Lecture-27_R

UEI407

Laplace Transform of Two Sided Functions (BLT):

Let us discuss about the Laplace transform of two sided function. Let us first consider a negative function and then positive function. Then we will consider the two sided function.

ROC for $f(t) = e^{bt}\, u(-t)$

Let us consider a function $f(t) = e^{bt}\, u(-t)$ which has been shown in Fig. 1. This a negative function of time and b is a positive quantity.

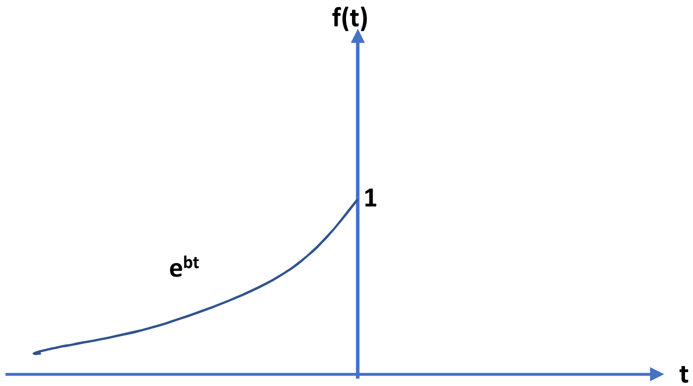

Fig.1 **$f(t) = e^{bt}u(-t)$**

The Laplace transform of this function is given by

$$LT\left[e^{bt}u(-t)\right] = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{0} f(t)e^{-st}dt + \int_{0}^{\infty} f(t)e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{bt}e^{-st}dt = \int_{-\infty}^{0} e^{-(s-b)t}dt$$

$$=\left[\frac{e^{-(s-b)t}}{-(s-b)}\right]_{-\infty}^{0}=\left[\frac{e^{-(s-b)t}}{(s-b)}\right]_{0}^{-\infty}$$

At t→-∞, $e^{-(s-b)t}=e^{-(\sigma+j\omega-b)t}\Big|_{t=-\infty}=e^{-(\sigma-b)t}e^{-j\omega t}\Big|_{t=-\infty}$

Therefore, $=e^{-(\sigma-b)t}e^{-j\omega t}\Big|_{t=-\infty}$ will be zero if and only if $\sigma < b$.

Hence $LT[f(t)]==e^{-(\sigma-b)t}e^{-j\omega t}\Big|_{t=-\infty}=\frac{e^{-\infty t}-e^{-0}}{(s-b)}=-\frac{1}{s-b}$ (1)

where Re(s) < b.

Figure 2 shows the ROC of $e^{bt}u(-t)$.

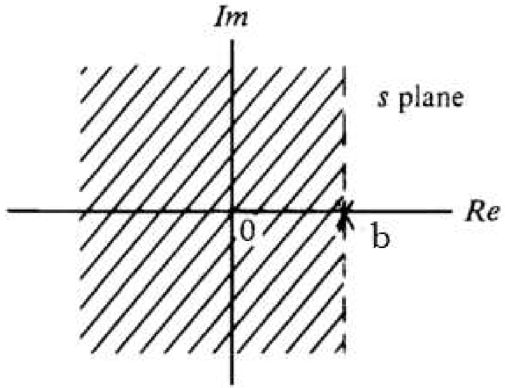

Fig. 2 ROC of $e^{bt}u(-t)$

ROC for $f(t) = e^{at}u(t)$

Let us consider a function $f(t) = e^{at}u(t)$ which has been shown in Fig.3.

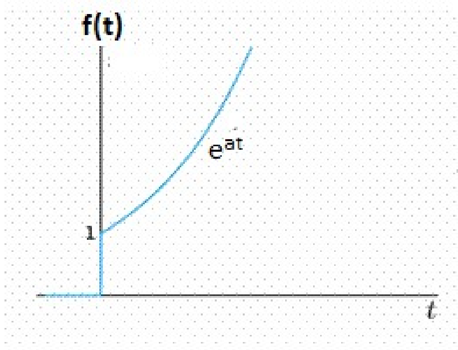

Fig.3 **$f(t) = e^{at}u(t)$**

The Laplace transform of this function is given by

$$LT\left[e^{at}u(t)\right] = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{0} f(t)e^{-st}dt + \int_{0}^{\infty} f(t)e^{-st}dt \quad = \int_{0}^{\infty} e^{-(s-a)t}dt = -\left[\frac{e^{-(s-a)t}}{s-a}\right]_{0}^{\infty}$$

At $t\to\infty$, $e^{-(s-a)t} = e^{-(\sigma+j\omega-a)t}\Big|_{t=\infty} = e^{-(\sigma-a)t}e^{-j\omega t}\Big|_{t=\infty}$

Therefore, $e^{-(\sigma-a)t}e^{-j\omega t}\Big|_{t=\infty}$ will be zero if and only if σ >a.

$$\text{Hence } LT[f(t)] = e^{-(\sigma-a)t}e^{-j\omega t}\Big|_{t=\infty} = \frac{e^{-\infty t} - e^{-0}}{-(s-a)} = \frac{1}{s-a} \tag{2}$$

where Re(s)>a.

Figure 4 shows the ROC of e ^{at}u(t).

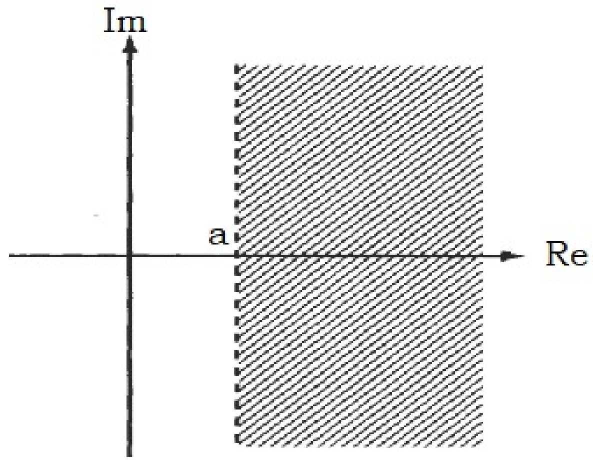

Fig. 4 ROC of e ^{at}u(t)

ROC for Two sided function

Let us consider the bilateral transfer function. Mathematically, we can write this function as

$f(t) = e^{\beta t}$ for $t < 0$ and

$f(t) = e^{\alpha t}$ for $t > 0$.

Figure 5 shows the function.

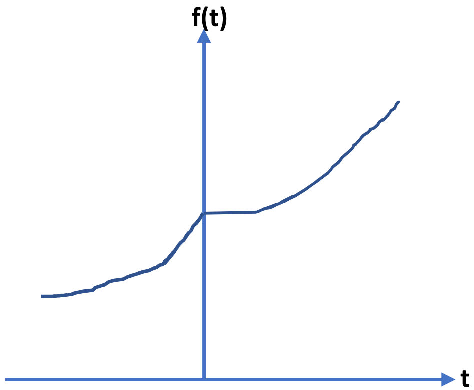

Fig. 5

$$LT[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{0} f(t)e^{-st}dt + \int_{0}^{\infty} f(t)e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{\beta t}e^{-st}dt + \int_{0}^{\infty} e^{\alpha t}e^{-st}dt = -\frac{1}{s-\beta} + \frac{1}{s-\alpha} = \frac{1}{s-\alpha} - \frac{1}{s-\beta} \qquad (3)$$

The first LT exits only if $\sigma < \beta$ and the second one exist only if $\sigma > \alpha$. Hence the both conditions will be satisfied if and only if $\alpha<\sigma<\beta$. Therefore, Re(s) must lie between α and β. For $\beta<\alpha$, the Laplace transform does not exist. Figure 6 shows the ROC of the two sided function.

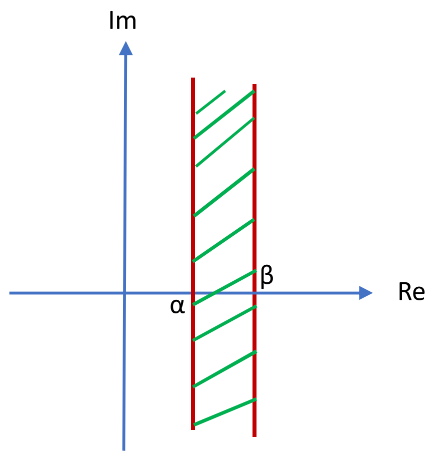

Fig. 6 ROC of Two Sided Function

Example 1 Find the Laplace transform of

$$f(t) = e^{3t}u(-t) + e^{t}u(t)$$

Solution:

We know that $LT\left[e^{\beta t}u(-t)\right] = -\dfrac{1}{s-\beta}$ with ROC $Re(s) < \beta$ and

$$LT\left[e^{\alpha t}u(t)\right] = \frac{1}{s-\alpha} \qquad \text{with ROC Re(s)} > \alpha.$$

Therefore, $LT\left[e^{3t}u(-t)\right] = -\frac{1}{s-3}$ with ROC Re(s) < 3 and $LT\left[e^{t}u(t)\right] = \frac{1}{s-1}$

with ROC Re(s) > 1.

Figure E1 shows the combined ROC for which the LT exists. The region of convergence is 1 < Re(s) < 3.

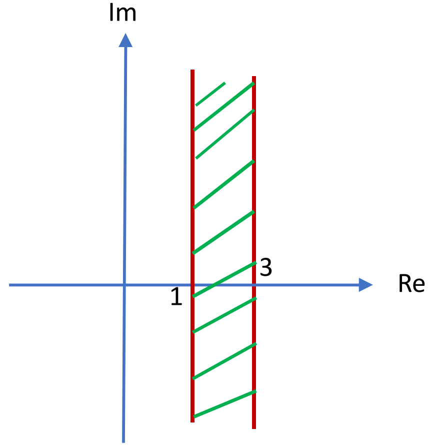

Fig. E1

$$\therefore F(s) = -\frac{1}{s-3} + \frac{1}{s-1} = \frac{-(s-1)+(s-3)}{(s-1)(s-3)} = \frac{-2}{(s-1)(s-3)}$$

Example 2 Determine the Laplace transform of $f(t) = e^{-6t}u(-t) + e^{5t}u(t)$

Solution: $f(t) = e^{\beta t}u(-t) + e^{\alpha t}u(t)$

Let Laplace transform of f(t) will exist if and only if $\alpha<\beta$.

Here $\beta = -6$ and $\alpha = 5$. Hence $\alpha > \beta$. Therefore, Laplace transform of f(t) does not exist.

Initial Value Theorem

Since $LT\left[\frac{df(t)}{dt}\right] = \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt \quad = s\,F(s) - f(0)$

Taking $s \to \infty$

$$\underset{s\to\infty}{Lt}\left[LT\left\{\frac{df(t)}{dt}\right\}\right] = \underset{s\to\infty}{Lt}\left[sF(s) - f(0)\right]$$

or, $$\lim_{s\to\infty}\int_0^{\infty}\frac{df(t)}{dt}e^{-st}dt = \lim_{s\to\infty}[sF(s)-f(0)]$$

At $s \to \infty$, $e^{-st} \to 0$

$$\lim_{s\to\infty}[sF(s)-f(0)] = 0$$

Since f (0) is a constant,

$$\therefore\ f(0) = \lim_{s\to\infty} sF(s)$$

$$\therefore\ \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad (4)$$

Equation (4) gives the initial value of the time domain solution f(t)

Final Value Theorem

Since LT $\left[\frac{df(t)}{dt}\right] = \int_0^{\infty}\frac{df(t)}{dt}e^{-st}dt \quad = s\,F(s) - f\,(0)$

Taking $s \to 0$

$$\underset{s \to 0}{Lt} \left[LT \left\{ \frac{df(t)}{dt} \right\} \right] = \underset{s \to 0}{Lt} [sF(s) - f(0)]$$

or, $$\underset{s \to 0}{Lt} \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \underset{s \to 0}{Lt} [sF(s) - f(0)]$$

Since at $s \to 0$, $e^{-st} \to 1$

or, $$f(\infty) - f(0) = -f(0) + \underset{s \to 0}{Lt} sF(s)$$

or, $$\underset{s \to 0}{Lt} sF(s) = f(\infty)$$

or, $$\underset{t \to \infty}{Lt} [f(t)] = \underset{s \to 0}{Lt} sF(s) \qquad (5)$$

Equation (5) gives the final value of the time domain solution f(t)

Example 3 Determine the initial and final value of the current where

$$I(s) = \frac{0.52}{s\left(s^2 + 0.45s + 0.818\right)}$$

Solution (i) Applying the initial value of theorem, it can be written that,

$$i(0) = \underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ sI(s)$$

$$= \underset{s\to\infty}{Lt}\ s\left[\frac{0.52}{s\left(s^2 + 0.45s + 0.818\right)}\right]$$

$$= \underset{s\to\infty}{Lt} \frac{0.52}{s^2 + 0.45s + 0.818}$$

$$= \frac{\underset{s\to\infty}{Lt} \frac{0.52}{s^2}}{\underset{s\to\infty}{Lt}\left[1 + \frac{0.45}{s} + \frac{0.188}{s^2}\right]}$$

$$= \frac{0}{1 + 0 + 0}$$

$= 0.$

(ii) Using final value of theorem, it can be written as

$$i(\infty) = \underset{s\to 0}{\mathrm{Lt}}\ sF(s) = \underset{s\to 0}{\mathrm{Lt}}\ sI(s)$$

$$= \underset{s\to 0}{\mathrm{Lt}}\ s\left[\frac{0.52}{s\left(s^2+0.45s+0.818\right)}\right]$$

$$= \underset{s\to 0}{\mathrm{Lt}}\ \frac{0.52}{s^2+0.45s+0.818} = \frac{0.52}{0.818} = 0.6357\ \text{A}$$

Therefore the initial current is 0 A and final current is 0.6357 A.

Example Determine f(t) with $F(s) = \frac{s+2}{(s+3)(s+4)}$ with

(i) Re(s) < –4

(ii) Re(s) > –3 and

(iii) Re(s) lying between –3 and –4.

Solution:

$$F(s) = \frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$\therefore A = \left[(s+3)\frac{s+2}{(s+3)(s+4)}\right]_{s=-3} = -1$$

and $B = \left[(s+4)\frac{s+2}{(s+3)(s+4)}\right]_{s=-4} = 2$

$$\therefore F(s) = \frac{-1}{s+3} + \frac{2}{s+4}$$

Re(s)< –4

Here both poles lie to the right of ROC. Therefore, both poles contribute to negative time function.

We know that $LT^{-1}\left[\frac{1}{s+3}\right] = -e^{-3t}u(-t)$.

$$\therefore LT^{-1}\left[\frac{-1}{s+3}\right] = e^{-3t}u(-t)$$

Again, $LT^{-1}\left[\frac{1}{s+4}\right] = -e^{-4t}u(-t)$.

$$\therefore LT^{-1}\left[\frac{2}{s+4}\right] = -2e^{-4t}u(-t)$$

$$LT^{-1}[F(s)] = e^{-3t}u(-t) - 2e^{-4t}u(-t)$$

Re(s) > –3

In this case both poles lie to the left of ROC. Therefore, both poles contribute to the positive time function.

We know that $LT^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}u(t)$.

$\therefore LT^{-1}\left[\frac{-1}{s+3}\right] = -e^{-3t}u(t)$

Again, $LT^{-1}\left[\frac{1}{s+4}\right] = e^{-4t}u(t)$.

$\therefore LT^{-1}\left[\frac{2}{s+4}\right] = 2e^{-4t}u(t)$

$$LT^{-1}[F(s)] = e^{-3t}u(t) + 2e^{-4t}u(t) = \left[e^{-3t} + 2e^{-4t}\right]u(t)$$

$-4 < Re(s) < -3$

Since the pole s = –3 lies to the right of ROC, hence it contributes to the negative function. On the other hand, s=–4 lies to the left of ROC, hence it contributes to the positive function.

We know that $LT^{-1}\left[\frac{1}{s+3}\right] = -e^{-3t}u(-t)$.

$\therefore LT^{-1}\left[\frac{-1}{s+3}\right] = e^{-3t}u(-t)$

Again, $LT^{-1}\left[\frac{1}{s+4}\right] = e^{-4t}u(t)$.

$\therefore LT^{-1}\left[\frac{2}{s+4}\right] = 2e^{-4t}u(t)$

$$LT^{-1}[F(s)] = e^{-3t}u(-t) + 2e^{-4t}u(t)$$

Example Determine f (t) for system whose F (s) = $\frac{s+3}{s(s+1)^2(s+2)}$. Use Heaviside theorem.

Solution F (s) $= \frac{s+3}{s(s+1)^2(s+2)}$

$$= \frac{a_1}{s} + \frac{a_{21}}{(s+1)^2} + \frac{a_{22}}{s+1} + \frac{a_3}{s+2}$$

$$\therefore\ a_1 = \left[s\frac{(s+3)}{s(s+1)^2(s+2)}\right]_{s=0} = \left[\frac{s+3}{(s+1)^2(s+2)}\right]_{s=0} = \frac{3}{2}$$

$$\therefore\ a_{21} = \left[(s+1)^2\frac{(s+3)}{s(s+1)^2(s+2)}\right]_{s=-1} = \left[\frac{s+3}{s(s+2)}\right]_{s=-1}$$

$$= \frac{-1+3}{(-1)(-1+2)} = \frac{2}{(-1)(1)} = -2$$

$$\therefore\ a_{22} = \left[\frac{d}{ds}\left\{(s+1)^2\frac{(s+3)}{s(s+1)^2(s+2)}\right\}\right]_{s=1} = \left[\frac{d}{ds}\left\{\frac{(s+3)}{s(s+2)}\right\}\right]_{s=1}$$

$$= \left[\frac{(s^2+2s)-(s+3)(2s+2)}{(s^2+2s)^2}\right]_{s=-1} = \frac{(1-2)-(-1+3)(-2+2)}{(1-2)^2}$$

$$= \frac{-1}{1} = -1$$

$$\therefore\ a_3 = \left[(s+2)\frac{(s+3)}{s(s+1)^2(s+2)}\right]_{s=-2} = \left[\frac{s+3}{s(s+1)^2}\right]_{s=-2}$$

$$\therefore\ = \frac{-2+3}{(-2)(-2+1)^2} = -\frac{1}{2}$$

$$\therefore\ F(s) = \frac{3/2}{s} + \frac{-2}{(s+1)^2} + \frac{-1}{(s+1)} + \frac{-\frac{1}{2}}{s+2}$$

Taking inverse Laplace transform, we have

$$f\ (t) = \frac{3}{2} - 2te^{-t} - e^{-t} - \frac{1}{2}e^{-2t}$$