Course: UMA 035 (Optimization Techniques)

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Remark:

The minimum/maximum value, obtained on solving a NLPP, may be local minimum/maximum instead of global minimum/maximum. However, if the considered NLPP is a convex programming problem. Then, the obtained minimum/maximum will be surely global minimum/maximum.

Convex programming problem

Minimize (f(X))

Subject to

g(X)<=0

where, f(X) and g(X) are convex functions.

OR

Minimize (f(X))

Subject to

g(X) >=0

where, f(X) is a convex function and g(X) is a concave function.

OR

Maximize (f(X))

Subject to

g(X) <=0

where, f(X) is a concave function and g(X) is a convex function.

OR

Max (f(X))

Subject to

g(X) >=0

where, f(X) and g(x) are concave functions.

Convex function

Let S be a convex subset in R^n . A function f: S→ R is said to be a convex function if

$f(\lambda x_1+(1-\lambda)\, x_2) <= \lambda f(x_1)+ (1-\lambda)\, f(x_2)$ for all x_1 , x_2 belongs to S and $0 \leq \lambda \leq 1$

Strictly convex function

Let S be a convex subset in R^n . A function f: S→ R is said to be a strictly convex function if

$f(\lambda x_1+(1-\lambda) x_2)< \lambda f(x_1)+ (1-\lambda) f(x_2)$ for all x_1 , x_2 belongs to S and $0 \leq \lambda \leq 1$

Example

Check that function $f(x)=|x|$ is a convex function or not.

Solution

$f(\lambda x_1+(1-\lambda) x_2)$

$= |\lambda x_1 + (1 - \lambda)x_2|$

$\leq |\lambda x_1| + |(1 - \lambda)x_2|$

$\leq \lambda|x_1| + (1 - \lambda)|x_2|$

$\leq \lambda f(x_1) + (1 - \lambda)f(x_2)$

Hence, f(x) is a convex function.

Concave function

Let S be a convex subset of R (set of real numbers). A function f: S→ R is said to be a concave function if

$f(\lambda x_1+(1-\lambda) x_2)>= \lambda f(x_1)+ (1-\lambda) f(x_2)$ for all x_1 , x_2 belongs to S and $0 \leq \lambda \leq 1$

Strictly Concave function

Let S be a convex subset in R^n (set of real numbers). A function f: S→ R is said to be a strictly concave function if

$f(\lambda x_1+(1-\lambda)\, x_2) > \lambda f(x_1)+ (1-\lambda)\, f(x_2)$ for all x_1 , x_2 belongs to S and $0 \leq \lambda \leq 1$

Example

Check that function $f(x)=-|x|$ is a concave function or not.

Solution

$f(\lambda x_1+(1-\lambda)\, x_2)$

$= -|\lambda x_1 + (1-\lambda)x_2|$

$\geq -|\lambda x_1| - |(1-\lambda)x_2|$

$\geq -\lambda|x_1| - (1-\lambda)|x_2|$

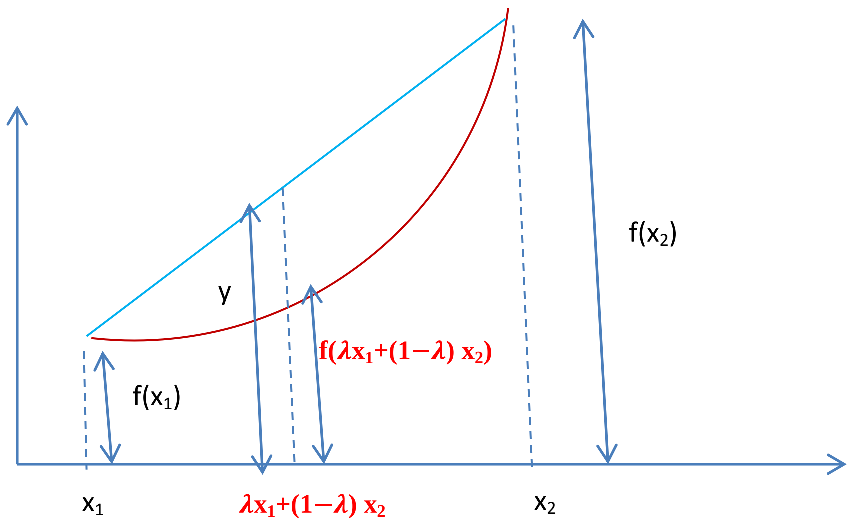
$\geq \lambda(-|x_1|) + (1-\lambda)(-|x_2|)$

$\geq \lambda f(x_1) + (1-\lambda)f(x_2)$

Hence, f(x) is a concave function.

Graphical interpretation o f a concave set

A chord, joining any two points on the curve, lies on or above the curve.



Equation of line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is

$$y - f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1)$$

The value of y at $x = \lambda x_1 + (1-\lambda) x_2$ is

$$y - f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + (1-\lambda) x_2 - x_1)$$

$$y = f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + (1-\lambda) x_2 - x_1)$$

It is obvious from the graph that

$$y >= f(\lambda x_1 + (1-\lambda) x_2)$$

$$\Rightarrow f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + (1-\lambda) x_2 - x_1) >= f(\lambda x_1 + (1-\lambda) x_2)$$

$$\Rightarrow f(x_1)+\frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1+x_2-\lambda x_2-x_1)>= f(\lambda x_1+(1-\lambda)\,x_2)$$

$$\Rightarrow f(x_1)+\frac{f(x_2)-f(x_1)}{x_2-x_1}(-\lambda(x_2-x_1)+(x_2-x_1))>= f(\lambda x_1+(1-\lambda)\,x_2)$$

$$\Rightarrow f(x_1)+f(x_2)(-\lambda+1)-f(x_1)(-\lambda+1)>= f(\lambda x_1+(1-\lambda)\,x_2)$$

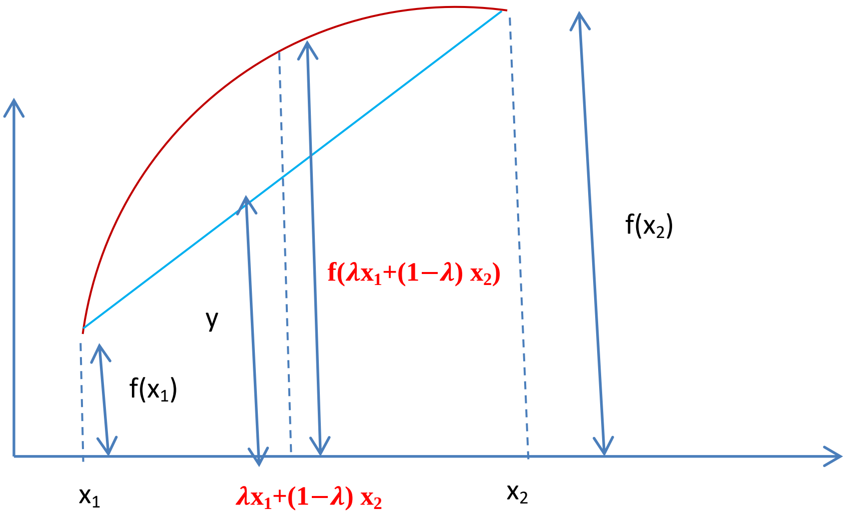
$$\Rightarrow f(x_1)+-\lambda f(x_2)+f(x_2)+\lambda f(x_1)-f(x_1)>= f(\lambda x_1+(1-\lambda)\,x_2)$$

$$\Rightarrow -\lambda f(x_2)+f(x_2)+\lambda f(x_1)>= f(\lambda x_1+(1-\lambda)\,x_2)$$

$$\Rightarrow (1-\lambda)f(x_2)+\lambda f(x_1)>= f(\lambda x_1+(1-\lambda)\,x_2)$$

Graphical interpretation o f a concave set

A chord, joining any two points on the curve, lies on or below the curve.



Equation of line joining the points (x_1, f(x_1)) and (x_2, f(x_2)) is

$$y - f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1)$$

The value of y at $x = \lambda x_1 + (1-\lambda)\, x_2$ is

$$y - f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + (1-\lambda)\, x_2 - x_1)$$

$$y = f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + (1-\lambda)\, x_2 - x_1)$$

It is obvious from the graph that

$y <= f(\lambda x_1 + (1-\lambda)\, x_2)$

$$\Rightarrow f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + (1-\lambda)\, x_2 - x_1) <= f(\lambda x_1 + (1-\lambda)\, x_2)$$

$$\Rightarrow f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(\lambda x_1 + x_2 - \lambda x_2 - x_1) <= f(\lambda x_1 + (1-\lambda)\, x_2)$$

$$\Rightarrow f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(-\lambda(x_2 - x_1) + (x_2 - x_1)) <= f(\lambda x_1 + (1-\lambda)\, x_2)$$

$\Rightarrow$ $f(x_1)+f(x_2)(-\lambda+1)-f(x_1)(-\lambda+1) <= f(\lambda x_1+(1-\lambda)\, x_2)$

$\Rightarrow$ $f(x_1)+-\lambda f(x_2)+f(x_2)+\lambda f(x_1)-f(x_1) <= f(\lambda x_1+(1-\lambda)\, x_2)$

$\Rightarrow$ $-\lambda f(x_2)+f(x_2)+\lambda f(x_1) <= f(\lambda x_1+(1-\lambda)\, x_2)$

$\Rightarrow$ $(1-\lambda)f(x_2)+\lambda f(x_1) <= f(\lambda x_1+(1-\lambda)\, x_2)$

Alternative method to check function is convex or not

Find the Hessian matrix of the given function. If the Hessian matrix is either positive definite or positive semi-definite then the function will be a convex function.

Find the Hessian matrix of the given function. If the Hessian matrix is either negative definite or negative semi-definite then the function will be a concave function.

Example:

Check that the function $f(x)=x_1^2+x_2^2+x_3^2$ is convex or not.

Solution

$$H=\begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_1 \partial x_3} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \frac{\partial^2 y}{\partial x_2 \partial x_3} \\ \frac{\partial^2 y}{\partial x_3 \partial x_1} & \frac{\partial^2 y}{\partial x_3 \partial x_3} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$D_1 = 1 > 0$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0$$

H is positive definite.

Hence, the given function is a convex function

Example:

Check that the function $f(x)=x_1^2+x_2^2+x_3^2-2x_1x_3$ is a convex function or not.

Solution

$$H=\begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_1 \partial x_3} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \frac{\partial^2 y}{\partial x_2 \partial x_3} \\ \frac{\partial^2 y}{\partial x_3 \partial x_1} & \frac{\partial^2 y}{\partial x_3 \partial x_3} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$D_1 = 1 > 0$

D$_2$=$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0$

D$_3$=$\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -3$

H is indefinite.

Hence, the given function is neither convex function nor concave function

Example:

Check that the function f(x)=$x_1^2 - 2x_2^2$ is convex or not.

Solution

H= $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

D$_1$= 1>0

D$_2$=$\begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2 < 0$

H is indefinite.

Hence, the given function is neither a convex function nor a concave function.

Example:

Check that the function $f(x) = -3x_1^2 + 2x_2^2 - 3x_3^2 - 10x_1x_2 + 4x_2x_3 + 6x_1x_3$ is convex or not.

Solution

$$H = \begin{bmatrix} -3 & -10 & 6 \\ -10 & 2 & 4 \\ 6 & 4 & -3 \end{bmatrix}$$

$D_1 = -3 < 0$

$$D_2 = \begin{vmatrix} -3 & -10 \\ -10 & 2 \end{vmatrix} = -106 < 0$$

H is indefinite.

Hence, the given function is neither a convex function nor a concave function.

Example:

Check that the following NLPP is a convex programming problem or not.

Minimize $f(x)= 4x_1^2+2x_2^2+x_3^2-4x_1x_2$

Subject to

$x_1+ x_2+ x_3=15,$

$2x_1- x_2+ 2x_3=20,$

$x_1>=0, x_2>=0, x_3>=0.$

Solution

Objective function

$f(x)=4x_1^2+2x_2^2+x_3^2-4x_1x_2$

Hessian matrix$=\begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$

$D_1=8>0$

$D_2=16 > 0$

Convex function

First and second constraints

$g_1(x)= x_1+ x_2+ x_3-15$ and $g_2(x)= 2x_1- x_2+ 2x_3-20$ are linear functions. Since, linear function is both convex and concave. So, $g_1(x)$ and $g_2(x)$ are convex functions.

Hence, the considered NLPP is a convex NLPP.

Non-linear programming problems with inequality constraints

Optimize (f(x))

Subject to

$g_i(x) \le 0$, i=1,2,…,m

KKT (Karush Kuhn-Tucker) conditions

- $$\begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} & \cdots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix} - \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{bmatrix} \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} & \cdots & \frac{\partial g_1(x))}{\partial x_n} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} & \cdots & \frac{\partial g_2(x))}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_m(x)}{\partial x_1} & \frac{\partial g_m(x)}{\partial x_2} & \cdots & \frac{\partial g_m(x))}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
- **$\lambda_i g_i(x)=0$, i=1,2,…,m**
- **$g_i(x) \le 0$, i=1,2,…,m**
- **$\lambda_i \le 0$ (for minimization problem)**

  OR

  $\lambda_i \ge 0$ (for maximization problem)

Example:

Solve the following NLPP using KKT conditions

Maximize $f(x)= 4x_1+6x_2-2x_1^2 -2x_1x_2-2x_2^2+49$

Subject to

$x_1+x_2<=2$

$x_1>=0, x_2>=0.$

Solution:

Maximize $f(x)= 4x_1+6x_2-2x_1^2 -2x_1x_2-2x_2^2+49$

Subject to

$x_1+x_2-2<=0$

$-x_1<=0,$

$-x_2<=0.$

$f(x)= 4x_1+6x_2-2x_1^2 -2x_1x_2-2x_2^2+49$

$g_1(x)= x_1+x_2-2$

$g_2(x)= -x_1$

$g_3(x)= -x_2$

- $$\begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} \end{bmatrix} - \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} \\ \frac{\partial g_3(x)}{\partial x_1} & \frac{\partial g_3(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $g_1(x)<=0$
  $g_2(x)<=0$
  $g_3(x)<=0$

- $\lambda_1 g_1(x)=0$
  $\lambda_2 g_2(x)=0$
  $\lambda_3 g_3(x)=0$

- $\lambda_1 \geq 0$
  $\lambda_2 \geq 0$
  $\lambda_3 \geq 0$

- $$\begin{bmatrix} 4-4x_1-2x_2 & 6-2x_1-4x_2 \end{bmatrix} - \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  $$\begin{bmatrix} 4-4x_1-2x_2 & 6-2x_1-4x_2 \end{bmatrix} - \begin{bmatrix} \lambda_1-\lambda_2 & \lambda_1-\lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  $4-4x_1-2x_2-\lambda_1+\lambda_2=0$ (1)
  $6-2x_1-4x_2-\lambda_1+\lambda_3=0$ (2)

- $x_1+x_2-2<=0$
  $-x_1<=0$
  $-x_2<=0$

- $\lambda_1(x_1+x_2-2)=0$ (3)
  $\lambda_2(-x_1)=0$ (4)
  $\lambda_3(-x_2)=0$ (5)

- $\lambda_1 \geq 0$
  $\lambda_2 \geq 0$

$\lambda_3 \geq 0$

Case (i)

$\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$

From (4) and (5)

$x_1 = 0$ and $x_2 = 0$

Putting in (1) and (2)

$4 + \lambda_2 = 0$

$6 + \lambda_3 = 0$

Solving

$\lambda_2 = -4$ **and** $\lambda_3 = -6$

Negative values of λ.

No solution.

Case (ii)

$\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$

From (3) and (5)

$x_1 + x_2 - 2 = 0$ and $x_2 = 0$

Therefore,

$x_1 = 2$

Putting in (1) and (2)

$-8 - \lambda_1 = 0$

$$6-4-\lambda_1+\lambda_3=0$$

Solving

$\lambda_1=-8$ **and** $\lambda_3=-10$

Negative values of λ.

No solution.

Case (iii)

$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3=0$

From (3) and (4)

$x_1+x_2-2=0$ **and** $x_1=0$

Therefore,

$x_2=2$

Putting in (1) and (2)

$$4-4-\lambda_1+\lambda_2=0$$
$$6-8-\lambda_1=0$$

Solving

$\lambda_1=-2$ **and** $\lambda_2=-2$

Negative values of λ.

No solution.

Case (iv)

$\lambda_1=0, \lambda_2=0, \lambda_3 \neq 0$

From (5)

$x_2=0$

Putting in (1) and (2)

$$\mathbf{4 - 4x_1 = 0}$$
$$\mathbf{6 - 2x_1 + \lambda_3 = 0}$$

Solving

$\mathbf{x_1 = 1}$ **and** $\mathbf{\lambda_3 = -4}$

Negative values of $\boldsymbol{\lambda_3}$**.**

No solution.

Case (v)

$\mathbf{\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0}$

From (4)

$\mathbf{x_1 = 0}$

Putting in (1) and (2)

$$\mathbf{4 - 2x_2 + \lambda_2 = 0}$$
$$\mathbf{6 - 4x_2 = 0}$$

Solving

$\mathbf{x_2 = \frac{3}{2}}$ **and** $\mathbf{\lambda_2 = -1}$

Negative values of $\boldsymbol{\lambda_2}$**.**

No solution.

Case (vi)

$\mathbf{\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0}$

From (3)

$\mathbf{x_1 + x_2 - 2 = 0}$

Putting in (1) and (2)

$$4 - 4x_1 - 2x_2 - \lambda_1 = 0$$
$$6 - 2x_1 - 4x_2 - \lambda_1 = 0$$

Solving

$x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$ and $\lambda_1 = -\frac{2}{3}$

Negative value of λ_1.

No solution.

Case (vii)

$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$

Putting in (1) and (2)

$$4 - 4x_1 - 2x_2 = 0$$
$$6 - 2x_1 - 4x_2 = 0$$

Solving

$x_1 = \frac{1}{3}$, $x_2 = \frac{4}{3}$

Optimal Solution

Optimal value is

$$4x_1+6x_2-2x_1^2-2x_1x_2-2x_2^2+49=4\left(\frac{1}{3}\right)+6\left(\frac{4}{3}\right)-2\left(\frac{1}{3}\right)^2-2\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)-2\left(\frac{4}{3}\right)^2+49$$

$$=\frac{483}{9}$$