Lecture 15: Numerical Analysis (UMA011)

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Multiple roots

for e.g.

$f(x) = (x^2-2x+1)(x-3)(x-4)$

Definition:

An equation $f(x) = 0$ has a root p with multiplicity m if for $x \neq p$, we can write $f(x) = (x-p)^m q(x)$, $q(p) \neq 0$.

$(x-1)^2$

$(x-1)(x-1)$

If $m = 1$, then equation $f(x) = 0$ has a simple root at p.

$f(x) = (x-1)^2(x-3)(x-4) = 0$

$x = 1, 1, 3, 4$

ie $f(x) = (x-p)^1 \, q(x)$, $q(p) \neq 0$

Root → equation

Zero → function.

Multiple roots

$f \in C^1[a,b] \Rightarrow f$ and f' are cont. on $[a,b]$.

Result:

The function $f \in C^1[a,b]$ has a simple zero at p in $[a,b]$ iff $f(p) = 0$ but $f'(p) \neq 0$.

$f(x) = (x-p)q(x)$, $q(p) \neq 0$

$\Rightarrow f(p) = 0$

& $f'(x) = (x-p)q'(x) + 1 \cdot q(x)$

$\Rightarrow f'(p) = 0 + q(p) \neq 0$

If we take p is root of $f(x)=0$ with multiplicity 2

then $f(x) = (x-p)^2 q(x)$, $q(p) \neq 0$

$\Rightarrow f(p) = 0$

$f'(p) = 0$

but $f''(p) \neq 0$

Generalized result:

The function $f \in C^1[a,b]$ has a zero of multiplicity m at p in $[a,b]$ iff $f(p) = 0$, $f'(p) = 0, \cdots, f^{m-1}(p) = 0$, but $f^m(p) \neq 0$.

Multiple roots

$f(x)=0$

p

Remarks:

(i) The Newton's method (in which $f'(p) \neq 0$ is required) works for those functions which has a simple zero not for multiplicity.

(ii) Newton's method gives quadratically convergent sequence but quadratic convergence might not occur if the zero is not simple.

N.M.

$$x_{n+1} = x_n - \frac{f(x_n)}{\boxed{f'(x_n)}}$$

$\neq 0$

$f'(p) \neq 0$

Multiple roots

Example:

Let $f(x) = e^x - x - 1$ $m=2$

a) Show that f has a zero of multiplicity 2 at $x = 0$. $p=0$

b) Show that Newton's method with $p_0 = 1$ converges to $x = 0$ but not quadratically.

Solution :- a)

$$f(x) = e^x - x - 1$$

$$f(0) = e^0 - 0 - 1 = 0$$

$$f'(x) = e^x - 1$$

$$f'(0) = e^0 - 1 = 0$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0 \neq 0 \Rightarrow f \text{ has a zero at } 0 \text{ with } m = 2$$

b)

$$f(x) = e^x - x - 1$$

Apply N.M. with $p_0 = 1$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$p_{n+1} = p_n - \frac{e^{p_n} - p_n - 1}{e^{p_n} - 1}$$

$$p_1 = p_0 - \frac{e^{p_0} - p_0 - 1}{e^{p_0} - 1} = 1 - \frac{e-2}{e-1} = \frac{\cancel{e} - 1 - \cancel{e} + 2}{e-1}$$

$$= \frac{1}{e-1} = 0.58198$$

$$p_2 = p_1 - \frac{e^{p_1} - p_1 - 1}{e^{p_1} - 1}$$

$$= 0.31906$$

$p_3 = 0.16800$, $p_4 = 0.08635$, $p_5 = 0.04380$

$p_6 = 0.02206$

check the order of convergence of $\langle p_n \rangle$

by $\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$

To check the linear order

$$\frac{|p_2 - p|}{|p_1 - p|} = \frac{|0.31906 - 0|}{|0.58198 - 0|} < 1$$

$$\frac{|p_3 - p|}{|p_2 - p|} = \frac{|0.16800 - 0|}{|0.31906 - 0|} < 1$$

$$\frac{|p_4 - p|}{|p_3 - p|} < 1 \qquad = - - \qquad \forall\, n$$

$\langle p_n \rangle$ is linear convergent sequence.

To check the 2nd order

$$\frac{|p_2 - p|}{|p_1 - p|^2} = \frac{|0.31906 - 0|}{|0.58198 - 0|^2} = 0.94201 < 1$$

$$\frac{|p_3 - p|}{|p_2 - p|^2} = \frac{|0.16800 - 0|}{|0.31906 - 0|^2} = 1.6503 > 1$$

$$\frac{|p_4 - p|}{|p_3 - p|^2} = \frac{|0.08635 - 0|}{|0.16800 - 0|^2} = 3.05945 > 1$$

$\langle \lambda_n \rangle$ is increasing sequence

$\Rightarrow \langle p_n \rangle$ is not quadratic convergence.

Multiple roots

Modified Newton's method (if multiplicity is not given)

Define a function $\mu(x) = \dfrac{f(x)}{f'(x)}$, $f(x)$ has a zero at p with multiplicity m.

$$\Rightarrow f(x) = (x-p)^m q(x), \qquad q(p) \neq 0$$

$$\Rightarrow \mu(x) = \frac{(x-p)^m q(x)}{(x-p)^m q'(x) + q(x) * m(x-p)^{m-1}}$$

$$= \frac{(x-p)\, q(x)}{(x-p)\, q'(x) + m\, q(x)}$$

$$\mu(x) = (x-p)^1 \, Q(x)$$

where $Q(x) = \dfrac{q(x)}{(x-p)q'(x) + mq(x)}$

Here $Q(p) = \dfrac{q(p)}{0 + mq(p)} = \dfrac{1}{m} \neq 0$

$\Rightarrow$ $\mu(x)$ has a simple zero at p

Apply N. M. on $\mu(x)$ to get the quadratic eqt

then $p_{n+1} = p_n - \dfrac{\mu(p_n)}{\mu'(p_n)}$, where $\mu(x) = \dfrac{f(x)}{f'(x)}$

$$\Rightarrow \quad p_{n+1} = p_n - \frac{\dfrac{f(p_n)}{f'(p_n)}}{\dfrac{f'(p_n)\, f'(p_n) - f(p_n)\, f''(p_n)}{\left(f'(p_n)\right)^2}}$$

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)\, f'(p_n)}{\left(f'(p_n)\right)^2 - f(p_n)\, f''(p_n)}} \rightarrow \text{Modified Newton's Method.}$$

Order of convergence:

Exercise:

1. Apply the Newton's method with $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$, and verify that the convergence is only of first-order. Further show that root $\alpha = 1$ has multiplicity 2.