

**Course: UMA 035 (Optimization Techniques)**

**Instructor: Dr. Amit Kumar,**

**Associate Professor,**

**School of Mathematics,**

**TIET, Patiala**

**Email: amitkumar@thapar.edu**

**Mob: 9888500451**

## Equation of a Plane

$$ax+by+cz=d$$

or

$$a_1x_1 + a_2x_2 + a_3x_3 = d$$

## Equation of a hyper plane

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = d$$

$$\text{Assuming } [a_1 \ a_2 \ \dots \ a_n] = A \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X,$$

$$AX=d$$

## Set of hyper planes

$$S=\{X: AX=d\}$$

## Equation of a closed half space

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n \leq d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leq d$$

$$\text{Assuming } [a_1 \ a_2 \ \dots \ a_n] = A \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X,$$

$$AX \leq d$$

**OR**

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n \geq d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq d$$

Assuming  $[a_1 \ a_2 \ \dots \ a_n] = A$  and  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X$ ,

$$AX \geq d$$

### Set of closed half spaces

$$S = \{X : AX \leq d\}$$

**OR**

$$S = \{X : AX \geq d\}$$

### Equation of an open half space

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n < d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} < d$$

Assuming  $[a_1 \ a_2 \ \dots \ a_n] = A$  and  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X$ ,

$$AX < d$$

**OR**

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n > d$$

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} > \mathbf{d}$$

Assuming  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n] = \mathbf{A}$  and  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{X}$ ,

$$\mathbf{AX} > \mathbf{d}$$

### Set of open half spaces

$$S = \{\mathbf{X} : \mathbf{AX} < \mathbf{d}\}$$

OR

$$S = \{\mathbf{X} : \mathbf{AX} > \mathbf{d}\}$$

### Matrix form of a LPP

Maximize  $(\mathbf{C}\mathbf{X})$

Subject to

$$\mathbf{AX} = \mathbf{b},$$

$$\mathbf{X} \geq \mathbf{0},$$

where,

➤  $\mathbf{C} = [c_1, c_2, \dots, c_n]$ ,

➤  $\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

➤  $\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ ,

$$\triangleright \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

**Set of feasible solution of a LPP**

$$S = \{\mathbf{X}: \mathbf{AX}=\mathbf{B}, \mathbf{X} \geq \mathbf{0}\}$$

**Set of optimal solutions of a LPP**

$$S = \{\mathbf{X}: \mathbf{CX}=\mathbf{p}, \mathbf{AX}=\mathbf{B}, \mathbf{X} \geq \mathbf{0}\}, \text{ where } p \text{ is the maximum value.}$$

**Method to check that a set is convex or not**

**Step 1:** Consider two general elements of the considered set.

**Step 2:** Assume that the considered elements satisfies the properties of the considered set (Conditions written after “ : ” in the set).

**Step 3:** Write the convex linear combination of the considered elements.

**Step 4:** Simplify the convex linear combination to obtain a general element by multiplying with scalars inside and adding the elements position wise.

**Step 5:** Check that for the general element, obtained in Step 4, the properties of the considered set (Conditions written after “ : ” in the set) are satisfying or not.

**Case 1:** If all the properties will be satisfied then the considered set will be convex.

**Case 2: If one or more properties will not be satisfied then the considered set will not be convex.**

**Prove that set of optimal solutions of a LPP is a convex set**

**Proof:**

Let  $S = \{X: CX=p, AX=b, X \geq 0\}$  (where  $p$  is the optimal value of the LPP) be the set of optimal solutions of a LPP.

Let  $X_1$  and  $X_2$  be two elements of the set  $S$ .

Since,  $X_1$  and  $X_2$  belongs to the set  $S$ . So, these will satisfy the following properties of the set  $S$ .

$C X_1=p, A X_1=b, X_1 \geq 0$

$C X_2=p, A X_2=b, X_2 \geq 0$

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$a_1 \geq 0,$

$a_2 \geq 0,$

$a_1 + a_2 = 1.$

Now,

$$(i) C(a_1 X_1 + a_2 X_2) = a_1(CX_1) + a_2(CX_2)$$

$$= a_1(p) + a_2(p) \quad (\text{since, } CX_1=p \text{ and } CX_2=p)$$

$$= (a_1 + a_2)p$$

$$= p \quad (\text{since, } a_1 + a_2 = 1)$$

$$(ii) A(a_1 X_1 + a_2 X_2) = a_1(AX_1) + a_2(AX_2)$$

$$= a_1(b) + a_2(b) \quad (\text{since, } AX_1=b \text{ and } AX_2=b)$$

$$= (a_1 + a_2)b$$

$$= b \quad (\text{since, } a_1 + a_2 = 1)$$

$$(iii) \text{ Since, } a_1 \geq 0, a_2 \geq 0, X_1 \geq 0, X_2 \geq 0. \text{ So, } a_1 X_1 + a_2 X_2 \geq 0$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of all optimal solutions of a LPP) is a convex set.

**Prove that set of feasible solutions of a LPP is a convex set**

**Proof:**

Let  $S = \{X: AX=b, X \geq 0\}$  be the set of feasible solutions of a LPP.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$$A \quad X_1 = b, \quad X_1 \geq 0$$

$$A \quad X_2 = b, \quad X_2 \geq 0$$

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$\begin{aligned} (i) \quad A \quad (a_1 X_1 + a_2 X_2) &= a_1 (AX_1) + a_2 (AX_2) \\ &= a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 = b \text{ and } AX_2 = b) \\ &= (a_1 + a_2)b \\ &= b \quad (\text{since, } a_1 + a_2 = 1) \end{aligned}$$

(ii) Since,  $a_1 \geq 0, a_2 \geq 0, X_1 \geq 0, X_2 \geq 0$ . So,  $a_1 X_1 + a_2 X_2 \geq 0$

**It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1X_1 + a_2 X_2$ .**

**Hence, the set S (set of all feasible solutions of a LPP) is a convex set.**

**Prove that set of hyper planes is a convex set**

**Proof:**

Let  $S = \{X : AX = b\}$  be the set of hyper planes.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$A X_1 = b$ ,

$A X_2 = b$ ,

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$a_1 \geq 0$ ,

$a_2 \geq 0$ ,

$a_1 + a_2 = 1$ .

Now,

$$\begin{aligned} \mathbf{A}(\mathbf{a}_1 \mathbf{X}_1 + \mathbf{a}_2 \mathbf{X}_2) &= \mathbf{a}_1 (\mathbf{AX}_1) + \mathbf{a}_2 (\mathbf{AX}_2) \\ &= \mathbf{a}_1 (\mathbf{b}) + \mathbf{a}_2 (\mathbf{b}) \quad (\text{since, } \mathbf{AX}_1 = \mathbf{b} \text{ and } \mathbf{AX}_2 = \mathbf{b}) \\ &= (\mathbf{a}_1 + \mathbf{a}_2)\mathbf{b} \\ &= \mathbf{b} \quad (\text{since, } \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{1}) \end{aligned}$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $\mathbf{a}_1 \mathbf{X}_1 + \mathbf{a}_2 \mathbf{X}_2$ .

Hence, the set S (set of hyper planes) is a convex set.

**Prove that set of closed half spaces is a convex set**

**Proof:**

Let  $S = \{X : \mathbf{AX} \leq \mathbf{b}\}$  be the set of closed half spaces.

Let  $\mathbf{X}_1$  and  $\mathbf{X}_2$  be two elements of the set S.

Since,  $\mathbf{X}_1$  and  $\mathbf{X}_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$\mathbf{A} \mathbf{X}_1 \leq \mathbf{b}$ ,

$\mathbf{A} \mathbf{X}_2 \leq \mathbf{b}$ ,

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$\leq a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 \leq b \text{ and } AX_2 \leq b)$$

$$\leq (a_1 + a_2)b$$

$$\leq b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of closed half spaces) is a convex set.

OR

Let  $S = \{X : AX \geq b\}$  be the set of closed half spaces.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

A  $X_1 \geq b$ ,

A  $X_2 \geq b$ ,

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$a_1 \geq 0$ ,

$a_2 \geq 0$ ,

$a_1 + a_2 = 1$ .

Now,

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$\geq a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 \geq b \text{ and } AX_2 \geq b)$$

$$\geq (a_1 + a_2)b$$

$$\geq b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of closed half spaces) is a convex set.

**Prove that set of open half spaces is a convex set**

**Proof:**

Let  $S = \{X : AX < b\}$  be the set of closed half spaces.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$A X_1 < b$ ,

$A X_2 < b$ ,

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$a_1 \geq 0$ ,

$a_2 \geq 0$ ,

$a_1 + a_2 = 1$ .

Now,

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$< a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 < b \text{ and } AX_2 < b)$$

$$< (a_1 + a_2)b$$

$$< b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of open half spaces) is a convex set.

OR

Let  $S = \{X : AX > b\}$  be the set of closed half spaces.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$AX_1 > b$ ,

$AX_2 > b$ ,

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$> a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 > b \text{ and } AX_2 > b)$$

$$> (a_1 + a_2)b$$

$$> b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of open half spaces) is a convex set.

**Prove that intersection of finite number of convex sets is a convex set**

**Proof:**

Let  $S_1 \cap S_2 \cap \dots \cap S_n$  be the intersection of "n" convex sets  $S_1, S_2, \dots, S_n$ .

Let  $X_1$  and  $X_2$  be two elements of the set  $S_1 \cap S_2 \cap \dots \cap S_n$ .

Since,  $X_1$  and  $X_2$  belongs to the set  $S_1 \cap S_2 \cap \dots \cap S_n$ .

So,

$X_1$  and  $X_2$  belongs to all the convex set  $S_1$

$X_1$  and  $X_2$  belongs to all the convex set  $S_2$

:

$X_1$  and  $X_2$  belongs to all the convex set  $S_n$

Furthermore as,

$S_1$  is a convex set. So,  $X_1$  and  $X_2$  belongs to the set  $S_1$  implies that  $a_1X_1 + a_2$

$X_2$  also belongs to  $S_1$

$S_2$  is a convex set. So,  $X_1$  and  $X_2$  belongs to the set  $S_2$  implies that  $a_1X_1 + a_2$

$X_2$  also belongs to  $S_2$

:

$S_n$  is a convex set. So,  $X_1$  and  $X_2$  belongs to the set  $S_n$  implies that  $a_1X_1 + a_2$

$X_2$  also belongs to  $S_n$

Finally,

$a_1X_1 + a_2 X_2$  belongs to  $S_1, S_2, \dots, S_n$  implies that  $a_1X_1 + a_2 X_2$  belongs to

$S_1 \cap S_2 \cap \dots \cap S_n$

Hence,  $S_1 \cap S_2 \cap \dots \cap S_n$  is a convex set.