17. 2-SAT (using Markov Chains)

2-SAT is a CNF over n variables

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_m, C_i = L_{i_1} \vee L_{i_2} \forall i \in [m]$$

We want to assign truth values to each of the variables to satisfy F (such an assignment may or may not exist).

Algorithm for 2-SAT

1. Start with an arbitrary truth assignment
2. Repeat until all clauses are satisfied or at most $2mn^2$ many iterations
   1. Choose an arbitrary clause that is not satisfied
   2. Choose a literal at random and flip the value of the variable corresponding to the literal.
3. If a valid truth assignment was found return that assignment, else return UNSAT.

Correctness & Guarantees of Algorithm

Let's take S to be a satisfying assignment for hte n variables.
A_i : Assignment to the variables after step i.
X_i : Number of variables having the same values in A_i and S

$$X_i = |\{j | A_i(x_j) = S(x_j)\}|$$

If $X_{i'} = n$, then $A_{i'} = S$ and that is a satisfying assignment.

Expected Time to Reach S

We know that

$$\mathbb{P}[X_{i+1} = 1 | X_i = 0] = 1$$

Suppose $1 \leq X_i \leq n - 1$

And let C be a clause that is not satisfied by A_i. If this is the case, then we know at least one of the two variable assignments in C is inconsistent with S.
If both variable assignments were consistent with S, then the clause should've been satisfied, else S wouldn't be a satisfying assignment.

If we randomly pick any one of the two variables and flip them, the probability that X_{i+1} increases is $\frac{1}{2}$.

$$\mathbb{P}[X_{i+1} = j + 1 | X_i = j] \geq \frac{1}{2}$$
$$\mathbb{P}[X_{i+1} = j - 1 | X_i = j] \leq \frac{1}{2}$$

Equivalence with Markov Chain

While the graph induced by $X_0, X_1, \ldots, X_n$ may not be a markov chain, we can study a very similar markov process and analyze the expected running time of this algorithm.

We look at $Y_0, Y_1, \ldots, Y_n$.

$$\mathbb{P}[Y_{i+1} = 1 | Y_i = 0] = 1$$
$$\mathbb{P}[Y_{i+1} = j + 1 | Y_i = j] = \frac{1}{2}$$
$$\mathbb{P}[Y_{i+1} = j - 1 | Y_i = j] = \frac{1}{2}$$

The transition probability matrix would be given as

$$P_{ij} = \begin{cases} \frac{1}{2} & \text{if } j = i + 1 \text{ or } i - 1 \ \forall 1 \leq i \leq n - 1 \\ 1 & \text{if } i = 0 \text{ and } j = i \\ 1 & \text{if } i = j = n \\ 0 & \text{otherwise} \end{cases}$$

Deriving a formula for expected number of steps

We have a random variable Z_i that represents the number of steps to reach n from j. Let $h_j = \mathbb{E}[Z_j]$.

Hence, we know that $h_n = 0$ and $h_0 = h_1 + 1$.

Further $h_j \geq$ expected number of steps to fully agree with S starting from A_0 which agrees with S in j locations.

$$Z_j = \begin{cases} 1 + Z_{j-1} & \text{with probability } \frac{1}{2} \\ 1 + Z_{j+1} & \text{with probability } \frac{1}{2} \end{cases}$$

Hence,

$$\begin{aligned} \mathbb{E}[Z_j] &= \frac{1}{2}(1 + \mathbb{E}[Z_{j-1}]) + \frac{1}{2}(1 + \mathbb{E}[Z_{j+1}]) \\ 2h_j &= h_{j+1} + h_{j-1} + 2 \\ h_{j+1} - h_j &= (h_j - h_{j-1}) - 2 \\ \sum_{j=1}^{k}(h_{j+1} - h_j) &= \sum_{j=1}^{k}((h_j - h_{j-1}) - 2) \end{aligned}$$

This gives us

$$\begin{aligned} h_{k+1} - h_1 &= h_k - h_0 - 2k \\ h_{k+1} &= h_k + (h_1 - h_0) - 2k \\ h_{k+1} &= h_k - 2k - 1 \end{aligned}$$

This can be rearranged to give us

$$h_k - h_{k+1} = 2k + 1$$

Expected Steps from h_0

Using this, we can derive the expected value for all indices,

$$\begin{aligned} h_n &= 0 \\ h_{n-1} &= 2(n-1) + 1 + h_n = 2(n-1) + 1 \\ \ldots \\ \sum_{k=0}^{n-1}(h_k - h_{k+1}) &= \sum_{k=0}^{n-1}(2k+1) \\ h_0 - h_n &= \frac{2n(n-1)}{2} + n = n^2 \\ h_0 &= n^2 \end{aligned}$$

Therefore, we know that the expected number of steps to find a satisfying assignment is $\leq n^2$

Sucess Probability

By Markov's Inequality

$$\mathbb{P}[Z_0 > 2n^2] \leq \frac{\mathbb{E}[Z_0]}{2n^2} = \frac{h_0}{2n^2} = \frac{n^2}{2n^2} = \frac{1}{2}$$

Now, we can boost the success probability by running this algorithm multiple times. We run this for $2mn^2$ steps, and analyze as if we run the experiment m times.

When we split this into m blocks each, the probability of failure in each block is at most $\frac{1}{2}$, with the condition that no satisfying assignment was found in the previous block.

For Markov Chains, the history does not matter, so if we aren't able to find a satisfying assignment after $(2n^2) \cdot i$ steps, it's as if we just started a new trial with an "arbitrary assignment".

Hence, the probability of success in finding a satisfying assignment after $2mn^2$ steps is

$$1 - \frac{1}{2^m}$$

If we take $m = n$, the runtime ends up to be $2n^3 = \mathcal{O}(n^3)$, with success probability $1 - \frac{1}{2^n}$.