Cross–correlation and Auto–correlation

There are two types of correlation. One is called cross–correlation and another is called auto–correlation. The correlation is said to be cross–correlation if correlation of two different sequences x(n) and y(n) is done. On the other hand, if sequences are same, the correlation is said to be auto–correlation. The cross–correlation is denoted by $r_{xy}(l)$ and expressed by

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \ , \ l = 0, \pm1, \pm2, \ldots\ldots\ldots \quad (1)$$

i.e., $$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n) \ , \ l = 0, \pm1, \pm2, \ldots\ldots\ldots \quad (2)$$

In Eq. (1) y(n) is delayed with respect to x(n) whereas in Eq. (2) x(n) is advanced with respect to y(n). These two operations are equivalent and provide identical cross–correlation sequences.

The cross–correlation sequence $r_{yx}(l)$ is defined below.

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l) \ , \ l = 0, \pm1, \pm2, \ldots\ldots\ldots \quad (3)$$

i.e., $$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n) \ , \ l = 0, \pm1, \pm2, \ldots\ldots\ldots \quad (4)$$

The auto–correlation x(n) denoted by $r_{xx}(l)$ is defined as follow:

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) \ , \ l = 0, \pm1, \pm2, \ldots\ldots\ldots \quad (5)$$

i.e., $$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)x(n) \ , \ l = 0, \pm1, \pm2, \ldots\ldots\ldots \quad (6)$$

Properties of Cross–correlation and Auto–correlation Sequences

The important properties of correlation sequence are discussed below.

1. The cross–correlation is not commutative.

Proof:

The cross–correlation of x(n) and y(n) is given by

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \tag{7}$$

The cross–correlation of x(n) and y(n) can also be expressed as

$$\begin{aligned} r_{xy}(l) &= \sum_{n=-\infty}^{\infty} x(n+l)\, y(n) = \sum_{n=-\infty}^{\infty} x[n-(-l)]\, y(n) \\ &= \sum_{n=-\infty}^{\infty} y(n)\, x[n-(-l)] = r_{yx}(-l)) \end{aligned} \tag{8}$$

Equation (8) proves that the correlation is not commutative.

2. The auto–correlation is an even function, i.e.,

$$r_{xx}(l) = r_{xx}(-l) \tag{9}$$

Proof:

The auto–correlation of x(n) is given by

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l) \tag{10}$$

The auto–correlation of x(n) and x(n) can also be expressed as

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, x(n) = \sum_{n=-\infty}^{\infty} x[n-(-l)]x(n) = r_{xx}(-l) \tag{11}$$

Equation (11) proves that the auto–correlation is an even function.

3. The cross–correlation is equivalent to convolution of one sequence with another folded sequence , i.e.,

$$r_{xy}(l) = x(l) * y(-l) \tag{12}$$

Proof:

The linear convolution of two sequences x(n) and y(n) is expressed by

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) \tag{13}$$

Replacing ‘n’ by ‘l’ we have from Eq. (13)

$$x(l) * y(l) = \sum_{k=-\infty}^{\infty} x(k)y(l-k) \tag{14}$$

Replacing ‘k’ by ‘n’, we have from Eq. (14)

$$x(l) * y(l) = \sum_{n=-\infty}^{\infty} x(n)y(l-n) \tag{15}$$

Similarly,

$$x(l) * y(-l) = \sum_{n=-\infty}^{\infty} x(n)y[-(l-n)] = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \tag{16}$$

The RHS of Eq. (16) is the cross–correlation of $x(l)$ and $y(l)$.

$$\therefore r_{xy}(l) = x(l) * y(-l) \tag{17}$$

Hence the cross–correlation is equivalent to convolution of one sequence with another folded sequence.

4. The cross–correlation satisfies the following:

$$\left| r_{xy}(l) \right| \le \sqrt{r_{xx}(0) r_{yy}(0)} = \sqrt{E_x E_y}$$

Proof:

Let us consider two finite energy sequences x(n) and y(n) and their combination be

$$a_1 x(n) + a_2 y(n-l)$$

where a_1 and a_2 are constants and l is time shift.

The energy of the signal is given by

$$E = \sum_{n=-\infty}^{\infty} [a_1 x(n) + a_2 y(n-l)]^2$$

$$= a_1^2 \sum_{n=-\infty}^{\infty} x^2(n) + 2a_1 a_2 \sum_{n=-\infty}^{\infty} x(n)y(n-l) + a_2^2 \sum_{n=-\infty}^{\infty} y^2(n-l) \tag{18}$$

$$= a_1^2 r_{xx}(0) + 2a_1 a_2 r_{xy}(l) + a_2^2 r_{yy}(0) \tag{19}$$

The energy of the signal $a_1 x(n) + a_2 y(n-l)$ is finite because the energy of the signal x(n) and y(n) is finite.

$$\therefore a_1^2 r_{xx}(0) + 2a_1 a_2 r_{xy}(l) + a_2^2 r_{yy}(0) \ge 0 \tag{20}$$

From Eq. (20), we say that $r_{xx}(0)$ is the energy of the sequence x(n).

$$\therefore E_x = r_{xx}(0) \tag{21}$$

Similarly,

$$\therefore E_y = r_{yy}(0) \tag{22}$$

From Eq. (20) we have

$$\left(\frac{a_1^2}{a_2^2}\right) r_{xx}(0) + 2\left(\frac{a_1}{a_2}\right) r_{xy}(l) + r_{yy}(0) \ge 0$$

i.e., $\left(\frac{a_1}{a_2}\right)^2 r_{xx}(0) + 2\left(\frac{a_1}{a_2}\right) r_{xy}(l) + r_{yy}(0) \geq 0$

i.e., $k^2 r_{xx}(0) + 2k r_{xy}(l) + r_{yy}(0) \geq 0$ (23)

where $k = \frac{a_1}{a_2}$ is a finite value.

Equation (23) can be rewritten as

$$\begin{bmatrix} k & 1 \end{bmatrix} \begin{bmatrix} r_{xx}(0) & r_{xy}(l) \\ r_{xy}(l) & r_{yy}(0) \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix} \geq 0 \tag{24}$$

For any finite value of 'k', we have from Eq. (24)

$$r_{xx}(0) r_{yy}(0) - r_{xy}^2(l) \geq 0 \tag{25}$$

i.e., $r_{xy}(l) \leq \sqrt{r_{xx}(0) r_{yy}(0)}$ (26)

i.e., $r_{xy}(l) \leq \sqrt{E_x E_y}$ (27)

Equation (26) and Eq. (27) prove the required relation.

5. The auto–correlation sequence attains maximum value at zero lag ($l = 0$), i.e.,

$$|r_{xx}(l)| \leq r_{xx}(0) = E_x \tag{28}$$

Proof:

If x(n) = y(n), we have from Eq. (26) and Eq. (27)

$$|r_{xx}(l)| \leq r_{xx}(0)$$

i.e., $|r_{xx}(l)| \leq E_x$

The normalized expression for $r_{xx}(l)$ is expressed by

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} \tag{29}$$

Similarly, the normalized cross–correlation sequences is

$$\rho_{xx}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}} \tag{30}$$

where $\rho_{xy}(l)$ is also called the cross–correlation coefficient.

6. The shape of cross–correlation sequence is totally dependent on the shapes of x(n) and y(n) and it does not depend on their instantaneous amplitudes.

Obtain the cross correlation sequence $r_{xy}(l)$ of the following sequences:

$$x(n) = \{3,-2,4,6,\underset{\uparrow}{1},3,-5\} \text{ and } y(n) = \{2,-2,3,-3,\underset{\uparrow}{5},1,-3,6\}.$$

Solution:

The sequences are

$$\begin{array}{ll} x(-4) = 3 & y(-4) = 2 \\ x(-3) = -2 & y(-3) = -2 \\ x(-2) = 4 & y(-2) = 3 \\ x(-1) = 6 & y(-1) = -3 \\ x(0) = 1 \leftarrow & y(0) = 5 \leftarrow \\ x(1) = 3 & y(1) = 1 \\ x(2) = -5 & y(2) = -3 \\ & y(3) = 6 \end{array}$$

The expression for cross correlation sequence $r_{xy}(l)$ is shown below.

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \tag{1}$$

For x(n), 'n' varies from – 4 to 2.

Eq. (1) becomes

$$r_{xy}(l) = \sum_{n=-4}^{2} x(n)y(n-l) \tag{2}$$

The Eq. (2) after expansion becomes

$$\begin{aligned} r_{xy}(l) &= x(-4)y(-4-l) + x(-3)y(-3-l) + x(-2)y(-2-l) + x(-1)y(-1-l) \\ &\quad + x(0)y(-l) + x(1)y(1-l) + x(2)y(2-l) \end{aligned}$$

Here $r_{xy}(l) = x(l) * y(-l)$

$$x(l) = \{3,-2,4,6,\underset{\uparrow}{1},3,-5\}$$

$$y(l) = \{2,-2,3,-3,\underset{\uparrow}{5},1,-3,6\}$$

$$y(-l) = \{6,-3,1,\underset{\uparrow}{5},-3,3,-2,2\}$$

The lower limit of $x(l)$ is -4 and that of $y(-l)$ is -3. Hence the overall lower limit becomes $-(4 + 3) = -7$. Hence l started from -7. The overall upper limit is $2 + 4 = 6$. Hence 'l' ends at 6.

$$\begin{aligned}\therefore r_{xy}(-7) &= x(-4)y(-4+7) + x(-3)y(-3+7) + x(-2)y(-2+7) + x(-1)y(-1+7)\\ &\quad + x(0)y(7) + x(1)y(1+7) + +x(2)y(2+7)\\ &= x(-4)y(3) + x(-3)y(4) + x(-2)y(5) + x(-1)y(6) + x(0)y(7) + x(1)y(8) + x(2)y(9)\\ &= x(-4)y(3) \quad [\because y(4) = y(5) = \cdots\cdots = y(9) = 0]\\ &= 3\times 6 = 18\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(-6) &= x(-4)y(-4+6) + x(-3)y(-3+6) + x(-2)y(-2+6) + x(-1)y(-1+6)\\ &\quad + x(0)y(6) + x(1)y(1+6) + +x(2)y(2+6)\\ &= x(-4)y(2) + x(-3)y(3) + x(-2)y(4) + x(-1)y(5) + x(0)y(6) + x(1)y(7) + x(2)y(8)\\ &= x(-4)y(2) + x(-3)y(3)\\ &= 3\times(-3) + (-2)\times 6 = -9 - 12 = -21\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(-5) &= x(-4)y(1) + x(-3)y(2) + x(-2)y(3)\\ &= 3\times 1 + (-2)\times(-3) + 4\times 6 = 3 + 6 + 24 = 33\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(-4) &= x(-4)y(0) + x(-3)y(1) + x(-2)y(2) + x(-1)y(3)\\ &= 3\times 5 + (-2)\times 1 + 4\times(-3) + 6\times 6 = 15 - 2 - 12 + 36 = 37\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(-3) &= x(-4)y(-1) + x(-3)y(0) + x(-2)y(1) + x(-1)y(2) + x(0)y(3)\\ &= 3\times(-3) + (-2)\times 5 + 4\times 1 + 6\times(-3) + 1\times 6 = -9 - 10 + 4 - 18 + 6 = -27\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(-2) &= x(-4)y(-2) + x(-3)y(-1) + x(-2)y(0) + x(-1)y(1) + x(0)y(2) + x(1)y(3)\\ &= 3\times 3 + (-2)\times(-3) + 4\times 5 + 6\times 1 + 1\times(-3) + 3\times 6 = 9 + 6 + 20 + 6 - 3 + 18 = 56\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(-1) &= x(-4)y(-3) + x(-3)y(-2) + x(-2)y(-1) + x(-1)y(0)\\ &\quad + x(0)y(1) + x(1)y(2) + x(2)y(3)\\ &= 3\times(-2) + (-2)\times 3 + 4\times(-3) + 6\times 5 + 1\times 1 + 3\times(-3) + (-5)\times 6\\ &= -6 - 6 - 12 + 30 + 1 - 9 - 30 = -32\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(0) &= x(-4)y(-4) + x(-3)y(-3) + x(-2)y(-2) + x(-1)y(-1)\\ &\quad + x(0)y(0) + x(1)y(1) + x(2)y(2)\\ &= 3\times 2 + (-2)\times(-2) + 4\times 3 + 6\times(-3) + 1\times 5 + 3\times 1 + (-5)\times(-3)\\ &= 6 + 4 + 12 - 18 + 5 + 3 + 15 = 27\end{aligned}$$

$$\begin{aligned}\therefore r_{xy}(1) &= x(-4)y(-5) + x(-3)y(-4) + x(-2)y(-3) + x(-1)y(-2)\\ &\quad + x(0)y(-1) + x(1)y(0) + x(2)y(1)\\ &= 3\times 0 + (-2)\times 2 + 4\times(-2) + 6\times 3 + 1\times(-3) + 3\times 5 + (-5)\times 1\\ &= -4 - 8 + 18 - 3 + 15 - 5 = 13\end{aligned}$$

$$\therefore r_{xy}(2) = x(-4)y(-6) + x(-3)y(-5) + x(-2)y(-4) + x(-1)y(-3)$$
$$+ x(0)y(-2) + x(1)y(-1) + x(2)y(0)$$
$$= 3\times 0 + (-2)\times 0 + 4\times 2 + 6\times(-2) + 1\times 3 + 3\times(-3) + (-5)\times 5$$
$$= 8 - 12 + 3 - 9 - 25 = -35$$

$$\therefore r_{xy}(3) = x(-4)y(-7) + x(-3)y(-6) + x(-2)y(-5) + x(-1)y(-4)$$
$$+ x(0)y(-3) + x(1)y(-2) + x(2)y(-1)$$
$$= 6\times 2 + 1\times(-2) + 3\times 3 + (-5)\times(-3)$$
$$= 12 - 2 + 9 + 15 = 34$$

$$\therefore r_{xy}(4) = x(-4)y(-8) + x(-3)y(-7) + x(-2)y(-6) + x(-1)y(-5)$$
$$+ x(0)y(-4) + x(1)y(-3) + x(2)y(-2)$$
$$= 1\times 2 + 3\times(-2) + (-5)\times 3$$
$$= 2 - 6 - 15 = -19$$

$$\therefore r_{xy}(5) = x(1)y(-4) + x(2)y(-3)$$
$$= 3\times 2 + (-5)\times(-2)$$
$$= 6 + 10 = 16$$

$$\therefore r_{xy}(6) = x(2)y(-4)$$
$$= -5\times 2$$
$$= -10$$

$$\therefore r_{xy}(l) = \{18,-21,33,37,-27,56,-32,\underset{\uparrow}{27},13,-35,34,-19,16,-10\}$$. Ans.

Alternative Method:

The correlation of x(n) and y(n) can be expressed as

$$r_{xy}(l) = x(l) * y(-l)$$

Here $x(l) = \{3,-2,4,6,\underset{\uparrow}{1},3,-5\}$ and $y(l) = \{2,-2,3,-3,\underset{\uparrow}{5},1,-3,6\}$.

$$\therefore y(-l) = \{6,-3,1,\underset{\uparrow}{5},-3,3,-2,2\}$$

$x(l) \Rightarrow$								3	−2	4	6	1	3	−5
												$\uparrow$		
$y(-l) \Rightarrow$							6	−3	1	5	−3	3	−2	2
										$\uparrow$				
								6	−4	8	12	2	6	−10
							−6	4	−8	−12	−2	−6	10	×
						9	−6	12	18	3	9	−15	×	×
					−9	6	−12	−18	−3	−9	15	×	×	×
				15	−10	20	30	5	15	−25	×	×	×	×
			3	−2	4	6	1	3	−5	×	×	×	×	×
		−9	6	−12	−18	−3	−9	15	×	×	×	×	×	×
	18	−12	24	36	6	18	−30	×	×	×	×	×	×	×
	18	−21	33	37	−27	56	−32	27	13	−35	34	−19	16	−10
								$\uparrow$						

$$\therefore r_{xy}(l) = \{18, -21, 33, 37, -27, 56, -32, \underset{\uparrow}{27}, 13, -35, 34, -19, 16, -10\}$$. Ans.