Lecture 12: Numerical Analysis (UMA011)

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F.P.I.

we can always find appropriate $g(x)$ from $f(x)=0$

by taking $g(x) = x - \dfrac{f(x)}{f'(x)}$, $f'(x) \neq 0$

So, if we have $x^3 - 7x + 2 = 0$

then $g(x) = x - \dfrac{x^3 - 7x + 2}{3x^2 - 7}$ → complicated.

and also $g(x) = \dfrac{x^3+2}{7}$ → easy calculation

Newton's method:

Newton- Raphson Method. → particular Case of f.P.I.

Importance:

well known and most power full method

Newton's method:

Importance:

well known and most power full method

Conditions for the convergence:

Suppose $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is small.

$f \in C[a,b]$

$f \in C^1[a,b]$

f is cont. on $[a,b]$

$f(x) = 0$

$[a, b]$ from IVT

$f \in C^2[a,b] \Rightarrow f'$ is cont. " "

f'' is cont. " "

Newton's method:

Derivation:

Taylor series

$f(x+h)$

$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) - - - -$

Let $f \in C^2[a,b]$. Suppose $p_0 \in [a,b]$ be an app. to p s.t. $f(p)=0, f'(p_0) \neq 0$ and $|p-p_0|$ is small.

Consider the Taylor's polynomial for $f(x)$ expanded about p_0 ie

$x+h=p$

$x=p_0$

$$f(p) = f(p_0) + (p-p_0) f'(p_0) + \frac{(p-p_0)^2}{2!} f''(p_0).$$

Since $|p-p_0|$ is small then $(p-p_0)^2$ is very small

$$f(p) \approx f(p_0) + (p - p_0) f'(p_0)$$

Now, p is exact root of $f(x) = 0$ ie $f(p) = 0$

$$0 \approx f(p_0) + (p - p_0) f'(p_0)$$

$$\frac{-f(p_0)}{f'(p_0)} \approx p - p_0 \qquad . \quad f'(p_0) \neq 0$$

$$\Rightarrow \quad p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \quad , \quad f'(p_0) \neq 0$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}, \quad f'(p_0) \neq 0$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}, \quad f'(p_1) \neq 0$$

$[a, b]$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)}, \quad f'(p_2) \neq 0$$

N.M. $\rightarrow$ $\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}}, \quad f'(p_n) \neq 0 \quad \forall n.$

Newton's method:

Graphical representation:

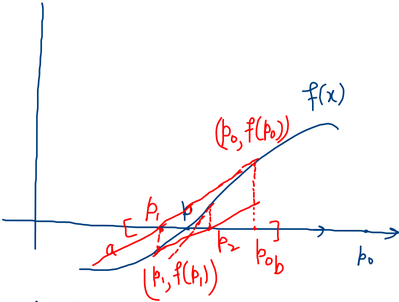
Eqn of tangent line.

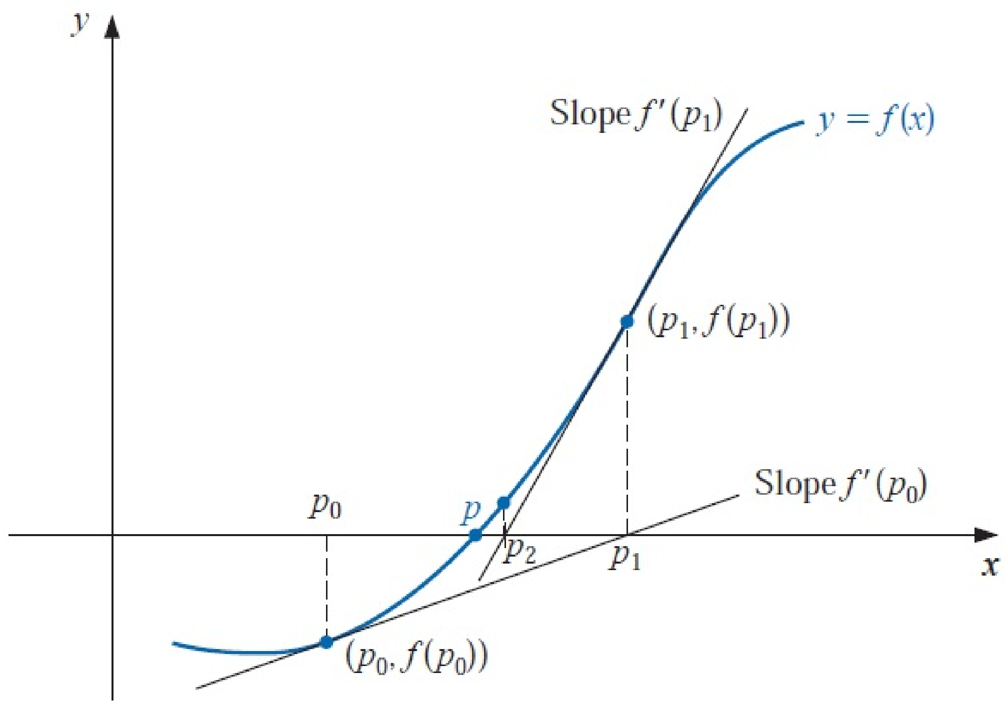
$$y - f(p_0) = f'(p_0)(x - p_0)$$

At x-axis $y = 0$

$$0 - f(p_0) = f'(p_0)(x - p_0)$$

$$\frac{-f(p_0)}{f'(p_0)} = x - p_0 \quad \Rightarrow p_1 = x = p_0 - \frac{f(p_0)}{f'(p_0)}$$





y
Slope $f'(p_1)$
$y = f(x)$
$(p_1, f(p_1))$
Slope $f'(p_0)$
p_0
p
p_2
p_1
x
$(p_0, f(p_0))$

ly

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

$$- \quad - \quad - \quad - \quad -$$

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}} \; , \qquad f'(p_n) \neq 0 \quad \forall n$$

N.M.

Newton's method:

Example:

Find the root of an equation $f(x) = \cos(x) - x = 0$

$f(x)$ is a cont. function

and $f(0) = +ve$, $f(1) = -ve$ $f(\frac{\pi}{2}) = -ve$ $\frac{3.14}{2} = 1.5\ldots$

By IVT, $\exists$ a root in $[0, \pi/2]$

By N.M. $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{\cos p_n - p_n}{-\sin p_n - 1}$

$$p_{n+1} = p_n + \frac{(\cos p_n - p_n)}{\sin p_n + 1}$$

let $p_0 = \frac{\pi}{4}$, $n = 0$

$$p_1 = p_0 + \frac{\cos p_0 - p_0}{\sin p_0 + 1}$$

$$= \frac{\pi}{4} + \frac{\cos \pi/4 - \pi/4}{\sin\frac{\pi}{4} + 1} = \frac{\pi}{4} + \frac{\frac{1}{\sqrt{2}} - \pi/4}{\frac{1}{\sqrt{2}} + 1} = 0.78539$$

$$p_2 = p_1 + \frac{\cos p_1 - p_1}{\sin p_1 + 1} = 0.739536$$

$p_3 = 0.739085$ $\quad$ $p_4 = 0.739085$ Ans.

Newton's method:

statement

Convergence result for Newton's method:

Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$. $\Leftrightarrow$ $|p-p_0|$ is small

$p-\delta$ $\quad$ δ $\quad$ p $\quad$ $p+\delta$

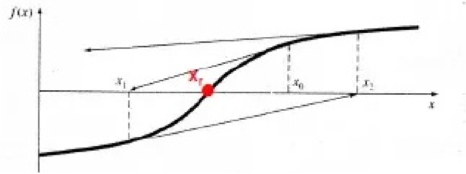
Newton's method:

Case of failure:

(i) When the initial guess is on the inflection of the function i.e. $f''(p_0) = 0$.

for e.g.

$y = x^{1/3}$



In this case, the sequence generated by N.M. diverges.

Newton's method:

Case of failure:

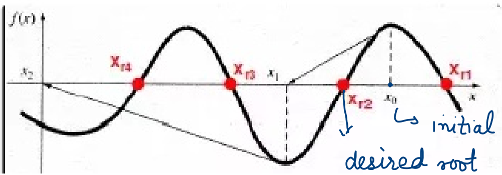
(ii) When there is an another slope near to the initial guess.

for e.g.

$y = \sin x$

or

$y = \cos x$



↳ initial guess

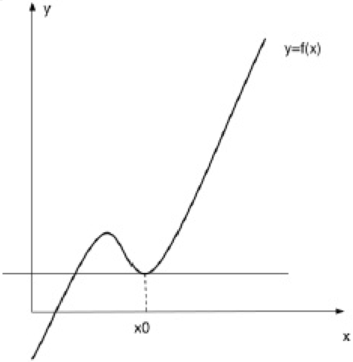
desired root

In this case, the sequence generated by N.M. converges to undesire root

Newton's method:

Case of failure:

(iii) When the initial guess or any iterative value of function never hits the $x-$axis i.e. $f'(x) = 0$.

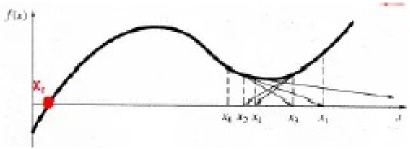


In this case, N.M. does not generate any sequence

Newton's method:

Case of failure:

(iv) When the initial guess is between local maximum or local minimum.



In this case, the sequence generated by N·M oscillates.

Newton's method:

Exercise:

1. Find the root of an equation $x - e^{-x} = 0$ by using Newton's method with the accuracy of 10^{-2}. Ans 0.5671 in 3 iterations.
2. The function $f(x) = \sin x$ has a zero on the interval $(3, 4)$ namely, $x = \pi$. Perform three iterations of Newton's method to approximate this zero, using $x_0 = 4$.