

## Lecture 7: Numerical Analysis (UMA011)

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## Root-finding problem

### Bisection method: Example

Show that  $f(x) = x^3 + 2x^2 - 3x - 1 = 0$  has a root in  $[1, 2]$  and use the bisection method to determine an approximation to the root i.e. is accurate to atleast within  $10^{-2}$ . = tolerance

$[a, b]$

**Solution:**  $f(x)$  is a cubic poly., so it is continuous

$$\text{at } [1, 2] \quad \text{and} \quad f(1) = 1 + 2 - 3 - 1 < 0$$

$$f(2) = 2^3 + 2(2)^2 - 3(2) - 1 = 8 + 8 - 6 - 1 > 0$$

$$\text{so} \quad f(1) * f(2) < 0$$

from I V T, we have  $f(x)=0$  has a root in  $[1, 2]$

-ve +ve  
[1, 2]

Bisection method

$$p_1 = \frac{1+2}{2} = 1.5$$

The root lie in bet<sup>n</sup> either [1, 1.5] or [1.5, 2]

check the sign of  $f(1.5) = +ve$

By IVT , the root lie in [1, 1.5].

so  $p_2 = \frac{1+1.5}{2} = 1.25$        $\overset{\uparrow}{[1, 1.25]}$  and  $[1.25, 1.5]$

$$|p_1 - p_2| < 10^{-2}$$

The root lie in bet<sup>n</sup> either [1, 1.25] or [1.25, 1.5]

check the sign of  $f(1.25) = +ve$

By IVT , the root lie in [1, 1.25]

$$p_3 = \frac{1+1.25}{2} = 1.125$$

$$|p_n - p_{n-1}| < 10^{-2}$$

Table

0.32 ~~④~~ ①

0.001

$10^{-2}$

u

$$|p_7 - p| < \frac{2^{-1}}{2^7}$$

$$\frac{1}{2^7} < 10^{-2}$$

$n$	$a$ $f(a) \approx u$	$b$ $f(b) \approx u$	$p_n$	$ p_n - p_{n-1} $	$f(p_n)$
1	1	2	1.5	$ 1.5 - 1.25  \leq 10^{-2}$	+ve
2	1	1.5	1.25	$ 1.25 - 1.125  \leq 10^{-2}$	+ve
3	1	1.25	1.125	$ 1.125 - 1.125  \leq 10^{-2}$	-ve
4	1.125	1.25	1.1875	$ 1.125 - 1.1875  \leq 10^{-2}$	-ve
5	1.1875	1.25	1.21875	$ 1.1875 - 1.21875  \leq 10^{-2}$	+ve
6	1.1875	1.21875	1.203125	$ 1.21875 - 1.203125  \leq 10^{-2}$	-ve
7	1.203125	1.21875	1.1953125	$ 1.21875 - 1.1953125  \leq 10^{-2}$	+ve

$$p_7 = 1.1953125$$

Approximate  
root

not

## Bisection method

### Maximum error bound

Suppose that  $f \in C[a, b]$  and  $f(a) * f(b) < 0$ . The Bisection method generates a sequence  $\{p_n\}_{n=1}^{\infty}$  approximating a zero  $p$  of  $f$  with  $|p_n - p| \leq \frac{b-a}{2^n}$ , when  $n \geq 1$ .

$$\begin{aligned} |p_n - p| \\ \leq C \end{aligned}$$

**Proof:**

length of  $[a, b]$  or  $[a_1, b_1]$

$$\begin{array}{c} +-----+ \\ a_1 = a \quad b = b_1 \\ | \qquad \qquad \qquad | \\ p_1 \end{array}$$

$$= |b_1 - a_1| = |b - a|$$

After applying bisection method and I.V.T, we get

the root lie in  $[a_2, b_2]$  (say)

$$\text{length of } [a_2, b_2] = |b_2 - a_2| = \frac{1}{2} |b_1 - a_1|$$

Again, applying bisection method on  $[a_2, b_2]$  and then applying IVT, we get  $[a_3, b_3]$  in which root of  $f(x) = 0$  lie.

$$\begin{aligned}\text{length of the interval } [a_3, b_3] &= |b_3 - a_3| = \frac{1}{2} |b_2 - a_2| \\ &= \frac{1}{2^2} |b_1 - a_1|\end{aligned}$$

Continue this process upto  $n$ -iterations, we get the root lie in  $[a_n, b_n]$

$$\begin{aligned}\text{The length of interval } [a_n, b_n] &= |b_n - a_n| \\ &= \frac{1}{2^{n-1}} |b_1 - a_1|\end{aligned}$$

$$|b_n - a_n| = \frac{1}{2^{n-1}} |b - a| \quad (*)$$

Now,  $|b_n - p| = \left| \frac{a_n + b_n}{2} - p \right|$

As  $p \in [a_n, b_n]$  then  $a_n \leq p \leq b_n$   
 $-a_n \geq -p$   
 $\Rightarrow -p \leq -a_n$

$$\Rightarrow |b_n - p| \leq \left| \frac{a_n + b_n}{2} - a_n \right| = \left| \frac{a_n + b_n - 2a_n}{2} \right|$$

$$|p_n - p| \leq \frac{|b_n - a_n|}{2}$$

from  $\textcircled{*}$ , we get

$$|p_n - p| \leq \frac{1}{2} \left( \frac{1}{2^{n-1}} |b_1 - a_1| \right)$$

$$|p_n - p| \leq \frac{1}{2^n} |b - a| .$$

## Bisection method

### Example

Determine the number of iterations necessary to solve

$f(x) = x^3 + 4x^2 - 10 = 0$  with accuracy  $10^{-3}$  using  $a_1 = 1$  and  $b_1 = 2$ .

**Solution:**

from prev. result , we have

$$\frac{|b_1 - a_1|}{2^n} \leq 10^{-3}$$

$$\frac{1}{2^n} \leq 10^{-3}$$

$$2^n \geq 10^3$$

$$n \log 2 \geq 3 \log 10$$

$$n \geq \frac{3 \log_{10}}{\log_2}$$

$$n \geq 9.965$$

$$n = 10$$

**Exercise:**

- 1 Use intermediate value theorem to get the first positive root of  $x - 2^{-x} = 0$  and hence apply bisection method to find the root accurate to within  $10^{-1}$ .
- 2 Using the bisection method, determine the point of intersection of the curves given by  $y = 3x$  and  $y = e^x$  in the interval  $[0, 1]$  with an accuracy 0.1.
- 3 Find a bound for the number of iterations needed to achieve an approximation of  $(25)^{1/3}$  by the bisection method with an accuracy  $10^{-2}$ . Hence find the approximation with given accuracy.