Eigen Values of $n \times n$ matrix A are obtained by solving its characteristic equation

$$\det(A - \lambda I) = 0$$

$$\Rightarrow C_n \lambda^n + C_{n-1}\lambda^{n-1} + C_{n-2}\lambda^{n-2} + \dots + C_0 = 0 \quad — ①$$

↳ This is polynomial in λ and gives you the eigen values corresponding to matrix A.

However, if n is very large, then the polynomial ① is difficult to solve and also very time-consuming which is sensitive to rounding errors.

∴ We need to develop an alternative method for approximating eigen values.

- The n roots of ① are n eigen values of A.

* For each eigenvalue λ_i, there exists a non-zero vector X_i which is a nonzero solution of the system of equations

$$(A - \lambda_i I) X_i = 0$$

Properties of Eigen Values

① A has a zero eigen value iff A is singular matrix

$$|A - 0.I| = 0 \Rightarrow |A| = 0 \Rightarrow A \text{ is singular.}$$

② A and A^T has same eigen values.

③ If the eigen values of matrix A are $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ are the eigen values of A^m where m is the positive integer. Both A and A^m have same set of eigen vectors.

④ If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of A, then $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}$ are eigen values of A^{-1}. Also both A and A^{-1} have same set of eigen vectors

⑤ Sum of all eigen values of A is equal to the trace of A, i.e $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$ = trace of A.

⑥ If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of A, then $\lambda_i - q$, $i = 1, 2, \dots, n$ are the eigen values of $A - qI$, where q is some constant.

<u>Dominant Eigen value:</u> Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of $n \times n$ matrix A. Then λ_1 is called dominant eigen value if $|\lambda_1| > |\lambda_i|$ for $i = 2, 3, \ldots, n$.

<u>Dominant Eigen Vector:</u> The eigen vectors corresponding to λ_1 is known as dominant eigen vectors of A.

Power Method

* This method is normally used to determine the largest eigenvalue (in magnitude) and the corresponding eigen vector of the system $AX = \lambda X$.
* This method of approximating eigen values is iterative method.
* In this method, we assume that the matrix A has a dominant eigenvalue with corresponding dominant eigen vectors.
* Choose an initial approximation X_0 to generate sequences $\{X_k\}$ and $\{C_k\}$ recursively or iteratively.

$$X_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})^T$$

Let us consider the eigen value problem $AX = \lambda X$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of A such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n|$$

· Assume ~~that~~ an initial approximation X_0 of one of the dominant eigen vectors of A. Here X_0 must be nonzero vector in $\mathbb{R}^n$.

Now, we can generate a sequence $\{X_k\}$ such that

$$X_1 = AX_0$$

$$X_2 = AX_1 = A(AX_0) = A^2X_0$$

$$\vdots$$

$$X_k = AX_{k-1} = A(A^{k-1}X_0) = A^kX_0$$

For large ~~values~~ powers of k and by scaling this sequence, we obtain a good approximation of ~~the~~ largest

Eigen value and dominant eigen vector.

Example: Compute six iterations of the power method to approximate a dominant eigen vector of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol. Consider initial non-zero approximation of $X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Y_1 = AX_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} = -10\begin{bmatrix} 1 \\ 0.4 \end{bmatrix} = K_1 X_1$$

$|-10| = 10$
$|-4| = 4$
$Max = 10$

$$= -10\,X_1 \quad \text{where} \quad X_1 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}$$

$$Y_2 = AX_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} -2.8 \\ -1 \end{bmatrix} = -2.8\begin{bmatrix} 1 \\ 0.3571 \end{bmatrix} = -2.8X_2 = K_2 X_2$$

$$Y_3 = AX_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.3571 \end{bmatrix} = \begin{bmatrix} -2.2852 \\ -0.7855 \end{bmatrix} = -2.2852\begin{bmatrix} 1 \\ 0.3437 \end{bmatrix}$$

$$= 2.284\;X_3 = K_3 X_3$$

$$Y_4 = AX_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.3437 \end{bmatrix} = \begin{bmatrix} -2.1244 \\ -0.7185 \end{bmatrix} = -2.1244\begin{bmatrix} 1 \\ 0.3382 \end{bmatrix}$$

$$= -2.128\,X_4 = K_4 X_4$$

$$Y_5 = AX_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.3382 \end{bmatrix} = \begin{bmatrix} -2.0584 \\ -0.691 \end{bmatrix} = -2.0584\begin{bmatrix} 1 \\ 0.3357 \end{bmatrix}$$

$$= -2.056\,X_5 = K_5 X_5$$

$$Y_6 = AX_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.3357 \end{bmatrix} = \begin{bmatrix} -2.0284 \\ -0.6785 \end{bmatrix} = -2.0284\begin{bmatrix} 1 \\ 0.3345 \end{bmatrix}$$

$$= -2.032\,X_6 = K_6 X_6$$

$|K_2 - K_1| = 7.2$, $|K_3 - K_2| = 0.516$, $|K_4 - K_3| = 0.156$, $|K_5 - K_4| = 0.072$, $|K_6 - K_5| = 0.024$

$$Y_7 = AX_6 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.3345 \end{bmatrix} = \begin{bmatrix} -2.014 \\ -0.6725 \end{bmatrix} = -2.014\begin{bmatrix} 1 \\ 0.3339 \end{bmatrix} = K_7 X_7 \qquad |K_7 - K_6| = 0.0144$$

Algorithm

1. Define matrix A and initial guess x ($\|x\|_\infty = 1$). ~~Assume K_old = 0~~
2. Calculate $y = Ax$
3. ~~Set K_old = K_new~~
4. Find the largest element (in magnitude) of matrix y and assign it to K_{new}
5. Now calculate fresh value of $x = \left(\frac{1}{K_{new}}\right) * y$
6. If $|K\_new - K\_old| > error$, goto step 3. ($|K_n - K_{n-1}| > error$)

else
STOP.

Q $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Estimate Dominant eigen value & vector and then the remaining eigen values. ($\boxed{tol = 0.001}$)

Sol. $y_1 = Ax_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 0.6 \\ 0.2 \\ 1 \end{bmatrix} = 5x_1 = \lambda_1 x_1$ $\quad \lambda_1 = 5$

Now $y_2 = Ax_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2.2 \end{bmatrix} = 2.2 \begin{bmatrix} 0.4545 \\ 0.4545 \\ 1 \end{bmatrix}$

$= 2.2\, x_2 = \lambda_2 x_2$ $\quad \boxed{|\lambda_2 - \lambda_1| = 2.8}$

$y_3 = Ax_2 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.4545 \\ 0.4545 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.3635 \\ 1.5455 \\ 2.818 \end{bmatrix} = 2.818 \begin{bmatrix} 0.4839 \\ 0.5484 \\ 1 \end{bmatrix}$

$= 2.818\, x_3 = \lambda_3 x_3$ $\quad \boxed{|\lambda_3 - \lambda_2| = 0.618}$

$y_4 = Ax_3 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.4839 \\ 0.5484 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5807 \\ 1.5806 \\ 3.129 \end{bmatrix} = 3.129 \begin{bmatrix} 0.5052 \\ 0.5051 \\ 1 \end{bmatrix}$

$= 3.129\, x_4 = \lambda_4 x_4$ $\quad \boxed{|\lambda_4 - \lambda_3| = 0.311}$

$y_5 = Ax_4 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5052 \\ 0.5051 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5156 \\ 1.4947 \\ 3.0205 \end{bmatrix} = 3.0205 \begin{bmatrix} 0.5018 \\ 0.4949 \\ 1 \end{bmatrix}$

$= 3.0205\, x_5 = \lambda_5 x_5$ $\quad \boxed{|\lambda_5 - \lambda_4| = 0.1085}$

$y_6 = Ax_5 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5018 \\ 0.4949 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4916 \\ 1.4913 \\ 2.9865 \end{bmatrix} = 2.9865 \begin{bmatrix} 0.4994 \\ 0.4993 \\ 1 \end{bmatrix} = 2.9863\, x_6 = \lambda_6 x_6$ $\quad \boxed{|\lambda_6 - \lambda_5| = 0.034}$

$y_7 = Ax_6 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.4994 \\ 0.4993 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.498 \\ 1.5005 \\ 2.9973 \end{bmatrix} = 2.9973 \begin{bmatrix} 0.4998 \\ 0.5002 \\ 1 \end{bmatrix} = 2.9973\, x_7 = \lambda_7 x_7$ $\quad \boxed{|\lambda_7 - \lambda_6| = 0.011}$

$y_8 = Ax_7 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.4998 \\ 0.5002 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5002 \\ 1.5006 \\ 3.0004 \end{bmatrix} = 3.0004 \begin{bmatrix} 0.5000 \\ 0.5001 \\ 1 \end{bmatrix} = 3.0004\, x_8 = \lambda_8 x_8$

$\boxed{|\lambda_8 - \lambda_7| = 0.0031 < tol}$

$y_9 = Ax_8 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5000 \\ 0.5001 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5002 \\ 1.5001 \\ 3.0003 \end{bmatrix} = 3.0003 \begin{bmatrix} 0.5000 \\ 0.5000 \\ 1 \end{bmatrix} = 3.0003\, x_9$

Here $|k_9 - k_8| = |3.0003 - 3.0004| = 0.0001 < 0.001$

$\therefore$ Dominant eigen value $= 3.0003 \approx 3$

Dominant eigenvector $= [0.5, 0.5, 1]^T$

\# Find smallest eigen value of A in magnitude.

$\lambda_1, \lambda_2, \lambda_3$ are eigen ~~values of A~~ such that

$\Rightarrow |\lambda_1| > |\lambda_2| > |\lambda_3| \quad \Rightarrow \lambda_1$ is dominant eigen value of A.

$\Rightarrow \left|\frac{1}{\lambda_1}\right| < \left|\frac{1}{\lambda_2}\right| < \left|\frac{1}{\lambda_3}\right| \quad \Rightarrow \left|\frac{1}{\lambda_3}\right|$ is the dominant eigen value of A^{-1}

$\Rightarrow \lambda_3$ is the smallest eigen value of A. (in magnitude)

e.g. Find smallest eigenvalue in magnitude of A

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}, \quad \text{or} = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}$$

Now $Y_1 = A^{-1}X_0 = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2 \\ 1.5 \end{bmatrix} = 2\begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix} = K_1 X_1$

$Y_2 = A^{-1}X_1 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}\begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 1.25 \end{bmatrix} = 1.75\begin{bmatrix} 0.7143 \\ 1 \\ 0.7143 \end{bmatrix} = K_2 X_2$

$|K_2 - K_1| = 0.25$

$Y_3 = A^{-1}X_2 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}\begin{bmatrix} 0.7143 \\ 1 \\ 0.7143 \end{bmatrix} = \begin{bmatrix} 1.2143 \\ 1.7143 \\ 1.2143 \end{bmatrix} = 1.7143\begin{bmatrix} 0.7083 \\ 1 \\ 0.7083 \end{bmatrix} = K_3 X_3$

$|K_3 - K_2| = 0.0357$

$$y_4 = A^{-1}X_3 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0.7083 \\ 1 \\ 0.7083 \end{bmatrix} = \begin{bmatrix} 1.2083 \\ 1.7083 \\ 1.2083 \end{bmatrix} = 1.708 \begin{bmatrix} 0.7073 \\ 1 \\ 0.7073 \end{bmatrix}$$

$$= k_4 x_4$$

$|k_4 - k_3| = 0.006 < 10^{-2}$

$\therefore$ Dominant eigen value of $A^{-1} = \mu = 1.7083$

$\Rightarrow \frac{1}{\lambda} = 1.7083 \Rightarrow \lambda = \frac{1}{1.7083} = 0.58537$

Smaller eigen value of $A = \cancel{}\, 2 - \sqrt{2} \approx 0.58\cancel{5}86$

Find eigen value of A nearest to q.

For this find largest eigen value (in magnitude) of $(A - qI)^{-1}$.

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigen values of A

$\lambda_1 - q, \lambda_2 - q, \ldots, \lambda_n - q$ " " " $A - qI$

$\frac{1}{\lambda_1 - q}, \frac{1}{\lambda_2 - q}, \ldots, \frac{1}{\lambda_n - q}$ — — $(A - qI)^{-1}$

$\mu = \frac{1}{\lambda - q}$: dominant eigen value of $(A - qI)^{-1}$

$\Rightarrow \lambda - q = \frac{1}{\mu} \Rightarrow \lambda = q + \frac{1}{\mu}$ is the dominant eigenvalue of A nearest to q.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find eigen value of A nearest to 3.

$\|X_1 - X_0\|_\infty = \max\{1, 0, 1\} = 1$

$\|X_2 - X_1\|_\infty = \max\{1, 0, 1\} = 1$

$\|X_3 - X_2\|_\infty = \max\{0.3333, 0, 0.3333\} = 0.3333$

$\cancel{\|X_4 - X_3\|_\infty = 0.0476}$

$\|X_5 - X_4\|_\infty = 0.0084$

$$A - 3I = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$(A - 3I)^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Y_1 = \cancel{Ax}\,(A-3I)^{-1} X_0 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = k_1 X_1$$

$$Y_2 = (A-3I)^{-1} X_1 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = k_2 X_2$$

$$Y_3 = (A-3I)^{-1} X_2 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} -2/3 \\ 1 \\ -2/3 \end{bmatrix} = 3 \begin{bmatrix} -0.6667 \\ 1 \\ -0.6667 \end{bmatrix} = k_3 X_3$$

$$Y_4 = (A-3I)^{-1} X_3 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -0.6667 \\ 1 \\ -0.6667 \end{bmatrix} = \begin{bmatrix} -1.6667 \\ 2.3334 \\ -1.6667 \end{bmatrix} = 2.3334 \begin{bmatrix} -0.7143 \\ 1 \\ -0.7143 \end{bmatrix} = k_4 X_4$$

$$Y_5 = (A-3I)^{-1} X_4 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -0.7143 \\ 1 \\ -0.7143 \end{bmatrix} = \begin{bmatrix} -1.7143 \\ 2.4286 \\ -1.7143 \end{bmatrix} = 2.4286 \begin{bmatrix} -0.7059 \\ 1 \\ -0.7059 \end{bmatrix} = k_5 X_5$$

$\cancel{\|\delta_6 - \delta_5\| = 0.0168 \times 10^{-2}}$

$\|X_5 - X_4\|_\infty = \max\{0.0084, 0, 0.0084\} = 0.0084 < 10^{-2}$

$\left.\mu = 2.4286\right\} \Rightarrow \dfrac{1}{\lambda - 3} = 2.4286$

$\Rightarrow \lambda - 3 = \dfrac{1}{2.4286} = 0.41176 \Rightarrow \lambda = 3 + 0.41176$
$= 3.41176$

$\lambda = 2 + \sqrt{2} = 3.41421$

Power Method Important Points

① If initial vector and given matrix have only real entries, then power method only give real eigen value (dominant) i.e power method will not give complex eigen value.

② If A is symmetric then it always has real eigen values. So power method converges for symmetric matrices

③ If A is symmetric and positive definite, then also its eigen values are positive number

④ If two eigen values ~~are~~ have equal magnitude, power method may not converge.
$\lambda_1 = \lambda_2$ or $\lambda_1 = -\lambda_2$ or $\lambda_1 = \bar{\lambda}_2$

⑤ Power method converges slowly if $\frac{|\lambda_2|}{|\lambda_1|}$ is close to 1.

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