# Lecture 20: Numerical Analysis (UMA011)

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## System of linear equations:

**Example:**

Suppose that

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
4x_1 + 6x_2 + 8x_3 &= 5 \\
6x_1 + \alpha x_2 + 10x_3 &= 5,
\end{aligned}
$$

with $|\alpha| < 10$. For which of the following values of $\alpha$ will there be no row interchange required when solving this system using scaled partial pivoting?

$a.)\ \alpha = 6,$ $b.)\ \alpha = 9,$ $c.)\ \alpha = -3.$

## System of linear equations

**Solution:**

$$[A:b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}\left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 4 & 6 & 8 & 5 \\ 6 & \alpha & 10 & 5 \end{array}\right]$$

To find the scaled factors

$$s_1 = \max\{|2|, |1|, |3|\} = 3$$

$$s_2 = 8$$

$$s_3 = \max\{|6|, |\alpha|, 10\} = 10$$

$$\max\left\{\frac{|a_{11}|}{s_1}, \frac{|a_{21}|}{s_2}, \frac{|a_{31}|}{s_3}\right\} = \max\left\{\frac{2}{3}, \frac{4}{8}, \frac{6}{10}\right\} = \frac{2}{3} = \frac{|a_{11}|}{s_1}$$

$$E_2 \to E_2 - 2E_1, \qquad E_3 \to E_3 - 3E_1$$

$$[A:b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 0 & 4 & 2 & : & 3 \\ 0 & \alpha-3 & 1 & : & 2 \end{bmatrix}$$

$$\max\left\{\frac{|a_{22}|}{s_2}, \frac{|a_{32}|}{s_3}\right\} = \max\left\{\frac{4}{8}, \frac{|\alpha-3|}{10}\right\}$$

$$= \max\left\{\frac{1}{2}, \frac{|\alpha-3|}{10}\right\}$$

If there is no row interchange, then

$$\frac{|\alpha-3|}{10} < \frac{1}{2}$$

$$|\alpha-3| < 5$$

$$-5 < \alpha-3 < 5$$

$$-5 < \alpha - 3 < 5$$

$$-2 < \alpha < 8.$$

a.) when $\alpha = 6$, then no row interchange required.

b) when $\alpha = 9$, then row interchange required.

c) when $\alpha = -3$, then row interchange required

## System of linear equations

### LU Factorization:

Procedure

$Ax = b$

$x = ?$

$$A = \begin{array}{c} E_1 \\ E_2 \\ \\ E_n \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = L * U$$

$L$ ↓ Lower triangular Matrix

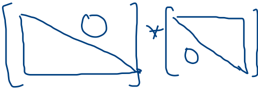
$U$ ↓ upper Triangular Matrix

$1^{st}$ step

Reduce A into U

If $a_{11} \neq 0$

$$\left.\begin{array}{l} E_2 \to E_2 - \dfrac{a_{21}}{a_{11}} E_1 \\ E_3 \to E_3 - \dfrac{a_{31}}{a_{11}} E_1 \\ - - - - - \\ E_n \to E_n - \dfrac{a_{n1}}{a_{11}} E_1 \end{array}\right\} E_j \to E_j - \frac{a_{j1}}{a_{11}} E_1$$



L U

$E_3 \to E_3 - \dfrac{a_{32}}{a_{22}} E_2$

$E_n \to E_n - \dfrac{a_{n2}}{a_{22}} E_2$

$$E_j \rightarrow E_j - \frac{a_{ji}}{a_{ii}} E_i, \quad i = 1, 2 \cdots n-1$$
$$j = i+1, i+2 \cdots n.$$

$$A \sim \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \cdots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = U$$

2nd step. construct L

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ l_{n1} & \cdots & & \cdots & l_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & - & 0 \\ l_{21} & 1 & 0 \; - & 0 \\ l_{31} & l_{32} & 1 & 0 \\ | & | & & \\ l_{n1} & l_{n2} & - & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}} \qquad l_{32} = \frac{a_{32}}{a_{22}}$$

Hly get all the entries of $L$

$$l_{31} = \frac{a_{31}}{a_{11}} \qquad l_{n2} = \frac{a_{n2}}{a_{22}}$$

In general, $l_{ji} = \dfrac{a_{ji}}{a_{ii}}$

$-$ $\quad$ $-$

$$l_{n1} = \frac{a_{n1}}{a_{11}}$$

How can we get the entries of L

$$L * U = \begin{bmatrix} l_{11} & 0 & \text{---} & 0 \\ l_{21} & l_{22} & - & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{n1} & \text{---} & & l_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & - - - & a_{1n} \\ 0 & a_{22} & - & a_{2n} \\ & - & & \text{---} \\ 0 & - & \text{---} & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & - & a_{1n} \\ a_{21} & a_{22} & - & a_{2n} \\ & \text{---} & & \\ & \text{---} & & \\ a_{n1} & \text{---} & & a_{nn} \end{bmatrix} = A$$

$$l_{11} a_{11} = a_{11}$$

$$l_{11} = 1 \qquad\qquad l_{22} = 1$$

$$l_{32} = \frac{a_{32}}{a_{22}}$$

$$l_{21} a_{11} = a_{21}$$

$$l_{21} = \frac{a_{21}}{a_{11}} \qquad\qquad - \; - \; -$$

$$l_{31} = \frac{a_{31}}{a_{11}} \qquad\qquad - \; - \; .$$

3rd step

$A = \check{L}\,\check{U}$

$AX = b$

$\Rightarrow (\check{L}\check{U})X = b$

$(AB)C$

$= A(BC)$

$L(UX) = b$

Put $\underset{n\times n}{U}\ \underset{n\times 1}{X} = \underset{n\times 1}{Y}$ (say)

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$$

$\check{L}Y = \check{b}$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

by using forward sub.

$1 * y_1 = b_1$ ✓

$l_{21} * y_1 + y_2 = b_2$

$y_n = \text{---}$

4th step

$$UX = Y$$

$$\begin{bmatrix} a_{11} & - & - & a_{1n} \\ 0 & a_{22} & - & a_{2n} \\ | & & & \\ 0 & 0 & - & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Use backward sub.

## System of linear equations

**Example:**

Determine the *LU* factorization for matrix $A$ in the linear system $Ax = b$, where $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$. Then use the factorization to solve the system.

## System of linear equations

**Solution:**

$$A = \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$$

$$E_2 \to E_2 - 2E_1, \quad E_3 \to E_3 - 3E_1, \quad E_4 \to E_4 \overset{-1}{+} E_1$$

$$A \sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{bmatrix} \qquad E_3 \to E_3 - 4E_2, \quad E_4 \to E_4 + \overset{3}{3}E_2$$

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}$$

$AX = b$

$(LU)X = b$

$L(UX) = b$

Let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ (say)

$\Rightarrow LY = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

$y_1 = 1$

$2y_1 + y_2 = 1$

$y_2 = -1$

$3y_1 + 4y_2 + y_3 = -3$

$y_3 = -2$

$-y_1 - 3y_2 + y_4 = 4$

$-1 + 3 + y_4 = 4$

$y_4 = 2$

$$\Rightarrow Y = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$$

Take $UX = Y \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$

Use backward sub.

$-13x_4 = 2$

$x_4 = -2/13,$

from $3^{rd}$ eqn.

$$3x_3 + 13x_4 = -2$$

$$3x_3 + \cancel{13} * \left(-\frac{2}{\cancel{13}}\right) = -2$$

$$\Rightarrow \quad x_3 = 0$$

from $2^{nd}$ eqn.

$$-x_2 - x_3 - 5x_4 = -1$$

$$-x_2 - 0 - 5 * \frac{-2}{13} = -1$$

$$-x_2 = -1 + \frac{10}{13} = \frac{-3}{13}$$

$$\Rightarrow \quad x_2 = \frac{3}{13},$$

from $1^{st}$ eqn

$$x_1 + x_2 + 3x_4 = 1$$

$$x_1 + \frac{3}{13} + 3 * -\frac{2}{13} = 1$$

$$x_1 = 1 + \frac{3}{13} = \frac{16}{13}$$

$$X = \begin{bmatrix} 16/13 \\ 3/13 \\ 0 \\ -2/13 \end{bmatrix} \sim \begin{bmatrix} 1.23308 \\ 0.23308 \\ 0 \\ -0.1538 \end{bmatrix}$$

## System of linear equations:

**Exercise:**

1. Modify the *LU* Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear system:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

2. Given the linear system

$$\begin{aligned} x_1 - x_2 + \alpha x_3 &= -2 \\ -x_1 + 2x_2 - \alpha x_3 &= 3 \\ \alpha x_1 + x_2 + x_3 &= 2. \end{aligned}$$

## System of linear equations:

**Exercise (continued):**

a. Find value(s) of $\alpha$ for which the system has no solution.
b. Find value(s) of $\alpha$ for which the system has infinite number of solutions.
c. Assuming a unique solution exists for a given $\alpha$, find the solution.