Course: UMA 035 (Optimization Techniques)

Instructor: Dr. Amit Kumar,

Associate Professor,

School of Mathematics,

TIET, Patiala

Email: amitkumar@thapar.edu

Mob: 9888500451

Example:

Solve the following LPP by the Simplex method.

Maximize $(2x_1+x_2)$

Subject to

$x_1-x_2<=10,$

$2x_1-x_2<=10,$

$x_1>=0, x_2>=0.$

Solution

Maximize $(2x_1+x_2)$

Subject to

$x_1-x_2+S_1=10,$

$2x_1-x_2 +S_2=10,$

$x_1>=0, x_2>=0, S_1>=0, S_2>=0.$

		2	1	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	Solution	Minimum Ratio
$Z_j - C_j =$		−2	−1	0	0		
0	S_1	1	−1	1	0	10	10/1=10
0	S_2	2	−1	0	1	40	40/2=20
$Z_j - C_j =$		0	−3	2	0		
2	x_1	1	−1	1	0	10	10/-
0	S_2	0	1	−2	1	20	20/1=20
$Z_j - C_j =$		0	0	−4	3		
2	x_1	1	0	−1	1	30	30/-
1	x_2	0	1	−2	1	20	20/-

Since, S_1 is entering variable and it is not possible to find any leaving variable. So, the problem has an unbounded optimal solution.

Row operations used to obtain second table

$R_1 \rightarrow R_1 - (-2) * (R_2/(1)) \Rightarrow R_1 \rightarrow R_1 + 2 R_2$

$R_2 \rightarrow R_2/(1) \Rightarrow R_2 \rightarrow R_2$

$R_3 \rightarrow R_3 - (2) * (R_2/(1)) \Rightarrow R_3 \rightarrow R_3 - 2 R_2$

Row operations used to obtain third table

$R_1 \rightarrow R_1 - (-3)(R_3 / (1)) \implies R_1 \rightarrow R_1 + 3R_3$

$R_2 \rightarrow R_2 - (-1)(R_3 / (1)) \implies R_2 \rightarrow R_2 + R_3$

$R_3 \rightarrow R_3 / (1) \implies R_3 \rightarrow R_3$

Example:

Solve the following LPP by the Simplex method.

Maximize $(4x_1 + x_2)$

Subject to

$x_1 - x_2 <= 1$,

$-2x_1 + x_2 <= 2$,

$x_1 >= 0, x_2 >= 0$.

Solution

Maximize $(4x_1 + x_2)$

Subject to

$x_1 - x_2 + S_1 = 1$,

$-2x_1 + x_2 + S_2 = 2$,

$x_1 >= 0, x_2 >= 0, S_1 >= 0, S_2 >= 0$.

		4	1	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	Solution	Minimum Ratio
$Z_j - C_j =$		−4	−1	0	0		
0	S_1	1	−1	1	0	1	1/1=1
0	S_2	−2	1	0	1	2	2/-
$Z_j - C_j =$		0	−5	4	0		
2	x_1	1	−1	1	0	1	1/-
0	S_2	0	−1	2	1	4	4/-

Since, x_2 is entering variable and it is not possible to find any leaving variable. So, the problem has an unbounded optimal solution.

Row operations used to obtain second table

$R_1 \longrightarrow R_1 - (-4)_* (R_2/(1)) \Longrightarrow R_1 \longrightarrow R_1 + 4 R_2$

$R_2 \longrightarrow R_2/(1) \Longrightarrow R_2 \longrightarrow R_2$

$R_3 \longrightarrow R_3 - (-2)_* (R_2/(1)) \Longrightarrow R_3 \longrightarrow R_3 + 2 R_2$

Example:

Check that the following LPP can be solved by the Simplex method or not.

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2 \quad =3,$

$4x_1+3x_2>=6,$

$x_1+2x_2<=4$

$x_1>=0,$

$x_2>=0.$

Solution

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2 \quad =3,$

$4x_1+3x_2-S_1 \quad =6,$

$x_1+2x_2+ \quad S_2 \quad =4,$

$x_1>=0, x_2>=0, S_1>=0, S_2>=0.$

Subject to

$3x_1+x_2$ =3,

$4x_1+3x_2-S_1$ =6,

x_1+2x_2+ S_2 =4,

$x_1>=0, x_2>=0, S_1>=0, S_2>=0.$

		−2	−1	0	0		
C_B	Basic Variables	x_1	x_2	S_1	S_2	Solution	Minimum Ratio
$Z_j - C_j =$							
		3	1	0	0		
		4	3	−1	0		
	S_2	1	2	0	1		

Not possible to find the first and second basic variables as

1

0

0

and

0

1

0

does not exist in the table.

So Simplex method cannot be used.

If Simplex method fails then we can use any of the following methods:

> **Big-M method**

> **Two-Phase method**

If it is not possible to find first basic variable then add a variable A_1 (artificial variable) in the first constraint as well as add $-MA_1$ in the objective function.

If it is not possible to find second basic variable then add a variable A_2 (artificial variable) in the second constraint as well as add $-MA_2$ in the objective function.

$\vdots$

If it is not possible to find m^{th} basic variable then add a variable A_m (artificial variable) in the mth constraint as well as add $-MA_m$ in the objective function.

Apply the simplex method after adding the missing columns in the Table

If in the optimal table, there exist one or more artificial variables in the column of basic variables. Then, the LPP has no solution.

Example:

Check that the following LPP can be solved by the Simplex method or not.

If not then solve by Big-M method.

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2 \qquad =3,$

$4x_1+3x_2>=6,$

$x_1+2x_2<=4$

$x_1>=0,$

$x_2>=0.$

Solution

Minimize $(2x_1+x_2)$

Subject to

$3x_1+x_2 \qquad =3,$

$4x_1+3x_2-S_1 \qquad =6,$

$x_1+2x_2+ \quad S_2 \quad =4,$

$x_1>=0, x_2>=0, S_1>=0, S_2>=0.$

Maximize $(-2x_1 - x_2)$

Subject to

$3x_1 + x_2 \qquad = 3,$

$4x_1 + 3x_2 - S_1 \qquad = 6,$

$x_1 + 2x_2 + \qquad S_2 \qquad = 4,$

$x_1 >= 0,\ x_2 >= 0,\ S_1 >= 0,\ S_2 >= 0.$

As discussed, it is not possible to find first and second basic variable.

Maximize $(-2x_1 - x_2 - MA_1 - MA_2)$

Subject to

$3x_1 + x_2 \qquad + A_1 \qquad = 3,$

$4x_1 + 3x_2 - S_1 \qquad + A_2 = 6,$

$x_1 + 2x_2 + \qquad S_2 \qquad = 4,$

$x_1 >= 0,\ x_2 >= 0,\ S_1 >= 0,\ S_2 >= 0.$

C_B	Basic Variables	-2 x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	Solution	Minimum Ratio
$Z_j - C_j =$									
		3	1	0	0	1	0		
		4	3	−1	0	0	1		
0	S_2	1	2	0	1	0	0		

C_B	Basic Variables	-2 x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	Solution	Minimum Ratio
$Z_j - C_j =$									
	A_1	3	1	0	0	1	0		
	A_2	4	3	−1	0	0	1		
0	S_2	1	2	0	1	0	0		

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$									
	A_1	3	1	0	0	1	0	3	
	A_2	4	3	-1	0	0	1	6	
0	S_2	1	2	0	1	0	0	4	

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$					0	0	0		
$-M$	A_1	3	1	0	0	1	0	3	
$-M$	A_2	4	3	-1	0	0	1	6	
0	S_2	1	2	0	1	0	0	4	

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$					0	0	0		
$-M$	A_1	3	1	0	0	1	0	3	
$-M$	A_2	4	3	-1	0	0	1	6	
0	S_2	1	2	0	1	0	0	4	

$$[(-M)(3) + (-M)(4) + (0)(1)] - (-2) = -7M + 2$$

$$[(-M)(1) + (-M)(3) + (0)(2)] - (-1) = -4M + 1$$

$$[(-M)(0) + (-M)(-1) + (0)(0)] - (0) = M$$

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		$-7M+2$	$-4M+1$		0	0	0		
$-M$	A_1	3	1	0	0	1	0	3	
$-M$	A_2	4	3	-1	0	0	1	6	
0	S_2	1	2	0	1	0	0	4	

Since, M is a large positive real number. So assuming M=100,

$-7M+2=-698$

$-4M+1=-399$

Out of these two negative values -698 is minimum. So, variable x_1 is entering variable.

		−2	−1	0	0	−M	−M		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		−7M +2	−4M+ 1		0	0	0		
−M	A_1	3	1	0	0	1	0	3	
−M	A_2	4	3	−1	0	0	1	6	
0	S_2	1	2	0	1	0	0	4	

		−2	−1	0	0	−M	−M		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		−7M +2	−4M+ 1		0	0	0		
−M	A_1	3	1	0	0	1	0	3	3/3=1
−M	A_2	4	3	−1	0	0	1	6	6/4=1.5
0	S_2	1	2	0	1	0	0	4	4/1=4

$R_1 \longrightarrow R_1 - (-7M+2)*(R_2/(3)) \implies$ **Not apply it due to presence of M**

$R_2 \longrightarrow R_2/(3) \implies R_2 \longrightarrow R_2/(3)$

$R_3 \longrightarrow R_3 - (4)_* R_2/(3) \implies R_3 \longrightarrow R_3 - \dfrac{4}{3} R_2$

$R_4 \longrightarrow R_4 - (1)_* R_2/(3) \implies R_4 \longrightarrow R_4 - \dfrac{1}{3} R_2$

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		0			0	*	0		
-2	x_1	1	$\dfrac{1}{3}$	0	0	*	0	1	
$-M$	A_2	0	$\dfrac{5}{3}$	-1	0	*	1	2	
0	S_2	0	$\dfrac{5}{3}$	0	1	*	0	3	

C_B	Basic Variables	-2 x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	Solution	Minimum Ratio
$Z_j - C_j =$		0			0	*	0		
-2	x_1	1	$\dfrac{1}{3}$	0	0	*	0	1	
$-M$	A_2	0	$\dfrac{5}{3}$	-1	0	*	1	2	
0	S_2	0	$\dfrac{5}{3}$	0	1	*	0	3	

$$[(-2)(\tfrac{1}{3}) + (-M)(\tfrac{5}{3}) + (0)(\tfrac{5}{3})] - (-1) = -\tfrac{5}{3}M + \tfrac{1}{3}$$

$$[(-2)(0) + (-M)(-1) + (0)(0)] - (0) = M$$

Remark: If an artificial variable is leaving variable. Then, no need to write its column in the next table.

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		0	$-\frac{5}{3}M + \frac{1}{3}$	M	0	$*$	0		
-2	x_1	1	$\frac{1}{3}$	0	0	$*$	0	1	
$-M$	A_2	0	$\frac{5}{3}$	-1	0	$*$	1	2	
0	S_2	0	$\frac{5}{3}$	0	1	$*$	0	3	

Since, M is a large positive real number. So assuming M=100,

$$-\frac{5}{3}M + \frac{1}{3} = -\frac{499}{3}$$

M=100

Since only one value of $Z_j - C_j$ corresponding to x_2 is negative. So, x_2 is entering variable.

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j$ =		0	$-\frac{5}{3}M + \frac{1}{3}$	M	0	$*$	0		
-2	x_1	1	$\frac{1}{3}$	0	0	$*$	0	1	
$-M$	A_2	0	$\frac{5}{3}$	-1	0	$*$	1	2	
0	S_2	0	$\frac{5}{3}$	0	1	$*$	0	3	

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j$ =		0	$-\frac{5}{3}M + \frac{1}{3}$	M	0	$*$	0		
-2	x_1	1	$\frac{1}{3}$	0	0	$*$	0	1	$1/(\frac{1}{3}) = 3$
$-M$	A_2	0	$\frac{5}{3}$	-1	0	$*$	1	2	$2/(\frac{5}{3}) = \frac{6}{5}$
0	S_2	0	$\frac{5}{3}$	0	1	$*$	0	3	$3/(\frac{5}{3}) = \frac{9}{5}$

Row operations

$R_1 \rightarrow R_1 - (-\frac{5}{3}M + \frac{1}{3})*(R_3 / (\frac{5}{3}))$ $\Rightarrow$ <mark>Not apply it due to presence of M</mark>

$R_2 \rightarrow R_2 - (\frac{1}{3}) * R_3 / (\frac{5}{3})$ $\Rightarrow R_2 \rightarrow R_2 - (\frac{1}{5}) * R_3$

$R_3 \rightarrow R_3 / (\frac{5}{3}) \Rightarrow R_3 \rightarrow (\frac{3}{5})R_3$

$R_4 \rightarrow R_4 - (\frac{5}{3})* (R_3 / (\frac{5}{3})) \Rightarrow R_4 \rightarrow R_4 - R_3$

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		0	0		0	*	*		
-2	x_1	1	0	$\frac{1}{5}$	0	*	*	$\frac{3}{5}$	
-1	x_2	0	1	$-\frac{3}{5}$	0	*	*	$\frac{6}{5}$	
0	S_2	0	0	1	1	*	*	1	

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		0	0		0	*	*		
-2	x_1	1	0	$\dfrac{1}{5}$	0	*	*	$\dfrac{3}{5}$	
-1	x_2	0	1	$-\dfrac{3}{5}$	0	*	*	$\dfrac{6}{5}$	
0	S_2	0	0	1	1	*	*	1	

$$[(-2)(\tfrac{1}{5}) + (-1)(-\tfrac{3}{5}) + (0)(1)] - (0) = \tfrac{1}{5}$$

		-2	-1	0	0	$-M$	$-M$		
C_B	Basic Variables	x_1	x_2	S_1	S_2	A_1	A_2	Solution	Minimum Ratio
$Z_j - C_j =$		0	0	$\dfrac{1}{5}$	0	$*$	$*$		
-2	x_1	1	0	$\dfrac{1}{5}$	0	$*$	$*$	$\dfrac{3}{5}$	
-1	x_2	0	1	$-\dfrac{3}{5}$	0	$*$	$*$	$\dfrac{6}{5}$	
0	S_2	0	0	1	1	$*$	$*$	1	

Optimal solution is

$x_1 = \dfrac{3}{5}$

$x_2 = \dfrac{6}{5}$

$S_2 = 1$

Remaining are 0 i.e., $S_1 = A_1 = A_1 = 0$

Putting the optimal solution in the objective function ($2x_1 + x_2$) of the given

LPP, the obtained minimum value is $2 * \dfrac{3}{5} + \dfrac{6}{5} = \dfrac{12}{5}$

C_B	Basic Variables	-2 x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	Solution	Minimum Ratio
$Z_j - C_j =$		$-7M+2$	$-4M+1$		0	0	0		
$-M$	A_1	3	1	0	0	1	0	3	$3/3=1$
$-M$	A_2	4	3	-1	0	0	1	6	$6/4=1.5$
0	S_2	1	2	0	1	0	0	4	$4/1=4$
$Z_j - C_j =$		0	$-\frac{5}{3}M+\frac{1}{3}$	M	0	$*$	0		
-2	x_1	1	$\frac{1}{3}$	0	0	$*$	0	1	$1/(\frac{1}{3})=3$
$-M$	A_2	0	$\frac{5}{3}$	-1	0	$*$	1	2	$2/(\frac{5}{3})=\frac{6}{5}$
0	S_2	0	$\frac{5}{3}$	0	1	$*$	0	3	$3/(\frac{5}{3})=\frac{9}{5}$
$Z_j - C_j =$		0	0	$\frac{1}{5}$	0	$*$	$*$		
-2	x_1	1	0	$\frac{1}{5}$	0	$*$	$*$	$\frac{3}{5}$	
-1	x_2	0	1	$-\frac{3}{5}$	0	$*$	$*$	$\frac{6}{5}$	
0	S_2	0	0	1	1	$*$	$*$	1	

Optimal solution is

$x_1 = \dfrac{3}{5}$

$x_2 = \dfrac{6}{5}$

$S_2 = 1$

Remaining are 0 i.e., $S_1 = A_1 = A_1 = 0$

Putting the optimal solution in the objective function ($2x_1 + x_2$) of the given LPP, the obtained minimum value is $2*\dfrac{3}{5} + \dfrac{6}{5} = \dfrac{12}{5}$