Lecture 35: Numerical Analysis (UMA011)

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Least Square Approximation Method:

Least square fit of a general function:

Suppose that the data points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Let $f(x, a, b, \cdots)$, where $a, b, \cdots$ are the constants to be determined to the given data. Now residuals is given by

$$e_i = y_i - f(x_i, a, b, \cdots) \Rightarrow E = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} (y_i - f(x_i, a, b, \cdots))^2.$$

We need to find $a, b, \cdots$ such that error E is minimum.

The necessary condition for minimum is $\dfrac{\partial E}{\partial a} = 0, \dfrac{\partial E}{\partial b} = 0, \cdots$

Least Square Approximation Method:

Example:

Use the method of least square to fit the curve $f(x) = c_0 + \frac{c_1}{\sqrt{x}}$ to the following data:

	x_0	x_1	x_2	x_3	x_4
x	0.2	0.4	0.6	0.8	1.0
$y = f(x)$	16	14	11	6	3
	y_0	y_1	y_2	y_3	y_4

i.e.: find c_0, c_1

Solution:

let $e_i = y_i - f(x_i) = y_i - \left(c_0 + \frac{c_1}{\sqrt{x_i}}\right)$

$$E = \sum_{i=0}^{n} (y_i - f(x_i))^2 = \sum_{i=0}^{4} \left(y_i - \left(c_0 + \frac{c_1}{\sqrt{x_i}}\right)\right)^2$$

To get E is minimum, $\frac{\partial E}{\partial c_0} = 0$, $\frac{\partial E}{\partial c_1} = 0$

$$2\sum_{i=0}^{4}\left(y_i-\left(c_0+\frac{c_1}{\sqrt{x_i}}\right)\right)(-1)=0$$

&

$$2\sum_{i=0}^{4}\left(y_i-\left(c_0+\frac{c_1}{\sqrt{x_i}}\right)\right)\left(\frac{-1}{\sqrt{x_i}}\right)=0$$

$$\Rightarrow \quad \sum_{i=0}^{4}y_i - c_0\sum_{i=0}^{4}1 - c_1\sum_{i=0}^{4}\frac{1}{\sqrt{x_i}}=0$$

$$5c_0-\sum_{i=0}^{4}y_i+c_1\sum_{i=0}^{4}\frac{1}{\sqrt{x_i}}=0 \quad — \quad 1$$

&

$$\sum_{i=0}^{4}\frac{y_i}{\sqrt{x_i}} - c_0\sum_{i=0}^{4}\frac{1}{\sqrt{x_i}} - c_1\sum_{i=0}^{4}\frac{1}{x_i}=0 \quad — \quad 2$$

i	x_i	y_i	$1/\sqrt{x_i}$	$y_i/\sqrt{x_i}$	$1/x_i$
0	0.2	16	2.2361	35.7776	5
1	0.4	14	1.5811	22.1354	2.5
2	0.6	11	1.2910	14.201	1.6667
3	0.8	6	1.1180	6.708	1.25
4	1.0	3	1	3	1
		50	7.2262	81.822	8.9167

$c_0 = -1.1886$

$c_1 = 7.5961$

from eqn 1, 2, we get

$$5c_0 - 50 + c_1(7.2262) = 0$$

$$\Rightarrow 5c_0 + 7.2262\,c_1 = 50 \quad -(3)$$

& $$81.822 - 7.2262\,c_0 - 8.9167\,c_1 = 0$$

$$\Rightarrow 7.2262\,c_0 + 8.9167\,c_1 = 81.822 \quad -(4)$$

By solving 3, 4 we can get c_0, c_1

Note:

If value of function $f(x)$ at $x=0$ is also given in the prev. example data

like

x:	$\boxed{0}$	0.2	0.4	0.6	0.8	1.0
$f(x)$:	$\boxed{20}$	16	14	11	6	3

and it is asked to fit a curve which has x in the denominator

like $f(x) = c_0 + \frac{c_1}{\sqrt{x}}$

Then??

✩ try to remove x from the denominator,

like $\sqrt{x}\, f(x) = \sqrt{x}\, c_0 + c_1$

$F(x) = X c_0 + c_1$ where $X = \sqrt{x}$, $F(x) = \sqrt{x}\, f(x)$

Least Square Approximation Method:

Example:

Use the method of least square to fit the curve $f(x) = ab^x$ to the following data:

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	16	14	11	6	3

Solution: The given curve is $f(x) = ab^x$

Taking log both sides.

$$\log(f(x)) = \log(ab^x)$$

$$\log(f(x)) = \underbrace{\log a}_{A} + x \underbrace{\log b}_{B}$$

$$F(x) = A + xB$$

To find normal eq^{n}s. for linear function

$e_i = y_i - (A + Bx_i)$

$\sum_{i=0}^{4} e_i^2 = E$ (say)

$= \sum_{i=0}^{4} (y_i - (A + Bx_i))^2$

$\frac{\partial E}{\partial A} = 0, \quad \frac{\partial E}{\partial B} = 0$

$\sum_{i=0}^{4} (y_i - A - Bx_i) = 0$

$\sum_{i=0}^{4} (y_i x_i - Ax_i - Bx_i^2) = 0$

Here, $y_i = F(x_i)$

i	x_i	$f(x_i)$	$\log(f(x_i)) = F(x_i)$	$x_i F(x_i)$	x_i^2
0	0.2	16	1.2041	0.2408	0.04
1	0.4	14	1.1461	0.4584	0.16
2	0.6	11	1.0414	0.6248	0.36
3	0.8	6	0.7782	0.6226	0.64
4	1.0	3	0.4771	0.4771	1.0
	3.0		4.6469	2.4237	2.2

For the linear function $A + Bx$, we use the least square approximation to x_i, $f(x_i)$, then the normal equations are

$$\sum_{i=0}^{4} F(x_i) - A(5) - B\sum_{i=0}^{4} x_i = 0 \quad \text{--- (1)}$$

&

$$\sum_{i=0}^{4} F(x_i)\, x_i - A\sum_{i=0}^{4} x_i - B\sum_{i=0}^{4} x_i^2 = 0 \quad \text{--- (2)}$$

The normal eqn's becomes after using all the values in table :-

$$4.6469 - 5A - 3B = 0$$

$$\Rightarrow \quad 5A + 3B = 4.6469$$

&

$$2.4237 - 3A - 2.2B = 0$$

$$3A + 2.2B = 2.4237$$

solve these equation's to get A, B

Now, $A = \log a \Rightarrow a = \text{antilog}(A)$

$B = \log b \qquad b = \text{antilog}(B)$

Least Square Approximation Method:

Exercise:

1. By the method of least square fit a curve of the form $y = ax^b$ to the following data.

x	2	3	4	5
y	27.8	62.1	110	161

Hint:-

$y = ax^b$

$\log y = \log a + b \log x$

$Y = A + bX$

2. Use the method of least squares to fit a curve $y = \frac{c_0}{x} + c_1\sqrt{x}$ to the following data.

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

Hint: either.

$xy = c_0 + c_1 x^{3/2}$

$f(x) = c_0 + c_1 X$

or $\sum_{i=0}^{5} e_i^2 = \sum_{i=0}^{5}\left(y_i - \left(\frac{c_0}{x_i} + c_1\sqrt{x_i}\right)\right)^2$

Least Square Approximation Method:

Exercise:

3. We are given the following values of a function f of the variable t :

t	0.1	0.2	0.3	0.4
f	0.76	0.58	0.44	0.35

Obtain a least square approximation of the form $f(t) = ae^{-3t} + be^{-2t}$.

Hint:– $E := \sum_{i=0}^{3} e_i^2 = \sum_{i=0}^{3} \left(y_i - \left(a e^{-3t_i} + b e^{-2t_i}\right)\right)^2$

find eqⁿs in a, b by taking $\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0$