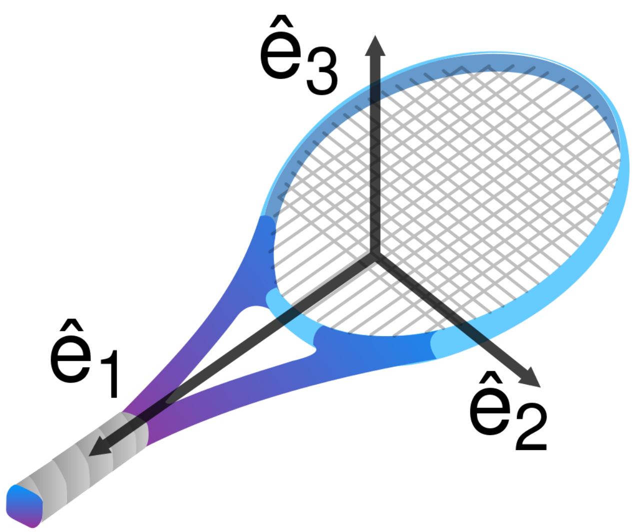
The Tennis Racquet Theorem

The Tennis Racket Theorem states that an object with three unique moments of inertia, rotation about the axis of intermediate moment of inertia is unstable, while rotation about the other two axis is stable.



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https://upload.wikimedia.org/wikipedia/commons/b/ba/Tennis_racket_theorem.ogv [5]

Problem Statement

The Dzhanibekov Effect, also known as the Tennis Racket Theorem, presents a perplexing phenomenon observed in the dynamics of rotating rigid bodies in microgravity conditions. This effect involves the erratic flipping and rotation of a rigid body after an initial asymmetrical perturbation, defying conventional expectations of rotational motion. Understanding the factors that influence the occurrence and behaviour of the Dzhanibekov Effect is vital for spacecraft design, robotics, and other applications in microgravity environments.

In this report, we aim to comprehensively investigate the Dzhanibekov Effect through numerical simulation and analysis. Our primary objective is to shed light on the complex and unpredictable motion of a rigid body subjected to an initial perturbation in microgravity. We will explore the impact of various parameters, such as moment of inertia, angular velocity, and initial conditions, on the occurrence and characteristics of this effect. Additionally, we will validate the accuracy of our numerical model by comparing simulated results with established analytical solutions and experimental data.

By addressing this challenge, we seek to provide a deeper understanding of the Dzhanibekov Effect's underlying mechanisms and its practical implications for microgravity applications. Our findings aim to contribute valuable insights to the fields of aerospace engineering, space science, and robotics, enabling more informed decision-making in designing and operating objects in microgravity environments.

Basically there are three axes where any object can rotate such as I1, I2, I3.The chronology follows that the object should be least stable at the I3 axis since I1 has highest stability but it is not the case here. Here we are also focusing on the problem why the object is least stable while rotating around the I2 axis. Here we have taken in depth our studies where we found out the moment of inertia about all the three axes and proved that I2 has least stability.

Theory

The tennis racket theorem can be qualitatively analysed with the help of Euler's equations. Under torque–free conditions, they take the following form:

$$I_1\dot{\omega}_1 = -(I_3 - I_2)\omega_3\omega_2 \quad [2.1]$$

$$I_2\dot{\omega}_2 = -(I_1 - I_3)\omega_1\omega_3 \quad [2.2]$$

$$I_3\dot{\omega}_3 = -(I_2 - I_1)\omega_2\omega_1 \quad [2.3]$$

Here I_1, I_2, I_3 denote the object's principal moments of inertia, and we assume $I_1 < I_2 < I_3$. The angular velocities around the object's three principal axes are $\omega_1, \omega_2, \omega_3$ and their time derivatives are denoted by $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$.

Stable rotation around the first and third principal axis

Consider the situation when the object is rotating around axis with moment of inertia I_1. To determine the nature of equilibrium, assume small initial angular velocities along the other two axes. As a result, according to equation (1), $\dot{\omega}_1$ is very small. Therefore, the time dependence of ω_1 may be neglected.

Now, differentiating equation (2) and substituting $\dot{\omega}_3$ from equation (3),

$$I_2\ddot{\omega}_2 = -(I_1 - I_3)\omega_1\dot{\omega}_3 \quad [2.4]$$

$$I_3 I_2\ddot{\omega}_2 = (I_1 - I_3)(I_2 - I_1)(\omega_1)^2\omega_2 \quad [2.5]$$

i.e. $\ddot{\omega}_2$ = (negative quantity).ω_2

Because $I_2 - I_1 > 0$ and $I_1 - I_3 < 0$

Note that ω_2 is being opposed and so rotation around this axis is stable for the object.

Similar reasoning gives that rotation around axis with moment of inertia I_3 is also stable.

Unstable rotation around the second principal axis

Now apply the same analysis to axis with moment of inertia I_2. This time $\dot{\omega}_2$ is very small. Therefore, the time dependence of ω_2 may be neglected.

Now, differentiating equation (1) and substituting $\dot{\omega}_3$ from equation (3),

$$I_1 I_3 \ddot{\omega}_1 = (I_3 - I_2)(I_2 - I_1)(\omega_2)^2 \omega_1 \quad [2.6]$$

i.e. $\ddot{\omega}_{21}$ = (positive quantity).ω_1

Note that ω_1 is *not* opposed (and therefore will grow) and so rotation around the second axis is *unstable*. Therefore, even a small disturbance along other axes causes the object to 'flip'. [5]

Assumptions

Rigid Body Assumption: The object is assumed to be a rigid body, meaning it does not deform during motion. This assumption simplifies the analysis and focuses on the object's rotational behaviour.

Idealised Geometry: The racquet's geometry is often simplified to basic shapes, such as a rectangular frame with a handle. Realistic details, such as the shape of the strings or frame design, may be ignored for simplicity.

Uniform Mass Distribution: The mass distribution within the object is assumed to be uniform. In reality, the object may have varying mass distributions, but this assumption simplifies calculations.

Neglecting Air Resistance: The analysis often neglects the effects of air resistance on the object's motion. While air resistance can affect the object's behaviour, it is usually considered a minor factor in simplified analyses.

Idealised Initial Conditions: Initial conditions, such as the object's initial angular velocities, are often idealised for simplicity. Real-world situations may involve more complex initial conditions.

Neglecting External Forces: Some analyses assume there are no external forces or torques acting on the racquet during its motion. This simplification allows for a focus on the intrinsic properties of the racquet.

Linear Behavior Near Equilibrium: The analysis may assume that the object's motion is linearized near equilibrium positions. This simplifies the study of stability around stable and unstable axes.

Constant Moments of Inertia: In some cases, constant moments of inertia may be assumed over the course of motion. In reality, moments of inertia can change due to racquet orientation changes, but this assumption simplifies calculations.

No Energy Losses: Energy losses due to friction or other dissipative forces may be neglected in simplified analyses.

Derivation

F = ma [4.1]

However in Rotational motion,

$$\tau = I\ddot{\theta} \quad [4.2]$$

Those familiar with matrices might wonder, what would it mean to diagonalise the moment of inertia matrix. If you diagonalise the moment of inertia matrix, you will get a matrix where the new coordinate system represents the principal axis of the object and the diagonal terms are the moment of inertia is every one of those axes. There are two ways to find the diagonalised moment of the inertia matrix. The easy way is to observe the geometry of the object, as sometimes (in most common shapes) the principal axes are obvious. However, to be rigorous in the derivation, you can use a standard matrix diagonalisation from elementary linear algebra. The procedure will be as follows

Find the eigenvalues of the inertia matrix.

$$\lambda := det(\lambda I_{identity} - I) = 0 \quad [4.3]$$

Find the eigenvector matrix P of the inertia matrix.

P := $[v_1, v_2, v_3]$

Use P^{-1}IP to get a diagonalised inertia matrix.

$$I_{diagonal} = P^{-1}IP \quad [4.4]$$

The values in the diagonal matrix are going to allow us to understand the Tennis Racket Theorem. For those of you who are sharp, you will notice that the axis of intermediate moment of inertia is the eigenvector of the intermediate eigenvalue.

Euler Equations

Newton's second law for rotational motion gets very complicated to work with very fast. So Euler used the diagonalisation as a way to simplify and separate the three equations in Newton's second law. The Euler equations are as follows

$$I\omega \; + \omega \; \times \; (\omega I) \; = \; \tau \qquad [4.5]$$

Breaking this component wise, in the principle axis

$$I_1\dot{\omega}_1 \; + \; (I_3 - I_2)\,\omega_3 \; = \; \tau_1 \qquad [4.6]$$
$$I_2\dot{\omega}_2 \; + (I_1 - I_3)\,\omega_1\omega_3 \; = \; \tau_2 \qquad [4.7]$$
$$I_3\dot{\omega}_3 \; + \; (I_2 - I_1)\,\omega_2\omega_1 \; = \; \tau_3 \qquad [4.8]$$

I will not go in depth into the derivation, but I will outline the chain of thought. Consider this alternate version of Newton's law

$$\tau = (\frac{dL}{dt})_{inertial}$$

$$(\frac{dL}{dt})_{inertial} = (\frac{dL}{dt})_{rot} + \omega \times L \; = \tau$$

Expanding the second equation will give you Euler's equation of motion for rigid bodies.

Understanding the Theorem

Consider an inertia matrix (diagonalised) with moment of inertia I1 and I2 and I3 such that I1 < I2 < I3.

Now, let the angular velocity vector be $\omega = \omega_x\vec{i} + \epsilon_1\vec{j} + \epsilon_2\vec{k}$; where the epsilons are small perturbations in the other two principal axes.

Now plugging this into Euler equations, we obtain

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\epsilon_1\epsilon_2$$

$\dot{\omega}_1 = -k \quad ; \; k << 1$

Now, differentiating the second Euler Equation

$$I_2\ddot{\omega}_2 = (I_3 - I_1)(\dot{\omega}_1\omega_3 + \omega_1\dot{\omega}_3)$$

Substituting in our expression for omega 1 and omega 3, and since multiplying the epsilons makes it small enough to ignore,

$$I_2\ddot{\omega}_2 = (I_3 - I_1)\left(\frac{I_1 - I_2}{I_3}\omega_1^2\omega_2\right)$$

This gives us a differential equation for omega 2 of the form

$$\ddot{\omega}_2 = -\lambda\omega_2, \quad \lambda \in \mathbb{R}^+$$

Whose solution is elementary

$$\omega_2 = c_1\sin(\sqrt{\lambda}t) + c_2\cos(\sqrt{\lambda}t)$$

Therefore, we know that perturbation of rotation in the omega 1 axis is stable and undergoes periodic motion, or in the terminology of rigid body motion, it undergoes precession.

The perturbation in omega 3 follows a similar argument as above, and I shall leave it to you as an exercise to work it through.

For the intermediate axis, we have $\omega = \epsilon_1\vec{i} + \omega_2\vec{j} + \epsilon_2\vec{k}$

Plugging into the Euler equations

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\epsilon_1\epsilon_2$$

Differentiating the third Euler equation

$$I_3\ddot{\omega}_3 = (I_1 - I_2)(\dot{\omega}_1\omega_2 + \omega_1\dot{\omega}_2)$$

Substituting our derived expressions

$$I_3\ddot{\omega}_3 = (I_1 - I_2)\left(\frac{I_2 - I_3}{I_1}\omega_2^2\omega_3\right)$$

Now, when we rearrange, we derive the following differential equation

$$\ddot{\omega}_3 = \lambda\omega_3, \quad \lambda \in \mathbb{R}^+$$

Notice that the coefficient is now positive, which therefore results in exponential solutions

$$\omega_3 = c_1 e^{\sqrt{\lambda}t} + c_2 e^{\sqrt{\lambda}t}$$

This solution shows that omega 3 is unstable under perturbation of omega 2 along the intermediate axis.

When there is a perturbation in the rotation along the major or minor axis, the system acts to decrease the perturbation, along with the perturbation in the two other axes forming a sine and cosine pair, negating each other. This causes precession. However, when there is a perturbation in the rotation along the intermediate axis, the system acts to increase the perturbation. In other words, the perturbation in the other two axes reinforce each other, and increase exponentially, creating unstable rotation.

Boundary Conditions

Boundary conditions define the constraints or requirements at the initial and final states of the racquet's motion:

- Initial Angular Velocities: You would specify the initial angular velocities (ω_1, ω_2, ω_3) for each of the principal axes of the racquet. These initial conditions determine how the racquet starts its motion.
- Steady-State Conditions: In some analyses, boundary conditions may involve reaching a steady-state or equilibrium condition. For example, you might study how the racquet behaves when it reaches a stable rotation around one of its axes.
- External Torques: If there are external torques acting on the racquet, you would specify the magnitude and direction of these torques as boundary conditions. This can be relevant when studying real-world scenarios where forces are applied to the racquet during its motion.
- Time Limits: You would specify the time interval or duration of the analysis, indicating when the racquet's motion starts (t = t0) and when it ends (t = t_end).
- Position or Orientation Constraints: Depending on the problem, you might impose constraints on the racquet's position or orientation at specific times. For example, you might analyse the racquet's behaviour when it is initially positioned at a certain angle.
- Energy Considerations: Boundary conditions related to energy can be important. You might specify the total energy of the racquet at the beginning of the motion or ensure that certain energy constraints are met.
- Convergence Conditions: When using numerical methods like the Newton-Raphson method, you would specify convergence criteria to determine when the solution has reached a stable state or when the analysis should terminate.
- Constraints on Principal Moments of Inertia: In some cases, you might impose constraints on the values of the principal moments of inertia (I_1, I_2, I_3) to simulate variations in racquet design.

The specific choice of boundary conditions depends on the research question or problem you aim to address.

Parameters

Principal axis 1 is one of these three axes, and it is typically the axis with the largest or smallest moment of inertia, depending on the object's shape.

While analysing the Tennis Racquet Theorem, we have considered a range of parameters and variables that describe the racquet's behaviour. Here are some key parameters and variables which we have included in our analysis:

Moments of Inertia (I_1, I_2, I_3):

These parameters represent the principal moments of inertia, defining how mass is distributed about the racquet's principal axes. They are essential in describing rotational behaviour and are typically determined experimentally or through modelling.

The moments of inertia are diagonal elements of the inertia tensor:

$$I_1 = \int r^2 \mathrm{dm} \quad [6.1]$$

$$I_2 = \int r^2 \mathrm{dm} \quad [6.2]$$

$$I_3 = \int r^2 \mathrm{dm} \quad [6.3]$$

External Torques (τ_1, τ_2, τ_3):

Parameters denoting external forces or torques acting on the racquet, potentially originating from a player's hand or wrist. These torques can influence the racquet's motion.

Mass Distribution Parameters:
Characteristics describing how mass is distributed within the racquet. These parameters influence the moments of inertia and overall racquet dynamics.

Racquet Geometry Parameters:
Factors such as the racquet's length (L), width (W), and shape parameters (e.gcross-sectional profile) can impact its rotational behaviour.

Time (t):
A fundamental variable representing the evolution of the racquet's motion over a given duration.

Angular Velocities ($\omega_1, \omega_2, \omega_3$):

Variables indicating the racquet's rotational rates around its principal axes. They describe how the racquet rotates over time.

The angular velocities may be related to angular displacements over time:

$$\omega_1 = \frac{d\theta_1}{dt} \quad [6.4]$$

$$\omega_2 = \frac{d\theta_2}{dt} \quad [6.5]$$

$$\omega_3 = \frac{d\theta_3}{dt} \quad [6.6]$$

Angular Accelerations ($\alpha_1, \alpha_2, \alpha_3$): Variables describing how angular velocities change over time due to torques and moments of inertia.

Angular accelerations can be calculated using Newton's second law for rotation:

- $\alpha_1 = \frac{\tau_1}{I_1}$ [6.8]
- $\alpha_2 = \frac{\tau_2}{I_2}$ [6.9]
- $\alpha_3 = \frac{\tau_3}{I_3}$ [6.10]

Position ($\theta_1, \theta_2, \theta_3$):
Variables indicating the racquet's orientation angles around its principal axes. They describe the racquet's angular position at any given time.

Energy (E):
The total energy of the racquet, including kinetic and potential energy, is a variable that can be calculated as the racquet's motion evolves.

Total energy can be expressed as the sum of kinetic and potential energy:

- $E=K+UE=K+U$ [6.11]
- $K=\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$ [6.12]

Angular Momentum (L_1, L_2, L_3):

Variables representing the angular momentum about each principal axis, providing insights into the racquet's rotational behaviour.
Angular momentum is calculated as the product of moment of inertia and angular velocity:

- $L_1 = I_1\omega_1$ [6.13]
- $L_2 = I_2\omega_2$ [6.14]
- $L_3 = I_3\omega_3$ [6.15]

External Forces (F_1, F_2, F_3): Variables denoting forces or torques applied externally to the racquet, which may originate from player interactions or other sources.

Impact Forces: Variables describing the forces and torques experienced during ball-racquet interactions, including parameters such as contact time and impact location.

Stress and Strain Variables: If considering racquet deformation or flex, variables related to stress and strain within the racquet material may be pertinent.

Solution Methodology

- The solution methodology for analysing the Tennis Racquet Theorem is a structured and systematic approach that begins by clearly defining the research objectives and problem scope. It involves identifying crucial parameters such as moments of inertia, initial conditions, and external torques that govern the racquet's rotational motion.
- These parameters are then used to formulate differential equations based on principles of rotational dynamics and Euler's equations and Runge-Kutta Method.
- Precise boundary conditions are established to define the racquet's initial state, including its angular velocities and any constraints.
- The results are visually represented through plots and charts, facilitating effective communication of the methodology and outcomes.
- Conclusions are drawn based on the analysis, potentially influencing racquet design or performance optimization.
- Throughout the process, an iterative approach may be employed to refine the analysis, models, or research parameters, allowing for a more comprehensive and nuanced exploration of the Tennis Racquet Theorem.

Algorithm

- **I1**, **I2**, and **I3** are the moments of inertia around the principal axes of the racquet.
- **omega1, omega2,** and **omega3** are the initial angular velocities around these axes.
- **t0** is the initial time, **t_end** is the final time, and **step_size** is the time step for the simulation.

Given equations:

$$I_1\dot{\omega}_1 = -(I_3 - I_2)\omega_3\omega_2 \qquad (1)$$

$$I_2\dot{\omega}_2 = -(I_1 - I_3)\omega_1\omega_3 \qquad (2)$$

$$I_3\dot{\omega}_3 = -(I_2 - I_1)\omega_2\omega_1 \qquad (3)$$

A. Using Euler's method:

Our ODE is of the for $\frac{d\omega}{dt} = f(\omega)$.
Our goal is to find $\omega(t)$.

Algorithm:

- While **t** is less than **t_end**, do the following steps:
- Calculate the angular accelerations (**alpha1, alpha2, alpha3**) using Euler's equations based on the current angular velocities (**omega1, omega2, omega3**) and the moments of inertia (**I1, I2, I3**).
- Update the angular velocities using Euler's method:

  *omega1 = omega1 + alpha1 * step_size*

  *omega2 = omega2 + alpha2 * step_size*

  *omega3 = omega3 + alpha3 * step_size*

- Increment **t** by **step_size**.
- Append the current time t and angular velocities (**omega1, omega2, omega3**) to the respective lists.

- After the loop completes, we will have collected the time and angular velocity data and then we plot it.

B. Using Runge-Kutta method:

Our ODE is of the for $\frac{d\omega}{dt} = f(\omega)$.

Our goal is to find $\omega(t)$.

Algorithm:

- First we Create lists (**t_values, omega1_values, omega2_values, omega3_values**) to store time and angular velocity values during the simulation.
- Define the *runge_kutta function*, which performs one step of the *Runge-Kutta 4th order* method:
- Calculate the values of **k1_1, k1_2, and k1_3** using the equations representing the angular accelerations for each axis.
- Calculate the values of **k2_1, k2_2,** and **k2_3** based on the previous values and midpoints.

  *k2_alpha1, k2_alpha2, k2_alpha3 = calculate_alpha(omega 1 + 0.5 * step_size * k1_alpha1,omega2 + 0.5 * step_size * k1_alpha2, omega 3 + 0.5 * step_size * k1_alpha3)*

- Calculate the values of **k3_1, k3_2,** and **k3_3** similarly.
- Calculate the values of **k4_1, k4_2,** and **k4_3** for the final evaluation.
- Update **omega1, omega2,** and **omega3** using weighted averages of these k values.

  *new_omega1 = omega1 + (step_size / 6) * (k1_alpha1 + 2 * k2_alpha1 + 2 * k3_alpha1 + k4_alpha1)*

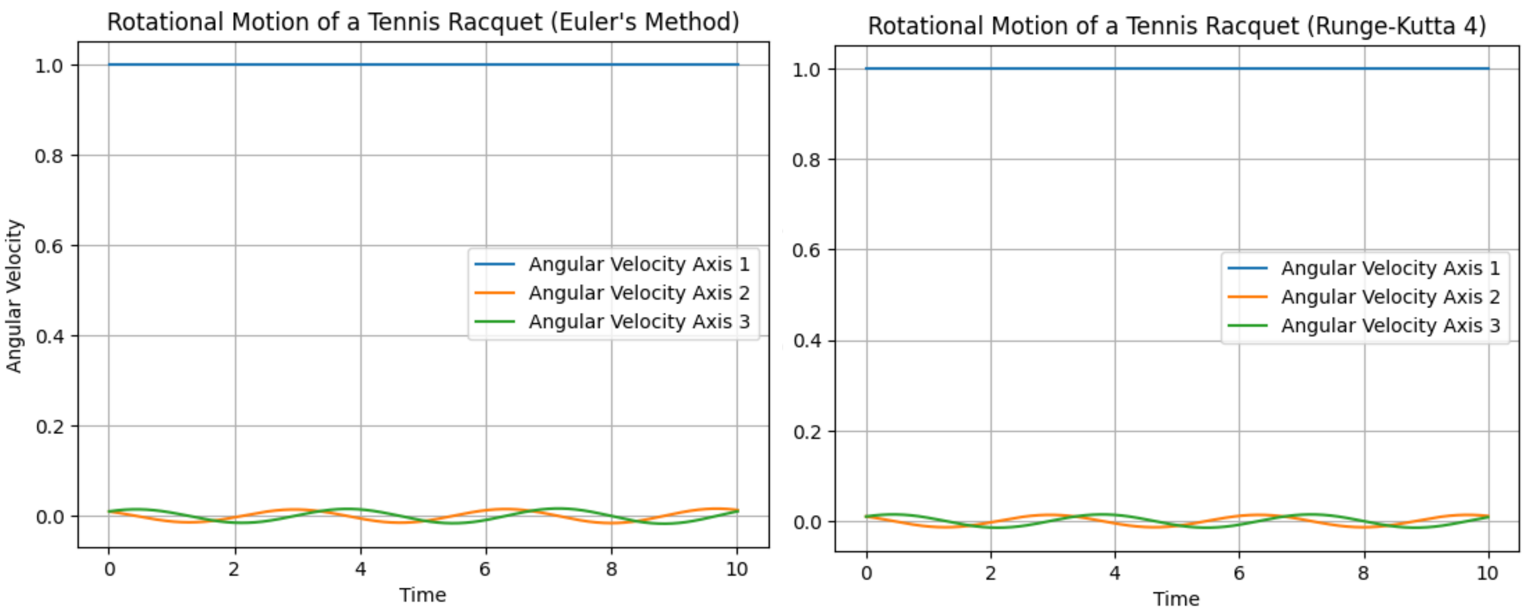
  And same for omega2 and omega3

- Advance the time t by the time step h.
- Perform the simulation in a loop:
- Append the current time t and angular velocities **omega1, omega2,** and **omega3** to the respective lists.
- Call the *runge_kutta* function with the specified time step **step_size** to update the angular velocities.
- Then we plot all the values omegas we get from the above algorithm.

Attached below is the colab notebook containing all the codes for the graphs obtained by both the two methods.

https://colab.research.google.com/drive/1LJwaEDXrz_yhEWBDqGAATvXTRJr8ua0N?usp=sharinghttps://colab.research.google.com/drive/1LJwaEDXrz_yhEWBDqGAATvXTRJr8ua0N?usp=sharing

Results and Discussion



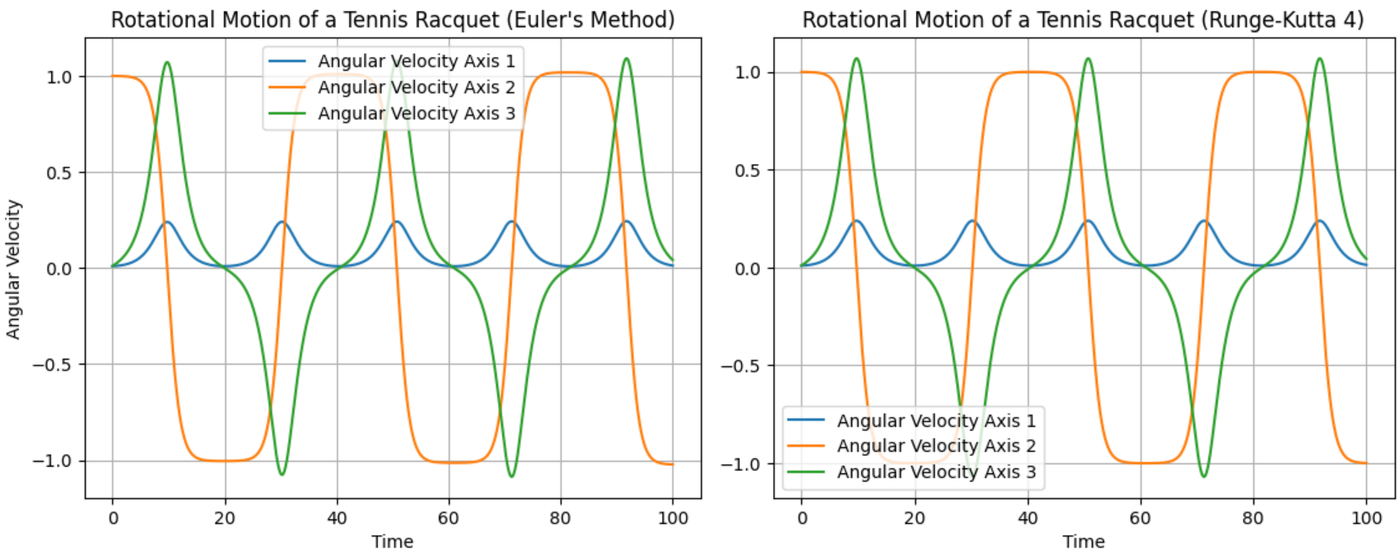
Graph 10.1 Angular Velocity Profiles for Rotational Motion about principal axis 1 by (a) euler's method (b) Runge-Kutta 4 method

Observation:

Angular velocity about the 1st axis is constant throughout the motion, also it keeps on oscillating about the other two axes, remaining small in magnitude.

Conclusion:

When an object having different moments of inertia along all three principal axes is rotated about 1st principal axis, any small perturbations along other two axes do not grow with time and the object keeps on oscillating about them (if energy loss is not considered, otherwise oscillations would be damped with time). Thus rotation about this axis is stable.



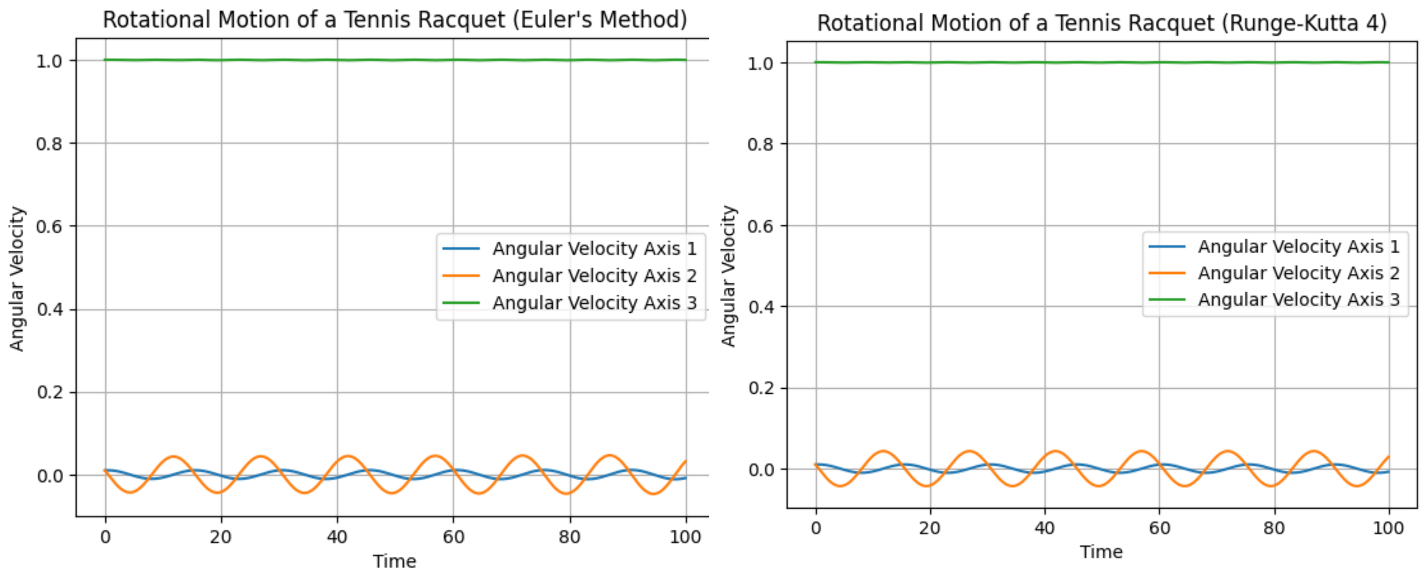
Graph 10.2 Angular Velocity Profiles for Rotational Motion about principal axis 2 by (a) euler's method (b) Runge-Kutta 4 method

Observation:

This graph shows how angular velocity about the 2nd axis changes its sign after a regular interval of time depending on the parameters.
Also, angular velocity about the 1st axis increases and decreases in a pattern and the direction of rotation does not change.
Whereas, angular velocity about the 3rd axis increases and decreases in a pattern; but the direction also changes.

Conclusion:

Perturbations in the other two directions increase which result in flipping the direction of rotation about the 2nd axis. Thus the object flips again and again if there is no energy loss from the system. Rotation about this axis is unstable.



Graph 10.3 Angular Velocity Profiles for Rotational Motion about principal axis 3 by (a) euler's method (b)Runge-Kutta 4 method

Observations and conclusions for rotation about this axis are similar to what we came across in rotation about the 1st axis.

References:

*[1]*A tumbling T-handle in space: the Dzhanibekov effect | Rotations. (n.d.). https://rotations.berkeley.edu/a-tumbling-t-handle-in-space/

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