Lecture 37: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Numerical Quadrature:

Simpson's Rule:

To derive Simpson's rule for approximating $\int_a^b f(x)dx$, we use second degree Lagrange's interpolating polynomials $P_2(x)$ with equally spaced nodes $x_0 = a, x_2 = b$ and $x_1 = a + h = \frac{a+b}{2}$, where $h = \frac{(b-a)}{2}$,

$$
\begin{aligned}
P_2(x) &= \overset{l_0(x)}{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}} f(x_0) + \overset{l_1(x)}{\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}} f(x_1) \\
&\quad + \overset{l_2(x)}{\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}} f(x_2) = \sum_{i=0}^{2} l_i(x) f(x_i) \\
\Rightarrow f(x) &= P_2(x) + e_2(x) \\
\Rightarrow f(x) &= \sum_{i=0}^{2} l_i(x) f(x_i) + \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2), \;\; \xi \in (a,b)
\end{aligned}
$$

Numerical Quadrature:

Simpson's Rule:

$$\Rightarrow \int_a^b f(x)dx$$

$$\begin{aligned} = \quad & \int_{a=x_0}^{b=x_2} \left(\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) \right. \\ & \left. + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right) dx \\ & + \int_{a=x_0}^{b=x_2} \frac{f'''(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2)dx \\ = \quad & \sum_{i=0}^{2} a_i f(x_i) + E(f), \ \text{(say), where } a_i = \int_{a=x_0}^{b=x_2} l_i(x)dx. \end{aligned}$$

Numerical Quadrature:

Simpson's Rule:

Now, $a_0 = \int_{a=x_0}^{b=x_2} l_0(x)dx = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx$

let $x = x_0 + ih$

$dx = h\,di$

Int. limits

lower limit if $x = x_0$

$x_0 = x_0 + ih$

$x_0 - x_0 = ih$

$i = 0$

upper limit

If $x = x_2$

$x_2 = x_0 + ih \Rightarrow i = 2$

$\Rightarrow a_0 = \int_0^2 \frac{((x_0+ih)-(x_0+h))\,(x_0+ih-(x_0+2h))}{(-h)\,(-2h)} h\,di$

$= \frac{1}{2h}\int_0^2 ((i-1)h \quad (i-2)h)\,di$

$= \frac{h}{2}\int_0^2 (i^2 - 3i + 2)\,di = \frac{h}{2}\left[\frac{i^3}{3} - \frac{3i^2}{2} + 2i\right]_0^2$

$= \frac{h}{2}\left[\frac{8}{3} - 3 * \frac{\cancel{4}^{2}}{\cancel{2}} + 4\right] = \frac{h}{2}\left[\frac{8}{3} - 2\right] = h/3$

||ly

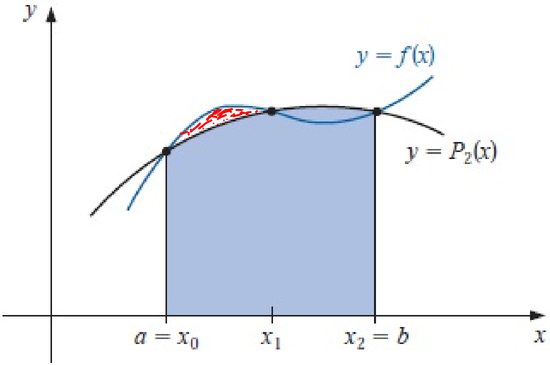
$$a_1 = \int_{x_0}^{x_2} l_1(x)\,dx, \qquad a_2 = \int_{x_0}^{x_2} l_2(x)\,dx$$

$$a_1 = \frac{4h}{3}, \qquad a_2 = \frac{h}{3}$$

$$\Rightarrow \int_{a=x_0}^{b=x_2} f(x)\,dx \simeq a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2)$$

$$\simeq \boxed{\frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]} \rightarrow \text{Simpson's-}\tfrac{1}{3}\text{rd rule.}$$

Simpson's Rule:



Numerical Quadrature:

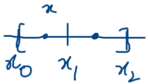
Simpson's Rule:

The error term is given by

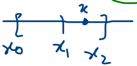
$$E(f) = \int_{a=x_0}^{b=x_2} \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)\,dx.$$ Since,

$(x-x_0)(x-x_1)(x-x_2)$ changes its sign in $[x_0, x_2]$ therefore we can not apply Weighted Mean Value Theorem.

Moreover, $$\int_{x_0}^{x_2} \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)\,dx = 0.$$



$(x-x_0)(x-x_1)(x-x_2)$
+ve -ve -ve
= +ve



$(x-x_0)(x-x_1)(x-x_2)$
+ve +ve -ve
= -ve

W.M.V.T

If f is cont. on $[a,b]$, g is integrable and does not change its sign on $[a,b]$ then $\exists\ c \in (a,b)$ s.t.

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

Simpson's Rule:

We repeat any one of the nodes $a = x_0, x_1, x_1, x_2$ in the interval. Suppose we repeat x_1 point then the interpolating points will be $a = x_0, x_1 = \frac{a+b}{2}, x_1 = \frac{a+b}{2}, x_2 = b$.

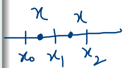
Thus error term becomes

$$E(f) = \int_{x_0}^{x_2} \underbrace{\frac{f^{iv}(\xi)}{4!}}_{= f(x)} \underbrace{(x - x_0)(x - x_1)^2(x - x_2)}_{= g(x)}\, dx$$

x_0 x_0

x_1

x_2



$(x - x_0) = +ve$

$(x - x_1)^2 = (-ve)^2$

$(x - x_2) = +ve$

$= +ve$

$+ve$

Here, $(x - x_0)(x - x_1)^2(x - x_2)$ does not change its sign over $[x_0, x_2]$, so by W.M.V.T, then $\exists$ $c \in (a,b)$ s.t.

$$\frac{f^{iv}(c)}{4!} \int_{a=x_0}^{b=x_2} (x-x_0)(x-x_1)^2(x-x_2)\,dx$$

Put $x = x_0 + ih$

$dx = h\,di$

If $x = x_0$ then $i = 0$

If $x = x_2$ then $i = 2$

$$\Rightarrow \frac{f^{iv}(c)}{4!} \int_0^2 (ih)(i-1)^2 h^2 (i-2)h \; h\,di = \boxed{\frac{-h^5}{90} f^{iv}(c)}$$

error in Simpson's rule

Numerical Quadrature:

Example:

Compare the Trapezoidal rule and Simpson's rule approximations to $\int_0^1 x\, e^x\, dx$. Find the absolute error and maximum bound for the errors.

Solution:

Exact value

$$\int_0^1 x e^x dx = \left(x e^x\right)_0^1 - \int_0^1 1\, e^x dx$$

$$= \left[x e^x - e^x\right]_0^1 = (e - e) - (-e^0) = \underline{1} \rightarrow \text{exact value}$$

By Trap.

$$\int_0^1 x e^x dx = \frac{h}{2}\left[f(0) + f(1)\right], \text{ where } h = 1 - 0 = 1$$

$f(x) = x e^x$

$$= \frac{1}{2}\left[0 + e^1\right] = \frac{e}{2} = \underline{1.3591}$$

Here

By simpson

$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$

$h = \frac{b-a}{2} = \frac{1-0}{2} = \frac{1}{2}$

$$\int_0^1 x e^x dx = \frac{h}{3}\left[f(0) + 4f(0.5) + f(1)\right]$$

$$= \frac{1/2}{3}\left[0 + \overset{2}{\cancel{4}} * \frac{1}{\cancel{2}} * e^{1/2} + e^1\right]$$

$$= \frac{1}{6}\left[2e^{1/2} + e\right] = 1.0026$$

Absolute errors

A.E. in Trap. $= |1.3591 - 1| = 0.3591$

A.E in Simpson's $= |1.0026 - 1| = 0.0026$

Max. error in Trap. $= \max\limits_{0<c<1} \left| -\frac{h^3}{12} f''(c) \right|$

Now,

$$f(x) = xe^x$$

$$f'(x) = xe^x + e^x = (x+1)e^x$$

$$f''(x) = (x+1)e^x + e^x = (x+2)e^x$$

$$f'''(x) = (x+2)e^x + e^x = (x+3)e^x$$

$$f^{iv}(x) = (x+4)e^x.$$

Max. error in Trap. $\max_{0<c<1} \left| -\frac{1^3}{12}(c+2)e^c \right| = \frac{1}{12} \max_{0<c<1} \left|(c+2)e^c\right|$

$h = b-a = 1-0 = 1$

$$= \frac{1}{12}(1+2)e^1 = \frac{3e}{12}$$

$$= 0.6796$$

Max error bound in Simpson's rule

where, $h = \frac{b-a}{2} = \frac{1}{2}$

$$\max_{0<c<1} \left| \frac{-h^5}{90} f^{iv}(c) \right| = \frac{\left(\frac{1}{2}\right)^5}{90} \max_{0<c<1} \left| (c+4) e^c \right|$$

$$= \frac{1}{32 \times \cancel{90}_{18}} \cancel{5} \times e^1 = \frac{e}{32 \times 18}$$

$$= 0.0047$$

Numerical Quadrature:

Exercise:

1. Approximate the integral $I = \int_0^2 \frac{1}{x+1}\, dx$ using the trapezoidal and Simpson's formulas and compare with exact values. Also, find the maximum bound for the errors.
2. The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?

Hint for Que. 2

by Trap. $\frac{h}{2}[f(0) + f(2)] = 4$, $h=2$ $\Rightarrow [f(0) + f(2)] = 4$

by Simpson's $\frac{h}{3}[f(0)+4f(1)+f(2)] = 2$, $h=1$ $\frac{1}{3}[f(0)+4f(1)+f(2)] = 2$