School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

End-Semester Examination, December 2018

B.E. IV Semester — UMA007 : Numerical Analysis

Time Limit: 03 Hours — Maximum Marks: 100

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Instructions: This question paper has two printed pages. You are expected to answer all the questions. Organize your work in a reasonably neat, organized, and coherent way. Mysterious or unsupported answers will not receive full credit. Calculator without graphing mode is permitted.

1. (a) Show that the computation of

$$f(x) = \frac{e^x - 1}{x}$$

is unstable for small value of x. Rewrite the expression to make it stable. [10 marks]

(b) Show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-2}. [10 marks]

2. (a) Using the four-digit arithmetic, solve the following system of equations by Gaussian elimination with partial pivoting

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000. \end{aligned}$$

[10 marks]

(b) Let us consider a system $Ax = b$ and apply an iterative method with initial guess $x^{(0)}$. Prove that the sequence of iterations $\{x^{(k)}\}$ defined by

$$x^{(k)} = Tx^{(k-1)} + c, \quad 0, 1, 2, \cdots$$

converges to the unique solution of $x = Tx + c$ if and only if spectral radius $\rho(T) < 1$. [10 marks]

3. (a) Let $f(x) = \sqrt{x - x^2}$ and $P_2(x)$ be the Lagrange interpolating polynomial on $x_0 = 0$, x_1 and $x_2 = 1$. Find the largest value of x_1 in $(0, 1)$ for which $f(0.5) - P_2(0.5) = -0.25$. [7 marks]

(b) Let $f \in C^n[a, b]$ and $x_0, x_1, x_2, \cdots, x_n$ are distinct numbers in $[a, b]$. Let $P_n(x)$ be the interpolating polynomial in Newton's form. Then prove that there exists a point $\xi \in (a, b)$ such that

$$f[x_0, x_1, x_2, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

[7 marks]

(c) In Neville's method, suppose $x_j = j$, for $j = 0, 1, 2, 3$ and it is known that $P_{0,1}(x) = x + 1$, $P_{1,2}(x) = 3x - 1$, and $P_{1,2,3}(1.5) = 4$. Find $P_{2,3}(1.5)$ and $P_{0,1,2,3}(1.5)$. —Neville's [6 marks]

4. (a) Determine the values of number of subintervals n and step-size h required to approximate

$$\int_0^1 \frac{1}{x+4}\, dx$$

to within 10^{-3} and hence compute the approximation using composite Trapezoidal rule. [10 marks]

(b) Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

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and hence derive the formula for

$$\int_a^b f(x)dx.$$

1.01848 [10 marks]

5. (a) Consider the initial value problem

$$\frac{dy}{dt} = 1 + ty, \quad y(0) = 1.$$

Show that the function $f(t, y) = 1 + ty$ satisfies a Lipschitz condition for region $0 \le t \le 2$. Also find first three approximations of the solution using Picard's method.

[10 marks]

(b) Transform the second-order initial-value problem

$$t^2y'' - 2ty' + 2y = t^3 \ln t, \; 1 \le t \le 1.1, \; y(1) = 1, \; y'(1) = 0$$

into a system of first order initial-value problems, and use the forth-order Runge-Kutta method with $h = 0.1$ to find the approximate solution $y(1.1)$.

[10 marks]