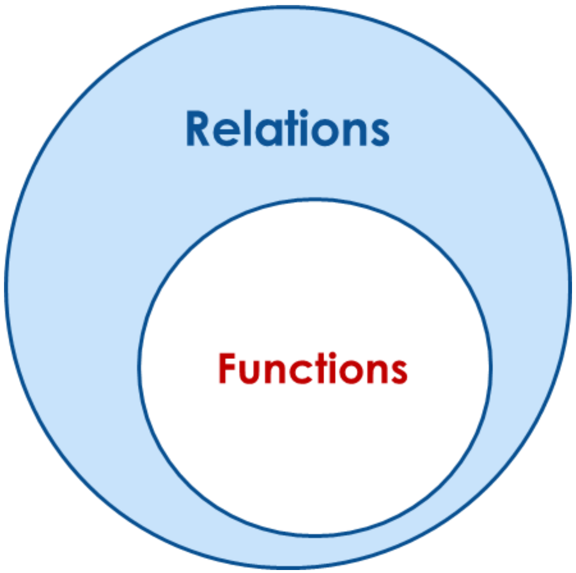

Relations

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Contents

What is a Relation?

In discrete mathematics, relation is a way of showing a relationship between any two sets.

- Relationship between any program and its variable.
- Relationship between pair of cities linked by railway in a network.

Necessity for studying Relation

- Relational Database model is based on the concept of relation.

Cartesian Product

- Given two sets A and B, their ***cartesian product*** $A \times B$, is defined as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Ordered Pairs

- The elements of $A \times B$ are called ***ordered pairs*** with the elements of A as the first entry and elements of B as the second entry.

- Order matters

Special Case:

$$A^2 = A \times A = \{(a_1, a_2) \mid a_1, a_2 \in A\}$$

Similarly,

$$A^n = A \times A \times \cdots \times A(n\ times) = \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in A\}$$

Relations

Relation is the subset of the cartesian product of the sets.

n-ary Relation

- Let $\{A_1, A_2, \ldots, A_n\}$ be n sets.
- An n-ary relation R on $A_1 \times A_2 \times \ \ldots \times A_n$ is a subset of $A_1 \times A_2 \times \ \ldots \times A_n$.
- If $A_i = \mathrm{A}\,; \forall i$, then R is called the *n-ary relation on A*.

Empty and Universal Relation

- If $R = \emptyset$, then R is called the ***empty*** or ***void relation***.

- If $R = A_1 \times A_2 \times \ldots \times A_n$, then R is called the ***universal relation***.

Definition (Binary Relation)

- Given two sets A and B, a relation between A and B is a subset of $A \times B$.
- If R is a relation on $A \times B$ (i.e., $R \subseteq A \times B$) and $(a, b) \in R$, we say "a is related to b".
- It can also be written as aRb.

Example:

Let $\boldsymbol{A} = \{\boldsymbol{a}, \boldsymbol{b}\}$ and $\boldsymbol{B} = \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$

$\boldsymbol{R} = \{(\boldsymbol{a}, \mathbf{3}), (\boldsymbol{b}, \mathbf{2}), (\boldsymbol{b}, \mathbf{4})\}$ is a relation from A to B.

Binary Relation on a set

- A binary relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a **subset of** $\boldsymbol{A} \times \boldsymbol{A}$.

Examples:

1. "Taller -than " is a relation on people.
   $(a, b) \in$ "Taller -than" if person a is taller than person b.

2. "$\geq$" is a relation on real set **R.**
   "$\geq$" $= \{(x, y) \in \mathbf{R} \mid x, y \in \mathbf{R}, x \geq y\}$

Examples (Cont..)

3. Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$.

If $R = \{(a, b) | \ a \ divides \ b)\}$ is a relation from A to B then ordered pairs in the relation R are

{(1,1,), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)}

Examples (Cont..)

Let A={1, 2, 3}

$$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

- Here, $A \times A$ is an universal relation on A.
- $\emptyset$ is an empty relation on A.

Examples (Cont..)

"=" = Δ= {(1,1), (2,2), (3,3)}

"<"= {(1,2), (1,3), (2,3)}

">"= {(2,1), (3,1), (3,2)}

"≤"= {(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

"≥"= {(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)}

"|"= {(1,1), (1,2), (1,3), (2,2), (3,3)}

"multiple of"= {(1,1), (2,1), (2,2), (3,1), (3,3)}

Representing Relations

Relations can be represented in two ways:

- Matrix
- Graph

Representation of Relations as Matrix

- If R is a relation on set $A = \{a_1, a_2, \dots, a_n\}$ and $|A| = n$, then it can be represented as $n \times n$ Boolean Matrix M_R.

M_R can be defined as:

$$M_R = [m_{ij}]_{n\times n}$$

where, $m_{ij} = \begin{cases} 0 & ; if\ (a_i, a_j) \notin R \\ 1 & ; if\ (a_i, a_j) \in R \end{cases}$

Examples

- Let A={1, 2, 3}
- Let $R = \{(1,1),(1,2),(2,1),(2,3),(3,2),(3,3)\}$ be a relation on A.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Examples (Cont..)

- Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

- Because R consists of those ordered pairs with $a_{ij} = 1$, it follows that:

$$\boldsymbol{R} = \{(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{3}),(\mathbf{2},\mathbf{4}),(\mathbf{2},\mathbf{5}),(\mathbf{3},\mathbf{1}),(\mathbf{3},\mathbf{3}),(\mathbf{3},\mathbf{5})\}.$$

Representation of Relations as a Digraph (Directed Graph)

- The graph of a relation R over A is a directed graph with nodes corresponding to the elements of A. There is an edge from node x to y if and only if $(x, y) \in R$.
- An edge of the form (x, x) is called a self-loop.

Examples

- Let A={1, 2, 3}
- Let $R_1 = \{(1,2), (1,3), (2,3)\}$ be a < relation on A.

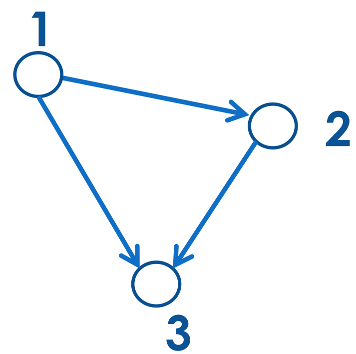


Figure 1

Examples (Cont..)

- Let $A = \{1,2,3\}$ and $B = \{a,b\}$
- Let $R_2 = \{(1, a), (1, b), (2,a), (3, b)\}$ be a relation from A to B.

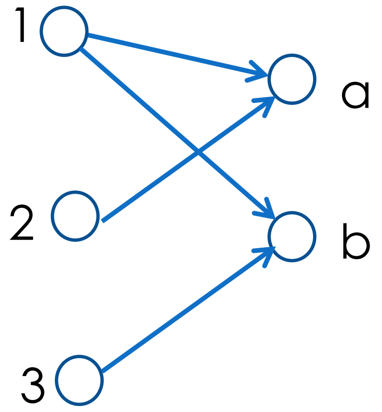


Figure 2

Domain and Range

Domain of Relation R= set of all first co-ordinates

Range of Relation R= set of all second co-ordinates

Example

“<”$= \{(1,2), (1,3), (2,3)\}$ on $A = \{1,2,3\}$

- Domain of “<”$= \{1,2\}$
- Range of “<”$= \{2,3\}$

Equality of Two relations

- Let R_1 be an n-ary relation on $A_1 \times A_2 \times \ldots \times A_n$.
- Let R_2 be an m-ary relation on $B_1 \times B_2 \times \cdots \times B_m$.

- Then, $\boldsymbol{R_1 = R_2}$
  If and only if

- **n=m**
- $\boldsymbol{A_i} = \boldsymbol{B_i}; \forall i,\ 1 \le i \le n$
- and, $\boldsymbol{R_1}$ **&** $\boldsymbol{R_2}$ **are equal set of ordered pairs**.

Example

- Let $A = \{a, b\}, B = \{1, 2\}, C = \{1, 2, 3\}$

- Let $R_1 = \{(a, 1), (b, 2)\}$ is a relation on $A \times B$
- Let $R_2 = \{(a, 1), (b, 2)\}$ is a relation on $A \times C$

$$R_1 = R_2?$$

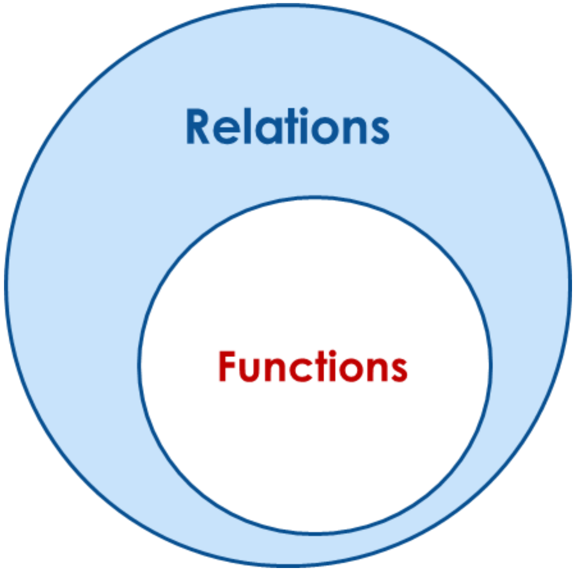
No

Relations

How many number of relations are there on a set A having n elements?

$$2^{n^2}$$


Thank you!!!
...



Relations

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Contents

Properties of Relations

- Reflexive
- Symmetric
- Transitive
- Irreflexive
- Asymmetric
- Antisymmetric

Reflexive Relations

- R is ***reflexive*** iff $(x, x) \in R$ for every element $x \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

   Is R_1 reflexive?

   No

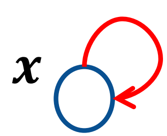
2. $=, A \times A, \leq, \geq, |, multiple\ of$ Reflexive? Yes
3. $\emptyset, <, >$ Reflexive? No

Reflexive Relation in Matrix and Graph

- If R is a **reflexive** relation, all the elements on the main diagonal of M_R are equal to 1.

$$M_R = \begin{bmatrix} 1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 1 \end{bmatrix}$$

- A loop must be present at all vertices in the graph.



Symmetric Relations

- R is ***symmetric*** iff $(y, x) \in R$ whenever $(x, y) \in R$ for all $x, y \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose $R_2 = \{(1,2), (2,1), (2,3)\}$ be a relation on A.

   Is R_1 Symmetric?

   No

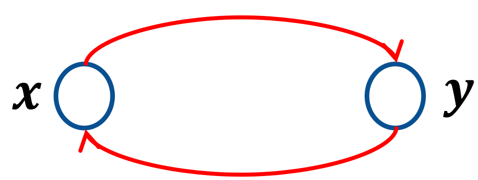
2. "sibling-of" is symmetric, but "sister-of" is not.
3. $A \times A, \emptyset, =$ Symmetric? Yes
4. $<, >, \leq, \geq, |, \textit{multiple of}$ Symmetric? No

Symmetric Relation in Matrix and Graph

- R is a symmetric relation if and only if $m_{ji} = 1$, whenever $m_{ij} = 1$.

$$M_R = \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

- If (x, y) is an edge in the graph, then there must be an edge (y, x) also.



Transitive Relations

- A relation R on a set A is called **transitive** if whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$, for all $x, y, z \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose R_3= {(1,3), (3,1)} be a relation on A.

   Is R_3 Transitive?

   No

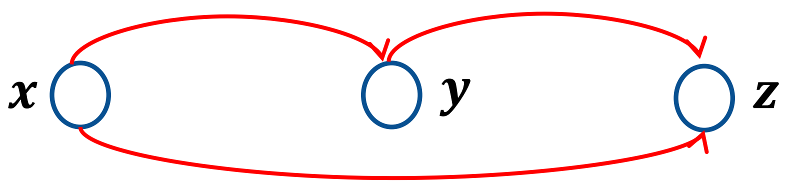
2. $A \times A,\ \emptyset, =, <, >, \leq,\ \geq,\ |,\ multiple\ of$

   Transitive?

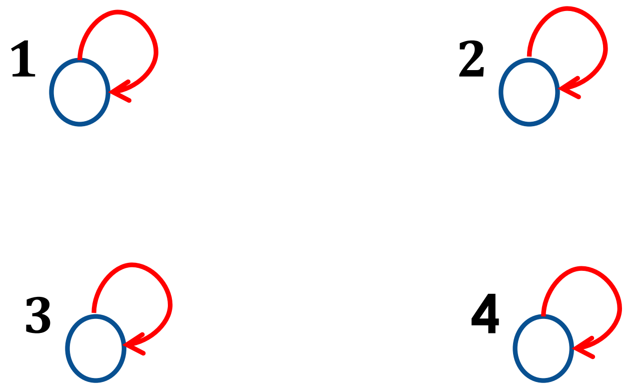
   Yes

Transitive Relations in Graph

- R is transitive iff in its graph, for any three nodes x, y and z such that there is an edge (x, y) and (y, z), there exists an edge (x, z).



Examples

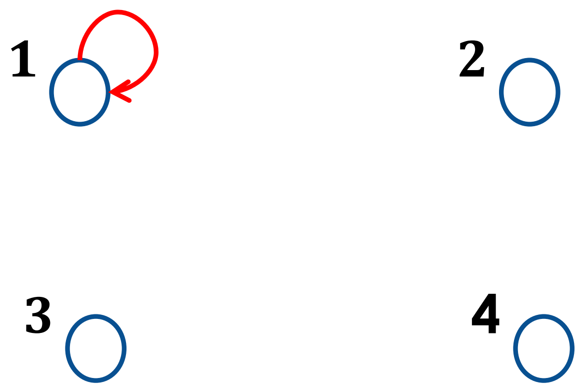


Equality Relation on A= {1, 2, 3, 4}

$$M_R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- Reflexive?
  Yes
- Symmetric?
  Yes
- Transitive?
  Yes

Examples (Cont..)



- Reflexive?
  No
- Symmetric?
  Yes
- Transitive?
  Yes

Examples (Cont..)

- Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- Reflexive?

  Yes
- Symmetric?

  Yes

How many number of Reflexive Relations are there on set A having n elements?

$$2^{n(n-1)}$$

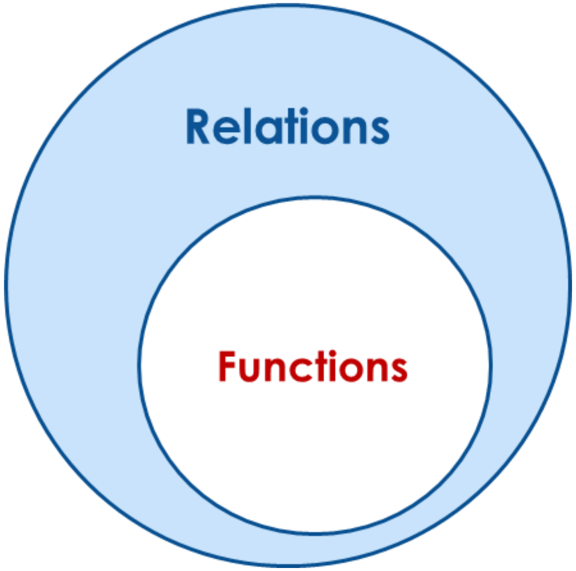
How many number of Symmetric Relations are there on set A having n elements?

$$2^{n(n+1)/2}$$

How many number of Transitive Relations are there on set A having n elements?

No closed form found


Thank you!!!
...



Relations

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Properties of Relations

- Reflexive
- Symmetric
- Transitive
- Irreflexive
- Asymmetric
- Antisymmetric

Irreflexive Relations

- *R* is ***irreflexive*** iff (x, x) ∉ *R* for every element x ∈ A.
- No Reflexive ordered pair should belong to the relation.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

   Is R_1 Irreflexive?

   No

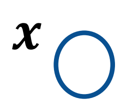
2. $\emptyset, <, >$ Irreflexive? Yes
3. $\Delta, A \times A, \leq, \geq, |, \textit{multiple of}$ Irreflexive? No

Irreflexive Relation in Matrix and Graph

- If R is an irreflexive relation, all the elements on the main diagonal of M_R are equal to 0.

$$M_R = \begin{bmatrix} 0 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 0 \end{bmatrix}$$

- No vertex should contain self-loop in the graph.



Asymmetric Relations

- A relation R on a set A such that for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \notin R$, is called ***asymmetric***.

Examples

Let A={1, 2, 3}

1. Suppose R_2= {(1,2)} be a relation on A.

Is R_2 Asymmetric?

Yes

2. Suppose R_3= {(1, 3), (3,1), (2,3)} be another relation on A.

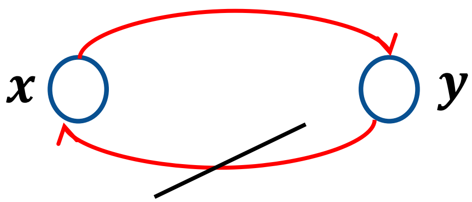
Is R_3 Asymmetric?

No

3. $\emptyset, <, >$ Asymmetric? Yes
4. $A \times A, \leq, \geq, |,$ *multiple of* Asymmetric? No

Asymmetric Relations in Graph

- If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
- There must also be no self loop.



Antisymmetric Relations

- A relation R on a set A such that for all $x, y \in A$, if $(x, y) \in R$ and if $(y, x) \in R$, then $x = y$, is called ***antisymmetric.***

If $x \neq y$ and if (x, y) is present, then (y, x) should not be present there.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,2), (2,1), (2,3)\}$ be a relation on A.

   Is R_1 Antisymmetric?

   No

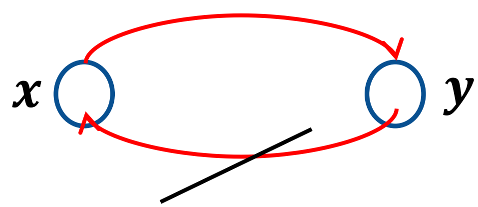
2. $\emptyset, \Delta, <, >, \leq, \geq, |, \textit{multiple of}$ Antisymmetric? Yes
3. $A \times A$ Antisymmetric? No

Antisymmetric Relation in Matrix and Graph

- *R* is a antisymmetric relation if and only if $m_{ji} = 0$, or $m_{ij} = 0$, when $i \neq j$.

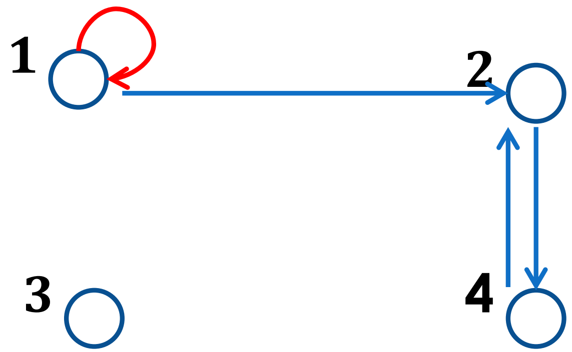
$$M_R = \begin{bmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{bmatrix}$$

- If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.



Self-loops can be there.

Example



- Reflexive?
  No
- Symmetric?
  No
- Transitive?
  No
- Irreflexive?
  No
- Antisymmetric?
  No
- Asymmetric ?
  No

Some Points to remember

- There can be a relation which is neither reflexive nor irreflexive.

Example

1. Let A={1, 2, 3}

   Suppose R_3= {(1,1), (2,2), (2,3)} be a relation on A.

Neither Reflexive nor Irreflexive

Some Points to remember (Cont..)

- There can be a relation which is neither symmetric nor antisymmetric.

Example

Let A={1, 2, 3}

Suppose $R_5 = \{(1,2), (2,3), (3,2)\}$ be a relation on A.

Neither Symmetric nor Antisymmetric

How many number of Irreflexive Relations are there on set A having n elements?

$$2^{n(n-1)}$$

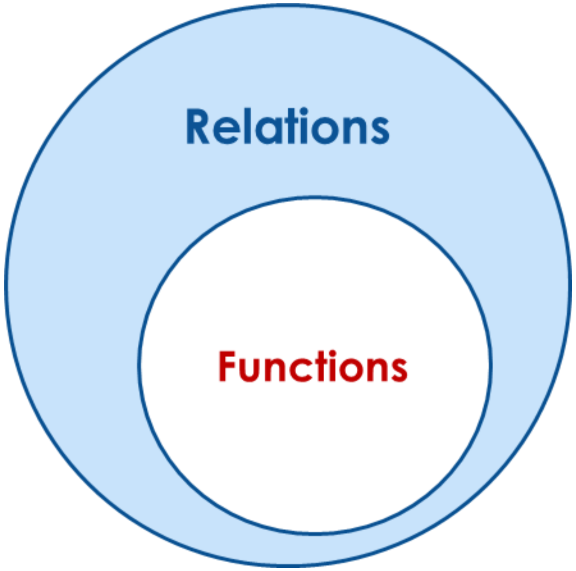
How many number of Asymmetric Relations are there on set A having n elements?

$$3^{n(n-1)/2}$$

How many number of Antisymmetric Relations are there on set A having n elements?

$$2^n 3^{n(n-1)/2}$$


Thank you!!!
...



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Contents

Inverse Relation

- If $R \subseteq A \times B$ then $R^{-1} \subseteq B \times A$, and is defined as:

$$R^{-1} = \{(b, a) | (a, b) \in R\}$$

R	R^{-1}
$<$	$>$
$\leq$	$\geq$
divides	*multiple of*
subset	*superset*

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$
  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$

$$R^{-1} = \{(3,1), (5,1), (4,2), (5,3)\}$$

Complementary Relations

- Let R be a relation from A to B, then complementary relation R^C is defined as:

$$R^C = \{(a, b) | (a, b) \notin R \; and \; (a, b) \in A \times B\}$$

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5))\}$$

$$R^C = \{(1,4), (2,3), (3,3), (3,4), (2,5)\}$$

Combining Relation

- Given two relations R_1 and R_2, these can be combined by using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

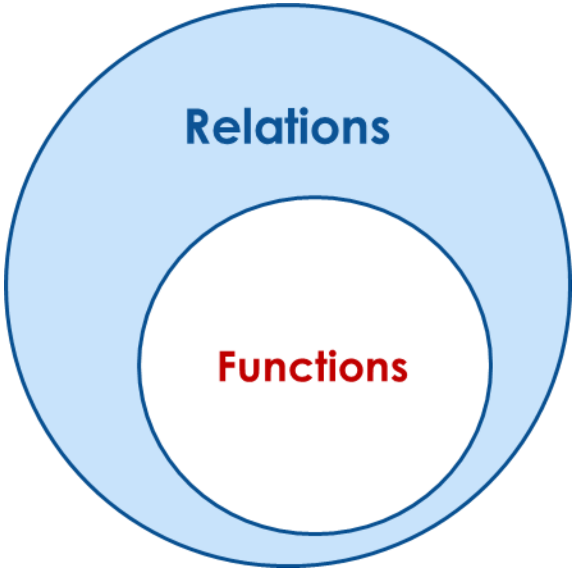
- $R_1 \cup R_2 = \{(a,b) |\ (a,b) \in R_1 or\ (a,b) \in R_2\ or\ both\}$
- $R_1 \cap R_2 = \{(a,b) |\ (a,b) \in R_1\ and\ (a,b) \in R_2\}$
- $R_1 - R_2 = \{(a,b) |\ (a,b) \in R_1\ and\ (a,b) \notin R_2\}$
- $R_2 - R_1 = \{(a,b) |\ (a,b) \in R_2\ and\ (a,b) \notin R_1\}$

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$
  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$,
  $R_1 = \{(1,4), (2,3), (2,5), (3,3), (3,5)\}$ and
  $R_2 = \{(1,3), (1,4), (2,3), (3,4), (3,5)\}$

- $R_1 \cup R_2 = \{(1,3), (1,4), (2,3), (2,5), (3,3), (3,4), (3,5)\}$
- $R_1 \cap R_2 = \{(1,4), (2,3), (3,5)\}$
- $R_1 - R_2 = \{(2,5), (3,3)\}$
- $R_2 - R_1 = \{(1,3), (3,4)\}$


Thank you!!!
...



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Contents

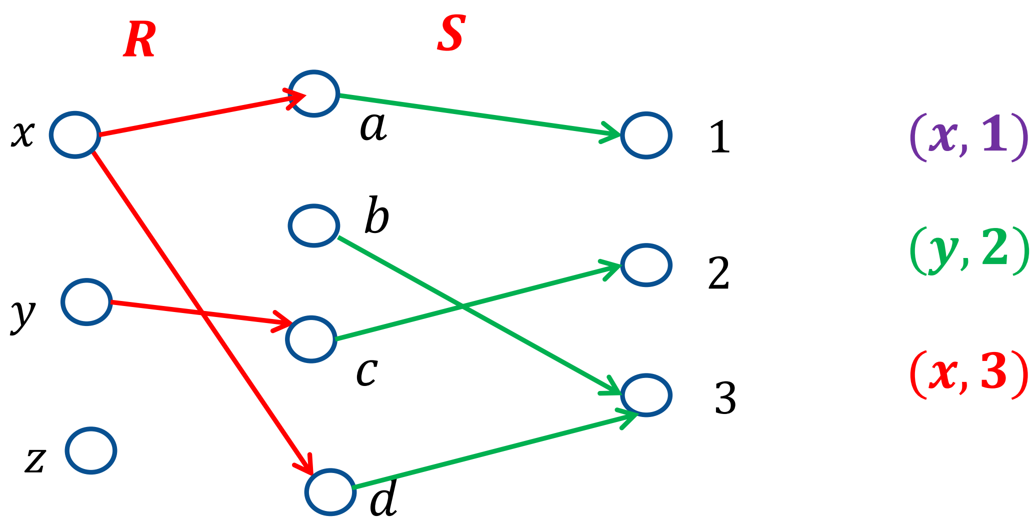
Composition of Relations

- If $R \subseteq \mathrm{A} \times B$ and $\mathrm{S} \subseteq B \times C$ are two relations, then the composition (or composite) of S with R is a relation from A to C and is defined as:

$$SoR = \{\{a, c)\big|\exists b \in B \; such \; that \; (a, b) \in R \; and \; (b, c) \in S\}$$

Representing the Composition of Relations

- Let $A = \{x, y, z\}$, $B = \{a, b, c, d\}$ and $C = \{1, 2, 3\}$.
- Suppose $R = \{(x, a), (x, d), (y, c)\}$ be a relation from $A\ to\ B$.
- Suppose $S = \{(a, 1), (b, 3), (c, 2), (d, 3)\}$ be a relation from $B\ to\ C$.



$$\boldsymbol{SoR} = \{(\boldsymbol{x}, \mathbf{1}), (\boldsymbol{y}, \mathbf{2}), (\boldsymbol{x}, \mathbf{3}\}$$

Power of Relations

- If $R \subseteq \mathrm{A} \times A$, then

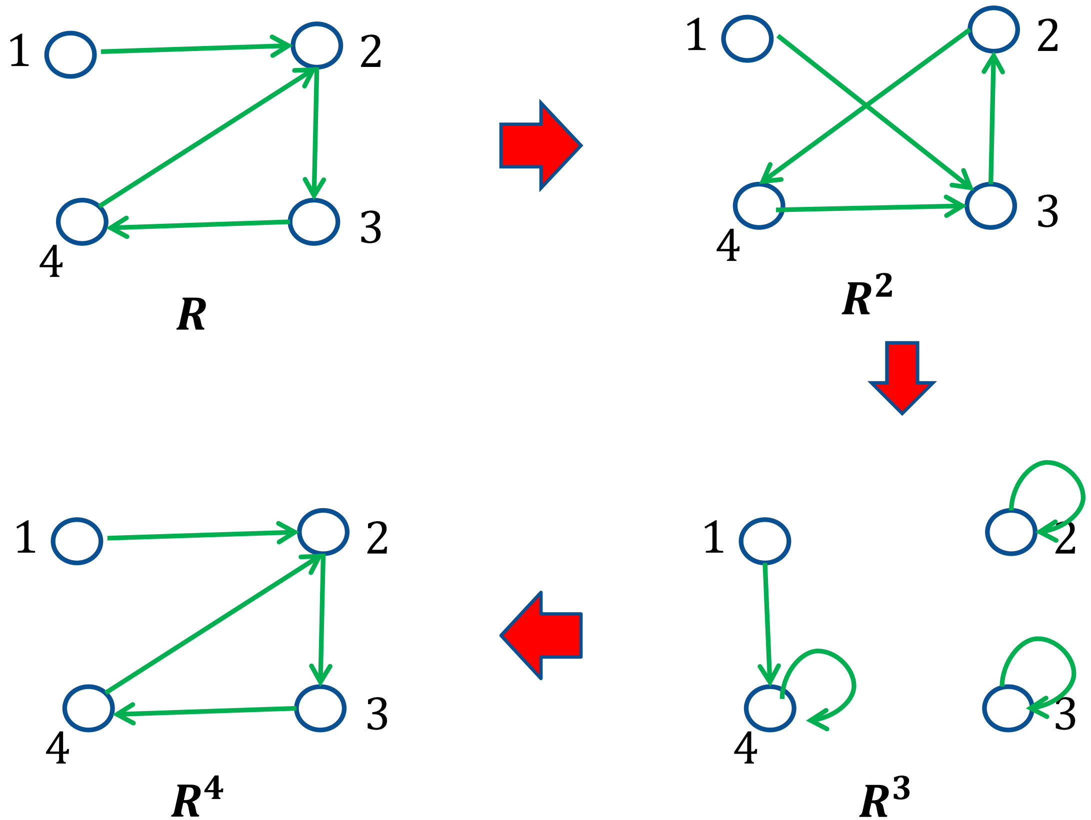
  $R^2 = RoR$

  $R^3 = R^2 oR$

  $\vdots$

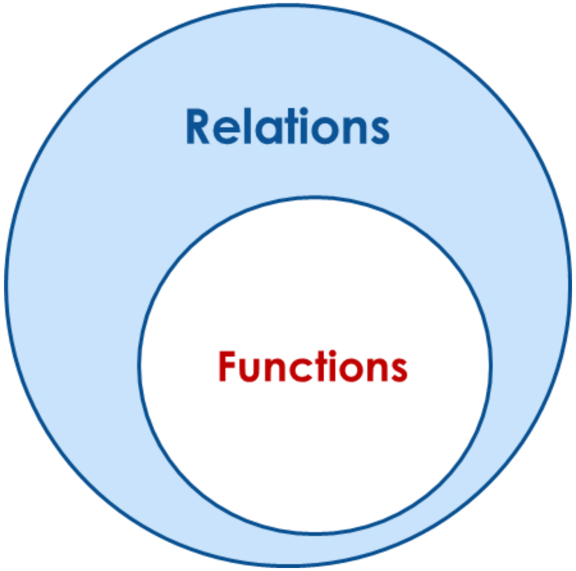
  $R^n = R^{n-1} oR$

Example



The pair (a, b) is in R^n if there is a path of length n from a to b in R.


Thank you!!!
...



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Contents

Equivalence Relation

- Let R be a relation on set A, then R is called equivalence relation if it is:

- Reflexive
- Symmetric
- Transitive

Examples

- Let $A = \{1, 2, 3\}$

1. $\emptyset$ $\boldsymbol{i.e.}$ **Empty Relation on** $\boldsymbol{A}$

Reflexive?

Symmetric?

Transitive?

Not an Equivalence Relation

2. $\Delta = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation on $\boldsymbol{A}$

Smallest Equivalence Relation on $\boldsymbol{A}$

Examples (Cont..)

- Let $A = \{1, 2, 3\}$

3. Universal Relation on A i.e. $A \times A$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation

Largest Equivalence Relation on A

4. Let $A = \{1, 2, 3, 4\}$

$$\boldsymbol{R} = \{(\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{2}), (\mathbf{2},\mathbf{1}), (\mathbf{2},\mathbf{2}), (\mathbf{3},\mathbf{3}), (\mathbf{3},\mathbf{4}), (\mathbf{4},\mathbf{3}), (\mathbf{4},\mathbf{4})\}$$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation on A

If R_1 *and* R_2 are two equivalence relations on A, then which of the following is always true?

I. $R_1 \cap R_2$ is an Equivalence Relation.

II. $R_1 \cup R_2$ is an Equivalence Relation.

(a) Only I

(b) Only II

(c) Both are true

(d) Both are false

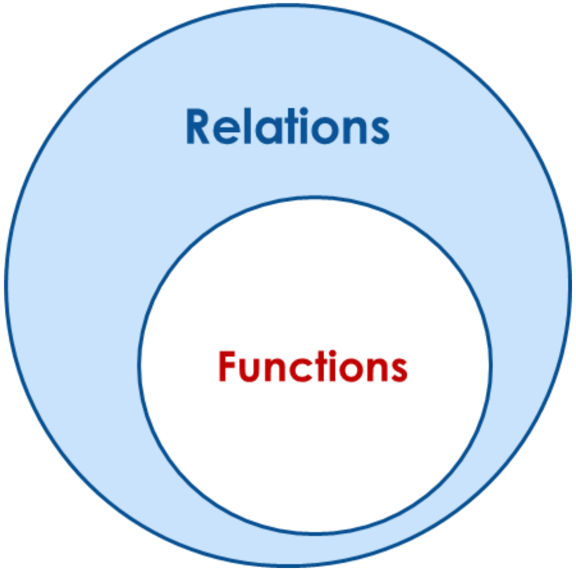
Exercise

- Let R be a relation defined on *set of integers* as:

$$xRy \text{ iff } x + y \ is \ even$$

Is R an equivalence relation?


Thank you!!!
...



Relations

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Equivalence Class

- Let R be an equivalence relation on A, and $a \in A$.
- The equivalence class of a, denoted as $[a]$ or $\overline{a}$, is defined as:

$$\overline{a} = [a] = \{b \in A | (a, b) \in R\}$$

Examples

- Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$

First Check whether R is an equivalence relation on A or not.

Reflexive?

Symmetric?

Transitive?

Equivalence Classes:-

$[1] = \{1, 2\}$
$[2] = \{1, 2\}$
$[3] = \{3, 4\}$
$[4] = \{3, 4\}$

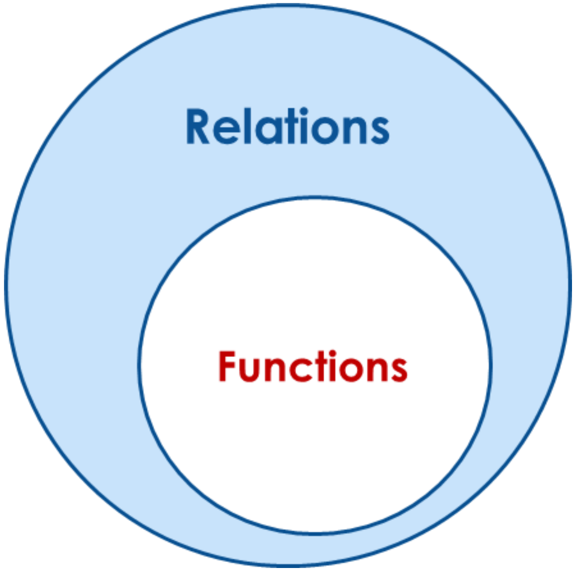
Examples

- Let $R = \{(1,1),(2,2),(3,3),(4,4)\}$ on $A = \{1,2,3,4\}$

Equivalence Classes:-

$[1] = \{1\}$
$[2] = \{2\}$
$[3] = \{3\}$
$[4] = \{4\}$


Thank you!!! ...



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Equivalence Relation to Partition

- Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$.
- R is an equivalence relation on A.

Equivalence Classes:-

$[1] = \{1, 2\} = [2]$

$[3] = \{3, 4\} = [4]$

- Partition $P = \{\{1, 2\}, \{3, 4\}\}$

Partition to Equivalence Relation

- Let $A = \{1, 2, 3, 4\}$ be a set and $P = \{\{1, 3\}, \{2, 4\}\}$ be a partition on A.
- Find Equivalence relation on A.

$\{1, 3\} \rightarrow \{(1, 1), (1,3), (3,1), (3,3)\}$
$\{2, 4\} \rightarrow \{(2, 2), (2,4), (4,2), (4,4)\}$

Therefore, the equivalence relation on A is:

- $\boldsymbol{R = \{(1, 1), (1, 3), (3, 1), (3, 3), (2, 2), (2, 4), (4, 2), (4, 4)\}}$

Result

- There is a one-to-one correspondence between partitions of A and Equivalence Relation on A.

- Therefore, if $|A| = n$, then

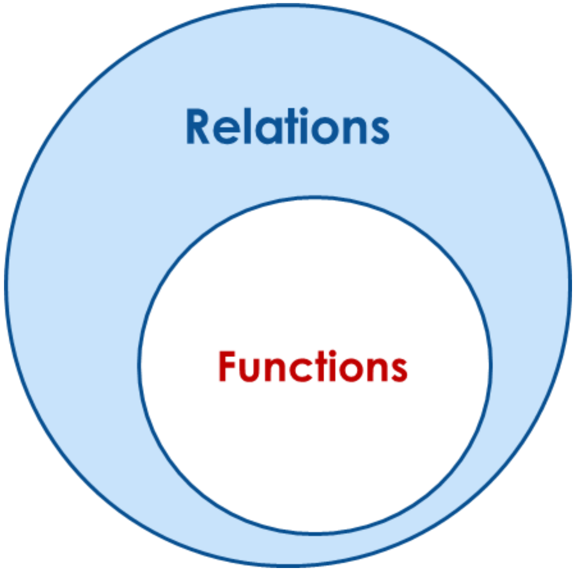
Number of Partitions of A= Number of Equivalence Relations on A
$= B_n$ (Bell Number)

Bell Number:

$$\boldsymbol{B_n = \sum_{k=0}^{n-1} n-1_{C_k}\, B_k}$$

where, $B_0 = 1$


Thank you!!!



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Closure of Relations

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

Reflexive Closure

- A relation is called **reflexive closure $\boldsymbol{R_r}$** of relation R if:

1) It is reflexive.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

Examples

1. Let A={1, 2, 3}

   Suppose R_1= {(1,1), (2,2), (2,3)} be a relation on A.

$$\boldsymbol{R_r = \{(1,1),(2,2),(2,3),(3,3)\}}$$

Examples (Cont..)

2. R is a relation defined on set of positive integers such that aRb if $a < b$.

Reflexive Closure?

Result:

- $\boldsymbol{R_r = R} \cup \Delta$
- $\boldsymbol{R_r = R}$ **iff $\boldsymbol{R}$ is Reflexive.**

Symmetric Closure

- A relation is called **symmetric closure $\boldsymbol{R_s}$** of relation R if:

1) It is symmetric.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

Examples

- Let A={1, 2, 3}

  Suppose R_1= {(1,1), (2,2), (2,3)} be a relation on A.

$$\boldsymbol{R_s = \{((1,1),(2,2),(2,3),(3,2))\}}$$

Result

- $R_s = R \cup R^{-1}$
- $R_r = R$ iff R is Symmetric.

Transitive Closure

- A relation is called Transitive closure R^* of relation R if:

1) It is transitive.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

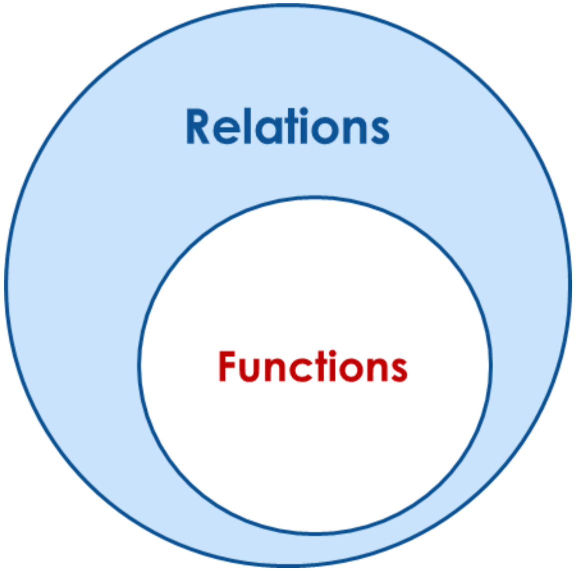
Result

1. Let $|A| = n$,

   $then, R^* = R^1 \cup R^2 \cup \cdots \cup R^n$

2. $R\ is\ transitive\ iff\ R^* = R$


Thank you!!! ...



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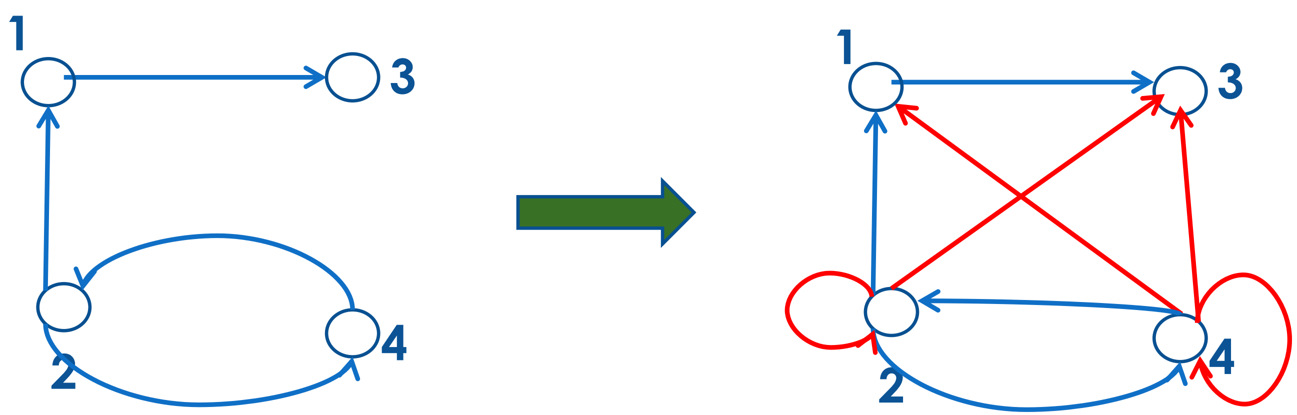
Warshall's Algorithm

- Computes the transitive closure of a relation

Example of transitive closure:



Warshall's Algorithm (Cont..)

- Main concept: a path exists between two vertices i, j, iff
  - there is an edge from i to j; or
  - there is a path from i to j going through vertex 1; or
  - there is a path from i to j going through vertex 1 and/or 2; or
  - there is a path from i to j going through vertex 1, 2, and/or 3; or
  - ...
  - there is a path from i to j going through any of the other vertices

Warshall's Algorithm (Cont..)

- On the k^{th} iteration, the algorithm determine if a path exists between two vertices i, j using vertices among $1, \ldots, k$ allowed as intermediate

$$\boldsymbol{W^{(k)}[i,j]} = \begin{cases} \boldsymbol{W^{(k-1)}[i,j]} \\ \boldsymbol{or} \\ \left(\boldsymbol{W^{(k-1)}[i,k]}\right) \boldsymbol{and} \left(\boldsymbol{W^{(k-1)}[k,j]}\right) \end{cases}$$

Warshall's Algorithm (Cont..)

- Recurrence relating elements $W^{(k)}$ to elements of $W^{(k-1)}$ is:

$$\boldsymbol{W^{(k)}[i,j] = W^{(k-1)}[i,j]} \text{ or } \boldsymbol{\left(W^{(k-1)}[i,k] \text{ and } W^{(k-1)}[k,j]\right)}$$

- It implies the following rules for generating $W^{(k)}$ from $W^{(k-1)}$ is:

1. If an element in row i and column j is 1 in $W^{(k-1)}$, it remains 1 in $W^{(k)}$.

2. If an element in row i and column j is 0 in $W^{(k-1)}$, it has to be changed to 1 in $W^{(k)}$ if and only if the element in its row i and column k and the element in its row k and column j are both $1's$ in $W^{(k-1)}$.

Warshall's Algorithm (Cont..)

- The procedure for computing $W^{(k)}$ from $W^{(k-1)}$ is as follows:

1. First transfer all $1's$ in $W^{(k-1)}$ to $W^{(k)}$.

2. List the locations $p_1, p_2, \ldots$, in column k of $W^{(k-1)}$, where the entry is 1, and the locations $q_1, q_2, \ldots$, in row k of $W^{(k-1)}$, where the entry is 1.

3. Put $1's$ in all the positions (p_i, q_i) of $W^{(k)}$ (if they are not already there).

Warshall's Algorithm (Cont..)

$$W^{(k-1)} = \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & \end{bmatrix} \end{array} \Rightarrow W^{(k)} = \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} \end{array}$$

Figure 1: Step for Changing zeros in Warshall's Algorithm

Example

1. Find transitive closure of relation R represented by following matrix (using Warshall's algorithm):

$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Solution 1:

$$W^{(0)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W^{(2)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W^{(3)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W^{(4)} = W^{(3)} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Answer is :

$$M_{R^*} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R^* = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$$


Thank you!!!
...