Course: UMA 035 (Optimization Techniques)

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Graphical method (For solving a LPP having one or two variables)

Step 1: **Find the feasible region of for the given LPP.**

Step 2: Check that the feasible region is bounded or unbounded.

Case (1): If the feasible region is bounded then go to Step 3.

Case (2): If the feasible region is unbounded then check that the maximum value of x_1 is infinite or the maximum value x_2 is infinite or the maximum value of both x_1 and x_2 is infinite.

Case (2a): If the maximum value of x_1 is infinite then add the constraint x_1<=M, where M is large positive real number in the given LPP.

Case (2b): If the maximum value of x_2 is infinite then add the constraint x_2<=M, where M is large positive real number in the given LPP.

Case (2c): If the maximum value of both x_1 and x_2 is infinite is infinite then add the constraints x_1<=M and x_2<=M, where M is large positive real number in the given LPP.

Step 3: Find the value of the objective function corresponding to each corner point (vertices) of the obtained feasible region.

Step 4: If the problem is of maximization then find the maximum of all the obtained values of the objective function and if the problem is of

minimization then find the minimum of all the obtained values of the objective function.

Step 5: The obtained maximum/minimum value is called the optimal value and the corner point corresponding to which the maximum/minimum value exist is called an optimal solution.

If the obtained optimal value is terms of M then the obtained solution is called unbounded solution.

If two optimal solutions exist then there will exist infinite number of optimal solutions which can be obtained as discussed earlier.

Example: Solve the following LPP by graphical method.

Maximize/Minimize ($3x_1 – 2x_2$)

Subject to

$x_1 – 4x_2 <= 5$,

$x_1 + 8x_2 >= 15$,

$x_1 >= 2$, $x_2 >= –8$.

Solution: Since, Minimum value of x_1 and x_2 are not 0. So, there is a need to transform these variables into new variables.

First variable

$x_1 >= 2$

may be written as

$x_1 - 2 >= 0$

Assume $x_1 - 2 = y_1$

i.e.,

$x_1 = y_1 + 2$

Replace x_1 with y_1+2 in the given LPP.

Second variable

$x_2 >= -8$

may be written as

$x_2 + 8 >= 0$

Assume $x_2 + 8 = y_2$

i.e.,

$x_2 = y_2 - 8$

Replace x_2 with y_2-8 in the given LPP.

Transformed LPP

Maximize/Minimize $(3 (y_1+2) - 2(y_2-8))$

Subject to

$(y_1+2)–4\ (y_2–8)\ < = 5,$

$(y_1+2) + 8(y_2–8) >= 15,$

$y_1+2>=2,\ y_2–8>= –8.$

Maximize/Minimize $(3y_1+6 – 2y_2+16))$

Subject to

$y_1+2–4y_2+32\ < = 5,$

$y_1+2 + 8y_2–64>= 15,$

$y_1>=2–2,\ y_2>= –8+8$

Maximize/Minimize $(3y_1– 2y_2+22))$

Subject to

$y_1–4y_2+34\ < = 5,$

$y_1+ 8y_2–62>= 15,$

$y_1>=0,\ y_2>= 0$

Maximize/Minimize $(3y_1– 2y_2+22))$

Subject to

$y_1–4y_2 < = 5–34,$

$y_1+ 8y_2>= 15+62,$

$y_1>=0,\ y_2>= 0$

Maximize/Minimize $(3y_1– 2y_2+22)$

Subject to

$y_1 – 4y_2 <= –29$,

$y_1 + 8y_2 >= 77$,

$y_1 >= 0$, $y_2 >= 0$

Draw First Cosnstraint

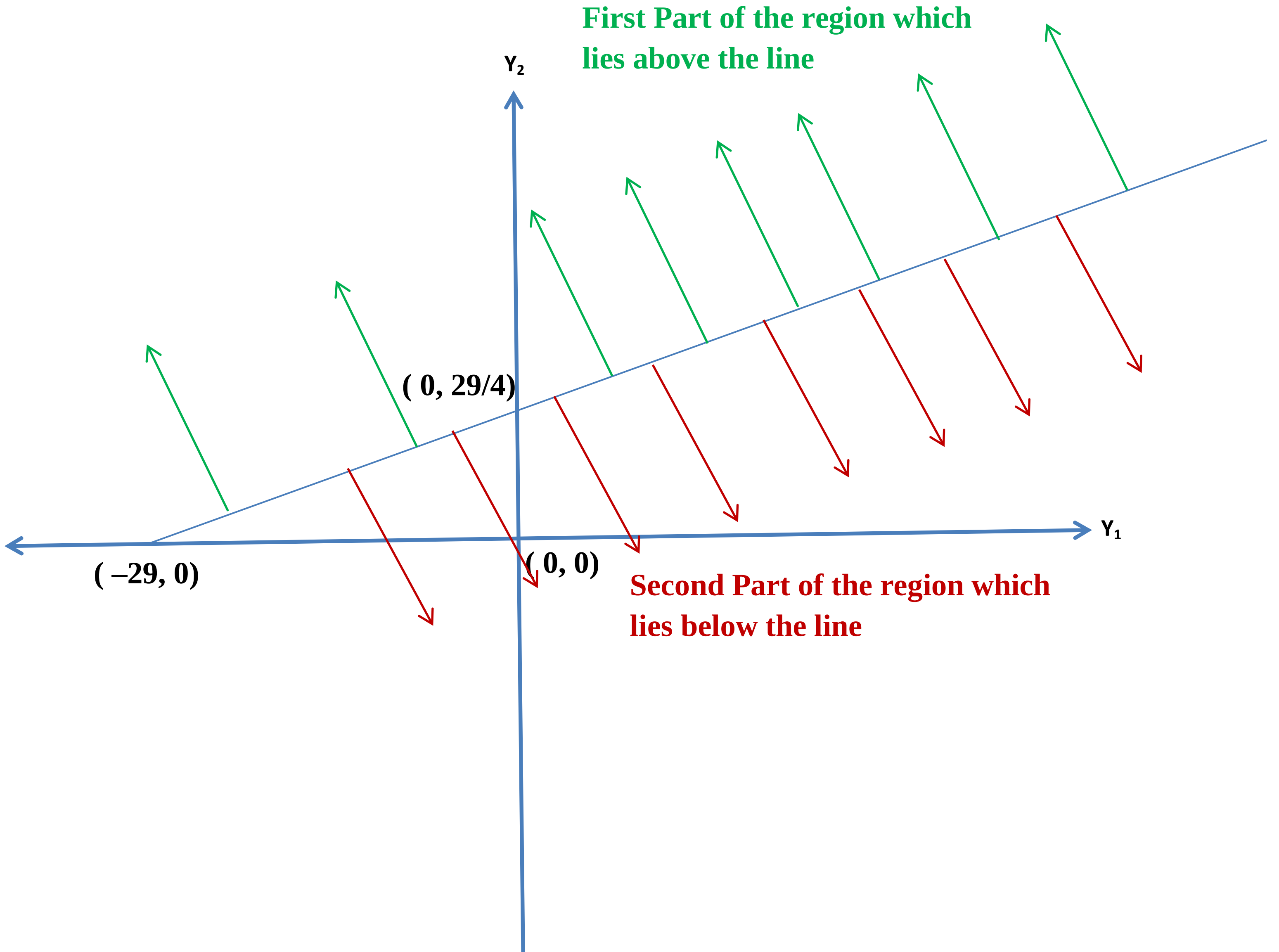
$y_1 – 4y_2 <= –29$

Assuming $y_1 = 0$, $y_1 – 4y_2 = –29$ implies $0 – 4y_2 = –29$ i.e., $y_2 = 29/4$

Therefore, first point is $(y_1, y_2) = (0, 29/4)$

Assuming $y_2 = 0$, $y_1 – 4y_2 = –29$ implies $y_1 – 0 = –29$ i.e., $y_1 = –29$

Therefore, second point is $(y_1, y_2) = (–29, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

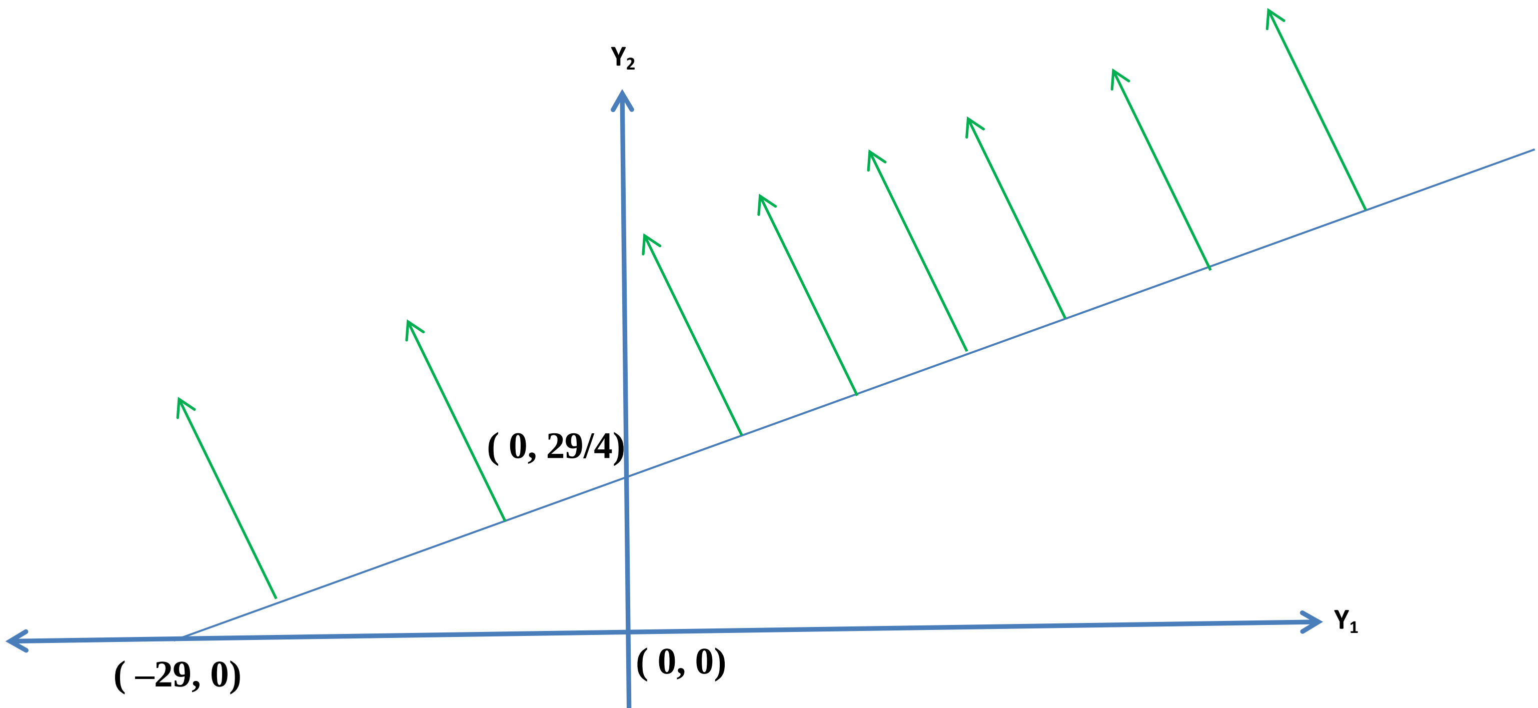
Putting (5,0) in the constraint

$y_1 - 4y_2 <= -29$, we have

$5 - 0 <= -29$

$5 <= -29$

It is obvious that the constraint is not satisfying. So, we consider the first part.



Draw Second Cosnstraint

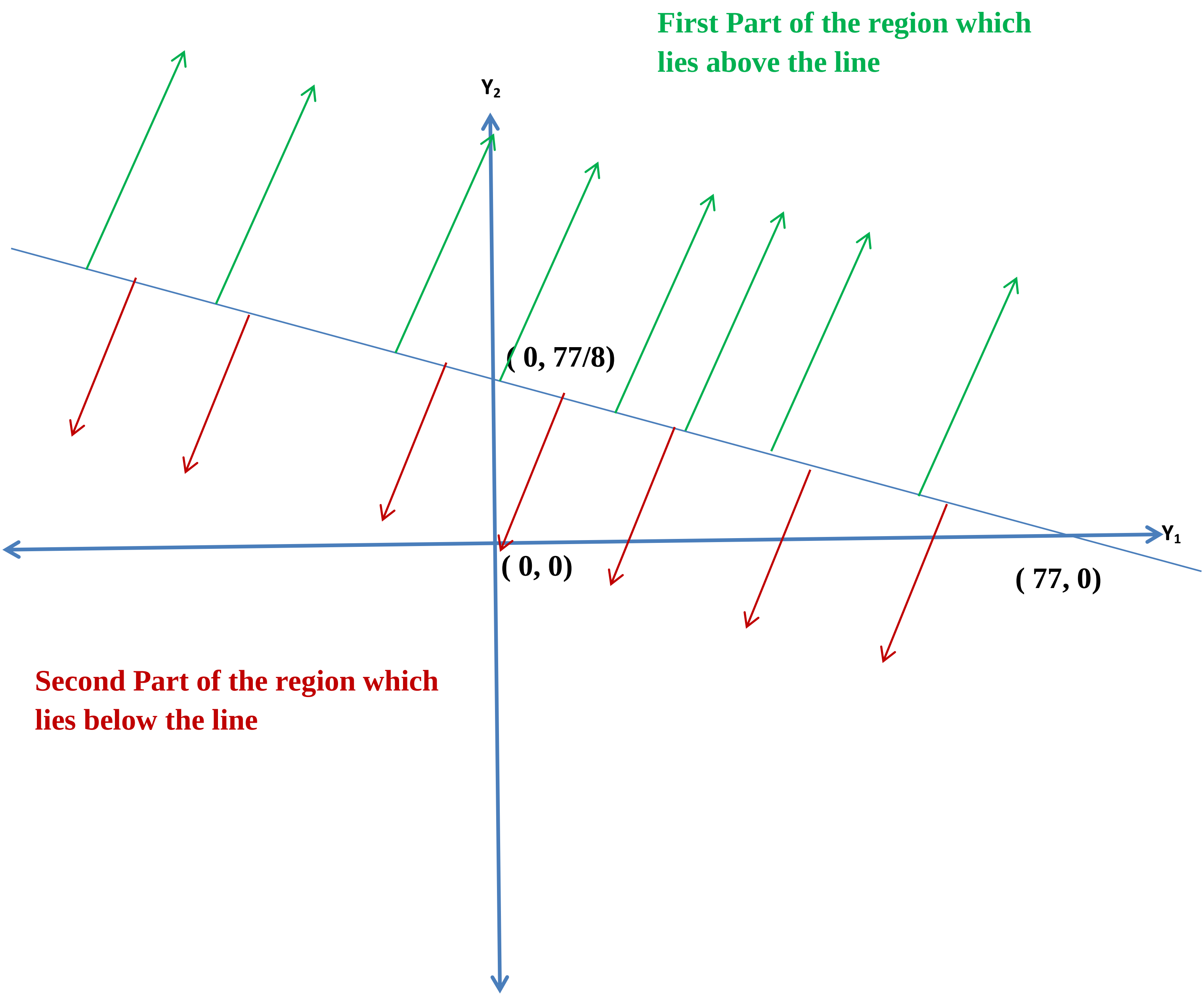
$y_1 + 8y_2 >= 77$

Assuming $y_1 = 0$, $y_1 + 8y_2 = 77$ implies $0 + 8y_2 = 77$ i.e., $y_2 = 77/8$

Therefore, first point is $(y_1, y_2) = (0, 77/8)$

Assuming $y_2 = 0$, $y_1+8y_2 = 77$ implies $y_1 = 77$ i.e., $y_1 = 77$

Therefore, second point is $(y_1,y_2)=(77, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint

then we will consider the second part otherwise the first part.

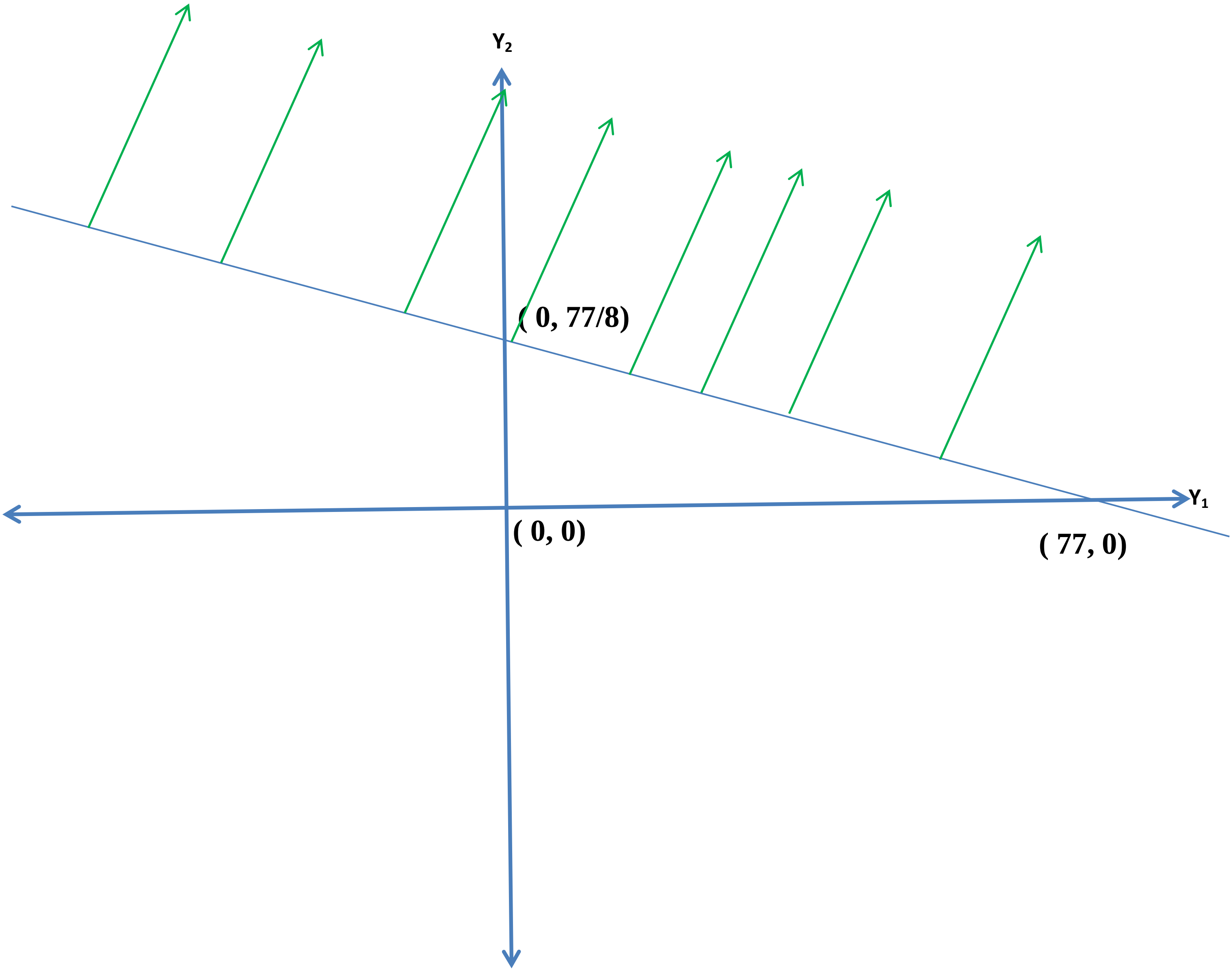
Putting (5,0) in the the constraint

$y_1+8y_2 >= 77$, we have

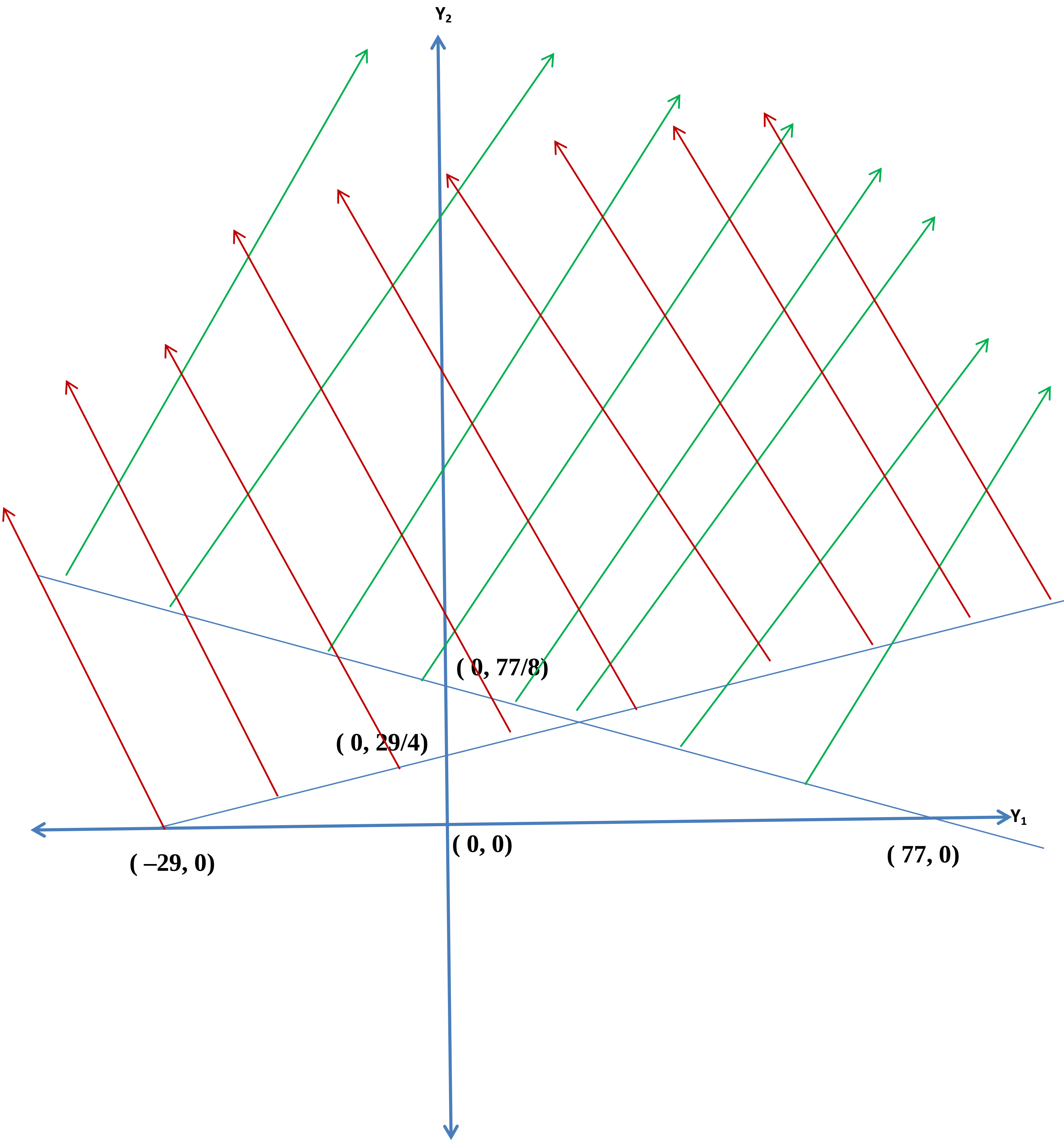
5+0> =77

5>= 77

It is obvious that the constraint is not satisfying. So, we consider the first part.

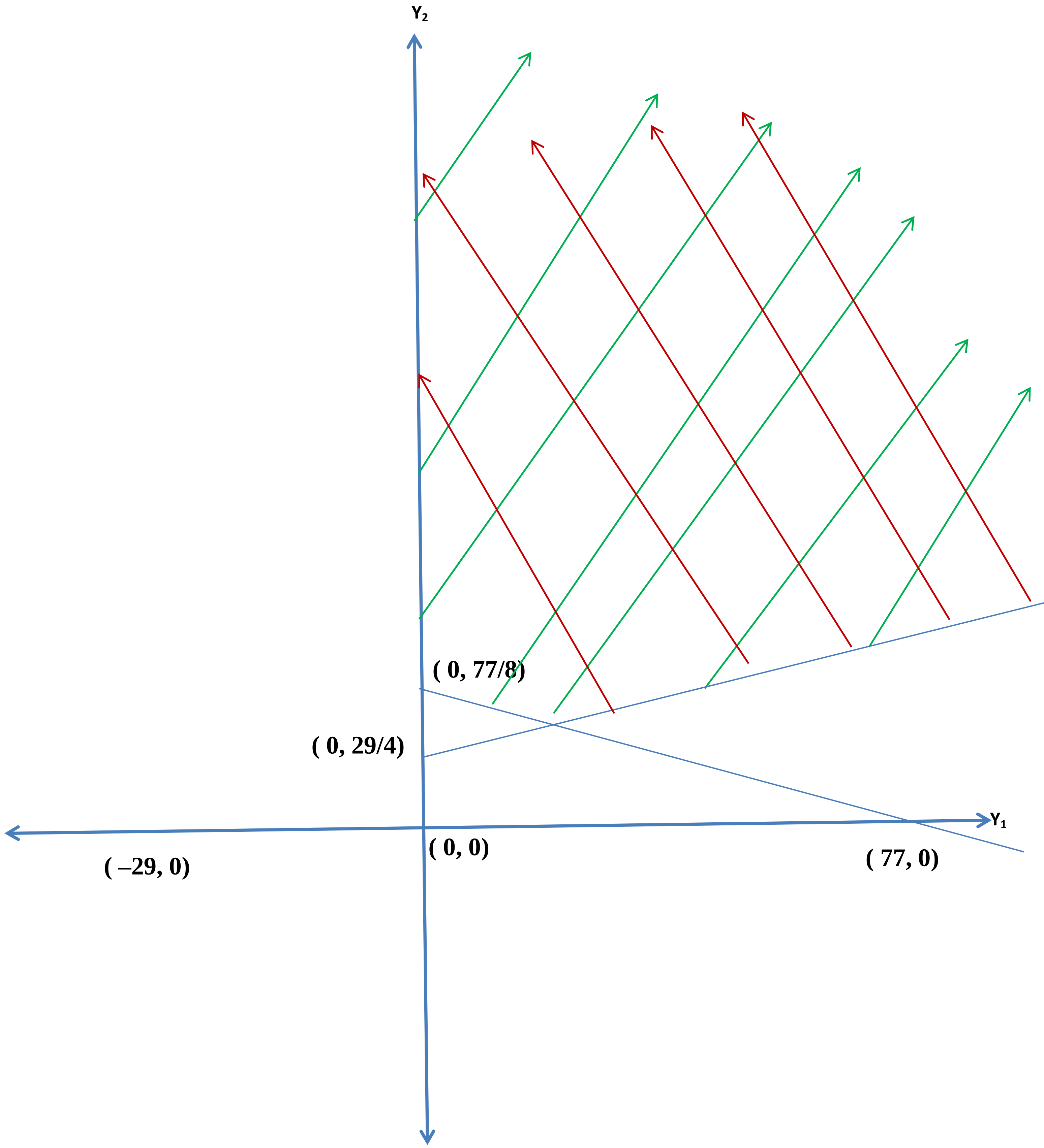


Common region of both the constraints



Common region of both the constraints in the first quadrant

(Feasible region)



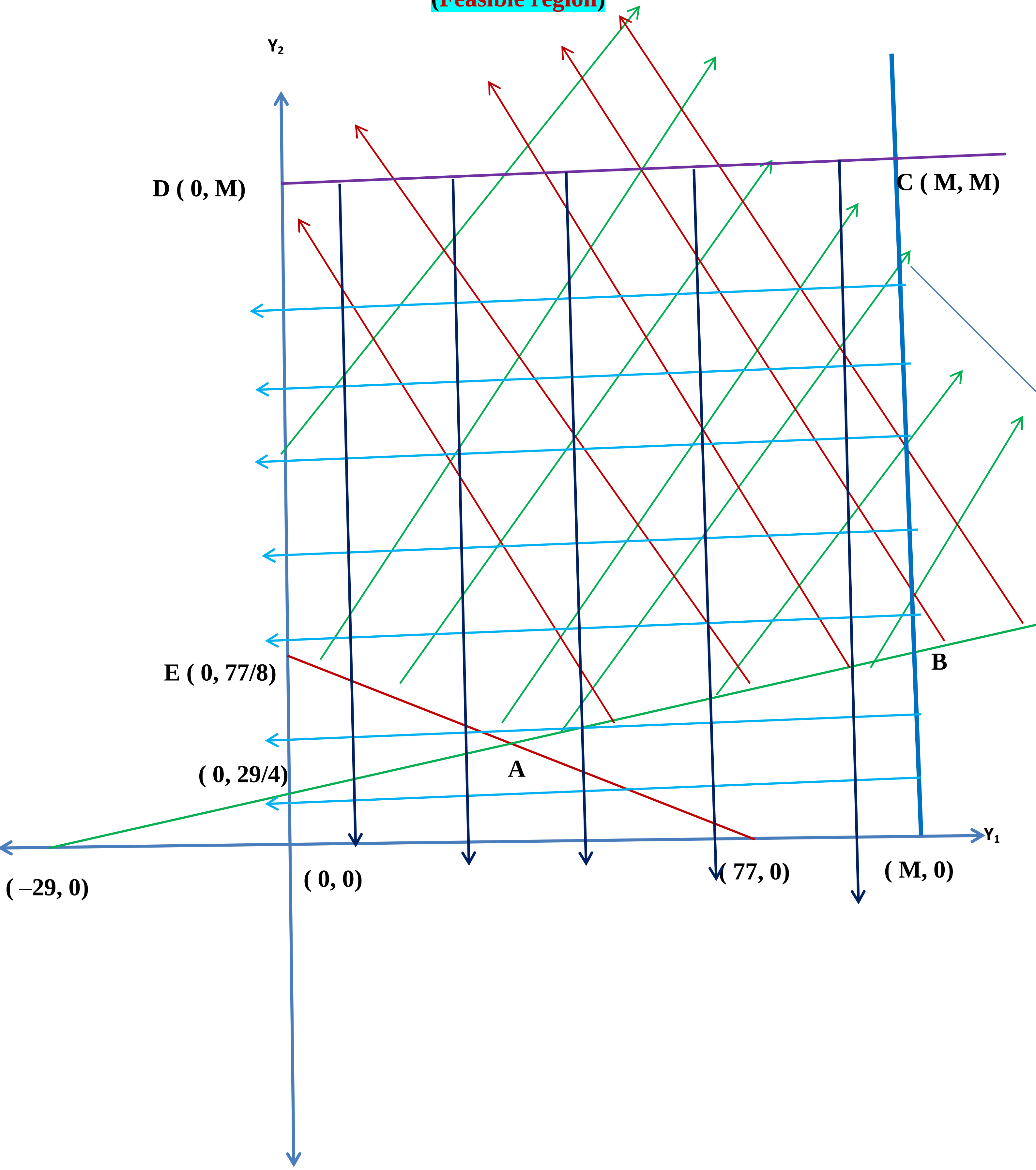
It is obvious that Maximum value of y_1 and the maximum value of y_2 in the feasible region is infinite. So, the feasible region is unbounded.

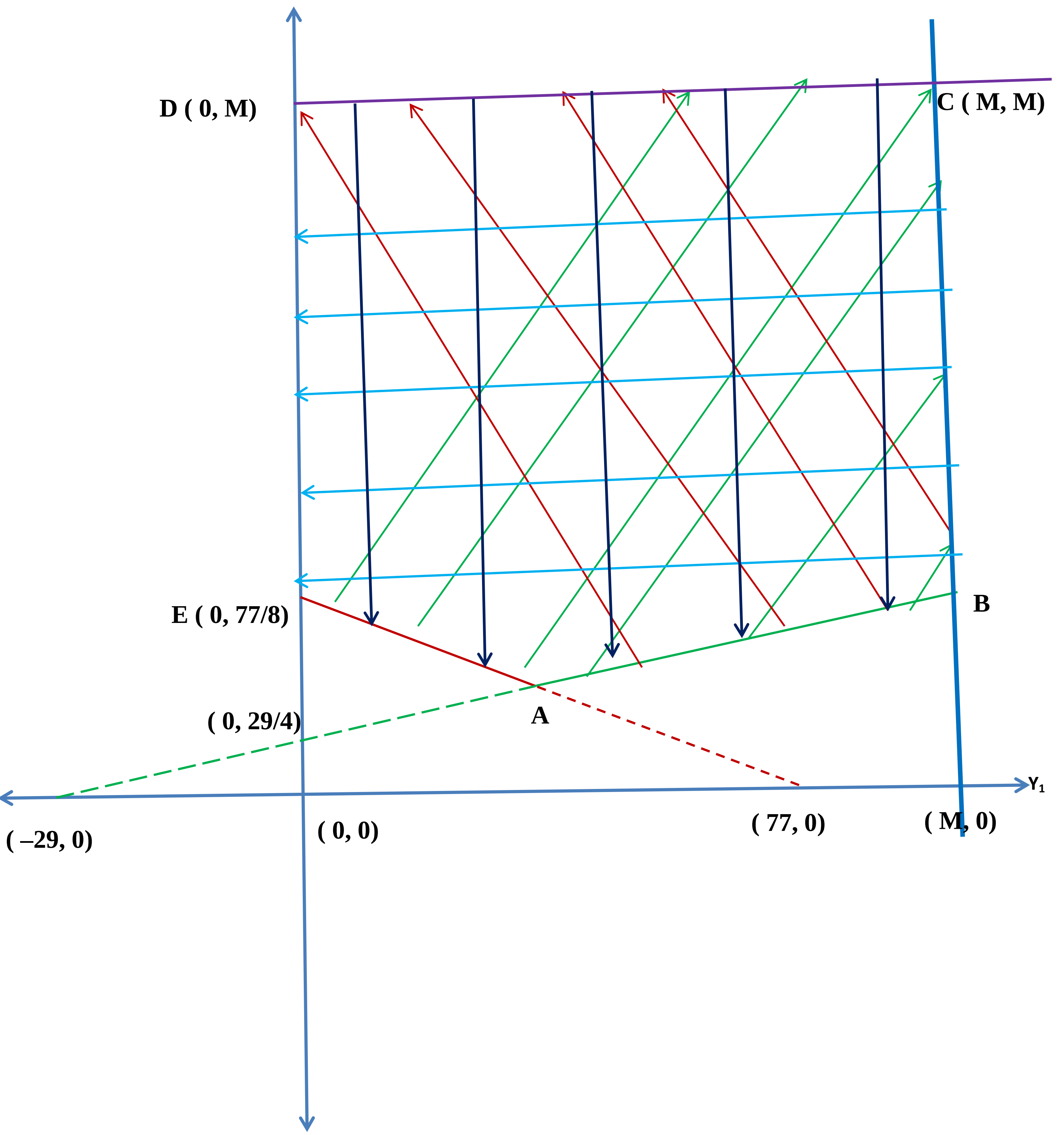
Since the maximum value of both y_1 and y_2 is infinite. So, there is need to add the constraints $y_1 \le M$ and $y_2 \le M$

Including these constraints in the feasible region, the new feasible region is

Common region of all the constraints in the first quadrant

(Feasible region)




D (0, M)
C (M, M)
E (0, 77/8)
B
A
(0, 29/4)
Y_1
(–29, 0)
(0, 0)
(77, 0)
(M, 0)

Extreme Points or Corner Points or Vertices

First point A

Intersection of $y_1 - 4y_2 = -29$ and $y_1 + 8y_2 = 77$

On Solving

$y_1 = 19/3$ and $y_2 = 53/6$

Therefore, A (19/3, 53/6)

Second point B

Intersection of $y_1 - 4y_2 = -29$ and $y_1 = M$

On Solving

$y_1 = M$ and $y_2 = (M+29)/4$

Therefore, B (M, (M+29)/4)

Third point C

Intersection of $y_1 = M$ and $y_2 = M$

Therefore, C (M, M)

Fourth point D

Intersection of $y_1 = 0$ and $y_2 = M$

Therefore, D (0, M)

Fifth point E

Intersection of $y_1 = 0$ and $y_2 = 77/8$

Therefore, E (0, 77/8)

Value of objective function $3y_1 - 2y_2 + 22$ at

- **A (19/3, 53/6) is $3(19/3) - 2(53/6) + 22 = 23.3333$**

- **B (M, (M+29)/4) is 3(M)– 2((M+29)/4)+22=(5M–15)/2=4085/2 (on**
- **assuming M=1000)**

- **C (M, M) is 3(M)– 2(M)+22=M+22=1022 (on assuming M=1000)**

- **D (0, M) is 3(0)– 2(M)+22=– 2(M)+22= – 1078 (on assuming M=1000)**

- **E (0, 77/8) is 3(0)– 2(77/8)+22=2.75**

Maximum {23.3333, 4085/2, 1022, – 1078, 2.75} = 4085/2 or (5M–15)/2

Since, maximum value is depending upon M. So, in case the problem is of maximization.

The optimal solution of the problem is unbounded.

Minimum {23.3333, 4085/2, 1022, – 1078, 2.75} = – 1078 or – 2(M)+22

Since, minimum value is depending upon M. So, in case the problem is of minimization.

The optimal solution of the problem is unbounded.

Example: **Solve the following LPP by graphical method**

Maximize/Minimize ($3x_1 - 2x_2$)

Subject to

$x_1 - 4x_2 <= 5,$

$x_1 + 8x_2 <= 15,$

$x_1 >= 2, x_2 >= -8.$

Solution: Since, Minimum value of x_1 and x_2 are not 0. So, there is a need to transform these variables into new variables.

First variable

$x_1 >= 2$

may be written as

$x_1 - 2 >= 0$

Assume $x_1 - 2 = y_1$

i.e.,

$x_1 = y_1 + 2$

Replace x_1 with $y_1 + 2$ in the given LPP.

Second variable

$x_2 >= -8$

may be written as

$x_2+8>=0$

Assume $x_2+8=y_2$

i.e.,

$x_2=y_2–8$

Replace x_2 with $y_2–8$ in the given LPP.

Transformed LPP

Maximize/Minimize $(3(y_1+2)-2(y_2-8))$

Subject to

$(y_1+2)-4(y_2-8) <= 5,$

$(y_1+2)+8(y_2-8) <= 15,$

$y_1+2>=2, y_2-8>= -8.$

Maximize/Minimize $(3y_1+6-2y_2+16))$

Subject to

$y_1+2-4y_2+32 <= 5,$

$y_1+2+8y_2-64>= 15,$

$y_1>=2-2, y_2>= -8+8$

Maximize/Minimize $(3y_1-2y_2+22))$

Subject to

$y_1 – 4y_2 + 34 \ < = 5,$

$y_1 + 8y_2 – 62 <= 15,$

$y_1 >= 0, y_2 >= 0$

Maximize/Minimize ($3y_1 – 2y_2 + 22$))

Subject to

$y_1 – 4y_2 < = 5 – 34,$

$y_1 + 8y_2 <= 15 + 62,$

$y_1 >= 0, y_2 >= 0$

Maximize/Minimize ($3y_1 – 2y_2 + 22$))

Subject to

$y_1 – 4y_2 < = –29,$

$y_1 + 8y_2 <= 77,$

$y_1 >= 0, y_2 >= 0$

Draw First Cosnstraint

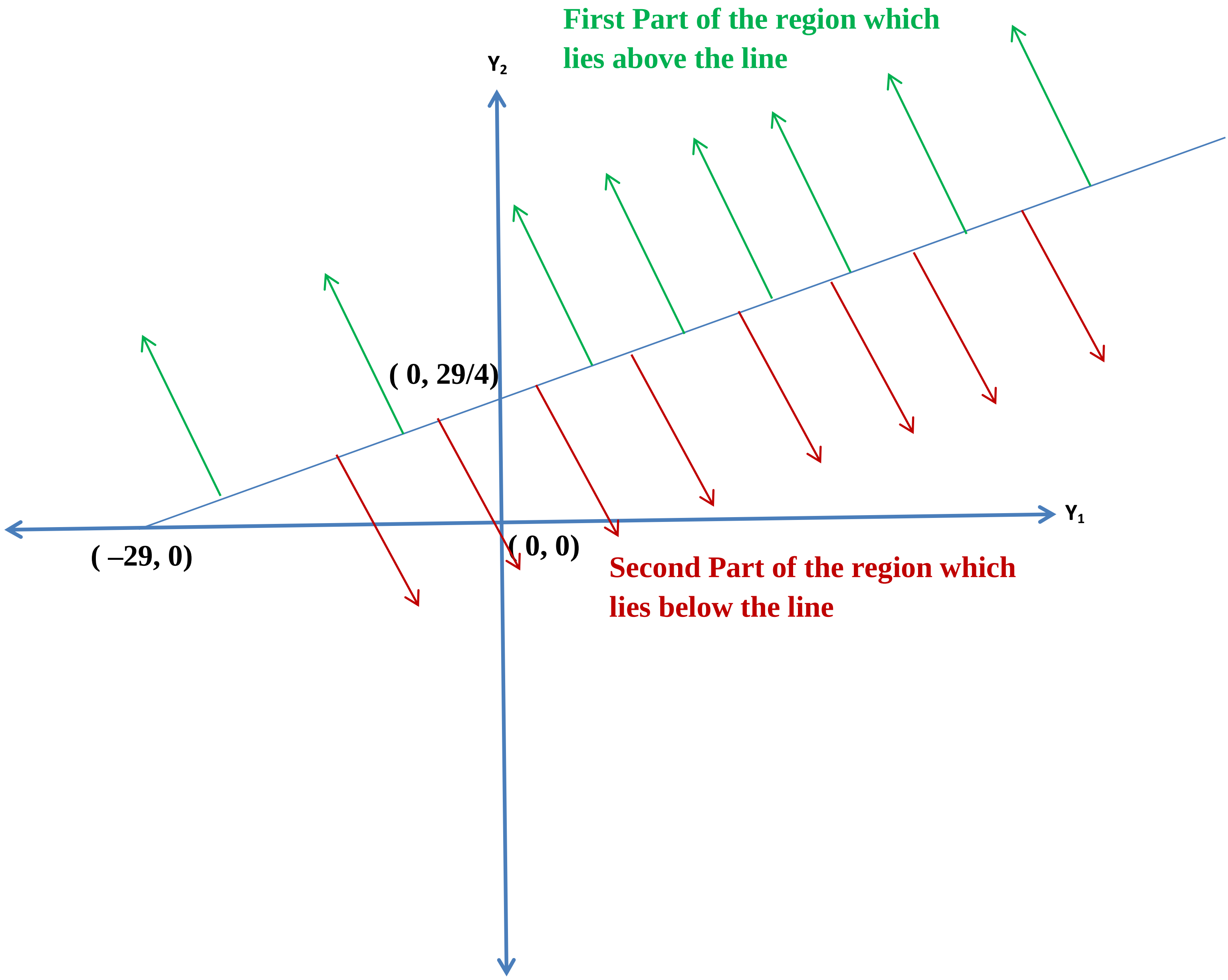
$y_1 – 4y_2 < = –29$

Assuming $y_1 = 0$, $y_1 – 4y_2 = –29$ implies $0 – 4y_2 = –29$ i.e., $y_2 = 29/4$

Therefore, first point is $(y_1, y_2) = (0, 29/4)$

Assuming $y_2 = 0$, $y_1 – 4y_2 = –29$ implies $y_1 – 0 = –29$ i.e., $y_1 = –29$

Therefore, second point is $(y_1, y_2) = (–29, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

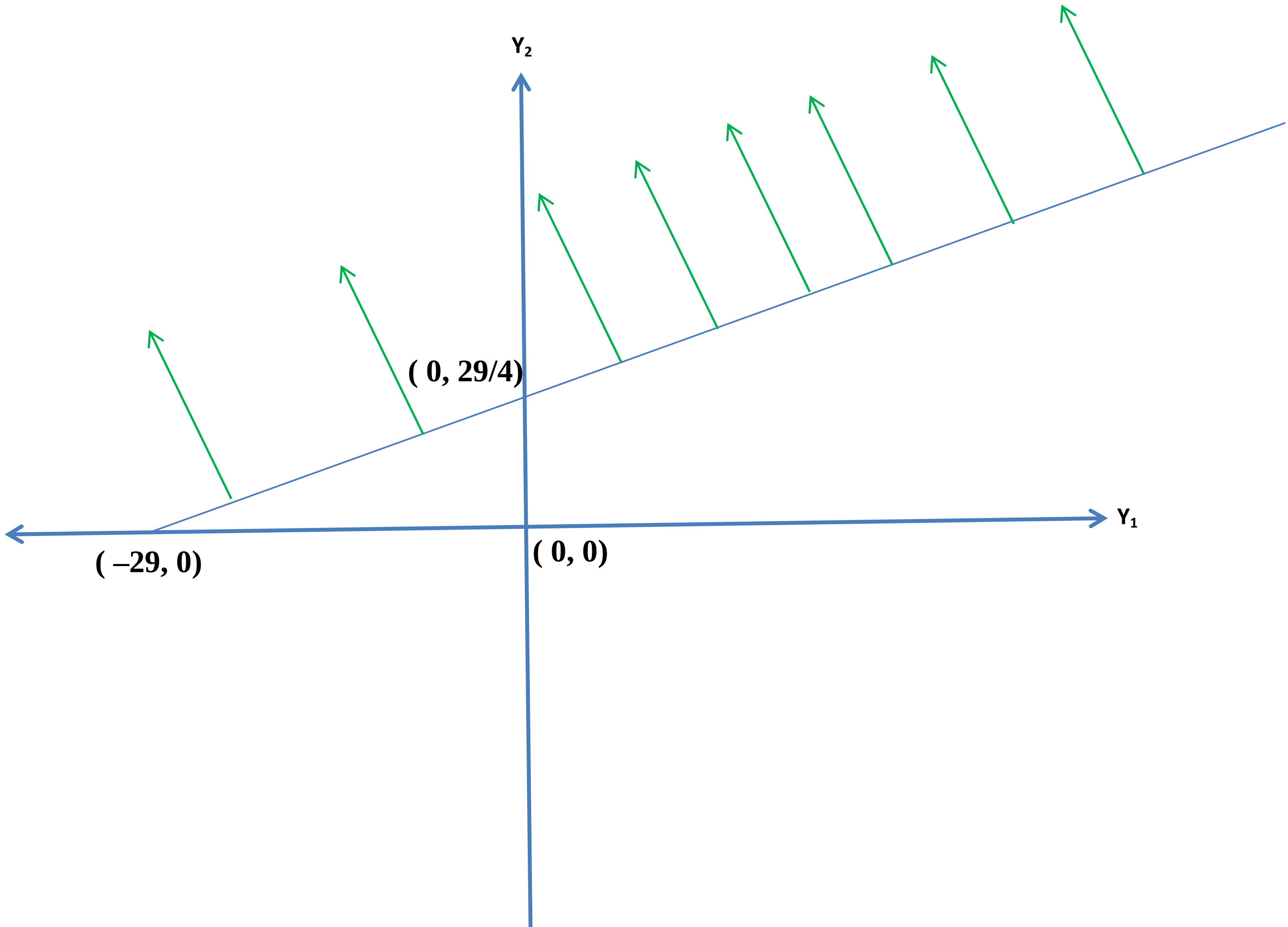
Putting (5,0) in the the constraint

$y_1 - 4y_2 <= -29$, **we have**

$5 - 0 <= -29$

$5 <= -29$

It is obvious that the constraint is not satisfying. So, we consider the first part.



Draw Second Cosnstraint

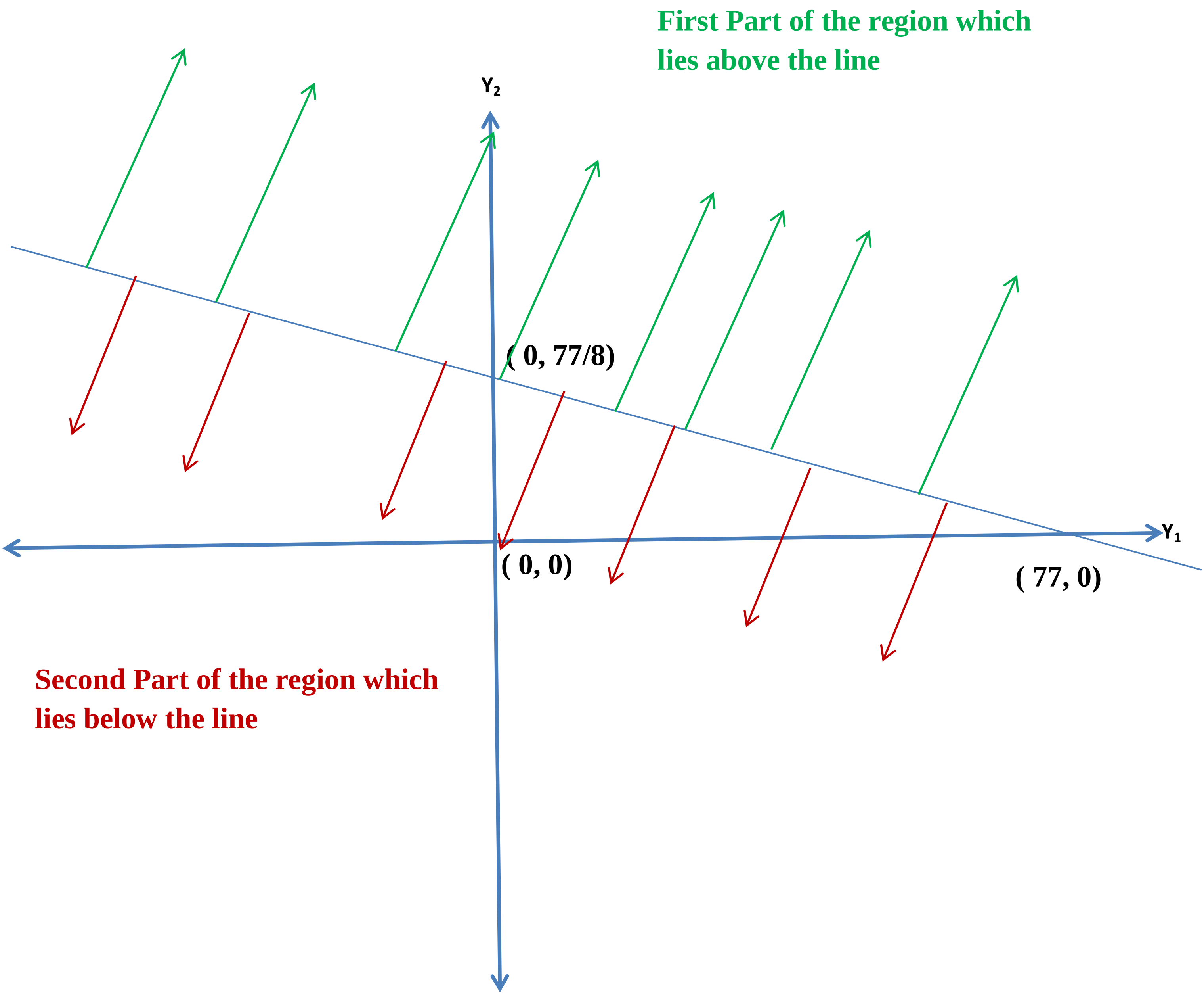
$y_1 + 8y_2 >= 77$

Assuming $y_1 = 0$, $y_1+8y_2 = 77$ implies $0 + 8y_2 = 77$ i.e., $y_2 = 77/8$

Therefore, first point is $(y_1,y_2)=(0, 77/8)$

Assuming $y_2 = 0$, $y_1+8y_2 = 77$ implies $y_1 = 77$ i.e., $y_1 = 77$

Therefore, second point is $(y_1,y_2)=(77, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

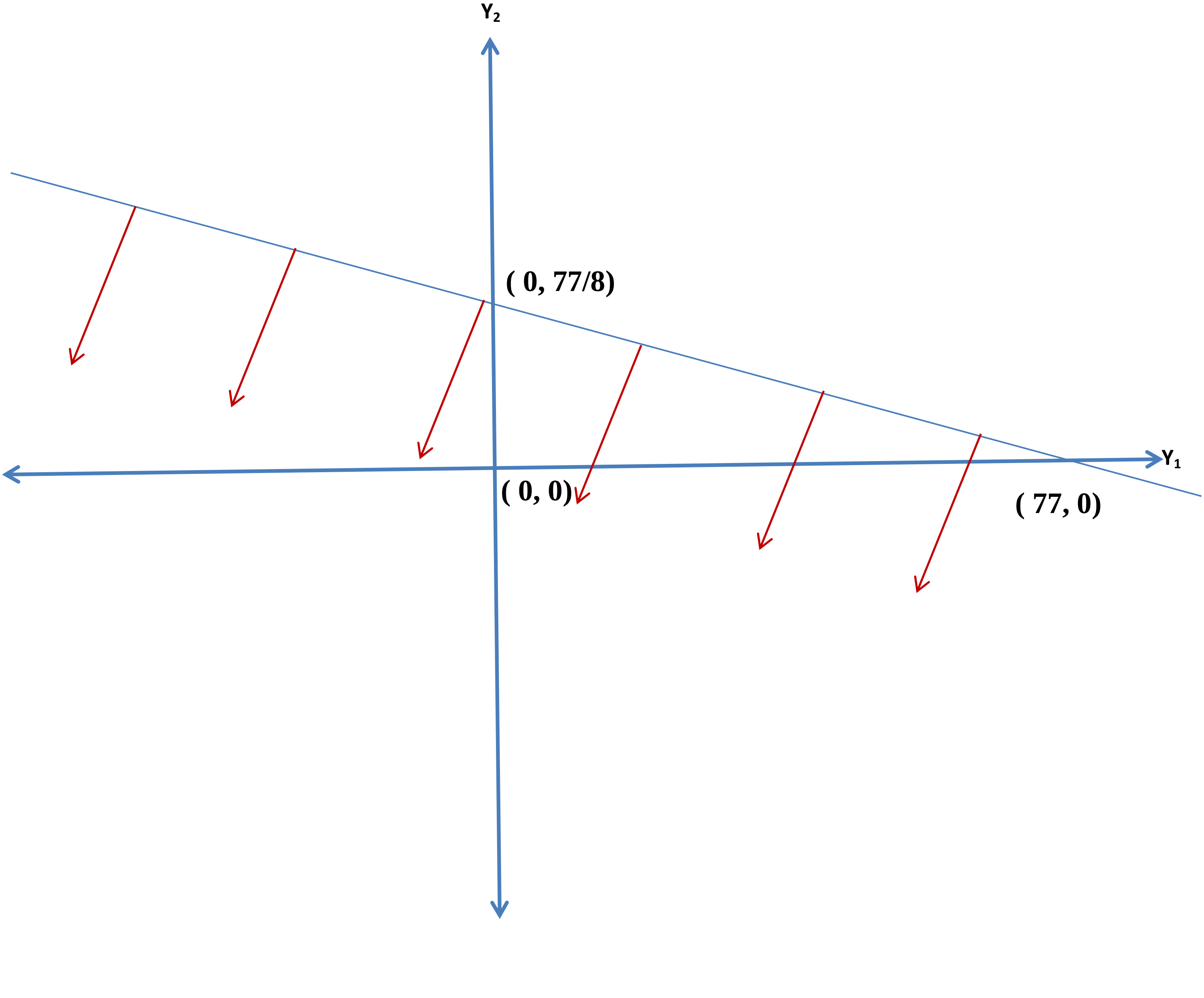
Putting (5,0) in the the constraint

$y_1+8y_2 <= 77$, we have

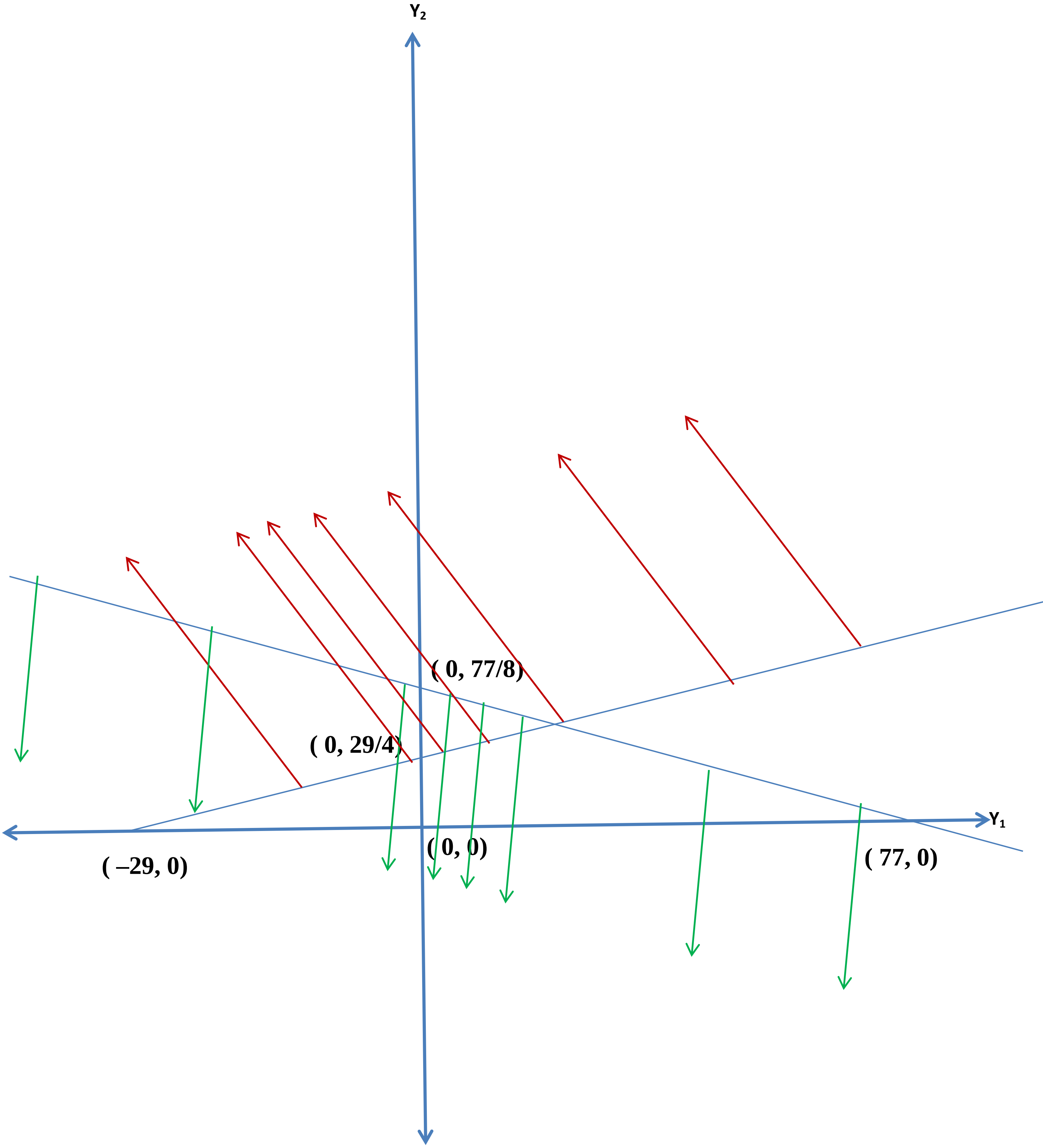
$5+0<=77$

5<= 77

It is obvious that the constraint is satisfying. So, we consider the second part.

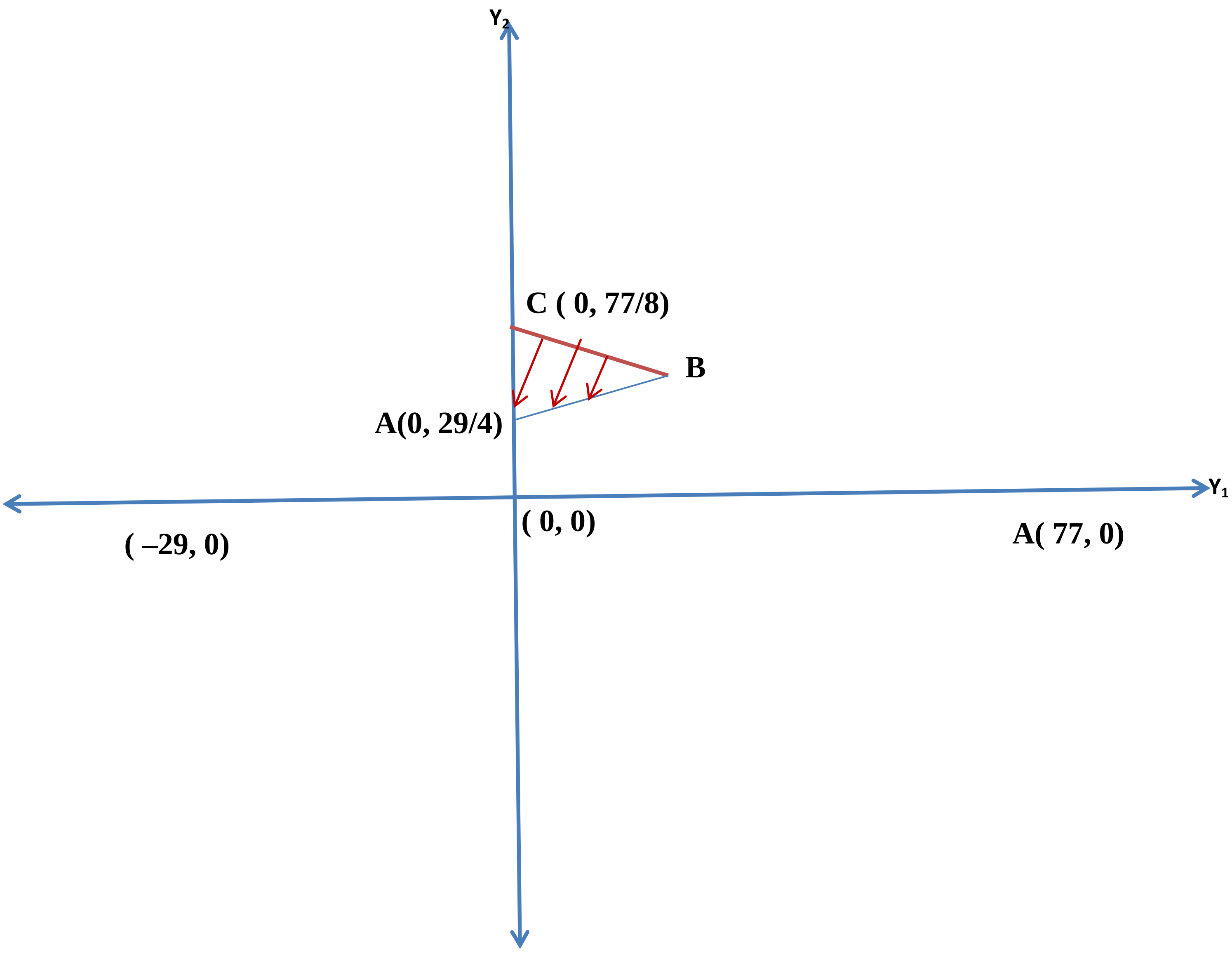


Common region of both the constraints



Common region of both the constraints in the first quadrant

(Feasible region)



It is obvious that Maximum value of y_1 and the maximum value of y_2 in the feasible region is finite. So, the feasible region is bounded.

Since, the feasible region is bounded so there is no need to add any constraint.

Extreme Points or Corner Points or Vertices

First point A

$y_1=0$ and $y_2= 29/4$

Therefore, A (0, 29/4)

Second point B

B is intersection of Intersection of $y_1-4y_2 = -29$ and $y_1+ 8y_2 = 77$

On Solving

$y_1=19/3$ and $y_2= 53/6$

Therefore, B (19/3, 53/6)

Third point C

B (0, 77/8)

Value of objective function $3y_1- 2y_2+22$ at

- **A (0, 29/4) is $3(0)- 2(29/4)+22=36.5$**
- **B (19/3, 53/6) is $3(19/3)- 2(53/6)+22=23.33$**
- **C (0, 77/8) is $3(0)- 2(77/8)+22=41.25$**

Maximum {36.5, 23.33, 41.25} =41.25

Maximum value is 41.25 which is corresponding to $y_1=0$ and $y_2= 77/8$

So, in case of the maximization problem, the optimal solution is $y_1=77$ and $y_2= 8$ and the optimal value is 41.25

Using the relations

$x_1-2=y_1$ and $x_2+8=y_2$

The optimal solution is

$x_1=0+2=2$ and $x_2=y_2-8=77/8-8= 13/8$

Minimum {36.5, 23.33, 41.25} = 23.33

Minimum value is 23..33 which is corresponding to $y_1=19/3$ and $y_2= 53/6$

So, in case of the minimization problem, the optimal solution is $y_1=19/3$ and $y_2= 53/6$ and the optimal value is 23.33

Using the relations

$x_1-2=y_1$ and $x_2+8=y_2$

The optimal solution is

$x_1=19/3+2=25/3$ and $x_2=y_2-8=(53/6)-8=5/6$

Example: Solve the following LPP by graphical method

Maximize/Minimize $(3x_1 - 2x_2)$

Subject to

$x_1 -4x_2 \; > = 5,$

$x_1 + 8x_2 \; <= 15,$

$x_1>=2, x_2>= -8.$

Solution: Since, Minimum value of x_1 and x_2 are not 0. So, there is a need to transform these variables into new variables.

First variable

$x_1>=2$

may be written as

$x_1 – 2 >= 0$

Assume $x_1 – 2 = y_1$

i.e.,

$x_1 = y_1 + 2$

Replace x_1 with y_1+2 in the given LPP.

Second variable

$x_2 >= –8$

may be written as

$x_2 + 8 >= 0$

Assume $x_2 + 8 = y_2$

i.e.,

$x_2 = y_2 – 8$

Replace x_2 with $y_2–8$ in the given LPP.

Transformed LPP

Maximize $(3(y_1+2) – 2(y_2–8))$

Subject to

$(y_1+2) – 4(y_2–8) >= 5,$

$(y_1+2) + 8(y_2–8) <= 15,$

$y_1+2>=2,\ y_2-8>=-8.$

Maximize/Minimize $(3y_1+6-2y_2+16))$

Subject to

$y_1+2-4y_2+32 \ > = 5,$

$y_1+2+8y_2-64<=15,$

$y_1>=2-2,\ y_2>=-8+8$

Maximize/Minimize $(3y_1-2y_2+22))$

Subject to

$y_1-4y_2+34 \ >= 5,$

$y_1+8y_2-62<=15,$

$y_1>=0,\ y_2>=0$

Maximize/Minimize $(3y_1-2y_2+22))$

Subject to

$y_1-4y_2 > = 5-34,$

$y_1+8y_2<=15+62,$

$y_1>=0,\ y_2>=0$

Maximize/Minimize $(3y_1-2y_2+22))$

Subject to

$y_1-4y_2 > = -29,$

$y_1+8y_2<=77,$

$y_1>=0, y_2>= 0$

Draw First Cosnstraint

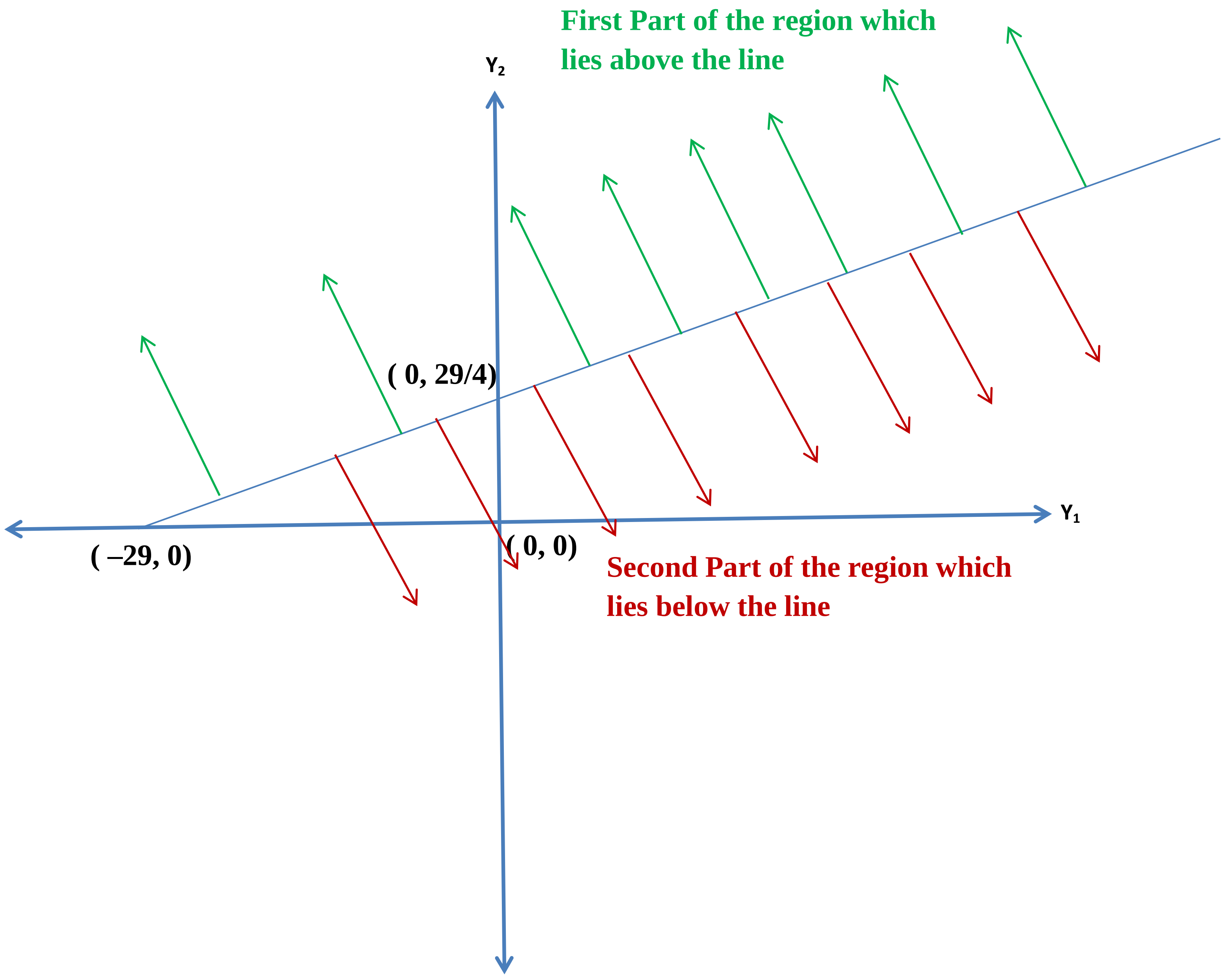
$y_1 - 4y_2 > = -29$

Assuming $y_1 = 0$, $y_1 - 4y_2 = -29$ implies $0 - 4y_2 = -29$ i.e., $y_2 = 29/4$

Therefore, first point is $(y_1, y_2) = (0, 29/4)$

Assuming $y_2 = 0$, $y_1 - 4y_2 = -29$ implies $y_1 - 0 = -29$ i.e., $y_1 = -29$

Therefore, second point is $(y_1, y_2) = (-29, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

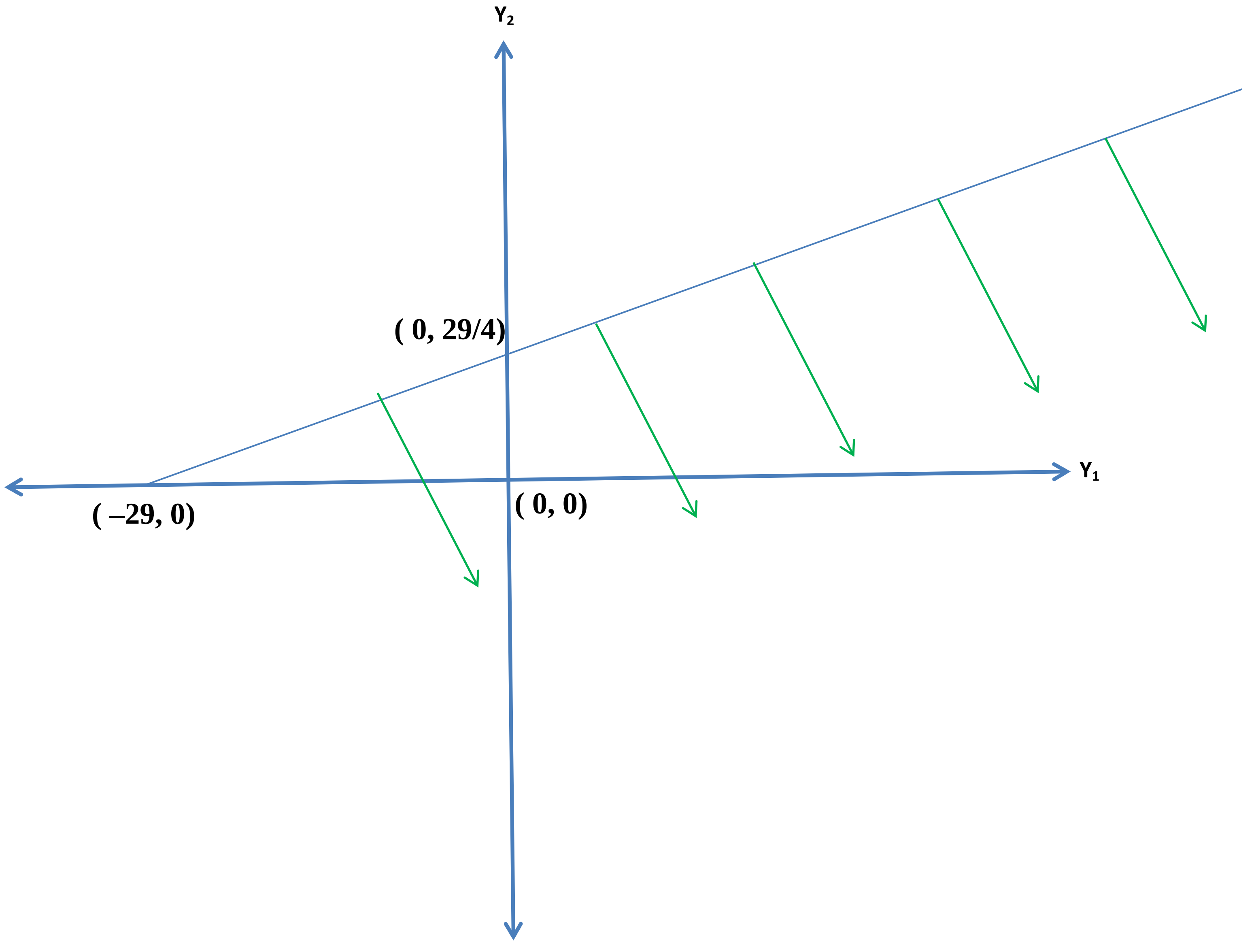
Putting (5,0) in the constraint

$y_1 - 4y_2 >= -29$, we have

$5 - 0 >= -29$

$5 >= -29$

It is obvious that the constraint is satisfying. So, we consider the second part.



Draw Second Cosnstraint

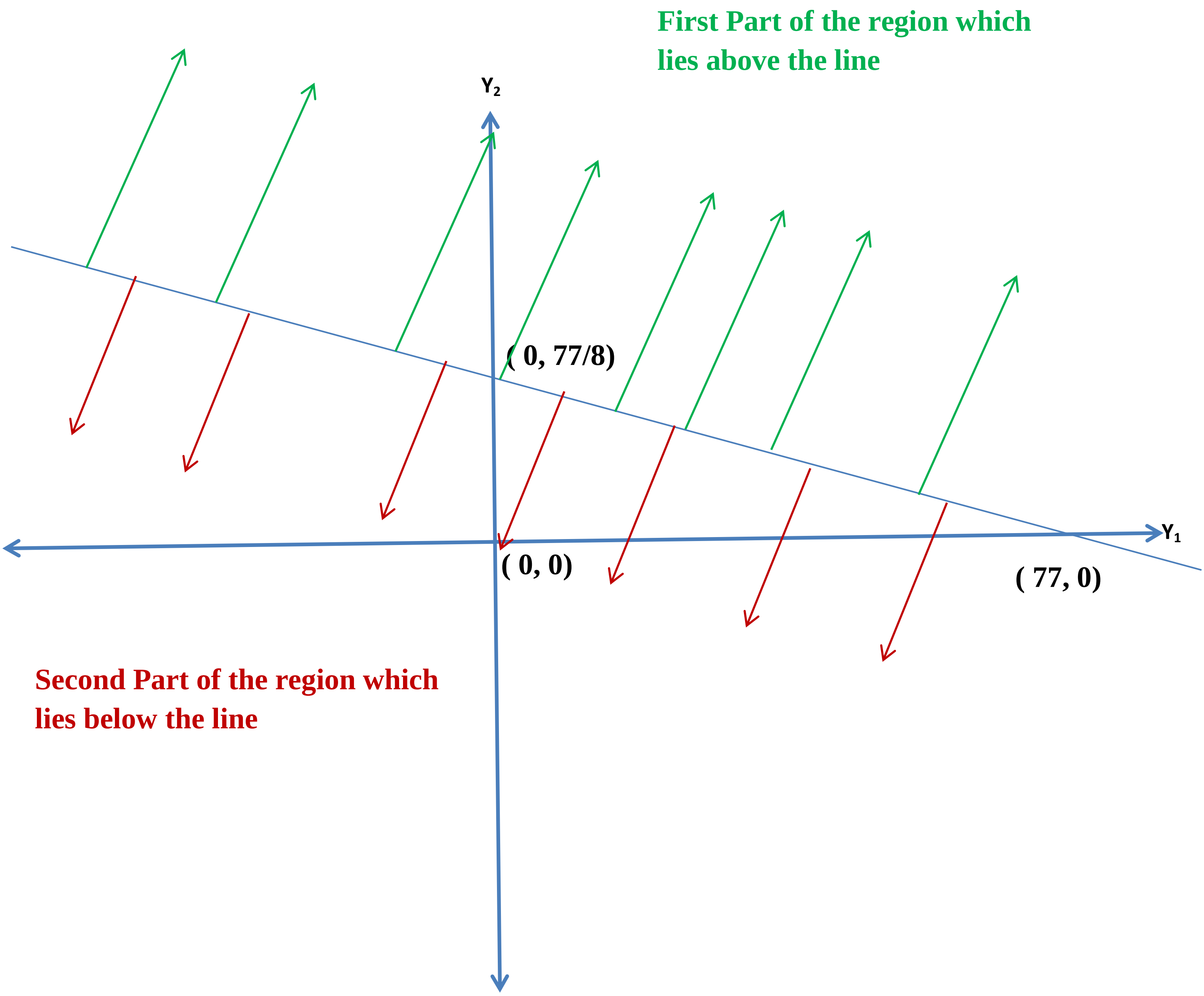
$y_1 + 8y_2 <= 77$

Assuming $y_1 = 0$, $y_1+8y_2 = 77$ implies $0 + 8y_2 = 77$ i.e., $y_2 = 77/8$

Therefore, first point is $(y_1,y_2)=(0, 77/8)$

Assuming $y_2 = 0$, $y_1+8y_2 = 77$ implies $y_1 = 77$ i.e., $y_1 = 77$

Therefore, second point is $(y_1,y_2)=(77, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

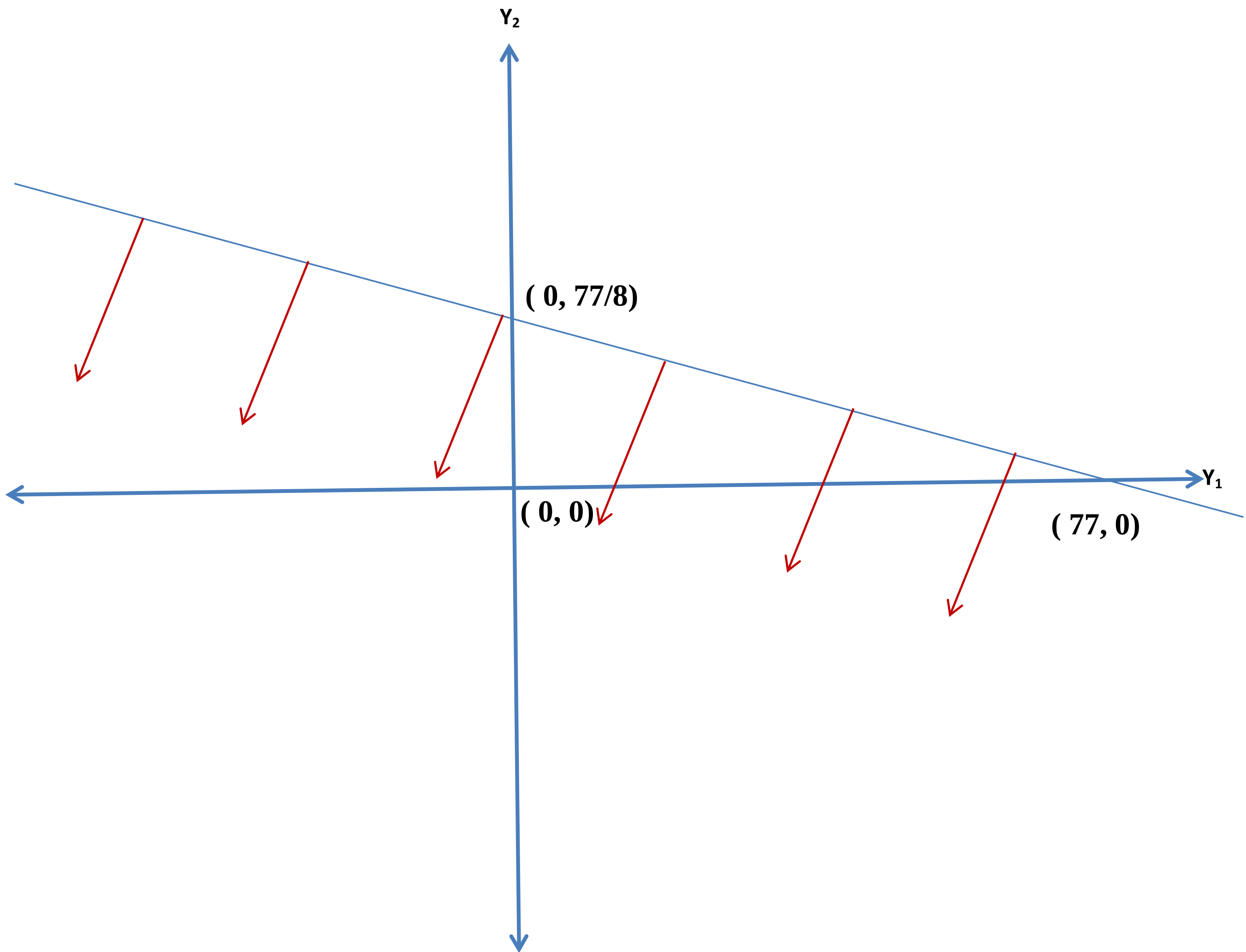
Putting (5,0) in the constraint

$y_1+8y_2 <= 77$,   we have

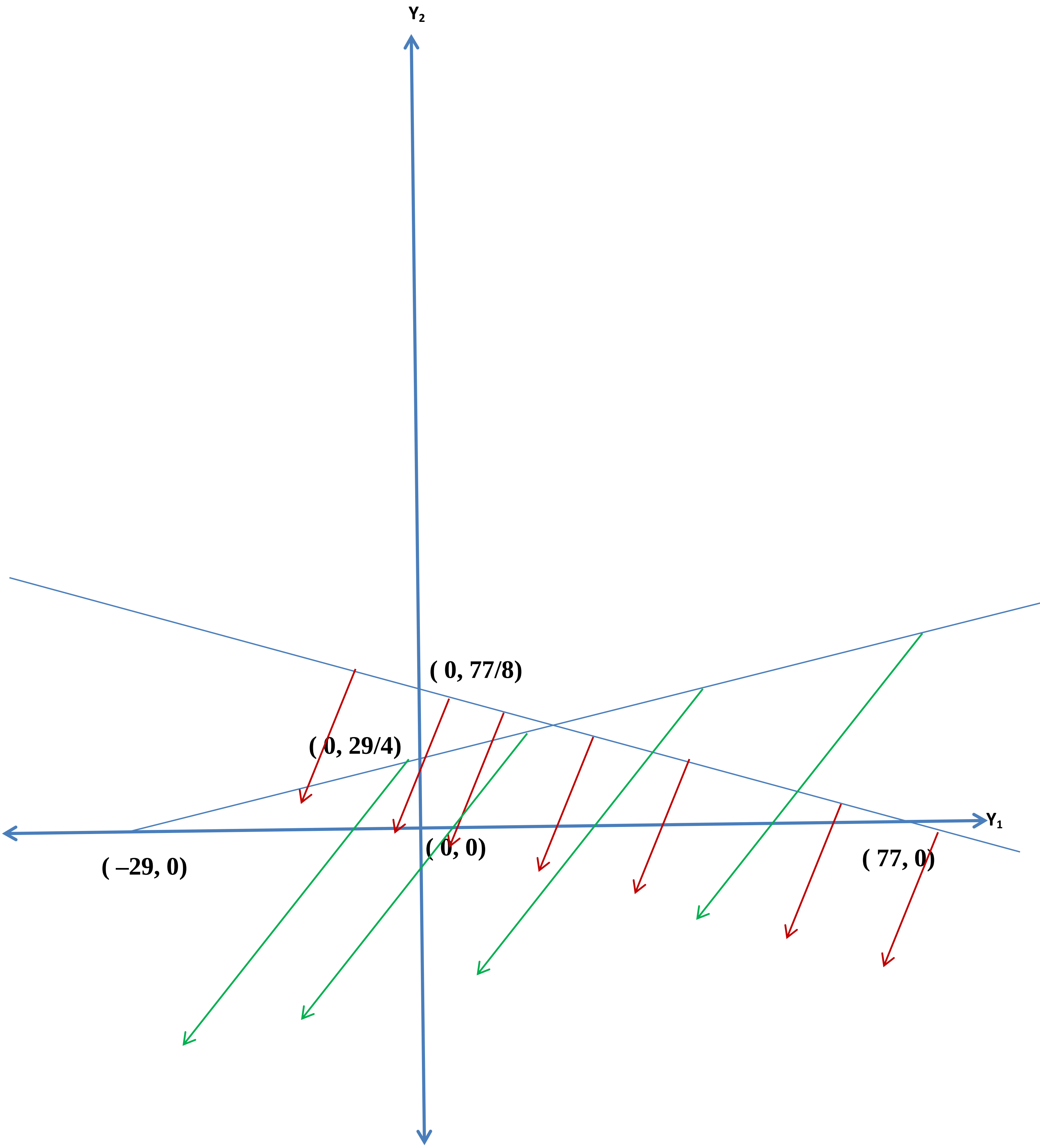
$5+0<=77$

$5<= 77$

It is obvious that the constraint is satisfying. So, we consider the second part.

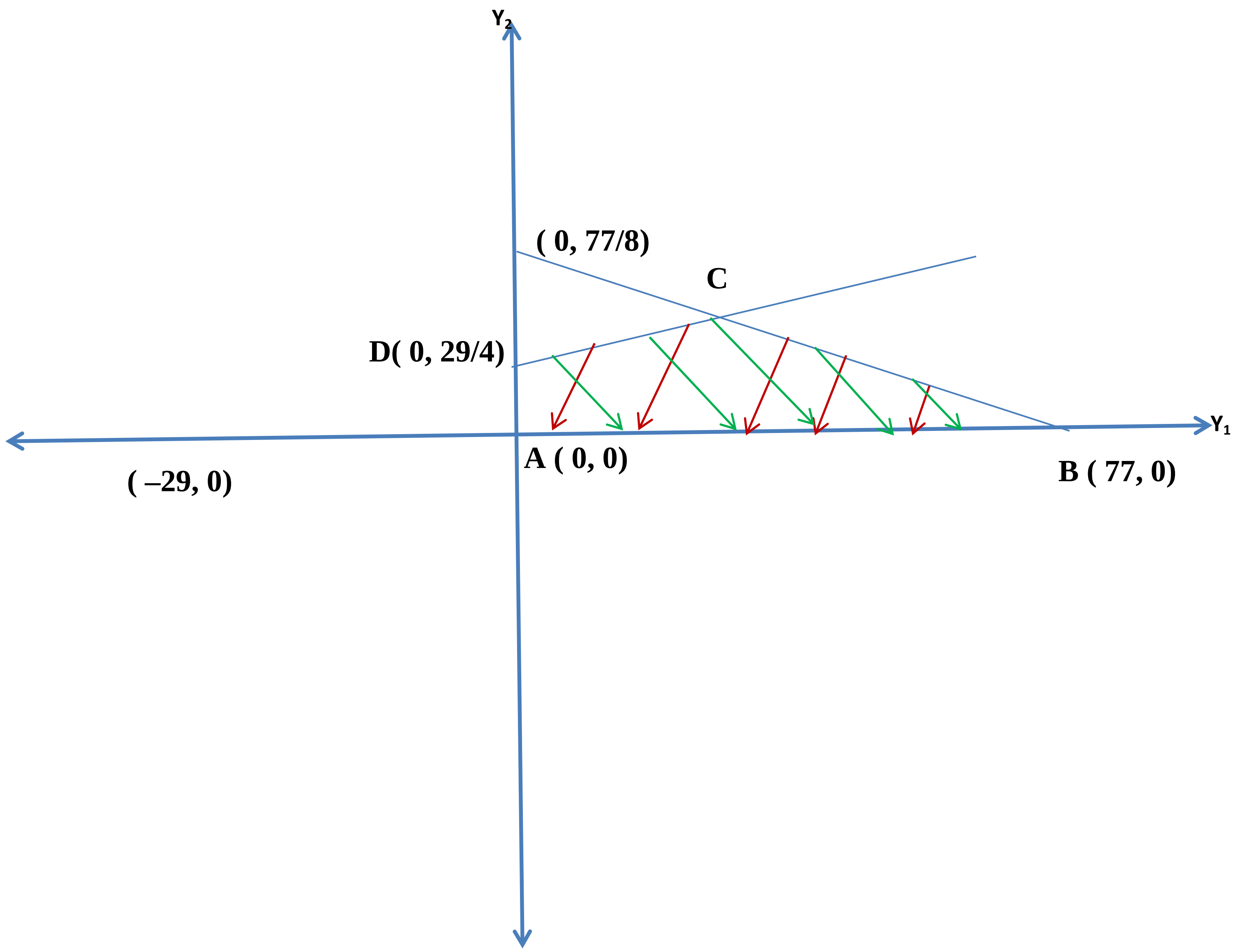


Common region of both the constraints



Common region of both the constraints in the first quadrant

(Feasible region)



It is obvious that Maximum value of y_1 and the maximum value of y_2 in the feasible region are finite. So, the feasible region is bounded.

Since, the feasible region is bounded so there is no need to add any constraint.

Extreme Points or Corner Points or Vertices

First point A

y_1=0 and y_2= 0

Therefore, A (0, 0)

Second point B

y_1=77 and y_2= 0

Therefore, B (77, 0)

Third point C

Intersection of $y_1 - 4y_2 = -29$ and $y_1 + 8y_2 = 77$,

On solving

y_1=19/3 and y_2= 53/6

Therefore, C (19/3, 53/6)

Fourth point D

Intersection of y_1= 0 and y_2 =29/4

Therefore, D (0, 29/4)

Value of objective function $3y_1 - 2y_2 + 22$ at

- **A (0, 0) is 3(0)– 2(0)+22=22**
- **B (77, 0) is 3(77)– 2(0)+22=253**
- **C (19/3, 53/6) is 3(19/3)– 2(53/6)+22=23.333**

- **D (0, 29/4) is 3(0)– 2(29/4)+22=14.75**

Maximum {22, 253, 23.33, 14.75} =253

Maximum value is 253 which is corresponding to $y_1=77$ and $y_2= 0$

So, in case of the maximization problem, the optimal solution is $y_1=77$ and $y_2= 0$ and the optimal value is 253

Using the relations

$x_1–2=y_1$ and $x_2+8=y_2$

The optimal solution is

$x_1=77+2=79$ and $x_2=y_2–8=0–8= –8$

Minimum {22, 253, 23.33, 14.75} =14.75

Minimum value is 14.75 which is corresponding to $y_1=0$ and $y_2= 29/4$

So, in case of the minimization problem, the optimal solution is $y_1=0$ and $y_2= 29/4$ and the optimal value is 14.75

Using the relations

$x_1–2=y_1$ and $x_2+8=y_2$

The optimal solution is

$x_1=0+2=2$ and $x_2=y_2–8=(29/4)–8= –0.75$

Maximize/Minimize $(3y_1 - 2y_2 + 22))$

Subject to

$y_1 - 4y_2 >= -29$,

$y_1 + 8y_2 >= 77$,

$y_1 >= 0$, $y_2 >= 0$

Draw First Cosnstraint

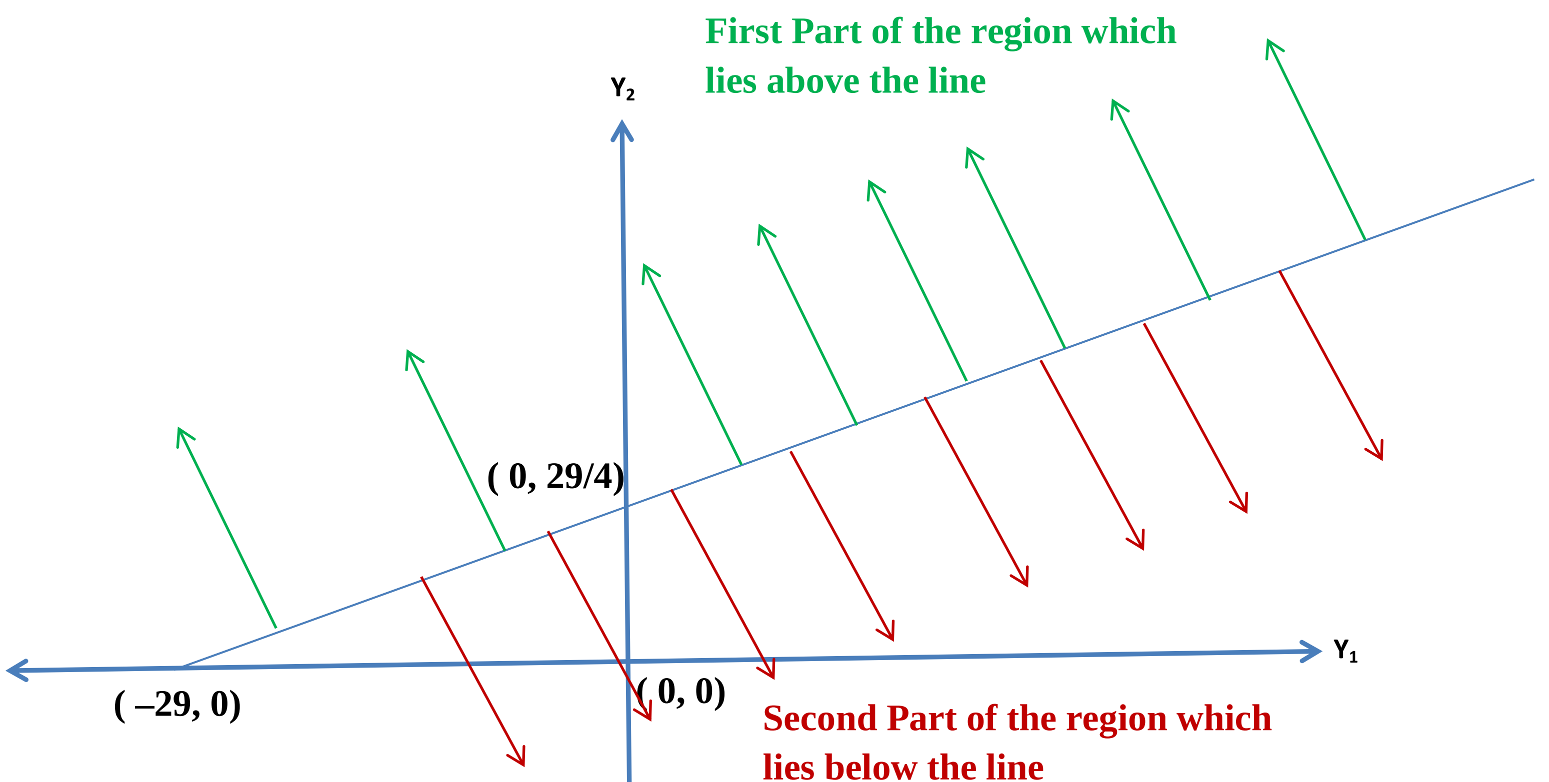
$y_1 - 4y_2 >= -29$

Assuming $y_1 = 0$, $y_1 - 4y_2 = -29$ implies $0 - 4y_2 = -29$ i.e., $y_2 = 29/4$

Therefore, first point is $(y_1, y_2) = (0, 29/4)$

Assuming $y_2 = 0$, $y_1 - 4y_2 = -29$ implies $y_1 - 0 = -29$ i.e., $y_1 = -29$

Therefore, second point is $(y_1, y_2) = (-29, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

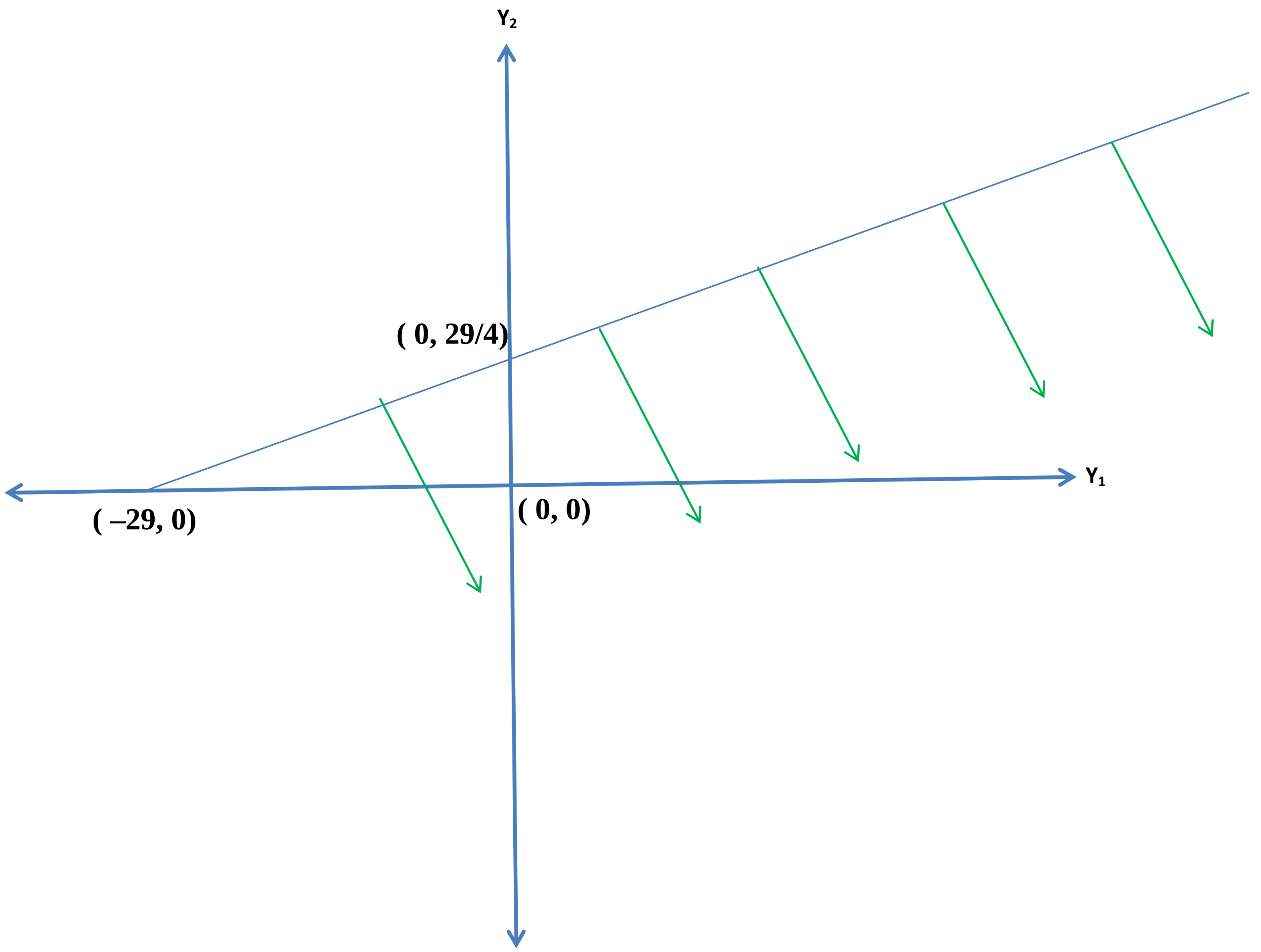
Putting (5,0) in the constraint

$y_1 - 4y_2 >= -29$, we have

$5 - 0 >= -29$

$5 >= -29$

It is obvious that the constraint is satisfying. So, we consider the second part.



Draw Second Cosnstraint

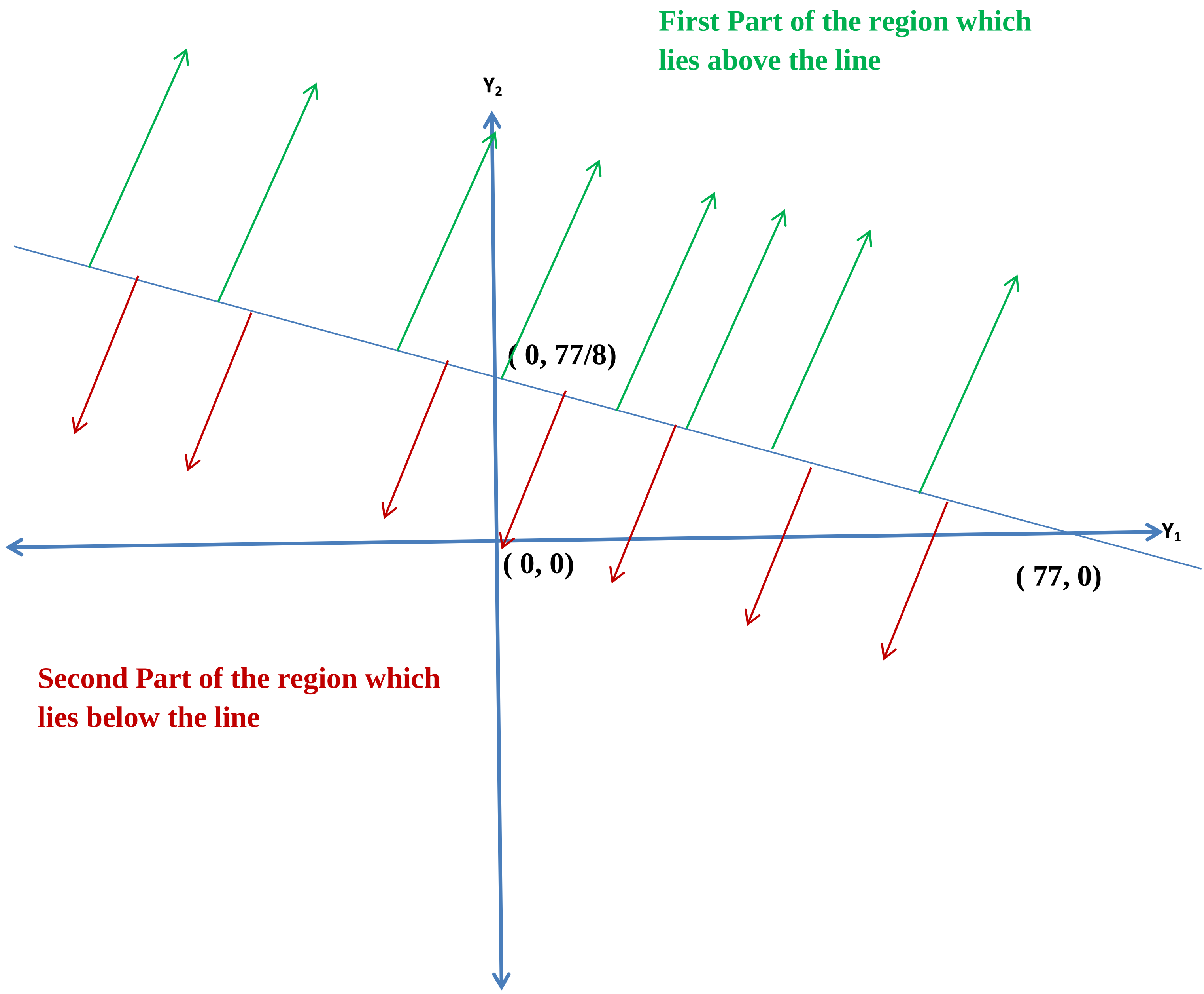
$y_1 + 8y_2 >= 77$

Assuming $y_1 = 0$, $y_1 + 8y_2 = 77$ implies $0 + 8y_2 = 77$ i.e., $y_2 = 77/8$

Therefore, first point is $(y_1, y_2) = (0, 77/8)$

Assuming $y_2 = 0$, $y_1 + 8y_2 = 77$ implies $y_1 = 77$ i.e., $y_1 = 77$

Therefore, second point is $(y_1, y_2) = (77, 0)$



(5, 0) lies in the second part of the region. If it will satisfy the constraint then we will consider the second part otherwise the first part.

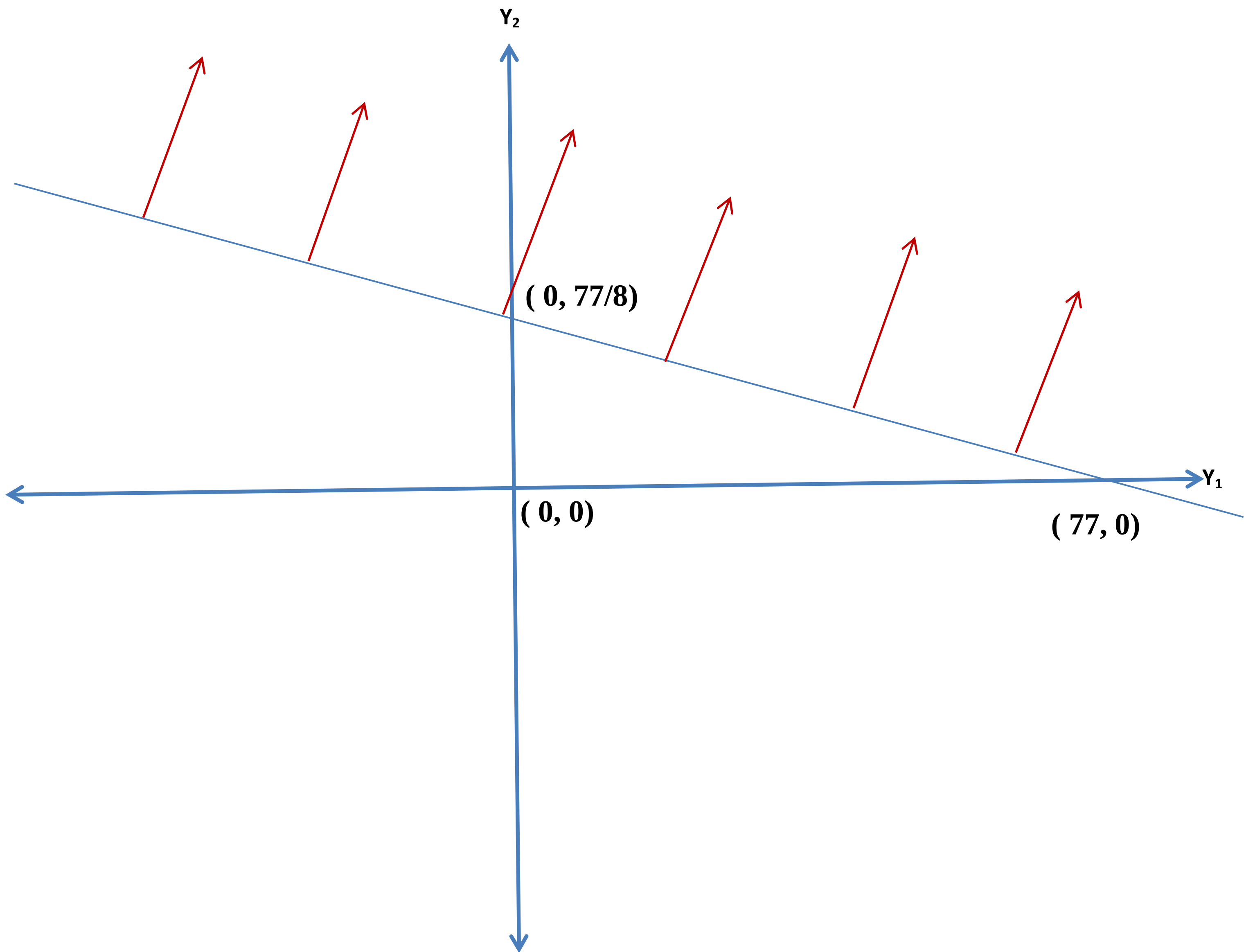
Putting (5,0) in the constraint

$y_1+8y_2 >= 77$, **we have**

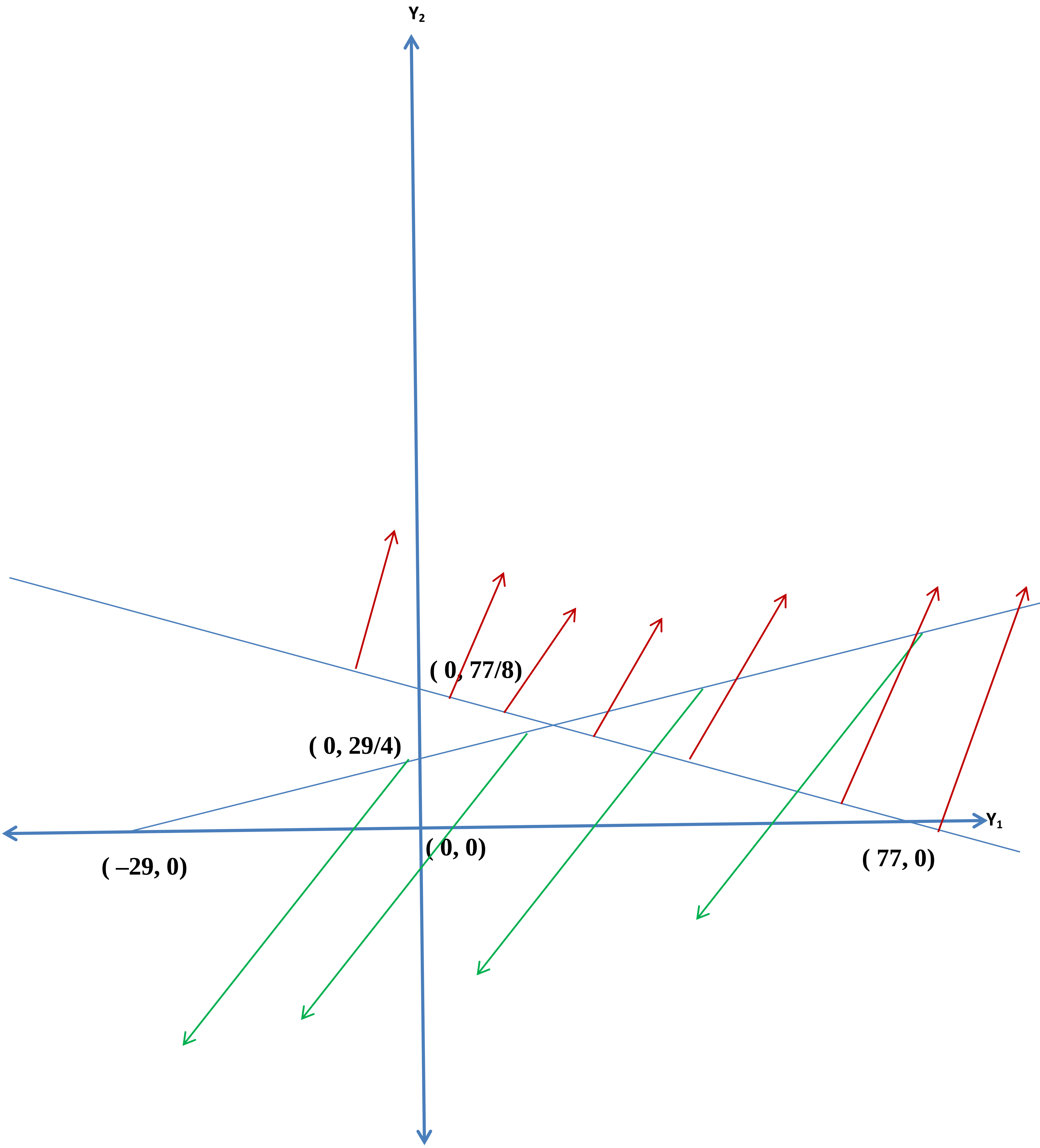
$5+0>=77$

$5>= 77$

It is obvious that the constraint is not satisfying. So, we consider the first part.

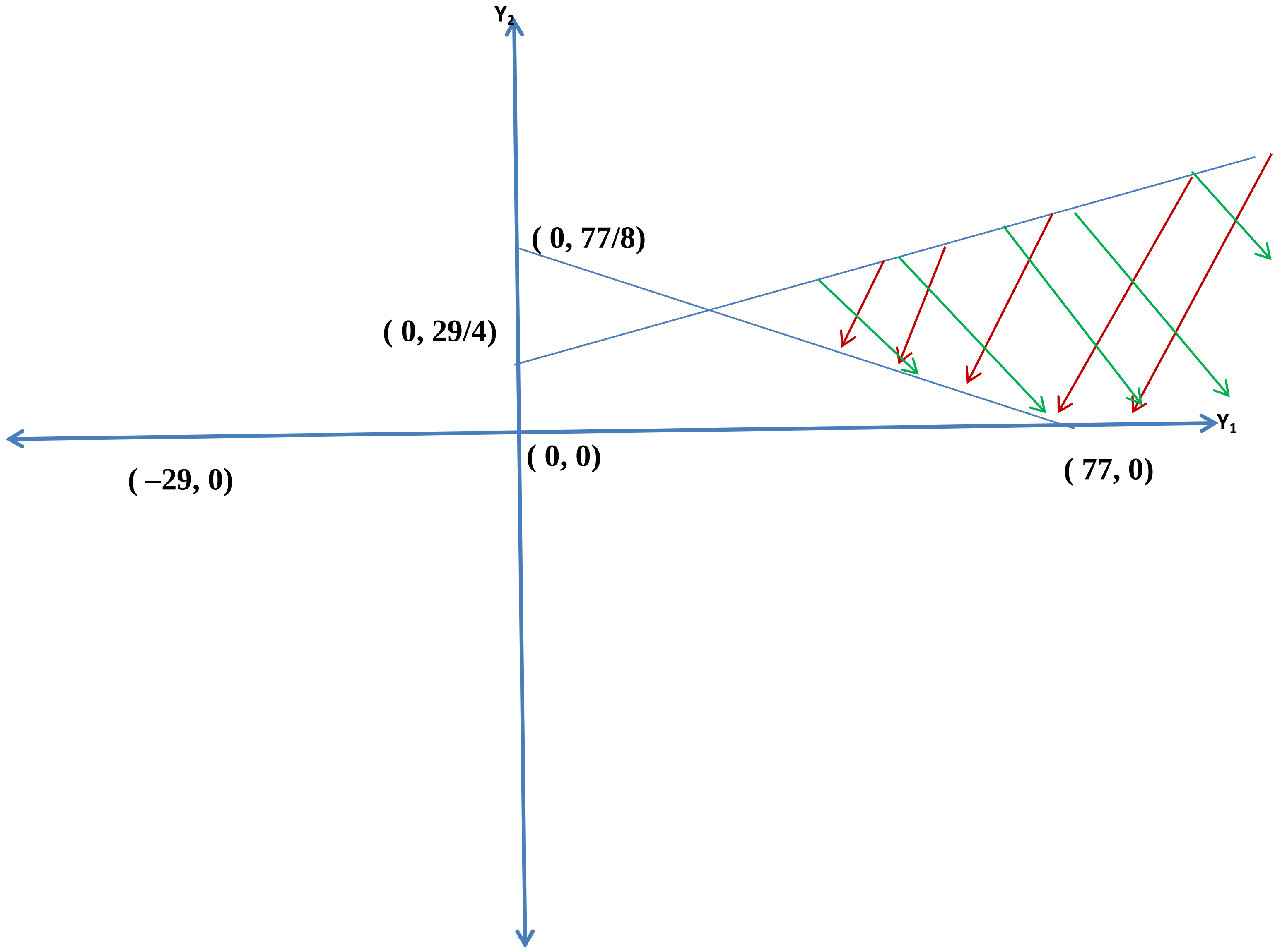


Common region of both the constraints



Common region of both the constraints in the first quadrant

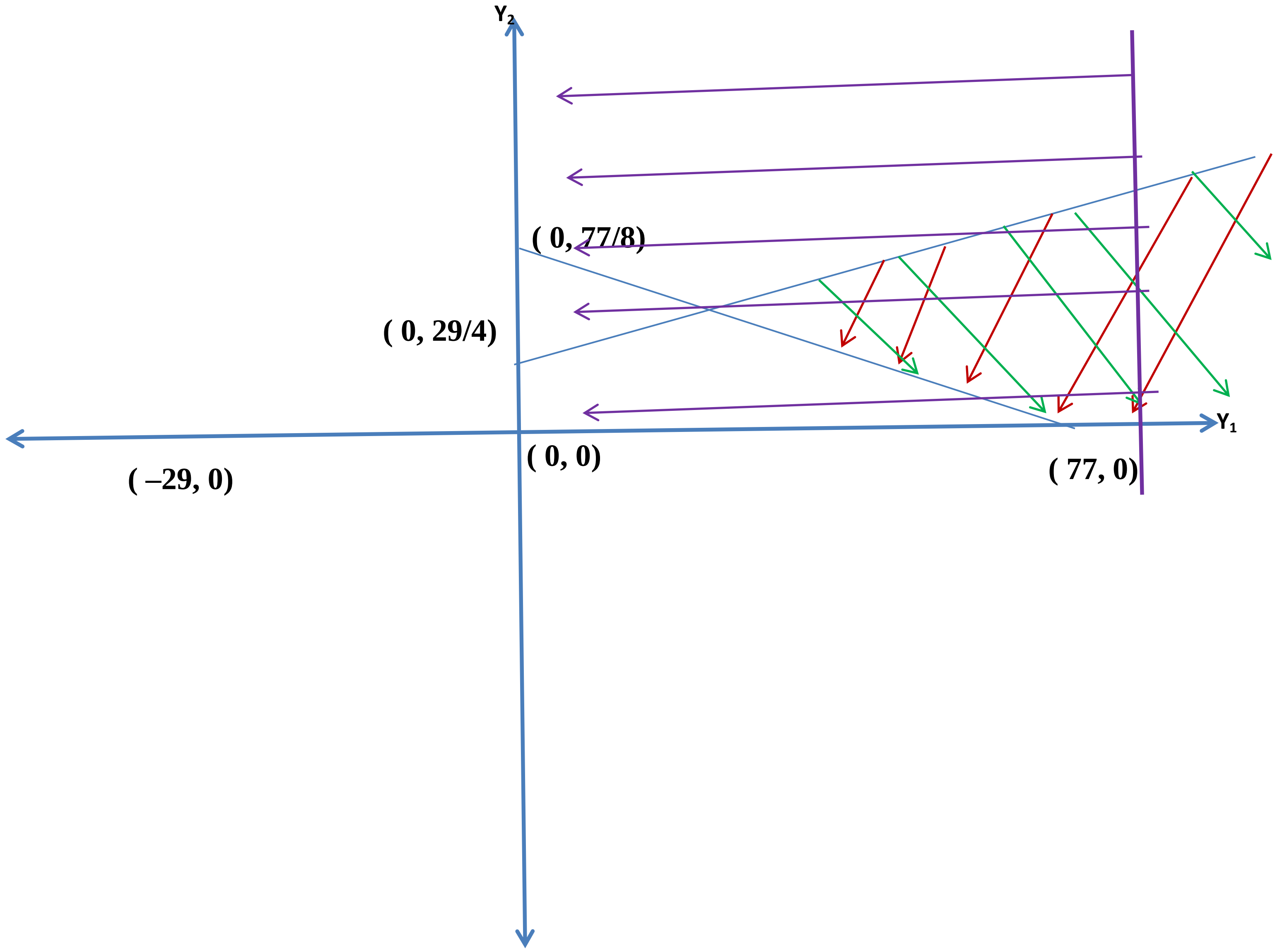
(Feasible region)

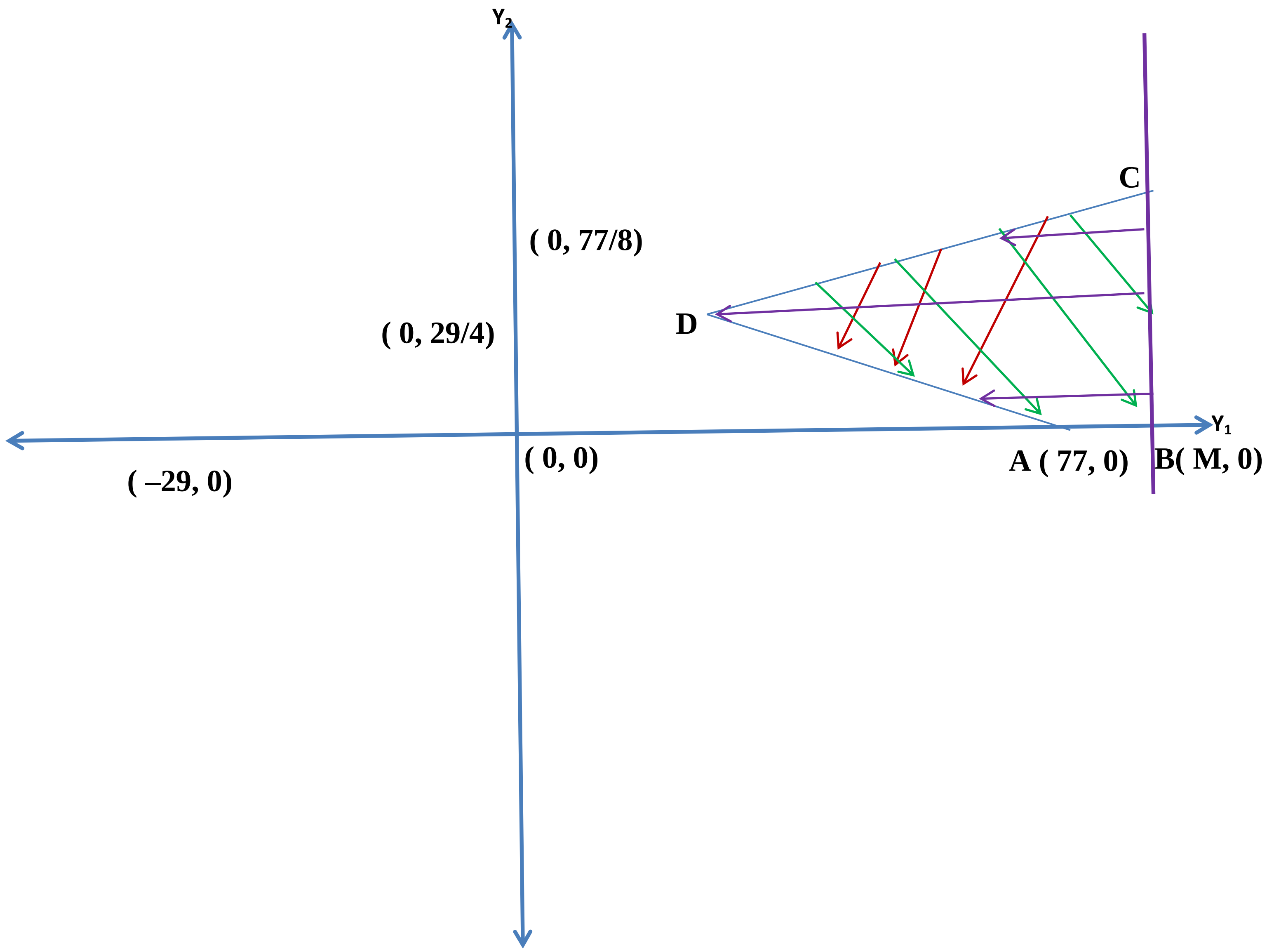


It is obvious that Maximum value of y_1 in the feasible region is infinite. So, the feasible region is unbounded.

Since, the maximum value of y_1 is infinite so there is need to add a constraint y_1<=M.

Including the constraint y_1<=M, the new feasible region is as follows:





Extreme Points or Corner Points or Vertices

First point A

y_1=77 and y_2= 0

Therefore, A (77, 0)

Second point B

y_1=M and y_2= 0

Therefore, B (M, 0)

Third point C

Intersection of $y_1 - 4y_2 = -29$ and y_1=M,

On solving

y_1=M and y_2= (M+29)/4

Therefore, C (M, (M+29)/4)

Fourth point D

Intersection of $y_1 - 4y_2 = -29$ and $y_1 + 8y_2 = 77$,

On solving

y_1=19/3 and y_2= 53/6

Therefore, D (19/3, 53/6)

Value of objective function $3y_1 - 2y_2 + 22$ at

- **A (77, 0) is 3(77)– 2(0)+22=253**
- **B (M, 0) is 3(M)– 2(0)+22=3M+22**
- **C (M, (M+29)/4) is 3(M)– 2((M+29)/4))+22=(5M+15)/2**
- **D (19/3, 53/6) is 3(19/3)– 2(53/6)+22=23.3333**

Maximum {253, 3M+22, (5M+15)/2, 23.33} =3M+22

Maximum value is 3M+22 which is depending upon M

So, in case of the maximization problem, the optimal solution is unbounded

Minimum {253, 3M+22, (5M+15)/2, 23.33} =23.33

Minimum value is 23.33 which is corresponding to y_1=19/3 and y_2= 53/6

So, in case of the minimization problem, the optimal solution is y_1=19/3 and y_2= 53/6 and the optimal value is 23.33

Using the relations

$x_1-2=y_1$ and $x_2+8=y_2$

The optimal solution is

$x_1=(19/3)+2=25/3$ and $x_2=(53/6)-8=5/6$