Lecture 29: Numerical Analysis (UMA011)

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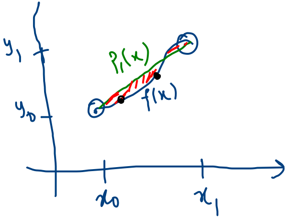
Polynomial Approximation

$f(x)$



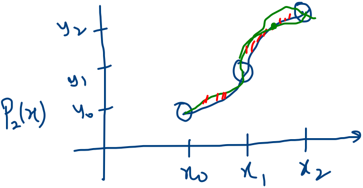
(x_0, y_0), (x_1, y_1)

$f(x_0)$ $\quad$ $f(x_1)$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$P_1(x_0) = f(x_0)$$

$$P_1(x_1) = f(x_1)$$



Polynomial interpolation:

Lagrange Interpolating polynomials:

Linear Interpolation: The linear Lagrange's interpolating polynomial passes through $(x_0, f(x_0)), (x_1, f(x_1))$ at which function $f(x)$ passes is

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1),$$

where $L_0(x) = \frac{x-x_1}{x_0-x_1}$ and $L_1(x) = \frac{x-x_0}{x_1-x_0}$.

s.t.

$P_1(x_0) = f(x_0)$

$P_1(x_1) = f(x_1)$

Lagrange Interpolating polynomials:

Quadratic Lagrange Interpolating polynomial:

Let function $f(x)$ passes through 3 points

$$(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)).$$

Consider the construction of a polynomial of degree at most 2 that passes through these 3 points.

For this, we define $L_{2,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2} \frac{x - x_i}{x_k - x_i}$.

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

The polynomial is given by

$P_2(x) = L_{2,0}(x)f(x_0) + L_{2,1}(x)f(x_1) + L_{2,2}(x)f(x_2)$.

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Lagrange Interpolating polynomials:

Generalization:

If $x_0, x_1, \cdots, x_n$ are $n+1$ distinct points and f is a function whose values are given at these numbers i.e. $f(x_0), f(x_1), \cdots, f(x_n)$, then a unique polynomial $P(x)$ of degree at most n exists with $f(x_k) = P(x_k)$, for each $k = 0, 1, 2, \cdots, n$. The polynomial is given by

$$\begin{aligned} P_n(x) &= L_{n,0}(x)f(x_0) + L_{n,1}(x)f(x_1) + \cdots + L_{n,n}(x)f(x_n) \\ &= \sum_{k=0}^{n} L_{n,k}(x)f(x_k), \end{aligned}$$

Lagrange Interpolating polynomials:

Generalization (continue):

where for each $k = 0, 1, 2, \cdots, n$

$$
\begin{aligned}
L_{n,k}(x) &= \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \\
&= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}.
\end{aligned}
$$

Lagrange Interpolating polynomials:

Example:

1. Use the numbers $x_0 = 2, x_1 = 2.75, x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.
2. Use this polynomial to approximate $f(3) = \frac{1}{3}$.

Solution: (i) Given points are $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$

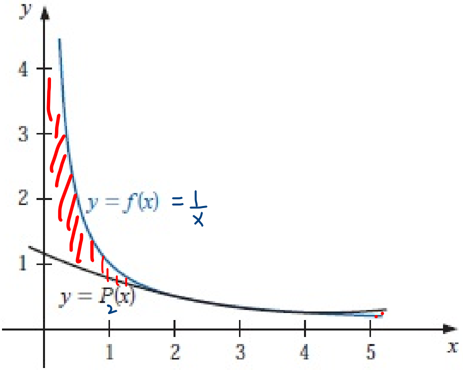
and $f(x_0) = \frac{1}{2}$, $f(x_1) = \frac{1}{2.75}$, $f(x_3) = \frac{1}{4}$.

The second degree Lagrange's Interpolating polynomial is given by $P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$

$$P_2(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} * \frac{1}{2} + \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} * \frac{1}{2.75} + \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} * \frac{1}{4}$$

$$P_2(x) = \frac{x^2}{22} - \frac{35}{88}x + \frac{49}{44}$$ ✓

(ii) $$P_2(3) = \frac{9}{22} - \frac{35}{88} * 3 + \frac{49}{44} = 0.32955 \approx \frac{1}{3} = f(3)$$


y
4
3
2
1
y = f(x) = 1/x
y = P_2(x)
1
2
3
4
5
x

Lagrange Interpolating polynomials:

Result (error term):

Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots, x_n$, and hence in (a, b), exists with

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n), \quad (1)$$

Exact   App.

where $P_n(x)$ is $n-$th degree Lagrange's interpolating polynomial.

$f(x) \to$ exact

$P_n(x) \to$ App.

$|f(x) - P_n(x)|$ = error function

Proof: Generalized Rolle's theorem :- If $f \in C^n[a,b]$ and f has zeros at $(n+1)$ distinct numbers, then there exists a no. ξ in (a,b) for which $f^{(n)}(\xi) = 0$.

If we have $x = x_k \ \forall \ k = 0, 1, 2 \cdots n$ in (1)

then $f(x_k) = P_n(x_k) \ \forall \ k = 0, 1, 2 \cdots n$

and for any $\xi(x)$

Now, if $x \neq x_k$, then we define a function g for t in $[a, b]$ by

$$\begin{cases} g(t) = f(t) - P_n(t) - \left[f(x) - P_n(x)\right] \dfrac{(t-x_0)(t-x_1) \cdots (t-x_n)}{(x-x_0)(x-x_1) \cdots (x-x_n)} \\ \quad\;\; = f(t) - P_n(t) - (f(x) - P_n(x)) \displaystyle\prod_{i=0}^{n} \frac{(t-x_i)}{(x-x_i)} \end{cases}$$

Since $f \in C^{n+1}[a,b]$ and $P_n \in C^{\infty}[a,b]$ then $g \in C^{n+1}[a,b]$

for $t = x_0$

$$g(x_0) = f(x_0) - P_n(x_0) - \left(f(x) - P_n(x)\right)(0)$$

$$= f(x_0) - P_n(x_0) = 0$$

$||^{ly}$ $g(x_1) = 0$, $g(x_2) = 0$ - - - $g(x_n) = 0$

$\Rightarrow$ $g(x_k) = 0 \quad \forall\ k = 0, 1, 2 - - - n$.

To be continued . . . .

Lagrange Interpolating polynomials:

Exercise:

1. For the given functions $f(x)$, let $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.6$. Construct Lagrange interpolation polynomials of degree at most one and at most two to approximate $f(1.4)$, and find the absolute error.
   a. $f(x) = \sqrt[3]{x-1}$.
   b. $f(x) = \log_{10}(3x-1)$.
2. Let $P_3(x)$ be the Lagrange interpolating polynomial for the data $(0,0)$, $(0.5,y)$, $(1,3)$ and $(2,2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.