Lecture 14: Numerical Analysis (UMA011)

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Order of convergence:

Definition:

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda, \checkmark$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

Remark (i) If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.
(ii) If $\alpha = 2$, the sequence is quadratically convergent.
In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.

$\alpha = 3$ cubic

$\alpha = 4$ 4th order

$p_n \to p$

$\frac{|p_2 - p|}{|p_1 - p|^2} = \lambda_1$

$\frac{|p_3 - p|}{|p_2 - p|^2} = \lambda_2$

λ_n

decreasing sequence

$\frac{1}{1000}$

$a_n = \frac{1}{n} \to 0$ when $n \to \infty$ $\qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$

$\frac{1}{(1000)^2}$

$b_n = \frac{1}{n^2} \to 0$ when $n \to \infty$ $\qquad 1, \frac{1}{4}, \frac{1}{9} \cdots \to 0$

$c_n = \frac{1}{n^3} \to 0$ when $n \to \infty$

Order of convergence:

Order of convergence of bisection method:

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence generated by Bisection method

The error bound by sequence $\{p_n\}$ converges to p in

$[a,b]$ is $|p_n - p| \leq \frac{b-a}{2^n}$, $n \geq 1$

Also, $|p_{n+1} - p| \leq \frac{b-a}{2^{n+1}}$

Take $\frac{|p_{n+1} - p|}{|p_n - p|^1} \leq \left(\frac{b-a}{2^{n+1}}\right) / \left((b-a)/2^n\right) = \frac{1}{2} < 1$

$\{p_n\}$ converges to p <u>linearly</u>.

Now,
$$\frac{|p_{n+1}-p|}{|p_n-p|^2} \le \frac{\frac{b-a}{2^{n+1}}}{\left(\frac{b-a}{2^n}\right)^2} = \frac{2^{2n}}{2^{n+1}(b-a)}$$

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} \le \lim_{n\to\infty} \frac{2^{n-1}}{b-a} \to \infty$$

$\{p_n\}$ is <u>not</u> converging to p <u>quadratically</u>.

Order of convergence:

Order of convergence of fixed point iteration method:

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, $\forall x \in [a,b]$. Suppose, in addition, that g' is continuous on $[a,b]$ and a positive constant $k < 1$ exists with $|g'(x)| \leq k < 1$, for all $x \in (a,b)$

(i) If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, then the sequence $p_n = g(p_{n-1})$, $n \geq 1$ converges only linearly to the unique fixed point p in $[a,b]$.

$g(x)$

$p_n \to p$

$p_n \to p$

$g(p) = p$

Order of convergence:

Order of convergence of fixed point iteration method:

Proof of (i)

$$f(x) = f(h) + (x-h)f'(h) + (x-h)^2 f''(h) + \cdots$$

Expand $g(p_n)$ in Taylor poly. about p

$$g(p_n) \approx g(p) + (p_n - p)\, g'(c_n), \quad p_n < c_n < p$$

$$p_{n+1} \approx p + (p_n - p)\, g'(c_n),$$

$$p_{n+1} - p = (p_n - p)\, g'(c_n)$$

$$\frac{|p_{n+1} - p|}{|p_n - p|} = g'(c_n)$$

Results

Sandwich thm

$$a_n < b_n < c_n$$

$$\begin{array}{ccc} a_n & b_n & c_n \\ \downarrow & \Downarrow & \downarrow \\ a & a & a \end{array}$$

If $a_n \to a$

f is Cont.

then $f(a_n) \to f(a)$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} |g'(c_n)| \quad , \quad p_n < c_n < p$$

$$\begin{array}{ccc} p_n & c_n & p \\ \downarrow & \Downarrow & \downarrow \\ p & p & p \end{array}$$ by sandwich thm.

$$= g'(p) < 1$$

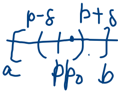
$\Rightarrow$ $\{p_n\}$ converges to p linearly.

Order of convergence:

Order of convergence of fixed point iteration method:

(ii) If $g'(p) = 0$ and $g''(x)$ is continuous function with $|g''(x)| < M$ on an open neighbourhood of p, the there exists a $\delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$ the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$



Order of convergence:

Order of convergence of fixed point iteration method:

Proof of (ii)

Expand $g(p_n)$ in Taylor poly. about p

$$g(p_n) \approx g(p) + (p_n - p)\, g'(p) + \frac{(p_n-p)^2}{2!} g''(c_n), \quad p_n < c_n < p$$

$$p_{n+1} = p + 0 + \frac{(p_n-p)^2}{2!} g''(c_n)$$

$$p_{n+1} - p = \frac{(p_n-p)^2}{2!} g''(c_n) \quad \text{— } (1)$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{|g''(c_n)|}{2}$$

$$p_n < c_n < p$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$p \qquad p \qquad p$$

$$= \frac{g''(p)}{2}$$

$\Rightarrow \{p_n\}$ converges to p atleast quadratically.

Also, by using $|g''(x)| < M$ on $x \in [p-\delta, p+\delta]$ in 1, we have

$$|p_{n+1} - p| < \frac{|p_n - p|^2}{2} M$$

Order of convergence:

Order of convergence of fixed point iteration method:

In general, if $g'(p) = 0$, $g''(p) = 0, \cdots, g^{m-1}(p) = 0$, then the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least of order m to p.

$g'(p) \neq 0$

$g'(p) = 0$

$g''(p) = 0$

for exact order

$g'(p) = 0$, $g''(p) = 0$ - - - - $g^{m-1}(p) = 0$

but $g^{m}(p) \neq 0$

then $\{p_n\}$ converges to p with order m.

Order of convergence:

Order of convergence of Newton's method:

Newton's Method is given by

Let p be the root of eq^n $f(x)=0$

i.e. $f(p)=0$

s.t. $f'(p)\neq 0$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = g(p_n)$$

$$p_{n+1} = g(p_n)$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$ then $g(p)=p$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}$$

$$= 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

$$= \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(p) = \frac{f(p)\,f''(p)}{(f'(p))^2} = \frac{0}{\text{non-zero finite value}} = 0$$

$\Rightarrow$ The sequence generated by N.M gives atleast quadratically convergent.

Order of convergence:

Example:

Given that the iterates $x_{n+1} = \frac{2}{3}x_n + \frac{a}{3x_n^2}$, $a \in \mathbb{R}$ converges to $p = a^{1/3}$. Find the order of convergence of the iteration scheme.

$a \neq 0$

Solution :-

$$x_{n+1} = \frac{2}{3}x_n + \frac{a}{3x_n^2} = g(x_n)$$

$$\Rightarrow g(x) = \frac{2}{3}x + \frac{a}{3x^2}$$

Here $g(p) = g(a^{1/3}) = \frac{2}{3}a^{1/3} + \frac{a}{3a^{2/3}} = a^{1/3} = p$.

$\Rightarrow g(p) = p$ ie p is a fixed pt. for g

Now, $g'(x) = \frac{2}{3} - \frac{2a}{3x^3}$

$$g'(a^{1/3}) = \frac{2}{3} - \frac{2a}{3a^{3/3}} = \frac{2}{3} - \frac{2}{3} = 0$$

$$\Rightarrow \quad g'(p) = 0$$

And $\quad g''(x) = \frac{6a}{3x^4} = \frac{2a}{x^4}$

$$g''(a^{1/3}) = \frac{2a}{a^{4/3}} \neq 0 \qquad \Rightarrow g''(p) \neq 0$$

$\Rightarrow$ The order of convergence of x_{n+1} is 2 ie quadratic cgt.

Order of convergence:

Exercise:

1. What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a}, \; a \in \mathbb{R}$$

as it converges to the fixed point $p = \sqrt{a}$?

2. The iterates $x_{n+1} = 2 - (1 + c)x_n + cx_n^3$ converges to $p = 1$ for some values of constant c (provided that x_0 is sufficiently close to p). For what values of c, if any, convergence is quadratic.