2. Inequalities & Distributions

Markov's Inequality Proof

Assuming the random variable X is non-negative.

$$
\begin{aligned}
E[X] &= \sum_x x \cdot Pr[X = x] \\
&\geq \sum_{x \geq a} x \cdot Pr[X = x] \\
&\geq a \cdot \sum_{x \geq a} Pr[X = x] \\
&= a \cdot Pr[X \geq a]
\end{aligned}
$$

Hence,

$$Pr[X \geq a] \leq \frac{E[X]}{a}$$

Alternative Proof of Markov's Inequality

This can also be proven using this method

$$
\begin{aligned}
E(X) = P(X < a) \cdot E(X|X < a) + P(X \geq a) \cdot E(X|X < a) \\
E(X) \geq P(X \geq a) \cdot E(X|X \geq A) \\
E(X) \geq a \cdot P(X \geq A) \\
P(X \geq a) \leq \frac{E(X)}{a}
\end{aligned}
$$

Chebyshev's Inequality Proof

We make use of Markov's Inequality to get a tighter inequality, known as the Chebyshev's inequality
Instead of taking $X \geq a, E[X]$, we can replace X, to work with $|x - \mu| \geq a, E[|x - \mu|]$.

Therefore

$$P(|x - \mu| \geq a) \leq \frac{E[|x - \mu|]}{a}$$

But, we know that

$$P(|x-\mu| \geq a) = P((x-\mu)^2 \geq a^2)$$

Hence, we have

$$P((x-\mu)^2 \geq a^2) \leq \frac{E[(x-\mu)^2]}{a^2}$$

We know that $E[(x-\mu)^2] = Var(X)$, we can rewrite this as

$$P(|x-\mu| \geq a) \leq \frac{Var(X)}{a^2}$$

Hence proved.

Geometric distribution

If the probability of success is p, the expected number of trials to get a success is

$$\frac{1}{p}$$

In a geometric series, the random variable is defined as

$$X = \begin{cases} 1 & \text{with probability } \ p \\ 0 & \text{with probability } \ 1-p \end{cases}$$

We can prove the expected value below

$$\begin{aligned} E[X] &= (1-p) \cdot (1 + E[x]) + p \\ E[X] &= 1 + E[x] - p - pE[x] + p \\ (1-(1-p))E[X] &= 1 \\ E[X] &= \frac{1}{p} \end{aligned}$$

Linearity of Expectation

The expected value of the sum of random variables is equal to the sum of their individual expected values **irrespective of whether they are independent or not.**

Proof

Assume there are 2 random variables X, Y which may or may not be independent.

$$
\begin{aligned}
E[X+Y] &= \sum_x \sum_y [(x+y) \cdot P(X=x, Y=y)] \\
&= \sum_x \sum_y [x \cdot P(X=x, Y=y)] + \sum_y \sum_x [y \cdot P(X=x, Y=y)] \\
&= \sum_x x \sum_y P(X=x, Y=y) + \sum_y y \sum_x P(X=x, Y=y) \\
&= \sum_x xP(X=x) + \sum_y yP(Y=y) \\
&= E[X] + E[Y]
\end{aligned}
$$

Hence, we've showed that linearity of expectation holds without using the property that X, Y are independent.