CHAPTER 2 (9 LECTURES)
ROOTS OF NON-LINEAR EQUATIONS IN ONE VARIABLE

1. Introduction

Finding one or more root (or zero) of the equation

$$f(x) = 0$$

is one of the more commonly occurring problems of applied mathematics. In most cases explicit solutions are not available and we must be satisfied with being able to find a root to any specified degree of accuracy. The numerical procedures for finding the roots are called iterative methods. These problems arise in variety of applications.

Definition 1.1 (Simple and multiple root)**.** *A zero (root) has a "multiplicity", which refers to the number of times that its associated factor appears in the equation. A root having multiplicity one is called a simple root. For example, $f(x) = (x-1)(x-2)$ has a simple root at $x = 1$ and $x = 2$, but $g(x) = (x-1)^2$ has a root of multiplicity 2 at $x = 1$, which is therefore not a simple root.*
A multiple root is a root with multiplicity $m \geq 2$ is called a multiple point or repeated root. For example, in the equation $(x-1)^2 = 0$, $x = 1$ is multiple (double) root.
If a polynomial has a multiple root, its derivative also shares that root.

Let α be a root of the equation $f(x) = 0$, and imagine writing it in the factored form

$$f(x) = (x - \alpha)^m \phi(x)$$

with some integer $m \geq 1$ and some continuous function $\phi(x)$ for which $\phi(\alpha) \neq 0$. Then we say that α is a root of $f(x)$ of multiplicity m.

Now we study some iterative methods to solve the non-linear equations.

2. The Bisection Method

The first technique, based on the Intermediate Value Theorem, is called the Bisection, or Binary-search, method.
Let $f(x)$ be a continuous function on some given interval $[a, b]$ and it satisfies the condition $f(a)\,f(b) < 0$, then by Intermediate Value Theorem the function $f(x)$ must have at least one root α in (a, b) with $f(\alpha) = 0$. The bisection method repeatedly bisects the interval $[a, b]$ by taking $c = \dfrac{a+b}{2}$ and then selects a subinterval in which a root must lie for further processing. It is a very simple and robust method, but it is also relatively slow. Usually $[a, b]$ is chosen to contain only root α.
To begin, set $a_1 = a$ and $b_1 = b$, and let c_1 be the midpoint of $[a, b]$, that is

$$c_1 = \frac{a_1 + b_1}{2}.$$

If $f(c_1) = 0$, then $\alpha = c_1$, and we are done.
If $f(c_1) \neq 0$, then if $f(a_1)f(c_1) < 0$ then we set $a_2 = a_1$ and $b_2 = c_1$. If $f(c_1)f(b_1) < 0$ then we set $a_2 = c_1$ and $b_2 = b_1$.
Then reapply the process to the interval $[a_2, b_2]$ and so on.

Stopping Criteria: Since this is an iterative method, we must determine some stopping criteria that will allow the iteration to stop. We can use the following criteria to stop in term of absolute error and relative error

$$|c_{n+1} - c_n| \leq \epsilon,$$

$$\frac{|c_{n+1} - c_n|}{|c_{n+1}|} \leq \epsilon,$$

provided $|c_{n+1}| \neq 0$.
Criterion $|f(c_n)| \leq \epsilon$ can be misleading since it is possible to have $|f(c_n)|$ very small, even if c_n is not close to the root.

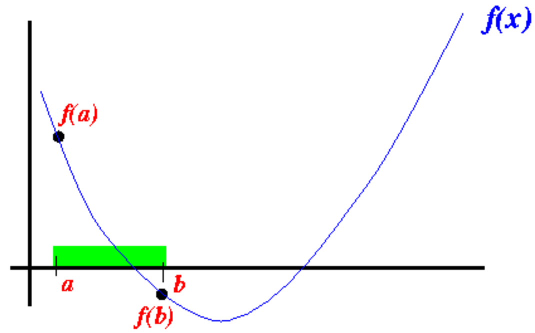


FIGURE 1. Bisection method

Example 1. *Use bisection method to find the solution of the equation $3x - e^x = 0$ in the interval $[1, 2]$ accurate within 10^{-2}.* [1]

Sol. Let $f(x) = 3x - e^x$. Root lies in the interval $[1, 2]$ as $f(1)f(2) < 0$.
We show the iteration in the following table.

TABLE 1. Bisection method iterations

n	a	b	$c = (a+b)/2$	$\lvert c_{n+1} - c_n \rvert$	sign$(f(a).f(c))$
1	1	2	1.5000		$+$
2	1.5	2	1.7500	0.25	$-$
3	1.5	1.75	1.6250	0.125	$-$
4	1.5	1.6250	1.5625	0.0625	$-$
5	1.5	1.5625	1.5313	0.0312	$-$
6	1.5	1.5313	1.5156	0.0157	$-$
7	1.5	1.5156	1.5078	$0.0078 < 0.01$	$-$

Thus the root with desired accuracy is 1.5078.

Example 2. *The sum of two numbers is 20. If each number is added to its square root, then the product of the resulting sums is 155.55. Perform five iterations of bisection method to determine the two numbers.*

Sol. Let x and y be the two numbers. Then,

$$x + y = 20.$$

Now $\sqrt{x}$ is added to x and $\sqrt{y}$ is added to y. The product of these sums is

$$\begin{aligned} (x+\sqrt{x})(y+\sqrt{y}) &= 155.55. \\ \therefore\ (x+\sqrt{x})(20-x+\sqrt{20-x}) &= 155.55. \end{aligned}$$

Write the above equation in to root finding problem

$$f(x) = (x+\sqrt{x})(20-x+\sqrt{20-x}) - 155.55 = 0.$$

[1]Choice of initial approximations: Initial approximations to the root are often known from the physical significance of the problem. Graphical methods are used to find the zero of $f(x) = 0$ and any value in the neighborhood of root can be taken as initial approximation.
If the given equation $f(x) = 0$ can be written as $f_1(x) = f_2(x) = 0$, then the point of the intersection of the graphs $y = f_1(x)$ and $y = f_2(x)$ gives the root of the equation. Any value in the neighborhood of this point can be taken as initial approximation.

As $f(6)f(7) < 0$, so there is a root in interval (6.7). Below are the iterations of bisection method for finding root. Therefore root is 6.53125.

n	a	b	c	sign$(f(a).f(c))$
1	6.000000	7.000000	6.500000	> 0
2	6.500000	7.000000	6.750000	< 0
3	6.500000	6.750000	6.625000	< 0
4	6.500000	6.625000	6.562500	> 0
5	6.500000	6.562500	**6.531250**	< 0

If $x = 6.53125$, then $y = 20 - 6.53125 = 13.4688$.
Further we discuss the convergence of approximate solution to exact solution. In this step firstly we define the usual meaning of convergence and order of convergence.

Definition 2.1 (Convergence)**.** *A sequence $\{x_n\}$ is said to be converge to a point α with order p if there is exist a constant c such that*

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = c, \quad n \geq 0.$$

The constant c is known as asymptotic error constant. If we write $e_n = |x_n - \alpha|$ where e_n denote the absolute error in n-th iteration then we can write in limiting case

$$e_{n+1} = c\, e_n^p.$$

Two cases are given special attention.
(i) If $p = 1$ (and $c < 1$), the sequence is linearly convergent.
(ii) If $p = 2$, the sequence is quadratically convergent.

Definition 2.2. *Let $\{\beta_n\}$ is a sequence which converges to zero and $\{x_n\}$ is any sequence. If there exists a constant $c > 0$ and an integer $N > 0$ such that*

$$|x_n - \alpha| \leq c|\beta_n|, \quad \forall n \geq N,$$

then we say that $\{x_n\}$ converges to α with rate $O(\beta_n)$. We write

$$x_n = \alpha + O(\beta_n).$$

Example: Define two sequences for $n \geq 1$,

$$x_n = \frac{n+1}{n^2}, \quad \text{and} \quad y_n = \frac{n+2}{n^3}.$$

Both the sequences has limit 0 but the sequence $\{y_n\}$ converges to this limit much faster than the sequence $\{x_n\}$.
Now

$$|x_n - 0| = \frac{n+1}{n^2} < \frac{n+n}{n^2} = 2\frac{1}{n} = 2\beta_n$$

and

$$|y_n - 0| = \frac{n+2}{n^3} < \frac{n+2n}{n^3} = 3\frac{1}{n^2} = 3\tilde{\beta}_n.$$

Hence the rate of convergence of $\{x_n\}$ to zero is similar to the convergence of $\{1/n\}$ to zero, whereas $\{y_n\}$ converges to zero at a rate similar to the more rapidly convergent sequence $\{1/n^2\}$. We express this by writing

$$x_n = 0 + O(\beta_n) \quad \text{and} \quad y_n = 0 + O(\tilde{\beta}_n).$$

2.1. **Convergence analysis.** Now we analyze the convergence of the iterations generated by the bisection method.

Theorem 2.3. *Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. Then the bisection method generates a sequence $\{c_n\}$ approximating a zero α of f with linear convergence.*

Proof. Let $[a_1, b_1]$, $[a_2, b_2]$, $\cdots$, $[a_n, b_n]$, $\cdots$, denote the successive intervals produced by the bisection algorithm. Thus

$$a = a_1 \leq a_2 \leq \cdots \leq b_1 = b$$

$$b = b_1 \geq b_2 \geq \cdots \geq a_1 = a.$$

This implies $\{a_n\}$ and $\{b_n\}$ are monotonic and bounded and hence convergent.
Since

$$\begin{aligned} b_1 - a_1 &= (b - a) \\ b_2 - a_2 &= \frac{1}{2}(b_1 - a_1) = \frac{1}{2}(b - a) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_n - a_n &= \frac{1}{2^{n-1}}(b - a). \end{aligned} \tag{2.1}$$

Hence

$$\lim_{n\to\infty}(b_n - a_n) = 0.$$

Here $b - a$ denotes the length of the original interval with which we started. Take limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha \text{ (say)}.$$

Since f is continuous function, therefore

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(\alpha).$$

The bisection method ensures that

$$f(a_n)f(b_n) \leq 0$$

which implies

$$\lim_{n\to\infty} f(a_n)f(b_n) = f^2(\alpha) \leq 0$$

$$\implies f(\alpha) = 0.$$

Thus limit of $\{a_n\}$ and $\{b_n\}$ is a zero of $[a, b]$.
Since the root α is in either the interval $[a_n, c_n]$ or $[c_n, b_n]$. Therefore

$$|\alpha - c_n| < c_n - a_n = b_n - c_n = \frac{1}{2}(b_n - a_n)$$

Combining with (2.1), we obtain the further bound

$$e_n = |\alpha - c_n| < \frac{1}{2^n}(b - a).$$

Therefore

$$e_{n+1} < \frac{1}{2^{n+1}}(b - a).$$

$$\therefore e_{n+1} < \frac{1}{2}e_n.$$

This shows that the iterates c_n converge to α as $n \to \infty$. By definition of convergence, we can say that the bisection method converges linearly with rate $\frac{1}{2}$. ■

Illustrations:

(1) Since the method brackets the root, the method is guaranteed to converge, however, can be very slow.

(2) Computing c_n : It might happen that at a certain iteration n, computation of $c_n = \dfrac{a_n + b_n}{2}$ will give overflow. It is better to compute c_n as:

$$c_n = a_n + \frac{b_n - a_n}{2}.$$

(3) We can find the minimum number of iterations N needed with the bisection method to achieve a certain desired accuracy ε. The interval length after N iterations is $\dfrac{b-a}{2^N}$. So, to obtain an accuracy of ϵ, we must have $\dfrac{b-a}{2^N} \le \epsilon$. That is,
$$2^{-N}(b-a) \le \varepsilon,$$
or
$$N \ge \frac{\log(b-a) - \log \varepsilon}{\log 2}.$$
Note the number N depends only on the initial interval $[a, b]$ bracketing the root.

(4) If a function is such that it just touches the x-axis, for example $f(x) = x^2$, then we don't have a and b such that $f(a)f(b) < 0$ but $x = 0$ is the root of $f(x) = 0$.

(5) For functions where there is a singularity and it reverses sign at the singularity, bisection method may converge on the singularity. An example include $f(x) = \dfrac{1}{x}$. We can chose a and b such that $f(a)f(b) < 0$. However, the function is not continuous and the theorem that a root exists is not applicable.

Example 3. *Use the bisection method to find solutions accurate to within* 10^{-2} *for* $x^3-7x^2+14x-6 = 0$ *on* $[0, 1]$.

Sol. Minimum number of iterations are given by
$$N \ge \frac{\log(1-0) - \log(10^{-2})}{\log 2} = 6.6439.$$
Thus, a minimum of 7 iterations will be needed to obtain the desired accuracy using the bisection method. This yields the following results for mid-points c_n and for absolute errors $E_n = |c_n - c_{n-1}|$.

n	a_n	b_n	c_n	$sign(f(a).f(c))$	E_n
1	0	1	0.5	> 0	
2	0.5	1	0.75	< 0	0.25
3	0.5	0.75	0.625	< 0	0.125
4	0.5	0.625	0.5625	> 0	0.0625
5	0.5625	0.625	0.59375	< 0	0.03125
6	0.5625	0.59375	0.578125	< 0	0.015625
7	0.578125	0.59375	**0.5859375**		$0.0078125 < 0.01(= 10^{-2})$

3. Fixed-point iteration method

Definition 3.1. *A point x is a fixed point for a given function g if $g(x) = x$. The terminology was first used by the Dutch mathematician L. E. J. Brouwer (1882-1962) in the early 1900s.*

In this section we consider the root finding problem $f(x) = 0$. We rewrite this problem as fixed-point problem $x = g(x)$. Root-finding problems and fixed-point problems are equivalent classes in the following sense:
Given a root-finding problem $f(x) = 0$, we can define functions g with a fixed point at x in a number of ways. Conversely, if the function g has a fixed point at α, then the function defined by $f(x) = x - g(x)$ has a zero at α.
Although the problems we wish to solve are in the root-finding form, the fixed-point form is easier to analyze, and certain fixed-point choices lead to very powerful root-finding techniques.

Example 4. *Determine any fixed points of the function $g(x) = x^2 - 2$.*

Sol. A fixed point x for g has the property that
$$x = g(x) = x^2 - 2$$
which implies that
$$0 = x^2 - x - 2 = (x+1)(x-2).$$

A fixed point for g occurs precisely when the graph of $y = g(x)$ intersects the graph of $y = x$, so g has two fixed points, one at $x = -1$ and the other at $x = 2$.

Fixed-point iterations: We now consider solving an equation $x = g(x)$ for a root α by the iteration

$$x_{n+1} = g(x_n), \quad n \geq 0,$$

with x_0 as an initial guess to α.
Each solution of $x = g(x)$ is called a fixed point of g. We illustrate the procedure by taking two examples.

Example 5. *Find the root of $x - \cos x = 0$ using fixed-point iterations.*

Sol. We write $x = \cos x$. Fixed point iterations are

$$x_{n+1} = g(x_n) = \cos(x_n), \ n = 0, 1, 2, \cdots.$$

Start with $x_0 = 1$ and iteration are given as

$$\begin{aligned}
x_1 = \cos(x_0) &= \cos(1) = 0.5403 \\
x_2 = \cos(x_1) &= \cos(0.5403) = 0.85755 \\
x_3 = \cos(x_2) &= \cos(0.85755) = 0.65429 \\
x_4 = \cos(x_3) &= \cos(0.65429) = 0.79348 \\
x_5 = \cos(x_4) &= \cos(0.79348) = 0.70137 \\
x_6 = \cos(x_5) &= \cos(0.70137) = 0.76396 \\
\ldots\ldots\ldots\ldots & \quad \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

After few more iterations, it converges to 0.73918.

Example 6. *Find the $\sqrt{3}$ by fixed-point iterations.*

Sol. Let $x = \sqrt{3}$ and consider the solving $f(x) = x^2 - 3 = 0$.
We can write $g(x)$ in many ways.

(1) $x = x^2 + x - 3$.
(2) $x = 3/x$.
(3) $x = \frac{1}{2}(x + 3/x)$.

Let us start with $x_0 = 2$ for all three choices. The iterations are given in the Table.

TABLE 2. Table for iterations in three cases

n	1	2	3
0	2.0	2.0	2.0
1	3.0	1.5	1.75
2	9.0	2.0	1.732143
3	87.0	1.5	1.732051

Now $\sqrt{3} = 1.73205$ and it is clear that third choice is correct but why other two are not working? Therefore which of the approximation is correct or not, we will answer after the convergence result (which require $|g'(\alpha) < 1|$ and $a \leq g(x) \leq b, \forall x \in [a, b]$ in the neighborhood of root α).

Lemma 3.2. *Let $g(x)$ be a continuous function on $[a, b]$ and assume that $a \leq g(x) \leq b, \ \forall x \in [a, b]$ then $x = g(x)$ has at least one solution in $[a, b]$.*

Proof. Let g be a continuous function on $[a, b]$.
Let assume that $a \leq g(x) \leq b, \ \forall x \in [a, b]$.
Now consider $\phi(x) = g(x) - x$.
If $g(a) = a$ or $g(b) = b$ then proof is trivial. Hence we assume that $a \neq g(a)$ and $b \neq g(b)$.
Now since $a \leq g(x) \leq b$

$\implies g(a) > a$ and $g(b) < b$.
Now

$$\phi(a) = g(a) - a > 0$$

and

$$\phi(b) = g(b) - b < 0.$$

Now ϕ is continuous and $\phi(a)\phi(b) < 0$, therefore by Intermediate Value Theorem ϕ has at least one zero in $[a, b]$, i.e. there exists some α s.t.

$$g(\alpha) = \alpha, \ \alpha \in [a, b].$$

Graphically, the roots are the intersection points of $y = x$ & $y = g(x)$ as shown in the Figure. ■

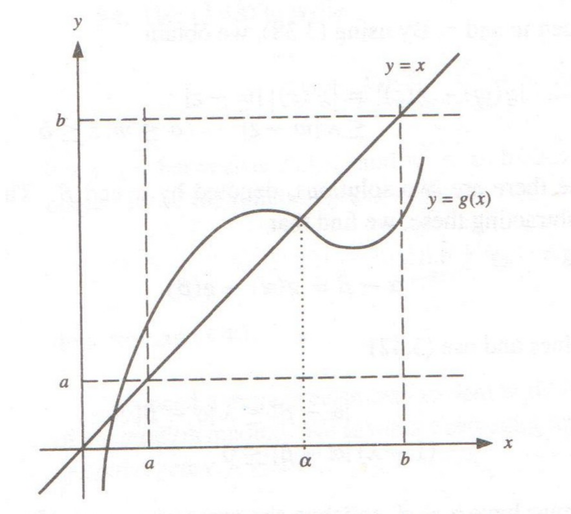


FIGURE 2. An example of Lemma

Theorem 3.3 (Contraction Mapping Theorem)**.** *Let g and g' are continuous functions on $[a, b]$ and assume that g satisfy $a \le g(x) \le b, \ \forall x \in [a, b]$. Furthermore, let $\lambda = \max\limits_{x\in(a,b)} |g'(x)|$ and $\lambda < 1$.*

Then prove that

(1) *The equation $x = g(x)$ has a unique solution α of $x = g(x)$ in the interval $[a, b]$.*
(2) *The iterates $x_{n+1} = g(x_n), \ n \ge 1$ converges linearly to α for any choice of $x_0 \in [a, b]$.*

(3)
$$|\alpha - x_n| \le \frac{\lambda^n}{1-\lambda}|x_1 - x_0|, \ n \ge 0.$$

Proof. Let g and g' are continuous functions on $[a, b]$ and assume that $a \le g(x) \le b, \ \forall x \in [a, b]$.
Let $\lambda = \max\limits_{x\in[a,b]} |g'(x)| < 1, \ \forall x \in (a, b)$.
By previous Lemma, there exists at least one solution to $x = g(x)$.
By Mean-Value Theorem

$$\begin{aligned} g(x) - g(y) &= g'(c)(x - y), \quad c \in (x, y) \\ |g(x) - g(y)| &\le \lambda|x - y|, \ 0 < \lambda < 1, \ \forall x \in [a, b]. \end{aligned} \tag{3.1}$$

(1) Let $x = g(x)$ has two solutions, say α and β in $[a, b]$ then $\alpha = g(\alpha)$, and $\beta = g(\beta)$. Now

$$\begin{aligned} |\alpha - \beta| &= |g(\alpha) - g(\beta)| \le \lambda|\alpha - \beta| \\ \implies (1 - \lambda)|\alpha - \beta| &\le 0 \\ \implies \alpha &= \beta, \quad \text{since} \quad 0 < \lambda < 1. \end{aligned}$$

Therefore $x = g(x)$ has a unique solution in $[a, b]$ which is α (say).

(2) To check the convergence of iterates $\{x_n\}$, we observe that they all remain in $[a,b]$ as $x_n \in [a,b],\ x_{n+1} = g(x_n) \in [a,b]$.
Now

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)| = |g'(c_n)||\alpha - x_n| \text{ as } \alpha = g(\alpha),\ x_{n+1} = g(x_n), \tag{3.2}$$

for some c_n between α and x_n.

$$\implies |\alpha - x_{n+1}| \leq \lambda|\alpha - x_n| \leq \lambda^2|\alpha - x_{n-1}|$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\leq \lambda^{n+1}|\alpha - x_0|$$

As $n \to \infty$, $\lambda^n \to 0$ which implies $x_n \to \alpha$. Also

$$|\alpha - x_n| \leq \lambda^n|\alpha - x_0|. \tag{3.3}$$

Now we prove that the convergence is linear. By equation (3.2)

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} = |g'(c_n)|,$$

for some c_n between α and x_n. Taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} = \lim_{n\to\infty} |g'(c_n)|$$

Now $x_n \to \alpha \implies c_n \to \alpha$.
Hence

$$\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} = |g'(\alpha)|.$$

If $|g'(\alpha)| < 1$, the above formula shows that iterates are linearly convergent with rate (asymptotic error constant) $|g'(\alpha)|$.

(3) Now we find the error bound. For this we can write

$$\begin{aligned}
|\alpha - x_0| &= |\alpha - x_1 + x_1 - x_0| \\
&\leq |\alpha - x_1| + |x_1 - x_0| \\
&\leq \lambda|\alpha - x_0| + |x_1 - x_0| \\
\implies (1-\lambda)|\alpha - x_0| &\leq |x_1 - x_0| \\
\implies |\alpha - x_0| &\leq \frac{1}{1-\lambda}|x_1 - x_0| \\
\implies \lambda^n|\alpha - x_0| &\leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|
\end{aligned}$$

Therefore using (3.3)

$$\begin{aligned}
|\alpha - x_n| &\leq \lambda^n|\alpha - x_0| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0| \\
\implies |\alpha - x_n| &\leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|.
\end{aligned}$$

■

Illustrations:

(1) In practice, it is difficult to find an interval $[a,b]$ for which $a \leq g(x) \leq b$ condition is satisfied. On the contrary if $|g'(\alpha)| > 1$, then the iteration method $x_{n+1} = g(x_n)$ will not converge to α. When $|g'(\alpha)| \approx 1$, no conclusion can be drawn and even if convergence occur, the method would be far too slow for the iteration method to be practical.

(2) **Stopping Criteria:** If

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0| < \varepsilon$$

where ε is desired accuracy. This bound can be used to find the number of iterations to achieve the accuracy ε.
Also from part theorem, $|\alpha - x_n| \leq \lambda^n|\alpha - x_0| \leq \lambda^n \max\{x_0 - a, b - x_0\} < \varepsilon$, can be used to find the number of iterations.

(3) The possible behavior of fixed-point iterates $\{x_n\}$ is shown in Figure 3 for various values of $g'(\alpha)$. To see the convergence, consider the case of $x_1 = g(x_0)$, the height of $y = g(x)$ at x_0. We bring the number x_1 back to the x-axis by using the line $y = x$ and the height $y = x_1$. We continue this with each iterate, obtaining a stair-step behavior when $g'(\alpha) > 0$. When $g'(\alpha) < 0$, the iterates oscillates around the fixed point α, as can be seen in the Figure. In first figure (on top) iterations are monotonic convergence, in second oscillatory convergent, in third figure iterations are divergent and in the last figure iterations are oscillatory divergent.

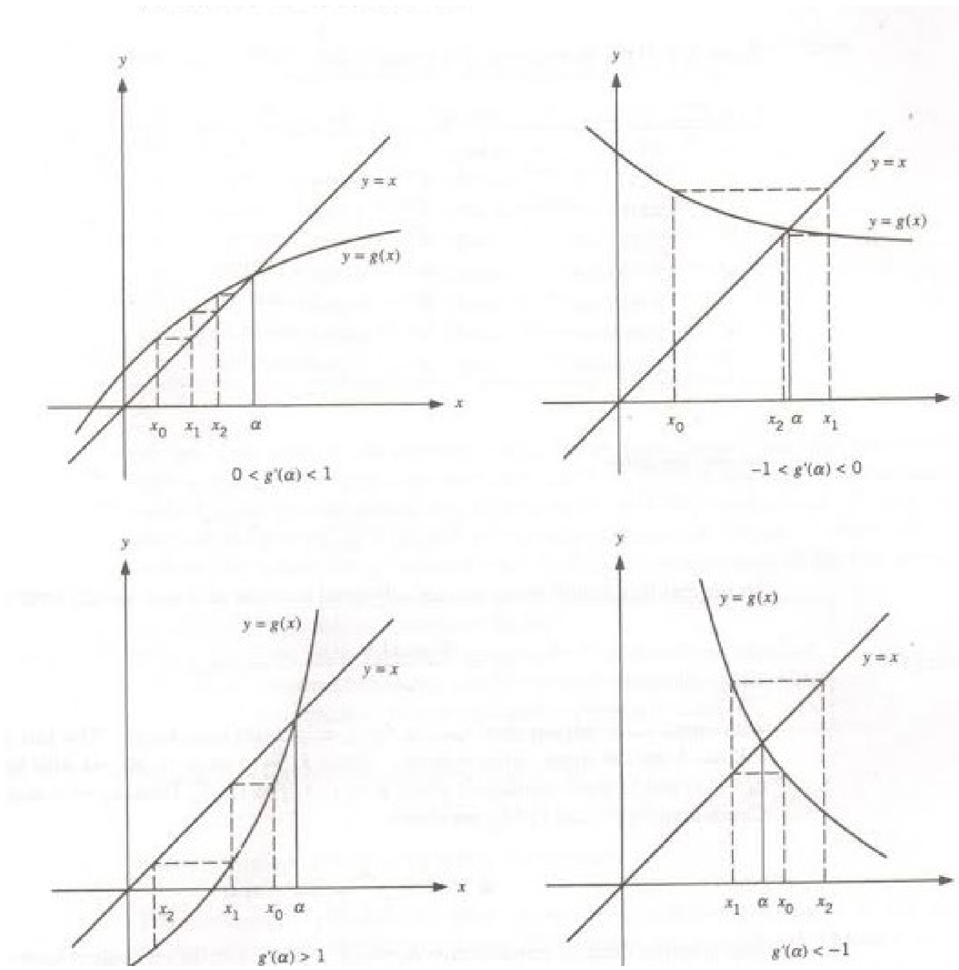


FIGURE 3. Convergent and non-convergent sequences $x_{n+1} = g(x_n)$

(4) Let g is p-times continuously differentiable function. If

$$g'(\alpha) = g''(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0, \; g^{(p)}(\alpha) \neq 0$$

then we can expect the higher order convergence. In this case, we expand $g(x_n)$ in a Taylor polynomial about α in the following way.

$$\begin{aligned}
x_{n+1} &= g(x_n) \\
&= g(\alpha + x_n - \alpha) \\
&= g(\alpha) + (x_n - \alpha)g'(\alpha) + \cdots + \frac{(x_n - \alpha)^{p-1}}{(p-1)!}g^{(p-1)}(\alpha) + \frac{(x_n - \alpha)^p}{p!}g^{(p)}(c_n),
\end{aligned}$$

for some c_n between x_n and α.
Using the values of derivatives and the fact $g(\alpha) = \alpha$, we obtain

$$\begin{aligned} x_{n+1} - \alpha &= \frac{(x_n - \alpha)^p}{p!} g^{(p)}(c_n) \\ \implies \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} &= \frac{g^{(p)}(c_n)}{p!} \\ \implies \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} &= (-1)^{p-1} \frac{g^{(p)}(c_n)}{p!}. \end{aligned}$$

Take limits $n \to \infty$ on both sides,

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = \lim_{n\to\infty} (-1)^{p-1} \frac{g^{(p)}(c_n)}{p!} = (-1)^{p-1} \frac{g^{(p)}(\alpha)}{p!}.$$

So by the definition of convergence, the iterations will have order of convergence p.

Example 7. *Consider the equation $x^3 - 7x + 2 = 0$ in $[0, 1]$. Write a fixed-point iteration which will converge to the solution.*

Sol. We rewrite the equation in the form $x = \dfrac{1}{7}(x^3 + 2)$ and define the fixed-point iteration

$$x_{n+1} = \frac{1}{7}(x_n^3 + 2).$$

Now $g(x) = \dfrac{1}{7}(x^3 + 2)$ is continuous function. Thus

$$\begin{aligned} g'(x) &= \frac{3x^2}{7},\ g''(x) = \frac{6x}{7} \\ g(0) &= \frac{2}{7},\ g(1) = \frac{3}{7} \\ g'(0) &= 0,\ g'(1) = \frac{6}{7}. \end{aligned}$$

Hence $\dfrac{2}{7} \le g(x) \le \dfrac{3}{7}$ and $|g'(x)| \le \dfrac{6}{7} < 1,\ \forall x \in [0, 1]$.
Hence by the fixed point theorem, the sequence $\{x_n\}$ defined above will converge to the unique solution of given equation. Starting with $x_0 = 0.5$, we can compute the solution as following.

$$x_1 = 0.303571429$$

$$x_2 = 0.28971083$$

$$x_3 = 0.289188016.$$

Therefore root correct to three decimals is 0.289.

Example 8. *An equation $e^x = 4x^2$ has a root in $[4, 5]$. Show that we cannot find that root using $x = g(x) = \frac{1}{2}e^{x/2}$ for the fixed-point iteration method. Can you find another iterative formula which will locate that root ? If yes, then find third iterations with $x_0 = 4.5$. Also find the error bound.*

Sol. Here $g(x) = \frac{1}{2}e^{x/2}$, $g'(x) = \frac{1}{4}e^{x/2} > 1$ for all $x \in (4, 5)$, therefore, the fixed-point iteration fails to converge to the root in [4, 5].
Now consider $x = g(x) = \ln(4x^2)$. Thus

$$g'(x) = \frac{2}{x} > 0,\ g''(x) = -\frac{2}{x^2} < 0,\ \forall x \in [4, 5].$$

Therefore g and g' are monotonic increasing and decreasing, respectively. Now

$$g(4) = 4.15888,\ g(5) = 4.60517,\ g'(4) = 0.5,\ g'(5) = 0.4.$$

Thus

$$4 \le g(x) \le 5,\ \lambda = \max_{4\le x\le 5} |g'(x)| = 0.5 < 1,\ \forall x \in [4, 5].$$

Using the fixed-point iteration method with $x_0 = 4.5$ gives the iterations as

$$\begin{aligned} x_1 &= g(x_0) = \ln(4 \times 4.5^2) = 4.3944 \\ x_2 &= 4.3469 \\ x_3 &= 4.3253. \end{aligned}$$

We have the error bound

$$|\alpha - x_3| \leq \frac{0.5^3}{1-0.5}|4.3944 - 4.5| = 0.0264.$$

Example 9. *Use a fixed-point method to determine a solution to within* 10^{-4} *for* $x = \tan x$, *for* x *in* $[4, 5]$.

Sol. Using $g(x) = \tan x$ and $x_0 = 4$ gives $x_1 = g(x_0) = \tan 4 = 1.158$, which is not in the interval $[4, 5]$. So we need a different fixed-point function.
If we note that $x = \tan x$ implies that

$$\begin{aligned} \frac{1}{x} &= \frac{1}{\tan x} \\ \implies x &= x - \frac{1}{x} + \frac{1}{\tan x}. \end{aligned}$$

Starting with x_0 and taking $g(x) = x - \dfrac{1}{x} + \dfrac{1}{\tan x}$, we obtain

$$x_1 = 4.61369, \; x_2 = 4.49596, \; x_3 = 4.49341, \; x_4 = 4.49341.$$

As x_3 and x_4 agree to five decimals, it is reasonable to assume that these values are sufficiently accurate.

Example 10. *The iterates* $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ *will converge to* $\alpha = 1$ *for some values of constant* c *(provided that* x_0 *is sufficiently close to* α*). Find the values of* c *for which convergence occurs? For what values of* c*, if any, convergence is quadratic?*

Sol. Fixed-point iteration

$$x_{n+1} = g(x_n)$$

with

$$g(x) = 2 - (1+c)x + cx^3.$$

If $\alpha = 1$ is a fixed point then for convergence $|g'(\alpha)| < 1$

$$\begin{aligned} &\implies |-(1+c) + 3c\alpha^2| < 1 \\ &\implies 0 < c < 1. \end{aligned}$$

For this value of c, $g''(\alpha) \neq 0$.
For quadratic convergence

$$g'(\alpha) = 0 \;\&\; g''(\alpha) \neq 0.$$

This gives $c = 1/2$.

Example 11. *Which of the following iteration*

(1) $x_{n+1} = \dfrac{1}{4}\left(x_n^2 + \dfrac{6}{x_n}\right)$

(2) $x_{n+1} = \left(4 - \dfrac{6}{x_n^2}\right)$

is suitable to find a root of the equation $x^3 = 4x^2 - 6$ *in the interval* $[3, 4]$*? Estimate the number of iterations required to achieve* 10^{-3} *accuracy, starting from* $x_0 = 3$.

Sol.

(1)

$$\begin{aligned} g(x) &= \frac{1}{4}\left(x^2 + \frac{6}{x}\right) \\ g'(x) &= \frac{1}{2}\left(x - \frac{3}{x^2}\right). \end{aligned}$$

g is continuous in $[3,4]$ and

$$g'(3) = \frac{4}{3} \quad g'(4) = \frac{61}{32}, \quad g'(x) \neq 0, \forall x \in (3,4)$$

Hence $|g'(x)| > 1$, for all $x \in (3,4)$. So this choice of $g(x)$ is not suitable.

(2)

$$\begin{aligned} g(x) &= \left(4 - \frac{6}{x^2}\right) \\ g'(x) &= \frac{12}{x^3}. \end{aligned}$$

Now g is continuous in $[3,4]$ and $g(x) \in [3,4]$, for all $x \in [3,4]$.
Also $|g'(x)| = \left|\frac{12}{x^3}\right| < 1$ for all $x \in (3,4)$.
Thus a unique fixed-point exists in $[3,4]$ by fixed point theory. To find an approximation of that root with an accuracy of 10^{-3} , we need to determine the number of iterations n so that

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0| < 10^{-3}.$$

Here $\lambda = \max\limits_{3 \leq x \leq 4} |g'(x)| = 4/9$ and using the fixed-point method by taking $x_0 = 3$, we have $x_1 = g(x_0) = 10/3$, we have

$$\begin{aligned} |\alpha - x_n| &\leq \frac{(4/9)^n}{1-4/9}|10/3 - 3| < 10^{-3} \\ n(\log 4 - \log 9) &= \log\left(\frac{10^{-3} \times 5}{3}\right) \\ n &> 7.8883 \approx 8. \end{aligned}$$

4. Iteration method based on first degree equation

4.1. **The Secant Method.** Let $f(x) = 0$ be the given non-linear equation.
Let $(x_0, f(x_0))$ and $(x_1, f(x_1))$ are two pints on the curve $y = f(x)$. Then the equation of the secant line joining two points on the curve $y = f(x)$ is given by

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1).$$

Let intersection point of the secant line with the x-axis is $(x_2, 0)$ then at $x = x_2,\ y = 0$. Therefore

$$\begin{aligned} 0 - f(x_1) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1) \\ x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1). \end{aligned}$$

Here x_0 and x_1 are two approximations of the root. The point $(x_2, 0)$ can be taken as next approximation of the root. This method is called the secant or chord method and successive iterations are given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\ f(x_n), \quad n = 1, 2, \ldots$$

Geometrically, in this method we replace the unknown function by a straight line or chord passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ and we take the point of intersection of the straight line with the x-axis as the next approximation to the root and continue the process.
Illustrations:

(1) **Stopping Criterion:** We can use the following stopping criteria

$$|x_{n+1} - x_n| < \varepsilon,$$

$$\text{or } \left|\frac{x_{n+1} - x_n}{x_{n+1}}\right| < \varepsilon,$$

where ε is prescribed accuracy.

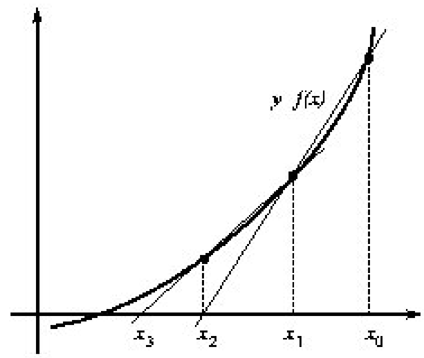


FIGURE 4. Secant method

(2) We can combine the secant method with the bisection method and bracket the root, i.e., we choose initial approximations x_0 and x_1 in such a manner that $f(x_0)f(x_1) < 0$. At each stage we bracket the root. The method is known as 'Method of False Position' or 'Regula Falsi Method'.
(3) Order of converegnce is almost 1.6.

Example 12. *Apply secant method to find the root of the equation $e^x = \cos x$ with relative error less than $< 0.5\%$.*

Sol. Let $f(x) = e^x - \cos x = 0$.
The successive iterations of the secant method are given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\, f(x_n), \quad n = 1, 2, \ldots$$

We take initial guesses $x_0 = -1.1$ and $x_1 = -1$, and let e_n denotes relative error at n-th step and we obtain

$$x_2 = -1.335205, \quad e_1 = \left|\frac{x_2 - x_1}{x_2}\right| \times 100\% = 10\%.$$

$$x_3 = -1.286223, \quad e_2 = \left|\frac{x_3 - x_2}{x_3}\right| \times 100\% = 25.01\%.$$

$$x_4 = -1.292594, \quad e_3 = \left|\frac{x_4 - x_3}{x_4}\right| \times 100\% = 3.68\%.$$

$$x_5 = -1.292696, \quad e_4 = \left|\frac{x_5 - x_4}{x_5}\right| \times 100\% = 0.49\%.$$

We obtain error less than 0.5% and accept $x_5 = -1.292696$ as root with prescribed accuracy.

4.2. **Newton's Method.** Let $f(x) = 0$ be the given non-linear equation.
Let the tangent line at point $(x_0, f(x_0))$ on the curve $y = f(x)$ intersect with the x-axis at $(x_1, 0)$. The equation of tangent is given by

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Here the number $f'(x_0)$ gives the slope of tangent at x_0. At $x = x_1$,

$$\begin{aligned} 0 - f(x_0) &= f'(x_0)(x_1 - x_0) \\ x_1 &= x_0 - \frac{f'(x_0)}{f(x_0)}. \end{aligned}$$

Here x_0 is the approximations of the root.
This is called the Newton's method and successive iterations are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots.$$

The method can be obtained directly from the secant method by taking limit $x_{n-1} \to x_n$. In the limiting case the chord joining the points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ becomes the tangent at $(x_n, f(x_n))$.
In this case problem of finding the root of the equation is equivalent to finding the point of intersection of the tangent to the curve $y = f(x)$ at point $(x_n, f(x_n))$ with the x-axis.
Note that once the Newton's method catches scent of the root, it usually hunts it down with amazing speed. But since the method is based on local information, namely $f(x_n)$ and $f'(x_n)$, the Newton's method sense of smell is deficient. If the initial estimate is not close enough to the root, the Newton's method may not converge, or may converge to the wrong root. Stopping criteria is same as we discussed in fixed point and secant methods.

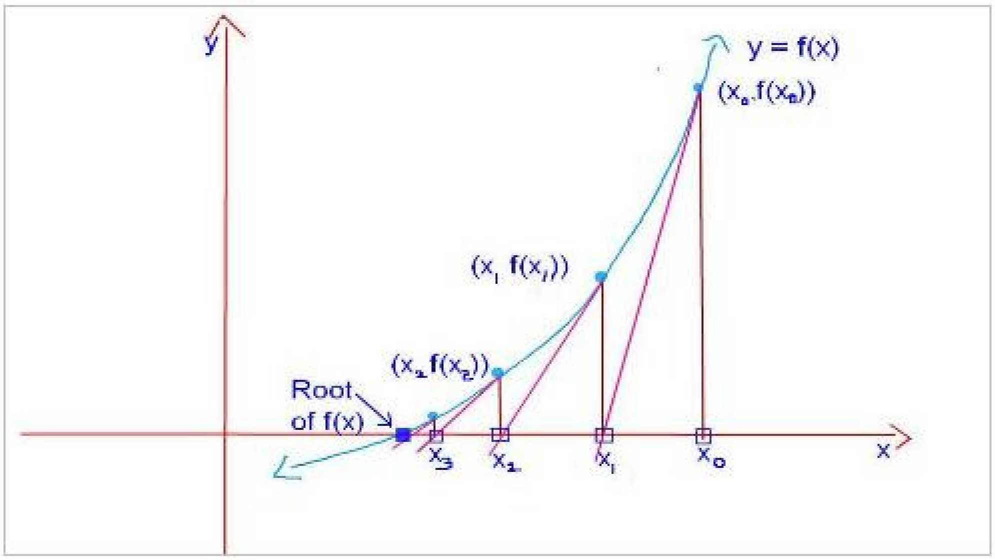


Figure 5. Newton's method.

Example 13. *A calculator is defective: it can only add, subtract, and multiply. Using the Newton's Method and the defective calculator to find* $1/1.37$ *correct to* 5 *decimal places.*

Sol. We consider

$$x = \frac{1}{1.37}, \quad f(x) = \frac{1}{x} - 1.37 = 0.$$

We have $f'(x) = -\dfrac{1}{x^2}$, and therefore the Newton's Method yields the iteration

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n(2 - 1.37x_n). \end{aligned}$$

Note that the expression $x_n(2 - 1.37x_n)$ can be evaluated on our defective calculator, since it only involves multiplication and subtraction. The choice $x_0 = 1$ can work and we get

$$x_1 = 0.63, \; x_2 = 0.716247, \; x_3 = 0.729670622, \; x_4 = 0.729926917, \; x_5 = 0.729927007.$$

Since the fourth and fifth iterates agree in to five decimal places, we assume that 0.729927007 is a correct solution to $f(x) = 0$, to at least five decimal places.

The following example illustrates very slow convergence and total failure of Newton's method in case of $f'(\alpha) = 0$ or not defined. Ordinarily the Newton's method is marvelously efficient, at least if the initial estimate is close enough to the root.

Example 14. (1) *Use Newton's method to solve the equation* $x^{100} = 0$*, using the initial estimate* $x_0 = 0.1$*. Calculate the next five iterations.*
(2) *The devotee then tried to use the method to solve* $3x^{1/3} = 0$*, using* $x_0 = 0.1$*. Calculate the next five estimates.*

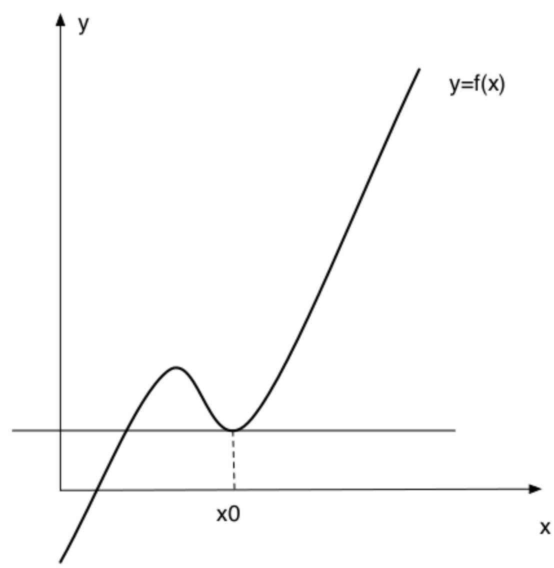


FIGURE 6. An example where Newton's method will not work.

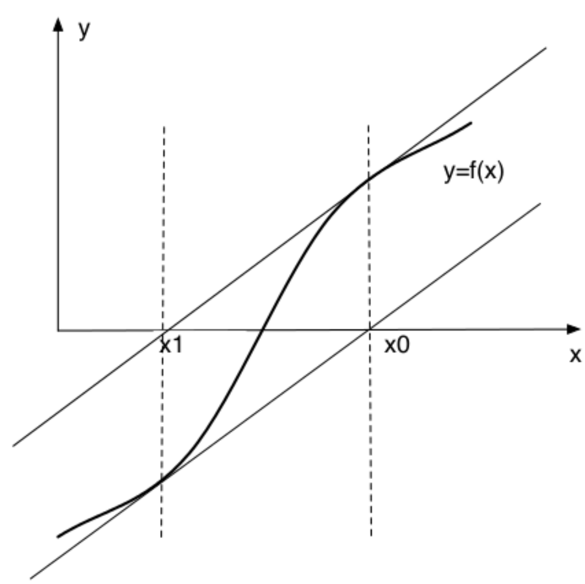


FIGURE 7. One more example of where Newton's method will not work.

Sol.

(1) Let $f(x) = x^{100}$. Then $f'(x) = 100x^{99}$ and the Newton's method iteration is

$$x_{n+1} = x_n - \frac{x^{100}}{100} x_n^{99} = \frac{99}{100} x_n.$$

Starting with $x_0 = 0.1$, we get

$$x_1 = 0.099,\ x_2 = 0.09801,\ x_3 = 0.0970299,\ x_4 = 0.096059601,\ x_5 = 0.095099004.$$

Note the slow progress rate. The root is 0, of course, but in 5 steps we have not reacher much closer to root.

(2) Let $f(x) = 3x^{1/3}$. Then $f'(x) = x^{-2/3}$, and the Newton's iteration becomes

$$x_{n+1} = x_n - \frac{3x^{1/3}}{x^{-2/3}} = x_n - 3x_n = -2x_n.$$

The next 5 estimates are $-0.2,\ 0.4, -0.8,\ 1.6,\ -3.2$. In fact, if we start with any non-zero estimate, the Newton's method estimates oscillate and hence no convergence.

The following example shows that choice of initial guess is very important for convergence.

Example 15. *Using Newton's Method to find a non-zero solution of $x = 2\sin x$.*

Sol. Let $f(x) = x - 2\sin x$.
Then $f'(x) = 1 - 2\cos x$, $f(1)f(2) < 0$, root lies in $(1, 2)$.
The Newton's iterations are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n} = \frac{2(\sin x_n - x_n\cos x_n)}{1 - 2\cos x_n};\ n \geq 0.$$

Let $x_0 = 1.1$. The next six estimates, to 3 decimal places, are:
$x_1 = 8.453$, $x_2 = 5.256$, $x_3 = 203.384$, $x_4 = 118.019$, $x_5 = -87.471$, $x_6 = -203.637$.
Therefore iterations diverges.
Note that choosing $x_0 = \pi/3 \approx 1.0472$ leads to immediate disaster, since then $1 - 2\cos x_0 = 0$ and therefore x_1 does not exist. The trouble was caused by the choice of x_0 as $f'(x_0) \approx 0$.
Let's see whether we can do better. Draw the curves $y = x$ and $y = 2\sin x$. A quick sketch shows that they meet a bit past $\pi/2$. If we take $x_0 = 1.5$. Here are the next five estimates
$x_1 = 2.076558$, $x_2 = 1.910507$, $x_3 = 1.895622$, $x_4 = 1.895494$, $x_5 = 1.895494$.

Example 16. *Find, correct to 5 decimal places, the x-coordinate of the point on the curve $y = \ln x$ which is closest to the origin. Use the Newton's Method.*

Sol. Let $(x, \ln x)$ be a general point on the curve, and let $S(x)$ be the square of the distance from $(x, \ln x)$ to the origin. Then

$$S(x) = x^2 + \ln^2 x.$$

We want to minimize the distance. This is equivalent to minimizing the square of the distance. Now the minimization process takes the usual route. Note that $S(x)$ is only defined when $x > 0$. We have

$$S'(x) = 2x + 2\frac{\ln x}{x} = \frac{2}{x}(x^2 + \ln x).$$

Our problem thus comes down to solving the equation $S'(x) = 0$. We can use the Newton's method directly on $S'(x) = 0$ which is equivalent to $x^2 + \ln x = 0$.
Let $f(x) = x^2 + \ln x$. Then $f'(x) = 2x + 1/x$ and we get the recurrence relation

$$x_{k+1} = x_k - \frac{x_k^2 + \ln x_k}{2x_k + 1/x_k}, \quad k = 0, 1, \cdots$$

We need to find a suitable starting point x_0. Experimentation with a calculator suggests that root lies in $(0, 1)$, so if we take $x_0 = 0.5$, then next iterations are

$$x_1 = 0.647716,\ x_2 = 0.652917,\ x_3 = 0.652919.$$

As x_2 agrees with x_3 to 5 decimal places, we can perhaps decide that, to 5 places, the minimum distance occurs at $x = 0.65292$.

4.3. **Convergence Analysis.**

Theorem 4.1. *Let $f \in C^2[a, b]$. If α is a simple root of $f(x) = 0$ and $f'(\alpha) \neq 0$, then Newton's method generates a sequence $\{x_n\}$ converging at least quadratically to root α for any initial approximation x_0 near to α.*

Proof. The proof is based on analyzing Newton's method as the fixed point iteration scheme

$$\begin{aligned} x_{n+1} &= g(x_n) \\ &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0 \end{aligned}$$

with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

We first find an interval $[\alpha-\delta, \alpha+\delta]$ such that $g(x) \in [\alpha-\delta, \alpha+\delta]$ and for which $|g'(x)| \leq \lambda$, $\lambda \in (0,1)$, for all $x \in (\alpha - \delta, \alpha + \delta)$.
Since f' is continuous and $f'(\alpha) \neq 0$, i.e., a continuous function is non-zero at a point which implies it will remain non-zero in a neighborhood of α.
Thus g is defined and continuous in a neighborhood of α. Also in that neighborhood

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{[f'(x)]^2}. \tag{4.1}$$

Now since $f(\alpha) = 0$, therefore

$$g'(\alpha) = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = 0.$$

Since g is continuous and $0 < \lambda < 1$, then there exists a number δ such that

$$|g'(x)| \leq \lambda, \ \forall x \in [\alpha - \delta, \alpha + \delta].$$

Now we will show that g maps $[\alpha - \delta, \alpha + \delta]$ into $[\alpha - \delta, \alpha + \delta]$.
If $x \in [\alpha - \delta, \alpha + \delta]$, the Mean Value Theorem implies that for some number c between x and α,

$$|g(x) - \alpha| = |g(x) - g(\alpha)| = |g'(c)|\, |x - \alpha| \leq \lambda |x - \alpha| < |x - \alpha|.$$

It follows that if $|x - \alpha| < \delta \implies |g(x) - \alpha| < \delta$.
Hence, g maps $[\alpha - \delta, \alpha + \delta]$ into $[\alpha - \delta, \alpha + \delta]$.
All the hypotheses of the Fixed-Point Convergence Theorem (Contraction Mapping) are now satisfied, so the sequence x_n converges to root α. Further from Eqs. (4.1)

$$g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} \neq 0,$$

which proves that convergence is of second-order provided $f''(\alpha) \neq 0$. ■

Remark 4.1. *Newton's method converges at least quadratically. If $g''(\alpha) = 0$, then higher order convergence is expected.*

Example 17. *The function $f(x) = \sin x$ has a zero on the interval $(3, 4)$, namely, $x = \pi$. Perform three iterations of Newton's method to approximate this zero, using $x_0 = 4$. Determine the absolute error in each of the computed approximations. What is the apparent order of convergence?*

Sol. Consider $f(x) = \sin x$. In the interval $(3, 4)$, f has a zero $\alpha = \pi$.
Also, $f'(x) = \cos x$.
Newton's iterations are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \ n \geq 0.$$

With $x_0 = 4$, we have

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \frac{\sin 4}{\cos 4} = 2.8422, \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8422 - \frac{\sin 2.8422}{\cos 2.8422} = 3.1509, \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 3.1509 - \frac{\sin 3.1509}{\cos 3.1509} = 3.1416. \end{aligned}$$

The absolute errors are:

$$\begin{aligned} e_1 &= |x_1 - x_0| = 1.1578, \\ e_2 &= |x_2 - x_1| = 0.3087, \\ e_3 &= |x_3 - x_2| = 0.0093. \end{aligned}$$

If p is the order of convergence then

$$\frac{e_3}{e_2} = \left(\frac{e_2}{e_1}\right)^p.$$

The corresponding order of convergence is

$$p = \frac{\ln(e_3/e_2)}{\ln(e_2/e_1)} = \frac{\ln(0.0093/0.3087)}{\ln(0.3087/1.1578)} = 2.6498.$$

We obtain a better than a second order of convergence (almost three).

4.4. **Newton's method for multiple roots.**

Definition 4.2. *Let α be a root of $f(x) = 0$ with multiplicity m. In this case we can write*

$$f(x) = (x - \alpha)^m \phi(x).$$

We can check

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0, f^{(m)}(\alpha) \neq 0.$$

Example 18. *Determine the multiplicity of the roots for the following equations*

(1) $(x-1)^2(x+2) = 0$
(2) $x^3 - 3x^2 + 3x - 1 = 0$
(3) $1 - \cos x = 0$.

Sol.

(1) Let $f(x) = (x-1)^2(x+2) = 0$. $x = 1$ is a multiple root with $m = 2$ and $x = -2$ is a simple root.
In this case $f(1) = f'(1) = 0$ and $f''(1) \neq 0$. Clearly we can write $f(x) = (x-1)^3$.
(2) Let $f(x) = x^3 - 3x^2 + 3x - 1 = 0$ which gives $x = 1$.
In this case $f(1) = f'(1) = f''(1) = 0$ and $f'''(1) \neq 0$. Thus $x = 1$ is a multiple root with $m = 3$.
(3) Let $f(x) = 1 - \cos x = 0$ which gives $x = 0$.
To determine the multiplicity of the root, we find that $f(0) = f'(0) = 0$ and $f''(0) \neq 0$. Thus multiplicity is 2. Also we can write

$$f(x) = 2\sin^2(x/2) = x^2 \frac{\sin^2(x/2)}{x^2} = x^2\phi(x).$$

So $f(x) = 0$ gives $x = 0, 0$.

Newton's method: Recall that we can regard Newton's method as a fixed point method:

$$x_{n+1} = g(x_n), \qquad g(x) = x - \frac{f(x)}{f'(x)}.$$

Then we substitute

$$f(x) = (x - \alpha)^m \phi(x)$$

to obtain

$$\begin{aligned} g(x) &= x - \frac{(x-\alpha)^m \, \phi(x)}{m(x-\alpha)^{m-1}\phi(x) + (x-\alpha)^m \, \phi'(x)} \\ &= x - \frac{(x-\alpha)\, \phi(x)}{m\phi(x) + (x-\alpha)\phi'(x)}. \end{aligned}$$

Therefore we obtain

$$g'(\alpha) = 1 - \frac{1}{m} \neq 0.$$

For $m > 1$, this is nonzero, and therefore Newton's method is only linearly convergent.
There are ways of improving the speed of convergence of Newton's method, creating a modified method that is again quadratically convergent. In particular, consider the fixed point iteration formula

$$x_{n+1} = g(x_n), \qquad g(x) = x - m\frac{f(x)}{f'(x)}$$

in which we assume to know the multiplicity m of the root α being sought. Then modifying the above argument on the convergence of Newton's method, we obtain

$$g'(\alpha) = 1 - m\frac{1}{m} = 0$$

and the iteration method will be quadratically convergent. But most of the time we don't know the multiplicity.
One method of handling the problem of multiple roots of a function f is to define

$$\mu(x) = \frac{f(x)}{f'(x)}.$$

If α is a zero of f of multiplicity m with $f(x) = (x-\alpha)^m\,\phi(x)$, then

$$\begin{aligned}\mu(x) &= \frac{(x-\alpha)^m\,\phi(x)}{m(x-\alpha)^{m-1}\,\phi(x) + (x-\alpha)^m\,\phi'(x)} \\ &= (x-\alpha)\frac{\phi(x)}{m\phi(x) + (x-\alpha)\phi'(x)}\end{aligned}$$

also has a zero at α. However, $\phi(\alpha) \neq 0$, so

$$\frac{\phi(\alpha)}{m\phi(\alpha) + (\alpha-\alpha)\phi'(\alpha)} = \frac{1}{m} \neq 0,$$

and α is a simple zero of $\mu(x)$. Newton's method can then be applied to $\mu(x)$ to give

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)/f'(x)}{\{[f'(x)]^2 - [f(x)][f''(x)]\}/[f'(x)]^2}$$

which simplifies to

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}.$$

If g has the required continuity conditions, functional iteration applied to g will be quadratically convergent regardless of the multiplicity of the zero of f. Theoretically, the only drawback to this method is the additional calculation of $f''(x)$ and the more laborious procedure of calculating the iterates. In practice, however, multiple roots can cause serious round-off problems because the denominator of the above expression consists of the difference of two numbers that are both close to 0.

Example 19. *Let $f(x) = e^x - x - 1$. Show that f has a zero of multiplicity 2 at $x = 0$. Show that Newton's method with $x_0 = 1$ converges to this zero but not quadratically.*

Sol. We have $f(x) = e^x - x - 1$, $f'(x) = e^x - 1$ and $f''(x) = e^x$.
Now $f(0) = 1 - 0 - 1 = 0$, $f'(0) = 1 - 1 = 0$ and $f''(0) = 1$.
Therefore f has a zero of multiplicity 2 at $x = 0$.
Starting with $x_0 = 1$, iterations are given by

$$\begin{aligned}x_{n+1} &= x_n - 2\frac{f(x_n)}{f'(x_n)} \\ x_1 &= 0.16395 \\ x_2 &= 0.0044781 \\ x_3 &= 0.000003422.\end{aligned}$$

Example 20. *The equation $f(x) = x^3 - 7x^2 + 16x - 12 = 0$ has a double root at $x = 2.0$. Starting with $x_0 = 1$, find the root correct to three decimals using modified Newton's method.*

Sol. Given

$$\begin{aligned} f(x) &= x^3 - 7x^2 + 16x - 12 = 0 \\ f'(x) &= 3x^2 - 14x = 16 \\ f''(x) &= 6x - 14. \end{aligned}$$

Now we apply modified Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}.$$

Starting with x_0, the iterations are given by

$$x_1 = 2.1111, \; x_2 = 2.0078, \; x_3 = 2.000003.$$

The root correct to three decimal places is 2.000.

Remark However, if we apply simple Newton's method then convergence is very slow. Starting with same initial guess, the successive iterations are given by

$$x_{n+1} = x_n - \frac{x_n^3 - 7x_n^2 + 16x_n - 12}{3x_n^2 - 14x_n + 16}, \; n = 0, 1, 2, \ldots$$

Start with $x_0 = 1.0$, we obtain

$$x_1 = 1.4, \; x_2 = 1.652632, \; x_3 = 1.806484, \; x_4 = 1.89586$$

$$x_5 = 1.945653, \; x_6 = 1.972144, \; x_7 = 1.985886, \; x_8 = 1.992894.$$

Even in eight iterations, we did not get desired accuracy.

We end this chapter by solving an example with all three methods studied previously.

Example 21. *The function $f(x) = \tan \pi x - 6$ has a zero at $\dfrac{\arctan 6}{\pi} \approx 0.447431543$. Use eight iterations of each of the following methods to approximate this root. Which method is most successful and why?*
a. Bisection method in interval [0,1].
b. Secant method with $x_0 = 0$ and $x_1 = 0.48$.
c. Newton's method with $x_0 = 0.4$.

Sol. It is important to note that f has several roots on the interval $[0, 5]$ (to see make a plot).
a. Since f has several roots in $[0, 5]$, the bisection method converges to a different root in this interval. Therefore, it would be a better idea to choose the interval to be $[0, 1]$. For such case, we have the following results: After 8 iterations answer is 0.447265625.

n	a	b	c
0	0	1	0.5
1	0	0.5	0.25
2	0.25	0.5	0.375
3	0.375	0.5	0.4375
4	0.4375	0.5	0.46875
5	0.4375	0.46875	0.453125
6	0.4375	0.46875	0.4453125
7	0.4375	0.4453125	0.44921875
8	0.4453125	0.44921875	0.447265625

b. The Secant method diverges for $x_0 = 0$ and $x_1 = 0.48$.
The Secant method converges for some other choices of initial guesses, for example, $x_0 = 0.4$ and $x_1 = 0.48$. Few iterations are given:

$$x_2 = 4.1824045, \; x_3 = 4.29444232, \; x_4 = 4.57230361, \; x_5 = 0.444112051,$$

$$x_6 = 0.446817663, \; x_7 = 0.447469928, \; x_8 = 0.447431099, \; x_9 = 0.447431543.$$

c. We have

$$f(x) = \tan(\pi x) - 6, \; \text{and} \; f'(x) = \frac{\pi}{\cos^2(\pi x)}.$$

Since the function f has several roots, some initial guesses may lead to convergence to a different root. Indeed, for $x_0 = 0$, Newton's method converges to a different root. For Newton's method, therefore, it is suggested that we use $x_0 = 0.4$ in order to converge to given root.
Starting with $x_0 = 0.4$, we obtain

$$x_1 = 0.488826408, \ x_2 = 0.480014377, \ x_3 = 0.467600335, \ x_4 = 0.455142852,$$

$$x_5 = 0.448555216, \ x_6 = 0.447455353, \ x_7 = 0.447431554, \ x_8 = 0.447431543.$$

We see that for these particular examples and initial guesses, the Newton's method and the Secant method give very similar convergence behaviors. The Newton's method converges slightly faster though. The bisection method converges much slower than the two other methods, as expected.

Exercises

(1) Use the bisection method to find solutions accurate to within 10^{-3} for the following problems.
   (a) $x - 2^{-x} = 0$ for $0 \leq x \leq 1$.
   (b) $e^x - x^2 + 3x - 2 = 0$ for $0 \leq x \leq 1$.

(2) Using the bisection method, determine the point of intersection of the curves given by $y = 3x$ and $y = e^x$ in the interval $[0, 1]$ with an accuracy 0.1.

(3) Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-2} using the bisection algorithm.

(4) Find a bound for the number of iterations needed to achieve an approximation by bisection method with accuracy 10^{-2} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

(5) Sketch the graphs of $y = x$ and $y = 2 \sin x$. Use the bisection method to find an approximation to within 10^{-2} to the first positive value of x with $x = 2 \sin x$.

(6) The function defined by $f(x) = \sin(\pi x)$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the bisection method converges to
   (a) 0, if $a + b < 2$
   (b) 2, if $a + b > 2$
   (c) 1, if $a + b = 2$.

(7) Show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-2}.

(8) For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-2}, and perform the calculations.
   (a) $x = \dfrac{5}{x^2} + 2$.
   (b) $2 + \sin x - x = 0$ in interval $[2, 3]$.
   (c) $3x^2 - e^x = 0$.

(9) Show that $g(x) = \pi + 0.5 \sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Also estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

(10) Use the fixed-point iteration method to find smallest and second smallest positive roots of the equation $\tan x = 4x$, correct to 4 decimal places.

(11) Find all the zeros of $f(x) = x^2 + 10 \cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-2}.

(12) Let A be a given positive constant and $g(x) = 2x - Ax^2$.
   (a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $\alpha = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.
   (b) Find an interval about $1/A$ for which fixed-point iteration converges, provided x_0 is in that interval.

(13) Consider the root-finding problem $f(x) = 0$ with root α, with $f'(x) \neq 0$. Convert it to the fixed-point problem

$$x = x + cf(x) = g(x)$$

with c a nonzero constant. How should c be chosen to ensure rapid convergence of

$$x_{n+1} = x_n + cf(x_n)$$

to α (provided that x_0 is chosen sufficiently close to α)? Apply your way of choosing c to the root-finding problem $x^3 - 5 = 0$.

(14) Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$. What happens if $x_0 < 0$?

(15) Use secant method to find root accurate to within 10^{-3} for $-x^3 - \cos x =$ with initial guesses -1 and 0.

(16) Use secant method to find root of $x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$ with initial guesses 0.02 and 0.05. Use the stopping criterion that the relative error is less than 0.5%.

(17) Use Newton's method to approximate the positive root of $2\cos x = x^4$ correct to six decimal places.

(18) Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

(19) (a) Apply Newton's method to the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$

with the root $\alpha = 0$. What is the behavior of the iterates? Do they converge, and if so, at what rate?

(b) Do the same but with

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & x \geq 0 \\ -\sqrt[3]{x^2}, & x < 0 \end{cases}$$

(20) Apply the Newton's method with $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$, and verify that the convergence is only of first-order. Further show that root $\alpha = 1$ has multiplicity 2 and then apply the modified Newton's method with $m = 2$ and verify that the convergence is of second-order.

(21) Use Newton's method and the modified Newton's method to find a solution of

$$\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0, \quad \text{for } -2 \leq x \leq -1$$

accurate to within 10^{-3}.

(22) A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

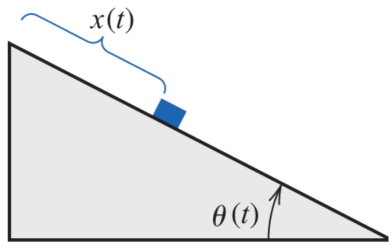
$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.

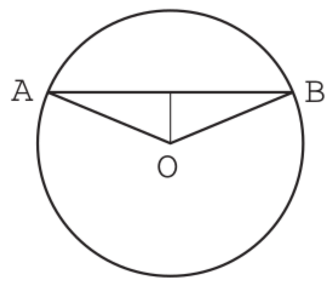
(23) An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb, and $k = 0.1$ lb-s/ft. Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.



(24) The circle below has radius 1, and the longer circular arc joining A and B is twice as long as the chord AB. Find the length of the chord AB, correct to four decimal places. Use Newton's method.



(25) It costs a firm $C(q)$ dollars to produce q grams per day of a certain chemical, where

$$C(q) = 1000 + 2q + 3q^{2/3}.$$

The firm can sell any amount of the chemical at \$4 a gram. Find the break-even point of the firm, that is, how much it should produce per day in order to have neither a profit nor a loss. Use the Newton's method and give the answer to the nearest gram.

Appendix A. Algorithms

Algorithm (Bisection):
To determine a root of $f(x) = 0$ that is accurate within a specified tolerance value ε, given values a and b such that $f(a)\ f(b) < 0$.
Define $c = \dfrac{a+b}{2}$.
If $f(a)\ f(c) < 0$, then set $b = c$, otherwise $a = c$.
End if.
Until $|a - b| \leq \varepsilon$ (tolerance value).
Print root as c.

Algorithm (Fixed-point):
To find a solution to $x = g(x)$ given an initial approximation x_0.
Input: Initial approximation x_0, tolerance value ϵ, maximum number of iterations N.
Output: Approximate solution α or message of failure.
Step 1: Set $i = 1$.
Step 2: While $i \leq N$ do Steps 3 to 6.
Step 3: Set $x_1 = g(x_0)$. (Compute x_i.)
Step 4: If $|x_1 - x_0| \leq \epsilon$ or $\dfrac{|x_1 - x_0|}{|x_1|} < \epsilon$ then OUTPUT x_1; (The procedure was successful.)
STOP.
Step 5: Set $i = i + 1$.
Step 6: Set $x_0 = x_1$. (Update x_0.)

Step 7: Print the output and STOP.

Algorithm (Secant):
1. Give inputs and take two initial guesses x_0 and x_1.
2. Start iterations

$$x_2 = x_1 - \frac{x_1 - x_0}{f_1 - f_0}\, f_0.$$

3. If

$$|x_2 - x_1| < \varepsilon \text{ or } \frac{|x_2 - x_1|}{|x_2|} < \varepsilon$$

then stop and print the root.
4. Repeat the iterations (step 2). Also check if the number of iterations has exceeded the maximum number of iterations.

Algorithm (Newton's method):
To find a solution to $f(x) = 0$, given an initial approximation x_0.
Input: Initial approximation x_0, tolerance value ϵ, maximum number of iterations N.
Output: Approximate solution x_1 or message of failure.
Step 1: Set $i = 1$.
Step 2: While $i \le N$ do Steps 3 to 6.
Step 3: Set $x_1 = x_0 - \dfrac{f(x_0)}{df(x_0)}$. (Compute x_i.)
Step 4: If $|x_1 - x_0| \le \epsilon$ or $\dfrac{|x_1 - x_0|}{|x_1|} < \epsilon$ then OUTPUT x_1; (The procedure was successful.) STOP.
Step 5: Set $i = i + 1$.
Step 6: Set $x_0 = x_1$. (Update x_0.)
Step 7: Output ('The method failed after Niterations, N=', N); (The procedure was unsuccessful.) STOP.

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