School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

UMA007 : Numerical Analysis
Assignment 9
Initial-Value Problems for Ordinary Differential Equations

1. Show that each of the following initial-value problems (IVP) has a unique solution, and find the solution.

(a) $y' = y\cos t,\ 0 \le t \le 1,\ y(0) = 1.$

(b) $y' = \dfrac{2}{t}y + t^2e^t,\ 1 \le t \le 2,\ y(1) = 0.$

2. Apply Picard's method for solving the initial-value problem generate $y_0(t),\ y_1(t),\ y_2(t)$, and $y_3(t)$ for the initial-value problem

$$y' = -y + t + 1,\ 0 \le t \le 1,\ y(0) = 1.$$

3. Consider the following initial-value problem

$$x' = t(x + t) - 2,\ x(0) = 2.$$

Use the Euler method with stepsize $h = 0.2$ to compute $x(0.6)$.

4. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2,\ y(1) = -1,$$

with exact solution $y(t) = -\dfrac{1}{t}$:

(a) Use Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

(b) Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.
i. y(1.052) ii. y(1.555) iii. y(1.978).

5. Solve the following IVP by second-order Runge-Kutta method

$$y' = -y + 2\cos t,\ y(0) = 1.$$

Compute $y(0.2), y(0.4)$, and $y(0.6)$ with mesh length 0.2.

6. Compute solutions to the following problems with a second-order Taylor method. Use step size $h = 0.2$.

(a) $y' = (\cos y)^2,\ 0 \le x \le 1,\ y(0) = 0.$

(b) $y' = \dfrac{20}{1 + 19e^{-x/4}},\ 0 \le x \le 1,\ y(0) = 1.$

7. A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8\ m/s$ is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8\ m/s^2$ and $k = 0.002\ kg/m$. The differential equation for the velocity v is given by

$$mv' = -mg - kv|v|.$$

(a) Find the velocity after $0.1, 0.2, \cdots, 1.0$ s.

(b) To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.

8. Using Runge-Kutta fourth-order method to solve the IVP at $x = 0.8$ for

$$\frac{dy}{dx} = \sqrt{x + y},\ y(0.4) = 0.41$$

with step length $h = 0.2$.

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9. Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2 \sqrt{2g}\frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft/s^2, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)$ ft^3 . Use the Runge-Kutta method of order four to find the following.

(a) The water level after 10 min with $h = 20$ s.

(b) When the tank will be empty, to within 1 min.

10. The following system represent a much simplified model of nerve cells

$$\begin{aligned} \frac{dx}{dt} &= x + y - x^3, \ x(0) = 0.5 \\ \frac{dy}{dt} &= -\frac{x}{2}, \ y(0) = 0.1 \end{aligned}$$

where $x(t)$ represents voltage across the boundary of nerve cell and $y(t)$ is the permeability of the cell wall at time t. Solve this system using Runge-Kutta fourth-order method to generate the profile up to $t = 0.2$ with step size 0.1.

11. Use Runge-Kutta method of order four to solve

$$y'' - 3y' + 2y = 6e^{-t}, \ 0 \leq t \leq 1, \ y(0) = y'(0) = 2$$

for $t = 0.2$ with stepsize 0.2.