

Numerical differentiation

Initial value Problem

- Problem of the form

$$\bullet \quad y'(t) = f(t, y) \quad a \leq t \leq b$$

$$y(a) = d$$

is called the initial value problem.

$$\text{e.g. } \frac{dy}{dt} = t^2 + y \quad 0 \leq t \leq 2$$

$$y(0) = 0$$

- Euler's Method

Divide $[a, b]$ into n equal intervals, i.e. $t_i = t_0 + ih$, $i = 0, 1, 2, \dots, n$, $h = \frac{b-a}{n}$.

$$\frac{dy}{dt}(t_i) = \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} = \frac{y(t_{i+1}) - y(t_i)}{h}$$

$$\therefore \frac{dy}{dt} = f(t, y)$$

$$\frac{dy_i}{dt} = f(t_i, y_i)$$

$$y(t_{i+1}) - y(t_i) = f(t_i, y_i)h$$

$$y(t_{i+1}) = y(t_i) + hf(t_i, y_i)$$

Q Solve IVP using Euler's method
with $h=1$

$$y' = \frac{x-y}{2}, \quad y(0) = 1 \quad \text{on } [0, 3]$$

Sol. $y_{i+1} = y_i + h(f(x_i, y_i))$

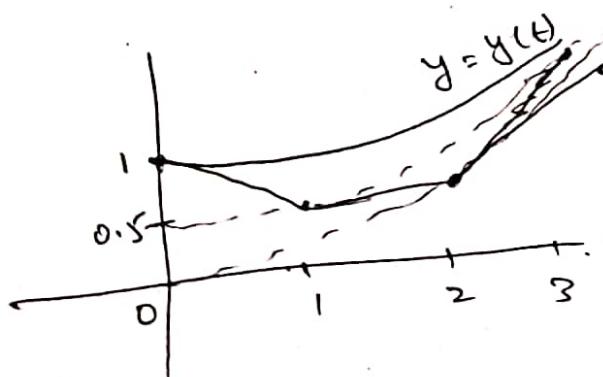
$$y(t_{i+1}) = y(t_i) + hf(x_i, y(t_i)) \quad t_{i+1} = t_i + 1$$

$$t_i = 0, \quad \text{then} \quad f(x, y) = \frac{x-y}{2}$$

$$\begin{aligned} y(1) &= y(0) + h f(0, y(0)) \\ &= 1 + \frac{0-1}{2} = 1 - 0.5 = 0.5 \end{aligned}$$

$$\begin{aligned} y(2) &= y(1) + h f(1, y(1)) \\ &= y(1) + f(1, 0.5) \\ &= 0.5 + \frac{1-0.5}{2} = 0.5 + \frac{0.5}{2} = 0.75 \end{aligned}$$

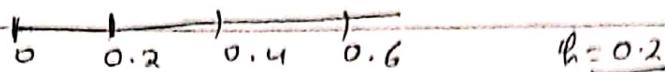
$$\begin{aligned} y(3) &= y(2) + h f(2, y(2)) \\ &= 0.75 + \frac{2-0.75}{2} = 1.375 \end{aligned}$$



$$x' = t(x+t) - 2$$

$$x(0) = 2$$

Use the Euler method with step size $h=0.2$ to compute $x(0.6)$?



$$f(t, x) = t(x+t) - 2$$

$$x(0.2) = x(0) + h \cdot f(0, x(0))$$

$$= 2 + 0.2 \cdot f(0, 2)$$

$$= 2 + 0.2(-2)$$

$$= 2 - 0.4 = 1.6$$

$$x(0.4) = x(0.2) + h f(0.2, x(0.2)) = x(0.2) + 0.2 f(0.2, 1.6)$$

$$= 1.6 + 0.2 \left(0.2(1.6 + 0.2) - 2 \right)$$

$$= 1.6 + 0.2(0.2(1.8) - 2)$$

$$= 1.6 + 0.2(0.36 - 2)$$

$$= 1.6 - 0.2 \times 1.64$$

$$= 1.6 - 0.328$$

$$= 1.272$$

$$x(0.6) = x(0.4) + h f(0.4, x(0.4))$$

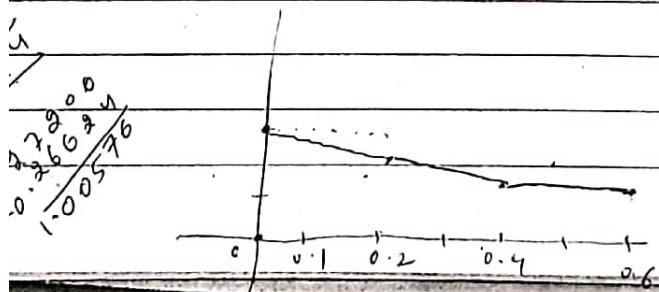
$$= 1.272 + 0.2 f(0.4, 1.272)$$

$$= 1.272 + 0.2 \left(0.4(1.272 + 0.4) - 2 \right)$$

$$= 1.272 + 0.2(0.4 \times 1.672 - 2)$$

$$= 1.272 + 0.2(-1.3312)$$

$$= 1.00576 \text{ Ans.}$$

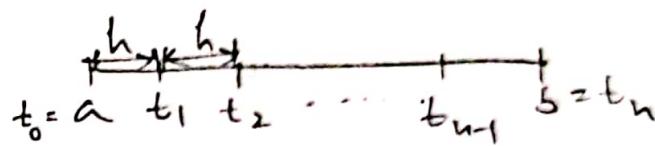


Modified Euler's Method

Ivl $y' = f(t, y), y(a) = \alpha, a \leq t \leq b$

Divide interval $[a, b]$ into n equal sub-intervals
using step length h

$$t_{i+1} = t_i + h$$



Prediction Step

$$y(t_{i+1}) = y(t_i) + h \cdot f(t_i, y(t_i))$$

Correction Step

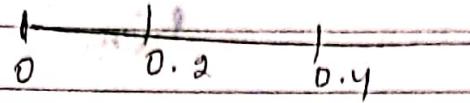
$$y(t_{i+1}) = y(t_i) + \frac{h}{2} \left[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1})) \right]$$

\downarrow
from Prediction Step

Ques Use Modified Euler's method to find $y(0.2)$ and $y(0.4)$ with $h=0.2$ for IVP

$$y' = y + e^y \quad y(0) = 0.$$

Solution



Prediction step

$$\begin{aligned} y(0.2) &= y(0) + h f(0, y(0)) \\ &= 0 + 0.2 f(0, 0) \\ &= 0 + 0.2 (0 + e^0) \\ &= 0.2 \end{aligned}$$

Correction step

$$\begin{aligned} y(0.2) &= y(0) + \frac{h}{2} (f(0, 0) + f(0.2, 0.2)) \\ &= 0 + 0.1 (1 + (0.2 + e^{0.2})) \\ &= 0.1 (1.2 + 1.2214) \end{aligned}$$

Next

$$\begin{aligned} \text{Prediction step} \quad y(0.4) &= y(0.2) + h f(0.2, y(0.2)) \\ &= 0.2421 + 0.2 (0.2421 + e^{0.2}) \\ &= 0.5348 \end{aligned}$$

$$\begin{aligned} \text{Correction step} \quad y(0.4) &= y(0.2) + \frac{h}{2} [f(0.2, y(0.2)) + f(0.4, y(0.4))] \end{aligned}$$

$$= 0.2421 + \frac{0.2}{2} [(0.2421 + e^{0.2}) + (e^{0.4} + 0.5348)]$$

$$= 0.2421 + 0.1 [(1.4635) + 2.0266]$$

$$= 0.5911$$

Runge-Kutta Method of Order 4 (R.K Method of order 4)

I.V.P : $y' = f(t, y)$, $y(a) = \alpha$, $a \leq t \leq b$

$h \rightarrow$ Step length

Divide $\{a, b\}$ into n equal parts with $\boxed{h = \frac{b-a}{n}}$

$$\boxed{t_{i+1} = t_i + h}$$

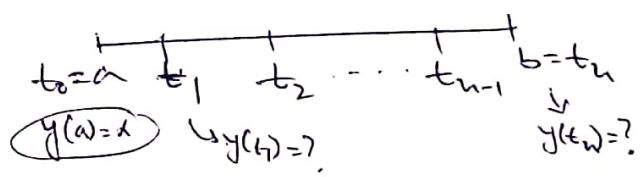
$$k_1 = h f(t_i, y(t_i))$$

$$k_2 = h f\left(t_i + \frac{h}{2}, y(t_i) + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(t_i + \frac{h}{2}, y(t_i) + \frac{k_2}{2}\right)$$

$$k_4 = h f(t_i + h, y(t_i) + k_3)$$

$$y(t_{i+1}) = y(t_i) + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$



Q Using Runge-Kutta Method of order 4 to solve IVP in $[0.4, 0.8]$
 for $y' = \sqrt{x+y}$, $y(0.4) = 0.41$ with step length 0.2

Sol. $f(x, y) = \sqrt{x+y}$, $x_0 = 0.4$, $y(x_0) = 0.41$, $h = 0.2$

To find $y(x_1) = y(0.6)$,

$x_1 = x_0 + h = 0.4 + 0.2 = 0.6$

$K_1 = h f(x_0, y(x_0)) = (0.2) f(0.4, 0.41) = (0.2) \sqrt{0.4 + 0.41} = 0.18$

$K_2 = h f\left(x_0 + \frac{h}{2}, y(x_0) + \frac{K_1}{2}\right) = (0.2) f\left(0.4 + \frac{0.2}{2}, 0.41 + \frac{0.18}{2}\right) = (0.2) f(0.5, 0.5) = 0.2$

$K_3 = h f\left(x_0 + \frac{h}{2}, y(x_0) + \frac{K_2}{2}\right) = (0.2) f\left(0.4 + \frac{0.2}{2}, 0.41 + \frac{0.2}{2}\right) = (0.2) f(0.5, 0.5) = 0.200996$

$K_4 = h f(x_0 + h, y(x_0) + K_3) = (0.2) f(0.4 + 0.2, 0.41 + 0.200996) = (0.2) f(0.6, 0.610996) = 0.22009$

Now $y(x_1) = y(x_0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$$= 0.41 + \frac{1}{6}(0.18 + 2(0.2) + 2(0.200996) + 0.22009) = 0.610347$$

$\Rightarrow \boxed{y(0.6) = 0.610347}$

$$\begin{aligned} \text{To find } y(x_2) &= y(0.8) \\ \text{Given } x_1 &= h f(0.6, y(0.6)) = (0.2) f(0.6, 0.610347) \\ &= 0.220032 \end{aligned}$$

$$\begin{aligned} k_2 &= (0.2) f\left(0.6 + \frac{0.2}{2}, 0.610347 + \frac{0.220032}{2}\right) \\ &= (0.2) f(0.7, 0.720363) = 0.238358 \end{aligned}$$

$$\begin{aligned} k_3 &= (0.2) f\left(0.7, 0.610347 + \frac{0.238358}{2}\right) \\ &= (0.2) f(0.7, 0.729526) = 0.239128 \\ k_4 &= (0.2) f\left(0.6 + 0.2, 0.610347 + 0.239128\right) \\ &= (0.2) f(0.8, 0.849473) = 0.256864 \end{aligned}$$

$$\begin{aligned} y(0.8) &= y(0.6) + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ &= 0.610347 + \frac{1}{6} (0.220032 + 0.476716) \\ &= 0.610347 + \frac{1}{6} (1.431884) \\ &= 0.848991 \end{aligned}$$

Example 4. Using Runge-Kutta fourth-order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y_0 = 1$ at $x = 0.2$ and 0.4 .

Sol.

$$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}, x_0 = 0, y_0 = 1, h = 0.2$$

$$K_1 = hf(x_0, y_0) = 0.2f(0, 1) = 0.200$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2f(0.1, 1.1) = 0.19672$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + 0.19599 = 1.196$$

Therefore $y(0.2) = 1.196$

Now $x_1 = x_0 + h = 0.2$

$$K_1 = hf(x_1, y_1) = 0.1891$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$y_2 = y(0.4) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.196 + 0.1792 = 1.3752.$$

Higher-order

① Use Ral's method of order 4 to find approximate value of $y(1.1)$ and $y'(1.1)$ with step length $h = 0.1$

$$\frac{d^2y}{dx^2} + y \frac{dy}{dx} + y = 2x, \quad y(1) = 1, \quad y'(1) = 1$$

Sol. Let $\frac{dy}{dx} = z = f_1(x, y, z)$ $y(1) = 1$

$$\frac{dz}{dx} + yz + y = 2x \Rightarrow \frac{dz}{dx} = 2x - yz - y = f_2(x, y, z), \quad z(1) = 1 \\ = y'(1)$$

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1$$

$$k_1 = h f_1(x_0, y_0, z_0) \\ = (0.1) f_1(1, 1, 1) = 0.1$$

$$k_2 = h f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) \\ = (0.1) f_1(1.05, 1.05, 1) \\ = (0.1) (1) = 0.1$$

$$k_3 = h f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) \\ = (0.1) f_1(1.05, 1.05, 1) = 0.1$$

$$k_4 = h f_1(x_0 + h, y_0 + k_3, z_0 + l_3) \\ = (0.1) f_1(1.1, 1.1, 1) \\ = 0.1$$

$$l_1 = h f_2(x_0, y_0, z_0) \\ = (0.1) f_2(1, 1, 1) = (0.1)[2 - 1 - 1] = 0$$

$$l_2 = h f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) \\ = (0.1) f_2(1.05, 1.05, 1) \\ = (0.1)[2(1.05) - (1.05 - 1.05)] = 0$$

$$l_3 = h f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) \\ = (0.1) f_2(1.05, 1.05, 1) = 0$$

$$l_4 = h f_2(x_0 + h, y_0 + k_3, z_0 + l_3) \\ = (0.1) f_2(1.1, 1.1, 1) = 0$$

$$y_1 = y(1.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}[0.1 + 0.2 + 0.2 + 0.1] = 1.1$$

$$z_1 = z(1.1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1 + \frac{1}{6}[0 + 0 + 0 + 0] = 1$$

$$\therefore y(1.1) = 1.1$$

$$y'(1.1) = z(1.1) = 1$$

Example 7. Solve by using fourth-order Runge-Kutta method for $x = 0.2$.

$$\frac{d^2y}{dx^2} = x \left(\frac{dy}{dx} \right)^2 - y^2, \quad y(0) = 1, \quad y'(0) = 0.$$

Sol. Let

$$\frac{dy}{dx} = z = f(x, y, z)$$

Therefore

$$\frac{dz}{dx} = xz^2 - y^2 = g(x, y, z)$$

Now $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$

$$K_1 = hf(x_0, y_0, z_0) = 0.0$$

$$L_1 = hg(x_0, y_0, z_0) = -0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right) = -0.02$$

$$L_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right) = -0.1998$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right) = -0.02$$

$$L_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right) = -0.1958$$

$$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.0392$$

$$L_4 = hg(x_0 + h, y_0 + K_3, z_0 + L_3) = -0.1905$$

Hence

$$y_1 = y(0.2) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.9801$$

$$z_1 = y'(0.3) = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) = -0.1970$$