Course: UMA 035 (Optimization Techniques)

Instructor: Dr. Amit Kumar,

Associate Professor,

School of Mathematics,

TIET, Patiala

Email: amitkumar@thapar.edu

Mob: 9888500451

Steepest Descent Technique

Minimization Problem

$$X_{n+1} = X_n - \lambda_n \left[\left(\frac{\partial y}{\partial x_1}, \quad \frac{\partial y}{\partial x_2}, \quad \cdots, \quad \frac{\partial y}{\partial x_n} \right) at\ X_n \right], n = 0, 1, 2, \ldots$$

Maximization Problem

$$X_{n+1} = X_n + \lambda_n \left[\left(\frac{\partial y}{\partial x_1}, \quad \frac{\partial y}{\partial x_2}, \quad \cdots, \quad \frac{\partial y}{\partial x_n} \right) at\ X_n \right], n = 0, 1, 2, \ldots.$$

Example:

Find the minimum of $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$ so that the error does not exceed by 0.5. The initial approximation is to be taken as $\left(1, \frac{1}{2}\right)$.

Solution

$$\left(\frac{\partial y}{\partial x_1}, \quad \frac{\partial y}{\partial x_2} \right) = (2x_1 - x_2, \quad -x_1 + 2x_2)$$

First iteration($n = 0$)

$$X_0 = \left(1, \frac{1}{2}\right)$$

$$X_1 = X_0 - \lambda_0 \left[\left(\frac{\partial y}{\partial x_1}, \quad \frac{\partial y}{\partial x_2} \right) at\ X_0 \right]$$

Since,

$\left(\frac{\partial y}{\partial x_1} \quad \frac{\partial y}{\partial x_2} \right) = (2x_1 - x_2, \quad -x_1 + 2x_2)$ at X_0 is

$$\left(2\times 1-\frac{1}{2}, \quad -1+2\times\frac{1}{2}\right)=\left(\frac{3}{2}, \quad 0\right)$$

So,

$$X_1=\left(1,\frac{1}{2}\right)-\lambda_0\left(\frac{3}{2}, \quad 0\right)=\left(1-\frac{3\lambda_0}{2}, \quad \frac{1}{2}\right)$$

Now, the value of the given function corresponding to X_1,

$$f(x_1,x_2)=x_1^2-x_1x_2+x_2^2=\left(1-\frac{3\lambda_0}{2}\right)^2-\left(1-\frac{3\lambda_0}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2$$

Since, $f(x_1,x_2)$ is only depending upon λ_0. So, for minimum value of $f(x_1,x_2)$,

$$\frac{df(x_1,x_2)}{d\lambda_0}=0$$

$$2\times-\frac{3}{2}\times\left(1-\frac{3\lambda_0}{2}\right)-\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)=0$$

$$2\times\left(1-\frac{3\lambda_0}{2}\right)-\left(\frac{1}{2}\right)=0$$

$$(2-3\lambda_0)-\left(\frac{1}{2}\right)=0$$

$$-3\lambda_0+\frac{3}{2}=0$$

$$\lambda_0=\frac{1}{2}$$

Hence,

$$X_1=\left(1,\frac{1}{2}\right)-\lambda_0\left(\frac{3}{2}, \quad 0\right)=\left(1-\frac{3\lambda_0}{2}, \quad \frac{1}{2}\right)=\left(1-\frac{3}{4}, \quad \frac{1}{2}\right)=\left(\frac{1}{4}, \quad \frac{1}{2}\right)$$

and

the value of the given function corresponding to X_1 is

$$f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 = \left(1 - \frac{3\lambda_0}{2}\right)^2 - \left(1 - \frac{3\lambda_0}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2$$
$$= \left(1 - \frac{3}{4}\right)^2 - \left(1 - \frac{3}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = \frac{1}{16} - \frac{1}{8} + \frac{1}{4} = \frac{3}{16}$$

While,

$$X_0 = \left(1, \frac{1}{2}\right)$$

and the value of the given function corresponding to X_0 is

$$f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 = (1)^2 - (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Since,

$\left|\frac{3}{16} - \frac{3}{4}\right| = 0.5625 > .5$ **. So, solution is not optimal.**

Second iteration ($n = 1$)

$$X_1 = \left(\frac{1}{4}, \frac{1}{2}\right)$$

$$X_2 = X_1 - \lambda_1 \left[\left(\frac{\partial y}{\partial x_1}, \quad \frac{\partial y}{\partial x_2}\right) \; at \; X_1\right]$$

Since,

$\left(\frac{\partial y}{\partial x_1} \quad \frac{\partial y}{\partial x_2}\right) = (2x_1 - x_2, \quad -x_1 + 2x_2)$ **at X_1 is**

$$\left(2 \times \frac{1}{4} - \frac{1}{2}, \quad -\frac{1}{4} + 2 \times \frac{1}{2}\right) = \left(0, \quad \frac{3}{4}\right)$$

So,

$$X_2 = \left(\frac{1}{4}, \frac{1}{2}\right) - \lambda_1\left(0, \quad \frac{3}{4}\right) = \left(\frac{1}{4}, \qquad \frac{1}{2} - \lambda_1 \times \frac{3}{4}\right)$$

Now, the value of the given function corresponding to X_2,

$$f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 = \left(\frac{1}{4}\right)^2 - \left(\frac{1}{2} - \lambda_1 \times \frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2} - \lambda_1 \times \frac{3}{4}\right)^2$$

Since, $f(x_1, x_2)$ is only depending upon λ_1. So, for minimum value of $f(x_1, x_2)$,

$$\frac{df(x_1, x_2)}{d\lambda_1} = 0$$

$$-\left(-\frac{3}{4}\right)\left(\frac{1}{4}\right) + 2 \times \left(-\frac{3}{4}\right) \times \left(\frac{1}{2} - \lambda_1 \times \frac{3}{4}\right) = 0$$

$$\lambda_1 = \frac{1}{2}$$

Hence,

$$X_2 = \left(\frac{1}{4}, \frac{1}{2}\right) - \lambda_1\left(0, \quad \frac{3}{4}\right) = \left(\frac{1}{4}, \qquad \frac{1}{2} - \lambda_1 \times \frac{3}{4}\right)$$

$$= \left(\frac{1}{4}, \qquad \frac{1}{2} - \frac{1}{2} \times \frac{3}{4}\right) = \left(\frac{1}{4}, \quad \frac{1}{8}\right)$$

and

the value of the given function corresponding to X_2 is

$$f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 = \left(\frac{1}{4}\right)^2 - \left(\frac{1}{2} - \lambda_1 \times \frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2} - \lambda_1 \times \frac{3}{4}\right)^2$$
$$= \left(\frac{1}{4}\right)^2 - \left(\frac{1}{2} - \frac{1}{2} \times \frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{2} \times \frac{3}{4}\right)^2 = \frac{3}{64}$$

While,

$$X_1 = \left(\frac{1}{4}, \quad \frac{1}{2}\right)$$

and

the value of the given function corresponding to X_1 is

$$f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 = \left(1 - \frac{3\lambda_0}{2}\right)^2 - \left(1 - \frac{3\lambda_0}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2$$

$$= \left(1 - \frac{3}{4}\right)^2 - \left(1 - \frac{3}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = \frac{1}{16} - \frac{1}{8} + \frac{1}{4} = \frac{3}{16}$$

Since,

$\left|\frac{3}{64} - \frac{3}{16}\right| = 0.140625 < .5$ **. So, solution is optimal.**