Maths Assignment - 2

Q.1) Find total number of possible isomorphisms of simple undirected graphs with 4 vertices. Also calculate number of graphs possible in each such isomorphism classes.

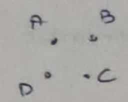
→ So for 4 vertices there will be ${}^4C_2 = 6$ possible edges.

$\therefore 2^6 = 64$ labelled simple graphs in total.

No. of unlabelled simple graphs = 11 classes (non-isomorphic)

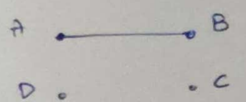
Isomorphic classes are as follows:-

1) Empty graph - 4 vertices 0 edges



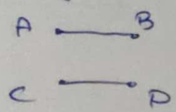
No. of labelled graphs = $\boxed{1}$

2) One edge (K_2 + 2 isolated)



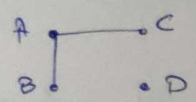
No. of labelled graphs = ${}^4C_2 = \boxed{6}$

3) 2 disjoint edges ($2K_2$)



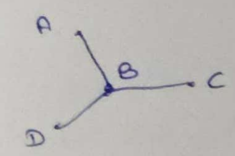
No. of labelled graphs = $\boxed{3}$

4) Path of length 2 + isolated vertex



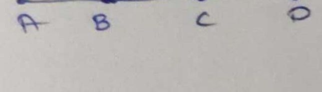
No. of labelled graphs = $4 \times {}^3C_2 = \boxed{12}$

5) Three edges - Star $K_{1,3}$



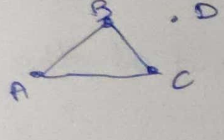
No. of labelled graphs = $\boxed{4}$

6) Three edges - Path on 4 vertices P_4



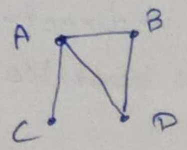
No. of labelled graphs = ${}^4C_2 \times 2 = \boxed{12}$

7) Three edges - Δ + one isolated



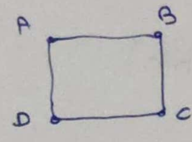
No. of labelled graphs = ${}^4C_3 = \boxed{4}$

8) Four edges (complement of P_3 + isolated class)



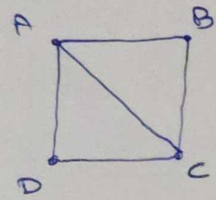
No. of labelled graphs = $\boxed{12}$

9) Four edges (C_4)



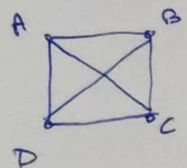
No. of labelled graph = $\boxed{3}$

10) Five edges (K_4 + minus one edge)



No. of labelled graphs = $\boxed{6}$

11) Six edges - Complete graph



No. of labelled graphs = $\boxed{1}$

$\therefore$ Total graphs = $\boxed{64}$

Q.2) Prove or disprove: If G is an Eulerian graph with edges e, f that share a common vertex, then G has an eulerian circuit in which e & f appears as consecutively.

→ Statement is true.

Let G be eulerian (connected & every vertex has even degree).

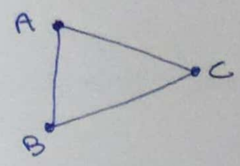
Let v be common endpoint of edges e & f.

i) Follow a cycle starting with e.

Start at 'v' & traverse edge e & then continue walking along unused edges, always taking unused edge out of the current edge.

2) If all edges are used, C_1 is an Eulerian circuit. Check if edge immediately after e in C_1 is f. If yes → Done. Else, you can rotate C_1 so that v-occurrence after e is the place where you splice in cycle containing f.

3) If unused edges remain, proceed as in pick any vertex that lies on current closed trail & that has unused incident edges.



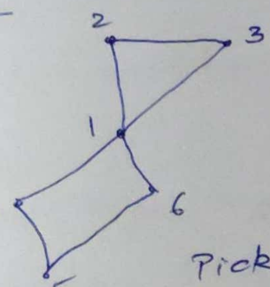
Eg 1

Eulerian Let e = AC f = AB

One eulerion circuit = B → C → A → B

Edges in order : $BC, \underbrace{CA}_{e}, \underbrace{AB}_{f}$

Eg 2



Deg (1) = 4, Rest Deg = 2

∴ Graph = Eulerian

Pick edges that share vertex 1:

e = 1 → 2 f = 1 → 4

∴ Eulerian circuit : [1 → 3 → 2 → 1 → 4 → 5 → 6 → 1]

8.3) Prove by induction on number of vertices n, that maximum number of edges in a simple non-Hamiltonian graph is $^{n-1}C_2 + 1$.

→ Let n = 3

∴ 3: $^2C_2 + 1 = 2$

On 3 vertices, a non-Hamiltonian simple graph is a path with 2 edges (K_3 Δ is Hamiltonian). So, maximum number of edges is 2 for non-Hamiltonian graph.

Let G be non-Hamiltonian graph

$|E(G)| \leq \binom{n-1}{2} + 1$ ... To prove

This is because G was chosen to be edge-maximal subject to non-Hamiltonian, adding any missing edges to G creates Hamiltonian cycle.

Let u, v be vertices in G, $\therefore$ G + uv = Hamiltonian

To find path P: P : $x = v_1, v_2, \ldots v_n = y$
n vertices

& x-y is path

Remove endpoint y & let $G' = G - y$.

G' = graph on n-1 vertices

By induction hypothesis,

$$|E(G')| \leq {}^{n-2}C_2 + 1$$

Now, since, y = not adjacent to x

$$\deg(y) \leq n-2$$

$$\therefore |E(G)| = |E(G')| + d(y) \leq {}^{n-2}C_2 + 1 + n-2$$

$${}^{n-2}C_2 + 1 + n-2 = \frac{(n-2)(n-3)}{2} + n-1$$

$$= \frac{(n-2)(n-3) + 2n-2}{2}$$

$$= \frac{n^2 - 3n + 4}{2} \quad \ldots \left(\frac{(n-1)(n-2)}{2} + \frac{2}{2}\right)$$

$$\left[\, |E(G)| \leq {}^{n-1}C_2 + 1 \,\right]$$

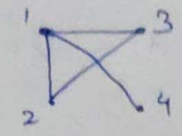
Hence, we found that for non-Hamiltonian graph G, no. of edges will be less than or equal to ${}^{n-1}C_2 + 1$.

Hence proved.

Let $n=4$

No. of edges $= {}^{3}C_2 + 1 = 4$.

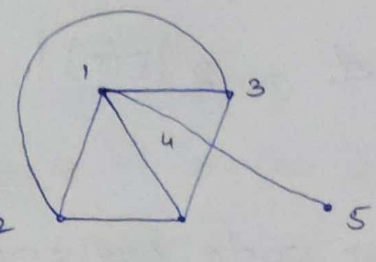
Now, non-Hamiltonian graph =



Let $n=5$

No. of edges $= {}^{4}C_2 + 1 = 7$

Now, non-Hamiltonian graph =



Q.4) Prove that bi-partite graph $K_{n,n}$ has $\frac{(n-1)!\,n!}{2}$ Hamiltonian cycles.

→ Let bipartition be $A = \{a_1, a_2, \dots a_n\}$
$B = \{b_1, b_2, \dots b_n\}$

Any Hamiltonian cycle in bipartite graph must alternate between A & B.

∴ Hamiltonian cycle is of form: $a_{i_1} - b_{j_1} - a_{i_2} - b_{j_2} - \dots - a_{i_n} - b_{j_n} - a_{i_1}$.

visiting each a-vertex & b-vertex only once.

Let a_1 be fixed starting vertex.

1) Now, to choose n vertices of B for filling all the b-positions $= n!$

2) For A vertices choices are: $(n-1)!$ since a_1 is already fixed

∴ Total ways $= n! \cdot (n-1)!$

But we also need to make sure undirected cycle is not counted twice, so we divide it by 2.

∴ Total ways $= \frac{n!\,(n-1)!}{2}$ Hence proved.

Q.57 Let $d_1, d_2, \ldots d_n$ be +ve integers with $n \geq 2$. Prove that there exists a tree with degree sequence $d_1, d_2, \ldots, d_n$ if & only if $\sum_{i=1}^{n} d_i = 2n-2$.

→ If T = Tree on n vertices

$\therefore |E(T)| = n-1$ ... By handshaking lemma

$$\therefore \sum_{i=1}^{n} d_i = 2|E(T)| \quad \text{where } E(T) = \text{edges of } T$$

Use the Prüfer code sequence bijection labelled between trees on $\{1, 2, \ldots n\}$ & length $n-2$ sequences over $\{1, \ldots n\}$

Under the Prüfer correspondence a vertex 'i' appears exactly $d_i - 1$ times in Prüfer sequence, where d_i = degree of i in that tree.

$\therefore$ Multiset of multiplicities in Prüfer sequence $= \{d_1 - 1, \ldots d_n - 1\}$

$$\text{length of sequence} = \sum_{i=1}^{n} d_i - 1 = \left(\sum_{i=1}^{n} d_i\right) - n$$

Now, if $\sum_{i=1}^{n} d_i = 2n-2$ then

$$\left[\sum_{i=1}^{n} d_i - 1 = n-2\right]$$

Multiplicities sum exactly to n-2 length ... the required length.

$\therefore$ The tree exists with degree of $i = 1 + (d_i - 1) = d_i$

Hence proved by Prüfer sequence.

Q. 6) Let T be labelled tree on n-vertices. Count total no. of possible trees that has i) 2 leaves ii) n-2 leaves.

→ We use Prüfer code that says vertex v has degree = 1 + (no. of times v appears in Prufer sequence).

i) For a leaf i.e degree = 1 :-

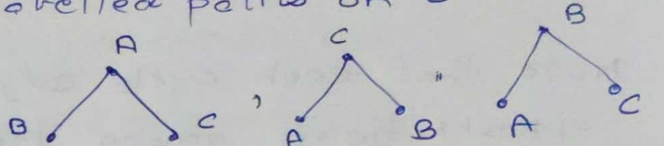
Exactly 2 labels must not appear in Prüfer sequence
(For 2 leaves)

So, the prüfer sequence will use exactly n-2 distinct labels.

No. of sequence of length n-2 = $n.(n-1)(n-3)\dots 3$

$$= \left[\frac{n!}{2}\right]$$

Eg: For n=3, we will have 3 labelled paths on 3 vertices which thus give 2 leaves.



ii) Exactly n-2 leaves:

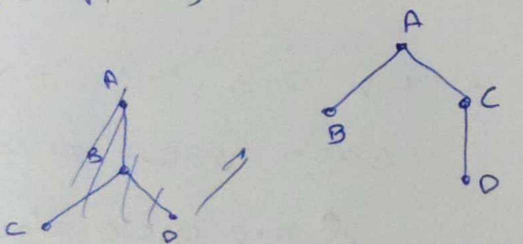
For exactly n-2 leaves, only 2 ~~leaves~~ nodes should appear in Prüfer sequence.

Choosing 2 labels (i, j) that appear in Prüfer sequence = $^nC_2$

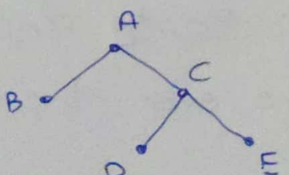
For length n-2 sequence, i, j appear = $2^{n-2} - 2$ times

$\therefore$ Total choices = $\left[(^nC_2) \times (2^{n-2} - 2) \right]$

Eg: n = 4, $\therefore$ Total 2 leaves trees are = $^4C_2 \times (2^2 - 2)$
= $\underline{12}$



For n=5, total 3 leaves trees are = $^5C_2 \times (2^3 - 2)$
= $\underline{60}$.



Q.7) Assign integer weights to each edges of the complete graph K_n. Prove that if the total weight on every Δ is even then total weight on every cycle is even.

→ Let us take a cycle $V_1 - V_2 - V_3 - \cdots V_k - V_1$

We break this cycle into Δs that share the first vertex v_1

↳ $\Delta_1 (v_1, v_2, v_3)$

$\Delta_2 (v_1, v_3, v_4)$

$\Delta_3 (v_1, v_4, v_5)$

$\vdots$

$\Delta_{k-2} (v_1, v_{k-1}, v_k)$

Each Δ uses one edge of main cycle & some edges that are not in cycle

Note that each cycle edge appears once & non-cycle edge appears twice among these triangles.

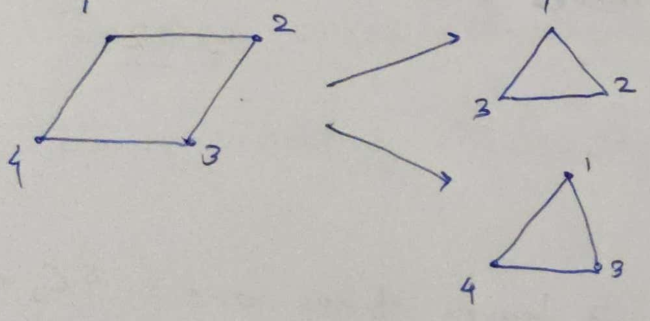
Total parity = cycle edges + non-cycle edges

↳ But this is even so we can ignore it.

∴ Total parity depends on cycle edges.

Thus, if total weight on every Δ is even then total weight on every edge will also be even. The converse is also true.

Eg: 1-2-3-4-1 be a cycle



∴ 1-2, 2-3, 4-1, 3-4 each appear once & are cycle edges.

∴ Parity depends only on cycle edges.

If both Δ's are even then cycles are even.

Q.8) Prove that every simple planar graph has a vertex of degree atmost 5.

→ Let G = simple planar graph with n vertices & m edges.

Euler's formula states for a simple planar graph with $n \geq 3$

$$m \leq 3n-6.$$

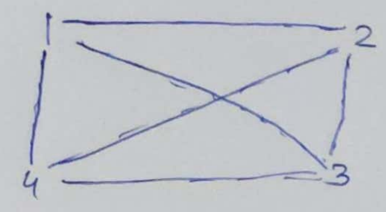
~~For~~ (Euler : $n - m + f = 2$ $\quad (2m \geq 3f)$)

$$\text{Average degree} = \frac{1}{n}\sum_{v \in V} \deg(v) = \frac{2m}{n} \leq \frac{2(3n-6)}{n} < \boxed{6}$$

Thus, average degree is $\underline{\underline{< 6.}}$

Max degree of vertex can be atmost 5, because if every vertex had degree ≥ 6, average would be > 6 which is not possible for simple planar graph.

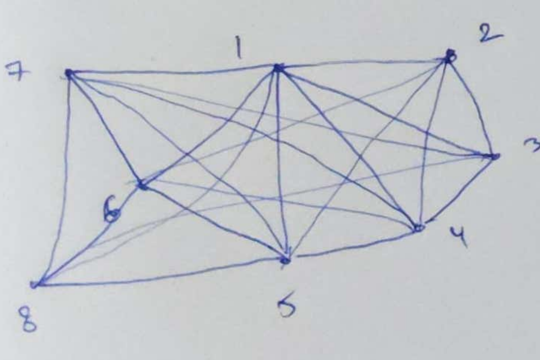
Eg:



All degrees are 3 (≤ 5)

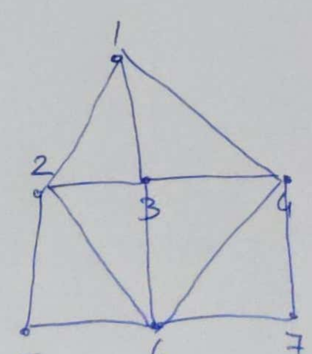
∴ Valid planar graph ✓

Eg: Lets try for a planar graph with 8 vertices, & every vertex ≥ 6 degree. Total edges = 18. But for maximum edges of planar graph = 12.



Not valid planar graph ×

Eg:



Valid planar graph ✓