Lecture 3: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Error Analysis: Significant figures

Significant figures

Significant figures of a number are those figures (digits) that carry meaning contributing to its precision. e.g. 123 120

Rules

1) All non-zero digits of a no. are significant
2) zeros between non-zero no.s are significant
1002
3) leading zeros to the left of the first non-zero digits are not S.D.

0.00123

4) Trailing zeros that are also to the right of a decimal pt. in a no. are S.D.

15 → 2 S.D.

15.0 → 3 S.D.

Error Analysis: Significant figures

Examples:

1.) Add the floating pt no's 0.4546e3 & 0.5433e7

$= 0.4546 \times 10^3$

0.5433×10^7

0.00004546×10^7

$0.5433 \quad \times 10^7$

0.54334546×10^7 → 0.5433×10^7.

2.) Subtract 0.5424e-99 from 0.5452e-99

0.5452×10^{-99}

$- 0.5424 \times 10^{-99}$ = 0.0028×10^{-99}

$= 0.28e-101$ Ans

0.2800×10^{-101}

Error Analysis: Significant figures

Examples:

3) Multiply 0.1111 e74 and 0.2000 e80

$0.02222 \times 10^{154} = 0.2222 \times 10^{153}$

Error Analysis: Loss of Significance

Loss of Significance:

one of most common error producing calculations involve the cancellation of S.D. due to subtractions of two nearly equal no's

e.g.

$$
\begin{array}{r l}
10002.254 & \rightarrow 8\ S.D \\
10002.264 & \quad 8\ S.D \\
\hline
0.010 & \rightarrow 2.S.D.
\end{array}
$$

loss of 6 S.D.

Error Analysis: Loss of Significance

Examples:

If $x = 0.3721478693$ and $y = 0.3720230572$, then what is the relative error in the computation of $x - y$ using 5−decimal digits of accuracy?

Solution $X = x - y = 0.3721478693 - 0.3720230572$

$= 0.0001248121$

$fl(x) = 0.37215$, $fl(y) = 0.37202$

✓ $x^* = fl(x) - fl(y) = 0.13000 \times 10^{-3} = 0.00013$

0.0001248121

A.E. $|X - x^*| = 0.00002 \ldots$

R.E. $\frac{|X - x^*|}{|X|} = \frac{0.00002 \ldots}{0.0001248121} = 0.04 = 4\%$ ✓

Error Analysis: Finite-digit-Arithmetic

Finite-digit-Arithmetic:

$x \oplus y$.

n-digit Arithmetic

$fl(x) + fl(y) = (\quad)$

$x \oplus y \circledast z = fl(x) + fl(y) * fl(z)$

$fl(x) + u$

$fl(x) + fl(u) = v = fl(v)$

ax^3+bx^2+cx+d

$d+x(c+bx+ax^2)$

$d+x(c+x(b+ax))$

Nested Arithmetic $f(x)$ evaluate $f(x)$ at $x=p$

$f(x) = ax^2+bx+c = c+x(b+ax)$

Error Analysis: Nested Arithmetic

Example:

Evaluate $f(x) = 1.5 + 3.2x - 6.1x^2 + x^3$ at $x = 4.71$ using 3-digit arithmetic directly and with nesting.

Solution. $f(x) = 1.5 + 3.2x - 6.1x^2 + x^3$

$x = \quad f(4.71) = 1.5 + 3.2(4.71) - 6.1(4.71)^2 + (4.71)^3$

$= -14.263899$

with 3-digit Arithmetic

$x_1 = f(4.71) = 1.5 + 3.2(4.71) - 6.1(4.71)^2 + (4.71)^3$

$= 1.5 + 15.1 - 6.1(22.2) + 4.71(22.2)$

$= 1.5 + 15.1 - 135 + 105 = -13.4$

With Nesting & 3-digit Arithmetic

$$f(x) = 1.5 + x(3.2 - 6.1x + x^2)$$

$$= 1.5 + x(3.2 + x(-6.1 + x))$$

$$X_2 = f(4.71) = 1.5 + 4.71(3.2 + 4.71(-6.1 + 4.71))$$

$$= 1.5 + 4.71(3.2 + 4.71(-1.39))$$

$$= 1.5 + 4.71(3.2 - 6.55)$$

$$= 1.5 + 4.71(-3.35) = 1.5 - 15.8 = -14.3$$

$$A.E. = |X - X_1| = |-14.263899 + 13.4| = 0.864$$

$$|X - X_2| = |-14.263899 + 14.3| = 0.03610$$

$$\Rightarrow |X - X_2| < |X - X_1|$$

Error Analysis: Nested Arithmetic

Example:

Evaluate $y \approx x - \sin(x)$, when x is small.

$x = 0.01$ ✓

$$y = x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots$$

$$= \frac{x^3}{3!}\left(1 - \frac{3!\,x^2}{5!} + \frac{3!\,x^4}{7!} - \cdots\right)$$

$$= \frac{x^3}{3!}\left(1 - \frac{3!}{5!}x^2\left(1 - \frac{5!\,x^2}{7!} - \cdots\right)\right)$$ ✓

Error Analysis

Exercise:

1. Let x be any real number we want to represent in a computer. Let $fl(x)$ be the representation by chopping with $n-$digits of x then what is largest possible values of $\frac{|x-fl(x)|}{|x|}$?
2. Evaluate $f(x) = x^3 - 3x^2 + 4x + 0.21$ at $x = 2.73$ using 3-digit arithmetic directly and with nesting. Also, find the absolute error and relative error.