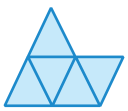
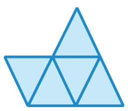
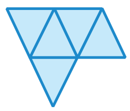
Investigation

Equipment required: 1 or 2 brains, isometric grid paper, isometric dot paper. (A computer drawing program may be used to produce the required shapes.)

Diamonds are forever!

A frieze is a decorative horizontal strip that consists of a repeating pattern. The repeating pattern is formed by combining shapes with translations, reflections and rotations. A frieze contains a high degree of symmetry.

Throughout this investigation you will be asked for different ways to connect your shapes. Mirror reflections or rotations are not considered to be different.

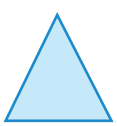
 and  and  are not different ways to connect five equilateral triangles.

The Big Question

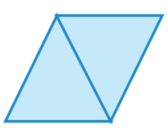
Can you make a frieze with at least five different shapes made from equilateral triangles, diamonds and trapeziums that includes translations, reflections and rotations?

Engage

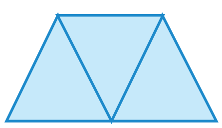
If an equilateral triangle or moniamond is reflected along an edge it becomes a diamond (rhombus). If the triangular shape is reflected along another edge it becomes a triamond (trapezium).



Moniamond



Diamond



Triamond

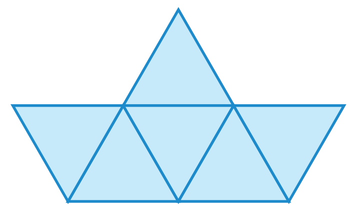
1 On isometric grid paper, draw the three different ways that four equilateral triangles can be connected to make a new shape. These are called tetriamonds.

2 On isometric grid paper, draw the four different ways that five equilateral triangles can be connected to make a new shape. These are called pentiamonds.

Explore

3 Six equilateral triangles can be connected to make a new shape. These are called hexiamonds. There are 12 different hexiamond shapes that can be made. Here is an example:

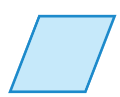
On isometric grid paper, copy the hexiamond on the right and find the 11 other hexiamond shapes.



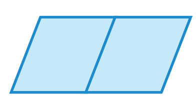
The 12 names given below might help you find the shape. Can you match each name to a shape?

- bar (parallelogram)
- crook (club)
- crown
- sphinx
- snake
- yacht
- chevron (bat)
- signpost (pistol)
- lobster
- hook (shoe)
- hexagon
- butterfly

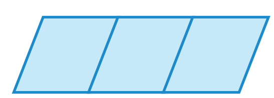
If a diamond shape (rhombus) is translated by a full side length and then joined to the original shape, it makes a shape we can call a straight bidiamond. If the diamond shape is translated again to make three diamond shapes joined together in a row, the result is a shape we can call a straight tridiamond.



Diamond



Straight bidiamond



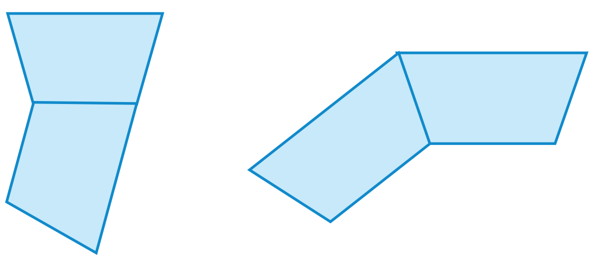
Straight tridiamond

Three diamonds can be connected to make tridiamonds of different shapes, including the straight tridiamond given above.

4 Nine of the 12 hexiamonds found in Question **3** can be made by reflecting, translating and rotating diamonds to make these tridiamonds. On isometric dot paper, draw the nine tridiamonds showing the diamonds that have been used to form them.

5 One of the hexiamond shapes when reflected along one of its sides will form a six-pointed star. Find the shape, reflect it to form the star and then translate the shape to make a row of three six-pointed stars. Draw your pattern on isometric dot paper.

6 There are nine ways that two triamonds (trapeziums made from three equilateral triangles) can be joined exactly edge to edge to make different shapes. On isometric dot paper, draw the nine triamonds (including the two given below) showing the triamonds that have been used to form them.



7 In how many ways can three triamonds be joined together? On isometric dot paper, draw at least three different shapes showing the triamonds that have been used to form them.

8 In how many ways can four triamonds be joined together? On isometric dot paper, draw at least three different shapes showing the triamonds that have been used to form them.

Strategy options

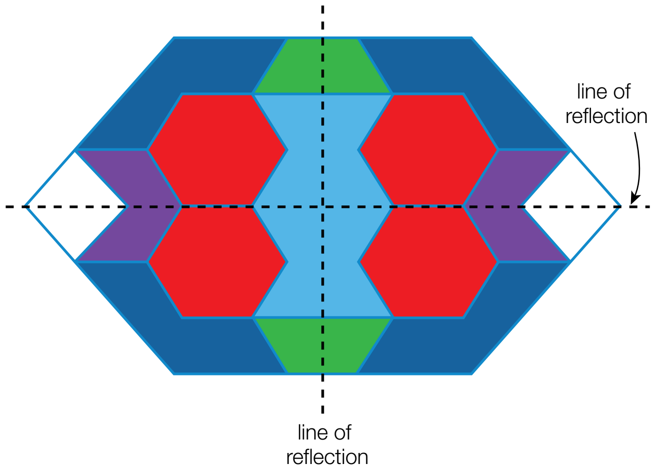
- Draw a diagram.
- Guess and check.
- Look for a pattern.

Explain

9 For two hexiamonds found in Question **3**, two tridiamonds found in Question **4** and two shapes found in Question **7**, explain how they were formed in terms of translation, reflection and rotation.

Elaborate

10 Answer the Big Question by making a frieze with at least five different shapes made from equilateral triangles, diamonds and trapeziums that includes translations, reflections and rotations. Your frieze pattern needs to be at least four triangles high and nine triangle lengths long. An example is given below. Indicate any lines of reflection.



Evaluate

11 Did you find it hard to find all the different shapes? What made it difficult?

12 Did the names help you to find some of the shapes in Question **3**?

Extend

13 There are at least 40 ways of using the 12 different shapes found in Question **3** to cover a rhombus of side 6 units. Can you find one?

10.4 Combined transformations

We can combine translations, reflections and rotational transformations into one transformation of a figure.

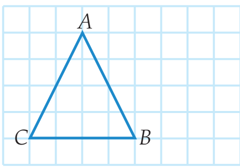
When combining transformations, the transformations are performed in sequence (one after the other) and, after each transformation, the vertices are labelled with an additional dash. For example, the vertex A''' has been transformed by three different transformations.

- Translations, reflections and rotations can be combined into a sequence of transformations.
- A side length can act as the line of reflection during a combined transformation.
- Different colours can be used to distinguish between several transformations.
- The properties of the three different types of transformations still hold true for combined transformations.

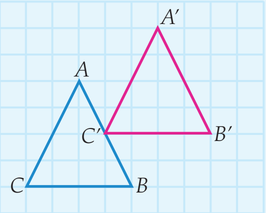
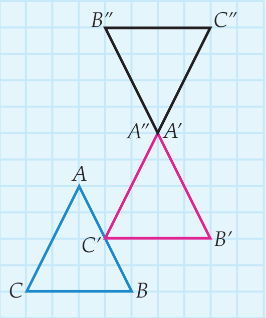
Worked Example 9

WE 9

Copy the following figure onto grid paper and draw the resulting image after the combined transformation has been carried out. Use a protractor to help you. Label each vertex appropriately after each transformation.



A translation of 3 units right and 2 units up, followed by a rotation of 180° in a clockwise direction about the vertex A'.

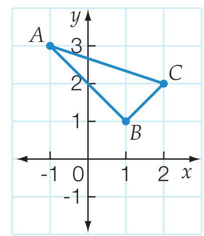
Thinking	Working
1 Perform the first identified transformation in the instructions. Make sure you use the correct notation with the transformed vertices.	
2 Perform each subsequent transformation on the next transformed image. You may want to distinguish each transformation by using a different colour.	

If a figure is drawn on a Cartesian plane, we can perform more than one type of transformation on it. For example, we can perform a rotation, then a translation. We could then reflect the figure in the x- or y-axis. We can then identify the coordinates of the final image.

Worked Example 10

WE10

For the figure given opposite:



(a) write the coordinates of each vertex on the given shape and copy the shape onto grid paper

(b) reflect the shape in the x-axis

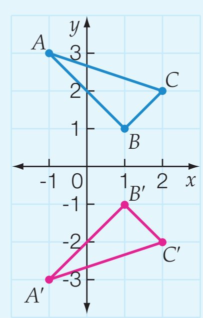
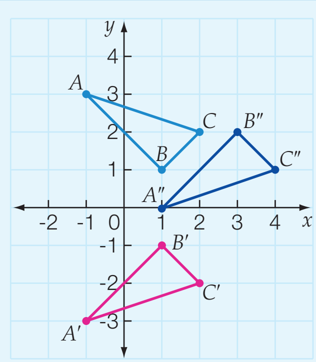
(c) write the coordinates of the reflected vertices

(d) translate this image 2 units right and 3 units up

(e) write the coordinates of the translated vertices

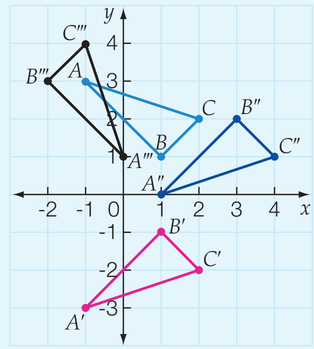
(f) rotate this translated image 90° in an anticlockwise direction about the origin

(g) write the coordinates of the vertices of the final image.

Thinking	Working
(a) Use the x- and y-axes to identify the coordinates of each of the vertices.	(a) $A = (-1, 3)$, $B = (1, 1)$, $C = (2, 2)$
(b) Reflect each vertex in the x-axis.	(b) 
(c) Identify the coordinates of each vertex on the resulting image.	(c) $A' = (-1, -3)$, $B' = (1, -1)$, $C' = (2, -2)$
(d) Translate each vertex with the given translation. (Here, we translate each vertex 2 units to the right and 3 units up.)	(d) 
(e) Identify the coordinates of each vertex on the resulting image.	(e) $A'' = (1, 0)$, $B'' = (3, 2)$, $C'' = (4, 1)$

(f) Rotate each vertex around the origin by the given rotation, labelling each vertex with image notation. (Here, we rotate each point in an anticlockwise direction by 90°.)

(f)



(g) Identify the coordinates of each vertex on the resulting image.

(g) $A''' = (0, 1)$, $B''' = (-2, 3)$, $C''' = (-1, 4)$

10.4 Combined transformations

Navigator

Answers page 691

Q1, Q2, Q3 Column 1, Q4, Q5, Q6, Q7, Q8 Column 1, Q9 Column 1, Q10, Q11, Q13, Q15	Q1, Q2, Q3 Column 2, Q4, Q5, Q6, Q7, Q8 Column 2, Q9 Column 2, Q10, Q11, Q12, Q13, Q15	Q1, Q2, Q3 Column 2, Q4, Q5, Q6, Q7, Q8 Column 2, Q9 Column 2, Q10, Q11, Q12, Q13, Q14, Q15

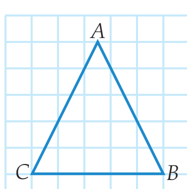
Equipment required: Protractor and grid paper for Questions **1–3**, **6**, **10 (b)** and **13–15**

Fluency

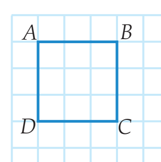
WE9

1 Copy each of the following figures onto grid paper and draw the resulting image after the combined transformation has been carried out. Use a protractor to help you. Label each vertex appropriately after each transformation.

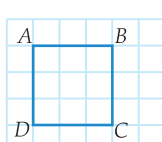
(a) A translation of 3 units left and 1 unit down, followed by a reflection in the line $B'C'$.



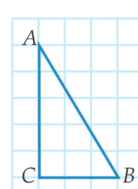
(b) A translation of 5 units right and 2 units down, followed by a reflection in the line $D'C'$.



(c) A translation of 5 units right, followed by a rotation of 90° about B' in a clockwise direction.

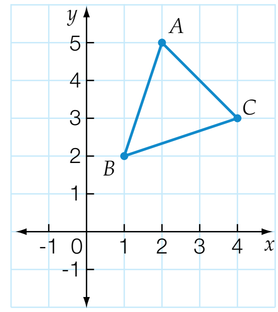


(d) A translation of 3 units left and 4 units down, followed by a rotation of 180° about C' in a clockwise direction.



WE10

2 For the figure given opposite:



(a) write the coordinates of each vertex on the given shape and copy the shape onto grid paper

(b) reflect the shape in the x-axis

(c) write the coordinates of the reflected vertices

(d) translate this image 5 units left and 3 units up

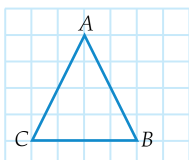
(e) write the coordinates of the translated vertices

(f) rotate this translated image 180° in a clockwise direction about the origin

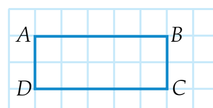
(g) write the coordinates of the vertices of the final image.

3 Copy each of the following figures onto grid paper and draw the resulting image after the combined transformation has been carried out. Use a protractor to help you. Label each of the vertices appropriately after each transformation (for **(e)–(h)**, point O is a point on or inside the figure and is reflected with the shape).

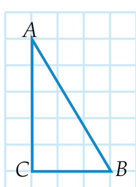
(a) Reflect along the line CB and translate 6 units right and 2 units up.



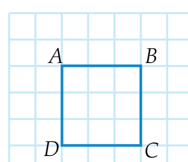
(b) Reflect along the line BC and translate 1 unit left and 3 units down.



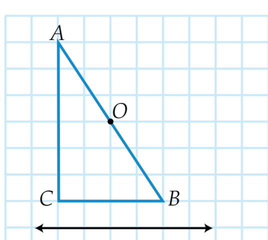
(c) Rotate 90° in a clockwise direction about vertex A and translate 8 units down.



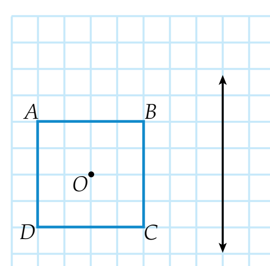
(d) Rotate 180° in an anticlockwise direction about vertex D and move 2 units left and 6 units up.



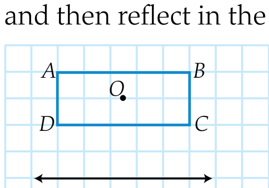
(e) Reflect along the line of reflection and then rotate 90° in an anticlockwise direction about the centre of rotation O'.



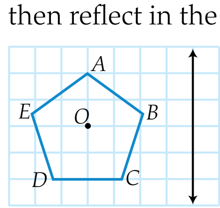
(f) Reflect along the line of reflection and then rotate 90° in a clockwise direction about the centre of rotation O'.



(g) Rotate 90° in an anticlockwise direction about the centre of rotation O and then reflect in the line of reflection.



(h) Rotate 180° in a clockwise direction about the centre of rotation O and then reflect in the line of reflection.



4 Which transformation reverses the order in which the vertices are labelled?

A rotation and translation **B** reflection

C translation **D** rotation

5 Which transformation can result in an image in the same position and orientation as the original figure?

A rotation and translation **B** translation

C reflection **D** rotation

Understanding

6 **(a)** Plot the points (-1, 2), (2, 3) and (3, -1) on a Cartesian plane and label them *A*, *B* and *C*.

(b) Join the points to form a triangle *ABC*, then translate the shape 3 units right and 1 unit down.

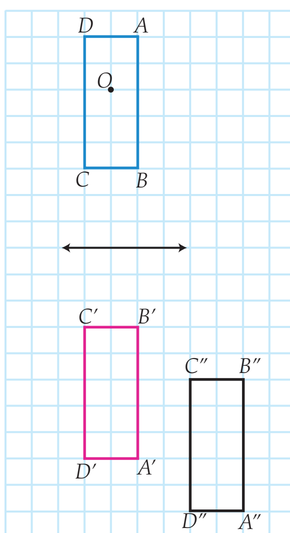
(c) Now, rotate the figure 90° in a clockwise direction about the origin.

(d) Write the coordinates of the vertices of the final image using image notation.

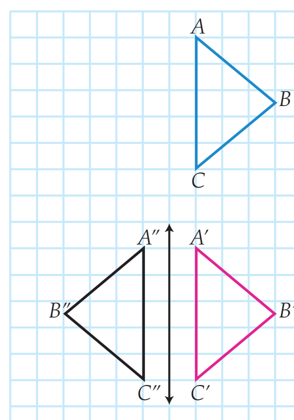
7 For the combined transformations in Question **1**, identify the necessary combined transformation that would move the image back to the position and orientation of the original figure.

8 Describe the transformations that have taken place to move the original figure in blue to the position in pink, then to its final position in black. List the relevant names of each transformation.

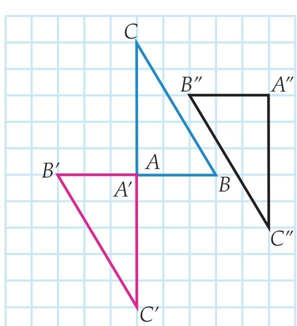
(a)



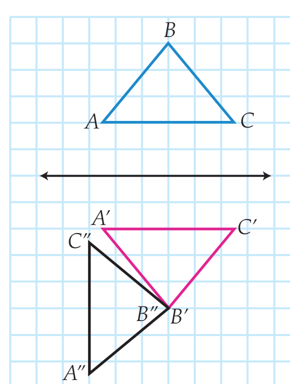
(b)



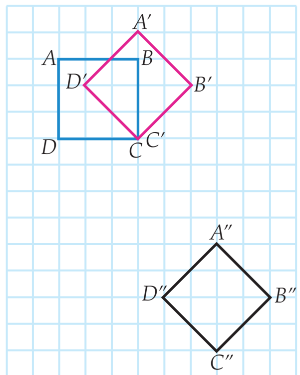
(c)



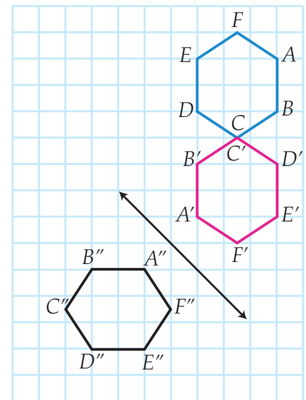
(d)



(e)



(f)



9 For the combined transformations stated in each part of Question **8**, identify the transformations required to move the final image back to the original position and orientation of the figure.

Reasoning

10 **(a)** Without plotting points, write the coordinates of the image after the line joining $A(-2, 3)$ and $B(4, -1)$ is translated 1 unit right and 5 units down, then reflected in the y-axis.

(b) Plot the line AB, the first image $A'B'$ and the final image $A''B''$ on a Cartesian plane to check your answer to part **(a)**.

11 Sarah has written a computer program to transform pictures of tiles. There are only two instructions in her program.

1 Reflect vertically in the line of reflection through the centre of each tile.

2 Rotate 90° clockwise around the given centre of rotation.

Sarah wants to transform the first pattern to the second pattern.

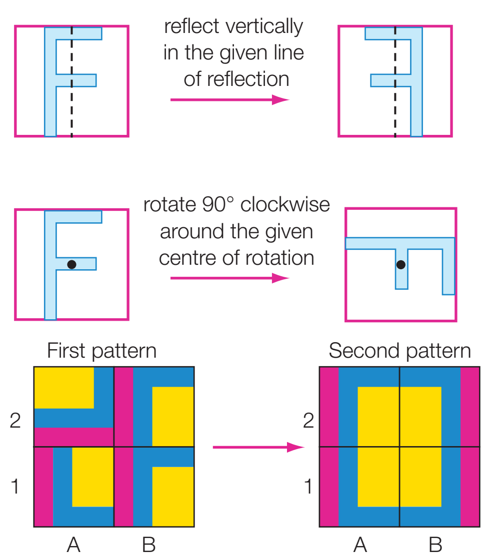
Copy and complete the following instructions to transform the tiles B1 and B2. You must only reflect vertically and rotate 90° clockwise through the centre of each tile.

A1 – Tile is in the correct position.

A2 – Reflect vertically, and then rotate 90° clockwise.

B1 – Rotate 90° clockwise and then …

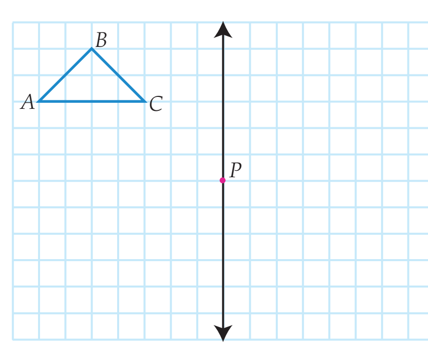
B2 – …



12 The following combined transformation is performed on triangle ABC.

- A reflection in the given line of reflection.
- A rotation of 180° about point P.
- A reflection in the given line of reflection.

Find a single transformation that will move the image $A'''B'''C'''$ back to the original position of the figure. Draw a diagram to show each required transformation.



Open-ended

13 Plot three points on a Cartesian plane and write their coordinates. Decide upon two particular transformations and use them to transform your figure. Draw the image after both transformations and write down the coordinates of the transformed vertices using image notation.

14 Select one of the following capital letters of the alphabet: *B*, *C*, *D*, *E*, *F*, *G*.

(a) Perform a translation and then a reflection in a vertical line of reflection. Draw the resulting images on grid paper and label each image appropriately.

(b) Find a single transformation that will move the final image back to the original position. Show this transformation on your diagram.

(c) Use your answer for part **(b)** to find a single transformation that would produce the same result as the two transformations in part **(a)**.

15 A shape is rotated through 180° about a point *O* and then its image is reflected in a horizontal line of reflection passing through *O*.

(a) Choose any shape and, on grid paper, carry out the transformations described above.

(b) Find a single transformation that would have the same results as the two transformations described above.

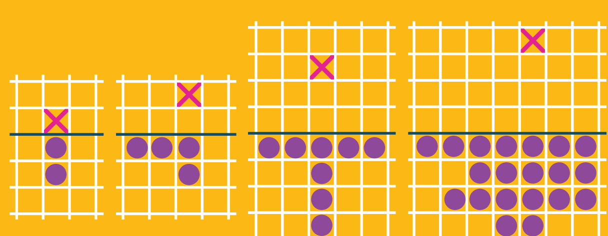
Outside the Square

Problem solving

Conway's army

Equipment required: 1–2 brains, grid paper, approximately 40 small counters

Conway's army is a one-person puzzle that was created by mathematician John Horton Conway in 1961. It is a variant of the peg solitaire game and uses a checkerboard of any dimension. The board is divided by a horizontal line; above it are empty cells and below it are a random number of game pieces, which are referred to as the 'soliders'. Examples of different boards and arrangements of soldiers is as follows.



How to play:

Play proceeds by jumping horizontally or vertically (not diagonally) over other pieces onto an empty space, where jumped-over pieces are then removed.

How to win:

The goal is to place a soldier as far above the horizontal line as possible. The cross in each of the diagrams at left is the maximum distance that a soldier can be placed from the horizontal line for the given number of original soldiers.

Questions

1 Create your own checkerboards with the horizontal line drawn. Begin by investigating how far four counters can be moved beyond the horizontal line. Try at least three different arrangements of the four counters. Is it possible to move a soldier past the second row?

2 Continue to investigate with 4, 5, 6, 7, 8, 9, 10, 15 and 20 counters to see how far past the horizontal line the counters can be moved.

3 Use your observations thus far to predict if it is possible to move a soldier to the fifth row above the horizontal line. Explain your answers.

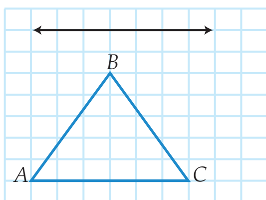
Strategy options

- Act it out.
- Test all possible combinations.

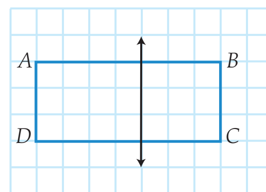
Half-time 10

1 Reflect each of the following figures in the line of reflection shown. **Ex. 10.2**

(a)

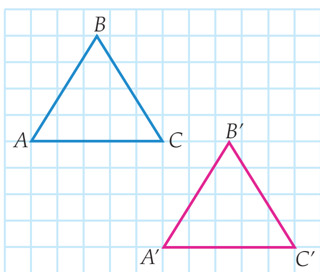


(b)

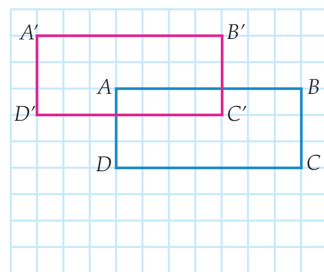


2 Describe the translation shown in each of the following diagrams. **Ex. 10.1**

(a)



(b)

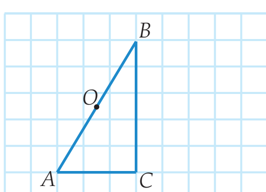


3 Show the image of each of the following figures after it is rotated about the given centre of rotation. **Ex. 10.3**

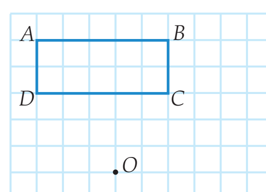
Rotate 180° in an anticlockwise direction about *O*.

Rotate 90° in a clockwise direction about *O*.

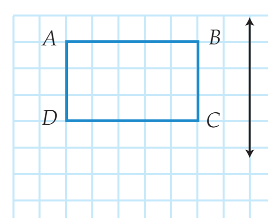
(a)



(b)



4 Reflect the following shape in the line of reflection shown, then translate it 2 units right and 1 unit down. **Ex. 10.4**



5 **(a)** Plot the points (1, 2), (3, 4) and (5, -2) on a Cartesian plane and join them to form a triangle. **Ex. 10.4**

(b) Draw the image after a translation of 6 units left and 1 unit up.

(c) Write the coordinates of the vertices of the translated points.

(d) Draw the image after the image drawn in part **(b)** is reflected in the x-axis.

(e) Write the coordinates of the vertices of this new image.

(f) Finally, rotate the image drawn in part **(d)** 180° clockwise about the origin.

(g) Write the coordinates of your final image.

Transformations with GeoGebra

Versions of this Exploration for other technologies are available in Pearson Reader.

Equipment required: 1 brain, 1 computer with GeoGebra

When you open the GeoGebra program, you will see seven menu options (File, Edit etc.) at the top of the screen. Below these are eleven icons called tools. By clicking on the small arrow in the bottom right-hand corner, a drop-down list of more tools appears. If you hover over a tool the name and how to use it will appear in the top right-hand corner of the screen.

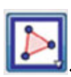
1 Click on the View menu. Deselect 'Axes' and 'Algebra View' by clicking on them. Select the 'Grid View' (the tick on the left of the word will appear or disappear when you select or deselect an option).

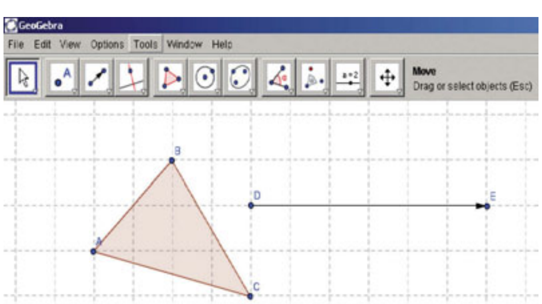
2 Click on the Options menu. Select 'Labelling', then 'New Points only'. Click on Options again, then select 'Point Capturing' and 'On (grid)'.

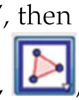
3 If a larger font is required, click on the Options menu and select 'Font Size'. Choose an appropriate size from the list provided.

Translating an object

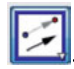
A translation is a slide. We will construct a triangle and translate it along a vector (a line with length and direction).

4 Click on the 'Polygon' tool, . Click on three points on the grid as shown in the diagram below. Notice that a line joins the points as you move around the triangle. To complete the triangle, click on the starting point again. We will be translating (sliding) the triangle ABC 6 units to the right. Can you picture where it will go?



5 Turn off the labelling by clicking on the Options menu, then selecting 'Labelling', then 'No New Objects'. Use the 'Polygon' tool, , to construct a triangle where you think the translated shape will appear.

6 Click on the drop-down arrow of the third tool from the left. Select  'Vector Between two Points'. Click on any grid point, then move the cursor 6 units to the right and click again to complete the vector.

7 Click on the drop-down arrow of the ninth tool from the left. Select the 'Translate Object by Vector' tool . Click anywhere inside the original triangle and then on the vector you have just made.

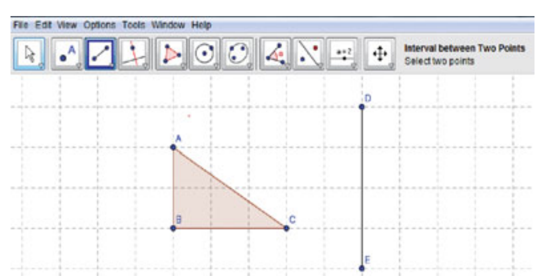
Does your translated image appear on top of the triangle you drew in step **5**? Notice that the points on the translated object are labelled as A', B' and C'. What is the relationship between a point on the original triangle and the translated point in the image?

8 Using the 'Select' tool , click on one of the end points of the vector (it does not matter which one). Drag the point about the page and notice how the translated image moves.

Reflecting an object in a line

A reflection is a flip. We will now construct a right triangle, and reflect it in a line.

9 Click on the File menu. Select 'New'. Unless instructed by your teacher, click 'Don't Save'. Construct a triangle as instructed in step **4**.

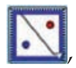


10 Construct an interval by clicking on the 'Line through Two Points' tool  (third tool from the left). Click on the drop-down arrow to select 'Interval between Two Points'. Construct a straight line segment 2 units to the right of your triangle by clicking on two points, one underneath the other.

We will be reflecting (flipping) the triangle ABC in the line segment DE. Can you picture where it will go?

11 Repeat step **5** to turn off the points labelling, then use the 'Polygon' tool to construct a triangle where you think the reflection will go.

12 To turn the labelling back on, click on the Options menu. Select 'Labelling', then 'New Points only'.

13 Select 'Reflect Object in Line' tool , the ninth tool from the left. Click anywhere inside the triangle and anywhere along the interval DE to reflect your object. Did your reflected image match up on top of the triangle you drew in step **11**? Notice that the points on the reflected object are labelled as A′, B′ and C′. What is the relationship between the points on the original triangle and the reflected points in the image?

14 Using the 'Select' tool  (you can get to this tool easily by pressing the 'Escape' key), click and drag one of the end points of the interval DE about the page and notice what happens to the reflected triangle A′B′C′. Keep dragging your point until you find another reflection that has all of the reflected points A′B′C′ on the grid. (Note: If your reflected object disappears from the page you can move the page using the 'Move Graphics View' tool . Press Escape to return to the 'Select' tool.)

Rotating an object

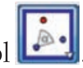
A rotation is a turn. We will now construct a triangle and rotate it by 180° in a clockwise direction.

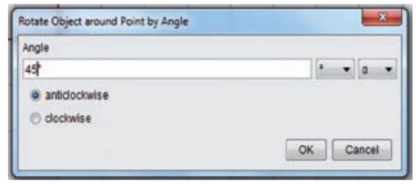
15 Click on the file menu. Select 'New'. Unless instructed by your teacher, click 'Don't Save'. Use the 'Polygon' tool to construct the same triangle you constructed in step **4**.

16 Select the 'New Point' tool , second from the left. Make a new point that is 1 unit to the right and 1 unit down from point C. This will be your 'centre of rotation'.

We will be rotating (turning) the triangle ABC by 180° about this centre of rotation. Can you picture where it will go?

17 Repeat step **5** to turn off the points labelling, then use the 'Polygon' tool to construct a triangle where you think the rotated triangle will appear.

18 Click on the drop-down arrow of the ninth tool from the left. Select the 'Rotate Object around Point by Angle' tool . Click anywhere inside the original triangle and then on the point you have just made. A pop-up window will appear. Delete the number 45 and type 180, but be sure to leave the degree symbol there. (If you delete it, select it again from the drop-down menu immediately to the right of the box.) Click clockwise. Click OK.



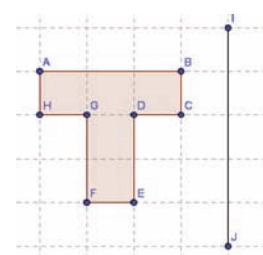
Does your rotated image match up on top of the triangle you drew in step **17**? Notice that the points on the rotated object are labelled as A′, B′ and C′. What is the relationship between a point on the original triangle and the rotated point in the image?

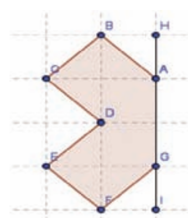
19 Repeat step **18** using angle sizes of 90°, 270° and 360°.

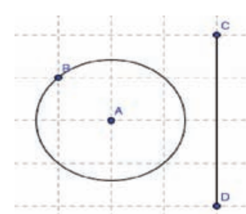
20 You should now have four triangles on the page (not including the unlabelled one you drew). To better understand how the points of the original figure are related to the rotated figures we will now use the 'Circle with Centre Through Point' tool  located as the sixth tool from the left. Click on the centre of rotation and then on one of the original vertices. Repeat this for all three of the original vertices so that you have three circles. What do you notice about the points on each circle?

Taking it further

21 Perform reflections, translations and rotations with various other shapes such as those shown. Be creative—see if you can create a pattern by performing a series of transformations on a shape, or combining it with other shapes.







10.5 Symmetry

Reflectional symmetry

We have seen how a line of reflection can produce a reversed or mirror image of an object, but many objects in our world, from flowers and insects in nature to company logos, flags and letters of the alphabet, have **reflectional symmetry** or mirror symmetry by themselves. Reflectional symmetry occurs when a line can be drawn through an object to divide it into two identical but reversed parts. Each part is the mirror image of the other. This line is called the **axis of symmetry**. In the photo below, a line can be drawn to divide the building into two identical but reversed parts.

Some objects have multiple axes of symmetry. The **order of reflectional symmetry** is the number of axes of reflectional symmetry that can be drawn through a shape. The Pentagon, as shown on the chapter title page, has five axes of symmetry, so a pentagon has symmetry of order 5. Objects with no line of symmetry are called **asymmetrical**. The letter R has no axes of symmetry and is therefore asymmetrical.

Is a human face symmetrical? Although we think our faces are symmetrical, if we reflect each side of our face separately, we get two different outcomes, neither of which is a perfect reproduction of our own face, one thinner and the other fatter. The face you see in a mirror is the mirror image or reversed view of your actual face.

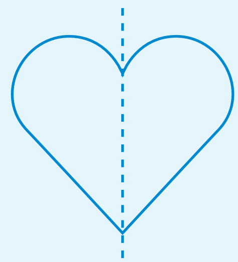
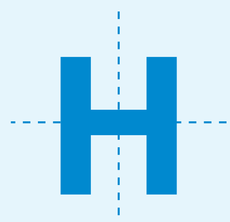
Worked Example 11

WE11

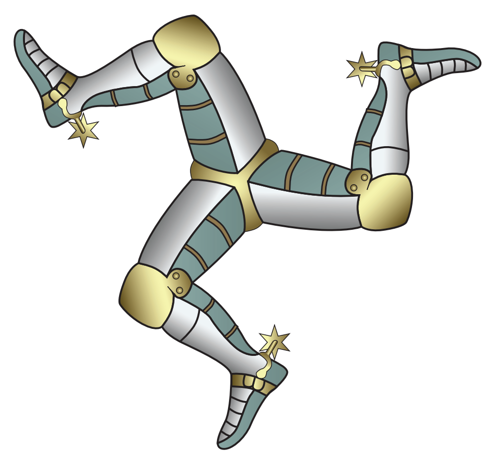
Copy each of the following figures and determine **(i)** whether it has reflectional symmetry by showing every possible axis of symmetry and **(ii)** the order of reflectional symmetry, if present.

(a)

(b)

Thinking	Working
(a) 1 Draw as many axes of symmetry as possible so that the object is divided into two identical but reversed halves.	(a) (i) 
2 Count the number of axes of symmetry. This gives us the order of reflectional symmetry.	(ii) Order of reflectional symmetry = 1.
(a) 1 Draw as many axes of symmetry as possible so that the object is divided into two identical but reversed halves.	(b) (i) 
2 Count the number of axes of symmetry. This gives us the order of reflectional symmetry.	(ii) Order of reflectional symmetry = 2.

Rotational symmetry



An object can also have **rotational symmetry**. If an object can be turned about a point less than 360 degrees and still look like the original object, that object is said to have rotational symmetry. The point about which the object is rotated is called the centre of rotation. An example of an object with rotational symmetry is the armoured 'triskelion', which appears on the Isle of Man's flag. The triskelion has three rotational symmetries referred to as 'three-fold symmetry', because every rotation of 120 degrees produces the original image.

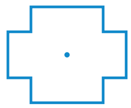
The **order of rotational symmetry** is the number of times in a complete 360° rotation that the original orientation of the shape appears. A pentagon has a rotational order of 5 because every rotation of 72° about the centre of the pentagon produces exactly the same image as the original. This occurs five times in a complete 360° rotation. A pentagon is therefore an interesting shape, as it has an order of 5 in both reflectional and rotational symmetry.

Worked Example 12

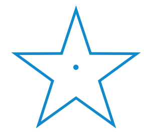
WE 12

Determine **(i)** which of the following figures has rotational symmetry, **(ii)** the order of rotational symmetry and **(iii)** the size of the angle of rotation required to produce an identical image.

(a)



(b)



Thinking

(a) Find the centre of rotation and place your pen/pencil on it.

Rotate the image around this point to determine whether the original image is reproduced.

Count the number of times in a complete 360° rotation that an identical image would be produced. This is the order of rotational symmetry. Divide this number into 360°. This gives the angle of rotation.

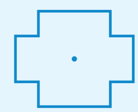
(b) Find the centre of rotation and place your pen/pencil on it.

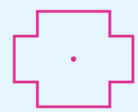
Rotate the image around this point, to determine whether the original image is reproduced.

Count the number of times in a complete 360° rotation that an identical image would be produced. This is the order of rotational symmetry. Divide this number into 360°. This gives the angle of rotation.

Working

(a)





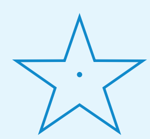
(i) Yes, it has rotational symmetry.

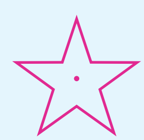
(ii) Order of rotational symmetry = 2.

(iii) $\dfrac{360}{2}$

$= 180°$

(b)





(i) Yes, it has rotational symmetry.

(ii) Order of rotational symmetry = 5

(iii) $\dfrac{360}{5}$

$= 72°$

- An axis of symmetry is a mirror line that can be drawn through a shape to divide it into two identical but reversed halves.
- The order of reflectional symmetry is related to the number of lines of reflectional symmetry.
- An asymmetrical object has no reflectional symmetry.
- If a figure can be turned less than 360° so that it matches the original figure, it has rotational symmetry.
- The order of rotational symmetry is the number of times in a complete 360° rotation that an identical image of the original would be produced.

10.5 Symmetry

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q10, Q11, Q12, Q13, Q15	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q10, Q11, Q12, Q13, Q14, Q16	Q1, Q2, Q3, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17

Answers page 694

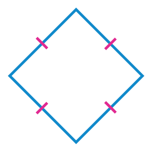
Equipment required: Grid paper for Question **12**

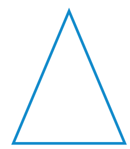
Fluency

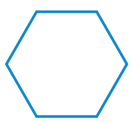
1 Copy each of the following figures and determine **(i)** whether it has reflectional symmetry by showing every possible axis of symmetry and **(ii)** the order of reflectional symmetry, if present.

WE11

(a)


(b)


(c)


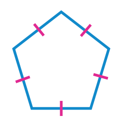
(d)


(e)


(f)


(g)

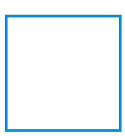

(h)

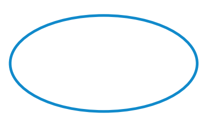

(i)


2 Determine **(i)** which of the figures from Question **1** has rotational symmetry, **(ii)** the order of rotational symmetry and **(iii)** the size of the angle of rotation required to produce an identical image.

WE12

3 How many axes of symmetry do the following objects have?

(a)


(b)


(c)


For Questions **4**, **5** and **6** refer to the letters of the word M A T H S.

4 Which letter does not have an axis of symmetry?

A M **B** A **C** H **D** S

5 How many of the letters have one axis of symmetry?

A 5 **B** 2 **C** 3 **D** 4

6 Which letter has two axes of symmetry?

A H **B** M **C** A **D** S

Understanding

7 Copy the lower case letters of the alphabet into your book.

a b c d e f g h i j k l m
n o p q r s t u v w x y z

(a) Which lower case letters are asymmetrical?

(b) Which lower case letters have one axis of symmetry?

(c) Which lower case letters have two or more axes of symmetry?

8 Copy the upper case letters of the alphabet into your book.

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z

(a) Which upper case letters are asymmetrical?

(b) Which upper case letters have one axis of symmetry?

(c) Which upper case letters have two or more axes of symmetry?

9 Do any of the capital letters from Question **8** have rotational symmetry? If so, which ones are they?

Reasoning

10 Symmetry is present everywhere in nature. For each of the following images, determine whether it has almost perfect reflectional and/or rotational symmetry and state the order of each.

(a) 

(b) 

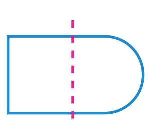
(c) 

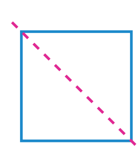
(d) 

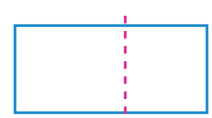
(e) 

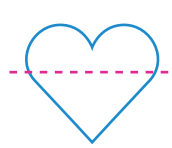
(f) 

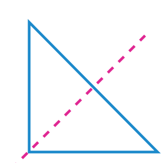
11 For each of the following shapes, state whether or not the dotted line is an axis of symmetry.

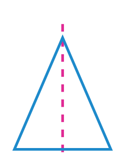
(a) 

(b) 

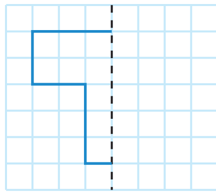
(c) 

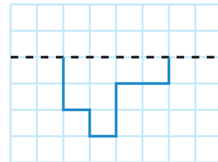
(d) 

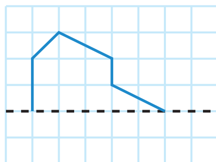
(e) 

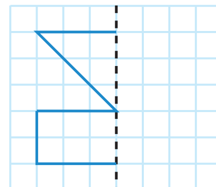
(f) 

12 For each of the following diagrams, use the dotted line as an axis of symmetry to complete the diagram using grid paper.

(a)


(b)


(c)


(d)


13 Which two quadrilaterals have exactly two lines of reflectional symmetry?

14 Which regular polygon has exactly three lines of reflectional symmetry?

Open-ended

15 WOW is a symmetrical word. Locate the axis of symmetry and find three more words that are also symmetrical.

16 The dates of some years have reflectional and/or rotational symmetry. An example of a year that has rotational symmetry is 1961 and a year that has reflectional symmetry is 1881.

Identify an example of a year that has:

(a) rotational symmetry

(b) reflectional symmetry

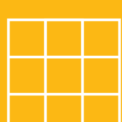
(c) both reflectional and rotational symmetry.

17 Describe when the following statement is true and when it is false.

'This parallelogram has reflectional symmetry.'

Outside the Square — Problem solving

Symmetry challenge



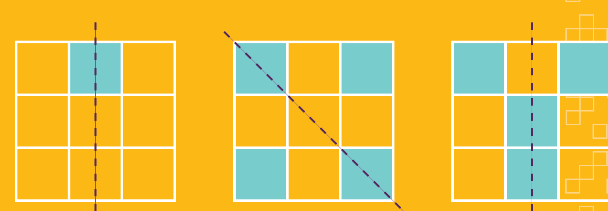
How many symmetrical designs of shaded squares can you find in a blank 3 by 3 square grid?

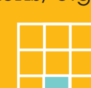
Hint: Before you shade any squares, can you identify the lines of symmetry present in this 3 by 3 grid shown at left? What symmetries are possible if you shade one square only?

Strategy options

- Draw a diagram.
- Look for a pattern.
- Break problem into manageable parts.

Here are three examples of possible symmetrical designs. Note the position of the line of symmetry.



Don't include any rotations; e.g.  is considered the same as .

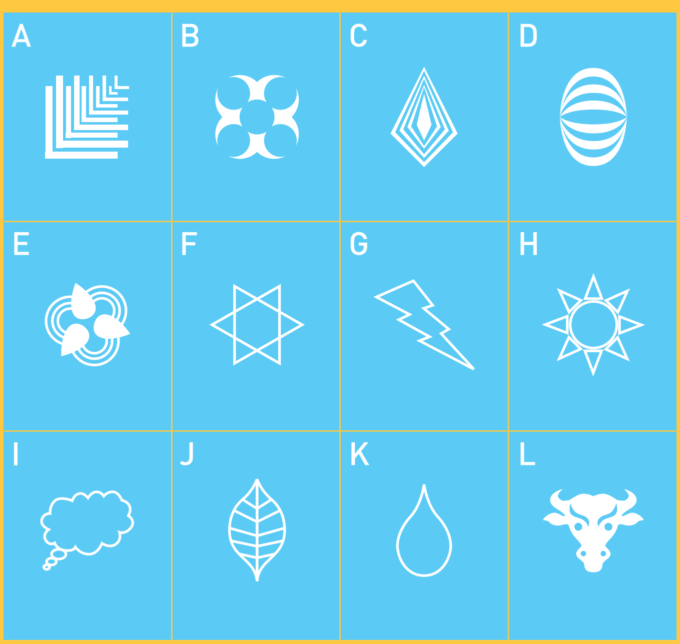
Maths 4 Real

SNOW LOGO

Symmetry is used in art, architecture, company logos and even in the design of national flags. We seem to be drawn to symmetrical objects, and designers and artists take advantage of our preference for symmetry.

Imagine you are a young marketing executive for a company that manufactures and sells snowboards. You know that many companies succeed or fail on the recognition of their brand by customers. The easiest method for achieving customer recognition is by integrating elements of symmetry into a company logo. Your manager, who is a symmetry fanatic, has given you the task of selecting the most symmetrical (reflectional and rotational symmetry) logo from a collection of sample logos developed by your company's graphic artists.

Sample logos



1 To assist you with this task, use the following table as a template and note down in your book the order of reflectional and rotational symmetry for each of the sample logos.

Logo	Order of reflectional symmetry	Order of rotational symmetry
A		
B		
C		

2 At the end of the selection process, you think that you may be able to develop a better-looking, more symmetrical logo for your company. Your task is to develop your own version of your new company's logo and identify all elements of its symmetry to show the CEO of the company.

3 Apart from the symmetry elements, write down two or three other criteria you would use when deciding which is the best logo for a company.

Research

- Find as many different examples as you can of well-known company logos or brands that have reflectional and/or rotational symmetry. (Car manufacturers are a good place to start.) Find some examples that are asymmetric (have no symmetry).
- Large companies spend a lot of money on getting the right design for their logo—the symbol by which the company is identified. Why do you think such importance is placed on the logo design?
- Which logo do you think is the most recognised throughout the world? Try to find out.

Mathspace

Ambigrams and symmetry

Equipment required: 1 brain, a small mirror

Letter ambigrams

An ambigram is a word that you can read in more than one direction or orientation. The word 'ambigram' is derived from *ambi* (meaning 'both', or 'on both sides') and *gram* (meaning 'something written down or drawn').

Ambigrams can have either reflectional symmetry or rotational symmetry.

Vertical reflectional symmetry

Use your mirror to check

Horizontal reflectional symmetry

COOKBOOK

Rotational symmetry

dollop

Turn your book to check the rotational symmetry

1 For each of the following ambigrams, complete the word (if required) and state whether it has rotational, vertical reflectional, or horizontal reflectional symmetry. (Use your mirror to check.)

(a) SWIMS (d) BID

(b) HIDE (e) bid

(c) MUM (f) su____

2 (a) Some pairs of letters, such as 'b' and 'q', have rotational symmetry when used together. (Rotate 'b q' or 'q b' to see how this works.) What other pairs of letters have rotational symmetry when used together?

(b) What single letters have rotational symmetry when used by themselves?

(c) Create your own ambigram with rotational symmetry or vertical reflectional symmetry.

Number ambigrams

Some numbers have symmetrical properties. For instance, the number 88 has vertical reflectional and horizontal reflectional symmetry, as well as rotational symmetry.

3 Find some number ambigrams that have the following symmetries.

(a) vertical reflectional symmetry

(b) horizontal reflectional symmetry

(c) rotational symmetry

4 Digital clocks often display numbers using a figure-of-eight digital number grid. This means that some digital numbers look the same upside down.

0123456789

(a) Which of these digital numbers have rotational symmetry?

(b) Which digital square numbers between 1 and 100 are ambigrams? What types of symmetry do they have?

(c) What is the largest 4-digit ambigram you can make that:

(i) uses only one of the digits 0 to 9

(ii) uses two of the digits 0 to 9

(d) What is the largest 5-digit ambigram you can make that:

(i) uses only one of the digits 0 to 9

(ii) uses three of the digits 0 to 9

Grid symmetry

Draw up a Cartesian grid that goes from 0 to 4 on the horizontal axis, and from 0 to 3 on the vertical axis.

Using the clues below, place the letters R, C, K, H, S, U, Y, M, and B at their correct places on the Cartesian grid.

Clues

- The letter at (1, 2) is not symmetrical in any way.
- The letters at (0, 0), (1, 1), (2, 2) and (3, 3) all have horizontal reflectional symmetry.
- The letters at (0, 2) and (3, 3) have rotational symmetry.
- The letters at (3, 0), (3, 1) and (3, 2) all have vertical reflectional symmetry.
- The letters at (1, 1) and (3, 1) contain only curved lines.
- The letters at (2, 2) and (3, 2) contain only straight lines.
- The letters at (3, 0), (3, 1) and (3, 2), in that order, spell a word that means 'tasty'.

What word is spelt by the letters at (1, 2), (3, 1), (0, 0) and (3, 0), in that order?

10.6 Drawing and visualising 3D shapes

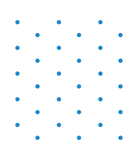
Objects such as buildings, machinery, car parts, people and play stations have three dimensions: length, width and height. It is often necessary to draw three-dimensional objects on a two-dimensional (2D) plane, such as a piece of paper or a computer screen.

The drawing of the house in this diagram is an isometric drawing used by an architect.

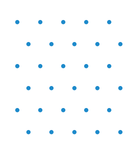
In Greek, 'iso' means 'equal' and 'metric' refers to 'measuring'. In isometric dot paper, the distances between each dot and the dots surrounding it are identical, be it a diagonal or a vertical direction.

Triangular dot paper, also called **isometric** paper can help to overcome the difficulties of drawing the three dimensions of real objects on a 2D plane. Objects represented on triangular dot paper are also known as isometric drawings.

It is recommended that isometric dot paper be held in this orientation:



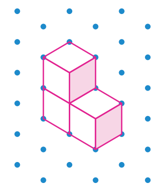
This orientation is very difficult to use (not recommended).

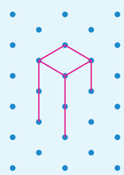
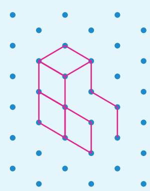
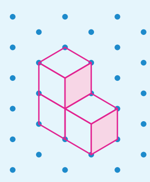


Worked Example 13

WE13

Use isometric dot paper to copy the following shape.



Thinking	Working
18 Join the points of a parallelogram that will represent the top face. Draw in all the necessary vertical lines from the corners of the face.	
19 Draw in all the lines from the bottom of the verticals you have just drawn in step **1** that represent horizontal lines. Draw in all additional required vertical lines from the ends of the lines you have just drawn.	
20 Join the ends of the lines you have just drawn. To create the illusion of 3D, choose one direction (front, side or top) and shade in all faces you would see from that direction.	

10.6 Drawing and visualising 3D shapes

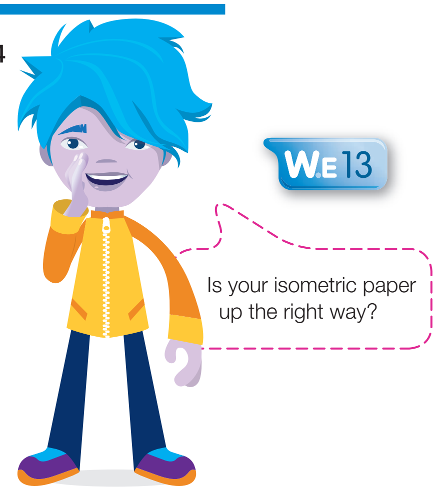
Navigator

Q1, Q2 (a)–(c), Q3, Q4, Q5, Q6 (a)–(c), Q8, Q9 (a), Q10, Q11, Q13	Q1, Q2 (b),(d),(f), Q4, Q5, Q6 (c)–(e), Q7, Q8, Q9 (b), Q10, Q11, Q14	Q1, Q2 (d)–(f), Q4, Q6 (d)–(f), Q7, Q8, Q9 (c), Q10, Q11, Q12, Q14

Answers page 696

Equipment required: Isometric dot paper for Questions **1**, **5**, **8**, **9** and **12–14**

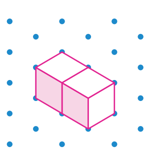
Fluency



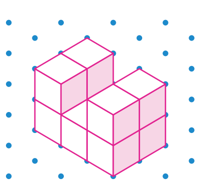
WE13

1 Use isometric dot paper to copy the following shapes.

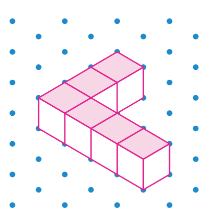
(a)



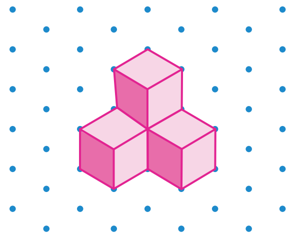
(b)

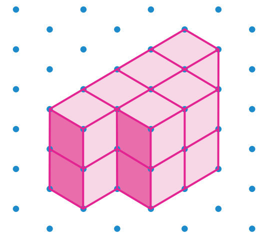


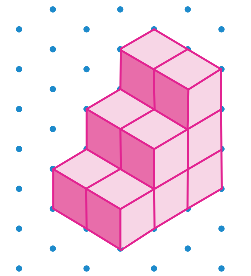
(c)

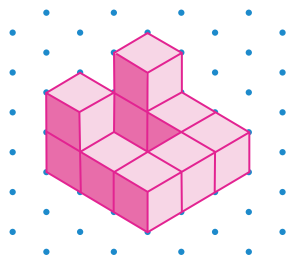


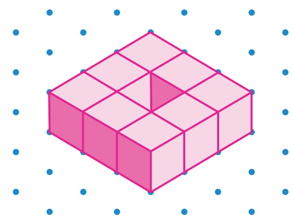
2 How many cubes would be required to build these solids? (Assume that there are no cubes missing at the back of the solids where you cannot see.)

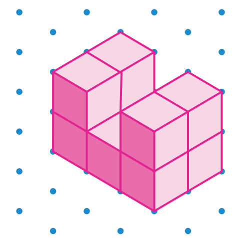
(a) 

(b) 

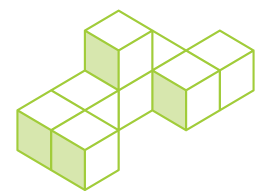
(c) 

(d) 

(e) 

(f) 

3 The number of cubes required to build this solid is:



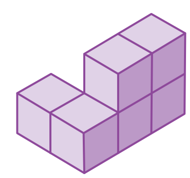
A 7

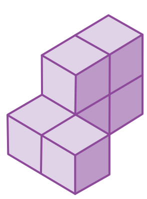
B 8

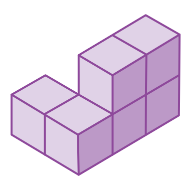
C 9

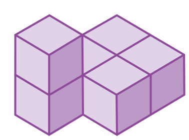
D 10

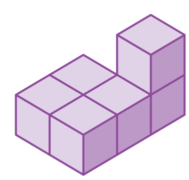
4 Which of the shapes below is *not* a rotation of the one shown?



A 

B 

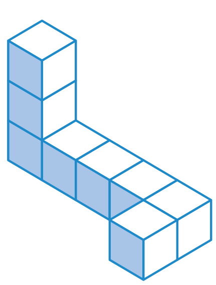
C 

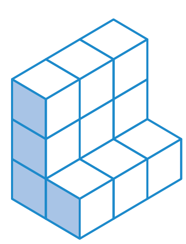
D 

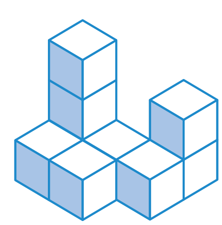
Understanding

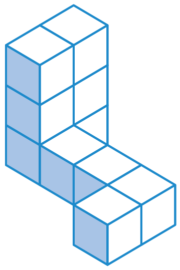
5 Using isometric dot paper, draw a 3D sketch of a Rubik's cube. (A Rubik's cube is made of 27 cubes stacked three layers high, three layers wide and three layers deep.)

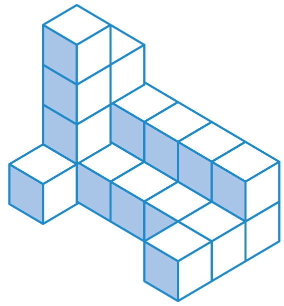
6 How many cubes are in each solid?

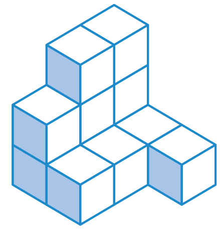
(a) 

(b) 

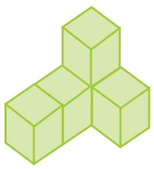
(c) 

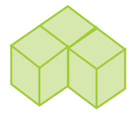
(d)


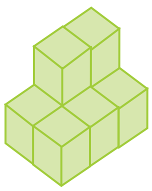
(e)


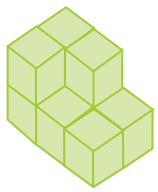
(f)


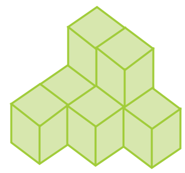
7 Which of the following shapes could result from joining the two shapes below?

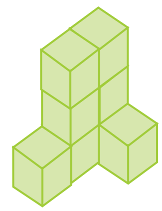




A


B


C


D


Reasoning

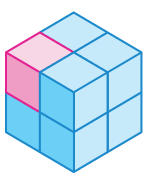
8 Using the aerial photograph on page 560 on the opening page of this chapter:

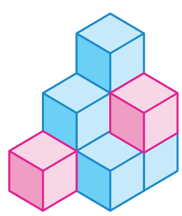
(a) draw a top view of the Pentagon

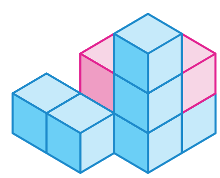
(b) draw a side view of the Pentagon.

(c) Does it matter which side you choose? Explain.

9 Draw the solid after the pink shaded blocks have been removed.

(a)


(b)


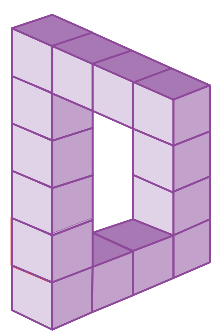
(c)


10 This shape is known as the impossible rectangle.

(a) Why is this shape known as the *impossible* rectangle? (You may like to try to make it with blocks.)

(b) Copy the impossible rectangle onto isometric dot paper.

(c) Draw a similar rectangle that is possible. Colour your drawing using three colours.



11 Draw the top view of the house drawn on page 604, the first page of this section.

Open-ended

12 A sculptor is designing a sculpture based on an arrangement of 24 cubes. Each cube has a width of 50 cm and the maximum height of the sculpture is to be 2.5 m. The sculptor wants to have a maximum of six cubes at the front and a maximum of four cubes in depth. Each cube must have at least one whole face touching another cube. Use isometric dot paper to design some sculptures that meet all these requirements.

13 Using isometric dot paper, draw three different solids that can be built from 10 cubes that have whole faces touching.

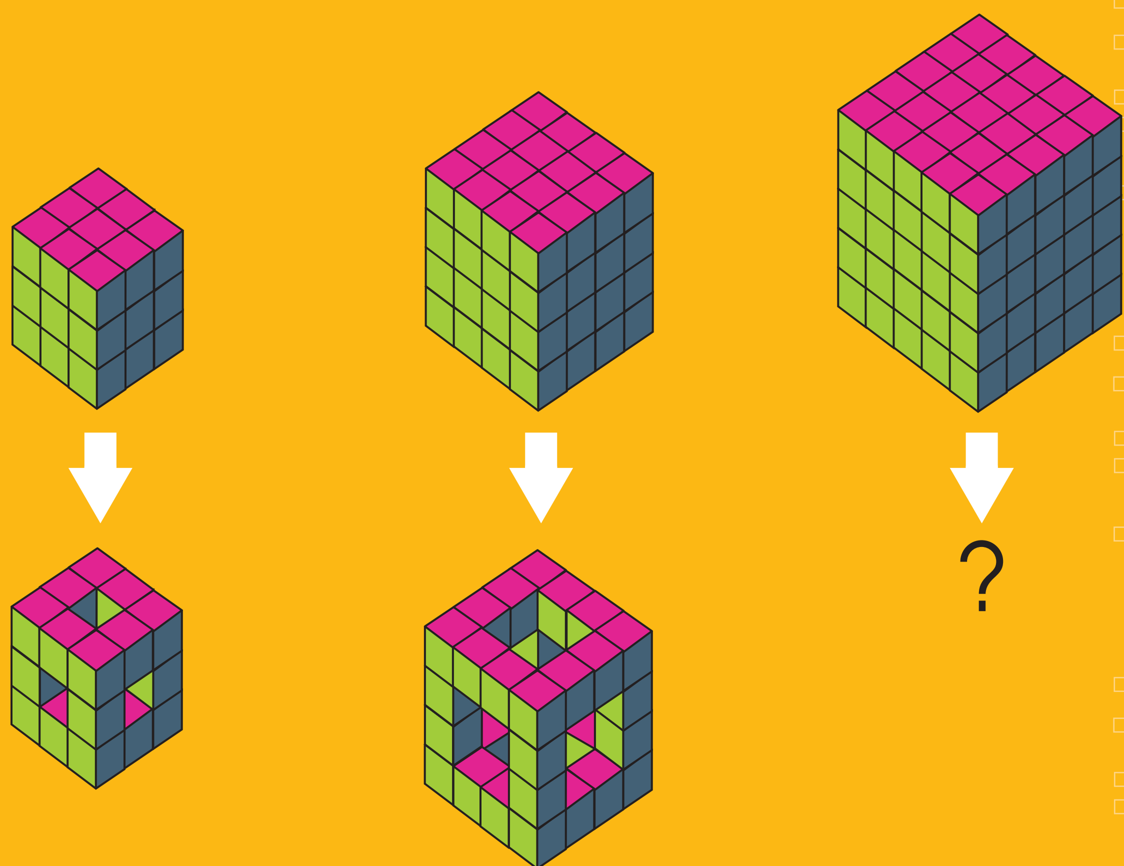
14 Draw three shapes on isometric dot paper that, when joined together, form a $3 \times 3 \times 3$ cube.

Outside the Square

Problem solving

What comes next?

Look at the pattern below.



A solid $3 \times 3 \times 3$ cube is made using 27 blocks. When one block is removed from the centre of each of the six faces and one block from the centre of the cube, a frame of the same cube is made using 20 cubes.

- Draw the next shape in the pattern.
- How many cubes are needed to make the frame of a $5 \times 5 \times 5$ cube?
- How many cubes are needed to make the frame of a $6 \times 6 \times 6$ cube?

Strategy options

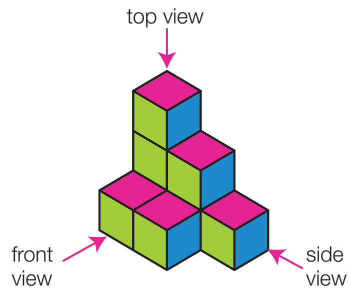
- Make a table.
- Look for a pattern.

Plan views and elevations

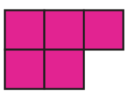
10.7

Three-dimensional objects can be viewed from various positions, or views. Objects looked at from the top, front and side can be drawn in two dimensions, and are known as **plan views**, or orthogonal projections.

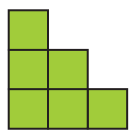
A 3D shape can look different when viewed from different points of view. A top view, or 'bird's eye view' is what you see when looking directly down on the shape, a side view is what you see when looking directly at the side and the front view is what you see when looking at the front.



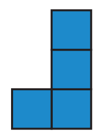
top view



front view



side view



Worked Example 14

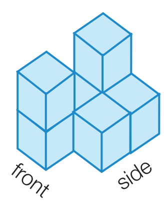
WE14

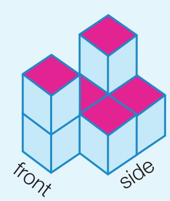
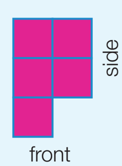
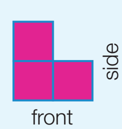
For the following solid, draw the:

(a) top view

(b) front view

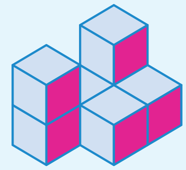
(c) side view.



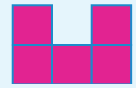
Thinking	Working
(a) 1 Highlight the faces seen from the top.	(a) 
2 Draw squares to make the shape you would see if you were looking down on the solid.	
(b) 1 Highlight the faces seen from the front. Do not highlight front faces that are behind another block.	(b) 
2 Draw squares to make the shape you would see if you were standing in front of the solid.	

(c) **1** Highlight the faces seen from the side.

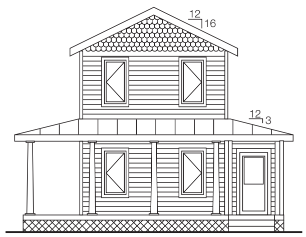
(c)



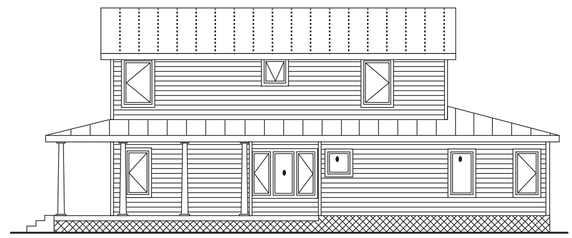
2 Draw squares to make the shape you would see if you were standing at the side of the solid.



Drawings of houses and buildings viewed from a side are known as **elevations**. Elevations may be referred to as front, rear and side elevations, or by compass direction, such as an east elevation.



FRONT ELEVATION



SIDE ELEVATION

10.7 Plans views and elevations

Navigator

Answers page 696

Q1, Q2, Q3, Q5, Q6, Q7, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Equipment required: Isometric dot paper for Questions **5**, **7**, **8**, **9**, and **10**

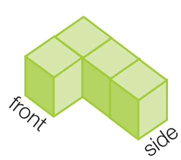
Fluency

WE14

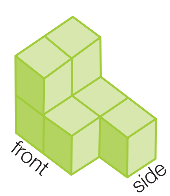
1 For each of the following solids, draw the:

(i) top view **(ii)** front view **(iii)** side view.

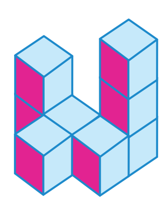
(a)



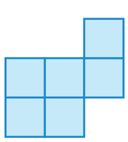
(b)



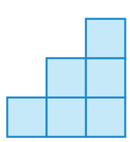
2 The front view of the solid shown has been shaded. The top view of this solid looks like:



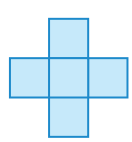
A



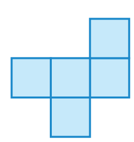
B



C

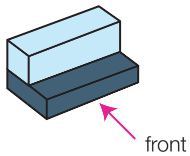


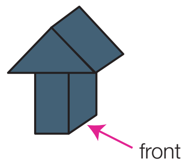
D

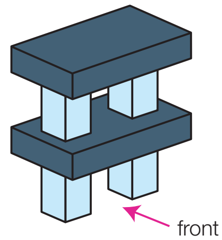


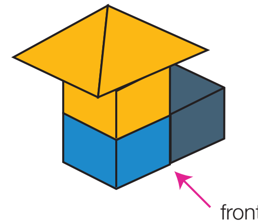
Understanding

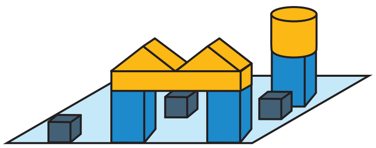
3 The following structures are made with coloured blocks. Draw the front elevation of each structure by imagining you are standing looking from the position of the arrow, then move 90° to the right and draw the side elevation.

(a)


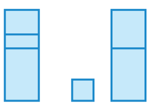
(b)


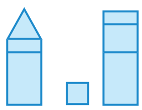
(c)


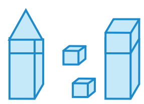
(d)


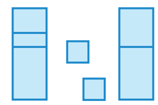
4


(a) Which diagram below shows a possible side elevation of the blocks above?

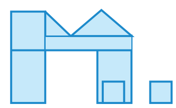
A


B


C


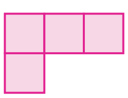
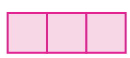
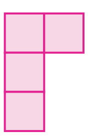
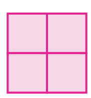
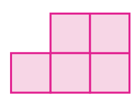
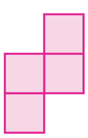
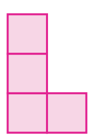
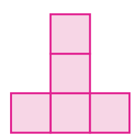
D


(b) Which elevation of the block structure above is shown opposite?

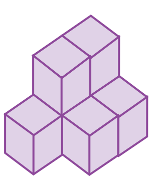


A front
B rear
C side
D top

5 For each set of the plan views below, draw the 3D solid on isometric dot paper.

	top view	front view	side view
(a)			
(b)			
(c)			

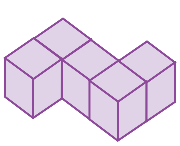
6 A *mat plan* shows the top view of a shape and the number of cubes in each position.

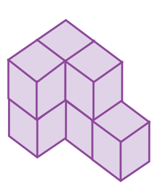

Isometric drawing

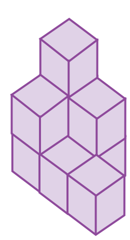
2	1
2	1
1	

mat plan

Draw the mat plan for each of the shapes below.

(a)


(b)


(c)


Reasoning

7 Draw each of the following mat plans as 3D shapes on isometric dot paper.

(a)

1	1
1	

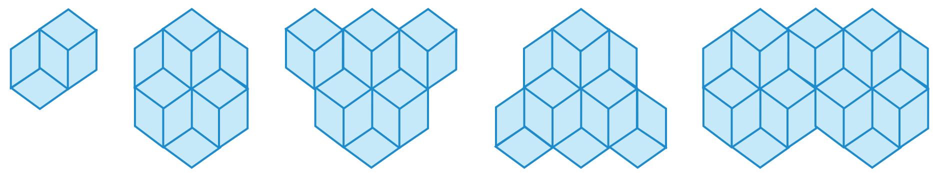
(b)

3	2
1	

(c)

3	3	3
2	2	
1		

8 (a) Copy the following diagrams onto isometric dot paper.



(b) On your drawings, use three colours to shade the different faces of the cubes.

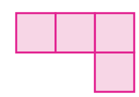
(c) How many cubes are in each of the diagrams?

(d) There is more than one way to colour the diagrams. If the cubes were coloured differently, would each diagram still look like it has the same number of cubes?

(e) Draw the top view of each shape.

Open-ended

9 A shape made with small cubes looks like this from above.



top view

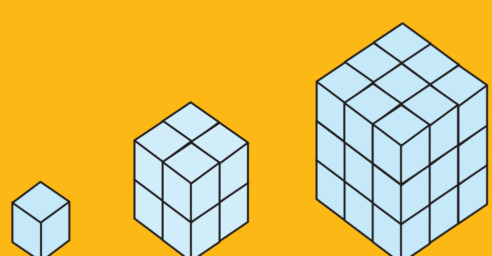
Draw three possible 3D drawings on isometric dot paper, stating the number of cubes in each diagram.

10 Select a three-dimensional object and draw the front, rear and side elevations.

Outside the Square — Problem solving

Bird's eye squares

Ezra glued some 1 cm cube blocks together to form different-sized cubes. He made four cubes of side length 2 cm, four cubes of side length 3 cm and had four 1 cm cubes left over.



Ezra made a solid with his different-sized cubes. The solid's top view was a square.

(a) What is the minimum number of cubes Ezra would need to build his solid?

(b) How many different solids can be made with the cubes he has made that have a top view of a square and are made with more than one cube? Draw the top view of each solid.

Strategy options

- Draw a diagram.
- Make a model.