Course: UMA 035 (Optimization Techniques)

Instructor: Dr. Amit Kumar,

Associate Professor,

School of Mathematics,

TIET, Patiala

Email: amitkumar@thapar.edu

Mob: 9888500451

Integer Linear Programming Problem

Example:

Solve the following integer linear programming problem by Branch and Bound method.

Max (x_1+x_2)

Subject to

$2x_1+5x_2<=16$

$6x_1+5x_2<=30$

x_1, x_2 >=0 and integers

Solution:

Firstly, there is a need to find an optimal solution by the graphical method.

$2x_1+5x_2=16$

Putting $x_1=0$, the value of x_2 is $\frac{16}{5}$.

$(x_1, x_2) = \left(0, \frac{16}{5}\right)$

Putting $x_2=0$, the value of x_1 is $\frac{16}{2}=8$.

$(x_1, x_2) = (8, 0)$

First line joins the points $(8, 0)$ and $\left(0, \frac{16}{5}\right)$.

$6x_1+5x_2=30$

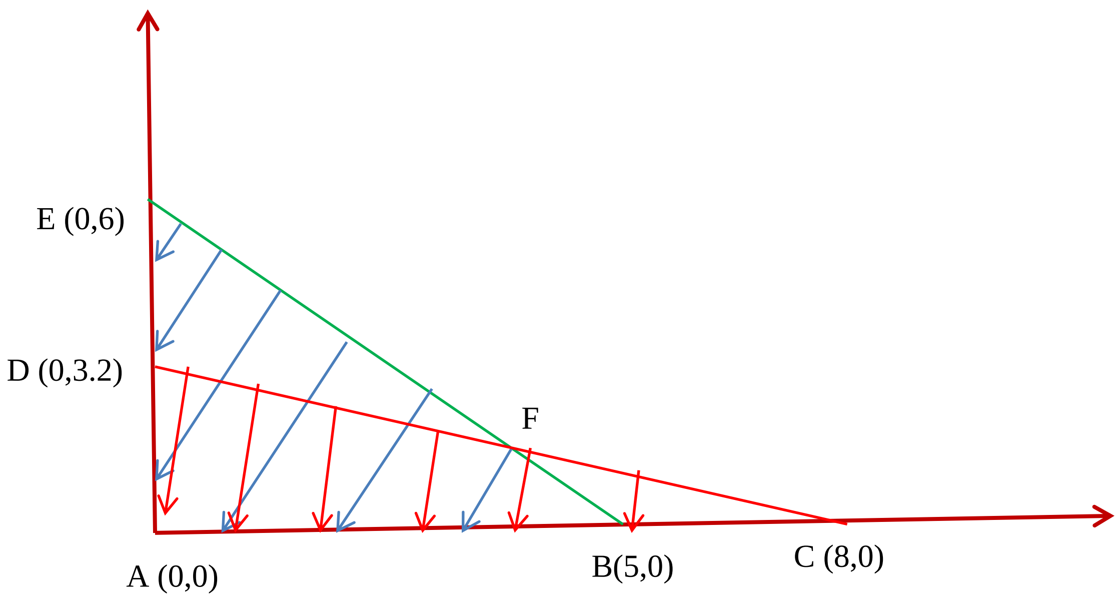
Putting $x_1=0$, the value of x_2 is $\frac{30}{5} = 6$.

$(x_1, x_2) = (0, 6)$

Putting $x_2=0$, the value of x_1 is $\frac{30}{6}=5$.

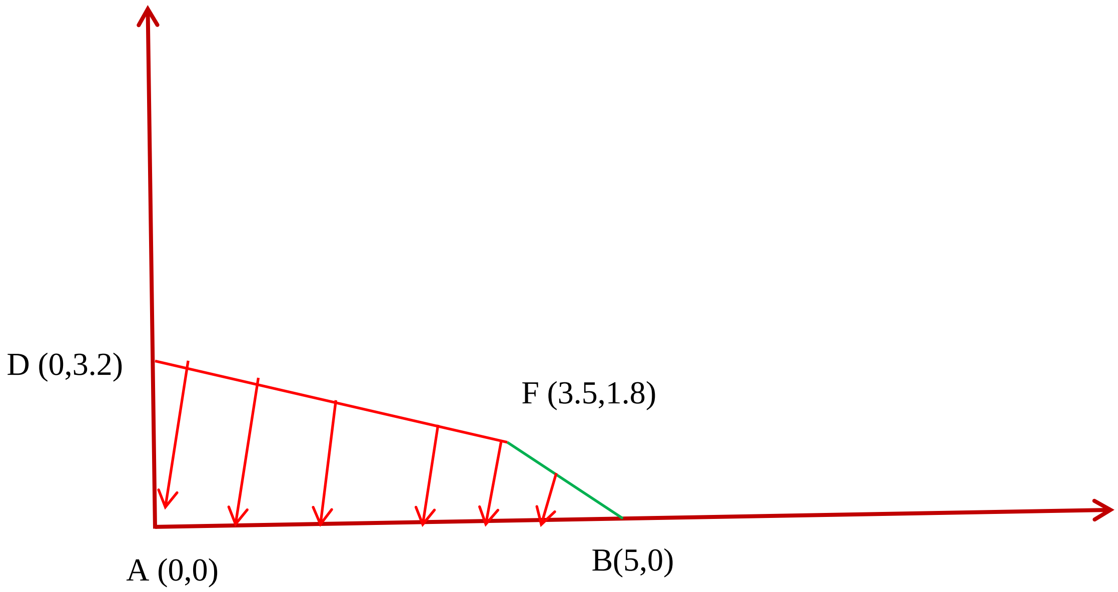
$(x_1, x_2) = (5, 0)$

Second line joins the points $(5, 0)$ and $(0, 6)$.



F is intersection of $2x_1+5x_2=16$ and $6x_1+5x_2=30$

On solving F is (3.5, 1.8).



Value of the objective function x_1+x_2 at

- **A (0,0) is 0**
- **B (5,0) is 5**
- **F (3.5, 1.8) is 5.3**
- **D (0, 3.2) is 3.2.**

Since, the problem is of maximization and the maximum value is 5.3 which is corresponding to $x_1=3.5$ and $x_2=1.8$. So, the initial optimal solution is $x_1=3.5$ and $x_2=1.8$.

But, as x_1 and x_2 are not integers. So, it is not required optimal solution.

We may start from x_1 or from $x_{2.}$

First method

The value of x_1 is 3.5.

The non-negative integers less than 3.5 are 0,1,2,3 and the non-negative integers greater than 3.5 are 4,5,6,…..

The required value of x_1 will be 0 or 1 or 2 or 3 i.e., <=3

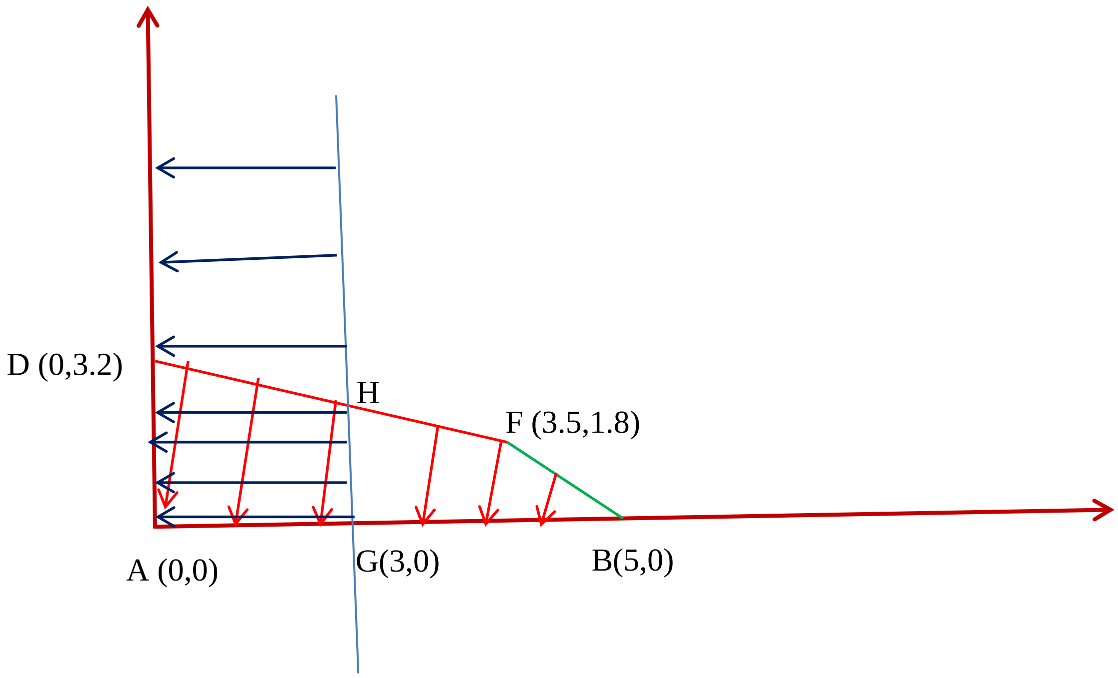
OR

The required value of x_1 will be 4 or 5 or 6 or ….. i.e., >=4

Hence,

x_1<=3 or x_1>=4.

Case (i) Including x_1<=3 in the graph,



The new feasible region is AGHD

where,

H is intersection of $x_1=3$ and $2x_1+5x_2=16$.

Solving

$x_1=3$ and $x_2=2$

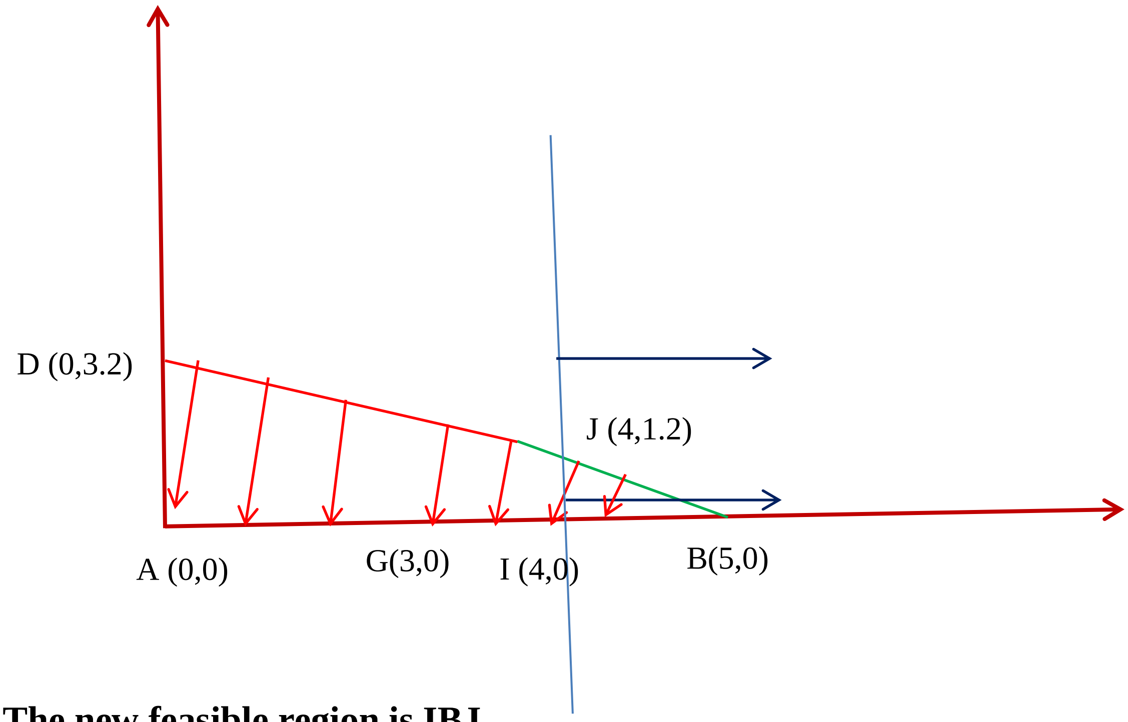
Therefore, H (3 , 2).

Value of the objective function x_1+x_2 at

- **A (0,0) is 0**
- **G (3,0) is 3**
- **H (3, 2) is 5**
- **D (0, 3.2) is 3.2.**

Since, the problem is of maximization and the maximum value is 5 which is corresponding to $x_1=3$ and $x_2=2$. So, the initial optimal solution is $x_1=3$ and $x_2=2$.

Case (ii) Including $x_1>=4$ in the graph,



The new feasible region is IBJ

where,

J is intersection of $x_1=4$ and $6x_1+5x_2=30$.

Solving

$x_1=4$ and $x_2=1.2$

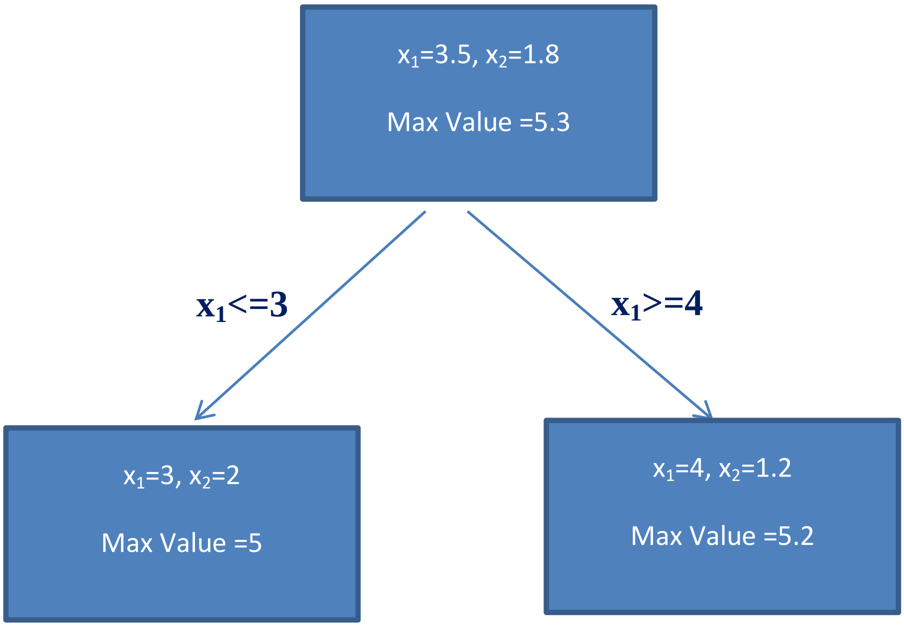
Therefore, J (4 , 1.2).

Value of the objective function x_1+x_2 at

- **I (4,0) is 4**
- **B (5,0) is 5**

- **J (4, 1.2) is 5.2**

Since, the problem is of maximization and the maximum value is 5.2 which is corresponding to x_1=4 and x_2=1.2. So, the initial optimal solution is x_1=4 and x_2=1.2.



The maximum value obtained in case 2 is more than the maximum value obtained in case 2.

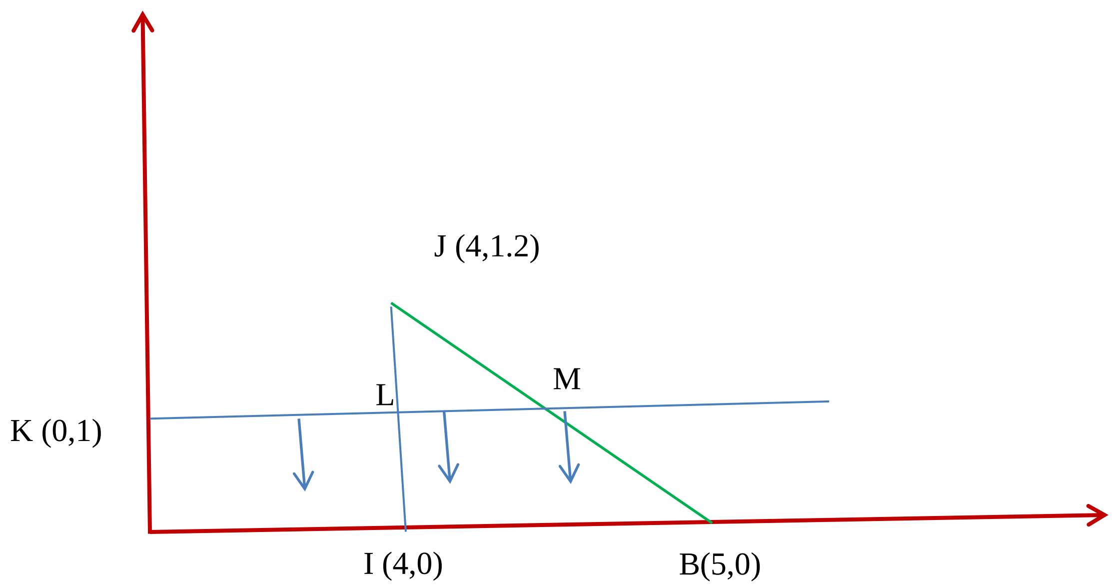
So, we will proceed with case 2.

The value of x_2=1.2

Case (i) x_2<=1

Case (ii) x_2>=2

Including x_2<=1 in the final graph of case (ii),



The new feasible region is IBML

where,

L is intersection of $x_2=1$ and $x_1=4$.

L (4 , 1)

M is intersection of $x_2=1$ and $6x_1+5x_2=30$.

Solving

x_2=1 and x_1=$\frac{25}{6}$

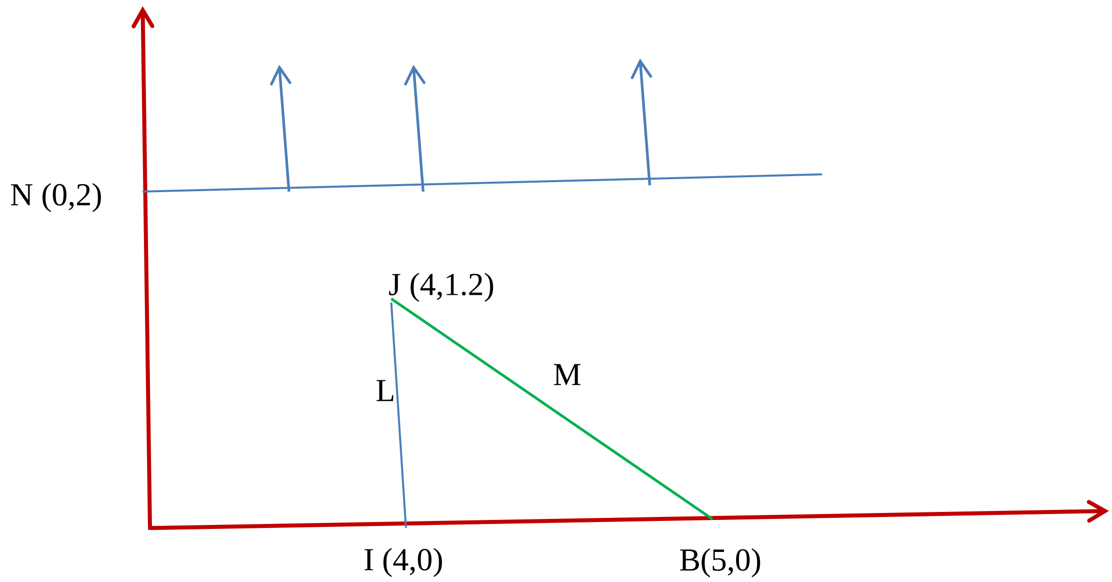
M ($\frac{25}{6}$, 1)

Value of the objective function x_1+x_2 at

- **I (4,0) is 4**
- **B (5,0) is 5**
- **M ($\frac{25}{6}$, 1) is $\frac{31}{6}$**
- **L (4 , 1) is 5**

Since, the problem is of maximization and the maximum value is $\frac{31}{6}$ which is corresponding to x_1=$\frac{25}{6}$ and x_2=1. So, the initial optimal solution is x_1=$\frac{25}{6}$ and x_2=1.

Including x_2>=2 in the final graph of case (ii),



No common region and hence no solution.

The new feasible region is IBML

where,

L is intersection of $x_2=1$ and $x_1=4$.

L (4 , 1)

M is intersection of $x_2=1$ and $6x_1+5x_2=30$.

Solving

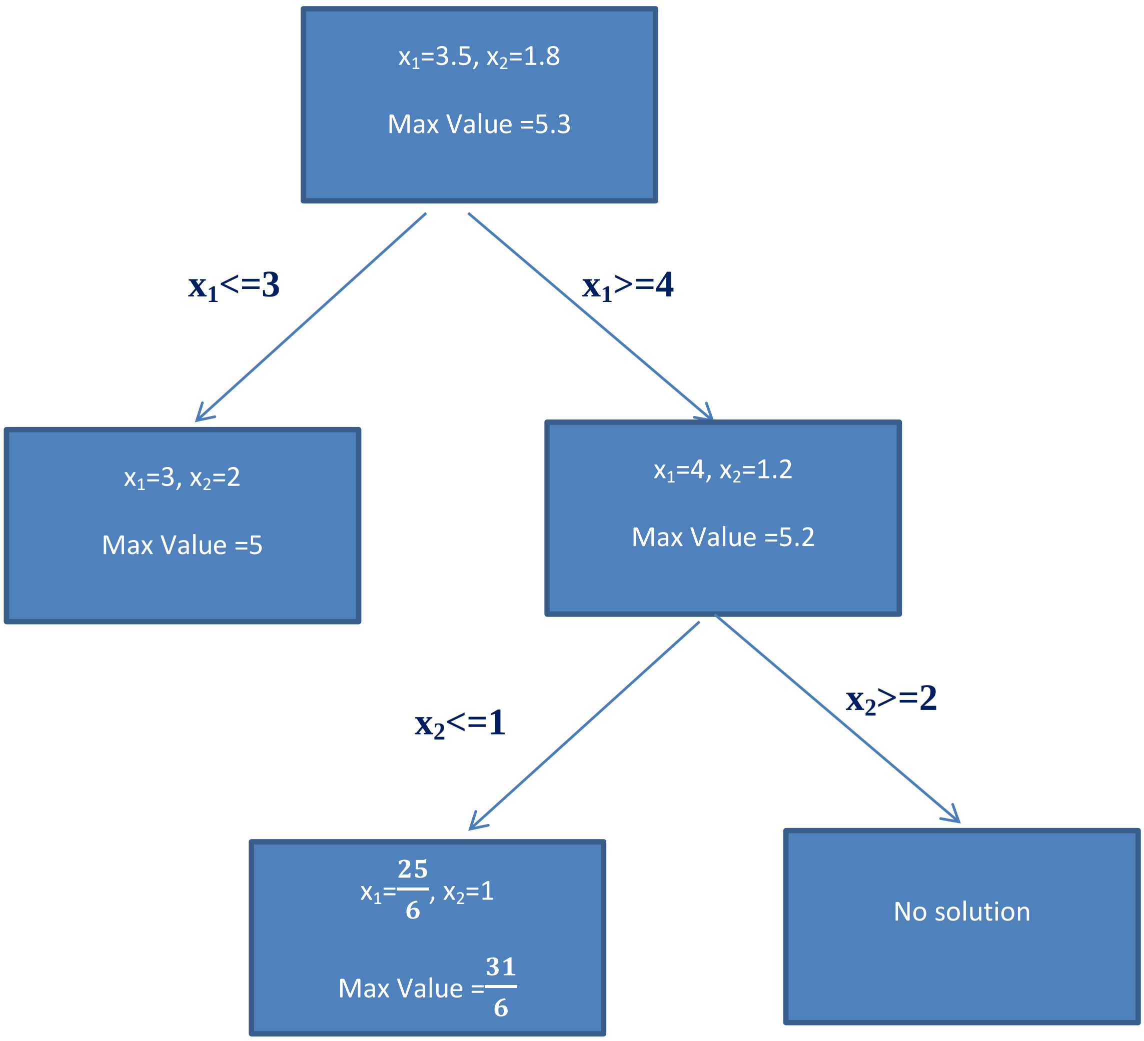
$x_2=1$ and $x_1=\frac{25}{6}$

M $(\frac{25}{6}, 1)$

Value of the objective function x_1+x_2 at

- **I (4,0) is 4**
- **B (5,0) is 5**
- **M $(\frac{25}{6}, 1)$ is $\frac{31}{6}$**
- **L (4 , 1) is 5**

Since, the problem is of maximization and the maximum value is $\frac{31}{6}$ which is corresponding to $x_1=\frac{25}{6}$ and $x_2=1$. So, the initial optimal solution is $x_1=\frac{25}{6}$ and $x_2=1$.

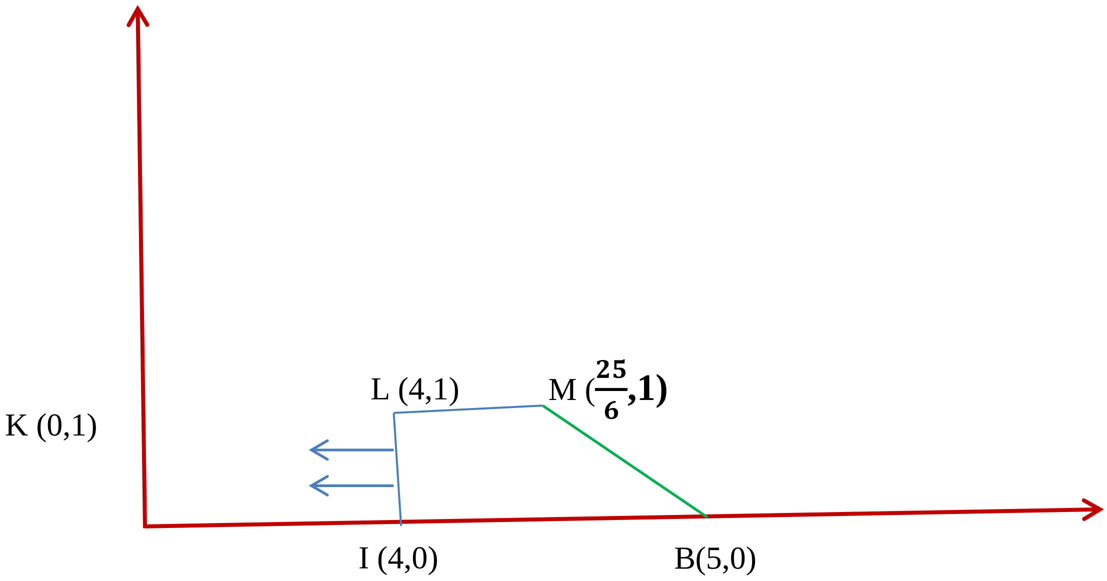


The value of x_1=4.16…

Case (i) x_1<=4

Case (ii) x_1>=5

Including x_1<=4 in the final graph of case (i),



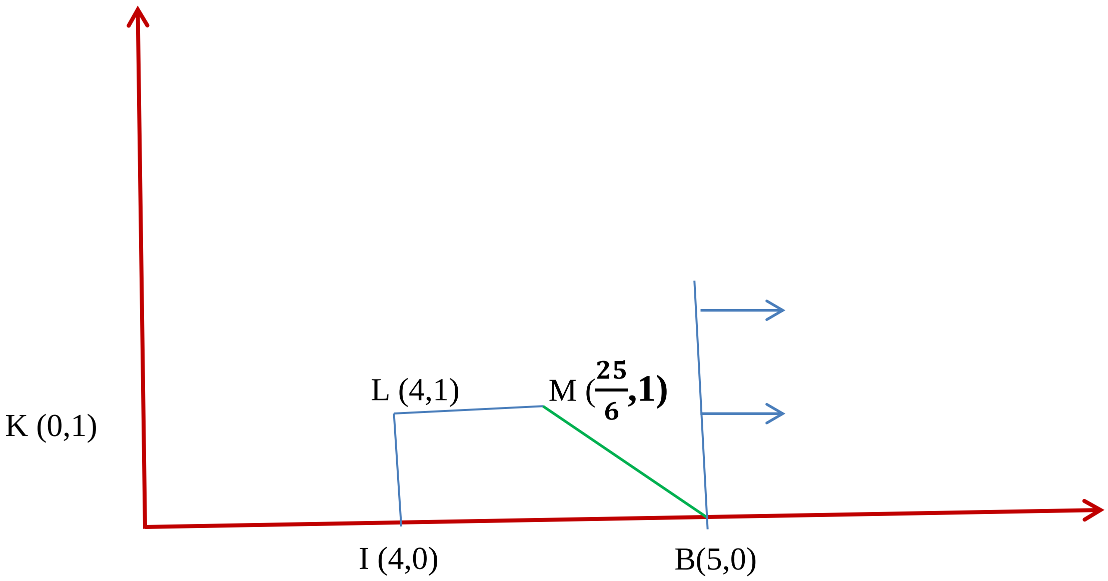
The new feasible region is line segment IL

Value of the objective function x_1+x_2 at

- **I (4,0) is 4**
- **L (4,1) is 5**

Since, the problem is of maximization and the maximum value is 5 which is corresponding to $x_1=4$ and $x_2=1$. So, the initial optimal solution is $x_1=4$ and $x_2=1$.

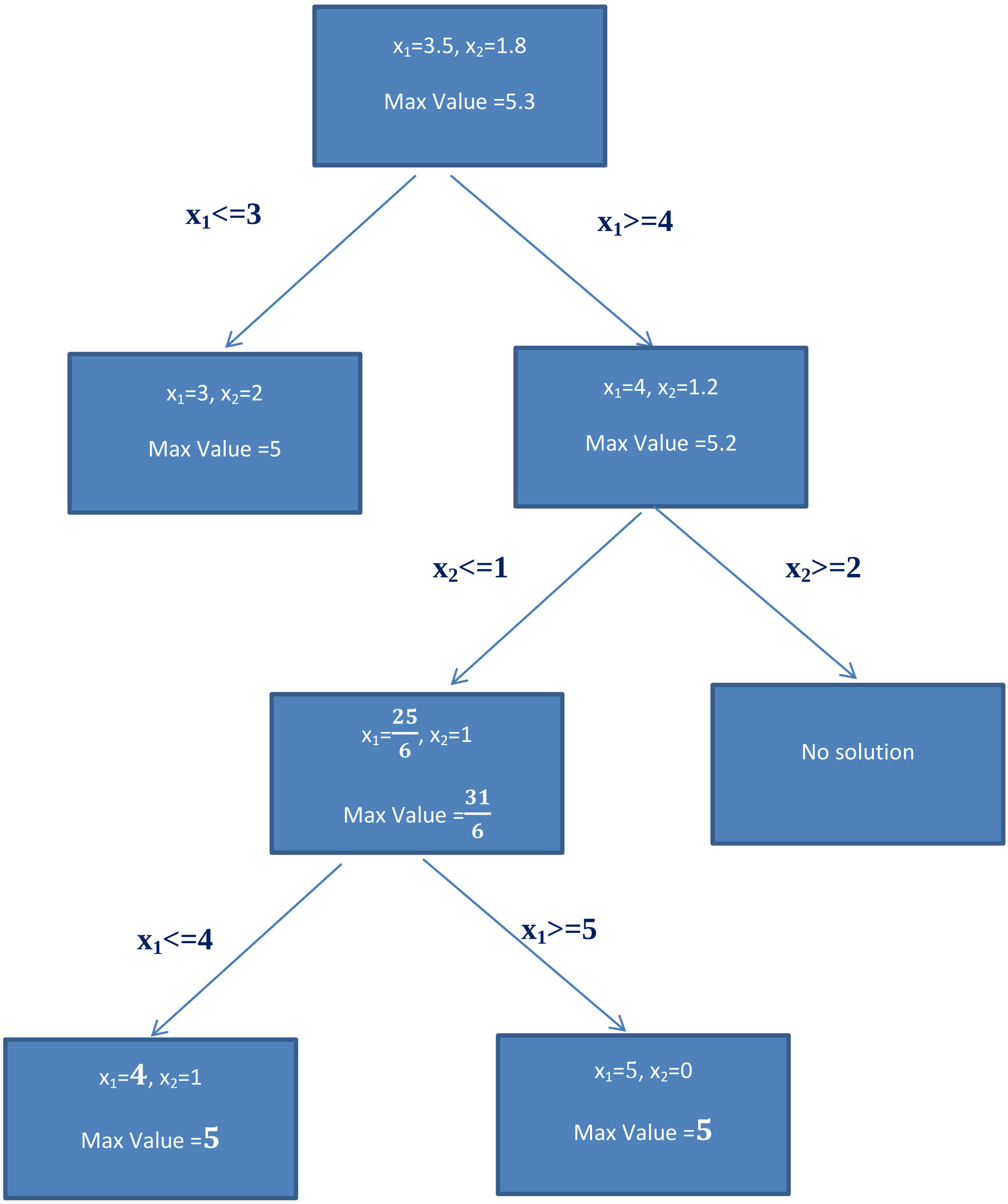
Including x_1>=5 in the final graph of case (i),



The new feasible region is the point B (5,0)

Value of the objective function x_1+x_2 at

- **B (5,0) is 5**



There are there optimal solutions

- **$x_1=4$ and $x_2=1$**

- $x_1=5$ and $x_2=0$
- $x_1=3$ and $x_2=2$

Second method

The value of x_2 is 1.8.

The non-negative integers less than 1.8 are 0,1 and the non-negative integers greater than 1.8 are 2,3,4,5,6,…..

The required value of x_2 will be 0 or 1 i.e., <=1

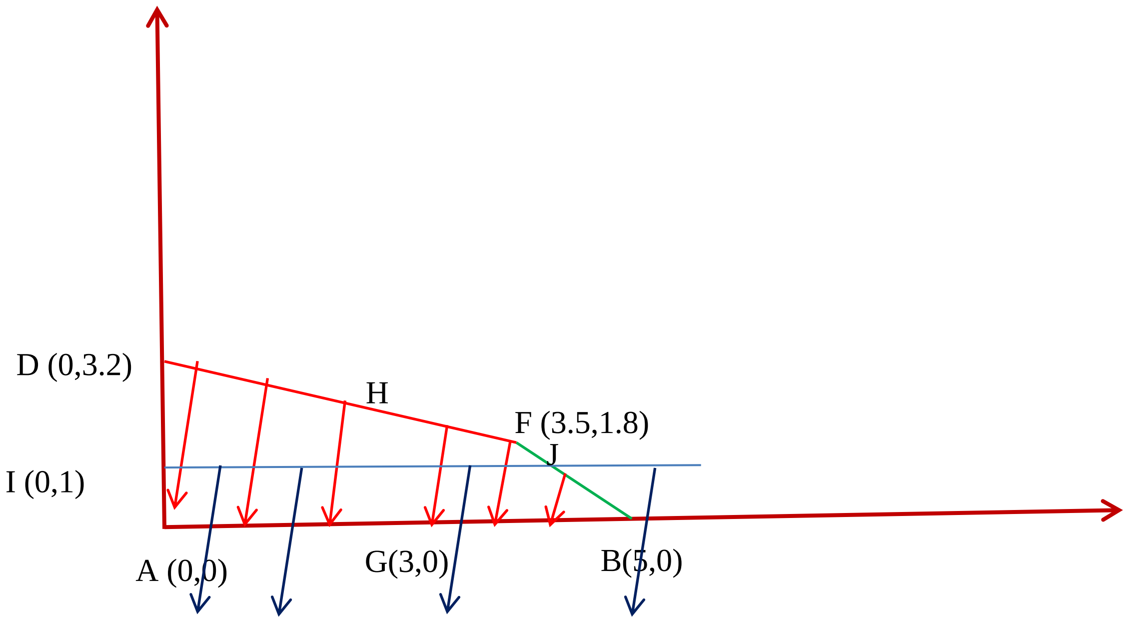
OR

The required value of x_2 will be 2,3,4 or 5 or 6 or ….. i.e., >=2

Hence,

x_2<=1 or x_2>=2.

Case (i) Including x_2<=1 in the graph,



The new feasible region is ABJI

where,

J is intersection of $x_2=1$ and $6x_1+5x_2=30$.

Solving

$x_1=\frac{25}{6}$ and $x_2=1$

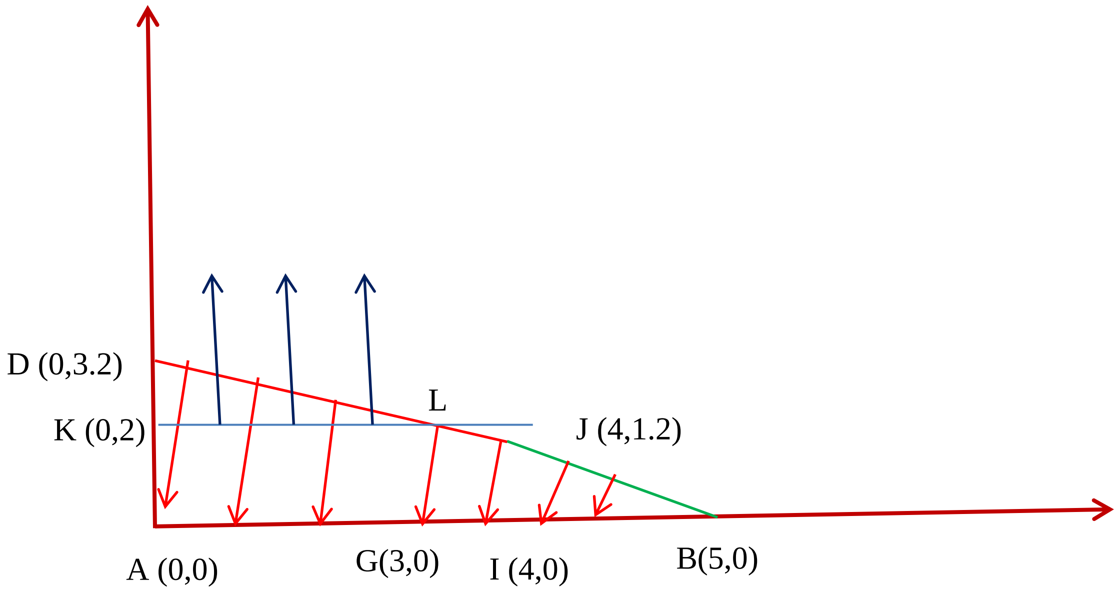
Therefore, J $(\frac{25}{6}, 1)$.

Value of the objective function x_1+x_2 at

- **A (0,0) is 0**
- **B (5,0) is 5**
- **J $(\frac{25}{6}, 1)$ is $\frac{31}{6}$**
- **I (0, 1) is 1.**

Since, the problem is of maximization and the maximum value is $\frac{31}{6}$ which is corresponding to $x_1=\frac{25}{6}$ and $x_2=1$. So, the initial optimal solution is $x_1=\frac{25}{6}$ and $x_2=1$.

Case (ii) Including $x_2>=2$ in the graph,



The new feasible region is KLD

where,

L is intersection of x_2=2 and $2x_1+5x_2$=16.

Solving

x_1=3 and x_2=2

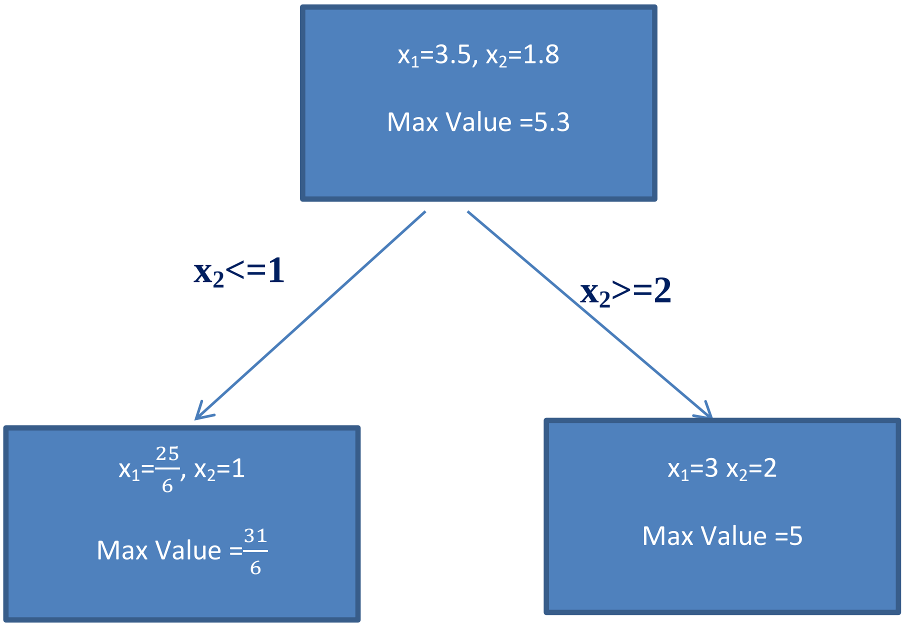
Therefore, L (3 , 2).

Value of the objective function x_1+x_2 at

- **K (0,2) is 2**
- **L (3,2) is 5**

- **D (0, 3.2) is 3.2**

Since, the problem is of maximization and the maximum value is 5 which is corresponding to x_1=3 and x_2=2. So, the initial optimal solution is x_1=3 and x_2=2.



The maximum value obtained in case 1 is more than the maximum value obtained in case 2.

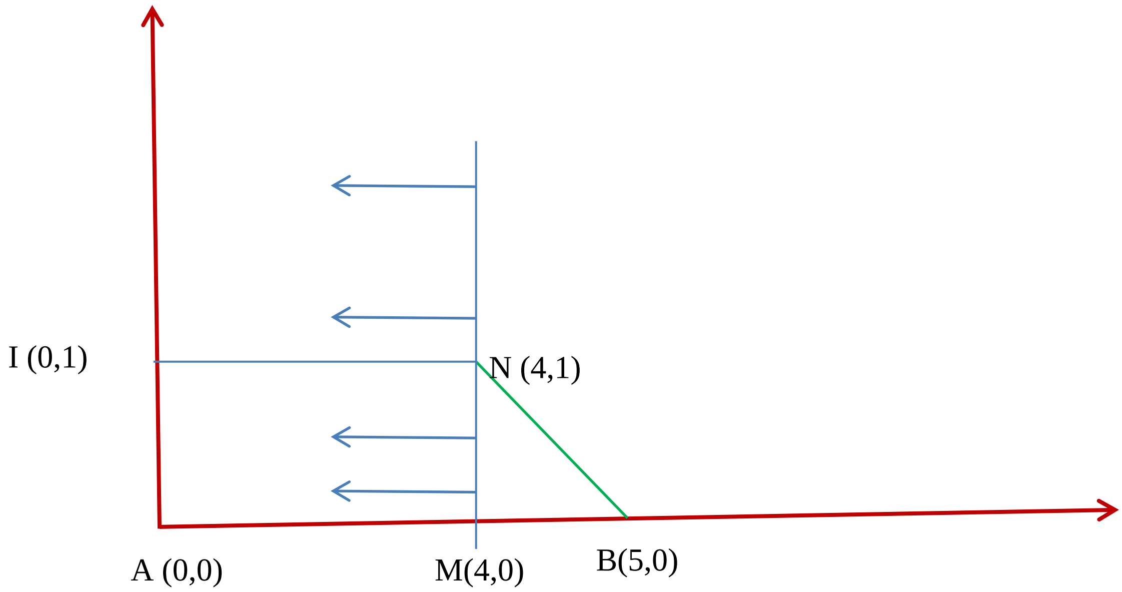
So, we will proceed with case 1.

The value of $x_1=\frac{25}{6}$

Case (i) x_1<=4

Case (ii) x_1>=5

Including $x_1<=4$ in the final graph of case (i),



The new feasible region is AMNI

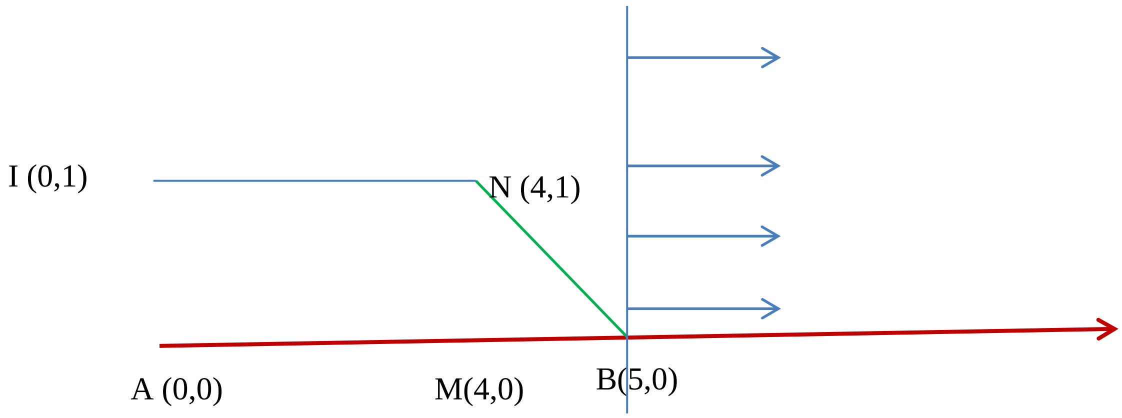
where,

Value of the objective function x_1+x_2 at

- **A (0,0) is 0**
- **M (4,0) is 4**
- **N (4, 1) is 5**
- **I (0 , 1) is 1**

Since, the problem is of maximization and the maximum value is 5 which is corresponding to $x_1=4$ and $x_2=1$. So, the initial optimal solution is $x_1=4$ and $x_2=1$.

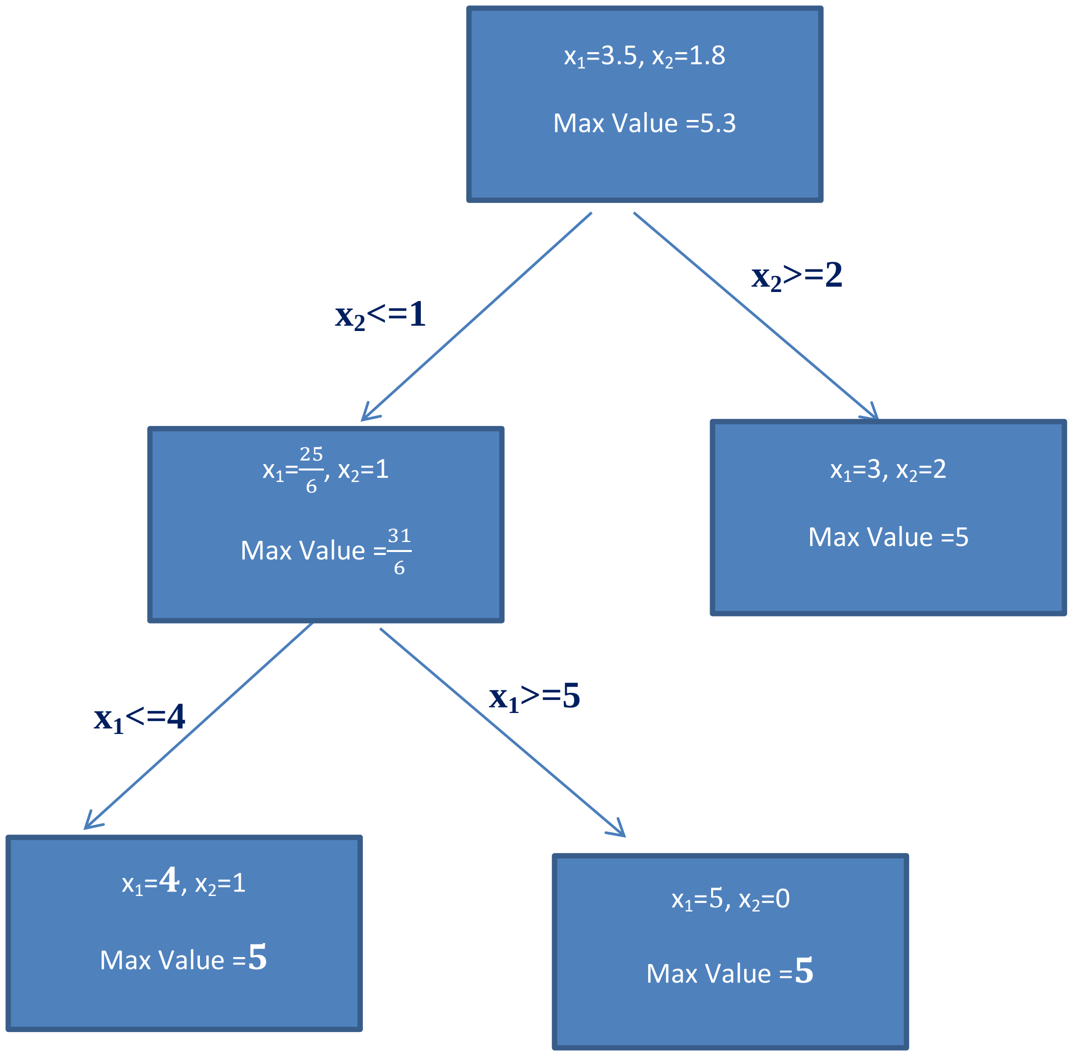
Including x_1>=5 in the final graph of case (i),



The new feasible region is the point B (5,0)

Value of the objective function x_1+x_2 at

- **B (5,0) is 5**



There are there optimal solutions

- **$x_1=4$ and $x_2=1$**
- **$x_1=5$ and $x_2=0$**
- **$x_1=3$ and $x_2=2$**

Example

Max ($5x_1+4x_2$)

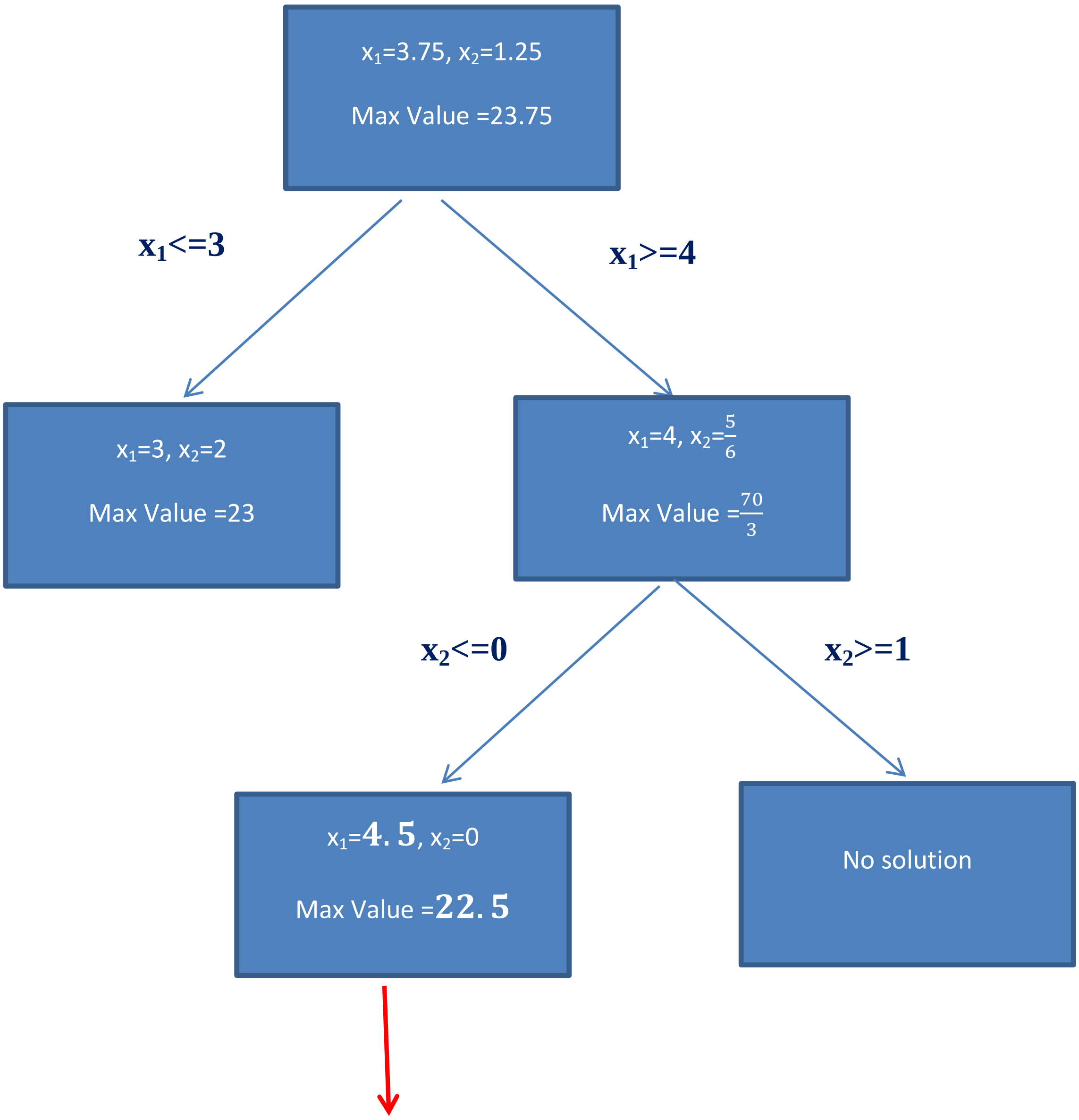
Subject to

$x_1+x_2<=5$

$10x_1+6x_2<=45$

$x_1, x_2 >=0$ and integers

DO YOURSELF



On proceeding below (in case of maximization problem), the maximum value decreases or remain same. Therefore, if we will consider the case $x_1 <= 4$ and $x_1 >= 5$ then the obtained value will be either 22.5 or less than 22.5. While, we have one integer solution $x_1=3$ and $x_2=2$ and the maximum value corresponding to this solution is 23. Therefore, no need to consider the cases $x_1 <= 4$ and $x_1 >= 5$.

The optimal solution is $x_1=3$ and $x_2=2$ and the maximum value is 23.