

**Course: UMA 035 (Optimization Techniques)**

**Instructor: Dr. Amit Kumar,**

**Associate Professor,**

**School of Mathematics,**

**TIET, Patiala**

**Email: [amitkumar@thapar.edu](mailto:amitkumar@thapar.edu)**

**Mob: 9888500451**

### Equation of a Plane

$$ax+by+cz=d$$

or

$$a_1x_1+ a_2x_2+ a_3x_3=d$$

### Equation of a hyper plane

$$a_1x_1+ a_2x_2+ a_3x_3+\dots+ a_nx_n= d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = d$$

$$\text{Assuming } [a_1 \ a_2 \ \dots \ a_n] = A \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X,$$

$$AX=d$$

### Set of hyper planes

$$S=\{X: AX=d\}$$

### Equation of a closed half space

$$a_1x_1+ a_2x_2+ a_3x_3+\dots+ a_nx_n \leq d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leq d$$

$$\text{Assuming } [a_1 \ a_2 \ \dots \ a_n] = A \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X,$$

$$AX \leq d$$

OR

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n \geq d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq d$$

$$\text{Assuming } [a_1 \ a_2 \ \dots \ a_n] = A \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X,$$

$$AX \geq d$$

Set of closed half spaces

$$S = \{X : AX \leq d\}$$

OR

$$S = \{X : AX \geq d\}$$

Equation of an open half space

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n < d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} < d$$

$$\text{Assuming } [a_1 \ a_2 \ \dots \ a_n] = A \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X,$$

$$AX < d$$

OR

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n > d$$

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} > d$$

Assuming  $[a_1 \ a_2 \ \dots \ a_n] = A$  and  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X$ ,

$$AX > d$$

### Set of open half spaces

$$S = \{X: AX < d\}$$

OR

$$S = \{X: AX > d\}$$

### Matrix form of a LPP

Maximize  $(CX)$

Subject to

$$AX = b,$$

$$X \geq 0,$$

where,

$$\triangleright C = [c_1, c_2, \dots, c_n],$$

$$\triangleright X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\triangleright A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

$$\triangleright b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

### Set of feasible solution of a LPP

$$S = \{X: AX=B, X \geq 0\}$$

### Set of optimal solutions of a LPP

$$S = \{X: CX=p, AX=B, X \geq 0\}, \text{ where } p \text{ is the maximum value.}$$

### Method to check that a set is convex or not

**Step 1:** Consider two general elements of the considered set.

**Step 2:** Assume that the considered elements satisfies the properties of the considered set (Conditions written after “ : ” in the set).

**Step 3:** Write the convex linear combination of the considered elements.

**Step 4:** Simplify the convex linear combination to obtain a general element by multiplying with scalars inside and adding the elements position wise.

**Step 5:** Check that for the general element, obtained in Step 4, the properties of the considered set (Conditions written after “ : ” in the set) are satisfying or not.

**Case 1:** If all the properties will be satisfied then the considered set will be convex.

**Case 2: If one or more properties will not be satisfied then the considered set will not be convex.**

**Prove that set of optimal solutions of a LPP is a convex set**

**Proof:**

**Let  $S = \{X: CX=p, AX=b, X \geq 0\}$  (where  $p$  is the optimal value of the LPP) be the set of optimal solutions of a LPP.**

**Let  $X_1$  and  $X_2$  be two elements of the set  $S$ .**

**Since,  $X_1$  and  $X_2$  belongs to the set  $S$ . So, these will satisfy the following properties of the set  $S$ .**

$$CX_1=p, AX_1=b, X_1 \geq 0$$

$$CX_2=p, AX_2=b, X_2 \geq 0$$

**The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$**

**where,**

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$(i) C (a_1 X_1 + a_2 X_2) = a_1 (CX_1) + a_2 (CX_2)$$

$$= a_1 (p) + a_2 (p) \quad (\text{since, } CX_1=p \text{ and } CX_2=p)$$

$$= (a_1 + a_2)p$$

$$= p \quad (\text{since, } a_1 + a_2 = 1)$$

$$(ii) A (a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$= a_1 (b) + a_2 (b) \quad (\text{since, } AX_1=b \text{ and } AX_2=b)$$

$$= (a_1 + a_2)b$$

$$= b \quad (\text{since, } a_1 + a_2 = 1)$$

$$(iii) \text{ Since, } a_1 \geq 0, a_2 \geq 0, X_1 \geq 0, X_2 \geq 0. \text{ So, } a_1 X_1 + a_2 X_2 \geq 0$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of all optimal solutions of a LPP) is a convex set.

**Prove that set of feasible solutions of a LPP is a convex set**

**Proof:**

Let  $S = \{X: AX=b, X \geq 0\}$  be the set of feasible solutions of a LPP.

Let  $X_1$  and  $X_2$  be two elements of the set  $S$ .

Since,  $X_1$  and  $X_2$  belongs to the set  $S$ . So, these will satisfy the following properties of the set  $S$ .

$$A X_1 = b, X_1 \geq 0$$

$$A X_2 = b, X_2 \geq 0$$

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$(i) A (a_1 X_1 + a_2 X_2) = a_1 (A X_1) + a_2 (A X_2)$$

$$= a_1 (b) + a_2 (b) \quad (\text{since, } A X_1 = b \text{ and } A X_2 = b)$$

$$= (a_1 + a_2)b$$

$$= b \quad (\text{since, } a_1 + a_2 = 1)$$

(ii) Since,  $a_1 \geq 0, a_2 \geq 0, X_1 \geq 0, X_2 \geq 0$ . So,  $a_1 X_1 + a_2 X_2 \geq 0$

**It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1X_1 + a_2X_2$ .**

**Hence, the set S (set of all feasible solutions of a LPP) is a convex set.**

**Prove that set of hyper planes is a convex set**

**Proof:**

**Let  $S = \{X: AX=b\}$  be the set of hyper planes.**

**Let  $X_1$  and  $X_2$  be two elements of the set S.**

**Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.**

$$AX_1=b,$$

$$AX_2=b,$$

**The convex linear combination of  $X_1$  and  $X_2$  is  $a_1X_1 + a_2X_2$**

**where,**

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$\begin{aligned}A(a_1 X_1 + a_2 X_2) &= a_1 (AX_1) + a_2 (AX_2) \\&= a_1 (b) + a_2 (b) \quad (\text{since, } AX_1=b \text{ and } AX_2=b) \\&= (a_1 + a_2)b \\&= b \quad (\text{since, } a_1 + a_2 = 1)\end{aligned}$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of hyper planes) is a convex set.

**Prove that set of closed half spaces is a convex set**

**Proof:**

Let  $S = \{X: AX \leq b\}$  be the set of closed half spaces.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$$AX_1 \leq b,$$

$$AX_2 \leq b,$$

**The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$**

**where,**

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

**Now,**

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$\leq a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 \leq b \text{ and } AX_2 \leq b)$$

$$\leq (a_1 + a_2)b$$

$$\leq b \quad (\text{since, } a_1 + a_2 = 1)$$

**It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .**

**Hence, the set S (set of closed half spaces) is a convex set.**

**OR**

**Let  $S = \{X: AX \geq b\}$  be the set of closed half spaces.**

Let  $X_1$  and  $X_2$  be two elements of the set  $S$ .

Since,  $X_1$  and  $X_2$  belongs to the set  $S$ . So, these will satisfy the following properties of the set  $S$ .

$$A X_1 \geq b,$$

$$A X_2 \geq b,$$

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$A (a_1 X_1 + a_2 X_2) = a_1 (A X_1) + a_2 (A X_2)$$

$$\geq a_1 (b) + a_2 (b) \quad (\text{since, } A X_1 \geq b \text{ and } A X_2 \geq b)$$

$$\geq (a_1 + a_2)b$$

$$\geq b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set  $S$  are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set  $S$  (set of closed half spaces) is a convex set.

**Prove that set of open half spaces is a convex set**

**Proof:**

Let  $S = \{X: AX \leq b\}$  be the set of closed half spaces.

Let  $X_1$  and  $X_2$  be two elements of the set  $S$ .

Since,  $X_1$  and  $X_2$  belongs to the set  $S$ . So, these will satisfy the following properties of the set  $S$ .

$$AX_1 \leq b,$$

$$AX_2 \leq b,$$

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$< a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 < b \text{ and } AX_2 < b)$$

$$< (a_1 + a_2)b$$

$$< b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of open half spaces) is a convex set.

OR

Let  $S = \{X: AX > b\}$  be the set of closed half spaces.

Let  $X_1$  and  $X_2$  be two elements of the set S.

Since,  $X_1$  and  $X_2$  belongs to the set S. So, these will satisfy the following properties of the set S.

$$AX_1 > b,$$

$$AX_2 > b,$$

The convex linear combination of  $X_1$  and  $X_2$  is  $a_1 X_1 + a_2 X_2$

where,

$$a_1 \geq 0,$$

$$a_2 \geq 0,$$

$$a_1 + a_2 = 1.$$

Now,

$$A(a_1 X_1 + a_2 X_2) = a_1 (AX_1) + a_2 (AX_2)$$

$$> a_1 (b) + a_2 (b) \quad (\text{since, } AX_1 > b \text{ and } AX_2 > b)$$

$$> (a_1 + a_2)b$$

$$> b \quad (\text{since, } a_1 + a_2 = 1)$$

It is obvious that all the conditions of the set S are satisfying for the convex linear combination  $a_1 X_1 + a_2 X_2$ .

Hence, the set S (set of open half spaces) is a convex set.

**Prove that intersection of finite number of convex sets is a convex set**

**Proof:**

Let  $S_1 \cap S_2 \cap \dots \cap S_n$  be the intersection of “n” convex sets  $S_1, S_2, \dots, S_n$ .

Let  $X_1$  and  $X_2$  be two elements of the set  $S_1 \cap S_2 \cap \dots \cap S_n$ .

Since,  $X_1$  and  $X_2$  belongs to the set  $S_1 \cap S_2 \cap \dots \cap S_n$ .

So,

$X_1$  and  $X_2$  belongs to all the convex set  $S_1$

$X_1$  and  $X_2$  belongs to all the convex set  $S_2$

$\vdots$

$X_1$  and  $X_2$  belongs to all the convex set  $S_n$

Furthermore as,

$S_1$  is a convex set. So,  $X_1$  and  $X_2$  belongs to the set  $S_1$  implies that  $a_1 X_1 + a_2$

$X_2$  also belongs to  $S_1$

$S_2$  is a convex set. So,  $X_1$  and  $X_2$  belongs to the set  $S_2$  implies that  $a_1 X_1 + a_2$

$X_2$  also belongs to  $S_2$

$\vdots$

$S_n$  is a convex set. So,  $X_1$  and  $X_2$  belongs to the set  $S_n$  implies that  $a_1 X_1 + a_2$

$X_2$  also belongs to  $S_n$

Finally,

$a_1 X_1 + a_2 X_2$  belongs to  $S_1, S_2, \dots, S_n$  implies that  $a_1 X_1 + a_2 X_2$  belongs to

$S_1 \cap S_2 \cap \dots \cap S_n$

Hence,  $S_1 \cap S_2 \cap \dots \cap S_n$  is a convex set.