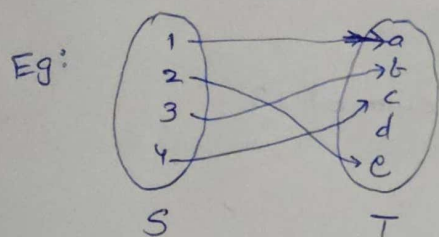


# Mathematical foundations of Computer Science (MA 714)

## Assignment - 1

Q.1) Let  $S$  be a set with  $n$  elements &  $T$  be a set with  $m$  elements. Calculate total number of one-one functions from  $S$  to  $T$ .

→ A function  $f: S \rightarrow T$  is called one-one or injective function if different elements of  $S$  map to different elements of  $T$  (no two elements of  $S$  should have same image in  $T$ ).



One-One functions

$S$  has  $n$  elements &  $T$  has  $m$  elements

So, there are 2 cases 1) If  $n > m$

2) If  $n \leq m$

1) If  $n > m$ : there are not enough elements in  $T$  to assign distinct images, so no one-one function exists.

□

2) If  $n \leq m$ :  
1. For first element of  $S$ , we have  $m$  choices  
2. For 2nd element we have  $m-1$  choices  
& so on  
n. For  $n^{\text{th}}$  element we will have  $m-n+1$  choices

∴ Total number of one-one functions =  $m \times (m-1) \times (m-2) \times \dots \times (m-n+1)$

$$= \frac{m!}{(m-n)!}$$

Ans: Number of one-one functions:  $\left[ \frac{m!}{(m-n)!} \right]$   
↳ if  $n \leq m$

&  
0 ↳ if  $n > m$ .

Q.27 How many positive integers less than  $10^6$  have the sum of digits equal to 19?

→ For any integer less than  $10^6$  will have at most 6 digits.  
Let us say the 6 digits are  $d_1, d_2, d_3, d_4, d_5$  &  $d_6$

We know that,  $[d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 19]$   
& each  $d_i \rightarrow 0 \leq d_i \leq 9$ .

17 Count all solutions for  $(\leq 9)$ :-

$$\text{Total integers} = \underset{\text{sum}}{19} + \underset{\text{digits}}{6} - 1 \binom{19}{19} \dots \left[ n+r-1 \binom{n}{r} \right]$$

↓  
to get sum of 19

$$\therefore {}^{24}C_{19} = \text{Total integers}$$

$$\therefore \text{Total integers} = \underline{42504}.$$

27 Now we need to subtract cases where digit  $\geq 10$ .

Suppose  $d_1 \geq 10$

$$\textcircled{1} [\text{Let } d_1 = d_1' + 10] \text{ with } d_1' \geq 0$$

$$\therefore [d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 19] \textcircled{2}$$

Substitute  $\textcircled{1}$  in  $\textcircled{2}$

$$\therefore d_1' + d_2 + d_3 + d_4 + d_5 + d_6 = 19 - 10$$

$$\therefore [d_1' + d_2 + d_3 + d_4 + d_5 + d_6 = 9]$$

$$\therefore \text{Total integers} = 6 \times {}^{(9+6-1)}C_9$$

$$\text{Total integers} = 6 \times {}^{14}C_9 = 12012$$

③ Now we calculate final count:

$$\begin{array}{l} \text{Final count} \\ \text{or} \\ \text{Total integers} \end{array} = 42504 - 12012 = \boxed{30492}$$

Ans: There are 30492 positive integers that have sum of digits equal to 19.

Q.3] Let  $n$  objects be given & you need to choose  $r$  objects from them with repetitions. Explain that the above process can be done in  $\binom{n+r-1}{r}$  ways.

→ We need to choose  $r$  objects from  $n$  different types where repetitions are allowed.

This is similar to placing ' $r$ ' identical stars into  $n$  labelled boxes (box 1 for  $x_1$ , box 2 for  $x_2, \dots$ ). The number  $x_i$  is simply how many stars in box  $i$ .

$$[x_1 + x_2 + \dots + x_n = r] \quad (x_i \geq 0)$$

Eg:- let  $n = 3$  (<sup>distinct</sup> stars)  $r = 4$

Suppose: string is (1, 2, 1)

\* | \*\* | \*

4 stars & (3-1) bars  
→ 2 bars

Every solution corresponds to unique string of length  $r + (n-1) \rightarrow$  which has ' $r$ ' stars exactly &  $n-1$  bars.  
Selecting ' $r$ ' items from this will result in:

$$\binom{n+r-1}{r} \text{ ways.}$$

Eg: Suppose we have a coin with H & T. So let  $r = 3$ .

Now possible 'n' can be: HHH HHT  
TTT THT

$$n = 4 \text{ \& } r = 3$$

$$\text{Bars} = n - 1 = 3.$$

H H H | T T T | T H T | H H T

Now, we choose 3 ( $r$ ) elements from  $n+r-1$  ways.

$$\therefore \binom{4+3-1}{3} \text{ ways}$$

Thus, for  $x_1 + x_2 + \dots + x_n = r$  There are  $(n-1)$  bars  
Total length =  $n-1 + r$   
To select ' $r$ ' objects

$$\therefore \left[ \binom{n+r-1}{r} \text{ ways or } \binom{n+r-1}{r} \text{ ways} \right]$$

Q.4) Prove that the coefficient of  $x^a y^b z^c w^d$  in expansion of  $(x+y+z+w)^n$  is  $\frac{n!}{a! b! c! d!}$  where numbers  $a, b, c, d \in \mathbb{N}$  satisfies  $a+b+c+d=n$ .

→ We know  $[a+b+c+d=n \dots \text{(given)}]$  ①

1) First we select 'a' elements of factors from 'n' for x.

$$\therefore {}^n C_a \dots x^a$$

2) Now for selecting 'b' from rest elements for y, it will be

$$\dots {}^{n-a} C_b \dots y^b \text{ (since a elements have been already picked)}$$

3) Now for selecting 'c' from rest elements for z, it will be

$$[{}^{n-a-b} C_c] \dots z^c \text{ (since (a+b) elements have been already picked)}$$

4) Now for selecting 'd' from rest elements for w, it will be

$$[{}^{n-a-b-c} C_d] \dots w^d \text{ (since (a+b+c) elements have already been picked)}.$$

$$\therefore \text{Total number of ways} = {}^n C_a \times {}^{n-a} C_b \times {}^{n-a-b} C_c \times {}^{n-a-b-c} C_d$$

$$= \frac{n!}{(n-a)! a!} \times \frac{(n-a)!}{(n-b)! b!} \times \frac{(n-a-b)!}{(n-a-b-c)! c!} \times \frac{(n-a-b-c)!}{1! d!}$$

$$\text{From ① } d = n-a-b-c \text{ (same)}$$

$$\therefore \text{Total ways} = \frac{n!}{(n-a)! a!} \times \frac{(n-a)!}{(n-b-a)! b!} \times \frac{(n-a-b)!}{c! (n-a-b-c)!} \times \frac{d!}{d!}$$

$$\text{Total ways} = \boxed{\frac{n!}{a! b! c! d!}}$$

Hence proved.

8.5) Show that in any list of ten non-negative integers  $(a_0, a_1, \dots, a_9)$  there is a string of consecutive items of list i.e.  $\{a_l, a_{l+1}, \dots\}$  where  $l \in 0, 1, \dots, 9$  whose sum  $a_l + a_{l+1} + \dots$  is divisible by 10.

Eg:  $m=10$  List  $\rightarrow \{32, 97, 82, 67, 44, 29, 10, 19, 25, 71\}$

List % 10 =  $\{2, 7, 2, 7, 4, 9, 0, 9, 5, 1\}$

$7 + 4 + 9 = 20$  which is divisible by 10.

So now  $67 + 44 + 29$  will also be divisible by 10.

Proof:

Sequence  $\rightarrow (a_0, a_1, \dots, a_9)$

Let us define prefix sums

$$\text{For } p_0 = 0 \quad p_k = \sum_{i=0}^{k-1} a_i \quad k=1, \dots, 10$$

There will be 11 prefix sums [10 for 0-9 & one for 10<sup>th</sup> element]

Modulo 10 will have (0-9) remainders

① Some  $p_k \equiv 0 \pmod{10}$

Sum of first  $k$  numbers is divisible by 10.

② ~~Some~~ No  $p_k \equiv 0$

By pigeonhole principle, two prefix sums are congruent

i.e.  $p_i \equiv p_j \pmod{10} \quad (i < j)$

Then,  $p_j - p_i = (a_i + a_{i+1} + \dots + a_{j-1}) \equiv 0 \pmod{10}$

So, sum of consecutive block is divisible by 10.

Thus, in every list of 10 integers, some consecutive sequence has sum divisible by 10.

Ans: There will always exist a consecutive block with sum divisible by 10.

8.6] Let there be  $n$  persons attending a gathering. Each of them knows certain number of persons (minimum 0 & maximum  $n-1$ ). Prove that there exists atleast two people who know the same number of persons.

→ ~~Let~~ There are ' $n$ ' persons

So, each person can know between 0 &  $n-1$  people.

But it is impossible to have both 0 &  $n-1$  at same time.

This is because if a person knows 0 persons at a gathering.

So for rest persons maximum persons they can know can be  $[(n-1)-1]$  → -1 for person having know 0 persons.

So, the range will be  $[0 \text{ to } n-2]$ .

Or

if a person knows  $n-1$  persons then there cannot be any person that knows 0 persons.

So, the range will be  $[1 \text{ to } n-1]$ .

By the pigeonhole principle, we can say that  $n-2$  persons may have distinct number of knowing people. But the rest person will have at least same count as other people.

Ans: By pigeonhole principle we prove that there will be atleast 2 people in the gathering knowing same number of people.