Write up on calculating degree exponent γ

The following fitting procedure has been described in detail in reference [1]. We start with a set of indegrees for N supplier nodes in the supply network of a focal company. So, for each supplier i the indegree is denoted by k_i. From this set of indegrees, we calculate the probability p_k^{real} of a supplier node having indegree k. This is done by counting the number of nodes with indegree k and divide by total number of nodes in the network. When calculated for all k ranging from maximum (k_{min}) to minimum (k_{min}), p_k^{real} describes the empirical indegree probability distribution. An indegree probability distribution that follows power-law has the form,

$$p_k \sim k^{-\gamma}$$

where k denotes the indegree and γ is called degree exponent. By fitting the empirical indegree probability distribution to the above describe formula, we aim to estimate degree exponent γ[2]. The data fitting process is done in following steps:

1. Choose a value K_{min} between Next, we estimate a value for degree exponent corresponding to this value of K_{min};

$$\gamma(K_{min}) = 1 + N\left[\sum_{i=1}^{N} ln\frac{k_i}{K_{\min} - 0.5}\right]^{-1}$$

2. For the pair of $(\gamma, K_{\min})$ parameters, the indegree probability distribution is given by

$$p_k^{fit} = \frac{1}{\zeta(\gamma, K_{\min})} k^{-\gamma}$$

   and, it's associated cumulative distribution function (CDF) is given by

$$P_k^{fit} = 1 - \frac{\zeta(\gamma, k)}{\zeta(\gamma, K_{\min})}$$

Note: $\zeta(\gamma, k)$ is called Hurwitz zeta function [3] which is a generalization of the Riemann zeta function, so it is also known as the generalized zeta function. It is defined by the formula

$$\zeta(\gamma, k) = \sum_{x=0}^{\infty} (x + k)^{-\gamma}$$

3. Next, we determine the maximum distance D between the CDF of the real data P_k^{real} and the fitted CDF P_k^{fit}. This in literature is called Kormogorov-Smirnov test,

$$D(K_{min}) = max_{k \geq K_{min}} |P_k^{real} - P_k^{fit}|$$

4. We calculate the maximum distance D for all possible values of K_{min} between maximum indegree (k_{min}) and minimum indegree (k_{min}). We find the K_{min} for which parameter D minimize and then fix that value as the small-indegree cut-off on the fitted indegree distribution.
5. Finally, we calculate the degree exponent γ as described in Step 1. using K_{min} that minimize D. Further, the standard error for the obtained degree exponent is,

$$\sigma_\gamma = \frac{1}{\sqrt{N\left[\frac{\zeta''(\gamma, K_{min})}{\zeta(\gamma, K_{min})} - \left(\frac{\zeta'(\gamma, K_{min})}{\zeta(\gamma, K_{min})}\right)^2\right]}}$$

As an example, we present result for 5-Tier supply network for "Company X" FactSet ID: '1228' for the Quarter- 1 [April 3rd, 2003 - July 1st, 2003]. The number of supplier nodes is N=3967 and L= 25264 number of links between the nodes. We only consider indegree which is the number of links coming into a node. Based on the fitting procedure we found the maximum distance D is minimized for $K_{min} = 2$ and the degree exponent γ is 2.872 with standard error, $\sigma_\gamma = 0.026$.

Reference

[1] Barabási, A. L. (2016). The scale-free property. *Network science*. Cambridge university press.

[2] A. Clauset, C.R. Shalizi, and M.E.J. Newman. Power-law distributions in empirical data. SIAM Review S1: 661-703, 2009.

[3] Weisstein, Eric W. "Hurwitz Zeta Function." Wolfram Research, Inc. (2002).