Theory Exercises

- **Taylor Polynomials (07-08-2018):**

  1. Produce the linear and quadratic Taylor polynomials for the following functions, $y = f(x)$. Plot graphs of both the functions and their Taylor polynomials.
     A. $y = \sqrt{x}$ at $a = 1$ B. $y = \sin x$ at $a = \pi/4$ C. $y = e^{\cos x}$ at $a = 0$
     D. $y = \ln(1 + e^x)$ at $a = 0$
  2. Produce a general formula for the degree n Taylor polynomials for the following functions, $y = f(x)$, using $a = 0$ as the point of approximations.
     A. $y = (1 - x)^{-1}$ B. $y = \sqrt{1 + x}$ C. $y = (1 + x)^{1/3}$
     Plot the above functions along with their respective fourth-degree Taylor polynomials.

- **The Bisection Method (09-08-2018):**

  1. Use the bisection method to find the real roots of the following functions, using an error tolerance of $\epsilon = 0.0001$.
     A. $x^3 - x^2 - x - 1 = 0$ B. $x = 1 + 0.3 \cos x$ C. The smallest positive root of $\cos x = 0.5 + \sin x$ D. $x = e^{-x}$ E. The two smallest positive roots of $e^{-x} = \sin x$
     F. $x^3 - 2x - 2 = 0$ G. All real roots of $x^4 - x - 1 = 0$
  2. Find the largest root of $f(x) = e^x - x - 2 = 0$, with $\epsilon = 0.0001$.
  3. Find the smallest positive root of $f(x) = 1 - x + \sin x = 0$, with $\epsilon = 0.0001$.
  4. Find the smallest non-zero positive root of $x = \tan x$, with an accuracy of $\epsilon = 0.0001$. Further solve for the root that is closest to $x = 100$.

- **The Newton-Raphson Method (10-08-2018):**

  1. Use the Newton-Raphson method to find the real roots of all the foregoing functions that you solved by the bisection method.
  2. The function $y = f(x) = ax(1 - x)$, with $a = \pm 1$, has two roots at $x = 0, 1$ and a turning point at $x = 1/2$. Use the Newton-Raphson method to test the convergence towards $x = 0$, starting from an initial value of 0.51. Similarly test the convergence towards $x = 1$, starting from 0.49. Perform this exercise for both signs of a. Repeat this entire exercise for initial values of 0.501, 0.5001 and 0.499, 0.4999, respectively.

- **The Secant Method (14-08-2018):**

  1. Use the secant method to find the real roots of all the foregoing functions that you solved by the bisection method.

- **Cubic Equations (17-08-2018):**

  1. In the set of problems on the bisection method, there are two cubic equations. Solve them by the Cardan method. Match your results with what you have got numerically.
  2. Solve the following cubic equations by Cardan's method and verify numerically.
     A. $x^3 + 72x - 1720 = 0$ B. $x^3 + 63x - 316 = 0$ C. $28x^3 - 9x^2 + 1 = 0$
     D. $x^3 - 15x^2 - 33x + 847 = 0$

- **Quartic Equations (21-08-2018):**

  1. In the set of problems on the bisection method, there is a quartic equation. Solve it by both the Ferrari and the Descartes methods. Match the results in both the cases, along with what you have got numerically.
  2. Solve the following quartic equations by either the Ferrari or the Descartes method, whichever is appropriate. Verify your results numerically.
     A. $x^4 - 3x^2 - 42x - 40 = 0$ B. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$ C. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$ D. $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$

- **Supplementary Problems on Foregoing Topics (07-09-2018):**

  1. Plot the following functions $y = f(x)$, discussing the quadrants in which they lie, their turning points (if any), the type of turning points and the asymptotic behaviour of $f(x)$.
     A. $y = (\ln x)/x$ B. $y = x - 1 + e^{-x}$ (Plot only for $x \geq 0$) C. $y = x + \sin x$ (Plot for $-2\pi \leq x \leq 2\pi$) D. $y = (e^x - e^{-x})/(e^x + e^{-x})$
  2. Produce the second-degree Taylor polynomial for $f(x) = \cos x$ using $a = \pi$ as the point of approximation. Plot the Taylor polynomial along with $f(x)$ within $0 \leq x \leq 2\pi$.
  3. Produce the third-order Taylor polynomial for $f(x) = \tan x$ using $a = 0$ as the point of approximation. Plot the Taylor polynomial along with $f(x)$ within $-\pi/2 \leq x \leq \pi/2$.
  4. Find the root of $f(x) = x + \ln x = 0$ by the bisection method. Apply an error tolerance of $\epsilon = 0.0010$ and provide your final answer correct up to 4 places of decimal. Present your numerical steps clearly in a table with full details.
  5. Find the larger root of $f(x) = x^2 - \exp(-x^2) = 0$ by the bisection method. Apply an error tolerance of $\epsilon = 0.0010$ and provide your final answer correct up to 4 places of decimal. Present your numerical steps clearly in a table with full details.
  6. Find the root of $f(x) = x^7 + 2(x - 1) = 0$ by the Newton-Raphson method. Tabulate your numerical steps clearly, and provide your answer correct up to 4 places of decimal.
  7. Find the root of $f(x) = x^5 + x - 1 = 0$ by the Newton-Raphson method. Tabulate your numerical steps clearly, and provide your answer correct up to 4 places of decimal.
  8. Find the root of $f(x) = 2 - x - e^x = 0$ by the secant method. Present your numerical steps clearly in a table, and provide your final answer correct up to 5 places of decimal.
  9. Find the roots of $x^4 - 3x^2 - 6x - 2 = 0$ by either the Descartes or the Ferrari method.

- **Polynomial Interpolation (07-09-2018):**

  1. Given the data points $(0, 2)$ and $(1, 1)$, find the following:
     (a) The straight line interpolating this data.
     (b) The function interpolating this data, $f(x) = a + be^x$. Also plot its graph.
  2. Find the function $P(x) = a + b\cos(\pi x) + c\sin(\pi x)$ that interpolates the data $(0, 2)$, $(0.5, 5)$ and $(1, 4)$. Approximate the function as a quadratic polynomial and check how well it interpolates the data.
  3. With the following data:

x	0.0	1.0	2.0	3.0	4.0	5.0	6.0
y	2.0000	2.1592	3.1697	5.4332	9.1411	14.406	21.303

     Carry out a linear interpolation between adjacent nodes.

4. Obtain a quadratic Lagrange polynomial that interpolates $(0,1)$, $(1,2)$ and $(2,3)$.
5. For $x_0 = 0.85$, $x_1 = 0.87$, $x_2 = 0.89$ and $f(x) = e^x$, estimate $f[x_0, x_1]$, $f[x_1, x_2]$ and $f[x_0, x_1, x_2]$. Also check the accuracy of the approximation $f[x_0, x_1] \simeq f'[(x_0+x_1)/2]$.
6. For three data points $(-2,-15)$, $(-1,-8)$ and $(0,-3)$,
   (a) Perform a linear Lagrange interpolation for $f(x)$ at $x = -0.5$.
   (b) A quadratic Lagrange interpolation for $f(x)$ at the same point.

- **Spline Interpolation (07-09-2018):**

1. For three data points $(0,1)$, $(1,1)$ and $(2,5)$,
   (a) Produce a quadratic Newton's divided-difference interpolation polynomial.
   (b) Find the natural cubic spline functions to interpolate the data.
2. With the following data:

x	1	2	3	4	5
y	3	1	2	3	2

   Find both the piecewise linear interpolation function and the natural cubic spline.
3. With the following data:

x	0	1/2	1	2	3
y	0	1/4	1	-1	-1

   Find both the piecewise linear interpolation function and the natural cubic spline.
4. With the following data:

x	0	1	2	2.5	3	4
y	1.4	0.6	1.0	0.65	0.6	1.0

   Find both the piecewise linear interpolation function and the natural cubic spline.
5. Verify that $s(x) = -5+8x-6x^2+2x^3$ for $1 \le x \le 2$ and $s(x) = 27-40x+18x^2-2x^3$ for $2 \le x \le 3$ define a natural cubic spline $s(x)$ on the interval $[1,3]$.
6. Verify that $s(x) = 2x^3$ for $0 \le x \le 1$, $s(x) = x^3 + 3x^2 - 3x + 1$ for $1 \le x \le 2$ and $s(x) = 9x^2 - 15x + 9$ for $2 \le x \le 3$ define a cubic spline $s(x)$ on $[0,3]$. Is it a natural cubic spline on this interval?

- **Numerical Integration and Differentiation (20-09-2018):**

1. Apply trapezoidal and Simpson's rules to the following three integrals:
$$\int_0^{\pi} e^x \cos(4x)\,\mathrm{d}x = \frac{e^{\pi}-1}{17}\,, \qquad \int_0^1 x^{5/2}\,\mathrm{d}x = \frac{2}{7}$$
$$\int_0^5 \frac{\mathrm{d}x}{1+(x-\pi)^2} = \arctan(5-\pi) + \arctan(\pi)$$
   Obtain $T_n(f)$ and $S_n(f)$ for $n = 2, 4, 8, ..., 512$. Also check how the error is reduced as n increases in value.
2. Apply trapezoidal and Simpson's rules with $n = 4, 8, ..., 512$ to find approximate values of the area under the curve for the following $f(x)$:
$$f(x) = \exp(-x^2),\ 0 \le x \le 10\,, \qquad f(x) = \arctan(1+x^2),\ 0 \le x \le 2$$
3. Given $f(x) = e^x\sqrt{x}$, find $S_n(f)$ for $n = 2$ and $n = 4$ over the interval $0 \le x \le 1$.

4. Given $f(x) = \sqrt{x}$,
   (a) First calculate the exact integral of $f(x)$ over the interval $0 \leq x \leq 1$.
   (b) Now estimate $T_n(f)$ over the same interval for $n = 2$ and find the percentage error.
5. Evaluate the following by Gaussian quadrature using the approximation with two nodes.

$$\int_0^5 (3x^2 + 2)\,\mathrm{d}x$$

   Also estimate the percentage error with respect to the exact integral.
6. Find the numerical derivative $D_h f(x)$ at $x = 1$ for the following $f(x)$:

$$f(x) = \arctan(x^2 - x + 1)\,, \qquad f(x) = \arctan(100x^2 - 199x + 100)$$

   Use $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$. Apply both the forward difference and the central-difference formulae, comparing the accuracy in both the cases.

- **Gaussian Elimination (09-10-2018):**

1. Solve the following systems by Gaussian elimination. Use an augmented matrix.

$$\begin{aligned} 2x_1 + x_2 + -x_3 &= 6 \\ 4x_1 - x_3 &= 6 \\ -8x_1 + 2x_2 + 2x_3 &= -8 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + 5x_2 + 4x_3 &= -3 \\ 2x_1 + 4x_2 + 29x_3 &= 15 \end{aligned}$$

2. Solve the following system by using an augmented matrix:

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= 5 \\ x_1 + 4x_2 + x_3 &= 12 \\ 2x_1 - x_2 + 4x_3 &= 12 \end{aligned}$$

- **Matrix Inversion (09-10-2018):**

Find the inverse of the following matrices using an augmented matrix.

1.
$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

2.
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

3. Find the inverse of

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

   by performing Gaussian elimination on an augmented matrix.

- **Nonlinear Systems (09-10-2018):**

  Solve the following nonlinear systems by Newton's method.

  1. (a) $x^2 + 4y^2 = 4\,, \qquad y = x^2 - 0.4x - 1.96$
     (b) $x^2 + y^2 = 1\,, \qquad 2y = 2x^3 + x + 1$
     (c) $x + y - 2xy = 0\,, \qquad x^2 + y^2 - 2x + 2y + 1 = 0$
  2. $x^2 + xy^3 = 9\,, \qquad 3x^2y - y^3 = 4$
     Use initial guess values $(x_0, y_0) = (1.2, 2.5), (-2, 2.5), (-1.2, -2.5), (2, -2.5)$.
  3. Consider the following system of nonlinear equations:

     $$x^2 + y^2 - 4 = 0\,, \qquad x^2 - y^2 - 1 = 0$$

     (a) Plot both the equations together on the x–y plane.
     (b) Obtain all the roots (where the equations intersect) by the Newton method.

- **Euler's Method and Taylor's Method of the Second Order (13-11-2018):**

  Compute the following initial-value problems by Euler's method and Taylor's method of the second order. For the first four problems, estimate the ten initial integrations by a hand calculator and then estimate the rest on a computer.

  1. $Y'(x) = [\cos(Y(x))]^2$, $0 \leq x \leq 10$, $Y(0) = 0$, $h = 0.2$
  2. $Y'(x) = (1 + x^2)^{-1} - 2Y^2$, $0 \leq x \leq 10$, $Y(0) = 0$, $h = 0.2$
  3. $Y'(x) = -Y^2$, $1 \leq x \leq 10$, $Y(1) = 1$, $h = 0.2$
  4. $Y'(x) = 0.25Y(1 - 0.05Y)$, $0 \leq x \leq 20$, $Y(0) = 1$, $h = 0.2$
  5. $Y'(x) = 1 - 2Y$, $Y(0) = 2$, $h = 0.25$. Find $Y(1)$.
  6. $Y'(x) = Y(\sin(x))/x$, $Y(0) = 2$, $h = 0.25$. Find $Y(1)$.

- **Backward Euler Method, Trapezoidal Method and Heun's Method (13-11-2018):**

  Apply the Euler method, the backward Euler method, the trapezoidal method and Heun's method on the following initial-value problems. Compare the respective accuracies. Use a hand calculator for a few early integrations, and then compute the rest on a computer.

  1. $Y'(x) = \lambda Y(x) + (1 - \lambda)\cos(x) - (1 + \lambda)\sin(x)$, $Y(0) = 1$, whose analytical solution is $Y(x) = \sin(x) + \cos(x)$. Use $\lambda = \pm 1$ and $h = 0.5, 0.1$ for $0 \leq x \leq 10$.
  2. $Y'(x) = \lambda Y(x) + (1 + x^2)^{-1} - \lambda \arctan(x)$, $Y(0) = 0$, whose analytical solution is $Y(x) = \arctan(x)$. Use $\lambda = -1, -10$ and $h = 0.5, 0.1$. Apply your judgement about the range of the integration.

- **Supplementary Problems on Differential Equations (13-11-2018):**

  1. Convert the second-order equation $Y'' + 4Y' + 13Y = 40\cos(x)$, $Y(0) = 3$, $Y'(0) = 4$, to a system of first-order equations.
  2. Obtain the integral solution of the differential equation $Y'(x) = -Y(x) + \sin(x) + \cos(x)$ for the initial value $Y(0) = 1$.
  3. Consider the initial-value problem $Y'(x) = Y(1 + e^{2x})$, $Y(0) = 1$.
     (a) Integrate by Euler's method up to $n = 3$, with a step-size of $h = 0.1$.
     (b) Integrate by the second-order Taylor method up to $n = 2$, with $h = 0.1$.