Assignments

- **Assignment 1: The Binary Search and Information Entropy (10-09-2018)**

1. Average information content is given by the formula, $\langle I \rangle = -k \sum_i P_i \log_2 P_i$, in which k is a constant and P_i is the probability of an event.
   (a) For a two-outcome problem (e.g. a coin toss), show that $\langle I \rangle$ peaks at $P = 1/2$.
   (b) Apply a very small perturbation as $P = (1/2) + \epsilon$, in which $\epsilon \ll 1/2$. Show that in this perturbative approach $\langle I \rangle \simeq a - b\epsilon^2$, with $a = k$ and $b = 4k/\ln 2$. Hint: $\ln(1 + x) \simeq x$, when $x \ll 1$.
   (c) Plot $\langle I \rangle$ versus P for both the actual function and the approximate function together and then compare the graphs for closeness on the line $\langle I \rangle = 0$. For plotting choose $k = 1$.

- **Assignment 2: An Astrophysical Inflow (10-09-2018)**

1. In the problem of spherically symmetric astrophysical accretion, interstellar fluid matter (a very thin gas) travels a great distance (almost from infinity) along radial lines and falls on to a massive star (or a neutron star or even a black hole) located at the origin of coordinates. The star can be treated as a point-like particle, and the rate of the fluid flow (matter flowing in unit time) on to it is given as

$$\dot{m} = \pi G^2 M^2 \frac{\rho_\infty}{c_s^3(\infty)} \left(\frac{2}{5 - 3\gamma} \right)^{(5-3\gamma)/2(\gamma-1)}$$

in which G is Newton's universal gravitational constant, M is the mass of the central astrophysical object, ρ_∞ is the constant density of the gas at infinity, $c_s(\infty)$ is the speed of sound at infinity, and γ is a dimensionless number called the polytropic exponent ($1 \leq \gamma \leq 5/3$). The velocity of the fluid flow v, as a function of the radial distance from the centre r, is given by the equation

$$f(v, r) \equiv \frac{v^2}{2} + n \left(\frac{\dot{\mu}}{vr^2} \right)^{1/n} - \frac{GM}{r} - nc_s^2(\infty) = 0 \tag{1}$$

with $\dot{\mu} = (\dot{m}/4\pi\rho_\infty){c_s}^{2n}(\infty)$ and $n = 1/(\gamma - 1)$. Solve Eq.(1) by the bisection method to find $v(r)$, using the values $M = 2 \times 10^{30}\,\mathrm{kg}$, $c_s(\infty) = 10\ \mathrm{km\ s^{-1}}$, $\rho_\infty = 10^{-21}\ \mathrm{kg\ m^{-3}}$ and $n = 2.5$. These values are typical of accretion of the interstellar medium on to a star. Each value of r in Eq.(1) will give a set of two real and physical roots of v. The plot of $v(r)$ is shown in Fig. 1, in which v is scaled as the Mach number, $v(r)/c_s(r)$. Obtain a similar plot. Values of $c_s(r)$ can be found from $c_s(r) = c_s(\infty)(\rho/\rho_\infty)^{(\gamma-1)/2}$ with $\rho \equiv \rho(r)$ being another function of r given by $\rho(r) = \dot{m}/(4\pi vr^2)$. Also scale the radial distance by $7 \times 10^8\,\mathrm{m}$, and then take its base-10 logarithm in the horizontal axis of the graph. For every value of r, run a bisection code going from $v = 0$ to $v = 10^5\,\mathrm{ms^{-1}}$, with a step-size of 0.0010. Converge on a root with a relative error tolerance of $|\epsilon| = 10^{-10}$.

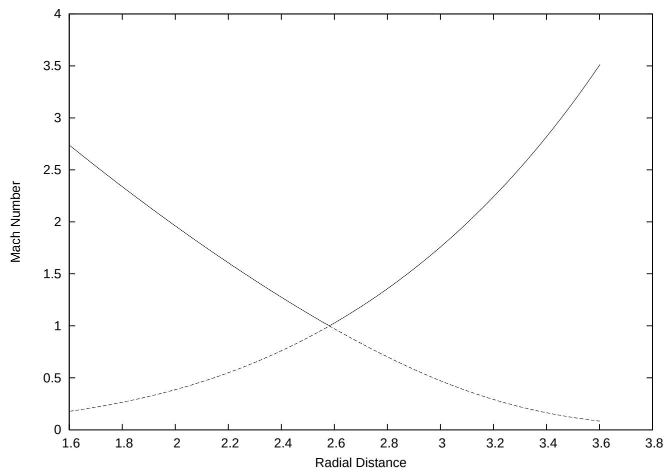

Figure 1: A plot of the Mach number, v/c_s, against the radial distance r (given on a base-10 logarithmic scale).

- **Assignment 3: A Nuclear Outflow (01-10-2018)**

1. High-energy impacts and collisions among elementary particles can result in an outflow of nuclear fluid. The rescaled equations of the steady outflow are

$$xyR^2 = 1, \tag{2}$$

$$y^2 + 3x^2 - 4x = B, \tag{3}$$

in which x is the density of the fluid, y is its velocity, R is the radial distance, and B is a constant.

(a) Using Eq. (2), eliminate x in Eq. (3) to get a quartic equation in $y \equiv y(R)$. Also find $\mathrm{d}y/\mathrm{d}R$ to check for turning and singular points of $y(R)$.

(b) Analytically solve the quartic equation $y \equiv y(R)$. For $B = 2$, plot R along the horizontal axis and y along the vertical axis of a graph.

(c) Further, the velocity of an acoustic wave in the nuclear matter is $u^2 = x(3x - 2)$. On the same graph now plot R versus u. Your complete graph should look like Fig. 2 below.

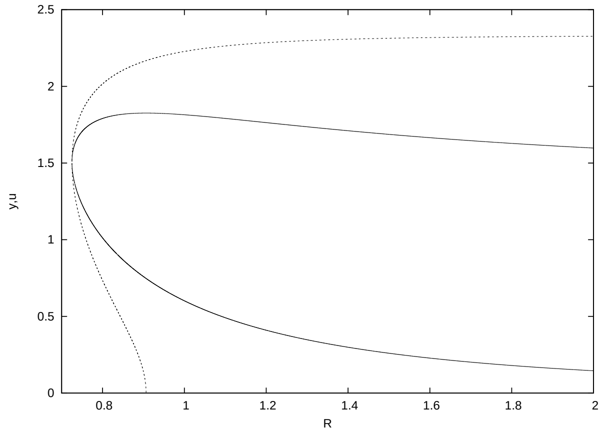

Figure 2: $y(R)$ is the inner continuous curve and $u(R)$ is the outer dotted curve.

- **Assignment 4: The Hydraulic Jump (01-10-2018)**

1. In a hydraulic jump, the height of a flowing liquid increases abruptly, without any pumping action. In a steady one-dimensional liquid flow, the rescaled equation of the flow is $4H - H^4 = 3\,(X - D)$, in which H is the flow height, X is the distance, and D is a constant.

(a) Restrict your study to the range $X \geq 0$, and first analyse all the implications of $\mathrm{d}H/\mathrm{d}X$.

(b) Analytically solve the quartic equation $H \equiv H(X)$. For the condition $X = H = 0$, plot X along the horizontal axis and H along the vertical axis of a graph. On the same graph, repeat the plotting exercise for $H = 1$ when $X = 2$. Your complete graph should look like Fig. 3 below.

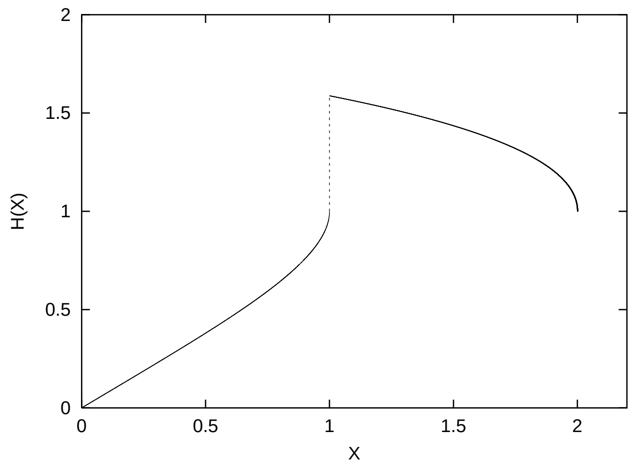

Figure 3: A plot of two solutions of H against X.

- **Assignment 5: The Liénard System (29-10-2018)**

1. The general mathematical form of a Liénard system is that of a nonlinear oscillator equation,

$$\ddot{\phi} + \epsilon \mathcal{H}(\phi, \dot{\phi})\dot{\phi} + \mathcal{V}'(\phi) = 0, \tag{4}$$

in which, $\mathcal{H}$ is a nonlinear damping coefficient and $\mathcal{V}$ is the "potential" of the system (with the prime on it indicating its derivative with respect to ϕ). Choose $\mathcal{H}(\phi, \dot{\phi}) = \mathcal{A}\phi + \mathcal{B}\dot{\phi}$ and $\mathcal{V}(\phi) = \mathcal{C}(\phi^2/2) + \epsilon\mathcal{D}(\phi^3/3)$, with the constant coefficients, $\mathcal{A} = \mathcal{B} = 0.03$, $\mathcal{C} = 1$ and $\mathcal{D} = -1$. Also choose the parameter $\epsilon = 1$. It is a "switch" parameter. When $\epsilon = 0$ nonlinearity disappears and when $\epsilon = 1$, nonlinearity is active. We decompose the second-order differential equation into a coupled first-order system. To that end, on introducing a new variable, ψ, equation (4) can be recast as

$$\begin{aligned} \dot{\phi} &= \psi \\ \dot{\psi} &= -\epsilon\left(\mathcal{A}\phi + \mathcal{B}\psi\right)\psi - \left(\mathcal{C}\phi + \epsilon\mathcal{D}\phi^2\right). \end{aligned} \tag{5}$$

Apply the fourth-order Runge-Kutta method on this system. Solve separately for the two initial conditions $(\phi, \psi) = (0.95, 0), (1.05, 0)$. Plot ϕ versus ψ. It should look like the figure below.

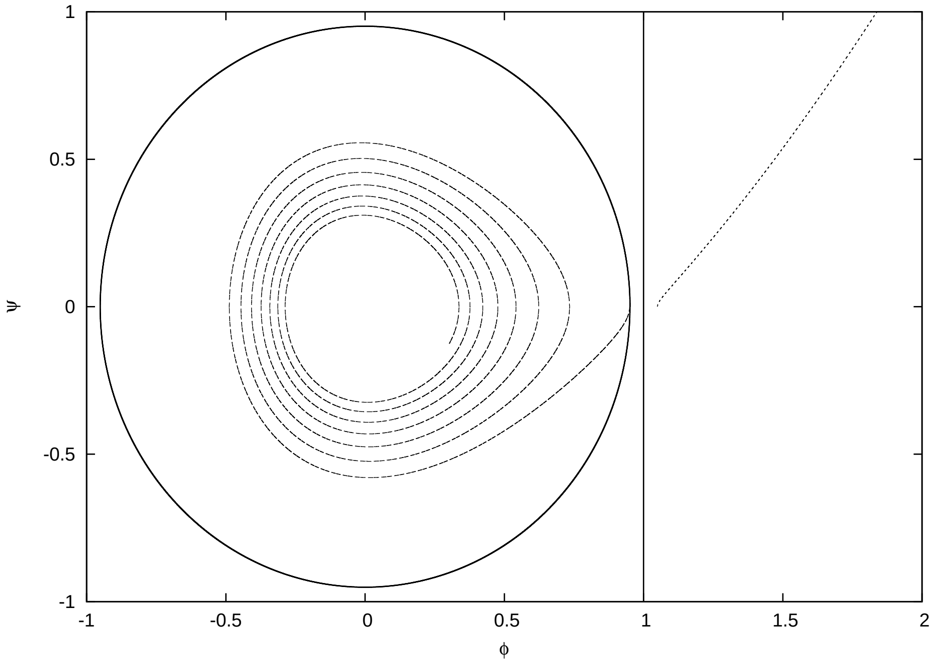

Figure 4: The closed ellipse is due to $\epsilon = 0$. The other curves are for $\epsilon = 1$. The initial condition $(0.95, 0)$ gives the spiral and the initial condition $(1.05, 0)$ gives the open curve.