Singular Value Decomposition :→

We know that every symmetric matrix A can be factored as $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix consisting of eigenvalues of A. If A is not symmetric, such a factorization is not possible. But again we know to factor a square matrix A as $A = PDP^{-1}$, here P is simply an invertible matrix.

Not every matrix is diagonalizable, but with the help of SVD (Singular value decomposition), we will show that every matrix has a factorization of the form $A = PDQ^T$, where P and Q are orthogonal matrices and D is a diagonal matrix.

Singular Values of a Matrix :→

If A is an $m\times n$ matrix, the singular values of A are the square ~~values of A~~ roots of the eigenvalues of A^TA and are denoted by $\sigma_1, \sigma_2, --, \sigma_n$. It is conventional to arrange the singular values so that $\sigma_1 \geq \sigma_2 \geq -- \geq \sigma_n$.

Eg: Find the singular values of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

C.E.: $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = 1$

thus $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1} = 1$ are the singular values of A.

Note → Since $[A]_{m\times n}$ and A^TA → symmetric and orthogonally diagonalizable.

$$\|Av_i\|^2 = (Av_i)^T A v_i = v_i^T A^T A v_i$$
$$= v_i^T \lambda_i v_i$$
$$= \lambda_i \quad (v_i \text{ is a unit vector})$$

So, the singular values of A are the lengths of the vectors $Av_1, Av_2, \dots Av_n$.

SVD :→

The decomposition of A involves an $m\times n$ diagonal matrix Σ of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \leftarrow m-r \text{ rows}$$

$\hookrightarrow$ $n-r$ columns

where D is an $r\times r$ diagonal matrix for some r not exceeding the smaller of m and n. If r equals m or n or both, some or all of the zero matrices do not appear.

Theorem:→ Let A be an $m\times n$ matrix with rank r. Then there exists an $m\times n$ matrix Σ for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, and there exists an $m\times m$ orthogonal matrix U and an $n\times n$ orthogonal matrix V such that

$$\boxed{A = U\Sigma V^T}$$

Note:- i) A factorization of $A = U\Sigma V^T$ is called SVD of A.

ii) The columns of U are called the left singular vectors of A, and the columns of V are called right singular vectors of A.

iii) The matrices U and V are not uniquely determined by A, but Σ must contain the singular values of A.

Example:- ① Find the singular value decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Soln :- It is divided into three steps:

Step ①: Find an orthogonal diagonalization of A^TA.

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

eigenvalues of A^TA are: $\lambda_1 = 360$, $\lambda_2 = 90$, $\lambda_3 = 0$

& its corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

as v_1, v_2 and v_3 are already unit vectors no need to normalize them.

Step ②: Set up V and Σ.

Arrange the eigenvalues of A^TA in decreasing order & find the singular values

$$\sigma_1 = 6\sqrt{10}, \quad \sigma_2 = 3\sqrt{10}, \quad \sigma_3 = 0.$$

The non zero singular values are the diagonal entries of D. The matrix Σ is the same size as A with D in its upper-left corner and 0's elsewhere

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \text{ and } \Sigma = [D \ 0] = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$V = [v_1 \ v_2 \ v_3]$$

Step ③: Construct U:

When A has rank r, the first r columns of U are the normalized vectors obtained from $Av_1, \dots Av_r$

$$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

Now singular value decomposition of A is

$$A = \underset{U}{\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}}$$

(11) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ — Find SVD

Solⁿ Step ①:- $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

eigen values $\lambda_1 = 3$ & $\lambda_2 = 1$ & corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Since eigenvectors are orthogonal, so we will normalize it

$$V = [q_1 \ q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Step 2 :- $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$

so; $D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} D \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

and $V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Step 3 :

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

~~Now singular value decomposition of A is~~

~~A~~ Now U should be a 3×3 matrix, but we are getting only 2 vectors, so we will extend $\{u_1, u_2\}$ to an orthonormal basis for $\mathbb{R}^3$.

say $\{u_1, u_2, e_3\}$ is linearly independent.

Apply the Gram-Schmidt Process:

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

so: $U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

so; $A = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (Ans.)

(11) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find SVD.

Ans.) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} =$

$= U \Sigma V^T$