

Group Theory

Dr. Smita Agrawal
Assistant Professor, CSED
TIET, Patiala
smita.agrawal@thapar.edu

Contents

- Symmetric Group-Definition
- Cyclic Notation
- Composition

Symmetric Group

- A one-to-one and onto mapping f of a set $X = \{1, 2, \dots, n\}$ onto itself is called a permutation.
- Such a permutation may be denoted as:

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}, \text{ where } j_i = f(i)$$

- The set of all such permutations is generally denoted by S_n .

40g

Symmetric Group (Cont..)

- ❑ Here, S_n is called the symmetric group of degree n under operation composition of function.
- ❑ Since there are $\boxed{n!}$ such permutation operations, the order (number of elements) of the symmetric group S_n is $n!$.

Example

- ❑ Let us consider the symmetric group S_3 . Let $X = \{1, 2, 3\}$.
- ❑ Then, S_3 has $3! = 6$ elements:

$$\left[\begin{array}{c} \left(\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 2 \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 2 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \right) \end{array} \right]$$

Cycle Notation

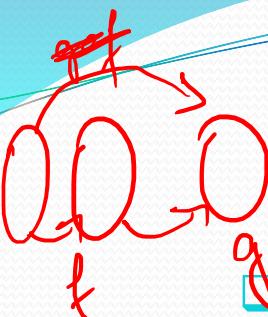
- Let $\{a_1, a_2, \dots, a_k\}$ be k distinct numbers between 1 and n . Then, the cycle (a_1, a_2, \dots, a_k) denotes the element of S_n that maps a_1 to a_2 , a_2 to a_3 , ..., a_{k-1} to a_k , a_k to a_1 , and leaves the remaining $n - k$ numbers fixed.
- The length of the cycle (a_1, a_2, \dots, a_k) is k .

Example

- let us consider the permutation $(315) \in S_5$.
- Then,

$$(315) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$$





Composition

$$\cancel{f \circ g(v) = f(g(v))}$$

- The group operation in a symmetric group is composition of function, denoted by the symbol \circ .
- The composition $f \circ g$ of permutations f and g maps any element $x \in X$ to $f(g(x))$.

Example

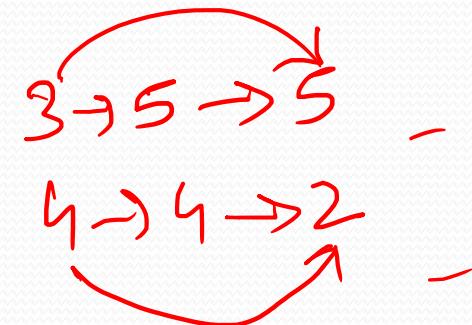
□ Let us consider S_6 .

□ Let $f = (1423) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 2 & 5 & 6 \end{pmatrix}$

□ And $g = (16235) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 4 & 1 & 2 \end{pmatrix}$

Then, $f \circ g = (1423)(16235) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 2 & 4 & 3 \end{pmatrix}$

$$g \circ f = (\underline{16235})(\underline{1423}) = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 2 & 4 & 3 \end{array} \right)$$



Composition (Cont..)

- A cycle of length $l = r \cdot p$, taken to the r^{th} power, will decompose into r cycles of length p .

Example

- Let us consider the permutation $(234561) \in S_6$. $\underline{l = 6}$
- $r = 2, p = 3$, then

$$\begin{aligned} \underline{(234561)^2} &= \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{array} \right) \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{array} \right) = \\ &= \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{array} \right) \quad \begin{array}{l} -1 \xrightarrow{3 \rightarrow 5} \\ -2 \xrightarrow{4 \rightarrow 6} \end{array} \quad \begin{array}{l} (135) \\ (246) \end{array} \end{aligned}$$

$= (135)(246)$

