Lecture 9
UEI407

Linear and Non-linear System

If a system satisfies the property of homogeneity and the property of superposition, the system is said to be linear.

- Homogeneity :

Let the system produces output y(t) for an input x(t). If the input is scaled ‘a’ times, the output will also be scaled by ‘a’ times for all values of ‘a’ and x(t). Therefore, we can write mathematically as follows:

$$x(t) \rightarrow y(t)$$

$$ax(t) \rightarrow ay(t)$$

- Superposition:

If two inputs $x_1(t)$ and $x_2(t)$ are applied to a system simultaneously, the overall output will be the effects of $y_1(t)$ and $y_2(t)$.

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

Combining these properties we can write the following for a linear system:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, then

$$a_1x_1(t) + a_2x_2(t) \rightarrow a_1y_1(t) + a_2y_2(t)$$

If any system does not satisfy the above two conditions, the system is called non–linear. Usually, non–linearity arises in a physical system due to non–linear input and output relation of its some components.

$$\frac{dy(t)}{dt} + 5x(t)\,y(t) = x(t)$$

represents a non–linear system whereas

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

represents a linear system.

Test the following systems for linearity:

(a) $y(t) = tx(t)$ (b) $y(t) = x(t^2)$ (c) $y(t)= x^2(t)$ (d) $y(t) = Ax(t) + B$

(a) Let H be the system operating on x(t) to produce

$y(t)=H\{x(t)\}=tx(t)$

Consider two signals $x_1(t)$ and $x_2(t)$

Let $y_1(t)$ and $y_2(t)$ be the respective responses of the system for two signals $x_1(t)$ and $x_2(t)$.

Continued:

$$\therefore \mathrm{y}_1(t) = H\{x_1(t)\} = tx_1(t)$$

$$and \;\; \mathrm{y}_2(t) = H\{x_2(t)\} = tx_2(t)$$

$$\therefore a_1 y_1(t) + a_2 y_2(t) = a_1 t x_1(t) + a_2 t x_2(t)$$

Let $\mathrm{y}_3(t)$ be the system response of the system H for the input $\mathrm{x}_3(t)$ *where* $\mathrm{x}_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

$$\therefore y_3(t) = H\{x_3(t)\} = H\{a_1 \mathrm{x}_1(t) + a_2 x_2(t)\}$$

$$= \mathrm{a}_1 H\{x_1(t)\} + \mathrm{a}_2 H\{x_2(t)\}$$

$$= \mathrm{a}_1 t x_1(t) + \mathrm{a}_2 t x_2(t)$$

$$= \mathrm{a}_1 y_1(t) + a_2 y_2(t)$$

Since $y_3(t) = \mathrm{a}_1 y_1(t) + a_2 y_2(t),$ the given system is linear.

(b) Let H be the system operating on x(t) to produce

$y(t)=H\{x(t)\}=x(t^2)$

Consider two signals $x_1(t)$ and $x_2(t)$

Let $y_1(t)$ *and* $y_2(t)$ be the respective responses of the system for two signals $x_1(t)$ and $x_2(t)$.

$\therefore\ y_1(t) = H\{x_1(t)\} = x_1(t^2)$

and $y_2(t) = H\{x_2(t)\} = x_2(t^2)$

$\therefore\ a_1 y_1(t) + a_2 y_2(t) = a_1 x_1(t^2) + a_2 x_2(t^2)$

Let $y_3(t)$ be the system response of the system H for the input $x_3(t)$ *where* $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

$$\begin{aligned}
\therefore\ y_3(t) = H\{x_3(t)\} &= H\{a_1 x_1(t) + a_2 x_2(t)\} \\
&= a_1 H\{x_1(t)\} + a_2 H\{x_2(t)\} \\
&= a_1 x_1(t^2) + a_2 x_2(t^2) \\
&= a_1 y_1(t) + a_2 y_2(t)
\end{aligned}$$

Since $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, the given system is linear.

(c) Let H be the system operating on x(t) to produce

$y(t)=H\{x(t)\}=x^2(t)$

Consider two signals $x_1^2(t)$ and $x_2^2(t)$

Let $y_1(t)$ *and* $y_2(t)$ be the respective responses of the system for two signals $x_1(t)$ and $x_2(t)$.

$\therefore y_1(t) = H\{x_1(t)\} = x_1^2(t)$

and $y_2(t) = H\{x_2(t)\} = x_2^2(t)$

$\therefore a_1 y_1(t) + a_2 y_2(t) = a_1 x_1^2(t) + a_2 x_2^2(t)$

Let $y_3(t)$ be the system response of the system H for the input $x_3(t)$ *where* $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

$$\therefore y_3(t) = H\{x_3(t)\} = x_3^2(t)$$

$$=\{a_1 x_1(t) + a_2 x_2(t)\}^2$$

$$=a_1^2 x_1^2(t) + a_2^2 x_2^2(t) + 2a_1 a_2 x_1(t) x_2(t)$$

Since $y_3(t) \neq a_1 y_1(t) + a_2 y_2(t)$, the given system is non-linear.

(d) Let H be the system operating on x(t) to produce

$y(t)=H\{x(t)\}=Ax(t)+b$

Consider two signals $x_1(t)$ and $x_2(t)$

Let $y_1(t)$ *and* $y_2(t)$ be the respective responses of the system for two signals $x_1(t)$ and $x_2(t)$.

$\therefore\ y_1(t) = H\{x_1(t)\} = Ax_1(t) + B$

and $y_2(t) = H\{x_2(t)\} = Ax_2(t) + B$

$\therefore\ a_1 y_1(t) + a_2 y_2(t) = A\{a_1 x_1(t) + a_2 x_2(t)\} + B(a_1 + a_2)$

Let $y_3(t)$ be the system response of the system H for the input $x_3(t)$ *where* $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

$$\therefore\ y_3(t) = H\{x_3(t)\} = Ax_3(t) + B$$

$$= A\{a_1 x_1(t) + a_2 x_2(t)\} + B$$

$$= Aa_1 x_1(t) + Aa_2 x_2(t) + B$$

Since $y_3(t) \neq a_1 y_1(t) + a_2 y_2(t)$, the given system is non-linear.

Linear Time Invariant System

A system is said to be linear time invariant (LTI) if the system is linear as well as also time invariant.