ADDITION CHAIN

Australian Curriculum: Mathematics (Year 7/8)

Australian Curriculum: Mathematics (Year 7)

ACMNA175: Introduce the concept of variables as a way of representing numbers using letters.

Australian Curriculum: Mathematics (Year 8)

ACMNA191: *Factorise algebraic expressions by identifying numerical factors.*
ACMNA192: *Simplify algebraic expressions using the four operations.*

Note: This resource uses algebra to prove generalisations made in arithmetic. Hence, although the algebraic operations involve no more than collecting like terms and identifying common numerical factors in linear expressions, the level of reasoning involved is sophisticated.

Lesson abstract

This resource prompts students to examine and generalise numerical relationships arising from “addition chains”. These are recursive sequences, such as the Fibonacci sequence, generated by summing the previous two terms. The teacher demonstrates a calculation shortcut to create a sense of curiosity about how it is done and why it works. This leads to a student search for similar relationships, using spreadsheets. The resource requires students to use algebra to justify the conditions for which these relationships hold true.

Mathematical purpose (for students)

To design addition chains and explore their generalised properties using algebraic notation.

Mathematical purpose (for teachers)

This resource focuses on mathematical reasoning using algebra. Students use spreadsheets to investigate potential arithmetic relationships and then use algebra to identify and justify which relationships are generally true. The task can be used as a springboard for an in-depth exploration of the Fibonacci sequence, and develops skills in using spreadsheets.

Lesson Length — Two to three lessons of approximately one hour each

Vocabulary Encountered
- Fibonacci sequence

Lesson Materials
- reSolve resource *Addition Chain* spreadsheet (optional)

We value your feedback after these lessons via <link to be advised>

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Demonstrating a mathematical magic trick

Set the scene by reminding students of other situations where they have created a mathematical "magic trick" (e.g. in other reSolve *Power of Algebra* resources), or by asking them to share their own examples of mathematical "magic tricks".

Present today's challenge to the students: All of these so-called magic tricks have a basis in logic – they can be explained! Our challenge is to go beyond the entertaining to understand and invent our own magic tricks. We are going to look at one example, understand why it works, and see if we can use out understanding to invent a similar one.

Ask each student to:

- write down two numbers vertically aligned (their starting numbers or 'seed numbers')
- add them
- Write the answer under the second number
- Add the last two numbers in the list
- Repeat until there are 10 numbers in the list
- Add the total of all 10 numbers in the list. In the example on the right the total will be 825.

7
5
12
17
29
…
…

Ask one student to come to the board, record their seed numbers and begin to write down their full list. As the student writes the 10th number, quickly tell them the total of their numbers. Make it look as though you have added all ten numbers very quickly. In fact, you have **multiplied the 7th number by 11** while the student has been writing the remaining numbers.

Teacher Notes

- To multiply by 11 start by writing the final digit of the number on the right had end, then successively add pairs of digits from the right, carrying if necessary, ending with the left hand digit or one more than the digit if carrying has been necessary.
- For example:
  - To multiply 312 by 11, the digits in the answer, starting from the right, will be 2, 2 + 1 = 3, 1 + 3 = 4, 3. Therefore 312 x 11 = 3432.
  - To multiply 467 x 11, the digits will be 7, 7 + 6 = 13 (3), 6 + 4 = 10 with one carried over makes 11 (1), 4 with 1 carried over makes 5. Therefore 467 x 11 = 5137.
  - 876 x 11: digits are 6, 6 + 7 = 13 (3), 7 + 8 = 15 with one carried over makes 16 (6), 8 with 1 carried over makes 9. 876 x 11 = 9636.
- With practice it is possible to work from the left, anticipating the carries.

Ask a second student to come to the board and write their full list, but stop after the 8th number (for example). Again tell them the total of all 10 numbers.

This poses the challenge of how it was possible to find the total so quickly, in fact without even seeing the last two numbers.

The relationships of numbers in addition chains

Give students time to look at their list of numbers, and at their total, to see if they can work out how it was possible to find the total without actually adding all their numbers.

Ideally someone will develop a hypothesis that the total is 11 times the 7th number.

Enabling Prompt

- *Subtract each of the numbers in your list from the total.*
- In the example above, the results will be:
  - 825 – 7 = 818
  - 825 - 5 = 820

- $825 - 12 = 813$
    - 825 - 17 = 808
    - 825 - 29 = 796
    - 825 - 46 = 779
    - **825 - 75 = 750**
    - 825 - 121 = 704
    - …
- The 7th subtraction always provides an interesting result: the answer is 10 times the number being subtracted. What does this mean about the relationship between the 7th number and the total of the 10 numbers?
    - Total sum - T_7 = $10T_7$, hence Total sum = $11T_7$.

Checking the relationship generalises

Ask a few students to come to the board and record just the seed numbers and the 7th number. Do not record the total at this stage.

Here is a sample board:

Student	David/Jemma	Sean	Kylie	Karen/Isaac
First number	4	3	8	6
Second number	7	2	10	2
…	…	…	…	…
Seventh number	76	31	120	46
Total of 10 numbers				

Complete the total for some of the columns and ask the students to verify the total to the rest of the class, e.g.:

- "David and Gemma, your total is 836. Is that correct?"
- "Sean, your total is 341. Is that correct?"

Ask the class to predict the total for the others on the board, and ask those students to verify the results.

Teacher Notes

- If students have completed the Tens and Units activity they may be able to multiply by 11 quickly. If not they can find the answer quickly by multiplying by 10 and then adding the original number. For example 47 x 11 = 470 + 47 = 517.

Investigating other relationships

Ask students what happens if we go beyond 10 numbers. Is there a quick way to calculate the sum of the first 11 numbers without actually doing the addition? Or 12? Or…?

Teacher Notes

- While this can be done by hand, it is much easier to get students to set up a spreadsheet, input their own seed numbers and fill down beyond 10 numbers.

	A	B	C
1		**Addition Chain**	**Factor of sum?**
2	**Seed a**	7	=B$13/B2
3	**Seed b**	5	=B$13/B3
4		=B2+B3	=B$13/B4
5		=B3+B4	=B$13/B5
…		…	…
12		=B10+B11	=B$13/B12
13	**Sum**	=SUM(B2:B12)	

  - Cell B4 contains a formula for adding the two previous numbers, which can then be filled down as far as desired. Cell B13 contains a formula for the sum of the numbers in the chain.
  - Column C checks to see if each of the numbers in the chain is an integer factor of the sum. The $sign before the 13 is an absolute cell reference that ensures that when the formula is filled down the number in row (the sum) is used in every formula.
- This spreadsheet is part of the reSolve *Addition Chain* spreadsheet resource and can be downloaded here. The Sheet labelled Addition Chain uses the formulas above.
- To create an addition chain of length 12, it will be necessary to insert a row before row 13, ensure that the formula sums the numbers B2 to B13, and that the reference in Column C is to B$14.
- Using the seed numbers 7 and 5 and filling down to sum 11 numbers, no integer factors appear. The same is true when 12 numbers are summed. The first integer factor that appears is when 14 numbers are summed, when the sum is 29 times the 9th number in the chain. This relationship holds true regardless of the seed numbers used.

Suggest that these are not the only relationships between numbers in the addition chain and the total of some of the numbers. We will explore this using a longer list and keeping a cumulative total.

Start a list on the board, similar to the below example:

Addition Chain	Cumulative Total
7	7
5	12
12	24
17	41
29	70
46	116

Ask students to find other relationships in this table that *appear* to be true.

For example, in the table above, the third number in the Cumulative Total [C(3)] appears to be twice the third number in the Addition Chain [A(3)]. Or C(5) = 14A(2). Or C(2) = 1A(3). However these may not be true for other pairs of seed numbers.

Ask students to generate some possible relationships to investigate using their own pair of seed numbers.

Teacher Notes

- Again, we recommend getting students to set up a spreadsheet and input their own seed numbers. Rows 2 and 3 in column A of the spreadsheet below contain two seed numbers. Cell B4 contains the formula to generate the addition chain as above. Column C gives one formula for the cumulative totals.

	A	B	C
1		**Addition Chain**	**Cumulative Total**
2	**Seed a**	7	=B2
3	**Seed b**	5	=C2+B3
4		=B2+B3	=C3+B4

- This spreadsheet is part of the reSolve *Addition Chain* spreadsheet resource and can be downloaded here. The Sheet labelled Addition Chain uses the formulas above.

Some of the relationships students will find are always true, and some are only true for special cases.

Make a list of possible relationships on the board. As a class refine the list by crossing out relationships that are only true in special cases (i.e. students identify and remove relationships that do not apply to their results).

Explain to students that one exception (as long as the arithmetic is correct!) suffices to show that a particular relationship is not generally true. However, even though we have several verifications of other relationships, it may be that all the pairs of seed numbers we have chosen happen to satisfy them but other seed numbers will not. To prove that a relationship is **always** true, we need to use algebra to generalise.

An algebraic explanation and exploration

Begin to create a generalised Addition Chain and Cumulative Total table as below:

Addition Chain	Cumulative Total
a	a
b	$a + b$
$a + b$	$2a + 2b$
$a + 2b$	$3a + 4b$
$2a + 3b$	$5a + 7b$
$3a + 5b$	$8a + 12b$

Ask students to use the generalised results to write down some relationships that must always be true. For example, this table shows that C(2) = 1A(3), and that C(6) = 4A(5), but C(5) ≠ 14A(2).

Justify the relationships by factorising 8a + 12b as 4(2a + 3b).

Teacher Notes

- Again, while this can be done by hand, it is much easier to get students to set up a spreadsheet. The spreadsheet below is part of the reSolve *Addition Chain* spreadsheet resource and can be downloaded here. The Sheet labelled Addition Chain uses the formulas above.
- Note that this spreadsheet also has a column labelled **n** to help keep track of which terms in the sequence or series are being considered.
- It is, of course, preferable to ask students to make their own spreadsheet.

	A	B	C	D	E
1		Addition Chain		Cumulative Total	
2	***n***	**# of *a***	**# of *b***	**# of *a***	**# of *b***
3	**1**	1	0	=B3	=C3
4	**=A3+1**	0	1	=D3+B3	=E3+C3
5	**=A4+1**	=B3+B4	=C3+C4	=D4+B4	=E4+C4
6	**=A5+1**	=B4+B5	=C4+C5	=D5+B5	=E5+C5

Ask students to use their spreadsheet to find other relationships that must always be true.

Following the two listed above, the next is C(10) = 11A(7), which is the one used in our original "trick". We have therefore shown that the trick will work for any pair of seed numbers.

Expected Student Response

- The first six relationships, excluding those for C(1) and C(3), are:
  - a + b = 1(a + b) or C(2) = 1A(3)
  - 8a + 12b = 4(2a + 3b) or C(6) = 4A(5)
  - 55a + 88b = 11(5a + 8b) or C(10) = 11A(7)
  - 377a + 609b = 29(13a + 21b) or C(14) = 29A(9)
  - 2584a + 4180b = 76(34a + 55b) or C(18) = 76A(11)
  - 17711a + 28656b = 199(89a + 144b) or C(22) = 199A(13)
- Note that the relationships for C(1) and C(3), while true, have been excluded as they do not fit the pattern.

Teacher Notes

- If students find these relationships interesting, they might like to investigate the sequence of ratios 1, 4, 11, 29, 76, 199 by typing into the Online Encyclopaedia of Integer Sequences at https://oeis.org/. The sequence of numbers was contributed by Lekraj Beedassy on 31 December, 2002. There is also a list of the first 200 numbers in the sequence at https://oeis.org/A002878/b002878.txt.

Further activities

Activity 1

Ask students to explain how they could use the relationship C(22) = 199A(13) as the basis of another addition "trick".

Activity 2

Investigate what happens in the Addition Chain using other seed numbers.

- What happens if one of the seed numbers is 0?
- What if one or both is negative? Will it always be the case that all the terms will eventually become all positive or all negative?

Activity 3

Find numbers for which the relationships that are not generally true will work.

For example, if C(5) = 14A(2), then $5a + 7b = 14b$. Hence $5a = 7b$ and $a = 7$, $b = 5$ is one possible pair of numbers for which this will be true. All others are multiples of $a = 7$ and $b = 5$.

Choose a relationship that you would "like" to work, and find a pair of seed numbers for which it will work. For example, if we want C(7) = 12A(4) to work, then $13a + 20b = 12(a + 2b)$ and $a = 4b$. Therefore we could use $a = 4$ and $b = 1$. Students can confirm these using the spreadsheet.

Note that this is an introduction to simple Diophantine equations.

Activity 4

Students could explore starting with three numbers and adding the previous three terms (a third order recursive relationship). For example:

Addition Chain	Cumulative Total
7	7
5	12
1	13
13	26
19	45
33	78
65	143
117	260
215	475

While we can observe several relationships between the numbers in the above example, the only relationship that holds generally is that C(8) = 4A(7).

Activity 5

Students could explore properties of Fibonacci numbers by using 1 and 1 as seed numbers. Some possibilities for investigation include:

- Explain why the pattern of odd and even numbers is O, O, E, O, O, E, …
- Investigate and prove a pattern involving multiples of 3.
- Explain why the sum of three consecutive Fibonacci numbers is twice the largest number in the sum.
- Find out about the history of the Fibonacci numbers and their link to the Golden Ratio. A useful link is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html.

A good source of Fibonacci puzzles is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibpuzzles.html.

Students could also explore Lucas numbers, the sequence commencing 2, 1, 3, 4, 7, 11, 18, 29,… It is interesting to note that every second Lucas number (1, 4, 11, 29,…) corresponds to the ratios found in the exploration of the relationships between the cumulative sums and the numbers in the addition chain. Students may also spot the fact that in the sequence 1, 4, 11, 29, …, $a_{n+1} = 3a_n - a_{n-1}$. A good reference for Lucas numbers is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/lucasNbs.html.

A Fibonacci-like sequence in which any two numbers can be used as seed numbers, such as those used in the activity, is called a G series or a generalised Fibonacci sequence. A good reference for the G sequence and its properties is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibGen.html.