanging the order of summation, we can write

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\left\{\sum_{n=-\infty}^{\infty} x_2(n-k)\right\} z^{-n} \tag{6.14}$$

The bracketed term in equation (6.14) is the z–transform of x(n–k). Using shifting property, we can write

$$Z[x(n-k)] = z^{-k}X(z) \tag{6.15}$$

Using equation (6.15), we can write equation (6.14) as follows

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} X(z) = \left[\sum_{k=-\infty}^{\infty} x_1(k) z^{-k}\right] \cdot X_2(z) = X_1(z) \cdot X_2(z) \tag{6.16}$$

where $X_1(z) = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k}$

Therefore, we can conclude from equation (6.16) that the convolution of two sequences in time domain is equivalent to multiplication of their z–transforms where the ROC of the product $X_1(z)$. $X_2(z)$ is the overlap or intersection of two individual sequences.

Example 6.2 Find the convolution of sequences

$$x_1 = \{\underset{\uparrow}{1}, -3, 2\} \quad \text{and} \quad x_2 = \{\underset{\uparrow}{1}, 2, 1\}$$

Solution:

Step I: Determine z–transform of individual signal sequences

$$X_1(z) = Z[x_1(n)] = \sum_{n=0}^{2} x_1(n)z^{-1} = x_1(0)z^0 + x_1(1)z^{-1} + x_1(2)z^{-2}$$

$$= 1z^0 - 3x_1(1)z^{-1} + 2z^{-2} = 1 - 3z^{-1} + 2z^{-2}$$

and $X_2(z) = Z[x_2(n)] = \sum_{n=0}^{2} x_2(n)z^{-1} = x_2(0)z^0 + x_2(1)z^{-1} + x_2(2)z^{-2}$

$$= 1z^0 + 2x_1(1)z^{-1} + 1z^{-2} = 1 + 2z^{-1} + 1z^{-2}$$

Step II: Multiplication of $X_1(z)$ and $X_2(z)$

$$X(z) = X_1(z)X_2(z) = \left(1 - 3z^{-1} + 2z^{-2}\right)\left(1 + 2z^{-1} + 1z^{-2}\right)$$

$$= 1 - 4z^{-1} + 0z^{-2} - z^{-3} + 2z^{-4}$$

Step III: Let us take inverse z–transform of X(z).

$$x(n) = \mathrm{IZT}\left[1 - 4z^{-1} + 0z^{-2} - z^{-3} + 2z^{-4}\right] = \left\{\underset{\uparrow}{1}, -4, 0, -1, 2\right\}$$

6.3.7 Correlation of Two Sequences

This property of z–transform states that if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$ then

$$\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-m) \overset{z}{\leftrightarrow} X_1(z)X_2\left(z^{-1}\right) \qquad (6.17)$$

Proof:

The following is the correlation of two sequences $x_1(n)$ and $x_2(n)$:

$$r_{x_1 x_2}(m) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-m)$$

Arranging the term $x_2(n-l)$ as $x_2[-(l-n)]$ in above equation we get

$$r_{x_1x_2}(m) = \sum_{n=-\infty}^{\infty} x_1(n)x_2[-(m-n)] \tag{6.18}$$

Therefore, the RHS of equation (6.18) represents the convolution of $x_1(n)$ and $x_2(-m)$. Equation (6.18) can be written as

$$r_{x_1x_2}(m) = x_1(m) * x_2(-m)$$

Taking z–transform of both sides of the above equation, we get

$$Z\left[r_{x_1x_2}(m)\right] = Z\left[x_1(m) * x_2(-m)\right] \tag{6.19}$$

Using the convolution property we can write

$$Z\left[r_{x_1x_2}(m)\right] = Z\left[x_1(m)\right] \cdot Z\left[x_2(-m)\right] \tag{6.20}$$

In equation (6.20), $Z\left[x_1(m)\right] = X_1(z)$ and $Z\left[x_2(-m)\right] = X_2\left(z^{-1}\right)$

Hence equation (6.20) can be written as follows

$$Z\left[r_{x_1x_2}(m)\right] = X_1(z)\cdot X_2\left(z^{-1}\right) \tag{6.21}$$

where the ROC of the z–transform of correlation is the intersection or overlap of ROC of two individual sequences.

6.3.8 Multiplication of Two Sequences

Multiplication of two sequences

This property of z–transform states that if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$ then

$$x_1(n) \cdot x_2(n) \overset{z}{\leftrightarrow} \frac{1}{2\pi j} \oint_c X_1(\nu)\, X_2\left(\frac{z}{\nu}\right) z^{-1} d\nu \tag{6.22}$$

where C is the closed contour which encloses the origin and lies in the ROC that is common to both $X_1(\nu)$ and $X_2\left(\frac{1}{\nu}\right)$.

Proof:

Let $x(n) = x_1(n).x_2(n)$ (6.23)

Inverse z–transform of $x_1(n)$ is given by

$$x_1(n) = \frac{1}{2\pi j} \oint_c X_1(v)\, v^{n-1} dv$$

(6.24)

Using equation (6.24) in equation (6.23), we get

$$x(n) = x_1(n) \cdot x_2(n) = \frac{1}{2\pi j} \oint_c X_1(v) v^{n-1} dv \;.x_2(n)$$

(6.25)

Taking z–transform of equation (6.25), we get

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint_c X_1(v) v^{n-1} dv \;.x_2(n) \right] z^{-n}$$

$$X(z) = \frac{1}{2\pi j} \oint_c X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{z}{v}\right)^{-n} \right] v^{-1} dv$$

(6.26)

The bracketed term in equation (6.26) is equal to $X_2\left(\frac{z}{v}\right)$.

Therefore, we can write equation (6.26) as follows:

$$X(z) = \frac{1}{2\pi j} \oint_c X_1(\nu) \cdot X_2\left(\frac{z}{\nu}\right) \nu^{-1} d\nu$$

(6.27)

Equation (6.27) gives the required result.

If the ROC of $X_1(\nu)$ and $X_2(\nu)$ be $a_1 < |\nu| < b_1$ and $a_2 < |z| < b_2$ respectively, then the ROC of

$X_2\left(\frac{z}{\nu}\right)$ will be $a_2 < \left|\frac{z}{\nu}\right| < b_2$.

Therefore, ROC of X(z) will be $a_2|\nu| < |z| < b_2|\nu|$ where the range of values of ν is

$a_1 < |\nu| < b_1$.

6.3.9 Conjugate of a Complex Sequence

This property of z–transform states that if x(n) is a complex sequence and if $x(n) \overset{Z}{\leftrightarrow} X(z)$,

$$\text{then } x^*(n) \overset{Z}{\leftrightarrow} \left[X\left(z^*\right)\right]^* \qquad (6.28)$$

Proof:

The z–transform of $x^*(n)$ is given by

$$Z\left[x^*(n)\right] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

i.e., $$Z\left[x^*(n)\right] = \left[\sum_{n=-\infty}^{\infty} x(n)\left(z^*\right)^{-n}\right]^* = \left[\sum_{n=-\infty}^{\infty} x(n)\left(z^*\right)^{-n}\right]^*$$

i.e., $$Z\left[x^*(n)\right] = \left[\sum_{n=-\infty}^{\infty} x(n)\left(z^*\right)^{-n}\right]^* \tag{6.29}$$

The bracketed term in equation (6.29) is equal to $X(z^*)$. Therefore, equation (6.29) can be written as

$$Z\left[x^*(n)\right] = \left[X\left(z^*\right)\right]^* = X^*\left(z^*\right) \tag{6.30}$$

The ROC of z–transform of conjugate sequence is identical to that of X(z).

6.3.10 z–transform of Real Part of Sequence

This property of z–transform states that if x(n) is a complex sequence and if $x(n) \overset{z}{\leftrightarrow} X(z)$, then

$$\mathrm{Re}[x(n)] \overset{z}{\leftrightarrow} \frac{1}{2}\left[X(z) + X^*\left(z^*\right)\right] \tag{6.31}$$

Proof:

Let us consider the RHS of equation (6.31).

$$\text{R.H.S.} = \frac{1}{2}\left[X(z) + X^*\left(z^*\right)\right] = \frac{1}{2}\left[X(z) + X^*\left(z^*\right)\right] \tag{6.32}$$

[Using conjugate property]

Again, z–transform satisfies the linearity property, hence we can write equation (6.32) as

$$\frac{1}{2}\left[X(z) + X^*\left(z^*\right)\right] = \frac{1}{2} Z\left[x(n) + x^*(n)\right] \tag{6.33}$$

Again, we know that $x(n) = \mathrm{Re}[x(n)] + j\mathrm{Im}[x(n)]$ (6.34)

and $x^*(n) = \mathrm{Re}[x(n)] - j\mathrm{Im}[x(n)]$ (6.35)

Using equations (6.34) and (6.35), we can write equation (6.33) as

$$\frac{1}{2}\left[X(z) + X^*\left(z^*\right)\right] = Z\{\mathrm{Re}[x(n)]\} \tag{6.36}$$

The ROC of X(z) is included in the ROC of above z–transform.

6.3.11 z–transform of Imaginary Part of Sequence

This property of z–transform states that if x(n) is a complex sequence and if $x(n) \overset{z}{\leftrightarrow} X(z)$, then

$$\mathrm{Im}[x(n)] \overset{z}{\leftrightarrow} \frac{1}{2j}\left[X(z) - X^*\left(z^*\right)\right] \tag{6.37}$$

Proof:

Let us consider the RHS of equation (6.31).

$$\text{R.H.S.} = \frac{1}{2j}\left[X(z) - X^*\left(z^*\right)\right] = \frac{1}{2j}\left[X(z) - X^*\left(z^*\right)\right] \tag{6.38}$$

[Using conjugate property]

Again, z–transform satisfies the linearity property, hence we can write equation (6.38) as

$$\frac{1}{2j}\left[X(z)-X^{*}\left(z^{*}\right)\right]=\frac{1}{2j}Z\left[x(n)-x^{*}(n)\right] \tag{6.39}$$

Again, we know that $x(n)=\mathrm{Re}[x(n)]+j\mathrm{Im}[x(n)]$ (6.40)

and $x^{*}(n)=\mathrm{Re}[x(n)]-j\mathrm{Im}[x(n)]$ (6.41)

Using equations (6.40) and (6.41), we can write equation (6.39) as

$$\frac{1}{2j}\left[X(z)-X^{*}\left(z^{*}\right)\right]=Z\{\mathrm{Im}[x(n)]\} \tag{6.42}$$

The ROC of X(z) is included in the ROC of above z–transform.

6.3.12 Initial Value Theorem

The initial value for a single sided sequence x(n) is x(0) whereas this is x(–∞) for double sided sequence. It is difficult to get this value from the knowledge of one sided z–transform. This theorem states that for a causal sequence x(n), x(0) can obtained by the knowledge of X(z), i.e., one–sided z–transform of x(n) i.e.,

$$x(0)=\underset{z\to\infty}{\mathrm{Lt}}\;X(z)$$

Proof:

From definition of z–transform, the z–transform of the sequence x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{6.43}$$

Since x(n) is causal sequence, x(n) = 0 for n < 0. Hence the equation (6.43) becomes

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots\cdots\cdots$$

taking the limits as z→∞, we can write

$$\underset{z \to \infty}{\mathrm{Lt}} X(z) = \underset{z \to \infty}{\mathrm{Lt}} \left[x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots\cdots\cdots \right]$$

$$= x(0) + 0 + 0 + \cdots\cdots\cdots$$

$$\text{i.e., } x(0) = \underset{z \to \infty}{\mathrm{Lt}} X(z) \tag{6.44}$$

Equation (6.44) gives the initial value theorem.

6.3.13 Final Value Theorem

The final value states that if a sequence x(n) has finite value as $n \to \infty$ called as $x(\infty)$, then this value can be determined by the knowledge of its one–sided z–transform i.e.,

$$\underset{n \to \infty}{\mathrm{Lt}}\ x(n) = x(\infty) = \underset{z \to 1}{\mathrm{Lt}}\ [(z-1)X(z)]$$

Proof:

For a one sided sequence x(n), we can write its z–transform as follows:

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots\cdots \qquad (6.45)$$

Again, we can show that

$$Z[x(n+1)u(n)] = zX(n) - zx(0) \qquad (6.46)$$

Subtracting equation (6.45) from equation (6.46), we get

$$Z[x(n+1)u(n)] - Z[x(n)] = (z-1)X(z) - zx(0)$$

i.e., $[x(1)-x(0)]z^0 + [x(2)-x(1)]z^1 + [x(3)-x(2)]z^2 + \cdots\cdots = (z-1)X(z) - zx(0)$ (6.47)

Taking z→1, we can write equation (6.47) as

$$[x(1)-x(0)]+[x(2)-x(1)]+[x(3)-x(2)]+\cdots+[x(\infty)-x(\infty-1)]$$

$$= -x(0)+\underset{z\to 1}{\mathrm{Lt}}\ (z-1)X(z)$$

i.e., $-x(0)+x(\infty)=-x(0)+\underset{z\to 1}{\mathrm{Lt}}\ (z-1)X(z)$

i.e., $x(\infty)=\underset{z\to 1}{\mathrm{Lt}}\ (z-1)X(z)$ (6.48)

Example 6.3 Find the initial value and final value of x(n) if its z–transform X(z) is given by

$$X(z)=\frac{0.5z^2}{(z-1)\left(z^2-0.85z+0.35\right)}$$

Solution:

The initial value x(0) is given by

$$x(0) = \underset{z \to \infty}{\text{Lt}} \; X(z) = \underset{z \to \infty}{\text{Lt}} \; \frac{0.5z^2}{(z-1)\left(z^2 - 0.85z + 0.35\right)} = \underset{z \to \infty}{\text{Lt}} \; \frac{0.5z^2}{z\left(z^2\right)} = 0$$

The final value or steady value of x(n) is given by

$$x(\infty) = \underset{z \to 1}{\text{Lt}} \; (z-1)X(z) = \frac{0.5}{1 - 0.85 + 0.35} = 1.0$$

6.3.13 z–transform of Partial Sum

If $x(n) \overset{z}{\leftrightarrow} X(z)$, then we can write that

$$\sum_{-\infty}^{\infty} x(n) \leftrightarrow \frac{X(z)}{1 - z^{-1}}$$

Proof:

Let $y(m) = \sum_{-\infty}^{m} x(n) = \cdots\cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots\cdots + x(m-1) + x(m)$

then we can write that

$$y(m-1) = \sum_{-\infty}^{m-1} x(n) = \cdots\cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots\cdots + x(m-1)$$

Therefore, subtracting the above two equations, we get

$$y(m)-y(m-1)=\sum_{-\infty}^{m}x(n)-\sum_{-\infty}^{m-1}x(n)=x(m)$$

The z–transform of the above equation gives

$$Y(z)-z^{-1}Y(z)=X(z)$$

i.e., $Y(z)\left[1-z^{-1}\right]=X(z)$

i.e., $$Y(z)=\frac{X(z)}{1-z^{-1}} \tag{6.49}$$

6.3.14 Parseval's Theorem

The Parseval's theorem states that

$$\sum_{n=-\infty}^{\infty}x_1(n)x_2^*(n)=\frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v}\right)v^{-1}dv$$

where $x_1(n)$ and $x_2(n)$ are complex valued sequences.

Proof:

The inverse z–transform of $x_1(n)$ is given as

$$x_1(n) = \frac{1}{2\pi j} \quad X_1(\nu)\nu^{n-1}d\nu \qquad (6.50)$$

Now, $\sum\limits_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \sum\limits_{n=-\infty}^{\infty} \frac{1}{2\pi j} \oint\limits_c X_1(\nu)\nu^{n-1}d\nu\, x_2^*(n)$ [using equation (6.50)]

$$= \frac{1}{2\pi j}\oint\limits_c X_1(\nu) \sum_{n=-\infty}^{\infty}\left[x_2^*(n)\nu^{n-1}d\nu\right]$$

$$= \frac{1}{2\pi j}\oint\limits_c X_1(\nu) \sum_{n=-\infty}^{\infty}\left[x_2^*(n)\nu^{n}\nu^{-1}d\nu\right]$$

$$= \frac{1}{2\pi j}\oint\limits_c X_1(\nu) \sum_{n=-\infty}^{\infty}\left[x_2^*(n)\left(\nu^{-n}\right)^{-1}\nu^{-1}\right]d\nu$$

$$= \frac{1}{2\pi j} \oint_c X_1(\nu) \sum_{n=-\infty}^{\infty} \left[x_2^*(n) \left(\nu^{-1} \right)^{-n} \nu^{-1} \right] d\nu$$

$$= \frac{1}{2\pi j} \oint_c X_1(\nu) \sum_{n=-\infty}^{\infty} \left[x_2^*(n) \left(\frac{1}{\nu} \right)^{-n} \nu^{-1} \right] d\nu$$

$$= \frac{1}{2\pi j} \oint_c X_1(\nu) \sum_{n=-\infty}^{\infty} \left[x_2(n) \left(\frac{1}{\nu^*} \right)^{-n} \nu^{-1} \right]^* d\nu$$

$$= \frac{1}{2\pi j} \oint_c X_1(\nu) \; X_2^*\left(\frac{1}{\nu} \right) \nu^{-1} \; d\nu \qquad (6.51)$$

6.4 z–transform of right sided exponential sequences

The right sided exponential sequence can be determined mathematically as

$$x(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

This sequence exists for positive values of n.

Figure 6.1 shows the plot of this sequence.

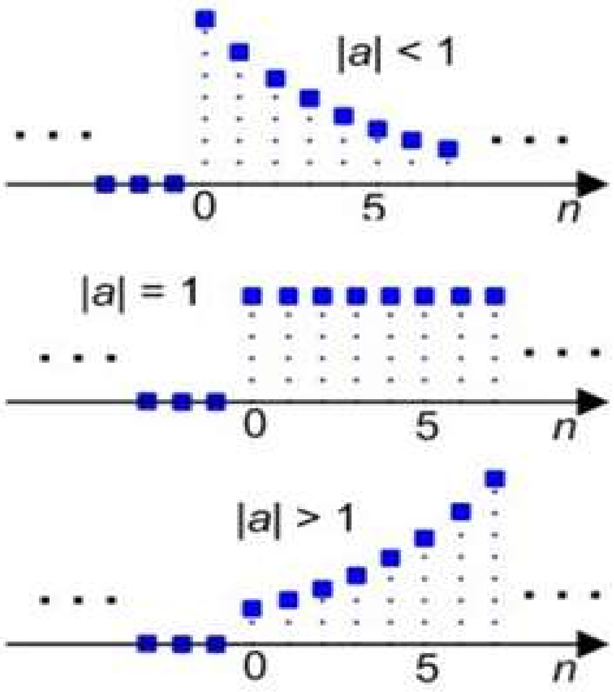

Figure 6.1

The z–transform of x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here $x(n) = a^n$ for $n \geq 0$ and hence the above equation becomes

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

Expanding the above equation we can write

$$X(z) = \left(az^{-1}\right)^1 + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 + \left(az^{-1}\right)^4 + \cdots\cdots\cdots$$

$$= \frac{1}{1 - az^{-1}} \tag{6.52}$$

X(z) will converge for $\left|az^{-1}\right| < 1$ i.e., $|z| > |a|$.

Hence the required z–transform is $= \dfrac{1}{1-az^{-1}}$ and ROC is $|z| > |a|$ shown in Fig. 6.2.

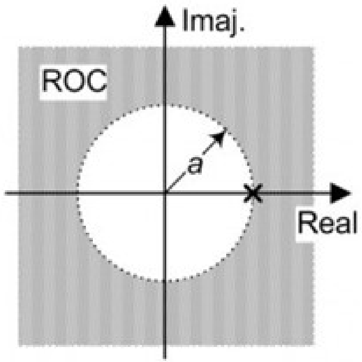

Fig. 6.2

6.5 z–transform of Left sided exponential sequences

The **left** sided exponential sequence can be determined mathematically as

$$x(n) = \begin{cases} -a^n u(-n-1) & \text{for } n \le 0 \\ 0 & \text{for } n > 0 \end{cases}$$

This sequence exists only for negative values of n.

Figure 6.3 shows the plot of this sequence.

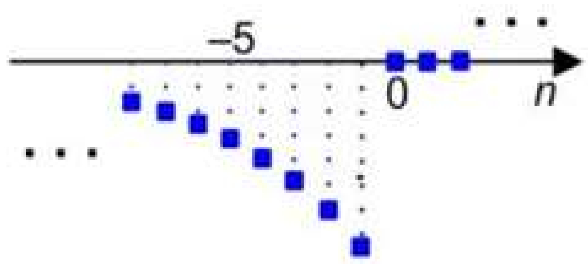

Figure 6.3 Plot of x(n)

The z–transform of x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here x(n) = a^n for n ≤ 0 and hence the above equation becomes

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{m=\infty}^{1} \left(a^{-1} z\right)^m$$

where m = –n.

Expanding the above equation we can write

$$X(z) = -\left[\left(a^{-1}z\right)^1 + \left(a^{-1}z\right)^2 + \left(a^{-1}z\right)^3 + \left(a^{-1}z\right)^4 + \cdots\cdots\right]$$

$$= -\left(a^{-1}z\right)^1\left[1 + \left(a^{-1}z\right)^1 + \left(a^{-1}z\right)^2 + \left(a^{-1}z\right)^3 + \cdots\cdots\right]$$

$$= -\frac{a^{-1}z}{1-a^{-1}z} = 1 - \frac{1}{1-a^{-1}z} \qquad (6.53)$$

X(z) will converge for $\left| a^{-1}z \right| < 1$ i.e., $|z| < |a|$.

Hence the required z–transform is $= \dfrac{1}{1-a^{-1}z}$ and ROC is $|z| < |a|$ shown in Fig. 6.4

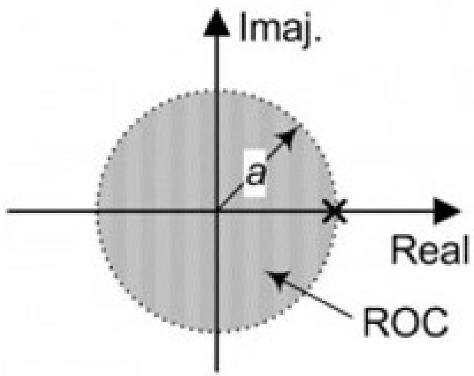

Fig. 6.4 ROC

6.6 Finite Length sequence:

Finite the z–transform of the following finite length sequence:

$$x(n)=\begin{cases}a^{n} & \text{for} 0 \le n \le N-1\\ 0 & \text{otherwise}\end{cases}$$

Solution:

The z–transform of x(n) is given by

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Let us put x(n) in the above equation.

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}=\sum_{n=-\infty}^{\infty} a^{n}z^{-n}$$

$$=\sum_{n=-\infty}^{-1}\left(a^{-1}z\right)^{-n}+\sum_{n=0}^{N-1}\left(a^{-1}z\right)^{-n}+\sum_{n=N}^{\infty}\left(a^{-1}z\right)^{-n}$$

$$=\sum_{n=0}^{N-1}\left(a^{-1}z\right)^{-n}$$

We know that

$$\sum_{m=N_1}^{N_2} \beta^m = \frac{\beta^{N_1} - \beta^{N_2+1}}{1-\beta} \quad N_2 > N_1$$

Using this result, we can write the above expression as follows:

$$X(z) = \frac{\beta^0 - \beta^{N-1+1}}{1-\beta} = \frac{1-\beta^N}{1-\beta} = \frac{1-\left(a^{-1}z\right)^N}{1-a^{-1}z} \tag{6.54}$$

This z–transform will be converged if $\sum_{n=0}^{N-1} \left(a^{-1}z\right)^{-n}$ has finite value i.e., $\sum_{n=0}^{N-1} \left(a^{-1}z\right)^{-n} < \infty$

.

The above sum will be finite because this sum will be finite if $a^{-1}z$ is finite i.e., $\left|a^{-1}z\right| < \infty$.

The above condition implies that $|a|<\infty$ and $z \neq 0$. If $|a|$ is finite the ROC is entire z–plane except z = 0.

This z–transform will be converged if $\sum_{n=0}^{N-1}\left(a^{-1}z\right)^{-n}$ has finite value i.e., $\sum_{n=0}^{N-1}\left(a^{-1}z\right)^{-n} < \infty$

.

The above sum will be finite because this sum will be finite if $a^{-1}z$ is finite i.e., $\left|a^{-1}z\right| < \infty$.

The above condition implies that |a|<∞ and z≠ 0. If |a| is finite the ROC is entire z–plane except z = 0.

6.7 z–transform of unit sample sequence

The unit sample sequence is given below:

$$\delta(n) = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{elsewhere} \end{cases}$$

The z–transform of x(n) by definition can be written as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1z^{0} = 1 \qquad (6.55)$$

Therefore, the z–transform of unit sample sequence is 1 which is constant and does not depend on values of z. Hence ROC is entire z–plane.

6.8 z–transform of delayed unit sample sequence

The delayed unit sample sequence is expressed by

$x_1(n) = \delta(n-k)$ for right shift and also by $x_2(n) = \delta(n+k)$ for left shift.

We have already shown that $Z[\delta(n)] = 1$ where ROC is entire z–plane.

Using shifting property of z–transform we can write

$$Z[x(n-k)] = z^{-k}X(z)$$

Applying this property to $x_1(n) = \delta(n-k)$, we have

$$Z[x_1(n)] = Z[\delta(n-k)] = z^{-k}Z[\delta(n)] = z^{-k} \cdot 1 = z^{-k} \qquad (6.56)$$

$Z[x_1(n)]$ has infinite value at $z = 0$ whereas it has finite values for all remaining values of z

Therefore the ROC of $Z[x_1(n)]$ is entire z–plane except at $z = 0$.

Applying this property to $x_2(n) = \delta(n+k)$, we have

$$Z[x_2(n)] = Z[\delta(n+k)] = z^{k}Z[\delta(n)] = z^{k} \cdot 1 = z^{k}$$

$Z[x_2(n)]$ has infinite value at $z = \infty$ whereas it has finite values for all remaining values of z Therefore the ROC of $Z[x_2(n)]$ is entire z–plane except at $z = \infty$.

6.9 z–transform of unit step sequence

The unit step sequence can be represented by

$$x(n) = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

The z–transform of $x(n) = a^n u(n)$ is given by

$$X(z) = \frac{1}{1 - az^{-1}} \qquad \text{ROC is } |z| > |a|$$

If a =1, X(z) becomes

$$X(z) = \frac{1}{1 - z^{-1}} \tag{6.57}$$

The ROC is $|z| > 1$ shown in Fig. 6.5

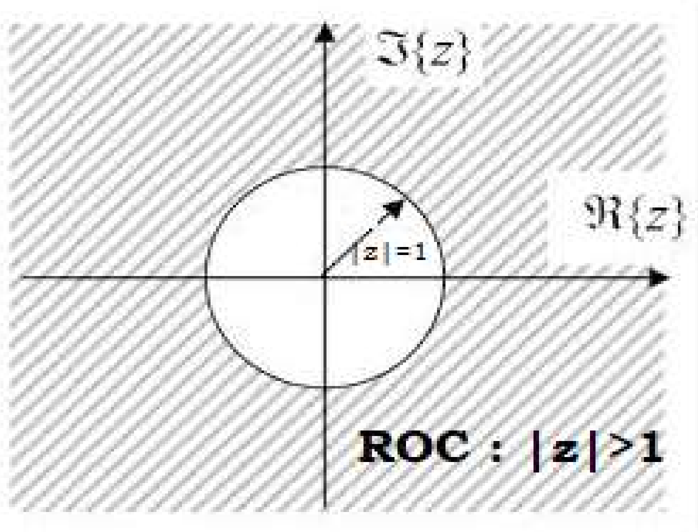

Fig.6.5　ROC

6.10 z–transform of folded unit step sequence

The folded unit step sequence can be represented by

$$x(n) = \begin{cases} 1 & \text{for} \quad n \le 0 \\ 0 & \text{elsewhere} \end{cases}$$

The z–transform of unit step sequence is given by

$$X(z) = \frac{1}{1 - z^{-1}} \qquad \text{where ROC is } |z| > 1.$$

Using the time reversal property of z–transform we can write hat

if $\quad x(n) \overset{z}{\leftrightarrow} X(z) \qquad \text{ROC is} \quad r_1 < |z| < r_2$

then $\quad x(-n) \overset{z}{\leftrightarrow} X\left(z^{-1}\right) \qquad \text{ROC is} \quad \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Therefore, the z–transform of u(–n) using the above property we can write

$$Z[u(-n)] = \frac{1}{1 - z} \qquad \text{ROC} : |z| < 1 \tag{6.58}$$

6.11 z–transform of the signal $x(n) = na^n u(n)$

Let us take $x_1(n) = a^n u(n)$

The z–transform of $x_1(n) = a^n u(n)$ is given by

$$Z\left[a^n u(n)\right] = \frac{1}{1 - az^{-1}} \qquad \text{ROC} \quad |z| > |a|$$

i.e., $X_1[z] = \dfrac{1}{1 - az^{-1}} \qquad \text{ROC} \quad |z| > |a|$

The given function is

$x(n) = na^n u(n) = nx_1(n)$ because $x_1(n) = a^n u(n)$.

$\therefore X[z] = Z[nx_1(n)]$

By differentiation property of z–transform, we can write

$$Z[nx_1(n)] = -z\frac{dX_1(z)}{dz} \qquad \text{ROC : } |z|>|a|$$

$$= -z\frac{d}{dz}\left[\frac{1}{1 - az^{-1}}\right] = \frac{az^{-1}}{\left(1 - az^{-1}\right)^2} \qquad \text{ROC : } |z|>|a| \qquad (6.59)$$

Therefore, the z–transform of $x(n) = na^n u(n)$ is $\dfrac{az^{-1}}{\left(1 - az^{-1}\right)^2}$ having ROC : $|z|>|a|$.

6.12 z–transform of unit ramp sequence

The unit ramp sequence is given by

$x(n) = nu(n)$

Hence, $X(z) = Z[nu(n)]$

Let us put a =1 in expression of z–transform of $na^n u(n)$, we get

$$Z[nu(n)] = \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \quad \text{ROC : } |z| > 1. \tag{6.60}$$

6.13 z–transform of causal cosine sequence

The causal cosine sequence is given by

$x(n) = \cos(n\omega)\, u(n)$

Using Euler's identity, we can write $\cos(n\omega)$ as follow:

$$\cos(n\omega) = \frac{e^{jn\omega} + e^{-jn\omega}}{2}$$

$$x(n) = \frac{1}{2}\left[e^{jn\omega} + e^{-jn\omega}\right]u(n) = \frac{1}{2}e^{jn\omega}u(n) + \frac{1}{2}e^{-jn\omega}u(n)$$

The z–transform of above equation is given below.

$$X(z) = Z\left[\frac{1}{2}e^{jn\omega}u(n) + \frac{1}{2}e^{-jn\omega}u(n)\right] = Z\left[\frac{1}{2}e^{jn\omega}u(n)\right] + Z\left[\frac{1}{2}e^{-jn\omega}u(n)\right]$$

Let us put $\alpha=e^{j\omega}$ and $\beta= e^{-j\omega}$ in the above expression, we get

$$X(z) = Z\left[\frac{1}{2}\alpha^n u(n)\right] + Z\left[\frac{1}{2}\beta^n u(n)\right] \tag{6.61}$$

We know that $Z\left[\alpha^n u(n)\right] = \dfrac{1}{1-\alpha z^{-1}}$ ROC : $|z| > |\alpha|$

where $\alpha=e^{j\omega} = \cos\omega + \sin\omega$

$$|\alpha| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$$

$$Z\left[\alpha^n u(n)\right] = \frac{1}{1-e^{j\omega}z^{-1}} \qquad \text{ROC : } |z| > 1 \tag{6.62}$$

Similarly, $Z\left[\beta^n u(n)\right] = \dfrac{1}{1-\beta z^{-1}} = \dfrac{1}{1-e^{-j\omega}z^{-1}}$ ROC : $|z| > 1$ (6.63)

Substituting the individual z–transform values from above equations (6.62) and (6.63) to equation (6.61), the expressions of X(z), we get

$$X(z) = \frac{1}{2}\left[\frac{1}{1-e^{j\omega}z^{-1}} + \frac{1}{1-e^{-j\omega}z^{-1}}\right] \qquad \text{ROC : } |z| > 1$$

$$X(z)=\frac{1}{2}\left[\frac{1}{1-e^{j\omega}z^{-1}}+\frac{1}{1-e^{-j\omega}z^{-1}}\right] \quad \text{ROC}: |z|>1$$

$$=\frac{1}{2}\left[\frac{1-e^{-j\omega}z^{-1}+1-e^{j\omega}z^{-1}}{\left(1-e^{j\omega}z^{-1}\right)\left(1-e^{-j\omega}z^{-1}\right)}\right]=\frac{1}{2}\left[\frac{2-\left(e^{-j\omega}+e^{j\omega}\right)z^{-1}}{1-\left(e^{-j\omega}+e^{j\omega}\right)z^{-1}+z^{-2}}\right]$$

$$=\frac{1}{2}\left[\frac{2-2z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}\right]=\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}} \quad \text{ROC}: |z|>1$$

$$\cos(n\omega)u(n)\overset{z}{\leftrightarrow}\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}} \quad \text{ROC}: |z|>1 \tag{6.64}$$

6.14 z–transform of causal sine sequence

The causal sine sequence is given by

$$x(n) = \sin(n\omega)\, u(n)$$

Using Euler's identity, we can write cos(nω) as follow:

$$\sin(n\omega)=\frac{e^{jn\omega}-e^{-jn\omega}}{2j}$$

$$x(n)=\frac{1}{2j}\left[e^{jn\omega}-e^{-jn\omega}\right]u(n)=\frac{1}{2j}e^{jn\omega}u(n)-\frac{1}{2j}e^{-jn\omega}u(n)$$

The z–transform of above equation is given below.

$$X(z) = Z\left[\frac{1}{2j}e^{jn\omega}u(n) - \frac{1}{2j}e^{-jn\omega}u(n)\right] = Z\left[\frac{1}{2j}e^{jn\omega}u(n)\right] - Z\left[\frac{1}{2j}e^{-jn\omega}u(n)\right]$$

Let us put $\alpha = e^{j\omega}$ and $\beta = e^{-j\omega}$ in the above expression, we get

$$X(z) = Z\left[\frac{1}{2j}\alpha^n u(n)\right] - Z\left[\frac{1}{2j}\beta^n u(n)\right] \tag{6.65}$$

We know that $Z\left[\alpha^n u(n)\right] = \dfrac{1}{1-\alpha z^{-1}}$ ROC : $|z| > |\alpha|$

where $\alpha = e^{j\omega} = \cos\omega + \sin\omega$

$$|\alpha| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$$

$$Z\left[\alpha^n u(n)\right] = \frac{1}{1-e^{j\omega}z^{-1}} \qquad \text{ROC : } |z| > 1 \tag{6.66}$$

Similarly, $$Z\left[\beta^n u(n)\right] = \frac{1}{1-\beta z^{-1}} = \frac{1}{1-e^{-j\omega}z^{-1}} \qquad \text{ROC : } |z| > 1 \tag{6.67}$$

Substituting the individual z−transform values from above equations (6.66) and (6.67) to equation (6.65), we get

$$X(z) = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega}z^{-1}} - \frac{1}{1-e^{-j\omega}z^{-1}}\right] \quad \text{ROC} : |z| > 1$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j\omega}z^{-1}-1+e^{j\omega}z^{-1}}{\left(1-e^{j\omega}z^{-1}\right)\left(1-e^{-j\omega}z^{-1}\right)}\right] = \frac{1}{2j}\left[\frac{\left(e^{j\omega}-e^{-j\omega}\right)z^{-1}}{1-\left(e^{-j\omega}+e^{j\omega}\right)z^{-1}+z^{-2}}\right]$$

$$= \frac{1}{2j}\left[\frac{2\,z^{-1}\sin\omega}{1-2\,z^{-1}\cos\omega\ +z^{-2}}\right] = \frac{z^{-1}\sin\omega}{1-2\,z^{-1}\cos\omega\ +z^{-2}} \quad \text{ROC: } |z| > 1$$

$$\sin(n\omega)u(n) \overset{z}{\leftrightarrow} \frac{z^{-1}\sin\omega}{1-2\,z^{-1}\cos\omega+z^{-2}} \quad \text{ROC: } |z| > 1 \qquad (6.68)$$

6.15 z−transform of $a^n\cos(n\omega)u(n)$

Let $x_1(n) = \cos(n\omega)u(n)$

Hence $X_1(z) = \dfrac{1-z^{-1}\cos\omega}{1-2\,z^{-1}\cos\omega+z^{-2}}$ ROC: $|z| > 1$

The given expression is $x(n) = a^n \cos(n\omega) u(n) = a^n\, x_1(n)$

$$\therefore X(z) = Z\left[a^n x_1(n)\right]$$

Using scaling property, we can write

$$X(z) = X_1\left(\frac{z}{a}\right) \qquad \text{ROC: } |a| r_1 < |z| < |a| r_2$$

$$\text{i.e., } X(z) = \frac{1 - \left(\frac{z}{a}\right)^{-1} \cos\omega}{1 - 2\left(\frac{z}{a}\right)^{-1} \cos\omega + \left(\frac{z}{a}\right)^{-2}} \qquad \text{ROC: } |z| > 1.|a|$$

Hence, the obtained z–transformation pair is

$$a^n \cos(n\omega) u(n) \overset{z}{\leftrightarrow} \frac{1 - \left(\frac{z}{a}\right)^{-1} \cos\omega}{1 - 2\left(\frac{z}{a}\right)^{-1} \cos\omega + \left(\frac{z}{a}\right)^{-2}} \qquad \text{ROC: } |z| > |a| \tag{6.69}$$

6.16 z–transform of $a^n \sin(n\omega)u(n)$

Let $x_1(n) = \sin(n\omega)u(n)$

Hence $X_1(z) = \dfrac{z^{-1}\sin\omega}{1 - 2z^{-1}\cos\omega + z^{-2}}$ ROC: $|z| > 1$

The given expression is $x(n) = a^n\cos(n\omega)u(n) = a^n x_1(n)$

$$\therefore X(z) = Z\left[a^n x_1(n)\right]$$

Using scaling property, we can write

$X(z) = X_1\left(\dfrac{z}{a}\right)$ ROC: $|a|r_1 < |z| < |a|r_2$

i.e., $X(z) = \dfrac{\left(\dfrac{z}{a}\right)^{-1}\sin\omega}{1 - 2\left(\dfrac{z}{a}\right)^{-1}\cos\omega + \left(\dfrac{z}{a}\right)^{-2}}$ ROC: $|z| > 1.|a|$

Hence, the obtained z–transformation pair is