16. Random Walks

Markov Chain

A generalized random walk on a directed graph.
$G = (V, E)$
$|V| = n, |E| = m$

We have a *transition probability matrix* $P_{n \times n}$
P_{ij} : Probability that j is the next state given i is the current state.
X_t : State of the markov chain at time t.
$p_i(t)$: Probability of being at state s_i at time t
Using this, we can define a probability vector

$$p_t = (p_1(t), p_2(t), \ldots, p_n(t))$$

Memory-Less

$$\mathbb{P}[X_t = a_t | X_{t-1} = a_{t-1}, \ldots, X_0 = a_0] = \mathbb{P}[X_t = a_t | X_{t-1} = a_{t-1}] = P_{a_{t-1}, a_t}$$

Hence, the future behaviour of the Markov Chain is only dependent upon the current state.

we know that

$$p_{t+1} = p_t P$$
$$\implies p_{t+m} = p_t \cdot P^m$$

Irreducible Markov Chain

A markov chain is irreducible if G is strongly connected, $\forall s_i \neq s_j$, $\exists$ a directed path from $s_i \to s_j$.

$$\forall i \neq j \;\; \mathbb{P}[\exists m > 0, X_{t+m} = s_j | X_t = s_i] > 0$$

Period of a State

We define the period of a state s_i in a markov chain as

$$d(s_i) = \gcd(\{t \in N | (P^t)_{ii} > 0, t > 0\})$$

Essentially, we find the time periods at which the probability of being at position i is non-zero, and find the gcd of the time periods to get the period of the state.

Say we get the period as $d(s_i) = T$ for some state, and at time t, we get $p_t(i) > 0$, then t belongs to the AP

$$\{a + i \cdot T | i \geq 0\}$$

Aperiodic -
A markov chain is aperiodic if $d(s_i) = 1 \, \forall s_i \in \{s_1, s_2, \ldots, s_n\}$

Other Properties
$p_t(i, j) = \mathbb{P}[X_t = s_j | X_0 = s_i]$ (Probability of being at s_j at time t given you start at s_i)
$r_t(i, j) = \mathbb{P}[X_t = s_j \text{ and } \forall i \leq u \leq t - 1, X_u \neq s_j | X_0 = s_i]$ (First visit to s_i from s_j)

Starting from s_i probability of a transition into state s_j is given by

$$f_{ij} = \sum_{t>0} r_t(i, j)$$

Expected number of time steps to reach s_j starting from s_i.

$$h_{ij} = \sum_{t>0} t \cdot r_t(i, j)$$

Theorem - If a markov chain is finite, irreducible and aperiodic, then $\exists T < \infty$ such that $(P^t)_{ij} > 0, \forall t \geq T, \forall i, j \in V$, $P_{t+1} = P_t$.

Stationary Distribution - A probability distribution π for a markov chain such that

$$\pi = \pi P$$

Fundamental Theorem of Markov Chains

For any finite, irreducible and aperiodic markov chain,

1. There exists a unique stationary distribution π such that $\forall 1 \leq i \leq n$, $\pi_i > 0$
2. $\forall 1 \leq i \leq n$, $f_{ii} = 1$, $h_{ii} = \frac{1}{\pi_i}$
3. Let $N(i, t)$ be the number of time the markov chain visits state i in t steps, then $\lim_{t \to \infty} \frac{N(i,t)}{t} = \pi_i$

Random Walks

We define the transition probability matrix as

$$P_{u,v} = \begin{cases} \frac{1}{deg(u)} & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

Given the graph $G = (V, E)$, we work with a non-bipartite, undirected & connected graph.
Claim - G is aperiodic
Proof -
We know cycles of length 2 exist already for every state. To ensure the period of every state is 1, we just need to prove every state has an odd cycle as well.
What non-bipartiteness ensures is that there exists a cycle in the graph with an odd number of nodes. Any state, whether inside the cycle or outside, can use this to create a cycle which is of odd length.
Going and coming back to the cycle for a state have the same length, hence making it even as we do it twice, and taking the cycle once gives us an odd length path.

Lemma - $\forall v \in V, \pi_v = \frac{deg(v)}{2m}$
Proof -
π is stationary if $\pi = \pi P$. Therefore, $\forall v \in V$

$$\pi_v = (\pi P)_v = \sum_u \pi_u P_{uv}$$

We are claiming that $\pi_u = \frac{deg(u)}{2m}$ is a solution.

$$\sum_u \pi_u P_{uv} = \sum_{u,(u,v)\in E} \frac{deg(u)}{2m} \cdot \frac{1}{deg(u)} = \frac{deg(v)}{2m} = \pi_v$$

Hence proved.

And from the fundamental theorem, we know this stationary distribution is unique.

Pagerank Algorithm

1. Each node in the graph is assigned a pagerank value of $\frac{1}{n}$.
2. Each node divides their page rank value by their out-degree, and send it to each neighbour.

3. Pagerank of each node is updated to sum of all values received in the previous iteration.

Can be modelled as a markov chain, with the transition probability matrix

$$P_{ij} = \begin{cases} \frac{1}{d_{\text{out}}(i)} & \text{if page } i \text{ links to } j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{pagerank}^t(j) = \sum_{i,(i,j)\in E} \frac{\text{pagerank}^{t-1}(i)}{d_{\text{out}}(i)}$$

If the web is finite, irreducible and aperiodic, we can apply the fundamental theorem to reach a unique stationary distribution.