Lecture 19: Numerical Analysis (UMA011)

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System of linear equations

Example:

Using four-digit arithmetic operations, solve the following system of equations by Gaussian elimination with partial pivoting

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000. \end{aligned}$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!

System of linear equations

Solution:

$$[A:b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 0.729 & 0.81 & 0.9 & 0.6867 \\ 1 & 1 & 1 & 0.8338 \\ 1.331 & 1.21 & 1.1 & 1 \end{array}\right]$$

$$\max\{|a_{11}|, |a_{21}|, |a_{31}|\} = 1.331 = a_{31}$$

$$E_1 \leftrightarrow E_3$$

$$[A:b] \sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 1.331 & 1.21 & 1.1 & 1 \\ 1 & 1 & 1 & 0.8338 \\ 0.729 & 0.81 & 0.9 & 0.6867 \end{array}\right]$$

$$E_2 \to E_2 - \frac{1}{1.331} E_1 , \quad E_3 \to E_3 - \frac{0.729}{1.331} E_1$$

$$E_2 - 0.7513\, E_1 \qquad E_3 \to E_3 - 0.5477\, E_1$$

$1 - 0.7513 * 1.21$
$= 1 - 0.9091 = 0.0909$
$1 - 0.7513 * 1.1$
$= 1 - 0.8264 = 0.1736$
$0.8338 - 0.7513 * 1$
$= 0.0825$

$$[A \vdots b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 1.331 & 1.21 & 1.1 & 1 \\ 0 & 0.0909 & 0.1736 & 0.0825 \\ 0 & 0.1473 & 0.2975 & 0.1390 \end{array}\right]$$

$0.81 - 0.5477 * 1.21$
$= 0.1473$
$0.9 - 0.5477 * 1.1$
$= 0.2975$
$0.6867 - 0.5477$
$= 0.1390$

Again using partial pivoting

$$\max \{ \underbrace{|0.0909|}_{=|a_{22}|}, \underbrace{|0.1473|}_{=|a_{32}|} \} = a_{32}$$

$$E_2 \leftrightarrow E_3$$

$$[A \vdots b] \sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 1.331 & 1.21 & 1.1 & 1 \\ 0 & 0.1473 & 0.2975 & 0.1390 \\ 0 & 0.0909 & 0.1736 & 0.0825 \end{array}\right] \qquad E_3 \to E_3 - \frac{0.0909}{0.1473} E_2$$

$$E_3 \rightarrow E_3 - 0.6171\,E_2$$

$$0.1736 - 0.6171 * 0.2975$$
$$= 0.1736 - 0.1836 = -0.01$$
$$0.0825 - 0.6171 * 0.1390$$
$$= 0.0825 - 0.08578$$
$$= -0.00328$$

$$[A:b] \sim \left[\begin{array}{ccc:c} 1.331 & 1.21 & 1.1 & 1 \\ 0 & 0.1473 & 0.2975 & 0.1390 \\ 0 & 0 & -0.01 & -0.00328 \end{array}\right]$$

Use backward sub.

$$-0.01\,x_3 = -0.00328$$

$$x_3 = 0.328$$

$$0.1473 x_2 + 0.2975 * 0.328 = 0.1390$$

$$0.1473 x_2 + 0.09758 = 0.1390$$

$$x_2 = 0.2812$$

$$1.331x_1 + 1.21(0.2812) + 1.1(0.328) = 1$$

$$x_1 = 0.2246$$

App. value $= x^* = \begin{bmatrix} 0.2246 \\ 0.2812 \\ 0.328 \end{bmatrix}$ Ans

Exact value $= X = \begin{bmatrix} 0.2245 \\ 0.2814 \\ 0.3279 \end{bmatrix}$ $\max\{|x_1 - x_1^*|, |x_2 - x_2^*|, |x_3 - x_3^*|\}$

= Absolute error.

System of linear equations

Scaled Partial Pivoting

It places the element in the pivot position that is largest relative to the entries in its row.

$$A = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Find scaled factors from the rows.

$$s_1 = \max\{|a_{11}|, |a_{12}| \cdots |a_{1n}|\}$$

$$s_2 = \max\{|a_{21}|, |a_{22}| \cdots |a_{2n}|\}$$

$$\cdots$$

$$s_n = \max\{|a_{n1}|, |a_{n2}| \cdots |a_{nn}|\}$$

$$s_i = \max\{|a_{i1}|, |a_{i2}|, \cdots |a_{in}|\}$$

$$i = 1, 2 \cdots n.$$

Take ratio of 1^{st} column entries and corresponding s_i

$$\text{Max.}\left\{\frac{|a_{11}|}{s_1}, \frac{|a_{21}|}{s_2}, \frac{|a_{31}|}{s_3} \cdots \frac{|a_{n1}|}{s_n}\right\} = \frac{|a_{p1}|}{s_p}$$

$$E_1 \leftrightarrow E_p \qquad s_1 \leftrightarrow s_p$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

$$\max\left\{\frac{|a_{22}|}{s_2}, \frac{|a_{32}|}{s_3}, \cdots \frac{|a_{p2}|}{s_1}, \cdots \frac{|a_{n2}|}{s_n}\right\}$$

Continue this procedure.

$$= \frac{|a_{q2}|}{s_q} \qquad E_2 \leftrightarrow E_q$$

Remarks :- 1) The scaled factors $s_1, s_2, \dots s_n$ are computed only once.

2) The scaled factors are row dependent, so they must also be interchanged when interchanges are performed.

System of linear equations

Example:

Using 3−digit arithmetic operations, solve the following system of equations by Gaussian elimination with scaled partial pivoting

$$
\begin{aligned}
2.11x_1 - 4.21x_2 + 0.921x_3 &= 2.01 \\
4.01x_1 + 10.2x_2 - 1.12x_3 &= -3.09 \\
1.09x_1 + 0.987x_2 + 0.832x_3 &= 4.21
\end{aligned}
$$

System of linear equations

Solution:

$$[A:b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 2.11 & -4.21 & 0.921 & 2.01 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 1.09 & 0.987 & 0.832 & 4.21 \end{array}\right]$$

Scaled factors.

$$s_1 = \max\{|2.11|, |-4.21|, |0.921|\} = 4.21.$$

$$s_2 = \max\{|4.01|, |10.2|, |-1.12|\} = 10.2$$

$$s_3 = \max\{|1.09|, |0.987|, |0.832|\} = 1.09$$

$$\max\left\{\frac{|a_{11}|}{s_1}, \frac{|a_{21}|}{s_2}, \frac{|a_{31}|}{s_3}\right\} = \max\left\{\frac{2.11}{4.21}, \frac{|4.01|}{10.2}, \frac{|1.09|}{1.09}\right\} = \frac{|a_{31}|}{s_3}$$

$$E_1 \leftrightarrow E_3 \qquad s_1 \leftrightarrow s_3$$

$$[A:b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & -0.921 & 2.01 \end{array}\right]$$

$$E_2 \to E_2 - \frac{4.01}{1.09} E_1, \quad E_3 \to E_3 - \frac{2.11}{1.09} E_1$$

$10.2 - 3.68 * 0.987$

$$E_2 \to E_2 - 3.68 E_1 \qquad E_3 \to E_3 - 1.94 E_1$$

$$\sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{ccc:c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & 6.57 & -4.18 & -18.6 \\ 0 & -6.12 & -0.689 & -6.16 \end{array}\right]$$

$-4.21 - 1.94 * 0.987$

$= -4.21 - 1.91$

$= -6.12$

$$\max\left\{\frac{|a_{22}|}{s_2}, \frac{|a_{32}|}{s_3}\right\} = \max\left\{\frac{6.57}{10.2}, \frac{|-6.12|}{4.21}\right\} = \frac{|a_{32}|}{s_3}$$

$$E_2 \leftrightarrow E_3$$

$$[A:b] \sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{cccc} 1.09 & 0.987 & 0.832 & : 4.21 \\ 0 & -6.12 & -0.689 & : -6.16 \\ 0 & 6.57 & -4.18 & : -18.6 \end{array}\right]$$

$$E_3 \to E_3 - \frac{6.57}{6.12} E_2 \sim E_3 \to E_3 - 1.07 E_2$$

$$\sim \left[\begin{array}{cccc} 1.09 & 0.987 & 0.832 & : 4.21 \\ 0 & -6.12 & -0.689 & : -6.16 \\ 0 & 0 & -4.92 & : -25.2 \end{array}\right]$$

By using back substitution, we get $x_3 = 5.12$, $x_2 = 0.430$, $x_1 = -0.436$

System of linear equations:

Exercise:

1. Use Gaussian elimination with scaled partial pivoting and three-digit chopping arithmetics to solve the following linear system, and compare the approximations with the actual solution $[0, 10, 1/7]^T$.

$$\begin{aligned} 3.03x_1 - 12.1x_2 + 14x_3 &= -119 \\ -3.03x_1 + 12.1x_2 - 7x_3 &= 120 \\ 6.11x_1 - 14.2x_2 + 21x_3 &= -139. \end{aligned}$$