14. MaxCut

Cut - A partition of vertices into 2 disjoint sets and value of the cut is the weight of all edges crossing the cut.

MAXCUT: Given a graph $G = (V, E)$, such that al the weight of edges in E is 1, find the maximum value of a cut attainable in G.

Randomized Algorithm to Approximate

This problem is NP-hard, but we can try approximating using randomness. We take each vertex in V, and with probability $\frac{1}{2}$ put it in A, else put it in B .

Theorem - Let $G = (V, E)$ be an undirected graph on n vertices and m edges. There exists a partition of V into disjoint sets A and B such that the cut value is at least $\frac{m}{2}$.
Proof - We make use of the approach mentioned above, and name the edges $e_1, e_2, \ldots, e_m$
We define X_i for each edge in the set

$$X_i = \begin{cases} 1 & \text{if edge } i \text{ connects } A \text{ to } B \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}[X_i = 1] = \mathbb{P}[\text{Endpoints are in different sets}] = \frac{1}{2}$$

Hence,

$$\mathbb{E}[X_i] = \frac{1}{2}$$

$$\mathbb{E}[|Cut(A, B)|] = \mathbb{E}\left[\sum_{i=1}^{m} X_i\right] = \sum_{i=1}^{m} \mathbb{E}[X_i] = \frac{m}{2}$$

Since the expected value is $\frac{m}{2}$, there exists a partition A, B such that at least $\frac{m}{2}$ edges connect A to B (proven using *reverse markov*).

Expected number of trials to find cut value $\geq \frac{m}{2}$

Take p to be the probability that a random cut has $\geq \frac{m}{2}$ edges.

$$
\begin{aligned}
\frac{m}{2} &= \mathbb{E}[|Cut(A,B)|] \\
&= \sum_{i \leq \frac{m}{2}-1} i \cdot \mathbb{P}[|Cut(A,B)| = i] + \sum_{i \geq \frac{m}{2}} i \cdot \mathbb{P}[|Cut(A,B)| = i] \\
&\leq \left(\frac{m}{2} - 1\right) \cdot (1-p) + m \cdot p \\
&= \frac{m}{2} - 1 - \frac{mp}{2} + p + mp \\
\implies p &\geq \frac{1}{\frac{m}{2}+1}
\end{aligned}
$$

Therefore, the expected number of trials is $\frac{1}{p} \leq \frac{m}{2} + 1$.

Derandomization using Conditional Expectation

Instead of placing vertices in A or B uniformly and independently like the earlier method, we now place vertices in a deterministic way, one at a time in an arbitrary order $v_1, v_2, \ldots, v_n$

Define x_i to be the set where v_i is placed

$$
x_i = \begin{cases} A & \text{if } v_i \text{ is in } A \\ B & \text{if } v_i \text{ is in } B \end{cases}
$$

Suppose the first k vertices are already placed. We then define the expected value of the cut as randomizing over the $n-k$ vertices left, while fixing the first k.

$$
\mathbb{E}[|Cut(A,B)||x_1, x_2, \ldots, x_k]
$$

Algorithm -
We'll use the algorithm that given you've placed k vertices, you decide the set to put the $k+1$ vertex based on which one gives a higher expected value, i.e pick the set such that

$$
\mathbb{E}[|Cut(A,B)||x_1, \ldots, x_k] \leq \mathbb{E}[|Cut(A,B)||x_1, \ldots, x_k, x_{k+1}]
$$

We claim this algorithm will give us a cut of size $\geq \frac{m}{2}$ by proving using induction.

Inductive Proof

Base Case

$$
\mathbb{E}[|Cut(A,B)|] = \mathbb{E}[|Cut(A,B)||x_1]
$$

Doesn't really matter which set you put v_1 in, as the case is symmetric.

Induction Step

We need to show that

$$\mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k] \leq \mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k,x_{k+1}]$$

If we placed v_{k+1} randomly in one of the sets A, B, then we'd get

$$\begin{aligned}\mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k,x_{k+1}] &= \frac{1}{2}\mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k,Y_{k+1}=A] \\ &+ \frac{1}{2}\mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k,Y_{k+1}=B] \\ &\leq \max\{I, II\}\end{aligned}$$

where

$$I = \mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k,Y_{k+1}=A]$$

and

$$II = \mathbb{E}[|Cut(A,B)||x_1,\ldots,x_k,Y_{k+1}=B]$$

If we compute I, II, we can just take the max of them, and increase our expected value accordingly.

Since we start with an expected value $\frac{m}{2}$, and only increase as we place more and more vertices, we'll end up with a deterministic cut size $\geq \frac{m}{2}$.

Greedy Algorithm - Place a vertex in the side with fewer neighbours and break ties arbitrarily.

This algorithm also always guarantees a cut with at least $\frac{m}{2}$ edges.