Lecture 16: Numerical Analysis (UMA011)

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overview

$$f(x) = 0$$

$$p$$

$$f(x) = (x-p)^1 q(x) - \ldots$$

for e.g.

$$f(x) = (x^2 - 3x + 2)(x-1)$$

$$= (x-1)(x-2)(x-1)$$

$$= (x-1)^2 (x-2)$$

1, 1, 2 → Simple

Multiple

$f(x)$

$f(x) = 0$

Modified Newton's method

if multiplicity is not given

$$p_{n+1} = p_n - \frac{\mu(p_n)}{\mu'(p_n)}$$

$$\mu(p_n) = \frac{f(p_n)}{f'(p_n)}$$

$$p_{n+1} = p_n - \frac{f(p_n) f'(p_n)}{(f'(p_n))^2 - f(p_n) f''(p_n)}$$

Multiple roots

Modified Newton's method (if multiplicity is given)

let $f(x)=0$ be an equation, then

Modified Newton method is

$$p_{n+1} = p_n - m\frac{f(p_n)}{f'(p_n)},$$

m is the multiplicity of required root p (say)

Multiple roots

Example:

Show that the modification of Newton's method improves the rate of convergence for $f(x) = e^x - x - 1$ at $x = 0$ with $p_0 = 1$.

Solution :-

$$f(x) = e^x - x - 1$$

$$f(0) = 0$$

$$f'(x) = e^x - 1$$

$$\Rightarrow f'(0) = 0 \qquad \Rightarrow f(x) \text{ has a zero at } x = 0 \text{ with } m = 2$$

$$f''(x) = e^x$$

$$f''(0) = 1 \neq 0$$

Apply modified Newton's method

$$p_{n+1} = p_n - 2\,\frac{e^{p_n} - p_n - 1}{e^{p_n} - 1}, \quad m = 2$$

let $p_0 = 1$

$$p_1 = 1 - 2\,\frac{e^1 - 1 - 1}{e - 1} = 1 - \frac{2(e-2)}{e-1} = \frac{3-e}{e-1}$$

$= 0.1639534$

$$p_2 = 0.16395 - 2\,\frac{\left(e^{0.16395} - 0.16395 - 1\right)}{\left(e^{0.16395} - 1\right)} = 0.0044779$$

$$p_3 = p_2 - 2\,\frac{e^{p_2} - p_2 - 1}{e^{p_2} - 1}$$

$$= 0.0044779 - 2\,\frac{e^{0.0044779} - 0.0044779 - 1}{e^{0.0044779} - 1}$$

$$= 0.0000033419$$

To check the order of convergence

$$\frac{|p_2 - 0|}{|p_1 - 0|^2} = \frac{0.0044779}{(0.1639534)^2} = 0.1665 < 1 \quad , \quad \frac{|p_3 - 0|}{|p_2 - 0|^2} = \frac{0.0000033419}{(0.0044779)^2} < 1$$

Multiple roots

Order of convergence of modified Newton's method:

Modified Newton's method

$$p_{n+1} = p_n - m\frac{f(p_n)}{f'(p_n)} = g(p_n),$$

Here $f(x) = 0$ has a root p with multiplicity m but $f'(p) \neq 0$

$$p_{n+1} = g(p_n)$$

Now, $g(x) = x - m\frac{f(x)}{f'(x)}$

ie $f(x) = (x-p)^m q(x)$.

$$g(x) = x - m \frac{(x-p)^m q(x)}{(x-p)^m q'(x) + q(x)\, m (x-p)^{m-1}}$$

$$g(x) = x - m \frac{(x-p)\, q(x)}{(x-p)\, q'(x) + m q(x)}$$

$$g'(x) = 1 - m \left[(x-p) q(x) \frac{d}{dx}\left(\frac{1}{(x-p)q'(x) + mq(x)} \right) + \frac{1}{(x-p)q'(x)+mq(x)} \left((x-p)q'(x) + q(x) \right) \right]$$

Put $x = p$

$$g'(p) = 1 - m \left[0 + \frac{\cancel{q(p)}}{m\, \cancel{q(p)}} \right] = 1 - \frac{m}{m} = 1 - 1 = 0$$

$\Rightarrow \quad g'(p) = 0$

The sequence generated by modified Newton's Method gives atleast quadratic convergent.

Multiple roots:

Exercise:

1. Use Newton's method and the modified Newton's method to find a solution of

$$(1-x)\sin(1-x)=0,$$

accurate to within 10^{-2}. Take initial approximation $x_0=0$.

2. Apply modified Newton's method with $m=2$ and $x_0=0.8$ to the equation $f(x)=x^3-x^2-x+1=0$, and verify that the convergence is of second-order.