Lecture 26: Numerical Analysis (UMA011)

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Error bounds in solutions of system of linear equations:

Residual vector property:

If $\hat{x}$ is an approximation to the solution x of $Ax = b$ and the residual vector $r = b - A\hat{x}$ has the property that $\|r\|$ is small, then $\|x - \hat{x}\|$ would be small as well.

$Ax = b$

$\hat{x}$

$Ax = b \qquad b - Ax = 0$

$A\hat{x} \approx b$

$r = b - A\hat{x}$

If $\|r\|_\infty$ is small

then $\|x - \hat{x}\|_\infty$ would be small.

Error bounds in solutions of system of linear equations

But, certain systems fail to have this property:

Example:

The linear system $Ax = b$ given by

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$$

has the unique solution $x = (1, 1)^t$. Determine the residual vector for the poor approximation $\hat{x} = (3, -0.0001)^t$.

Solution:

$$r = b - A\hat{X} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -0.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix} - \begin{bmatrix} 2.9998 \\ 3.0001 \end{bmatrix} = \begin{bmatrix} 0.0002 \\ 0 \end{bmatrix}$$

$$\|r\|_\infty = \max\{|0.0002|, |0|\} = 0.0002$$

$$x - \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ -0.0001 \end{bmatrix} = \begin{bmatrix} -2 \\ 1.0001 \end{bmatrix}$$

$$\|x - \hat{x}\|_\infty = \max\{|-2|, |1.0001|\} = 2$$

is not small.

Error bounds in solutions of system of linear equations

We can obtain this information by considering the norms of the matrix A and its inverse.

Condition number:

The condition number of the non-singular matrix A relative to maximum norm $\|.\|$ is

$$K(A) = \|A\| \, \|A^{-1}\|.$$

$\|\cdot\|_\infty$

Error bounds in solutions of system of linear equations

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\|I\| = \|A \times A^{-1}\|$$

Condition number:

For any non-singular matrix A and infinity norm $\|.\|$

$$1 = \|I\|_{\infty} = \|A.\,A^{-1}\| \le \|A\|\,\|A^{-1}\| = K(A).$$

$\Rightarrow K(A) \geq 1$ ✓

Note: A matrix A is well-conditioned if $K(A)$ is close to 1, and is ill-conditioned when $K(A)$ is significantly greater than 1.

Error bounds in solutions of system of linear equations:

Example:

Determine the condition number for the matrix:

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$$

let $A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$ then $A^{-1} = \dfrac{1}{2 - 2.0002} \begin{bmatrix} 2 & -1.0001 \\ -2 & 1 \end{bmatrix}^t$

$$= \frac{1}{-0.0002} \begin{bmatrix} 2 & -2 \\ -1.0001 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10000 & 10000 \\ 5000.5 & -5000 \end{bmatrix}$$

$K(A) = \|A\| \, \|A^{-1}\|$

$$\|A\|_\infty = \max\{|3|, |3.0001|\} = 3.0001$$

$$\|A^{-1}\|_\infty = \max\{20000, 10000.5\} = 20000$$

$$K(A) = 60002 >>> 1$$

$\Rightarrow$ A is ill-conditioned.

Error bounds in solutions of system of linear equations

Result:

Suppose that $\hat{x}$ is an approximation to the solution of $Ax = b$, A is a nonsingular matrix, and r is the residual vector for $\hat{x}$. Then for any natural norm,

$r = b - A\hat{x}$

$$\|x - \hat{x}\| \leq \|r\| \|A^{-1}\|$$

A.E.

and if $x \neq 0$ and $b \neq 0$,

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \frac{\|r\|}{\|b\|}. = K(A) * \frac{\|r\|}{\|b\|}$$

R.E.

Proof

Here $r = b - A\hat{X} = AX - A\hat{X}$ $\quad (\because b = AX)$

$$r = A(X - \hat{X})$$

$$A^{-1} r = X - \hat{X}$$

Now, $\|X - \hat{x}\| = \|A^{-1} r\| \le \|A^{-1}\| \, \|r\|$

$$\|X - \hat{x}\| \le \|r\| \, \|A^{-1}\| \quad \text{--- (1)}$$

To find bound in R.E. $\quad b = AX$

$$\|b\| = \|AX\| \le \|A\| \, \|X\|$$

$$\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|} \quad -(2)$$

Multiply (1) & (2)

$$\frac{\|x-\hat{x}\|}{\|x\|} \leq \frac{\|r\| \, \|A^{-1}\| * \|A\|}{\|b\|}$$

$$\frac{\|x-\hat{x}\|}{\|x\|} \leq K(A) \frac{\|r\|}{\|b\|}$$

Error bounds in solutions of system of linear equations:

Exercise:

1. Estimate the condition number for the following matrices:
   a.
   $$\begin{bmatrix} 5 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$
   b.
   $$\begin{bmatrix} 1 & 2 \\ 0.9999 & 2 \end{bmatrix}$$

   Are these matrices ill-conditioned?

error bounds in solutions of system of linear equations:

Exercise(continued):

2 Suppose $\hat{x} = \begin{bmatrix} 0.98 \\ 1.1 \end{bmatrix}$ is an approximate solution for the linear system $Ax = b$, where

$$A = \begin{bmatrix} 3.9 & 1.6 \\ 6.8 & 2.9 \end{bmatrix}, \quad b = \begin{bmatrix} 5.5 \\ 9.7 \end{bmatrix}.$$

$r = b - A\overset{n}{x}$

Find a bound for the relative error $\frac{\|x - \hat{x}\|}{\|x\|}$.