

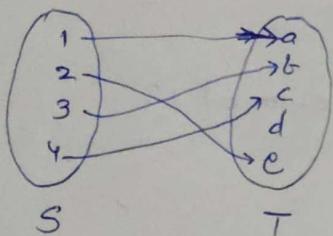
Mathematical foundations of Computer Science (MA 714)

Assignment - 1

Q.1) Let S be a set with n elements & T be a set with m elements. Calculate total number of one-one functions from S to T .

→ A function $f: S \rightarrow T$ is called one-one or injective function if different elements of S map to different elements of T (no two elements of S should have same image in T).

Eg:



One-One functions

S has n elements & T has m elements

So, there are 2 cases \Rightarrow If $n > m$

\Rightarrow If $n \leq m$

\Rightarrow If $n > m$: there are not enough elements in T to assign distinct images, so no one-one function exists.

\Rightarrow If $n \leq m$: 1. For first element of S , we have m choices

2. For 2nd element we have $m-1$ choices

& so on

2. For n^{th} element we will have $m-n+1$ choices

$$\therefore \text{Total number of one-one functions} = m \times (m-1) \times (m-2) \times \dots \times (m-n+1)$$

$$= \frac{m!}{(m-n)!}$$

Ans: Number of one-one functions: $[m! / (m-n)!]$

\hookrightarrow if $n \leq m$

&

\rightarrow if $n > m$.

Q.27 How many positive integers less than 10^6 have the sum of their digits equal to 19?

→ For any integer less than 10^6 will have at most 6 digits.
Let us say the 6 digits are d_1, d_2, d_3, d_4, d_5 & d_6 .
We know that, $[d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 19]$
& each $d_i \rightarrow 0 \leq d_i \leq 9$.

1) Count all solutions for (≤ 9):-

$$\text{Total integers} = {}^{19+6-1}_{19} C_r \dots [n+r-1 C_r]$$

sum digits
↓
to get sum of 19

$$\therefore {}^{24}C_{19} = \text{Total integers}$$

$$\therefore \text{Total integers} = \underline{42504}.$$

2) Now we need to subtract cases where digit ≥ 10 .

Suppose $d_1 \geq 10$

① [Let $d_1 = d_1' + 10$] with $d_1' \geq 0$

$$\therefore [d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 19] \quad ②$$

Substitute ① in ②

$$\therefore d_1' + d_2 + d_3 + d_4 + d_5 + d_6 = 19 - 10$$

$$\therefore [d_1' + d_2 + d_3 + d_4 + d_5 + d_6 = 9]$$

$$\therefore \text{Total integers} = {}^{9+6-1}_{9} C_9$$

$$\text{Total integers} = {}^{14}C_9 = 12012$$

③ Now we calculate final count:

$$\begin{array}{l} \text{Final count} \\ \text{or} \\ \text{Total integers} \end{array} = 42504 - 12012 = \boxed{30492}$$

Ans: There are 30492 positive integers that have sum of digits equal to 19.

Q.37 Let n objects be given & you need to choose r objects from them with repetitions. Explain that the above process can be done in $\binom{n+r-1}{r}$ ways.

→ We need to choose r objects from n different types where repetitions are allowed.

This is similar to placing ' r ' identical stars into n labelled boxes (box 1 for x_1 , box 2 for x_2 , ...). The number x_i is simply how many stars in box i .

$$[x_1 + x_2 + \dots + x_n = r] \quad (x_i \geq 0)$$

Eg:- Let $n = 3$ (stars) $r = 4$

Suppose: String is (1, 2, 1)

* | ** | *

4 stars & (3-1)
↓ 2 bars

Every solution corresponds to unique string of length $r + (n-1) \rightarrow$ which has ' r ' stars exactly & $n-1$ bars.

Selecting ' r ' items from this will result in:

$$\binom{n+r-1}{r} \text{ ways.}$$

Eg: Suppose we have a coin with H & T. So let $r = 3$.

Now possible ' n ' can be: HHH HHT
 TTT THT

$$n=4 \text{ & } r=3$$

$$\text{Bars} = n-1 = 3.$$

HHHH | TTTT | THT | HHT

Now, we choose 3(r) elements from $n+r-1$ ways.

$$\therefore \binom{n+r-1}{r} \text{ ways}$$

Thus, for $x_1 + x_2 + \dots + x_n = r$ There are $(n-1)$ bars

$$\text{Total length} = n-1+r$$

To select ' r ' objects

$$\therefore \left[\binom{n+r-1}{r} \text{ ways or } \binom{n+r-1}{r} \text{ ways} \right]$$

Q.4) Prove that the coefficient of $x^a y^b z^c w^d$ in expansion of $(x+y+z+w)^n$ is $\frac{n!}{a! b! c! d!}$ where numbers a, b, c and $d \in \mathbb{N}$ satisfies $a+b+c+d = n$.

→ We know $[a+b+c+d = n] \dots (\text{given}) \quad \textcircled{1}$

1) First we select 'a' elements or factors from 'n' for x .

$$\therefore [{}^n C_a] \dots x^a$$

2) Now for selecting 'b' from rest elements for y , it will be

$$\therefore [{}^{n-a} C_b] \dots y^b \quad (\text{since } a \text{ elements have been already picked})$$

3) Now for selecting 'c' from rest elements for z , it will be

$$[{}^{n-a-b} C_c] \dots z^c \quad (\text{since } (a+b) \text{ elements have been already picked})$$

4) Now for selecting 'd' from rest elements for w , it will be

$$[{}^{n-a-b-c} C_d] \dots w^d \quad (\text{since } (a+b+c) \text{ elements have already been picked}).$$

$$\therefore \text{Total number of ways} = {}^n C_a \times {}^{n-a} C_b \times {}^{n-a-b} C_c \times {}^{n-a-b-c} C_d$$

$$= \frac{n!}{(n-a)! a!} \times \frac{(n-a)!}{(n-b)! b!} \times \frac{(n-a-b)!}{(n-c)! c!} \times \frac{(n-a-b-c)!}{1! d!}$$

$$\text{From } \textcircled{1} \quad d = n - a - b - c$$

$$\therefore \text{Total ways} = \frac{n!}{(n-a)! a!} \times \frac{(n-a)!}{(n-b-a)! b!} \times \frac{(n-a-b)!}{c! (n-a-b-c)!} \times \frac{d!}{d!} \quad (\text{some})$$

$$\text{Total ways} =$$

$$\boxed{\frac{n!}{a! b! c! d!}}$$

Hence proved.

Q.5) Show that in any list of ten non-negative integers (a_0, a_1, \dots, a_9) there is a string of consecutive items of list i.e. a_l, a_{l+1}, \dots where $l \in \{0, 1, \dots, 9\}$ whose sum $a_l + a_{l+1} + \dots$ is divisible by 10.

\rightarrow Eg: $m=10$ List $\rightarrow \{32, 97, 82, 67, 44, 29, 10, 13, 25, 71\}$

List $\% 10 = \{2, 7, 2, \underbrace{7, 4, 9, 0, 9, 5, 1}\}$

$$7+4+8=20 \text{ which is divisible by } \underline{\underline{10}}.$$

So now $67+44+29$ will also be divisible by 10.

Proof:

Sequence $\rightarrow (a_0, a_1, \dots, a_9)$

Let us define prefix sums

$$\text{For } p_0 = 0 \quad p_k = \sum_{i=0}^{k-1} a_i \quad k=1, \dots, 10$$

There will be 11 prefix sums [10 for 0-9 & one for 10th element]

Modulo 10 will have (0-9) remainders

① Some $p_k \equiv 0 \pmod{10}$

Sum of first k numbers is divisible by 10.

② ~~Some~~ No $p_k \equiv 0$

By pigeonhole principle, two prefix sums are congruent

i.e. $p_i \equiv p_k \quad (i < j)$

$$\text{Then, } p_j - p_i = (a_i + a_{i+1} + \dots + a_{j-1}) \equiv 0 \pmod{10}$$

So, sum of consecutive block is divisible by 10.

Thus, in every list of 10 integers, some consecutive sequence has sum divisible by 10.

Ans: There will always exist a consecutive block with sum divisible by 10.

Q.67 Let there be n persons attending a gathering. Each of them knows certain number of persons (minimum 0 & maximum $n-1$). Prove that there exists atleast two people who know the same number of persons.

→ ~~Let~~ There are ' n ' persons

So, each person can know between 0 & $n-1$ people.

But it is impossible to have both 0 & $n-1$ at same time.

This is because if a person knows 0 persons at a gathering.

So for rest persons maximum persons they can know can

be $(n-1)-1$ → -1 for person having know 0 persons.

So, the range will be $[0 \text{ to } n-2]$.

Or

if a person knows $n-1$ persons then there cannot be any person that knows 0 persons.

So, the range will be $[1 \text{ to } n-1]$.

By the pigeonhole principle, we can say that $n-2$ persons may have distinct number of knowing people. But the rest person will have atleast same count as other people.

Ans: By pigeonhole principle we prove that there will be atleast 2 people in the gathering knowing same number of people.