Lecture 34: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Least Square Approximation Method:

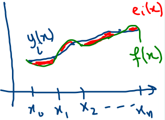
$f(x)$

$\downarrow$

$P_n(x)$

$|f(x) - P_n(x)| = e(x)$

$y_i = f(x_i)$



Least Square Approximation Method:

Suppose that the data points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, where x_i are the independent variable and y_i are the dependent variable.

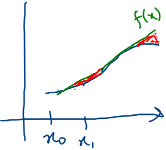
Let $e_i = y_i - f(x_i)$ be the error at each data points.

According to the method of least squares, the best fitting curve has the property that $\sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} (y_i - f(x_i))^2$ is minimum.

let $E = \sum_{i=0}^{n} e_i^2$ be minimized.

$= E(a, b, c - - -)$

Least Square Approximation Method:



Least Square fit of a straight line:

Suppose that the data points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Let $f(x) = a + bx$, where a, b are the constants to be determined to the given data.

Now residuals is given by

$$e_i = y_i - f(x_i) = y_i - (a + bx_i) \Rightarrow E = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} (y_i - (a + bx_i))^2.$$

We need to find a and b such that error E is minimum.

The necessary condition for minimum is $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$

Here E is $E(a,b)$

$$\frac{\partial E}{\partial a} = 2\sum_{i=0}^{n}(y_i - (a+bx_i))(-1) = 0, \quad \frac{\partial E}{\partial b} = 2\sum_{i=0}^{n}(y_i - (a+bx_i))(-x_i) = 0$$

$$\frac{\partial E}{\partial a} = 0 \Rightarrow \sum_{i=0}^{n} \left(y_i - (a + bx_i)\right) = 0$$

$$\Rightarrow -(n+1)a + \sum_{i=0}^{n} y_i - b\sum_{i=0}^{n} x_i = 0 \quad -\ (1)$$

$$\because \left.\begin{aligned} &\sum_{i=0}^{n} a \\ &= a\sum_{i=0}^{n} 1 \\ &a(n+1) \end{aligned}\right\}$$

& $$\frac{\partial E}{\partial b} = 0 \Rightarrow \sum_{i=0}^{n} \left(x_i y_i - a x_i - b x_i^2\right) = 0$$

$$= \sum_{i=0}^{n} x_i y_i - a\sum_{i=0}^{n} x_i - b\sum_{i=0}^{n} x_i^2 = 0 \quad -\ (2)$$

$$\left[-a(n+1) + \sum_{i=0}^{n} y_i - b\sum_{i=0}^{n} x_i = 0,\ \sum_{i=0}^{n} x_i y_i - a\sum_{i=0}^{n} x_i - b\sum_{i=0}^{n} x_i^2\right] \to \text{Normal equations}$$

Solve these equations to get a, b.

Least Square Approximation Method:

Least Square fit of a quadratic polynomial line:

Suppose that the data points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Let $f(x) = a + bx + cx^2$, where a, b, c are the constants to be determined to the given data.

Now residuals is given by $e_i = y_i - f(x_i) = y_i - (a + bx_i + cx_i^2) \Rightarrow E = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} (y_i - (a + bx_i + cx_i^2))^2$.

Here $E(a, b, c)$

We need to find a, b and c such that error E is minimum.

The necessary condition for minimum is $\frac{\partial E}{\partial a} = 0$, $\frac{\partial E}{\partial b} = 0$ and $\frac{\partial E}{\partial c} = 0$

$$\frac{\partial E}{\partial a} = 0 \Rightarrow 2 \sum_{i=0}^{n} \left(y_i - (a + bx_i + cx_i^2) \right) (-1) = 0$$

$$-(n+1)a + \sum_{i=0}^{n} y_i - b \sum_{i=0}^{n} x_i - c \sum_{i=0}^{n} x_i^2 = 0 \tag{1}$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow 2 \sum_{i=0}^{n} \left((y_i - (a + bx_i + cx_i^2) \; (-x_i) \right) = 0$$

$$\Rightarrow \sum_{i=0}^{n} x_i y_i - a \sum_{i=0}^{n} x_i - b \sum_{i=0}^{n} x_i^2 - c \sum_{i=0}^{n} x_i^3 = 0 \tag{2}$$

$$\frac{\partial E}{\partial c} = 0 \Rightarrow 2 \sum_{i=0}^{n} \left(y_i - (a + bx_i + cx_i^2) \right) (-x_i^2) = 0$$

$$\Rightarrow \sum_{i=0}^{n} x_i^2 y_i - a \sum_{i=0}^{n} x_i^2 - b \sum_{i=0}^{n} x_i^3 - c \sum_{i=0}^{n} x_i^4 = 0 \tag{3}$$

①, ②, ③ are called normal equations.

& Solve these eqⁿs to get a, b & c.

Least Square Approximation Method:

Example:

Obtain the least square straight line and quadratic polynomial fit to the following data:

$a+bx$ $\qquad$ $a+bx+cx^2$

$a+bx+cx^2$

x	5	10	15	20
$f(x)$	16	19	23	26

Solution:

To find straight line

i	x_i	$f(x_i)=y_i$	$x_i y_i$	x_i^2
0	5	16	80	25
1	10	19	190	100
2	15	23	345	225
3	20	26	520	400
	$\sum x_i = 50$	$\sum y_i = 84$	$\sum x_i y_i = 1135$	$\sum x_i^2 = 750$

By using least square app. method

$$-a(n+1) + \sum_{i=0}^{n} y_i - b\sum_{i=0}^{n} x_i = 0, \quad \sum_{i=0}^{n} x_i y_i - a\sum_{i=0}^{n} x_i - b\sum_{i=0}^{n} x_i^2 = 0$$

$$-4a + \sum_{i=0}^{3} y_i - b\sum_{i=0}^{3} x_i = 0, \quad \sum_{i=0}^{3} x_i y_i - a\sum_{i=0}^{3} x_i - b\sum_{i=0}^{3} x_i^2 = 0$$

$$-4a + 84 - 50b = 0, \quad 1135 - 50a - 750b = 0$$

$$4a + 50b = 84, \quad 50a + 750b = 1135.$$

on solving these eq^{n}s, we get $a = 0.68$
$b = 12.5$.

So, best fitting line is $f(x) = 0.68 + 12.5x$.

To find 2nd degree poly.

i	x_i	y_i	$x_i y_i$	x_i^2	x_i^3	x_i^4	$x_i^2 y_i$
0	5	16	80	25	125	625	400
1	10	19	190	100	1000	10000	1900
2	15	23	345	225	3375	50625	5175
3	20	26	520	400	8000	160000	10400
	$\Sigma x_i = 50$	84	1135	750	12500	221250	17875

Use least square app. normal eqn for quadratic poly.

$$-4a + 84 - b(50) - c(750) = 0 \Rightarrow 4a + 50b + 750c = 84 \quad -(1)$$

$$1135 - a(50) - b(750) - c(12500) = 0 \Rightarrow 50a + 750b + 12500c = 1135 \quad -(2)$$

& $17875 - a(750) - b(12500) - c(221250) = 0$

$$\Rightarrow 750a + 12500b + 221250c = 17875$$

Solve these three eqn's to get a, b, c

Least Square Approximation Method:

Exercise:

1. Use the method of least squares to fit the linear and quadratic polynomial to the following data.

x	-2	-1	0	1	2
$f(x)$	15	1	1	3	19