Let an orbit cross the horizontal axis at some $\rho_1 < K$. Since the path is symmetrical about the ρ axis, consider only half of it as ρ goes from zero to ρ_1. It covers a distance $L/2$. Then, as in Exercise 2.5.6,

$$L = 2\int_0^{L/2} dx = 2\int_0^{\rho_1} \frac{d\rho}{\rho} = \sqrt{2}\int_0^{\rho_1} \frac{d\rho}{\sqrt{C - U(\rho)}}. \tag{5.27}$$

We assume here that ρ' is not zero. Otherwise, ρ is a constant (namely zero, because of the boundary conditions), which is the trivial solution. Because of relation 5.24, one can evaluate the constant C as $U(\rho_1)$ since the derivative of ρ is zero at that point. Therefore,

$$L = \sqrt{2}\int_0^{\rho_1} \frac{d\rho}{\sqrt{U(\rho_1) - U(\rho)}}. \tag{5.28}$$

Figure 5.1 shows that $U(\rho)$ increases with ρ for ρ less than K. Hence, the term inside the square root in 5.28 is positive. Also the sign of this root is positive since as ρ goes from zero to ρ_1, the derivative $\rho'(x)$ remains nonnegative. It is also apparent from the figure that ρ_1 increases with C. That L also increases with C is less obvious.

Relation 5.28 shows that if a nontrivial steady state orbit is to exist on an interval of length L, then L must not be less than a quantity which is roughly $\pi\sqrt{\nu/r}$, which is the same qualitative conclusion reached before in the simpler model 5.13. To see this note that if ρ is small then $U(\rho)$ is roughly $r\rho^2/2\nu$, and so 5.28 is readily integrated to be $\pi\sqrt{\nu/r}$ (Exercise 5.8.4). The same conclusion can be ascertained from 5.23 since this represents an equation for harmonic motion with period $2\pi\sqrt{\nu/r}$ when ρ is small. Therefore, a half period has length $\pi\sqrt{\nu/r}$.

Figure 5.1 demonstrates that ρ increases to a maximum value at the point $x = L/2$ and then decreases again to zero at $x = L$.

5.3 Pollution in Rivers

An organic pollutant, such as human and animal fecal wastes, is thoroughly mixed in the water of a river which is moving downstream at a constant velocity c. The concentration ρ of the pollutant in the river is homogeneous in all directions except that of the downstream flow, which we take to be from left to right along the x axis. The river is thereby modeled by an advective flow in one dimension. Diffusive effects due to river turbulence and irregularities in its contours as it meanders downstream are all ignored. However, the pollutant is allowed to decay in the water due to bacterial action, which gradually decomposes it. This represents a sink term.

Let k be the rate at which the pollutant density is degraded. We assume it to be proportional to the density itself:

$$k(x, t) = -\mu\rho(x, t),$$

where μ is a proportionality constant that measures the efficiency of bacterial action. Since an advective model is appropriate, we adopt 5.8 in the special instance of $u = c$, a constant. The equation is then

$$\frac{\partial \rho}{\partial t} = -c\frac{\partial \rho}{\partial x} - \mu\rho. \tag{5.29}$$

This is a linear first-order partial differential equation, which can be solved by a simple device. First of all, let us reduce it to a form in which the sink term does not appear explicitly. This can be done by letting

$$v(x, t) = \rho(x, t)e^{\mu x/c}.$$

Substituting into 5.29 quickly gives

$$\frac{\partial v}{\partial t} = -c\frac{\partial v}{\partial x}. \tag{5.30}$$

Since 5.30 contains no sources or sinks one expects that however the density v is distributed initially on the river, this same distribution will persist over time except that it should be continuously translated to the right to account for the movement of the river. This leads us to surmise that $v(x, t)$ can be written in terms of a single variable

$$s = x - ct, \tag{5.31}$$

which is a translation of position x by an amount ct. Define a function $H(x, t)$ by

$$H(s, t) = v(s + ct, t) = v(x, t).$$

Then

$$\frac{dH}{dt} = c\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0$$

by virtue of 5.30. Therefore, H does not depend explicitly on t:

$$H(s, t) = f(s),$$

for some (as yet unknown) function f. It follows that

$$v(x, 0) = f(x). \tag{5.32}$$

This shows that f is in fact the initial distribution of the density v. Denote this known initial value by $v_0(x)$. Since 5.32 is true for all possible real numbers x including the value s, the solution is given as

$$v(x, t) = f(s) = v_0(x - ct). \tag{5.33}$$

This is called a *traveling wave* solution to 5.30 because the initial density distribution is propagated downstream as a wave front. Figure 5.2 gives two pictorial representations of this. Traveling wave solutions are to be encountered again in this chapter.

Since v is constant for any fixed value of $x - ct$, it has the same value for each x, t that lie on the straight line $x - ct = s$, a different value for each choice of x. There is therefore a family of straight lines, called *characteristics*, along each of which v is constant. Figure 5.3 shows the characteristics as lines having slope c in the x, t plane. They are the level

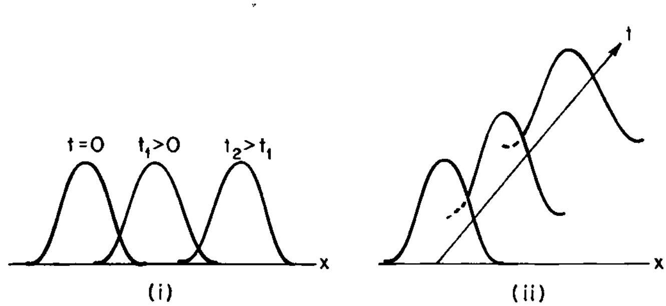

Figure 5.2 Two representations of the traveling wave solution of Equation 5.30. Case (i) shows how the initial density propagates downstream over time while (ii) displays the same thing on the x, t plane.

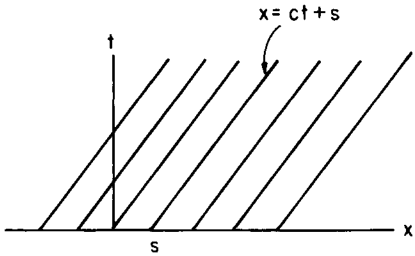

Figure 5.3 Characteristics corresponding to the traveling wave solution of Equation 5.30. The straight lines of slope c in the x, t plane are level curves of the density distribution in Figure 5.2ii.

curves of the surface shown in Figure 5.2ii. Once the characteristics are known, we have an explicit solution to an equation for all x, t provided the value of v is known at $t = 0$. It maintains the same value along the entire characteristic emanating from the x axis at the initial time. This will prove to be a powerful idea in Section 5.4.

Returning now to the original Equation 5.29, assume that $\rho(x,0) = \rho_0(x)$ is given. Then, since $v(x, t) = \rho(x, t)\, e^{\mu x/c}$, it follows that $v_0(x) = \rho_0(x)\, e^{\mu x/c}$, and therefore

$$\rho(x, t) = \rho_0(x - ct)e^{-\mu t},$$

in which the traveling wave is damped as t increases to account for the decay of pollutant over time.

Let us now modify the model to account for the practical situation in which there is a source of pollution at some given point on the river, which we take to be $x = 0$. A typical source is the discharge of a sewage treatment plant having an outfall on the river. Before the plant begins operating at $t = 0$, the river is clean. We want to determine the density of pollution downstream at any future time and location.

Pollutants are added to the river at a rate γ by the dumping. Define U by

$$U(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Then we can write down a boundary condition:

$$\rho(0, t) = \gamma U(t). \tag{5.34}$$

This is the way in which a point source is introduced into our model. Since this source exists at only one location, we do not have a smooth source term to work with. Using a boundary condition gets us around this difficulty. However, in order to find $\rho(x, t)$ a slightly altered traveling wave solution must be sought. This is because we do not have an initial density distribution available. Instead, 5.34 provides a density distribution for all time at a single point. So this time define a variable η by

$$\eta = t - \frac{x}{c},$$

and a function

$$H(x, \eta) = v\left(x, \eta + \frac{x}{c}\right) = v(x, t).$$

Proceeding, then, as we did above shows (Exercise 5.8.6) that the traveling wave solution to 5.30 is

$$v(x, t) = \gamma U\left(t - \frac{x}{c}\right), \tag{5.35}$$

and so

$$\rho(x, t) = \gamma U\left(t - \frac{x}{c}\right)e^{-\mu x/c}. \tag{5.36}$$

Observe that $\rho(0, t) = \gamma U(t)$, as required.

The solution 5.36 shows that if $x > ct$, then $\rho(x, t) = 0$, which is reasonable since it takes a time t for the pollutant to reach a downstream location $x = ct$ from the outfall. The exponential factor indicates that the pollutant density decreases as the substance is decomposed by bacteria during its journey down the river (Figure 5.4).

Now let us consider a more interesting model. The bacterial decomposition of the pollutant requires an uptake of dissolved oxygen in the water. As the pollutant is degraded, oxygen is used up. Let $\delta(x, t)$ be the density of dissolved oxygen in the river. Its maximum value, which depends on temperature, is δ_m. We assume it to be a known fixed quantity.

The rate at which the oxygen dissipates is the same as that of pollutant decay and is proportional to the pollutant concentration. This

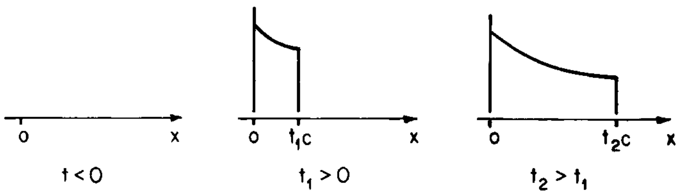

Figure 5.4 Pollutant density obtained from 5.36 and shown for three different times. Since the river is unpolluted upstream from the source at $x = 0$, we define $\rho(x, t)$ to be zero for $x < 0$.

is a sink term given as

$$k_1(x, t) = -\mu\rho(x, t).$$

There is also a source term due to the fact that the river surface, in contact with air above, draws oxygen in from the atmosphere by a process known as re-oxygenation. This happens at a rate proportional to the difference between the saturation level δ_m and the actual δ. Thus,

$$k_2(x, t) = \mu_1[\delta_m - \delta(x, t)].$$

Otherwise, the flow of oxygen is like that of the pollutant, and the advective model 5.8 is again the one to use with, however, the addition of a source and sink term:

$$\frac{\partial\delta}{\partial t} = -c\frac{\partial\delta}{\partial x} - \mu\rho + \mu_1(\delta_m - \delta). \tag{5.37}$$

We assume that $\mu_1 > \mu$ to ensure that self-purification of the river can overcome the degenerative effects of pollution.

The significance of dissolved oxygen is that it is sometimes used by sanitary engineers and marine scientists as a measure of the health of the water body. Oxygen poor waters are unhealthy for marine life and human consumption. It is therefore of considerable interest to correlate oxygen levels with the amount of pollutants that are present. Equation 5.37 expresses this relationship since it depends on ρ. To obtain ρ, one must also know the solution to 5.29, and so we have a pair of coupled partial differential equations to consider simultaneously. There are two boundary conditions, one for ρ (relation 5.34) and another for δ:

$$\delta(0, t) = \delta_m U(t). \tag{5.38}$$

This last condition expresses the fact that the oxygen level is at its highest value right at the source since the upstream water arrives untainted at $x = 0$. It begins to change as the pollutant moves downstream from there.

A plausible assumption is that the process of oxygen decay has reached a steady state after a sufficient lapse of time so that the pair of equations are unaffected by time, but depend only on position along the river. It is left to the reader to determine δ under this assumption. This leads to what is known as an *oxygen-sag curve* for a reason which is apparent after the solution is found (Exercise 5.8.7).

5.4 Highway Traffic

Cars move down a long single-lane highway from left to right. Because they have a finite size the density of cars at a given point is not well defined. However, by viewing the traffic flow from a far distance their size becomes negligible and it becomes plausible to approximate the density as a smooth function ρ.

It appears that an advective model is called for, as in the previous section. However, there is a notable difference in that velocity is not assumed to be constant. The cars should be able to move at different speeds if the model is to give interesting results. Also, there will be no smooth sources or sinks. Cars are neither created or destroyed on our road except through entrances and exits, which are point sources and sinks. These can be modeled by boundary conditions, much as in the water pollution model, and are not considered further.

From observations of actual traffic patterns, it is known that car velocity depends on traffic density in a roughly linear way. Suppose that u_m is the maximum speed any car is capable of on the highway (determined, perhaps, by speed limits and weather conditions) and ρ_m is the maximum possible density at which there is bumper to bumper traffic. Then u is related to ρ by

$$u(\rho) = u_m\left(1 - \frac{\rho}{\rho_m}\right). \tag{5.39}$$

This relation implies that each driver moves at the same velocity at any given density. As density increases, the spacing between cars diminishes,