Lecture 5: Numerical Analysis (UMA011)

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$$C.N. = \left| \frac{x f'(x)}{f(x)} \right|$$

If $C.N. \lesssim \lesssim \lesssim 1$

then $f(x)$ is well-conditioned.

If $C.N. >>> 1$

then $f(x)$ is ill-conditioned

Algorithms

Creating Algorithms:

$f(x) = \sin x + \cos \frac{x}{2}$ at a given value of x

$$\begin{cases} x_0 : x = f_0(x_0) \\ x_1 = \sin x_0 \\ x_2 : \frac{x_0}{2} \\ x_3 : \cos x_2 \\ x_4 : x_1 + x_3 \end{cases}$$

C.N.

$$= \left| \frac{x f_0'(x)}{f_0(x)} \right|$$

$$= \left| \frac{x \cdot 1}{x} \right| = 1$$

$f(x)$

$x_0 : x$

$x_1 \quad f_0(x_0)$ C.N $>>> 1$

$x_2 : f_1(x_1)$

$x_3 : f_2(x_2)$

x_4 — —

Algorithms and Stability

Example:

Write an algorithm to calculate the expression $f(x) = \sqrt{x+1} - \sqrt{x}$, when x is quite large. By considering the condition number of the subproblem of evaluating the function, show that such a function evaluation is not stable. Suggest a modification which makes it stable.

Solution :-

x_0 : $x = 12345 = f^*(x)$

x_1 : $x_0 + 1 = f_0(x_0) = 12346$

x_2 : $\sqrt{x_1} = f_1(x_1) = 111.113$

x_3 : $\sqrt{x_0} = f_2(x_0) = 111.108$

x_4 : $x_2 - x_3 = f_3(x_2)$ or $f_3(x_3)$

Step1

C.N. of $f^*(x)$ $= \left| \dfrac{x\,(f^*(x))'}{f^*(x)} \right| = \left| \dfrac{x\,(1)}{x} \right| = 1$

Step2

C.N. of $f_0(x_0) = x_0+1$ $= \left| \dfrac{x_0\, f_0'(x_0)}{f_0(x_0)} \right| = \left| \dfrac{x_0\,(1)}{x_0+1} \right| < 1$

Step3

C.N. of $f_1(x_1) (= \sqrt{x_1})$ $= \left| \dfrac{x_1\, f_1'(x_1)}{f_1(x_1)} \right| = \left| \dfrac{\dfrac{x_1}{2\sqrt{x_1}}}{\sqrt{x}} \right|$

$= \left| \dfrac{x_1}{2x_1} \right| = \dfrac{1}{2} < 1$

Step 4 C.N. of $f_4(x_0) = \sqrt{x_0}$

$$= \left| \frac{x_0 \, f_4'(x_0)}{f_4(x_0)} \right| = \left| \frac{x_0 \frac{1}{2\sqrt{x_0}}}{\sqrt{x_0}} \right| = \frac{1}{2} < 1$$

Step 5 C.N. of $f_5(x_2) = x_2 - x_3$ $f_5(x_3)$

$$= \left| \frac{x_2 \, f_5'(x_2)}{f_5(x_2)} \right| = \left| \frac{x_2 (1)}{x_2 - x_3} \right| = \left| \frac{111.113}{111.113 - 111.108} \right| \Rightarrow \text{subtraction of nearly equal no's.}$$

$$>>> 1 \qquad = \left| \frac{111.113}{0.005} \right| = |22222.6| >>>> 1$$

$\Rightarrow$ Algorithm is unstable

Modification of $f(x)$

$$= \sqrt{x+1} - \sqrt{x} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$$

$$= \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$$F(x) = \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$x4$: $x_2 + x_3 = f_5(x_2)$ or $f_5(x_3)$

$x5$: $\frac{1}{x4} = f_6(x_4)$

$$\text{C.N. of } f_5(x_2) = x_2 + x_3$$

$$= \left| \frac{x_2 \, f_5'(x_2)}{f_5(x_2)} \right| = \frac{x_2 \,(1)}{x_2 + x_3} < 1$$

or

$f_5(x_3)$

$$\text{C.N} = \left| \frac{x_3 \, f_5'(x_3)}{x_2 + x_3} \right|$$

$$= \left| \frac{x_3 \,(1)}{x_2 + x_3} \right|$$

$$< 1$$

$$\text{C.N. of } f_6(x_4) = \frac{1}{x_4}$$

$$= \left| \frac{x_4 \; \frac{-1}{x_4^2}}{1/x_4} \right| = 1$$

$\Rightarrow$ Algorithm is Stable.

Error Analysis

Exercise:

1. Consider the stability (by calculating the condition number) of $\sqrt{x+1} - 1$ when x is near 0. Rewrite the expression to rid it of subtractive cancellation.
2. Write an algorithm to calculate the expression $f(x) = ln(x+1) - ln(x)$, for large values of x using six digit rounding arithmetic. By considering the condition number of the subproblem of evaluating the function, show that such a function evaluation is not stable. Also propose the modification of function evaluation so that algorithm will become stable.